Unit 3: Spectral theory II

Prof.Dr.P.M.Bajracharya

School of Mathematical Sciences
T.U., Kirtipur

April 2, 2024

Theorem

Let A be a square matrix with eigenvector u belonging to eigenvalue λ.

(a) If m is a natural number then λ^m is an eigenvalue of the matrix A^m with the same eigenvector u.

(b) If the matrix A is invertible then the eigenvalue of the inverse matrix A^{-1} is
$$\frac{1}{\lambda} = \lambda^{-1}$$
with the same eigenvector u.

We prove the theorem by using mathematical induction. The three steps of mathematical induction are:

Step 1: Check the result for some base case

$$m = m_0.$$

Step 2: Assume that the result is true for $m = k$.
Step 3: Prove the result for $m = k + 1$.

Proof of (a). **Step 1:** Using the definition of eigenvalues and eigenvectors, we have $Au = \lambda u$ which means the result holds for $m = 1$:

$$Au = \lambda u.$$

Proof of (a). Step 1: Using the definition of eigenvalues and eigenvectors, we have $Au = \lambda u$ which means the result holds for $m = 1$:

$$Au = \lambda u.$$

Step 2: Assume that the result is true for $m = k$:

$$A^k u = \lambda^k u.$$

Step 3: Required to prove the case $m = k + 1$; that is, we need to prove

$$A^{k+1}u = \lambda^{k+1}u.$$

Step 3: Required to prove the case $m = k + 1$; that is, we need to prove

$$A^{k+1}u = \lambda^{k+1}u.$$

We have

$$A^{k+1}u = A(A^k u) = A(\lambda^k u)$$

Step 3: Required to prove the case $m = k + 1$; that is, we need to prove

$$A^{k+1}u = \lambda^{k+1}u.$$

We have

$$\begin{aligned} A^{k+1}u &= A(A^k u) = A(\lambda^k u) \\ &= \lambda^k (Au) = \lambda^k (\lambda u) \end{aligned}$$

Step 3: Required to prove the case $m = k + 1$; that is, we need to prove

$$A^{k+1}u = \lambda^{k+1}u.$$

We have

$$\begin{aligned} A^{k+1}u &= A(A^k u) = A(\lambda^k u) \\ &= \lambda^k (Au) = \lambda^k (\lambda u) \\ &= \lambda^{k+1} u \end{aligned}$$

Step 3: Required to prove the case $m = k + 1$; that is, we need to prove

$$A^{k+1}u = \lambda^{k+1}u.$$

We have

$$\begin{aligned} A^{k+1}u &= A(A^k u) = A(\lambda^k u) \\ &= \lambda^k(Au) = \lambda^k(\lambda u) \\ &= \lambda^{k+1}u \end{aligned}$$

Therefore by mathematical induction

$$A^m u = \lambda^m u$$

for any natural number m.

Proof of (b).

Again, using the definition of eigenvalues and eigenvectors, we have

$$Au = \lambda u$$

Proof of (b).

Again, using the definition of eigenvalues and eigenvectors, we have

$$Au = \lambda u$$

Left multiplying both sides of this by A^{-1} gives

$$A^{-1}(Au) = A^{-1}(\lambda u) = \lambda A^{-1}u.$$

Proof of (b).

Again, using the definition of eigenvalues and eigenvectors, we have

$$Au = \lambda u$$

Left multiplying both sides of this by A^{-1} gives

$$A^{-1}(Au) = A^{-1}(\lambda u) = \lambda A^{-1}u.$$

This gives

$$A^{-1}u = \frac{1}{\lambda}u = \lambda^{-1}u.$$

Proof of (b).
Again, using the definition of eigenvalues and eigenvectors, we have

$$Au = \lambda u$$

Left multiplying both sides of this by A^{-1} gives

$$A^{-1}(Au) = A^{-1}(\lambda u) = \lambda A^{-1} u.$$

This gives

$$A^{-1} u = \frac{1}{\lambda} u = \lambda^{-1} u.$$

Therefore, the eigenvalue of the inverse matrix A^{-1} is

$$\frac{1}{\lambda} = \lambda^{-1}$$

with the same eigenvector u. □

Example

Find the eigenvalues of A^7 where
$$A = \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{pmatrix}.$$

Solution. Because A is an upper triangular matrix, the eigenvalues are the entries on the leading diagonal, that is

$$\lambda = 1, 2, 3.$$

Solution. Because A is an upper triangular matrix, the eigenvalues are the entries on the leading diagonal, that is

$$\lambda = 1, 2, 3.$$

By the above theorem (a) we can see that the eigenvalues of A^7 are

$$\lambda = 1^7, 2^7, 3^7,$$

Solution. Because A is an upper triangular matrix, the eigenvalues are the entries on the leading diagonal, that is

$$\lambda = 1, 2, 3.$$

By the above theorem (a) we can see that the eigenvalues of A^7 are

$$\lambda = 1^7, 2^7, 3^7,$$

that is,

$$\lambda = 1, 128, 2187.$$

Question: For a square matrix A, is there an invertible matrix P such that $P^{-1}AP$ produces a diagonal matrix?

Example

Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. Determine $P^{-1}AP$.

Solution. We have

$$P^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Solution. We have

$$P^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Then

$$P^{-1}AP = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Solution. We have

$$P^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Then

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Similar matrices

A square matrix B is **similar** to a matrix A if there exists an invertible matrix P such that $P^{-1}AP = B$.

Similar matrices

A square matrix B is **similar** to a matrix A if there exists an invertible matrix P such that $P^{-1}AP = B$.

Theorem

Let A and B be similar matrices. The eigenvalues of these matrices are identical.

Proof. Since A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$.

Proof. Since A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$. We have to prove that the eigenvalues of A and B are identical,

Proof. Since A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$. We have to prove that the eigenvalues of A and B are identical, i.e.,

$$|B - \lambda I| = |A - \lambda I|$$

Proof. Since A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$. We have to prove that the eigenvalues of A and B are identical, i.e.,

$$|B - \lambda I| = |A - \lambda I|$$

We have

$$|B - \lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - P^{-1}\lambda IP|$$

Proof. Since A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$. We have to prove that the eigenvalues of A and B are identical, i.e.,

$$|B - \lambda I| = |A - \lambda I|$$

We have

$$\begin{aligned} |B - \lambda I| &= |P^{-1}AP - \lambda I| = |P^{-1}AP - P^{-1}\lambda I P| \\ &= |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| \end{aligned}$$

Proof. Since A and B are similar, there exists an invertible matrix P such that $P^{-1}AP = B$. We have to prove that the eigenvalues of A and B are identical, i.e.,

$$|B - \lambda I| = |A - \lambda I|$$

We have

$$\begin{aligned} |B - \lambda I| &= |P^{-1}AP - \lambda I| = |P^{-1}AP - P^{-1}\lambda IP| \\ &= |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| \\ &= |A - \lambda I|. \end{aligned}$$

Theorem

Eigenvectors v_1 and v_2 that correspond to distinct eigenvalues λ_1 and λ_2 of a 2×2 matrix are linearly independent.

Proof. Suppose that

$$c_1v_1 + c_2v_2 = 0. \tag{1}$$

Proof. Suppose that

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiplying both sides of Equation (1) by A,

Proof. Suppose that

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiplying both sides of Equation (1) by A, we get

$$c_1 A v_1 + c_2 A v_2 = 0$$

Proof. Suppose that

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiplying both sides of Equation (1) by A, we get

$$\begin{aligned} c_1 A v_1 + c_2 A v_2 &= 0 \\ \Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 &= 0. \end{aligned}$$

Proof. Suppose that

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiplying both sides of Equation (1) by A, we get

$$\begin{aligned} c_1 A v_1 + c_2 A v_2 &= 0 \\ \Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 &= 0. \end{aligned} \tag{2}$$

Now, multiplying both sides of Equation (1) by λ_2,

Proof. Suppose that

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiplying both sides of Equation (1) by A, we get

$$\begin{aligned} c_1 A v_1 + c_2 A v_2 &= 0 \\ \Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 &= 0. \end{aligned} \tag{2}$$

Now, multiplying both sides of Equation (1) by λ_2, we get

$$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0 \tag{3}$$

Proof. Suppose that

$$c_1 v_1 + c_2 v_2 = 0. \tag{1}$$

Multiplying both sides of Equation (1) by A, we get

$$\begin{aligned} c_1 A v_1 + c_2 A v_2 &= 0 \\ \Rightarrow c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 &= 0. \end{aligned} \tag{2}$$

Now, multiplying both sides of Equation (1) by λ_2, we get

$$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = 0 \tag{3}$$

Subtracting (3) from (2), we get

$$(\lambda_1 - \lambda_2) c_1 v_1 = 0.$$

Since $\lambda_1 \neq \lambda_2$ and $v_1 \neq 0$, we must have $c_1 = 0$.

Since $\lambda_1 \neq \lambda_2$ and $v_1 \neq 0$, we must have $c_1 = 0$. Similarly, $c_2 = 0$.

Since $\lambda_1 \neq \lambda_2$ and $v_1 \neq 0$, we must have $c_1 = 0$.
Similarly, $c_2 = 0$.
Therefore, v_1 and v_2 are linearly independent. $\square$

Generalizing the above theorem we obtain the following theorem.

Generalizing the above theorem we obtain the following theorem.

Theorem

Eigenvectors $v_1, ..., v_n$ that correspond to distinct eigenvalues $\lambda_1, ..., \lambda_n$ of an $n \times n$ matrix are linearly independent.

Proof. Suppose that

$$c_1 v_1 + \cdots + c_n v_n = 0. \tag{4}$$

Proof. Suppose that

$$c_1 v_1 + \cdots + c_n v_n = 0. \tag{4}$$

Multiplying both sides of Equation (4) by A, we get

$$c_1 A v_1 + \cdots + c_n A v_n = 0$$

Proof. Suppose that

$$c_1 v_1 + \cdots + c_n v_n = 0. \tag{4}$$

Multiplying both sides of Equation (4) by A, we get

$$\begin{aligned} c_1 A v_1 + \cdots + c_n A v_n &= 0 \\ \Rightarrow c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n &= 0. \end{aligned}$$

Proof. Suppose that

$$c_1 v_1 + \cdots + c_n v_n = 0. \tag{4}$$

Multiplying both sides of Equation (4) by A, we get

$$\begin{aligned} c_1 A v_1 + \cdots + c_n A v_n &= 0 \\ \Rightarrow c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n &= 0. \end{aligned} \tag{5}$$

Now, multiplying both sides of Equation (4) by λ_n, we get

$$c_1 \lambda_n v_1 + \cdots + c_n \lambda_n v_n = 0 \tag{6}$$

Proof. Suppose that

$$c_1 v_1 + \cdots + c_n v_n = 0. \tag{4}$$

Multiplying both sides of Equation (4) by A, we get

$$\begin{aligned} c_1 A v_1 + \cdots + c_n A v_n &= 0 \\ \Rightarrow c_1 \lambda_1 v_1 + \cdots + c_n \lambda_n v_n &= 0. \end{aligned} \tag{5}$$

Now, multiplying both sides of Equation (4) by λ_n, we get

$$c_1 \lambda_n v_1 + \cdots + c_n \lambda_n v_n = 0 \tag{6}$$

Subtracting (6) from (5), we get

$$(\lambda_1 - \lambda_n) c_1 v_1 + \cdots + (\lambda_{n-1} - \lambda_n) c_{n-1} v_{n-1} = 0.$$

Thus, we have eliminated the term containing v_n.

Thus, we have eliminated the term containing v_n. Similarly, we can eliminate the terms containing $v_2, ..., v_{n-1}$ to get

$$(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) c_1 v_1 = 0.$$

Thus, we have eliminated the term containing v_n. Similarly, we can eliminate the terms containing $v_2, ..., v_{n-1}$ to get

$$(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) c_1 v_1 = 0.$$

Since eigenvectors are distinct and $v_1 \neq 0$, we must have $c_1 = 0$.

Thus, we have eliminated the term containing v_n. Similarly, we can eliminate the terms containing $v_2, ..., v_{n-1}$ to get

$$(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) c_1 v_1 = 0.$$

Since eigenvectors are distinct and $v_1 \neq 0$, we must have $c_1 = 0$.
In this way, we can show that each $c_i = 0$.

Thus, we have eliminated the term containing v_n.
Similarly, we can eliminate the terms containing $v_2, ..., v_{n-1}$ to get

$$(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) c_1 v_1 = 0.$$

Since eigenvectors are distinct and $v_1 \neq 0$, we must have $c_1 = 0$.
In this way, we can show that each $c_i = 0$.
Therefore, eigenvectors $v_1, ..., v_n$ are linearly independent. $\square$

Diagonalizable matrix

An $n \times n$ matrix A is said to **diagonalizable** if it is similar to a diagonal matrix D.

Diagonalizable matrix

An $n \times n$ matrix A is said to **diagonalizable** if it is similar to a diagonal matrix D.

Theorem

An $n \times n$ matrix A is **diagonalizable** iff it has n linearly independent eigenvectors.

Theorem

If an $n \times n$ matrix A has n distinct eigenvalues, then it is diagonalizable.

Theorem

If a matrix A is symmetric, then any two distinct eigenvectors corresponding to different eigenvalues are orthogonal.

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2.

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\lambda_1 v_1 \cdot v_2$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\lambda_1 v_1 \cdot v_2 = (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned} \lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (A v_1)^T v_2 \\ &= (v_1^T A^T) v_2 \end{aligned}$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned} \lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (A v_1)^T v_2 \\ &= (v_1^T A^T) v_2 = v_1^T (A^T v_2) \end{aligned}$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned}\lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\ &= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (Av_2)\end{aligned}$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned} \lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\ &= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (Av_2) \\ &= v_1^T (\lambda_2 v_2) \end{aligned}$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned} \lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\ &= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (A v_2) \\ &= v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2 \end{aligned}$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned}
\lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\
&= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (Av_2) \\
&= v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2 \\
&= \lambda_2 v_1 \cdot v_2.
\end{aligned}$$

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned}\lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\ &= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (Av_2) \\ &= v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2 \\ &= \lambda_2 v_1 \cdot v_2.\end{aligned}$$

Hence $(\lambda_1 - \lambda_2) v_1 \cdot v_2 = 0$,

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned}
\lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\
&= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (Av_2) \\
&= v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2 \\
&= \lambda_2 v_1 \cdot v_2.
\end{aligned}$$

Hence $(\lambda_1 - \lambda_2) v_1 \cdot v_2 = 0$, but $\lambda_1 - \lambda_2 \neq 0$,

Proof. Let v_1 and v_2 be eigenvectors that correspond to eigenvalues λ_1 and λ_2. To show that $v_1 \cdot v_2 = 0$, compute

$$\begin{aligned}
\lambda_1 v_1 \cdot v_2 &= (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 \\
&= (v_1^T A^T) v_2 = v_1^T (A^T v_2) = v_1^T (Av_2) \\
&= v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2 \\
&= \lambda_2 v_1 \cdot v_2.
\end{aligned}$$

Hence $(\lambda_1 - \lambda_2) v_1 \cdot v_2 = 0$, but $\lambda_1 - \lambda_2 \neq 0$, so

$$v_1 \cdot v_2 = 0.$$

We mention the following result without proof.

Theorem

If A is a symmetric matrix, then all eigenvalues of A are real.

Example

Consider

$$A = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

Compute its eigenvalues. Verify that

$$\begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

are orthogonal eigenvectors of A.

Solution. Note that this matrix is symmetric. First, we compute 3 eigenvalues.

To do so, consider

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 4 & 3 \\ 4 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{pmatrix}$$

In this case,

$$|A - \lambda I| = (1 - \lambda)((1 - \lambda)^2 - 0) - 4(4(1 - \lambda) - 0)$$

In this case,

$$|A-\lambda I| = (1-\lambda)((1-\lambda)^2 - 0) - 4(4(1-\lambda) - 0) \\ + 3(0 - 3(1-\lambda))$$

In this case,

$$
\begin{aligned}
|A - \lambda I| &= (1-\lambda)((1-\lambda)^2 - 0) - 4(4(1-\lambda) - 0) \\
&\quad + 3(0 - 3(1-\lambda)) \\
&= (1-\lambda)^3 - 25(1-\lambda)
\end{aligned}
$$

In this case,

$$
\begin{aligned}
|A - \lambda I| &= (1-\lambda)((1-\lambda)^2 - 0) - 4(4(1-\lambda) - 0) \\
&\quad + 3(0 - 3(1-\lambda)) \\
&= (1-\lambda)^3 - 25(1-\lambda) \\
&= (1-\lambda)((1-\lambda)^2 - 25)
\end{aligned}
$$

In this case,

$$\begin{aligned}|A-\lambda I| &= (1-\lambda)((1-\lambda)^2-0)-4(4(1-\lambda)-0)\\ &\quad +3(0-3(1-\lambda))\\ &= (1-\lambda)^3-25(1-\lambda)\\ &= (1-\lambda)((1-\lambda)^2-25)\\ &= (1-\lambda)(\lambda-6)(\lambda+4).\end{aligned}$$

In this case,

$$\begin{aligned}|A-\lambda I| &= (1-\lambda)((1-\lambda)^2-0)-4(4(1-\lambda)-0)\\ &\quad +3(0-3(1-\lambda))\\ &= (1-\lambda)^3-25(1-\lambda)\\ &= (1-\lambda)((1-\lambda)^2-25)\\ &= (1-\lambda)(\lambda-6)(\lambda+4).\end{aligned}$$

We see that if $\lambda=-4,1,6$, then

$$|A-\lambda I|=0.$$

In this case,

$$
\begin{aligned}
|A-\lambda I| &= (1-\lambda)((1-\lambda)^2-0)-4(4(1-\lambda)-0) \\
&\quad +3(0-3(1-\lambda)) \\
&= (1-\lambda)^3-25(1-\lambda) \\
&= (1-\lambda)((1-\lambda)^2-25) \\
&= (1-\lambda)(\lambda-6)(\lambda+4).
\end{aligned}
$$

We see that if $\lambda=-4,1,6$, then

$$|A-\lambda I|=0.$$

So, the eigenvalues are $\lambda=-4,1,6$.

We now verify that the given vectors are eigenvectors.

We now verify that the given vectors are eigenvectors. Here they are:

$$v_1 = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} : \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

We now verify that the given vectors are eigenvectors. Here they are:

$$v_1 = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} : \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 24 \\ 18 \end{pmatrix} = 6 \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

We now verify that the given vectors are eigenvectors. Here they are:

$$v_1 = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} : \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 30 \\ 24 \\ 18 \end{pmatrix} = 6 \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}.$$

Thus, the eigenvector v_1 is associated with $\lambda_1 = 6$.

$$v_2 = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} : \quad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} : \quad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}.$$

$$v_2 = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} : \quad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} = 1 \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}.$$

Thus, the eigenvector v_2 is associated with $\lambda_2 = 1$.

$$v_3 = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}: \quad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} : \quad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = -4 \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

$$v_3 = \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} : \quad \begin{pmatrix} 1 & 4 & 3 \\ 4 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} = -4 \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

Thus, the eigenvector v_3 is associated with $\lambda_3 = -4$.

To verify that the eigenvectors v_1, v_2, v_3 are orthogonal

To do so, let $V_O = (v_1 \ v_2 \ v_3)$ and let's multiply it by it's transpose:

To verify that the eigenvectors v_1, v_2, v_3 are orthogonal

To do so, let $V_O = (v_1\ v_2\ v_3)$ and let's multiply it by it's transpose:

$$V_O^T V_O = \begin{pmatrix} 5 & 4 & 3 \\ 0 & -3 & 4 \\ -5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & -5 \\ 4 & -3 & 4 \\ 3 & 4 & 3 \end{pmatrix}$$

To verify that the eigenvectors v_1, v_2, v_3 are orthogonal

To do so, let $V_O = (v_1 \; v_2 \; v_3)$ and let's multiply it by it's transpose:

$$V_O^T V_O = \begin{pmatrix} 5 & 4 & 3 \\ 0 & -3 & 4 \\ -5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & -5 \\ 4 & -3 & 4 \\ 3 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 50 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 50 \end{pmatrix}$$

To verify that the eigenvectors v_1, v_2, v_3 are orthogonal

To do so, let $V_O = (v_1 \; v_2 \; v_3)$ and let's multiply it by it's transpose:

$$V_O^T V_O = \begin{pmatrix} 5 & 4 & 3 \\ 0 & -3 & 4 \\ -5 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 & -5 \\ 4 & -3 & 4 \\ 3 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 50 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 50 \end{pmatrix}$$

So, these three vectors are eigenvectors, and they are orthogonal, and so they provide a basis for $\mathbb{R}^3$.

To normalize them, we divide by their norms, the square of which are the diagonal elements of $V_O^T V_O$.

Here are the normalized eigenvectors.

$$\lambda_1 = 6: \qquad v_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

$$\lambda_1 = 1: \qquad v_2 = \frac{1}{\sqrt{25}} \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$$

$$\lambda_1 = -4: \qquad v_1 = \frac{1}{\sqrt{50}} \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}.$$

Let's now work with these normalized eigenvectors. In this case, the 3×3 matrix of normalized eigenvectors (the columns of which form an orthonormal basis for $\mathbb{R}^3$) is

$$V = (v_1 \ v_2 \ v_3) = \begin{pmatrix} 5/\sqrt{50} & 0 & -5/\sqrt{50} \\ 4/\sqrt{50} & -3/\sqrt{25} & 4/\sqrt{50} \\ 3/\sqrt{50} & 4/\sqrt{25} & 3/\sqrt{50} \end{pmatrix}.$$

Clearly, $V^T V = I$. Moreover,

$$AV = V\Lambda.$$

Here is how do we express the matrix in two standard forms.