4 Quadratic Forms III: Connections with conic sections

Prof.Dr.P.M.Bajracharya

School of Mathematical Sciences
T.U., Kirtipur

April 26, 2024

Summary

1. A Geometric View of Principal Axes
2. Conics in Standard Form
3. General Conic Sections
4. Eliminating Cross-product Terms

We can now state the theorem on the change of variable in a quadratic form.

The Principal Axes Theorem

Let A be an $n \times n$ symmetric matrix. Then there are an orthogonal matrix P and a diagonal matrix D such that the mapping defined by

$$x = Py$$

transforms the quadratic form $x^T Ax$ into a quadratic form $y^T Dy$ with no cross-product term.

We can now state the theorem on the change of variable in a quadratic form.

The Principal Axes Theorem

Let A be an $n \times n$ symmetric matrix. Then there are an orthogonal matrix P and a diagonal matrix D such that the mapping defined by

$$x = Py$$

transforms the quadratic form $x^T Ax$ into a quadratic form $y^T Dy$ with no cross-product term.

The columns of P in the theorem are called the **principal axes** of the quadratic form $x^T Ax$. The vector y is the coordinate vector of x relative to the orthonormal basis of $\mathbb{R}^n$ given by these principal axes.

Proof. Let A be an $n \times n$ symmetric matrix.

Proof. Let A be an $n \times n$ symmetric matrix. Then by the spectral theorem there is an orthogonal matrix P of eigenvectors of A. such that

$$P^T AP = D,$$

where D is the diagonal matrix of eigenvalues of A. Since P is orthogonal,

$$P^{-1} = P^T.$$

Proof... Put

$$x = Py.$$

Proof... Put

$$x = Py.$$

Then

$$\begin{aligned} x^T A x &= (Py)^T A (Py) \\ &= y^T P^T A P y \end{aligned}$$

Proof... Put

$$x = Py.$$

Then

$$\begin{aligned} x^T A x &= (Py)^T A (Py) \\ &= y^T P^T A P y \\ &= y^T (P^T A P) y \end{aligned}$$

Proof... Put

$$x = Py.$$

Then

$$\begin{aligned} x^T A x &= (Py)^T A (Py) \\ &= y^T P^T A P y \\ &= y^T (P^T A P) y \\ &= y^T D y. \end{aligned}$$

Proof... Put

$$x = Py.$$

Then

$$\begin{aligned} x^T A x &= (Py)^T A (Py) \\ &= y^T P^T A P y \\ &= y^T (P^T A P) y \\ &= y^T D y. \end{aligned}$$

Since D is a diagonal matrix, the quadratic form $y^T D y$ does not contain cross-product terms. $\square$

A Geometric View of Principal Axes

We saw earlier that subspaces, linear dependence, etc. generalize the intuitive geometric ideas that we have about lines through the origin, planes through the origin, etc. The quadratic forms too generalize intuitive geometric ideas that are related to conic sections and their generalizations.

A Geometric View of Principal Axes

Suppose $Q(x) = x^T Ax$, where A is an invertible 2×2 symmetric matrix, and let c be a constant. It can be shown that the set of all x in $\mathbb{R}^2$ that satisfy

$$x^T Ax = c \tag{1}$$

either corresponds to

an ellipse (or circle), a hyperbola, two intersecting lines, or a single point, or contains no points at all.

If A is a diagonal matrix, the graph is in standard position, such as in the Figure below.

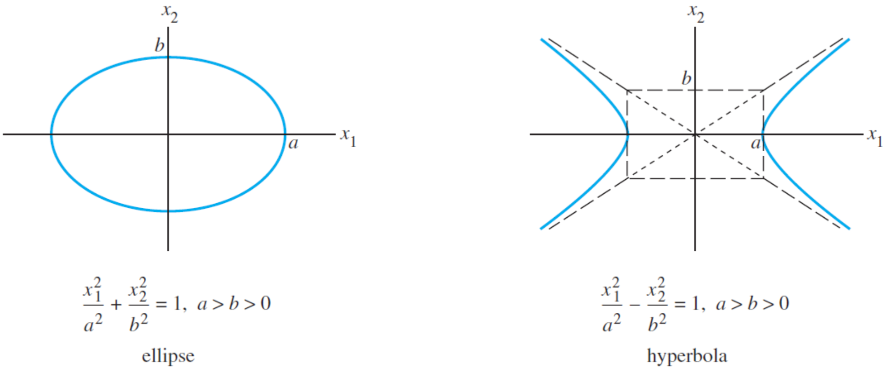


Figure: An ellipse and a hyperbola in standard position.

If A is not a diagonal matrix, the graph of equation (1) is rotated out of standard position, as in the Figure below.

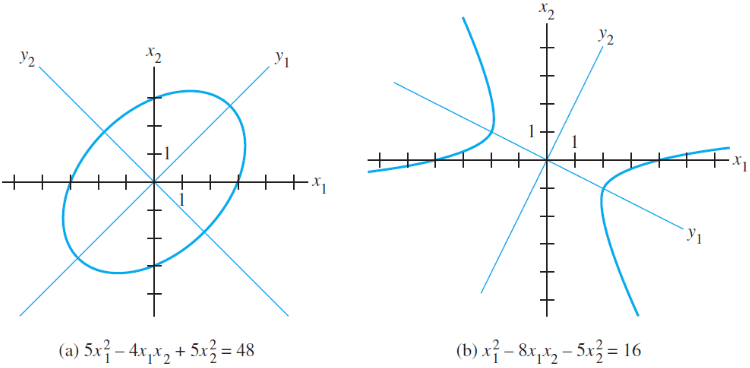


Figure: An ellipse and a hyperbola not in standard position.

Finding the principal axes (determined by the eigenvectors of A) amounts to finding a new coordinate system with respect to which the graph is in standard position.

Steps for Making a Change of Variables

(1) **Make the matrix** A of the quadratic form.

(2) **Find eigenvalues** by solving the characteristic equation.

(3) **Find the eigenvectors** that correspond to each eigenvalue.

(4) **Check for orthogonality**, and if necessary, apply the **Gram-Schmidt process**.

(5) **Normalize** to create matrix P that is orthonormal.

(6) **Write** P and D matrices and use them to create the transformed quadratic equation.

From the previous discussion, we can rewrite the above example as follows:

Example

Make a change of variable that transforms the quadratic form

$$Q(x) = 8x_1^2 + 6x_1x_2$$

into a quadratic form with no cross-product term.

Conics in Standard Form

- **Ellipse.** The equation in standard form is given by
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$
Note that the major and minor axes of an ellipse in standard form are just the canonical axes. Without loss of generality, let's assume that $a \geq b \geq 0$, otherwise it is longer along the other axis.

- **Ellipse.** The equation in standard form is given by
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$
Note that the major and minor axes of an ellipse in standard form are just the canonical axes. Without loss of generality, let's assume that $a \geq b \geq 0$, otherwise it is longer along the other axis.
Remark. A circle is just an ellipse with $a = b$.

- **Ellipse.** The equation in standard form is given by
$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1.$$
Note that the major and minor axes of an ellipse in standard form are just the canonical axes. Without loss of generality, let's assume that $a \geq b \geq 0$, otherwise it is longer along the other axis.
Remark. A circle is just an ellipse with $a = b$.
- **Hyperbola.** The equation in standard form is given by
$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1.$$

Degenerate cases.

- **Parabola.** This is the set of points in the plane that are equidistant from a fixed point and a fixed line.
- **Line.** This arises when the quadratic and constant terms are zero.
- **Point** This arises when the quadratic and linear and constant terms are zero.

Degenerate cases.

- **Parabola.** This is the set of points in the plane that are equidistant from a fixed point and a fixed line.
- **Line.** This arises when the quadratic and constant terms are zero.
- **Point** This arises when the quadratic and linear and constant terms are zero.

Remark.

Note that the a and b above basically correspond to stretching the units of the x_1 and x_2 axes.

General Conic Sections

Let's say that, instead of being given a conic section in standard form, we are given some quadratic function expression and we have to determine the type of conic section to which it corresponds.

Example

Identify and sketch the conic whose equation is

$$9x_1^2 - 4x_2^2 - 72x_1 + 8x_2 + 176 = 0.$$

Example

Identify and sketch the conic whose equation is

$$9x_1^2 - 4x_2^2 - 72x_1 + 8x_2 + 176 = 0.$$

Solution. First write terms in x_1 and x_2 separately.

$$9(x_1^2 - 8x_1) - 4(x_2^2 - 2x_2) + 176 = 0.$$

Solution... Then, complete the square to get

$$9(x_1-4)^2-4(x^2-1)^2+176-144+4=0$$

Solution... Then, complete the square to get

$$9(x_1 - 4)^2 - 4(x^2 - 1)^2 + 176 - 144 + 4 = 0$$

and then simplify this to get

$$9(x_1 - 4)^2 - 4(x^2 - 1)^2 + 36 = 0.$$

Solution... Then, complete the square to get

$$9(x_1 - 4)^2 - 4(x^2 - 1)^2 + 176 - 144 + 4 = 0$$

and then simplify this to get

$$9(x_1 - 4)^2 - 4(x^2 - 1)^2 + 36 = 0.$$

Then we get

$$-\frac{(x_1 - 4)^2}{4} + \frac{(x_2 - 1)^2}{9} = 1.$$

Solution... Then, complete the square to get

$$9(x_1 - 4)^2 - 4(x^2 - 1)^2 + 176 - 144 + 4 = 0$$

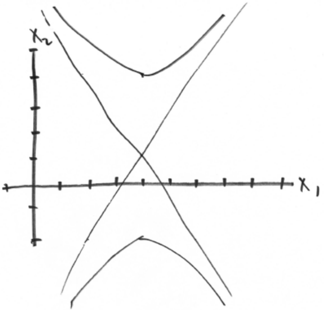
and then simplify this to get

$$9(x_1 - 4)^2 - 4(x^2 - 1)^2 + 36 = 0.$$

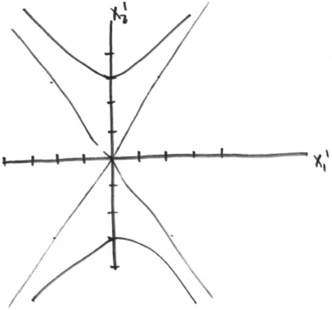
Then we get

$$-\frac{(x_1 - 4)^2}{4} + \frac{(x_2 - 1)^2}{9} = 1.$$

This is a hyperbola. See the figure given below.



(a) Original.



(b) Transformed.

There are two things to note about this example.

(a) Since the negative sign is on the x_1 and not the x_2, the hyperbola opens up-down and not left-right.

(a) Since the negative sign is on the x_1 and not the x_2, the hyperbola opens up-down and not left-right.

(b) Having those other linear and constant terms simply amounts to shifting the origin.

(a) Since the negative sign is on the x_1 and not the x_2, the hyperbola opens up-down and not left-right.

(b) Having those other linear and constant terms simply amounts to shifting the origin. That is, if we define a new set of coordinates

$$x_1' = x_1 - 4$$
$$x_2' = x_2 - 1,$$

which simply amounts to shifting the origin, then we get the equation

$$-\frac{(x_1')^2}{4} + \frac{(x_2')^2}{9} = 1.$$

Example

Identify and sketch the conic whose equation is

$$9x_1^2 + 4x_2^2 - 72x_1 - 8x_2 + 112 = 0.$$

Example

Identify and sketch the conic whose equation is

$$9x_1^2 + 4x_2^2 - 72x_1 - 8x_2 + 112 = 0.$$

Solution. First write terms in x_1 and x_2 separately.

$$9(x_1^2 - 8x_1) + 4(x_2^2 - 2x_2) + 112 = 0.$$

Example

Identify and sketch the conic whose equation is

$$9x_1^2 + 4x_2^2 - 72x_1 - 8x_2 + 112 = 0.$$

Solution. First write terms in x_1 and x_2 separately.

$$9(x_1^2 - 8x_1) + 4(x_2^2 - 2x_2) + 112 = 0.$$

Then, complete the square to get

$$9(x_1 - 4)^2 + 4(x^2 - 1)^2 + 112 - 144 - 4 = 0$$

Example

Identify and sketch the conic whose equation is

$$9x_1^2 + 4x_2^2 - 72x_1 - 8x_2 + 112 = 0.$$

Solution. First write terms in x_1 and x_2 separately.

$$9(x_1^2 - 8x_1) + 4(x_2^2 - 2x_2) + 112 = 0.$$

Then, complete the square to get

$$9(x_1 - 4)^2 + 4(x^2 - 1)^2 + 112 - 144 - 4 = 0$$

and then simplify this to get

$$9(x_1 - 4)^2 + 4(x^2 - 1)^2 - 36 = 0.$$

Solution ... Then we get

$$\frac{(x_1-4)^2}{4}+\frac{(x_2-1)^2}{9}=1.$$

Solution ... Then we get

$$\frac{(x_1-4)^2}{4}+\frac{(x_2-1)^2}{9}=1.$$

This is an ellipse.

Solution ... Then we get

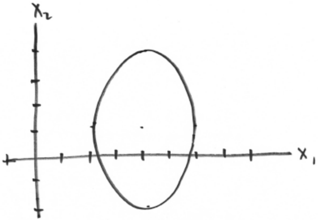
$$\frac{(x_1-4)^2}{4}+\frac{(x_2-1)^2}{9}=1.$$

This is an ellipse. If we define a new set of coordinates

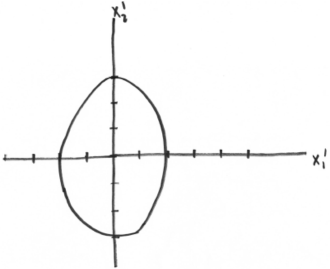
$$\begin{aligned} x_1' &= x_1 - 4 \\ x_2' &= x_2 - 1, \end{aligned}$$

which simply amounts to shifting the origin, then we get the equation

$$\frac{(x_1')^2}{4}+\frac{(x_2')^2}{9}=1.$$



(a) Original.



(b) Transformed.

Eliminating Cross-product Terms

Question: What if there is an x_1x_2 term?

Question: What if there is an x_1x_2 term?

Answer: We still complete the square, but using the other variables.

Question: What if there is an x_1x_2 term?

Answer: We still complete the square, but using the other variables. In this case, we will see that A has a rotation, i.e., we don't have just a scaling and translation shift of the axes like in the previous example.

If a central conic is rotated out of standard position, then it can be identified by first rotating the coordinate axes to put it in standard position.

To find a rotation that eliminates the cross product term in the equation

$$ax^2 + 2hxy + by^2 = k$$

To find a rotation that eliminates the cross product term in the equation

$$ax^2 + 2hxy + by^2 = k$$

we express it in the matrix form

$$x^T Ax = (x\ y)\begin{pmatrix} a & h \\ h & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = k$$

To find a rotation that eliminates the cross product term in the equation

$$ax^2 + 2hxy + by^2 = k$$

we express it in the matrix form

$$x^T A x = (x\ y)\begin{pmatrix} a & h \\ h & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = k$$

and look for a change of variable

$$x = Px'$$

that diagonalizes A and for which $\det(P) = 1$.

We have discussed earlier that the principal axes can always be found by rotating the x_1 and x_2 axes counterclockwise about the origin through an angle θ.

We have discussed earlier that the principal axes can always be found by rotating the x_1 and x_2 axes counterclockwise about the origin through an angle θ. This rotation has matrix

$$P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The determinant of any orthogonal matrix P is either 1 or -1 (because $PP^T = I$).

The determinant of any orthogonal matrix P is either 1 or -1 (because $PP^T = I$). The orthogonal matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

arising from rotations all have determinant 1.

The determinant of any orthogonal matrix P is either 1 or -1 (because $PP^T = I$). The orthogonal matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

arising from rotations all have determinant 1.More generally, given any quadratic form $q = x^T Ax$, the orthogonal matrix P such that $P^T AP$ is diagonal can always be chosen so that $|P| = 1$ by interchanging two eigenvalues (and hence the corresponding columns of P).

The determinant of any orthogonal matrix P is either 1 or -1 (because $PP^T = I$). The orthogonal matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

arising from rotations all have determinant 1.More generally, given any quadratic form $q = x^T Ax$, the orthogonal matrix P such that $P^T AP$ is diagonal can always be chosen so that $|P| = 1$ by interchanging two eigenvalues (and hence the corresponding columns of P). It is worthwhile to note that orthogonal 2×2 matrices with determinant 1 correspond to rotations.

Example

(a) Identify and sketch the conic whose equation is

$$5x^2 - 4xy + 8y^2 - 36 = 0$$

by rotating the xy-axes to put the conic in standard position.

(a) Find the angle θ through which you rotated the xy-axes in part (a).

Solution (a) The given equation can be written in the matrix form

$$u^T A u = 36,$$

where

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$$

The characteristic polynomial of A is

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 8-\lambda \end{vmatrix} = (9-\lambda)(4-\lambda)$$

so the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 9$.

Solution (a) ... $\underline{\underline{\lambda_1 = 4}}$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Solution (a) ... $\underline{\underline{\lambda_1 = 4}}$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

This gives

$$x - 2y = 0$$

Solution (a) ... $\underline{\underline{\lambda_1 = 4}}$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

This gives

$$x - 2y = 0$$

By observation, the eigenvector associated with $\lambda_1 = 4$ is

$$v_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Solution (a) ... $\underline{\underline{\lambda_2 = 9}}$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

Solution (a) ... $\underline{\underline{\lambda_2 = 9}}$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

This gives

$$2x + y = 0$$

Solution (a) ... $\underline{\underline{\lambda_2 = 9}}$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

This gives

$$2x + y = 0$$

By observation, the eigenvector associated with $\lambda_2 = 9$ is

$$v_2 = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

Solution (a) ... Thus, A is orthogonally diagonalized by

$$P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

Solution (a) ... Thus, A is orthogonally diagonalized by

$$P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

Moreover, it happens that $\det(P\ ) = 1$, so we are assured that the substitution

$$u' = Pu, \text{ where } u' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

performs a rotation of axes.

Solution (a) ... It follows that the equation of the conic in the $x'y'$-coordinate system is

$$(x' \ y') \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = 36,$$

which we can write as

$$4x'^2 + 9y'^2 = 36$$

We can now see that the conic is an ellipse.

Solution (b) ... Since $|P| = 1$, the matrix P is a rotation matrix.

Solution (b) ... Since $|P| = 1$, the matrix P is a rotation matrix. Hence

$$P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Solution (b) ... Since $|P| = 1$, the matrix P is a rotation matrix. Hence

$$P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This implies that

$$\cos\theta = 2/\sqrt{5}, \ \sin\theta = 1/\sqrt{5}, \ \tan\theta = 1/2.$$

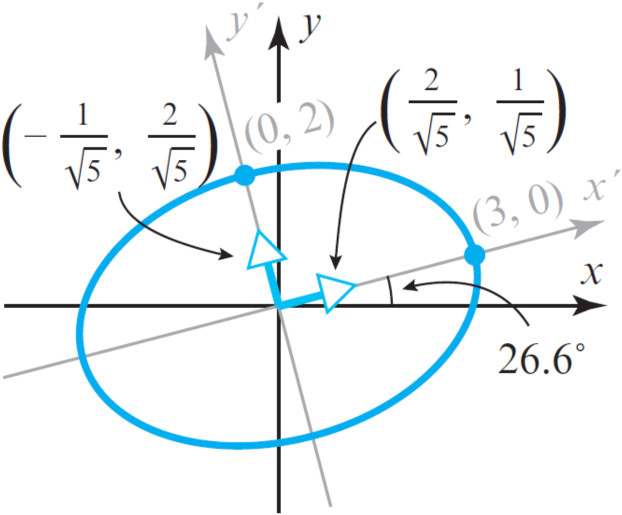
Solution (b) ... Since $|P| = 1$, the matrix P is a rotation matrix. Hence

$$P = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This implies that

$$\cos\theta = 2/\sqrt{5},\ \sin\theta = 1/\sqrt{5},\ \tan\theta = 1/2.$$

Thus, $\theta = \tan^{-1}(1/2) \approx 26.6^\circ$ (see the figure given below).


y'
y
$\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$
$(0, 2)$
$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$
$(3, 0)$
x'
x
$26.6°$