## Dual Methods & ADMM:-

**Problem:-** $\min_{x,z} \left(f(x)+g(z)\right)$, s.t $h(x,z)=0$

**Augmented Lagrangian:-**

$$F(x,z;y) = f(x) + g(z) + y^T h(x,z) + \frac{\rho}{2}\|h(x,z)\|_2^2$$

$$\bar{x} = \operatorname{argmin}_x F(x,z;y)$$

$$\bar{z} = \operatorname{argmin}_z F(\bar{x},z;y)$$

$$\bar{y} = y + \rho h(\bar{x},\bar{z})$$

$y \to$ lagrange multiplier

$\rho \to$ penalty constraint

**Lasso Problem:-**

$$\min_x \left[\frac{1}{2}\|Ax-b\|_2^2 + \lambda\|x\|_1\right]$$

$$= \min_{x,z} \left[\frac{1}{2}\|Ax-b\|_2^2 + \lambda\|z\|_1\right) \text{ s.t. } x-z=0$$

$$F(x,z;y) = \min_{x,z} \left[\frac{1}{2}\|Ax-b\|_2^2 + \lambda\|z\|_1\right] + y^T(x-z) + \frac{\rho}{2}\|x-z\|_2^2$$

Update eq$^n$:-

$$\bar{x} = \underset{x}{\operatorname{argmin}}\ F(x, z; y)$$

$$\bar{x} = \underset{x}{\operatorname{argmin}} \left[ \frac{1}{2}\|Ax-b\|_2^2 + \lambda\|z\|_1 \right] + y^T(x-z) + \frac{\rho}{2}\|x-z\|_2^2$$

$$\frac{\partial F(x,z;y)}{\partial x} = 0$$

$$\frac{1}{\cancel{2}}\left(\cancel{2}(Ax-b)\cdot A + y^T + \frac{\rho}{\cancel{2}} \times \cancel{2}(x-z) = 0\right.$$

$$(Ax-b)\cdot A + y^T + \rho(x-z) = 0$$

$$A^TAx - A^Tb + y^TI + \rho I x - \rho z = 0$$

$$(A^TA + \rho I)x = A^Tb + \rho z - y^T$$

$$\boxed{\bar{x} = (A^TA + \rho I)^{-1}(A^Tb + \rho z - y^T)}$$

$$\bar{z} = \underset{z}{\operatorname{argmin}}\ F(\bar{x}, z; y)$$

$$\bar{z} = \underset{z}{\operatorname{argmin}} \left[ \frac{1}{2}\|A\bar{x}-b\|_2^2 + \lambda\|z\| + y^T(\bar{x}-z) + \frac{\rho}{2}\|\bar{x}-z\|_2^2 \right]$$

$$\frac{\partial F}{\partial z} = 0$$

$$\lambda - y^T + \frac{\rho}{\cancel{2}}\left(\cancel{2}(\bar{x}-z)\right)(-1) = 0$$

$$\lambda - y^T - \rho\bar{x} + \rho z = 0$$

$$\rho z = \rho\bar{x} + y^T - \lambda$$

$$\boxed{\bar{z} = \bar{x} + \frac{y^T}{\rho} - \lambda/\rho}$$

$$\bar{y} = y + \rho\, h(\bar{x}, \bar{z})$$

$$\boxed{\bar{y} = y + \rho(\bar{x} - \bar{z})}$$