```
In [1]: import numpy as np
    import matplotlib.pyplot as plt
    import pandas as pd
    import ipysheet as ips
    import panel as pn
    pn.extension("katex", "mathjax")
```

# **Tutorial 2**

- · tutorial problems on Darcy's law and intrinsic permeability
- · homework problems on Darcy's law and intrinsic permeability

## **Tutorial Problems on**

- · Darcy's Law and
- Intrinsic Permeability

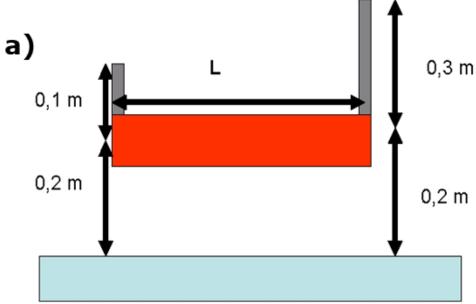
## **Tutorial Problem 7: Flow Direction and Hydraulic Gradient**

Indicate the direction of flow shown in the figure in next slides, and determine the hydraulic gradient for a Darcy column with length L = 50 cm! (Figures not to scale.)

#### **Tutorial Problem 7 – Solution**

The relevant topic is covered in Lecture 04, slide 8

```
In [100]: png_pane = pn.pane.PNG("images/T02_TP7_a1.png", width=600)
png_pane
Out[100]:
```



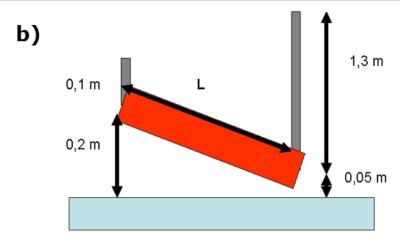
```
In [101]: # Image (a)
L = 0.5 # m, length of the column
Ea_hl = 0.2 #, m, elevation head, left
Pa_hl = 0.1 #, m pressure head, left
Ea_hr = 0.2 #, m, elevation head, right
Pa_hr = 0.3 #, m pressure head, right
Ha_hl = Ea_hl + Pa_hl # m, hydraulic head, left
Ha_hr = Ea_hr + Pa_hr # m, hydraulic head, right
DH_a = Ha_hr - Ha_hl
H_ga = DH_a/L#, no unit, hydraulic gradient

print("Hydraulic head LEFT: {0:1.1f}".format(Ha_hl),"m"); print("Hydraulic head RIGHT:: {0:1.1f}".format(Ha_hr),"m")
print("Hydraulic Head Difference: {0:1.1f}".format(DH_a),"m");print("Hydraulic gradient: {0:1.1f}".format(H_ga))
png_pane.object = "images/T02_TP7_a2.png"
```

Hydraulic head LEFT: 0.3 m Hydraulic head RIGHT:: 0.5 m Hydraulic Head Difference: 0.2 m Hydraulic gradient: 0.4

```
In [92]: png_pane2 = pn.pane.PNG("images/T02_TP7_b1.png", width=500)
png_pane2
```

Out[92]:

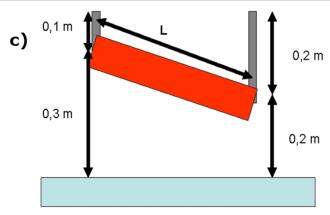


```
In [93]: # Image (b)
L = 0.5 # m, length of the column
Eb_hl = 0.2 #, m, elevation head, left
Pb_hl = 0.1 #, m pressure head, left
Eb_hr = 0.05 #, m, elevation head, right
Pb_hr = 1.3 #, m pressure head, right
Hb_hl = Eb_hl + Pb_hl # m, hydraulic head, left
Hb_hr = Eb_hr + Pb_hr # m, hydraulic head, right
DH_b = Hb_hr - Hb_hl
H_gb = DH_b/L#, no unit, hydraulic gradient
print("Hydraulic head LEFT: {0:1.1f}".format(Hb_hl),"m");print("Hydraulic gradient: {0:1.1f}".format(Hb_hr),"m")
print("Hydraulic Head Difference: {0:1.1f}".format(DH_b),"m");print("Hydraulic gradient: {0:1.1f}".format(H_gb))
png_pane2.object = "images/T02_TP7_b2.png"
```

Hydraulic head LEFT: 0.3 m Hydraulic head RIGHT:: 1.4 m Hydraulic Head Difference: 1.1 m Hydraulic gradient: 2.1

```
In [94]: png_pane3 = pn.pane.PNG("images/T02_TP7_c1.png", width=400)
png_pane3
```

#### Out[94]:



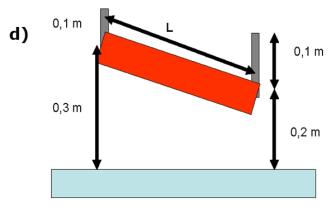
```
In [95]: # Image (c)
L = 0.5 # m, length of the column
Ec_hl = 0.3 #, m, elevation head, left
Pc_hl = 0.1 #, m pressure head, left
Ec_hr = 0.2 #, m, elevation head, right
Pc_hr = 0.2 #, m pressure head, right
Hc_hl = Ec_hl + Pc_hl # m, hydraulic head, left
Hc_hr = Ec_hr + Pc_hr # m, hydraulic head, right
DH_c = Hc_hr - Hc_hl
H_gc = DH_c/L#, no unit, hydraulic gradient
print("Hydraulic head LEFT: {0:1.1f}".format(Hc_hl),"m");print("Hydraulic head RIGHT:: {0:1.1f}".format(Hc_hr),"m")
print("Hydraulic Head Difference: {0:1.1f}".format(DH_c),"m");print("Hydraulic gradient: {0:1.1f}".format(H_gc))
png_pane3.object = "images/T02_TP7_c2.png"
```

Hydraulic head LEFT: 0.4 m Hydraulic head RIGHT:: 0.4 m Hydraulic Head Difference: 0.0 m

Hydraulic gradient: 0.0

```
In [96]: png_pane4 = pn.pane.PNG("images/T02_TP7_d1.png", width=400)
png_pane4
```

#### Out[96]:



```
In [97]: # Image (d)
L = 0.5 # m, length of the column
Ed_hl = 0.3 #, m, elevation head, left
Pd_hl = 0.1 #, m pressure head, left
Ed_hr = 0.2 #, m, elevation head, right
Pd_hr = 0.1 #, m pressure head, right
Hd_hl = Ed_hl + Pd_hl # m, hydraulic head, left
Hd_hr = Ed_hr + Pd_hr # m, hydraulic head, right
DH_d = Hd_hr - Hd_hl
H_gd = DH_d/L#, no unit, hydraulic gradient
#output
print("Hydraulic head LEFT: {0:1.1f}".format(Hd_hl),"m");print("Hydraulic head Right:: {0:1.1f}".format(Hd_hr),"m")
print("Hydraulic Head Difference: {0:1.1f}".format(DH_d),"m");print("Hydraulic gradient: {0:1.1f}".format(H_gd))
png_pane4.object = "images/T02_TP7_d2.png"
```

Hydraulic head LEFT: 0.4 m Hydraulic head Right:: 0.3 m Hydraulic Head Difference: -0.1 m Hydraulic gradient: -0.2

#### **Tutorial Problem 8**

The hydraulic conductivity of a fine sand sample was found to be equal to  $1.36 \times 10^{-5}$  m/s in a Darcy experiment using water at a temperature of  $20^{\circ}$  C. What is the intrinsic permeability of this sample? Give results in cm<sup>2</sup> and D. (density of water at  $20^{\circ}$  C: 998.2 kg/m<sup>3</sup>; dynamic viscosity of water at  $20^{\circ}$  C:  $1.0087 \times 10^{-3}$  Pa·s; 1 D =  $0.987 \times 10^{-12}$  m<sup>2</sup>)

#### **Tutorial Problem 8 - Solution**

Relevant topics are covered in Lecture 04 slides 18-20.

Relationship between hydraulic conductivity K and intrinsic permeability k from lecture notes:

$$K_{water} = k \cdot rac{
ho_{water} \cdot g}{\eta_{water}}$$

Solve for . k

$$k = rac{\eta_{water} \cdot K_{water}}{
ho_{water} \cdot g}$$

```
In [10]: #Given
    Darcy = 0.987 * 10**-12 # m^2, 1D = 0.987*10^-12 m^2
    nu_w = 1.00087*10**-3 # Pa-S dynamic viscosity of water
    K_w = 1.36*10**-5 # m/s Conductivity of water
    g = 9.81 # m/s^2 accln due to gravity
    rho_w = 998.2 # kg/m^3, density of water

# Solution
    k = (nu_w*K_w)/(rho_w*g)#, m^2, permeability of water
    k_D = k/Darcy

print("The permeability is {0:1.1E}".format(k),"m\u00b2")
    print("The permeability in Darcy unite is: {0:1.2f}".format(k_D),"D")
```

The permeability is 1.4E-12 m<sup>2</sup> The permeability in Darcy unite is: 1.41 D

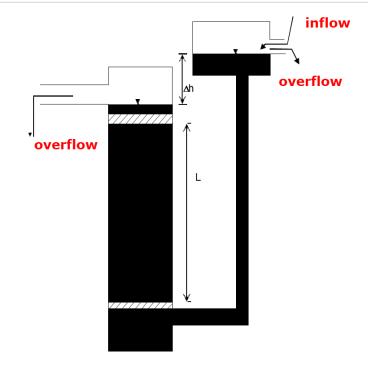
In [45]: ## Tutorial Problem 9: Constant-Head Permeameter r1 = pn.pane.LaTeX(r"""(a). Derive the expression for \$K\$ given below.  $K = \frac{QL}{A(h_{in}-h_{out})}$ \$\$ (b). The hydraulic conductivity of a medium sand sample (length 15 cm, cross-sectional area 25 cm\$^2\$) is to be determined. The hydraulic head difference is 5 cm and a water volume of 100 cm<sup>3</sup> pas-sed the sample during an experimental period of 12 min. <br><br><br>> (c). How long would 100 cm $^3$ \$ diesel (density: 0.85 g/cm $^3$ \$, dynamic viscosity: 3.5 $^4$ cdot 10 $^{-3}$ \$ Pa\$\cdot\$s at 20\$^\circ\$C) need to pass the sample under a head difference of 5 cm (density and dynamic viscosity of water at 20\$\circ\$C: 998.2 kg/m\$\square\$ and 1.0087\$\cdot  $10^{-3}$ \$ Pa\$\cdot\$s, resp.)? """, width=400, style={'font-size': '13pt'}) spacer = pn.Spacer(width=100) r2 =pn.pane.PNG("images/T02 TP9.png", width=400) pn.Row(r1,spacer, r2)

### Out[45]:

(a). Derive the expression for K given below.

$$K = rac{QL}{A(h_{in} - h_{out})}$$

- (b). The hydraulic conductivity of a medium sand sample (length 15 cm, cross-sectional area 25 cm<sup>2</sup>) is to be determined. The hydraulic head difference is 5 cm and a water volume of 100 cm<sup>3</sup> pas-sed the sample during an experimental period of 12 min.
- (c). How long would 100 cm $^3$  diesel (density: 0.85 g/cm $^3$ , dynamic viscosity:  $3.5 \cdot 10^{-3}$  Pa·s at 20°C) need to pass the sample under a head difference of 5 cm (density and dynamic viscosity of water at 20°C: 998.2 kg/m $^3$  and 1.0087· $10^{-3}$  Pa·s, resp.)?



```
In [40]: ### Tutorial Problem 9 - Solution ###
         r1 = pn.pane.LaTeX(r"""
         The relevant topic can be found in lecture 04, slides 15, 18-20
         <br><br>>
         Let the reference datum be at the bottom. Then from Darcy's Law:
         $$Q= -A\cdot K\frac{h {out}-h {in}}{L}$$
         With,
         Q = discharge [L\$^3\$/T] < br >
         L =column length [L]<br>
         A = cross-sectional area of column [L$^2$]<br>
         K = hydraulic conductivity [L/T]<br>
         h$_{in}$ =hydraulic head at column inlet [L]<br>
         h$ {out}$ = hydraulic head at column outlet [L]<br>
         <br>
         Solve for $K$:
         K= \frac{Q\cdot L}{A\cdot(h_{in}-h_{out})}
         $$
         If the reference level (z = 0) is located at the downgradient overflow, then set h \{out\} = 0.
         """, width= 500, style={'font-size': '13pt'})
         spacer = pn.Spacer(width=100)
         r2 =pn.pane.PNG("images/T02 TP9a.png", width=300)
         pn.Row(r1,spacer, r2, width=1000)
```

Out [40]: The relevant topic can be found in lecture 04, slides 15, 18-20

Let the reference datum be at the bottom. Then from Darcy's Law:

$$Q = -A \cdot K rac{h_{out} - h_{in}}{L}$$

With, Q = discharge  $[L^3/T]$ 

L =column length [L]

A = cross-sectional area of column  $[L^2]$ 

K = hydraulic conductivity [L/T]

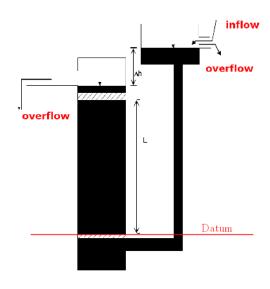
 $h_{in}$  =hydraulic head at column inlet [L]

 $h_{out}$  = hydraulic head at column outlet [L]

Solve for K:

$$K = rac{Q \cdot L}{A \cdot (h_{in} - h_{out})}$$

If the reference level (z=0) is located at the downgradient overflow, then set  $h_{out}=0$ .



```
In [42]: #Given (solution on 9b)
         L = 15 \# cm, length of column
         A = 25 \# cm^2, surface area of column
         h diff = 5 # cm, h in-h out
         0 = 100/12 \# cm^3/min discharge per min
         # Solution using derived equation in first part of the problem
         \# K = QL/A(h in-h out)
         K = (0*L)/(A*h_diff) \# cm/min, required conductivity
         K 1 = K*10**-2/60 \#, m/s, conductivity in m/s
         #output
         print("The conductivity in column is {0:1.2E}".format(K),"cm/min")
         print("The conductivity in column is {0:1.2E}".format(K 1),"m/s \n")
         if K 1 <= 1.67*10**-4:
             print("Fine to medium sand")
         else:
             print("to check further") # to be completed later.
```

The conductivity in column is 1.00E+00 cm/min The conductivity in column is 1.67E-04 m/s

Fine to medium sand

Continue solution on 9c

Discharge and Darcy's law: 
$$Q_{water} = rac{V}{t_{water}} = -A \cdot K_{water} \cdot rac{\Delta h}{L}$$

Solve for 
$$t_{water}$$
:  $t_{water} = rac{V}{Q_{water}} = -rac{V}{A \cdot K_{water} \cdot \Delta h/L} = -rac{V \cdot L}{A \cdot K_{water} \cdot \Delta h}$ 

Same step for 
$$t_{diesel}$$
 :  $t_{diesel} = - rac{V \cdot L}{A \cdot K_{diesel} \cdot \Delta h}$ 

time ratio: 
$$rac{t_{diesel}}{t_{water}} = rac{-rac{V\cdot L}{A\cdot K_{diesel}\cdot \Delta h}}{-rac{V\cdot L}{A\cdot K_{-diesel}\cdot \Delta h}} = rac{K_{water}}{K_{diesel}}$$

Use relationship between conductivity K and permeability k from lecture notes (slides 18)

$$rac{K_{water}}{K_{diesel}} = rac{k \cdot rac{
ho_{water} \cdot g}{\eta_{water}}}{k \cdot rac{
ho_{diesel} \cdot g}{\eta_{diesel}}} = rac{
ho_{water} \cdot \eta_{diesel}}{
ho_{diesel} \cdot \eta_{water}}$$

Solve for  $t_{diesel}$ 

```
In [14]: # Given data

rho_w = 920.2 # kg/m^3, density of water at 20°C
  eta_w = 1.0087*10**-3#, Pa-S dynamic viscosity of water
  rho_d = 0.85 # g/cm^3, density of diesel at 20°C
  eta_d = 3.5*10**-3#, Pa-S dynamic viscosity of diesel
  V_d = 100 # cm^3 volume of diesel
  t_w = 12 # min, time taken by water

# Calculations

t_d = (rho_w*eta_d)/(rho_d*1000*eta_w)*t_w

# multiplied by 1000 to convert unit g/cm^3 to kg/m^3

print("The time required for diesel will be: {0:0.2f}".format(t_d), "min")
```

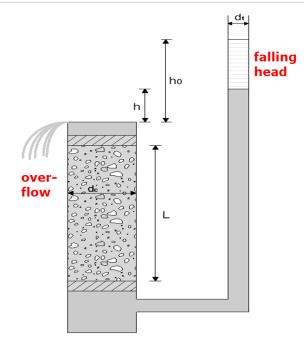
The time required for diesel will be: 45.08 min

# **Tutorial Problem 10: Falling-Head Permeameter**

#### Out[52]:

$$K = rac{d_t^2 L}{d_c^2 L} \cdot \ln rac{h_{in}(0) - h_{out}}{h_{in}(t) - h_{out}}$$

- 1. Derive the expression for K given above.
- 2. The hydraulic conductivity of a fine sand sample (length 15 cm, diameter 10 cm) is to be determined. The hydraulic head difference at the be-ginning and at the end of the experiment after 528 min is 5 cm and 0.5 cm, resp. The inner tube diameter is 2 cm.



# **Tutorial Problem 10: Solution**

Relevant information can be found in Lecture 04, Slides 14 and 16

```
In [53]: # Tutorial Problem 10: Solution
         r10 a1 = pn.pane.LaTeX(r"""
         Darcy's Law:
         $$
         Q(t) = -A_c \c \c K \cdot frac\{h_{out} - h_{in}(t)\}\{L\}
         Volumetric budget for standpipe:
         Q(t) = -\frac{dV_t}{dt}(t) = -A_t \cdot \frac{dh_{in}}{dt}(t)
         with <br>
         $A t$ = cross-sectional area of standpipe [L$^2$]<br>
         $V t$ = water volume in standpipe [L$^3$]<br>
         combine Darcy's law and the volumetric budget:
         -A_t \cdot dh_{in}{dt}(t) = -A_c \cdot dh_{in}{dt} - h_{in}(t){L}
         $$
         solve for $dh_{in}/dt$:
         \frac{dh_{in}}{dt} = \frac{K}{L}\frac{A_c}{A_t}(h_{out}-h_{in}) = \frac{K}{L}\frac{d_c}{d_t}\frac{A_c}{A_t}(h_{out}-h_{in})
         """, style={'font-size': '13pt'})
         r10 a2 =pn.pane.PNG("images/T02 TP10.png", width=300)
         pn.Row(r10_a1, r10_a2)
```

Out[53]: Darcy's Law:

$$Q(t) = -A_c \cdot K \cdot rac{h_{out} - h_{in}(t)}{L}$$

Volumetric budget for standpipe:

$$Q(t) = -rac{dV_t}{dt}(t) = -A_t \cdot rac{dh_{in}}{dt}(t)$$

with

 $A_t$  = cross-sectional area of standpipe [L $^2$ ]

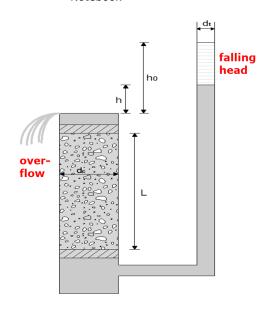
 $V_t$  = water volume in standpipe [L<sup>3</sup>]

combine Darcy's law and the volumetric budget:

$$-A_t \cdot rac{dh_{in}}{dt}(t) = -A_c \cdot K \cdot rac{h_{out} - h_{in}(t)}{L}$$

solve for  $dh_{in}/dt$ :

$$rac{dh_{in}}{dt} = rac{K}{L}rac{A_c}{A_t}(h_{out}-h_{in}) = rac{K}{L}igg(rac{d_c}{d_t}igg)^2(h_{out}-h_{in})$$



In [54]: # r10 a3 = pn.pane.LaTeX(r""" Equation for falling head:  $\frac{d_{in}}{dt} = \frac{K}{L} \frac{d_c}{d_t} \frac{2(h_{in})}$ This equation is an ordinary differential equation of first order. Providing hydraulic head  $h \{in\}(0)$  at the beginning of the experiment (t = 0), it may be solved by separation of variables: \$\$  $\frac{d}{dt} = \frac{K}{L}\Big(\frac{dt}{dt}\Big)^2 dt$ integrations on both sides by considering the initial condition:  $\inf \{h \{in\}(0)\}^{h \{in\}(t)} \frac{\sinh \{in\}}{h \{out\}-h \{in\}} = \inf 0^t \frac{K}{L} \frac{d t}{bigg}^2 dt = \frac{K}{L} \frac{h}{h} \frac{h}$  $g(\frac{d c}{d t} \frac{d t}{d t})^2 \in 0^t dt$ determine integral functions:  $\left[-\ln(h \{out\} - h \{in\}) \Big] \{h \{in\}(0)\}^{h \{in\}(t)\} = \frac{K}{L} \left(\frac{d t}{bigg}\right)^2 [t] 0^t$ insert limits of integration:  $-\ln \frac{h_{in}(t)}{h_{in}(t)} = \frac{K}{L} \frac{d_c}{d_t} \frac{2}{t}$ solve for K:  $K = \frac{d_c}{d_t}\bigg)^2 \frac{L}{t}\ln\frac{(0)-h_{out}}{h_{in}(t)-h_{out}}$ """, style={'font-size': '13pt'}) r10 a3

 $^{ extsf{Out}}$  [54] : Equation for falling head:  $rac{dh_{in}}{dt}=rac{K}{L}ig(rac{d_c}{d_t}ig)^2(h_{out}-h_{in})$ 

This equation is an ordinary differential equation of first order. Providing hydraulic head  $h_{in}(0)$  at the beginning of the experiment (t=0), it may be solved by separation of variables:

$$rac{dh_{in}}{h_{out}-h_{in}}=rac{K}{L}igg(rac{d_c}{d_t}igg)^2dt$$

integrations on both sides by considering the initial condition:

$$\int_{h_{in}(0)}^{h_{in}(t)} rac{dh_{in}}{h_{out}-h_{in}} = \int_0^t rac{K}{L} igg(rac{d_c}{d_t}igg)^2 dt = rac{K}{L} igg(rac{d_c}{d_t}igg)^2 \int_0^t dt$$

determine integral functions:

$$\left[-\ln(h_{out}-h_{in})
ight]_{h_{in}(0)}^{h_{in}(t)}=rac{K}{L}igg(rac{d_c}{d_t}igg)^2[t]_0^t$$

insert limits of integration:

$$-\lnrac{h_{out}-h_{in}(t)}{h_{out}-h_{in}(0)}=rac{K}{L}igg(rac{d_c}{d_t}igg)^2t$$

solve for K:

$$K = \left(rac{d_c}{d_t}
ight)^2 rac{L}{t} \ln rac{h_{in}(0) - h_{out}}{h_{in}(t) - h_{out}}$$

```
In [102]: # Given
          L = 15 \# cm, length
          L m = L/100 \# m, length
          d c = 10 # cm, diameter column
          d t = 2 # cm, diameter tube
          h d0 = 5 \# cm, head difference at start
          h dt = 0.5 \# cm, head difference at time t
          t = 528 # min, total time
          t s = 528*60 # sec, total time in seconds
          #solution using the developed equation
          K = (d_t/d_c)^{**2} * ((L_m)/t_s)^{*np.log}(h_d0/h_dt)
          #Output
          print("The conductivity in column is {0:1.2E}".format(K),"m/s \n")
          if K < 1.67*10**-5:
              print("Silt or silty sand")
          else:
              print("to check further") # to be completed later.
```

The conductivity in column is 4.36E-07 m/s Silt or silty sand

# **HOME WORK PROBLEMS**

**Darcy's Law and Intrinsic Permeability** 

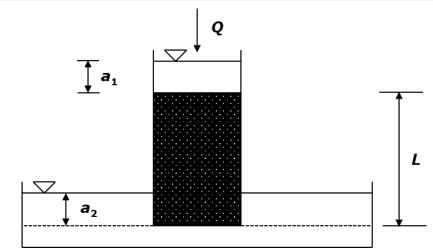
There is no obligation to solve homework problems!

Try to submit within next 2 weeks.

Out[103]:

## **Homework Problem 3**

- **A**. Derive an expression for hydraulic conductivity *K* for the constant-head permeameter shown in the figure.
- **B**. The hydraulic conductivity of a sample (length 10 cm, diameter 4 cm) is to be determined. The water depths  $a_1$  and  $a_2$  equal 6 cm and 3 cm, resp. A water volume of 250 ml passed the sample during an experimental period of 36 s.
- **C**. Which material could be contained in the sample?



**Homework Problem 4** 

In [66]: # r h4 = pn.pane.Markdown(""" A Darcy experiment is performed by a falling-head permeameter using water at 20°C. Length and diameter of the sample are 20 cm and 6 cm, resp. The inner tube diameter is 4 cm. The following data are available for the time-dependent hydraulic head difference : """, style={'font-size': '13pt'}) r h4b = pn.pane.PNG("images/T02 TH4a.png", width=400) r h4c = pn.pane.Markdown(""" \*\*A.\*\* Convert times to seconds and plot the logarithm of the ratios of head differences  $\ln(\Delta h(0)/\Delta h(t))$  vs. time t. \*\*B.\*\* Determine the slope of the corresponding regression line.<br/> \*\*C.\*\* Determine hydraulic conductivity K.<br><br> \*\*D.\*\* Determine intrinsic permeability k.<br> """, style={'font-size': '13pt'}) r h4d = pn.Column(r h4, r h4b, r h4c)r h4e = pn.pane.PNG("images/T02 TP10.png", width=400) spacer2=pn.Spacer(width=50) pn.Row(r\_h4d, spacer2, r\_h4e)

Out[66]:

A Darcy experiment is performed by a falling-head permeameter using water at 20°C. Length and diameter of the sample are 20 cm and 6 cm, resp. The inner tube diameter is 4 cm. The following data are available for the time-dependent hydraulic head difference:

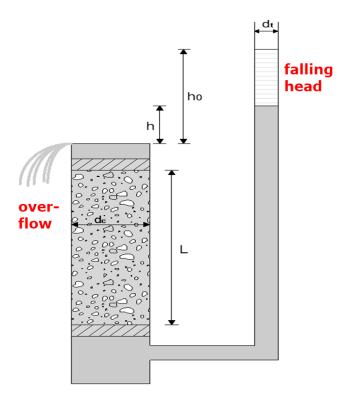
t (min)	0	5	18	23	27	29
<i>∆h</i> (cm)	36.9	33.6	26.3	23.9	22.1	21.3

**A.** Convert times to seconds and plot the logarithm of the ratios of head differences  $ln(\Delta h(0)/\Delta h(t))$  vs. time t. (Use the coordinate system on next page).

**B.** Determine the slope of the corresponding regression line.

**C.** Determine hydraulic conductivity K.

**D.** Determine intrinsic permeability k.



```
In [67]: #
    fig, ax = plt.subplots(figsize=(8, 6))
    plt.grid(axis='y', linestyle='--')
    plt.xlim((0, 1800)); plt.ylim((0,0.7))
    plt.xlabel("t(s)", fontsize=12 )
    plt.ylabel(r"ln($\Delta h(0)/\Delta h(t)$)(-)", fontsize=12);
```

