Unit 2D: Vector spaces and Dot Product

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Summary

1. Vector spaces
   - Subspaces of a vector space
   - Examples of subspaces and not-subspaces in two dimensions
   - Subspaces in $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, and beyond
2. Dot product
3. Basis vectors

Vector spaces

Vector spaces

A nonempty set V is a **vector space** over $\mathbb{R}$ if for every $x, y, z \in V$ and $a, b \in \mathbb{R}$

I. $x + y \in V$ with the properties:

1. $x + y = y + x$
2. $(x + y) + z = y + (x + z)$
3. $\exists\ 0 \in V$ such that $x + 0 = x$
4. $\exists\ -x \in V$ such that $x + (-x) = 0$

II $ax \in V$ with the properties:

1. $a(x + y) = ax + ay$
2. $(a + b)x = ax + bx$
3. $a(bx) = (ab)x$
4. $1x = x$.

Elements of a vector space are called **vectors**.

Using only these axioms, one can show that

- The element 0, called the **zero vector** in Axiom 1 (c) is unique.
- The element $-x$, called the **negative** of x, in Axiom 1 (d) is unique for each x in V.

In many problems, a vector space consists of an appropriate subset of vectors from some larger vector space. In this case, only two of the ten vector space axioms need to be checked:

For every $x, y \in V$ and $a \in \mathbb{R}$

$$x + y \in V, \;\; ax \in V.$$

The rest are automatically satisfied.

Subspaces

Let S be a non-empty subset of a vector space V. The set S is called a **subspace** of V if S is closed under the same vector addition and scalar multiplication as V, that is, for all $x, y \in S$ and for all $a \in \mathbb{R}$

$$x + y \in S \text{ and } ax \in S.$$

Example

The L_2 ball $B(0;1) = \{x \in \mathbb{R}^2 : \ \|x\|_2 \leq 1\}$ is not a subspace.

Solution.

- It is not closed under addition: for example,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in B(0;1),$$

but their sum

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin B(0;1).$$

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- Also, it is not closed under scalar multiplication: for example,

$$2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \notin B(0;1).$$

The same is true for the L_2 sphere. Similarly, L_1 and L_∞ balls and spheres are not subspaces of $\mathbb{R}^2$.

Example

The **positive orthant** is not a subspace.

Example

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Solution: It is not closed under scalar multiplication, for example: the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in positive orthant (first quadrant), but

$$-2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

has negative entry and so, it is not in prositive orthant (first quadrant). $\square$

Remark

The positive orthant is, however, closed under multiplication of nonnegative scalars, a fact that is sometimes of interest.

Example

The positive simplex is not a subspace.

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Solution: It is not closed under the addition of two vectors, and it is nor closed under scalar multiplication. $\square$

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It is, however, closed under the special class of vector additions of the form $z = ax + by$, where x, y, z are vectors, and a, b are numbers such that $a, b \geq 0$ and $a + b = 1$. We will see later why this is of interest.

Example

The line L given by the equation $x_1 + x_2 = 1$ is not a subspace.

Solution. For example,

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$$\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \notin L.$$

□

Examples of subspaces.

The set $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ is a "trivial" subspace of $\mathbb{R}^2$

since if we multiply it by any scalar or add it to itself, then the output is still the 0 vector.

We can see that $\mathbb{R}^2$ itself is a "trivial" subspace of $\mathbb{R}^2$ since $\mathbb{R}^2$ is a vector space and a subset of itself.

Problem

The line $x_2 = ax_1 + b$ (perhaps more familiar as $y = ax + b$) is not a subspace $\mathbb{R}^2$ for $b \neq 0$.

Solution

Let u, v be points on the line $x_2 = ax_1 + b$ and

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Solution

Let u, v be points on the line $x_2 = ax_1 + b$ and

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then

$$u_2 = au_1 + b$$
$$v_2 = av_1 + b.$$

Solution ...

Consider the point

$$w = u + v = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}.$$

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Then we have

$$u_2 + v_2 = a(u_1 + v_1) + 2b.$$

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Then we have

$$u_2 + v_2 = a(u_1 + v_1) + 2b.$$

This line is not on the line $x_2 = ax_1 + b$, the intercepts being different. $\square$

Problem

The line $x_2 = ax_1$ (in usual notations, $y = ax$) is a subspace $\mathbb{R}^2$.

Solution

Let u, v be two points on the line $x_2 = ax_1$ and

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Then

$$u_2 = au_1$$
$$v_2 = av_1.$$

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Then we have

$$u_2 + v_2 = a(u_1 + v_1).$$

This line is on the line $x_2 = ax_1$. $\square$

Problem

The set of points on two lines through the origin is not a subspace.

Proof.

Let

$$\Omega_x = \{(x_1, x_2) : \; x_1 = ax_2)\},$$
$$\Omega_y = \{(y_1, y_2) : \; y_1 = ay_2)\}.$$

Then $\Omega = \Omega_x \cup \Omega_y$ is not a subspace of $\mathbb{R}^2$.

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Then $\Omega = \Omega_x \cup \Omega_y$ is not a subspace of $\mathbb{R}^2$. In fact, the set Ω is not closed under addition of two vectors, since adding two vectors lying on two different lines results a vector that is not on either of those lines (except in the degenerate case when the two lines are the same). $\square$

Remark

The last two examples indicate that the union of two subspaces may not be a subspace.

Subspaces in $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, and beyond

There are two types of subspaces of $\mathbb{R}$:

$$\mathbb{R}, \quad \{0\}.$$

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Both of these are "trivial," in the sense that there is not too much interesting going on, and so one typically does not spend much time discussing the subspace aspects of $\mathbb{R}$, but it's good to understand such "extreme cases" in the definitions of vector spaces and subspaces.

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- The singleton set $\{0\}$ is a subspace of dimension 0.

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We see that there are four kinds of subspaces for $\mathbb{R}^3$:

- $\mathbb{R}^3$ is a three-dimensional subspace of $\mathbb{R}^3$.
- A plane through the origin is a two-dimensional subspace of $\mathbb{R}^3$.
- A line through the origin is a one-dimensional subspace of $\mathbb{R}^3$.
- The set $\{0\}$ is a zero-dimensional subspace of $\mathbb{R}^3$.

Dot product

Dot product on $\mathbb{R}^2$

If $x = (x_1, x_2)$ and $y = (y_1, y_2))$ are points on the plane $\mathbb{R}^2$, then the **dot product** or **inner product** between those two vectors is

$$x \cdot y = \sum_{1}^{2} x_i y_i = x_1 y_1 + x_2 y_2.$$

The following relation establishes the relationship between the dot poduct and the L_2 norm:

$$\|x\|_2 = \left(\sum_1^2 x_i^2 \right)^{1/2} = \sqrt{x \cdot x}.$$

We can easily generalize the dot product on $\mathbb{R}^2$ to that on $\mathbb{R}^n$.

Dot product on $\mathbb{R}^n$

If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n))$ are points in $\mathbb{R}^n$, then the dot product or inner product between those two vectors is

$$x \cdot y = x_1 y_1 + ... + x_n y_n = \sum_1^n x_i y_i.$$

The following relation establishes the relationship between the dot poduct on $\mathbb{R}^n$ and the L_2 norm on $\mathbb{R}^n$:

$$\|x\|_2 = \left(\sum_1^n x_i^2\right)^{1/2} = \sqrt{x \cdot x}.$$

The dot product allows us to define the perpendicularity (or orthogonality).

Orthogonality

Let x and y be two vectors in $\mathbb{R}^n$. We say that x is **perpendicular** (or **orthogonal**) to y, if

$$x \cdot y = 0.$$

Orthogonal compliment

Let x be vector in $\mathbb{R}^n$. The set of vectors perpendicular to a vector $x \in \mathbb{R}^n$ is denoted by $x^\perp$ and is defined by

$$x^\perp = \{y \in \mathbb{R}^n : \; x \cdot y = 0\}.$$

The set $x^\perp$ is called the **orthogonal compliment** of $\{x\}$.

Theorem

Given a vector $x \in \mathbb{R}^2$, such that $x \neq 0$, let $x^\perp$ be the set of vectors that are perpendicular to x. Then, $x^\perp$ is a subspace.

Proof.

If $x = 0$, then $x^{\perp} = \mathbb{R}^2$, which we know is a subspace.

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Suppose that $x \neq 0$. Let $u, v \in x^{\perp}$. If $x = (x_1, x_2), u = (u_1, u_2)$ and $v = (v_1, v_2)$, then we have

$$x \cdot (u + v) = x_1(u_1 + v_1) + x_2(u_2 + v_2)$$

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Suppose that $x \neq 0$. Let $u, v \in x^{\perp}$. If $x = (x_1, x_2), u = (u_1, u_2)$ and $v = (v_1, v_2)$, then we have

$$\begin{aligned} x \cdot (u + v) &= x_1(u_1 + v_1) + x_2(u_2 + v_2) \\ &= (x_1u_1 + x_2u_2) + (x_1v_1 + x_2v_2) = 0 \end{aligned}$$

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$$\begin{aligned} x \cdot (u + v) &= x_1(u_1 + v_1) + x_2(u_2 + v_2) \\ &= (x_1u_1 + x_2u_2) + (x_1v_1 + x_2v_2) = 0. \end{aligned}$$

Therefore,
$$u + v \in x^{\perp}.$$

Proof ...

Now, if $a \in \mathbb{R}$, then

$$x \cdot (au) = x_1(au_1) + x_2(au_2)$$

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Now, if $a \in \mathbb{R}$, then

$$\begin{aligned} x \cdot (au) &= x_1(au_1) + x_2(au_2) \\ &= a(x_1u_1 + x_2u_2) = 0. \end{aligned}$$

Proof ...

Now, if $a \in \mathbb{R}$, then

$$\begin{aligned} x \cdot (au) &= x_1(au_1) + x_2(au_2) \\ &= a(x_1u_1 + x_2u_2) = 0. \end{aligned}$$

Therefore,

$$au \in x^{\perp}.$$

Thus, by definition, $x^{\perp}$ is a subspace of $\mathbb{R}^2$. $\square$

Basis vectors

The span of vectors

Let $\{v_1, v_2, ..., v_k\}$ be a set of vectors in a vector space V. The span of $v_1, v_2, ..., v_k$ is denoted by span$\{v_1, v_2, ..., v_k\}$ and is defined by

$$\text{span}\{v_1, v_2, ..., v_k\} = \left\{ \sum_{1}^{k} a_i v_i : \text{ each } a_i \in \mathbb{R} \right\}$$

Theorem.

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Proof.

It is clear that $\text{span}\{v_1, v_2, ..., v_k\} \subseteq V$. Let us demonstrate that it is closed under addition and scalar multiplication.

Proof...

Addition: Let $x, y \in \text{span}\{v_1, v_2, ..., v_k\}$.

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$$x = \sum_{1}^{k} a_i v_i, \quad y = \sum_{1}^{k} b_i v_i,$$

where $a_i, b_i \in \mathbb{R}$.

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$$x = \sum_1^k a_i v_i, \quad y = \sum_1^k b_i v_i,$$

where $a_i, b_i \in \mathbb{R}$. Thus,

$$x + y = \sum_1^k (a_i + b_i) v_i \in \text{span}\{v_1, v_2, ..., v_k\}.$$

Proof.

Scalar multiplication: Let $a \in \mathbb{R}$. Then

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$$ax = a\sum_{1}^{k} a_i v_i$$

$$= \sum_{1}^{k}(aa_i)v_i \in \text{span}\{v_1, v_2, ..., v_k\}.$$

Therefore, span$\{v_1, v_2, ..., v_k\}$ is a subspace of V. □

Standard basis vectors

The **standard basis vectors** (or **standard unit vectors**) in $\mathbb{R}^n$, denoted e_k, have n entries, with a 1 in the kth position and a 0 in all the other positions.

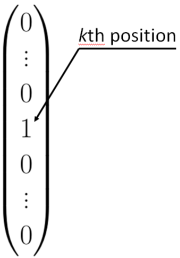

Figure: kth standard basis vector

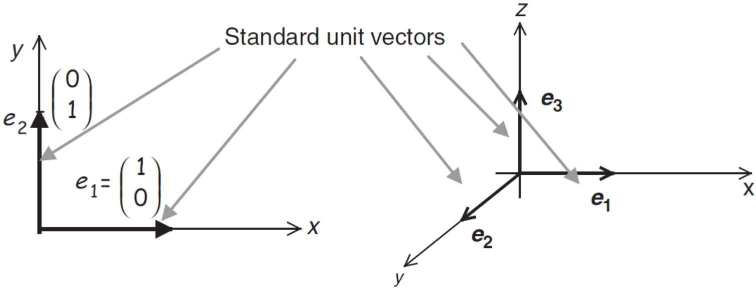

Figure: Standard basis vectors in the plane and in the space.

Question: Why are these standard basis vectors important?

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Ans. Every vector in the space can be expressed as a linear combination of the basis vectors of the space. It is easier to work with linear combinations.

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$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This gives

$$x = x_1 e_1 + x_2 e_2,$$

where $x_1 = 2$ and $x_2 = 3$.

Thus, we have expressed x as a *linear combination* of the basis vectors e_1 and e_2, Thus, we have expressed x as a *linear combination* of the basis vectors e_1 and e_2, and the coefficients involved in the linear combination are the components of the vector x.

Further, consider a linear combination:

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Now the question is:

Under what conditions is this linear combination zero?

that is,

When does the equality $x_1e_1 + x_2e_2 = 0$ hold?

To answer to this question, put

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Then

$$x = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
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Then

$$\begin{aligned} x &= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

If $x_1 e_1 + x_2 e_2 = 0$, then

$$x = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow x_1 = x_2 = 0.$$

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On the otherhand,

$$x_1 = x_2 = 0 \Rightarrow x_1 e_1 + x_2 e_2 = 0.$$

Therefore,

$$x_1 e_1 + x_2 e_2 = 0 \Leftrightarrow x_1 = x_2 = 0.$$

Problem

1. Show that any vector in $\mathbb{R}^3$ can be expressed as a linear combination of the three unit basis vectors in $\mathbb{R}^3$. Also, show that a linear combination of the three unit basis vectors in $\mathbb{R}^3$ equals to 0 if and only if all coefficients in the linear combination are zeros.
2. Do the above problem for $\mathbb{R}^n$.

The above discussion motivates the following definitions:

Linearly independent vectors

Vectors $v_1, v_2, ..., v_n$ in a vector space V are called **linearly independent vectors** if

$$\sum_{1}^{k} a_i v_i = 0 \Rightarrow a_i = 0 \text{ for all } i \in \{1, ..., k\}.$$

Now, we come to the following definition:

Basis for a vector space

A set $\{v_1, v_2, ..., v_n\}$ of vectors in a vector space V is called a **basis** for V if

1. The vectors $v_1, v_2, ..., v_n$ are linearly independent.
2. Every vector in V can be expressed as a linear combination of the vectors $v_1, v_2, ..., v_n$.

Not-standard basis vectors.

If we rotate e_1 by the angle θ, then

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \to \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix} = e_1'.$$

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If we rotate e_2 by the same angle θ, then

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \to \begin{pmatrix} -\tan\theta \\ 1 \end{pmatrix} = \begin{pmatrix} -a \\ 1 \end{pmatrix} = e_2'.$$

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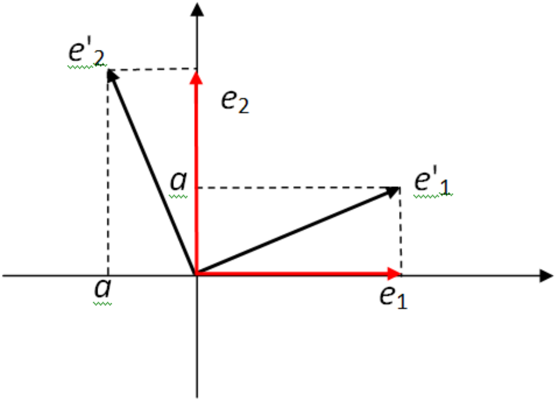
Note that just as $e_1 \cdot e_2 = 0$, since we have rotated both vectors by the same angle, so too $e_1' \cdot e_2' = 0$, i.e., e_1' and e_2' are also perpendicular.

With respect to these new basis vectors: any point on the line can be described by one number (the magnitude along e'_1, and in the same way we might be able to consider only one coordinate axis, here we might be able to consider only one of the new coordinate axes e'_1 and ignore the other e'_2 and still be able to do something useful with the data.


e'2
e2
a
e'1
a
e1

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Summarizing this discussion, there are several points to note.

- The vector $\begin{pmatrix} 1 \\ a \end{pmatrix}$ seems more natural to describe the data in Figure (c) given below.
- Using this more natural description it takes 1 number rather than 2 numbers.
- The line through the origin defined by $\begin{pmatrix} 1 \\ a \end{pmatrix}$ – as well as the line through the origin defined by the $\begin{pmatrix} -a \\ 1 \end{pmatrix}$ perpendicular to it – is a one-dimensional subspace of $\mathbb{R}^2$.

In the same way that any point on the plane can be expressed in terms of the standard basis vectors, so too any point on the plane can be expressed in terms of the two vectors $\begin{pmatrix} 1 \\ a \end{pmatrix}$ and $\begin{pmatrix} -a \\ 1 \end{pmatrix}$. We will study this later in detail.

Usefulness of basis in data science.

(a) Data points on a two dimensional plane, scattered in a round manner.

(b) Data points on a two dimensional plane, scattered in an elongated manner.

In Figure (a), different data points have different values for the two variables, x_1 and x_2, but both variables seem important to capture properties of the data.

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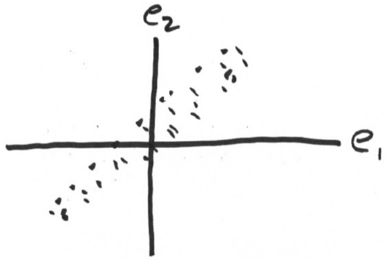
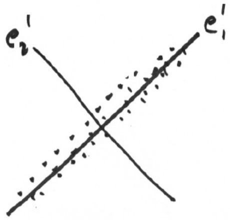
In Figure (b), the elongation is along the x-axis. It could be due to one feature being more "important" in some sense. That means, most of the information of interest is captured by x_1, while x_2 might be less important or simply random noise. In this case, we might hope or expect get very similar results by considering only (x_1), rather than (x_1, x_2), for each data point.



(c) Data points on a two dimensional plane, scattered in a different elongated manner.



(d) Same data points on a two dimensional plane, scattered in an elongated manner, but with rotated axes.

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By changing the axes, the data set plotted in Figure (c) can be visualized as in Figure(d).Now, the information from the data set can be obtained as in Figure (b) with respect to new axes e_1' and e_2'.

Summarizing this discussion, there are several points here.

- The vector $e_1' = \begin{pmatrix} 1 \\ a \end{pmatrix}$ seems more natural to describe the data in Figure (c).

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Summarizing this discussion, there are several points here.

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