Solution Sets Of Linear Systems

Prabhat Ale
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Linear System & Its Types

- **Linear systems typically refer to systems of linear equations. In mathematics, a system of linear equations consists of two or more equations involving two or more variables. Each equation in the system is linear, meaning that the highest power of any variable is 1.**

- **Types:**
    - **Consistent Linear Systems**
    - **Inconsistent Linear Systems**
    - **Overdetermined Linear Systems**
    - **Underdetermined Linear System**
    - **Homogeneous Linear Systems**
    - **Non Homogeneous Linear Systems**

Consistent Linear Systems

- A linear system is consistent if it has at least one solution.

- Example:

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 4 & -2 & 10 \end{array}\right]$$

Solving them we get,

$$\left[\begin{array}{cc|c} 2 & 0 & 5 \\ 0 & 1 & 0 \end{array}\right]$$

Inconsistent Linear Systems

- A linear system is inconsistent if it has no solution.
- Example:

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -2 & 3 & 7 \end{array}\right]$$

Solving them we get,

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{array}\right]$$

OverDetermined Linear Systems

- These systems have more equations than unknowns. Overdetermined systems often arise in real-world applications.

- Example:

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 3 & -2 & 4 \\ 5 & 1 & 12 \end{array}\right]$$

Solving this we get,

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 0 & 1 & 20/13 \\ 0 & 0 & 0 \end{array}\right]$$

UnDetermined Linear Systems

- **These systems have fewer equations than unknowns. Underdetermined systems typically have infinitely many solutions.**

- **Examples:**

$$\boxed{\begin{aligned} &2x + 3y - z = 5 \\ &4x - 6y + 2z = 10 \end{aligned}}$$

Homogeneous Linear Systems

- A system of linear equations is said to be homogeneous if it can be written in the form Ax = 0.
  where A is an m*n matrix and 0 is the zero vector in R^n.

- Such a system Ax = 0 always has at least one solution, namely x = 0 (the zero vector in R^n)

- The solution is usually called the trivial solution.

- The homogeneous equation Ax=0, the important question is whether there exists a non-trivial solution that is, a non zero vector x that satisfies Ax = 0.

Example1: Homogeneous Equation

3x1 + 5x2 - 4x3 = 0
-3x1 - 2x2 + 4x3 = 0
6x1 + x2 - 8x3 = 0

SOLUTION Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[\,A \quad \mathbf{0}\,]$ to echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is a free variable, $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions (one for each choice of x_3). To describe the solution set, continue the row reduction of $[\,A \quad \mathbf{0}\,]$ to *reduced* echelon form:

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= 0 \\ x_2 \qquad\quad &= 0 \\ 0 &= 0 \end{aligned}$$

Example1: Homogeneous Equation

Solve for the basic variables x_1 and x_2 and obtain $x_1 = \frac{4}{3}x_3, x_2 = 0$, with x_3 free. As a vector, the general solution of $A\mathbf{x} = \mathbf{0}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} = x_3\mathbf{v}, \quad \text{where } \mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Here x_3 is factored out of the expression for the general solution vector. This shows that every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v}$. The trivial solution is obtained by choosing $x_3 = 0$. Geometrically, the solution set is a line through $\mathbf{0}$ in $\mathbb{R}^3$. See Figure 1. ■

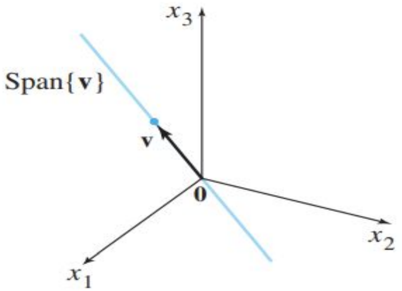

FIGURE 1

EXAMPLE 2 A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous "system"

$$10x_1 - 3x_2 - 2x_3 = 0 \tag{1}$$

SOLUTION There is no need for matrix notation. Solve for the basic variable x_1 in terms of the free variables. The general solution is $x_1 = .3x_2 + .2x_3$, with x_2 and x_3 free. As a vector, the general solution is

$$\begin{aligned}
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix} \\
&= x_2 \underset{\substack{\uparrow \\ \mathbf{u}}}{\begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underset{\substack{\uparrow \\ \mathbf{v}}}{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}} \quad (\text{with } x_2, x_3 \text{ free})
\end{aligned} \tag{2}$$

Geometrical Interpretation Of Figure 2

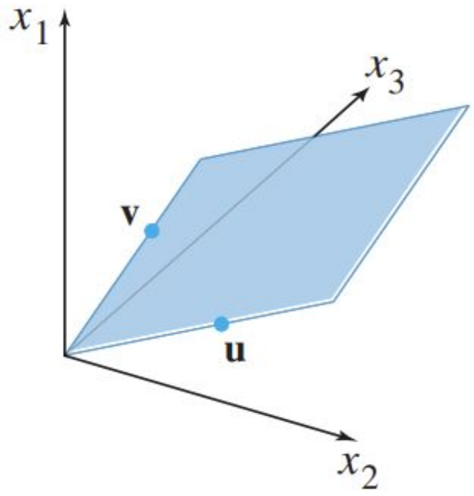

FIGURE 2

- **Examples 1 and 2, illustrate the fact that the solution set of a homogeneous equation Ax = 0 can always be expressed explicitly as span for suitable vectors.**

- **If the equation Ax = 0 has only one free variable, the solution set is a line through the origin, as in Figure 1.**

- **If the equation Ax = 0 has two or more free variables, then the solution set is a plane passing through the origin, as in Figure 2.**

Parametric Vector Form

- **Equation (2) presents a parametric vector equation of the plane, often expressed as x = su + tv, where s and t vary over all real numbers.**

- **Example 1 illustrates a similar approach, where the equation x = x3v (x3 unrestricted) or x = tv (with t $\in \mathbb{R}$) represents a parametric vector equation for a line.**

- **When solution sets are described explicitly using vectors, as shown in Examples 1 and 2, they're referred to as being in parametric vector form.**

Non Homogeneous Systems

Solutions of Nonhomogeneous Systems

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

EXAMPLE 3 Describe all solutions of $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

SOLUTION Here A is the matrix of coefficients from Example 1. Row operations on $[\,A \quad \mathbf{b}\,]$ produce

$$\left[\begin{array}{rrr:r} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \sim \left[\begin{array}{rrr:r} 1 & 0 & -\frac{4}{3} & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right], \qquad \begin{aligned} x_1 \qquad - \tfrac{4}{3}x_3 &= -1 \\ x_2 \qquad\qquad &= 2 \\ 0 &= 0 \end{aligned}$$

Thus $x_1 = -1 + \frac{4}{3}x_3$, $x_2 = 2$, and x_3 is free. As a vector, the general solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underset{\uparrow \atop \mathbf{p}}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} + x_3 \underset{\uparrow \atop \mathbf{v}}{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}$$

The equation $\mathbf{x} = \mathbf{p} + x_3\mathbf{v}$, or, writing t as a general parameter,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{3}$$

describes the solution set of $A\mathbf{x} = \mathbf{b}$ in parametric vector form. Recall from Example 1 that the solution set of $A\mathbf{x} = \mathbf{0}$ has the parametric vector equation

$$\mathbf{x} = t\mathbf{v} \quad (t \text{ in } \mathbb{R}) \tag{4}$$

[with the same $\mathbf{v}$ that appears in (3)]. Thus the solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector $\mathbf{p}$ to the solutions of $A\mathbf{x} = \mathbf{0}$. The vector $\mathbf{p}$ itself is just one particular solution of $A\mathbf{x} = \mathbf{b}$ [corresponding to $t = 0$ in (3)]. ■

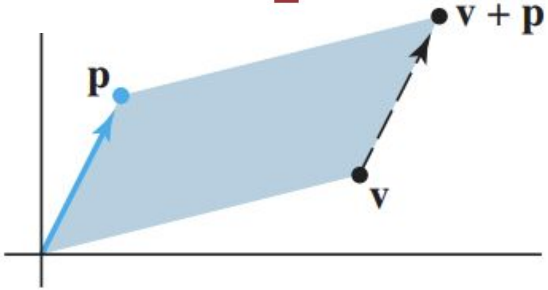

FIGURE 3

Adding $\mathbf{p}$ to $\mathbf{v}$ translates $\mathbf{v}$ to $\mathbf{v} + \mathbf{p}$.

To describe the solution set of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a *translation*. Given $\mathbf{v}$ and $\mathbf{p}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to *move* $\mathbf{v}$ in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. We say that $\mathbf{v}$ is **translated by $\mathbf{p}$ to $\mathbf{v} + \mathbf{p}$**. See Figure 3. If each point on a line L in $\mathbb{R}^2$ or $\mathbb{R}^3$ is translated by a vector $\mathbf{p}$, the result is a line parallel to L. See Figure 4.

Suppose L is the line through $\mathbf{0}$ and $\mathbf{v}$, described by equation (4). Adding $\mathbf{p}$ to each point on L produces the translated line described by equation (3). Note that $\mathbf{p}$ is on the line in equation (3). We call (3) **the equation of the line through $\mathbf{p}$ parallel to $\mathbf{v}$**. Thus *the solution set of $A\mathbf{x} = \mathbf{b}$ is a line through $\mathbf{p}$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$*. Figure 5 illustrates this case.

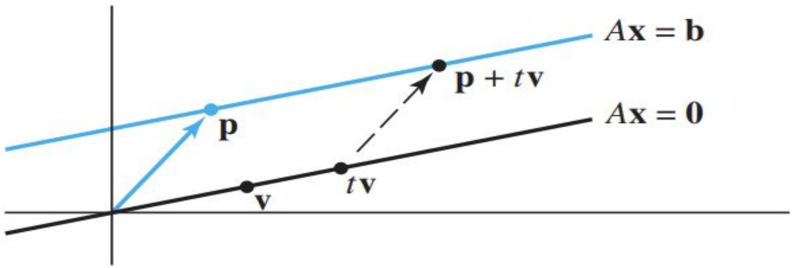

FIGURE 5 Parallel solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

THEOREM

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Conclusion

- **A homogeneous system $Ax = 0$ always has at least one solution, namely $x = 0$ (the zero vector in R^n). Such solution is usually called the trivial solution.**

- **If the equation $Ax = 0$ has only one free variable, the solution set is a line through the origin, as in Figure 1. If the equation $Ax = 0$ has two or more free variables, then the solution set is a plane passing through the origin, as in Figure 2.**

- **For non homogeneous linear system with equation $Ax=b$, the solution set can often be expressed as the sum of a particular solution to the non-homogeneous system and the general solution to the corresponding homogeneous system $Ax=0$ such that they are parallel to each other.**


thank you