Previous Lecture

steady-state 3D groundwater flow in three dimensions:

- **Darcy's law in isotropic aquifers**
- **streamlines and flow nets**
- **isochrones and protection zones**
- **Darcy's law in anisotropic aquifers**

questions?

Today

- **quantification of three-dimensional groundwater flow**
- **two-dimensional groundwater flow in confined aquifers**
- **two-dimensional groundwater flow in unconfined aquifers**
- **complete formulation of groundwater flow problems**

Quantification of Three-dimensional Groundwater Flow

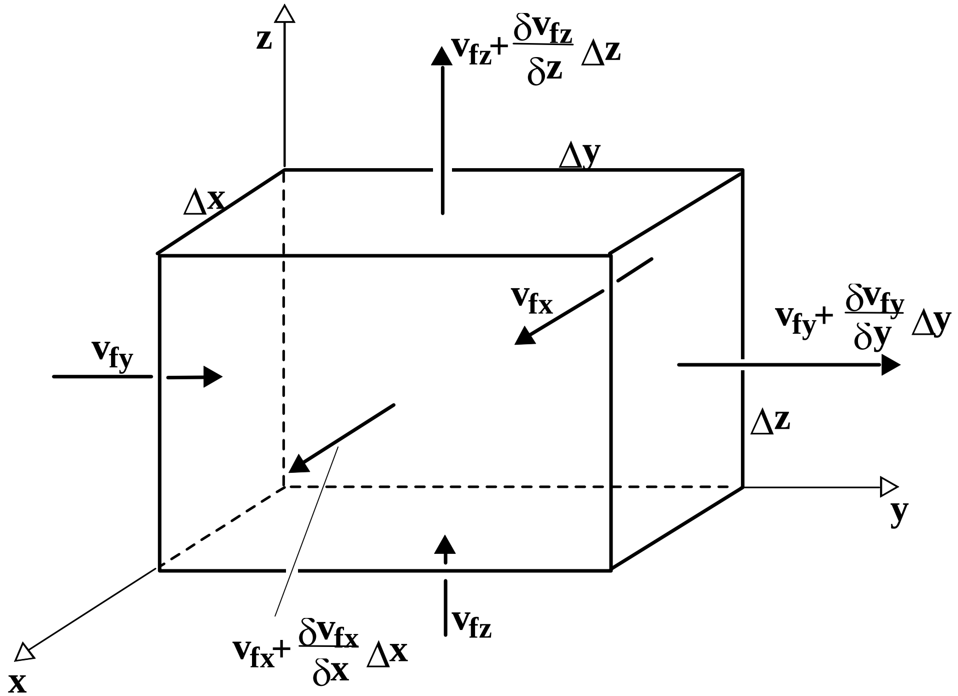
Control Volume

- **A <u>control volume</u> is a (fictitious) portion of an aquifer which is**
  - **much smaller than the region of investigation**
  - **much bigger than individual grains or pores (such that Darcy's law can be applied)**
- **The shape of the control volume is advantageously adjusted to the coordinate system used (e.g. rectangular for Cartesian coordinates)**

Classification

- **Groundwater flow regimes / aquifers can be classified according to**
  - 1D, 2D or 3D
  - steady-state or transient
  - homogeneous or heterogeneous
  - isotropic or anisotropic
  - confined or unconfined
  - with or without sources / sinks
- **This amounts to 96 possible combinations. (And further scenarios may exist.)**
- **Each combination corresponds to a certain equation governing groundwater flow. Sometimes these versions only differ with respect to details.**
- **It is important to keep in mind that all of them are based on just two principles: the conservation of volume and Darcy's law.**
  (Things can be quite a bit more complicated in consolidated systems which are not covered in detail in this course.)
- **Only these two principles are therefore needed to quantify groundwater flow in unconsolidated settings.**

Conservation of Volume



Scenario considered here:

- **3D**
- **transient**
- **isotropic**
- **heterogeneous**
- **confined**
- **without sources / sinks**

Volume budget:

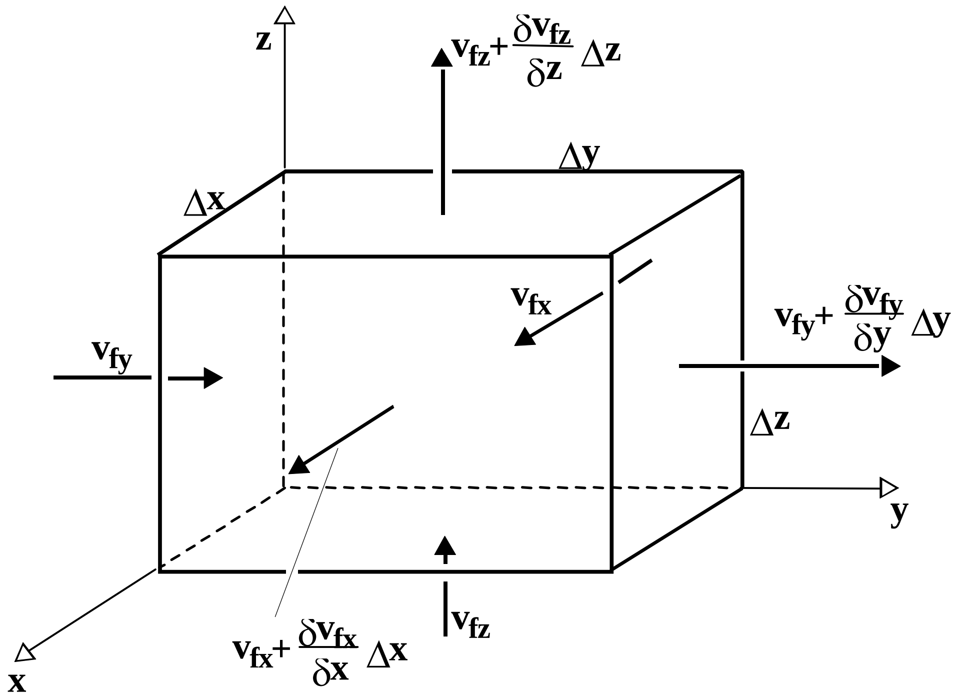
$$\frac{\Delta V_w}{\Delta t} = Q_{in} - Q_{out}$$

Darcy‘s Law

$$v_f = -K \operatorname{grad} h$$

- **The gradient vector [-] is oriented in the direction of the steepest *in*crease in hydraulic head.**
- **The minus sign indicates that groundwater flow is directed from “large“ to “small“ head values.**
- **Hydraulic conductivity is a scalar for isotropic aquifers but a tensor for anisotropic aquifers.**

Inflow and Outflow



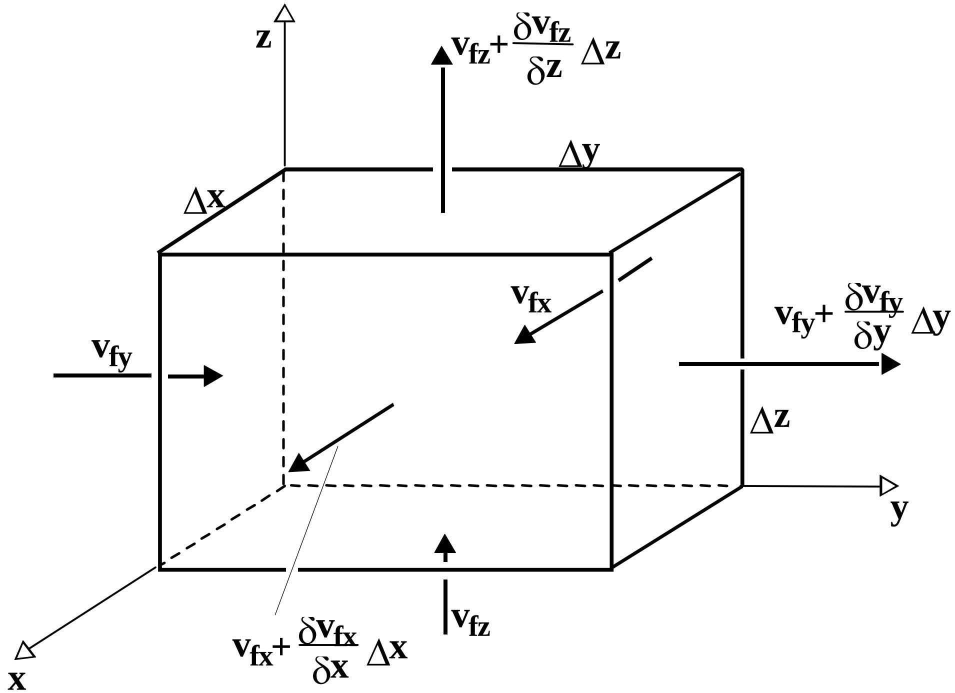
- **As stated before, the control volume is assumed to be small as compared to the region of investigation.**
- **Changes of Darcy velocity components across the control volume can therefore be regarded linear.**
- **Linear changes are obtained by multiplying first-order derivatives with corresponding distances.**

$$Q_{in} = v_{fx} \Delta y \Delta z + v_{fy} \Delta x \Delta z + v_{fz} \Delta x \Delta y$$

$$Q_{out} = \left(v_{fx} + \frac{\partial v_{fx}}{\partial x} \Delta x \right) \Delta y \Delta z + \left(v_{fy} + \frac{\partial v_{fy}}{\partial y} \Delta y \right) \Delta x \Delta z + \left(v_{fz} + \frac{\partial v_{fz}}{\partial z} \Delta z \right) \Delta x \Delta y$$

difference: $$Q_{in} - Q_{out} = -\frac{\partial v_{fx}}{\partial x} \Delta x \Delta y \Delta z - \frac{\partial v_{fy}}{\partial y} \Delta x \Delta y \Delta z - \frac{\partial v_{fz}}{\partial z} \Delta x \Delta y \Delta z$$

Temporal Change in Water Volume



$$\frac{\Delta V_w}{\Delta x \Delta y \Delta z} \propto \Delta h$$

$$\Delta V_w = S_s \Delta h \, \Delta x \, \Delta y \, \Delta z$$

The value of the specific storage coefficient corresponds to the change in water volume within a unit control volume if hydraulic head is increased / decreased by one unit.

Budgeting

- **insert expressions for Q_{in} - Q_{out} and ΔV_w into the volume budget relationship:**

$$S_s \frac{\Delta h}{\Delta t} = -\frac{\partial v_{fx}}{\partial x} - \frac{\partial v_{fy}}{\partial y} - \frac{\partial v_{fz}}{\partial z}$$

- **transition $\Delta t \to 0$:**

$$S_s \frac{\partial h}{\partial t} = -\frac{\partial v_{fx}}{\partial x} - \frac{\partial v_{fy}}{\partial y} - \frac{\partial v_{fz}}{\partial z}$$

- **Darcy‘s law by vector components:**

$$v_{fx} = -K \frac{\partial h}{\partial x} \quad , \quad v_{fy} = -K \frac{\partial h}{\partial y} \quad , \quad v_{fz} = -K \frac{\partial h}{\partial z}$$

- **groundwater flow equation:**

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y}\left(K \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z} \right)$$

Further Versions of the 3D Groundwater Flow Equation

- **with sources / sinks:**

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y}\left(K \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z} \right) + q$$

q **represents the volumetric rate of the source / sink per unit volume [1/T].**

Examples: **water injection / extraction through wells**
water transfer from / to rivers

- **with anisotropy:**

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial h}{\partial z} \right) + q$$

whereby principal axes of anisotropy are assumed to be in parallel with the coordinate axes

Special Cases

- **groundwater flow equation from the previous page:**
  (transient, heterogeneous, isotropic, confined, with sources / sinks)

$$S_s \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y}\left(K \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z}\left(K \frac{\partial h}{\partial z} \right) + q$$

- **<u>Poisson equation</u>:**
  (steady-state, homogeneous, isotropic, confined, with sources / sinks)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = -\frac{q}{K}$$

Siméon-Denis Poisson (1781 – 1840)

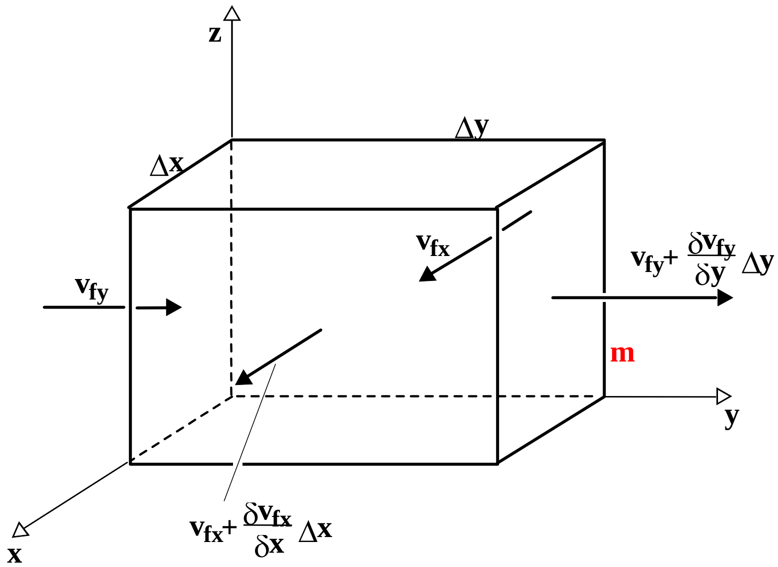
- **<u>Laplace equation</u>:**
  (steady-state, homogeneous, isotropic, confined, without sources / sinks)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0$$

Pierre-Simon Marquis de Laplace (1749 – 1827)

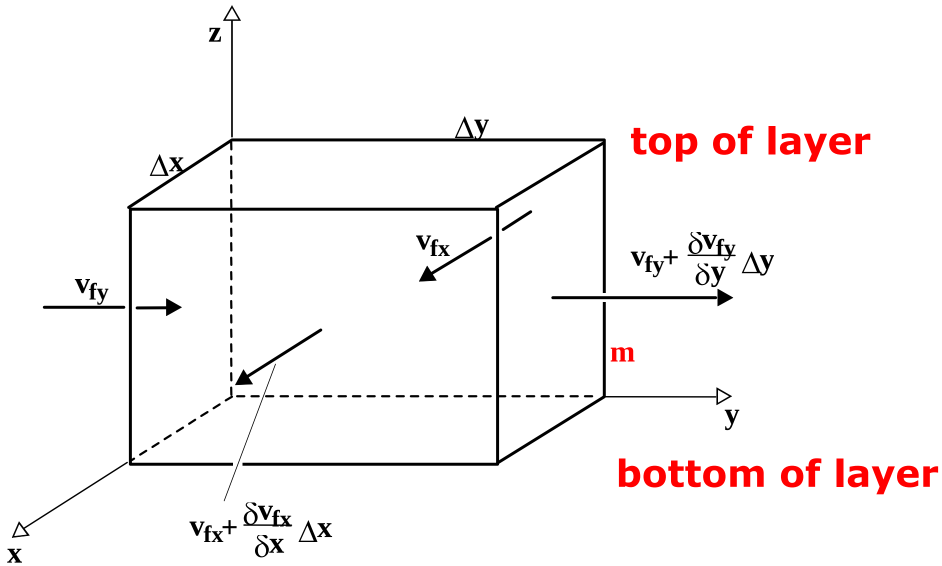
Two-dimensional Groundwater Flow in Confined Aquifers

Control Volume



- **For most unconsolidated aquifers it is observed that groundwater flow components perpendicular to layering are negligible.**
- **Accordingly, groundwater flow problems are frequently treated as two-dimensional, i.e. employing only two space coordinates.**
- **If an aquifer consists of several distinct major layers, the two-dimensional approach is used for each layer separately and, in addition, water transfer between layers is handled by appropriate source / sink terms (not shown in figure).**
- **In this case the control volume extends over the entire layer thickness *m*.**

Transmissivity



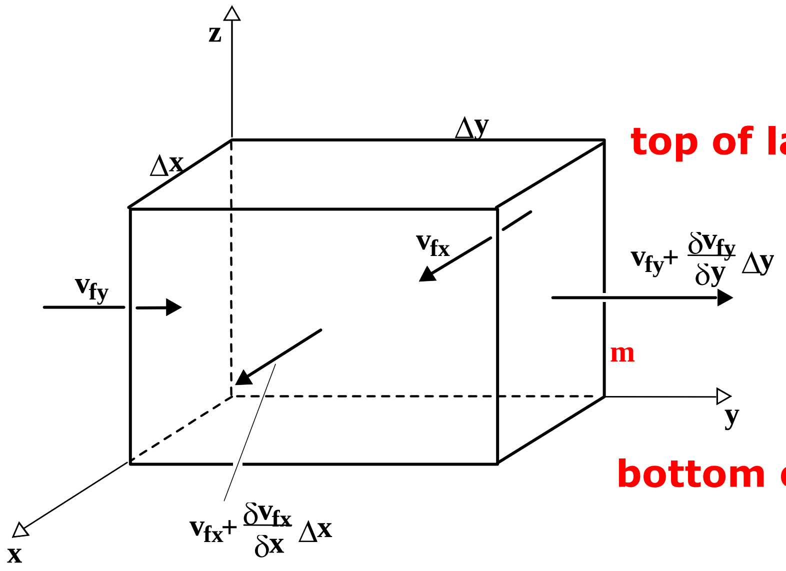
- **The step from three to two dimensions requires to "sum up" hydraulic conductivity values over the entire layer thickness in order to correctly quantify two-dimensional groundwater flow.**
- **This is done by introducing <u>transmissivities</u> T_x and T_y [L²/T] as follows:**

$$T_x = K_x \cdot m \qquad T_y = K_y \cdot m$$

where K_x and K_y denote vertically averaged hydraulic conductivities [L/T] along the x- and y-coordinate, respectively.

- **For a confined aquifer which is horizontally isotropic we simply have $T = K \cdot m$.**

2D Groundwater Flow Equations for Confined Conditions



- **As mentioned in a previous lecture, the storage coefficient *S* [-] is to be used instead of S_s [1/L] if vertical flow compo-nents are neglected.**
- **This results in the following 2D groundwater flow equation without sources / sinks:**

$$S\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)$$

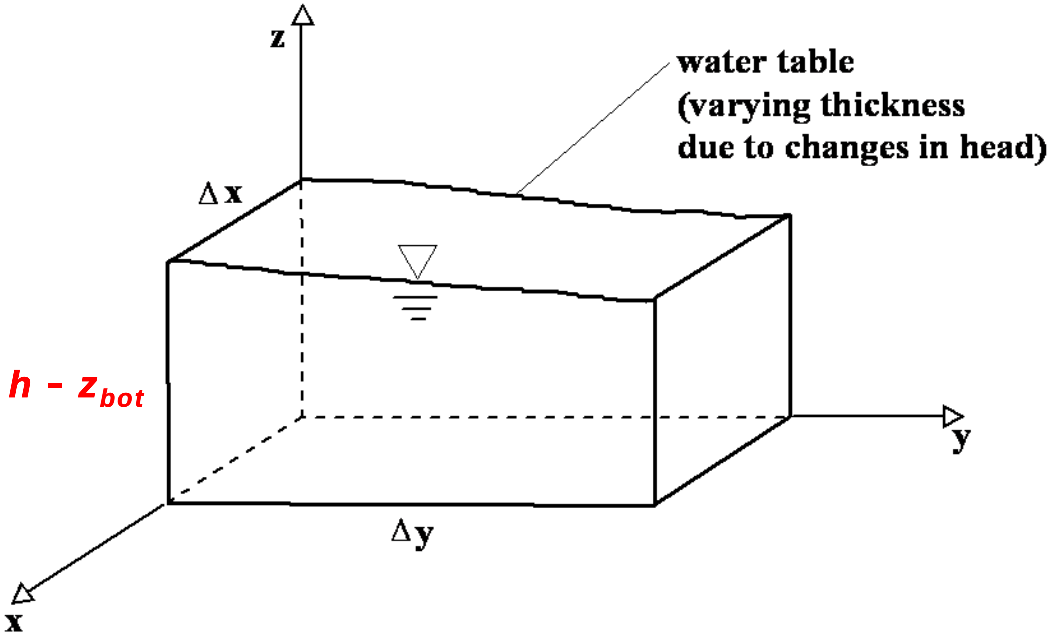
- **2D groundwater flow equation with sources / sinks:**

$$S\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)+N$$

where *N* denotes the volume-tric flux due to sources / sinks per unit surface area [L/T].

Two-dimensional Groundwater Flow in Unconfined Aquifers

Control volume



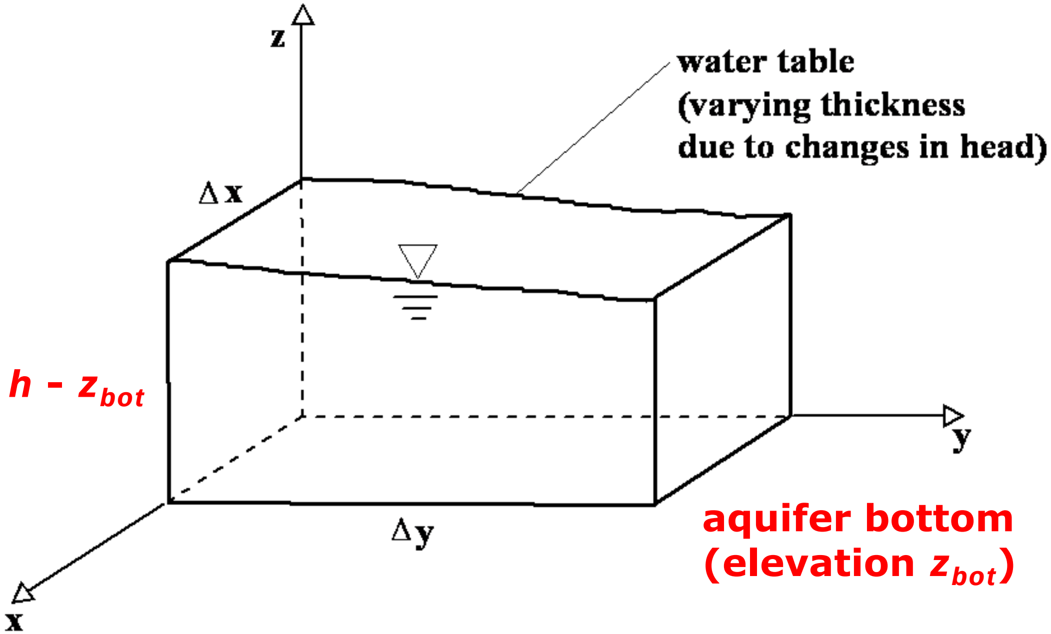
- **For unconfined layers, the control volume ex-tends from the aquifer bottom to the ground-water table.**
- **This implies that the height of the control vo-lume depends on the flow behaviour.**
- **For the same reason, transmissivities are defi-ned by using hydraulic head *h* to account for the saturated thickness:**

$$T_x = K_x \cdot (h - z_{bot})$$

$$T_y = K_y \cdot (h - z_{bot})$$

where z_{bot} represents the elevation of the aquifer bottom [L].

2D Groundwater Flow Equation for Unconfined Conditions



Joseph Boussinesq (1842 – 1929)

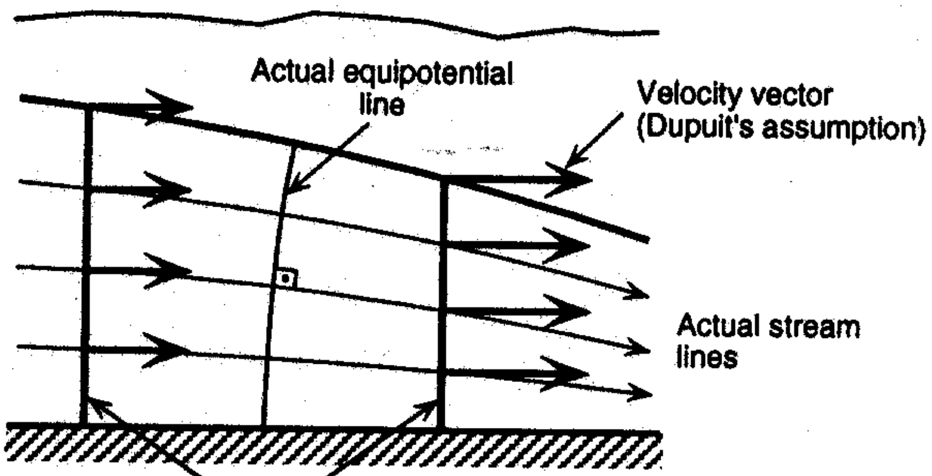
- **Formally, the 2D groundwater flow equation for unconfined aquifers is the same as for confined conditions, i.e.:**

$$S\frac{\partial h}{\partial t}=\frac{\partial}{\partial x}\left(T_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(T_y\frac{\partial h}{\partial y}\right)+N$$

- **However, transmissivities are computed in a different way!**
- **The above equation is also termed <u>Boussinesq equation</u>.**

Dupuit Assumptions

Jules Dupuit (1804 – 1866)



The Boussinesq equation for 2D groundwater flow in unconfined aquifers is based on the following assumptions which are due to Dupuit (1863):

- **Groundwater flow is horizontal (no vertical flow component).**
- **The flow velocity does not vary with depth.**
- **Darcy‘s law also holds at the water table.**

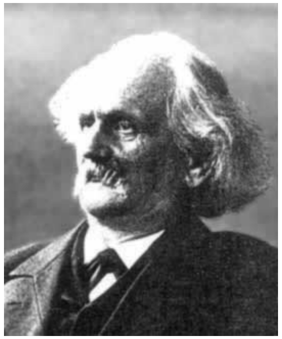
Complete Formulation of Groundwater Flow Problems

Steps

- **specify the geometric properties of the region of interest (dimensionality, shape)**
- **specify values of aquifer parameters (hydraulic conductivity, storage coefficient) by considering spatial variability and anisotropy, if necessary**
- **select the appropriate flow equation**
- **specify the initial condition (IC):**
  - head values at time t = 0
  - This step is not required for steady-state problems.
- **specify boundary conditions (BCs):**
  - BCs have to be given along the complete boundary (also at infinity if regions are assumed to be unbounded).
  - BCs may be time-dependent.
  - There are three major types of BCs (see next page).

Boundary Conditions

Peter Lejeune Dirichlet (1805 – 1859)



Carl Neumann (1832 – 1925)

Augustin Cauchy (1789 – 1857)

- **The number of boundary conditions required corresponds to the highest space derivative for each coordinate in the flow equation.**
- **<u>Boundary condition of the first kind</u> or <u>Dirichlet boundary condition</u>: The head value is given.**
- **<u>Boundary condition of the second kind</u> or <u>Neumann boundary condition</u>: The component of the head gradient, which is perpendicular to the boundary, is given.**
- **<u>Boundary condition of the third kind</u> or <u>Cauchy boundary condition</u> or <u>Robin boundary condition</u> (for completeness only): A relationship between the head value and the component of the head gradient, which is perpendicular to the boundary, is given.**

Relationships

aquifer / flow property	mathematical formulation
transient	with time derivative
confined	linear partial differential equation
anisotropic	$T_x \neq T_y$ or $K_x \neq K_y$, resp. (tensor)
heterogeneous (inhomogeneous)	coefficients depend on space coordinate(s)
with sources / sinks	inhomogeneous differential equation (contains a term without h)
fixed-head boundary condition	boundary condition of the first kind (Dirichlet)
flux boundary condition (in particular: “no flow”)	boundary condition of the second kind (Neumann)