Unit 3: Spectral theory III

Prof.Dr.P.M.Bajracharya

School of Mathematical Sciences
T.U., Kirtipur

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Summary

Spectral decompositions

Example. A matrix with non-orthogonal eigenvectors.

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Observe that $v_{\lambda=5}$ and $v_{\lambda=-1}$ are linearly independent, Observe that $v_{\lambda=5}$ and $v_{\lambda=-1}$ are linearly independent, but they are not orthogonal to each other.

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Thus, the given matrix is an example of a real-valued matrix, i.e., a matrix whose elements consist of only real numbers, that has eigenvalues that are imaginary/complex numbers and eigenvectors that contain imaginary/complex entries.

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Thus, the given matrix is an example of a real-valued $n \times n$ matrix that has fewer than n eigenvectors.

The matrices considered above are “not nice” in the sense that they do not have have a full set of n orthogonal real-valued eigenvectors and associated real-valued eigenvalues.

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$u_{\lambda=0} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. Then the corresponding unit vector is given by

$$v_{\lambda=0} = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$$

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Further, put

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Let us compute

$$V\Lambda V^T.$$

We have

$$V\Lambda V^T = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

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&= \begin{pmatrix} \sqrt{5} & 0 \\ 2\sqrt{5} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}
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&= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.
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Therefore,

$$A = 5 \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} + 0 \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$

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Outer product

The product

$$\lambda_i v_i v_i^T$$

is called an **outer product**.

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Basically, this will correspond to expressing A in terms of linear combinations of these special vectors and numbers.

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By nice properties, we mean that for a given $n \times n$ matrix A we can find a full set of eigenvectors that form an orthonormal basis of $\mathbb{R}^n$.

Problem

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Expressing A as a product of 3 matrices.

We have

$$V\Lambda V^T = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

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We have

$$V\Lambda V^T = \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Expressing A as a product of 3 matrices.

We have

$$\begin{aligned} V\Lambda V^T &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = A. \end{aligned}$$

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Therefore,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Expressing A as a sum of 2 outer products. We have

$$\sum_{i=1}^{2} \lambda_i v_i v_i^T = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T$$

Expressing A as a sum of 2 outer products. We have

$$\begin{aligned}\sum_{i=1}^{2}\lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T\\ &= 3\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + 1\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}\end{aligned}$$

Expressing A as a sum of 2 outer products. We have

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\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

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&= \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = A.
\end{aligned}
$$

Therefore,

$$
A = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}.
$$

For the time being, we will describe it in the context of the simple 2×2 matrices.

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To do this, let's enumerate the eigenvectors and eigenvalues in a way that will make it easier to generalize beyond 2×2 matrices.

- We enumerate eigenvalues in decreasing order.
- If a matrix has two (or more) identical eigenvalues, then we repeat i in λ_i according to the multiplicity, and we can enumerate their associated orthogonal eigenvectors arbitrarily.

Given this numbering convention, for a 2×2 symmetric matrices, we have seen that there are two eigenvalue-eigenvector pairs (λ_i, v_i) that satisfy:

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$$Av_i = \lambda_i v_i, \tag{1}$$

where $i \in \{1, 2\}$.

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$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & v_2 \end{pmatrix}.$$

Here, both are 2×2 matrices. If the ith eigenvalue is Λ_{ii}, then the corresponding ith eigenvector is the ith column of V.

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Note that

Since the eigenvectors are unit-length and pair-wise orthogonal, V is an orthogonal matrix.

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Note that

Since the eigenvectors are unit-length and pair-wise orthogonal, V is an orthogonal matrix. Hence

$$VV^T = V^T V = I \in \mathbb{R}^{2\times 2}$$

with $V^T = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}$.

Spectral decomposition: Expressing A as a product of three matrices.

Theorem (Spectral decomposing I)

Let v_1, v_2 be the eigenvectors associated with the eigenvalues λ_1, λ_2 of a 2×2 symmetric matrix A respectively. If

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & v_2 \end{pmatrix},$$

then

$$A = V \Lambda V^T. \tag{2}$$

Proof.

We have

$$\begin{aligned} AV &= \begin{pmatrix} Av_1 & Av_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} \end{aligned}$$

Proof.

We have

$$\begin{aligned} AV &= (Av_1 \quad Av_2) \\ &= (\lambda_1 v_1 \quad \lambda_2 v_2) \\ &= \begin{pmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} \\ \lambda_1 v_{21} & \lambda_2 v_{22} \end{pmatrix} \end{aligned}$$

Proof.

We have

$$\begin{aligned} AV &= \begin{pmatrix} Av_1 & Av_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} \\ \lambda_1 v_{21} & \lambda_2 v_{22} \end{pmatrix} \\ &= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \end{aligned}$$

Proof.

We have

$$
\begin{aligned}
AV &= \begin{pmatrix} Av_1 & Av_2 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} \\ \lambda_1 v_{21} & \lambda_2 v_{22} \end{pmatrix} \\
&= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\
&= V\Lambda.
\end{aligned}
$$

Proof.

We have

$$\begin{aligned} AV &= \begin{pmatrix} Av_1 & Av_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_1 & \lambda_2 v_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 v_{11} & \lambda_2 v_{12} \\ \lambda_1 v_{21} & \lambda_2 v_{22} \end{pmatrix} \\ &= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= V\Lambda. \end{aligned}$$

Therefore, from this in view of $VV^T = I$, we obtain

$$A = V\Lambda V^T.$$

□

This theorem provides a decomposition of the matrix A into the product of three matrices:

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- an orthogonal matrix V (consisting of the eigenvectors of A),

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This theorem provides a decomposition of the matrix A into the product of three matrices:

- an orthogonal matrix V (consisting of the eigenvectors of A),
- a diagonal matrix Λ (consisting of the eigenvalues of A), and
- the transpose of that orthogonal matrix, V^T.

We can write Equation (2) in terms of individual elements as follows:

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$$A = V\Lambda V^T$$

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We can write Equation (2) in terms of individual elements as follows:

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This simple eigendecomposition is the first example of the *spectral theorem.*

This decomposition of the form $A = V\Lambda V^T$ holds for general $n \times n$ symmetric matrices, and generalizations of it hold more generally.

This decomposition of the form $A = V\Lambda V^T$ holds for general $n \times n$ symmetric matrices, and generalizations of it hold more generally. In particular, if we consider computing $y = Ax$, this is the same as

$$y = V\Lambda V^T x = V(\Lambda(V^T x))$$

Spectral decomposition: expressing A as a sum of outer products.

We can write Equation (2) in terms of the columns of V (which recall are the eigenvectors, which are also the rows of V^T) and elements of Λ (which are the eigenvalues).

Spectral decomposition: expressing A as a sum of outer products.

We can write Equation (2) in terms of the columns of V (which recall are the eigenvectors, which are also the rows of V^T) and elements of Λ (which are the eigenvalues). This gives the following theorem:

Theorem (Spectral decomposition II)

Let v_1, v_2 be the eigenvectors associated with the eigenvalues λ_1, λ_2 of a 2×2 symmetric matrix A respectively. Then

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T. \tag{3}$$

Proof.

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & v_2 \end{pmatrix},$$

Proof.

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = (v_1 \quad v_2),$$

Then by Spectral theorem I, we have

$$\begin{aligned} A &= V\Lambda V^T \\ &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\ &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 v_1^T \\ \lambda_2 v_2^T \end{pmatrix}. \end{aligned}$$

Proof.

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad V = (v_1 \quad v_2),$$

Then by Spectral theorem I, we have

$$\begin{aligned} A &= V\Lambda V^T \\ &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\ &= (v_1 \quad v_2) \begin{pmatrix} \lambda_1 v_1^T \\ \lambda_2 v_2^T \end{pmatrix}. \end{aligned}$$

This gives

$$A = \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T.$$

□

Equation (3) says that if we do this for every i and sum up the corresponding matrices, then we get the original matrix A.

Equation (3) says that if we do this for every i and sum up the corresponding matrices, then we get the original matrix A. Thus, the decomposition given by Equation (3) expresses the matrix A in terms of the sum of 2 terms, each of which is the outer product of an eigenvector with its transpose, multiplied/scaled by the corresponding eigenvalue.

This decomposition holds for general $n \times n$ symmetric matrices, in which case

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T,$$

and generalizations of it hold more generally.

This decomposition holds for general $n \times n$ symmetric matrices, in which case

$$A = \sum_{i=1}^{n} \lambda_i v_i v_i^T,$$

and generalizations of it hold more generally. We will discuss this later.
In particular, by the spectral theorem II, the equation $y = Ax$ is equivalent to the equation

$$y = \lambda_1 v_1 v_1^T x + \lambda_2 v_2 v_2^T x. \tag{4}$$

Identity matrix.

How do we express the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in two standard forms?

Solution: For this matrix, we have computed
Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1$,
Eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Expressing A as a sum of 2 outer products. We have

$$\begin{aligned}\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\ &= 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A.\end{aligned}$$

Therefore,

$$A = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Alternatively, if we choose $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$, then we obtain

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&\quad + 1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A.
\end{aligned}$$

Therefore,

$$\begin{aligned} A &= 1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &\quad + 1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}. \end{aligned}$$

If, on the other hand, we choose $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, then we obtain

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + 1 \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \neq A.
\end{aligned}$$

Expressing A as a product of 3 matrices. If we choose $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then we obtain

$$\begin{aligned} V \Lambda V^T &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A. \end{aligned}$$

Therefore,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

If we choose $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$, then we obtain

$$\begin{aligned} V\Lambda V^T &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A. \end{aligned}$$

Therefore,

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

If we choose $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, then we obtain

$$\begin{aligned} V\Lambda V^T &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \neq A. \end{aligned}$$

Reflection matrix. How do we express the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in two standard forms?

Solution: For this matrix, we have computed
Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$,
Eigenvectors: $v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, $v_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$.

Expressing A as a sum of 2 outer products.

We have

$$\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&\qquad - 1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} - \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
A = 1 \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
- 1 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.
\end{aligned}
$$

Expressing A as a product of 3 matrices. We have

$$
\begin{aligned}
V\Lambda V^T &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A.
\end{aligned}
$$

Therefore,

$$A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

Example.

How do we express the matrix $A = \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix}$ in two standard forms?

Solutuion: For this matrix, we have computed
Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = 10$,
Eigenvectors: $v_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$, $v_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$.

Expressing A as a sum of 2 outer products. We have

$$
\begin{aligned}
\sum_{i=1}^{2} \lambda_i v_i v_i^T &= \lambda_1 v_1 v_1^T + \lambda_2 v_2 v_2^T \\
&= 5 \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \\
&\qquad + 10 \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 9 & -2 \\ -2 & 6 \end{pmatrix} = A.
\end{aligned}
$$

Therefore,

$$A = 5 \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} + 10 \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$

Expressing A as a product of 3 matrices. We have

$$
\begin{aligned}
V\Lambda V^T &= \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} \\
&= \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 & 10 \\ -20 & 10 \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} 45 & -10 \\ -10 & 30 \end{pmatrix} \\
&= \begin{pmatrix} 9 & 2 \\ -2 & 6 \end{pmatrix} = A
\end{aligned}
$$

Therefore,

$$A = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}.$$