# Unit 2J: Orthonormal Bases

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# Summary

# Orthogonal and orthonormal sets

## Orthogonal set

A set $S$ of vectors in $\mathbb{R}^n$ is called **orthogonal** if every pair of distinct vectors in $S$ are orthogonal.

A set $S$ of vectors in $\mathbb{R}^n$ is called **orthonormal** if $S$ is orthogonal and every vector in $S$ is a unit vector.

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## Example.

It is easy to check that the standard basis $E = \{e_1, e_2, ..., e_n\}$ for $\mathbb{R}^n$ is an orthonormal set.

## Example.

Let $u_1 = (2, 0, 0)$, $u_2 = (0, 1, 1)$ and $u_3 = (0, 1, -1)$. Find the orthonormal set associated with the set $S = \{u_1, u_2, u_3\}$.

**Solution:**

We have

$$u_1 \cdot u_2 = 2 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0$$

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$$u_2 \cdot u_3 = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot (-1) = 0.$$

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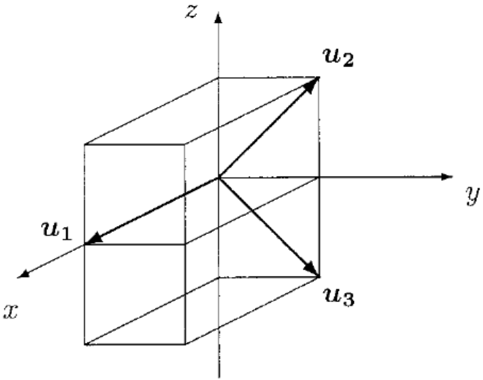
$$u_1 \cdot u_3 = 2 \cdot 0 + 0 \cdot 1 + 0 \cdot (-1) = 0$$
$$u_2 \cdot u_3 = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot (-1) = 0.$$

Hence the set $S$ is orthogonal.

# Solution...



**Solution...**

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Then for $i = 1, 2, 3$

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Therefore, the set $\{v_1, v_2, v_3\}$ is orthonormal. $\square$

**Note that**

The process of converting an orthogonal set to an orthonormal set by multiplying each vector $u$ by $\frac{1}{\|u\|}$ is called **normalizing**.

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## Theorem

An orthogonal set of nonzero vectors in a vector space is linearly independent.

**Proof.**

Let $S = \{v_1, v_2, ..., v_k\}$.

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Since $S$ is orthogonal, $v_i \cdot v_j = 0$ for $i \neq j$.

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Let $S = \{v_1, v_2, ..., v_k\}$. Consider the equation

$$c_1 v_1 + c_2 v_2 + ... + c_k v_k = 0.$$

Since $S$ is orthogonal, $v_i \cdot v_j = 0$ for $i \neq j$. For $i = 1, 2, ..., k$,

$$0 = (c_1 v_1 + c_2 v_2 + ... + c_k v_k) \cdot v_i$$

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$$\begin{aligned} 0 &= (c_1v_1 + c_2v_2 + ... + c_kv_k) \cdot v_i \\ &= (c_1v_1) \cdot v_i + (c_2v_2) \cdot v_i + ... + (c_kv_k) \cdot v_i \end{aligned}$$

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$$\begin{aligned} 0 &= (c_1v_1 + c_2v_2 + ... + c_kv_k) \cdot v_i \\ &= (c_1v_1) \cdot v_i + (c_2v_2) \cdot v_i + ... + (c_kv_k) \cdot v_i \\ &= c_1(v_1 \cdot v_i) + c_2(v_2 \cdot v_i) + ... + c_k(v_k \cdot v_i) \end{aligned}$$

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Let $S = \{v_1, v_2, ..., v_k\}$. Consider the equation

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$$\begin{aligned} 0 &= (c_1 v_1 + c_2 v_2 + ... + c_k v_k) \cdot v_i \\ &= (c_1 v_1) \cdot v_i + (c_2 v_2) \cdot v_i + ... + (c_k v_k) \cdot v_i \\ &= c_1 (v_1 \cdot v_i) + c_2 (v_2 \cdot v_i) + ... + c_k (v_k \cdot v_i) \\ &= c_i (v_i \cdot v_i) \end{aligned}$$

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Given that $v_i \neq 0$ for each $i$, we have $v_i \cdot v_i \neq 0$.

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Given that $v_i \neq 0$ for each $i$, we have $v_i \cdot v_i \neq 0$. This means, $c_i = 0$ for each $i$. Therefore, $S$ is linearly independent. $\square$

# Orthogonal and orthonormal bases

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- We have shown that the set $S = \{u_1, u_2, u_3\}$ is orthogonal. It is clear that each vector in $S$ is nonzero. Hence by the previous theorem, $S$ is linearly independent.
- We also know that any linearly independent set of $n$ elements of an $n$-dimensional vector space $V$ must span $V$ and thus is a basis.

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- We also know that any linearly independent set of $n$ elements of an $n$-dimensional vector space $V$ must span $V$ and thus is a basis. Therefore, $S$ is an orthogonal basis for $\mathbb{R}^3$.

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- We have shown that the set $S = \{u_1, u_2, u_3\}$ is orthogonal. It is clear that each vector in $S$ is nonzero. Hence by the previous theorem, $S$ is linearly independent.
- We also know that any linearly independent set of $n$ elements of an $n$-dimensional vector space $V$ must span $V$ and thus is a basis. Therefore, $S$ is an orthogonal basis for $\mathbb{R}^3$.
- Moreover, the orthonormal set associated with $S$ is an orthonormal basis for $\mathbb{R}^3$.

## Theorem

Let $B = \{v_1, v_2, ..., v_k\}$ be an orthogonal basis for a vector space $V$. Then for any vector $w \in V$,

$$w = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + ... + \frac{w \cdot v_k}{v_k \cdot v_k} v_k,$$

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that is,

$$w = \left( \frac{w \cdot v_1}{v_1 \cdot v_1}, \frac{w \cdot v_2}{v_2 \cdot v_2}, ..., \frac{w \cdot v_k}{v_k \cdot v_k} \right)$$

with respect to Basis $B$.

## Theorem...

Moreover, if $B = \{v_1, v_2, ..., v_k\}$ is an orthonormal basis for a vector space $V$, then for any vector $w \in V$,

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that is,

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with respect to Basis $B$.

**Proof.**

Let $B = \{v_1, v_2, ..., v_k\}$ be an orthogonal basis for a vector space $V$ and $w \in V$.

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Let $B = \{v_1, v_2, ..., v_k\}$ be an orthogonal basis for a vector space $V$ and $w \in V$. Then

$$w = c_1 v_1 + c_2 v_2 + ... + c_k v_k$$

for some real numbers $c_1, c_2, ..., c_k$.

## Proof.

Let $B = \{v_1, v_2, ..., v_k\}$ be an orthogonal basis for a vector space $V$ and $w \in V$. Then

$$w = c_1v_1 + c_2v_2 + ... + c_kv_k$$

for some real numbers $c_1, c_2, ..., c_k$. Since $S$ is orthogonal, $v_i \cdot v_j = 0$ for $i \neq j$.

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for some real numbers $c_1, c_2, ..., c_k$. Since $S$ is orthogonal, $v_i \cdot v_j = 0$ for $i \neq j$. For $i = 1, 2, ..., k$,

$$w \cdot v_i = (c_1 v_1 + c_2 v_2 + ... + c_k v_k) \cdot v_i$$

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for some real numbers $c_1, c_2, ..., c_k$. Since $S$ is orthogonal, $v_i \cdot v_j = 0$ for $i \neq j$. For $i = 1, 2, ..., k$,

$$\begin{aligned} w \cdot v_i &= (c_1v_1 + c_2v_2 + ... + c_kv_k) \cdot v_i \\ &= (c_1v_1) \cdot v_i + (c_2v_2) \cdot v_i + ... + (c_kv_k) \cdot v_i \end{aligned}$$

**Proof.**

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$$\begin{aligned} w \cdot v_i &= (c_1 v_1 + c_2 v_2 + ... + c_k v_k) \cdot v_i \\ &= (c_1 v_1) \cdot v_i + (c_2 v_2) \cdot v_i + ... + (c_k v_k) \cdot v_i \\ &= c_1 (v_1 \cdot v_i) + c_2 (v_2 \cdot v_i) + ... + c_k (v_k \cdot v_i) \end{aligned}$$

## Proof.

Let $B = \{v_1, v_2, ..., v_k\}$ be an orthogonal basis for a vector space $V$ and $w \in V$. Then

$$w = c_1 v_1 + c_2 v_2 + ... + c_k v_k$$

for some real numbers $c_1, c_2, ..., c_k$. Since $S$ is orthogonal, $v_i \cdot v_j = 0$ for $i \neq j$. For $i = 1, 2, ..., k$,

$$\begin{aligned} w \cdot v_i &= (c_1 v_1 + c_2 v_2 + ... + c_k v_k) \cdot v_i \\ &= (c_1 v_1) \cdot v_i + (c_2 v_2) \cdot v_i + ... + (c_k v_k) \cdot v_i \\ &= c_1 (v_1 \cdot v_i) + c_2 (v_2 \cdot v_i) + ... + c_k (v_k \cdot v_i) \\ &= c_i (v_i \cdot v_i) \end{aligned}$$

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Given that $v_i \neq 0$ for each $i$, we have $v_i \cdot v_i \neq 0$.

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Given that $v_i \neq 0$ for each $i$, we have $v_i \cdot v_i \neq 0$.
Therefore,

$$c_i = \frac{w \cdot v_i}{v_i \cdot v_i}$$

for each $i$.

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Given that $v_i \neq 0$ for each $i$, we have $v_i \cdot v_i \neq 0$. Therefore,

$$c_i = \frac{w \cdot v_i}{v_i \cdot v_i}$$

for each $i$. That means,

$$w = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + \ldots + \frac{w \cdot v_k}{v_k \cdot v_k} v_k.$$

## Proof ...

That is,

$$w = \left( \frac{w \cdot v_1}{v_1 \cdot v_1}, \frac{w \cdot v_2}{v_2 \cdot v_2}, ..., \frac{w \cdot v_k}{v_k \cdot v_k} \right)$$

with respect to Base $B$.

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with respect to Base $B$.

If $B = \{v_1, v_2, ..., v_k\}$ is an orthonormal basis for a vector space $V$, then the result follows from the previous result, because $v_i \cdot v_i = \|v_i\|_2 = 1$ for all $i$. □

## Example

Let
$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$
Show that $B = \{v_1, v_2\}$ is an orthonormal basis for $\mathbb{R}^2$. Find a vector $x \in \mathbb{R}^2$ with respect to the basis $B$.

**Solution:**

We have

$$v_1 \cdot v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

**Solution:**

We have

$$v_1 \cdot v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2}(1 \cdot 1 + 1 \cdot (-1)) = 0.$$

**Solution:**

We have

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Also, we have

$$v_1 \cdot v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

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We have

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Similarly,

$$v_2 \cdot v_2 = 1.$$

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Therefore, $B = \{v_1, v_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

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To find a vector $x \in \mathbb{R}^2$ with respect to the basis $B$, assume that $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with respect to the standard basis for $\mathbb{R}^2$.

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$$v_1 \cdot x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$v_1 \cdot x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}(x_1 + x_2)$$

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and

$$v_2 \cdot x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}(x_1 - x_2).$$

**Solution...**

Therefore,

$$x = \left( \frac{1}{\sqrt{2}}(x_1 + x_2), \frac{1}{\sqrt{2}}(x_1 - x_2) \right)$$

with respect to the basis $B$.

## Example

Let $V$ be a plane in $\mathbb{R}^3$ defined by the equation $ax + by + cz = 0$. Let $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. For any vector $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in V$, we have

$$n \cdot u = ax + by + cz = 0.$$

Thus $n$ is orthogonal to $V$.

## Example ...

In fact,

$$\begin{aligned} V &= \{(x, y, z) \in \mathbb{R}^3 : \; ax + by + cz = 0\} \\ &= \{u \in \mathbb{R}^3 | \; n \cdot u = 0\}. \end{aligned}$$

The vector $n$ is a normal vector of $V$.

## Example

Let $V = \text{span}\{u_1, u_2\}$ be a subspace of $\mathbb{R}^4$, where $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$. Find all vectors that are orthogonal to $V$.

**Solution:**

Let $v = (w, x, y, z)$ be a vector in $\mathbb{R}^4$ .

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Let $v = (w, x, y, z)$ be a vector in $\mathbb{R}^4$ . Then

$$v \cdot (au_1 + bu_2) = 0 \text{ for all } a, b \in \mathbb{R}$$

**Solution:**

Let $v = (w, x, y, z)$ be a vector in $\mathbb{R}^4$ . Then

$$
\begin{aligned}
& v \cdot (au_1 + bu_2) = 0 \text{ for all } a, b \in \mathbb{R} \\
\Leftrightarrow \quad & v \cdot u_1 = 0, \;\; v \cdot u_2 = 0 \\
\Leftrightarrow \quad & \begin{cases} w + x + y = 0 \\ -x - y + z = 0 \end{cases}
\end{aligned}
$$

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\Leftrightarrow \quad & v \cdot u_1 = 0, \;\; v \cdot u_2 = 0 \\
\Leftrightarrow \quad & \begin{cases} w + x + y = 0 \\ -x - y + z = 0 \end{cases} \\
\Leftrightarrow \quad & (w, x, y, z) = (-t, -s + t, s, t) \text{ for some } s, t \in \mathbb{R}.
\end{aligned}
$$

**Solution ...**

So a vector $v$ is orthogonal to $V$ if and only if
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So a vector $v$ is orthogonal to $V$ if and only if

$$v = (-t, -s+t, s, t) = s(0, -1, 1, 0) + t(-1, 1, 0, 1)$$

for some $s, t \in \mathbb{R}$,

**Solution ...**

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Therefore, span$\{(0,-1,1,0), (-1,1,0,1)\}$ is orthogonal to $V$. $\square$

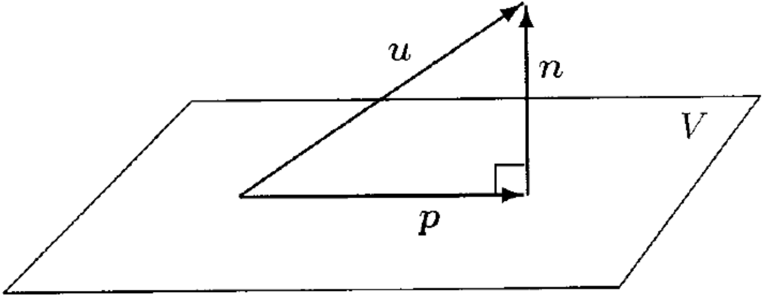
# Orthogonal projection

## Orthogonal projection

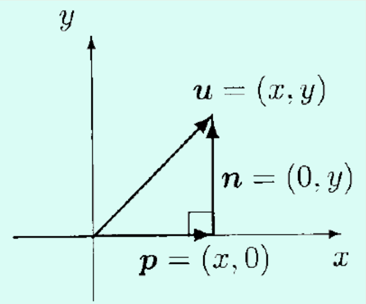
Let $V$ be a subspace of $\mathbb{R}^n$. Every vector $u \in \mathbb{R}^n$ can be written uniquely as

$$u = n + p$$

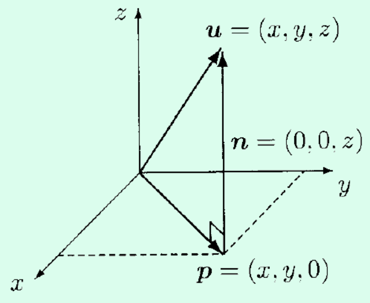
such that $n$ is a vector orthogonal to $V$ and $p$ is a vector in $V$. The vector $p$ is called the (**orthogonal**) **projection** of $u$ onto $V$.


$u$
$n$
$V$
$p$

The projection of
$u = \begin{pmatrix} x \\ y \end{pmatrix}$
onto the $x$-axis is
$p = \begin{pmatrix} x \\ 0 \end{pmatrix}$.
Here,
$n = \begin{pmatrix} 0 \\ y \end{pmatrix}$.



The projection of $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ onto the $xy$-plane is $p = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. Here, $n = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$.



## Theorem

Let $V$ be a subspace of $\mathbb{R}^n$ and $w$ a vector in $\mathbb{R}^n$.

**(a)** If $\{v_1, v_2, ..., v_k\}$ be an orthogonal basis for $V$, then

$$\frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + ... + \frac{w \cdot v_k}{v_k \cdot v_k} v_k,$$

is the projection of $w$ onto $V$.

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**(b)** If $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis for $V$, then

$(w \cdot v_1)v_1 + (w \cdot v_2)v_2 + ... + (w \cdot v_k)v_k$,

is the projection of $w$ onto $V$.

**Proof.**

**(a)** Let

$$p = \frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 + \ldots + \frac{w \cdot v_k}{v_k \cdot v_k} v_k, \qquad n = w - p.$$

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This means, $n$ is orthogonal to $V$. Since $w = n + p$ where $n \perp V$ and $p \in V$,

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This means, $n$ is orthogonal to $V$. Since $w = n + p$ where $n \perp V$ and $p \in V$, p is the projection of $w$ onto $V$.

**Proof...**

**(b)** This is a consequence of part (a), becasuse

$$v_i \cdot v_i = \|v_i\|_2^2 = 1.$$

$\square$

## Example

Let $V$ be a subspace of $\mathbb{R}^3$ spanned by the orthogonal vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Find the projection of $w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ onto $V$.

**Solution:**

The projection of $w = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ onto $V$ is equal to

$$\frac{w \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{w \cdot v_2}{v_2 \cdot v_2} v_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

# Gram-Schmidt Process

**Discussion**

**(a)** Let $\{u_1, u_2\}$ be a basis for a vector space $V$ where $V$ is either $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$ containing the origin.

# Gram-Schmidt Process

**Discussion**

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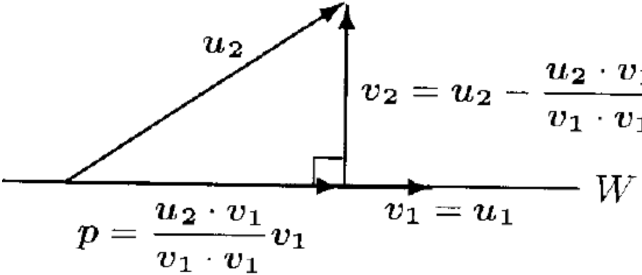
# Gram-Schmidt Process

## Discussion

**(a)** Let $\{u_1, u_2\}$ be a basis for a vector space $V$ where $V$ is either $\mathbb{R}^2$ or a plane in $\mathbb{R}^3$ containing the origin. Let $W$ be the subspace of $V$ spanned by $u_1$. (W is a line through the origin.) Then the projection of $u_2$ onto $W$ is

$$p = \frac{u_2 \cdot u_1}{u_1 \cdot u_1} u_1$$

Define $v_1 = u_1$ and $v_2 = u_2 - p = u_2 - \dfrac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$.




Then $\{v_1, v_2\}$ is an orthogonal basis for $V$.

**(b)** Let $\{u_1, u_2, u_3\}$ be a basis for $\mathbb{R}^3$ and let $V$ be the subspace of $\mathbb{R}^3$ spanned by $u_1, u_2$. Let $W$ be the subspace of $V$ spanned by $u_1$. ($V$ is a plane containing the origin.) Define

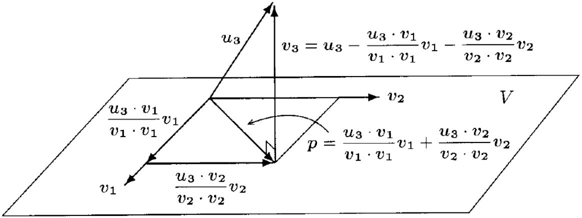
$$v_1 = u_1, \quad v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1.$$

By Part (a), $\{v_1, v_2\}$ is an orthogonal basis for $V$. Then the projection of $u_3$ onto $V$ is

$$p = \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2.$$

Define

$$v_3 = u_3 - p = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2.$$




Then $\{v_1, v_2, v_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

In general, we have the following process, known as **Gram-Schmidt Process**:

**Gram-Schmidt Process:**

Let $\{u_1, u_2, ..., u_k\}$ be a basis for a vector space $V$. Define

$$\begin{aligned}
v_1 &= u_1, \\
v_2 &= u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1, \\
v_3 &= u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2, \\
&\vdots \\
v_k &= u_k - \frac{u_k \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_k \cdot v_2}{v_2 \cdot v_2} v_2 - \ldots - \frac{u_k \cdot v_{k-1}}{v_{k-1} \cdot v_{k-1}} v_{k-1}
\end{aligned}$$

Then $\{v_1, v_2, v_3\}$ is an orthogonal basis for $V$.
Furthermore, let

$$w_1 = \frac{1}{\|v_1\|}v_1, w_2 = \frac{1}{\|v_2\|}v_2, ..., w_k = \frac{1}{\|v_k\|}v_k.$$

Then $\{w_1, w_2, w_3\}$ is an orthonormal basis for $V$.

# Orthonormality and Matrices

Let $\{v_1, v_2, ..., v_k\}$ be an orthonormal basis for vector space $V$. Then

$$v_i \cdot v_j = v_i^T v_j = 0 \text{ if } i \neq j$$

$$\|v_i\| = \sqrt{v_i^T v_i} = 1 \text{ for all } i.$$

If we use the relatively common notation that

$$\delta_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

then the two normality conditions can be written compactly as

$$v_i \cdot v_j = \delta_{ij} \quad \text{for all } i \in \{1, 2, ..., k\}.$$

To express the two orthonormality conditions in terms of conditions on matrices, let's define an $n \times k$ matrix $A$ to be

$$A = \begin{pmatrix} v_1 & v_2 & \cdots & v_k \end{pmatrix},$$

where $v_i = A_{:i}$ is the $i$th column of $A$.

Next, let's consider the matrix $A^T A$. Observe that $A^T A$ is a $k \times k$ matrix. Let's ask what information is contained in the $(ij)$ element of this matrix, i.e., in the element $(A^T A)_{ij}$ ? That is,

$$(A^T A)_{ij} = v_i^T v_j,$$

which, in this case, equals 1 or 0, depending on whether or not $i = j$.

That is, in this case,

$$(A^T A)_{ij} = \delta_{ij}$$

and we can write the matrix product as

$$A^T A = I_k,$$

where $I_k$ is the identity matrix of dimension $k$. This is the matrix way to express that a matrix has columns that form an orthonormal basis.

Given an $n \times n$ matrix $A$, recall that the inverse matrix $A^{-1}$ is the matrix such that

$$A^{-1}A = AA^{-1} = I_n.$$

So, we have the following definition:

**Orthogonal matrix**

A square matrix $A$ is called **orthogonal** if

$$A^{-1} = A^T.$$

## Example

Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Determine if $A$ is an orthogonal matrix.

## Example

Let

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Determine if $B$ is an orthogonal matrix.

## Practice Problems

1. Determine if the matrix
$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$
is an orthogonal matrix.

2. Prove that the product of two orthogonal matrices is orthogonal.

3. If $A = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{pmatrix}$ is an orthogonal matrix, then find the values of $x$ and $y$.

From the above discussion, we have the following result:

Let $A$ be a square matrix of order $n$. The following statements are equivalent:

1. $A$ is orthogonal.
2. The columns of $A$ form an orthonormal basis for $\mathbb{R}^n$.
3. The rows of $A$ form an orthonormal basis for $\mathbb{R}^n$.

Suppose that we have a set of vectors $v_1, v_2, ..., v_k \in \mathbb{R}^n$, such that any two different vectors $v_i$ and $v_j$ in that set are orthogonal to each other. To organize this information into a matrix, let's define an $n \times k$ matrix $A$ to be

$$A = (v_1 \quad v_2 \quad \dots \quad v_k).$$

The requirement that the basis vectors be orthogonal to each other means that

$$A^T A = D,$$

where $D$ is a $k \times k$ diagonal matrix, all the diagonal entries of which are positive.

The matrix $D$ is diagonal since the off-diagonal elements are the dot products between different basis vectors, which equal zero, since they are orthogonal; and the diagonal entries are all non-zero since each vector in the basis is non-zero and thus has a non-zero non-negative norm.