4 QF IV: Definiteness of Quadratic Forms

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Summary

1. Classification of Quadratic Forms

2. Constrained optimization

Classification of Quadratic Forms

- The taxonomy of conic sections into ellipses, hyperbolas, and parabolas, as well as lines and points, is very useful for quadratic forms in 2 variables.

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- However, it quickly becomes awkward for quadratic forms in more than 2 variables.
- For quadratic functions in n variables, a related but slightly different classification is more convenient.

Definition.

Let A be an $n \times n$ symmetric matrix, and recall that $Q(x) = x^T A x$ is the corresponding quadratic form. Then A (as well as Q) is

(a) **positive definite** if $x^T A x > 0$, for all $x \neq 0$

(b) **negative definite** if $x^T A x < 0$, for all $x \neq 0$

(c) **indefinite** if $x^T A x > 0$ for some x's and $x^T A x < 0$ for others.

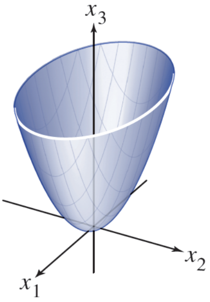
(d) **positive semidefinite** if $x^T A x \geq 0$, for all $x \neq 0$

(e) **negative semidefinite** if $x^T A x \leq 0$, for all $x \neq 0$.

Example 1. The following QF is *positive definite*:

$$f(x, y) = x^2 + y^2$$

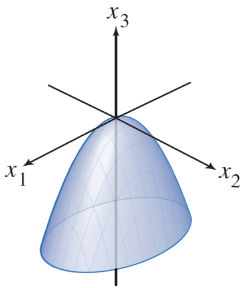
because $f(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ and that $f(x, y) = 0$ if and only if $x = 0$ and $y = 0$.



Example 2. The following QF is *negative definite*:

$$f(x, y) = -x^2 - y^2$$

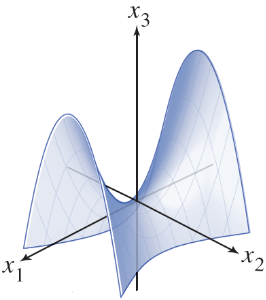
because $f(x, y) \leq 0$ for all $x, y \in \mathbb{R}$ and that $f(x, y) = 0$ if and only if $x = 0$ and $y = 0$.



Example 3. The following QF is *indefinite*:

$$f(x,y) = x^2 + xy - y^2$$

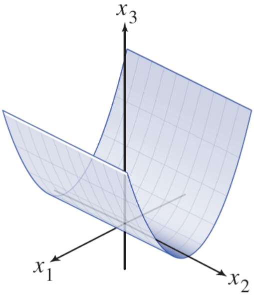
because $f(1,1) = 1 > 0$, but $f(-1,1) = -1 < 0$.



Example 4. The following QF is *positive semidefinite1* if

$$f(x,y) = x^2 + 2xy + y^2$$

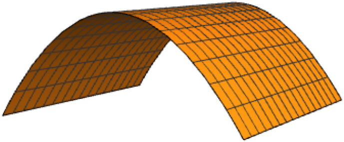
because $f(x,y) = (x+y)^2 \geq 0$ for all $x, y \in \mathbb{R}$. Observe that $f(x,y) = 0$ whenever $x = -y$.



Example 5. The following QF is *negative semidefinite* it

$$f(x, y) = -x^2 - 2xy - y^2$$

because $f(x, y) = -(x + y)^2 \leq 0$ for all $x, y \in \mathbb{R}$. Observe that $f(x, y) = 0$ whenever $x = -y$.



Theorem (Quadratic Forms and Eigenvalues).

Let A be a 2×2 symmetric matrix. Then a quadratic form $x^T A x$ is

(a) positive definite if and only if the eigenvalues of A are both positive,

(b) negative definite if and only if the eigenvalues of A are both negative,

(c) indefinite if and only if A has both positive and negative eigenvalues,

(d) positive semidefinite if and only if the eigenvalues of A are nonnegative,

(e) negative semidefinite if and only if the eigenvalues of A are nonpositive.

Consequently, if $|A| = 0$, then the quadratic form is neither positive definite nor negative definite.

Note that

This theorem can be generalized for an $n \times n$ matrix.

Example.

Determine the definiteness of the quadratic form $Q(x) = x_1^2 + 2x_1x_2 + x_2^2$.

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Clearly, the eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 0$. Hence the quadratic form is positive semi-definite.

Example.

For which real numbers k is the quadratic form $Q(x) = kx_1^2 - 6x_1x_2 + kx_2^2$ positive semi-definite?

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Solution. To determine the definiteness of this form we'll need to consider the matrix

$$\begin{pmatrix} k & -3 \\ -3 & k \end{pmatrix},$$

Solution...

Its characterstic polynomial is

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$$\begin{aligned} |A - \lambda I| &= \begin{pmatrix} k-\lambda & -3 \\ -3 & k-\lambda \end{pmatrix} \\ &= (k-\lambda)^2 - 9 \\ &= (k-\lambda+3)(k-\lambda-3). \end{aligned}$$

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Clearly, the eigenvalues of A are $\lambda_1 = k+3$ and $\lambda_2 = k-3$. In order for Q to be positive definite, both of these eigenvalues must be positive.

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Clearly, the eigenvalues of A are $\lambda_1 = k+3$ and $\lambda_2 = k-3$. In order for Q to be positive definite, both of these eigenvalues must be positive. So $k > 3$ is a necessary and sufficient condition for Q to be a positive definite quadratic form.

Constrained optimization

Let $x \in \mathbb{R}^n$ and A be an $n \times n$ matrix. A quadratic form $Q(x) = x^T Ax$ may not have a maximum: given any vector x, we can increase the value of $Q(x)$ by considering the vector $x' = 2x$.

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Example

Find $x \in \mathbb{R}^n$ to attain $\max x^T Ax$ subject to $x^T x = 1$.

The requirement that a vector x in $\mathbb{R}^n$ be a unit vector can be stated in several equivalent ways:

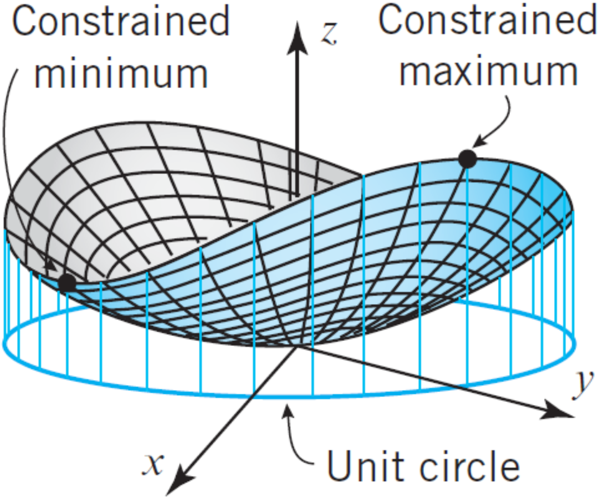
$$\|x\| = 1, \ \|x\|^2 = 1, \ x^T x = 1$$

and

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \tag{1}$$

The expanded version (1) of $x^T x = 1$ is commonly used in applications.

Geometrically, the problem of finding the maximum and minimum values of $x^T x$ subject to the requirement $x^T x = 1$ amounts to finding the highest and lowest points on the intersection of the surface with the right circular cylinder determined by the unit circle. See the figure below.


Constrained minimum
z
Constrained maximum
y
x
Unit circle

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Example.

Find the maximum and minimum values of

$$Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$$

subject to the constraint $x^Tx = 1$.

Solution. Since x_2^2 and x_3^2 are nonnegative, observe that

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and hence

$$\begin{aligned} Q(x) &= 9x_1^2 + 4x_2^2 + 3x_3^2 \\ &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \end{aligned}$$

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$$4x_2^2 \leq 9x_2^2 \; 3x_3^2 \leq 9x_3^2$$

and hence

$$\begin{aligned} Q(x) &= 9x_1^2 + 4x_2^2 + 3x_3^2 \\ &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\ &= 9(x_1^2 + x_2^2 + x_3^2) \end{aligned}$$

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$$\begin{aligned} Q(x) &= 9x_1^2 + 4x_2^2 + 3x_3^2 \\ &\leq 9x_1^2 + 9x_2^2 + 9x_3^2 \\ &= 9(x_1^2 + x_2^2 + x_3^2) \\ &= 9x^T x = 9. \end{aligned}$$

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To find the minimum value of $Q(x)$, we observe that

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So the minimum value of $Q(x)$ cannot be less than 9 when x is a unit vector. Furthermore, $Q(x) = 3$ when $x = (1, 0, 0)$. Thus, 3 is the minimum value of $Q(x)$ for $x^T x = 1$. $\square$

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It is easy to see in the above example that the matrix of the quadratic form Q has eigenvalues 9, 4, and 3 and that the greatest and least eigenvalues equal, respectively, the (constrained) maximum and minimum of $Q(x)$. The same holds true for any quadratic form, as well. This is what the next theorem claims.

Theorem

Let A be a symmetric matrix, and

$$m = \min\{x^T Ax : \|x\| = 1\}, \; M = \max\{x^T Ax : \|x\| = 1$$

Then M is the greatest eigenvalue λ_1 of A and m is the least eigenvalue of A. The value of $x^T Ax$ is M when x is a unit eigenvector u_1 corresponding to M. The value of $x^T Ax$ is m when x is a unit eigenvector corresponding to m.

Proof. Orthogonally diagonalize A as $V \Lambda V^T$. We observe that if $x = Vy$, then

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and for all y,

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$$\|Vy\| = (Vy)^T (Vy) = y^T V^T V y = y^T y = \|y\|^2.$$

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In particular, $\|y\| = 1$ if and only if $\|x\| = 1$. Thus $x^T Ax$ and $y^T \Lambda y$ assume the same set of values as x and y range over the set of all unit vectors.

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$$\Lambda = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

Proof... We have

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&\leq a y_1^2 + a y_2^2 + a y_3^2 \\
&= a(y_1^2 + y_2^2 + y_3^2)
\end{aligned}
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Proof... We have

$$\begin{aligned} y^T \Lambda y &= ay_1^2 + by_2^2 + cy_3^2 \\ &\leq ay_1^2 + ay_2^2 + ay_3^2 \\ &= a(y_1^2 + y_2^2 + y_3^2) \\ &= \|y\|^2 = a. \end{aligned}$$

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$$x = Ve_1 = (u_1 \; u_2 \; u_3) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = u_1$$

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which proves the statement about M. A similar argument shows that m is the least eigenvalue and this value of $x^T Ax$ is attained when

$$x = Ve_3 = u_3.$$

Example

Find the maximum and minimum values of

$$Q(x) = 5x_1^2 + 5x_2^2 + 4x_1x_2$$

subject to the constraint $x_1^2 + x_2^2 = 1$.

Solution. The given quadratic form can be expressed in matrix notation as

$$Q(x) = (x_1 \ x_2) \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x^T Ax.$$

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\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} \\
&= (5-\lambda)^2 - 4 \\
&= (7-\lambda)(3-\lambda).
\end{aligned}
$$

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$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}5-\lambda & 2\\ 2 & 5-\lambda\end{vmatrix}\\ &= (5-\lambda)^2-4\\ &= (7-\lambda)(3-\lambda).\end{aligned}$$

Therefore, the eigenvalues of A are

$$\lambda_1 = 7,\ \lambda_2 = 3.$$

Solution... To find the eigenvector associated with $\lambda_1 = 7$, we have

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$$x_1 - x_2 = 0.$$

So, the eigenvector associated with $\lambda_1 = 7$ is

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

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$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

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$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.$$

This gives the equation

$$x_1 + x_2 = 0.$$

Solution... So, the eigenvector associated with $\lambda_2 = 3$ is

$$v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.$$

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Thus, the constrained extrema are
Constrained maximum:

$$Q(x) = 7 \text{ at } (x_1, x_2) = (1/\sqrt{2}, 1/\sqrt{2})$$

Constrained minimum:

$$Q(x) = 3 \text{ at } (x_1, x_2) = (-1/\sqrt{2}, 1/\sqrt{2}).$$

□

Example

A rectangle is to be inscribed in the ellipse

$$4x^2 + 9y^2 = 36.$$

Use eigenvalue methods to find nonnegative values of x and y that produce the inscribed rectangle with maximum area.

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so the problem is to maximize the quadratic form $z = 4xy$ subject to the constraint $4x^2 + 9y^2 = 36$. In this problem, the graph of the constraint equation is an ellipse rather than the unit circle as required in the theorem,

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so the problem is to maximize the quadratic form $z = 4xy$ subject to the constraint $4x^2 + 9y^2 = 36$. In this problem, the graph of the constraint equation is an ellipse rather than the unit circle as required in the theorem, but we can remedy this problem by rewriting the constraint as

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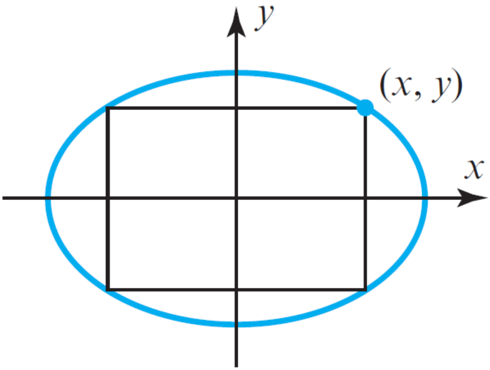
and defining new variables, x_1 and y_1, by the equations

$$x = 3x_1 \text{ and } y = 2y_1.$$

This enables us to reformulate the problem as follows:

Maximize $z = 4xy = 24x_1y_1$
subject to the constraint $x_1^2 + y_1^2$.



To solve this problem, we will write the quadratic form $z = 24x_1y_1$ as

$$z = x^T Ax = [x_1 \ y_1] \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

To find the eigenvalues of A, we have

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$$x_1 - x_2 = 0.$$

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Thus, the maximum area is $z = 12$, and this occurs at $(x_1, y_1) = (1/\sqrt{2}, 1/\sqrt{2})$. $\square$

Theorem

Let A, λ_1, and u_1 be as in the earlier theorem. Then the maximum value of $x^T Ax$ subject to the constraints

$$x^T x = 1, \quad x^T u_1 = 0$$

is the second greatest eigenvalue,λ_2, and this maximum is attained when x is an eigenvector u_2 corresponding to λ_2.

This theorem can be proved by an argument similar to the one above in which the theorem is reduced to the case where the matrix of the quadratic form is diagonal.

This theorem can be proved by an argument similar to the one above in which the theorem is reduced to the case where the matrix of the quadratic form is diagonal. The next example gives an idea of the proof for the case of a diagonal matrix.

Problem

Find the maximum value of

$$Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$$

subject to the constraint $x^T x = 1$ and $x^T u_1 = 0$, where $u_1 = (1, 0, 0)$.
Here u_1 is a unit eigenvector corresponding to the greatest eigenvalue $\lambda = 9$ of the matrix of the quadratic form.

Solution. If the coordinates of x are x_1, x_2, x_3, then the constraint $x^T u_1 = 0$ means simply that $x_1 = 0$.

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$$9x_1^2 + 4x_2^2 + 3x_3^2 = 4x_2^2 + 3x_3^2$$

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Thus the constrained maximum of the quadratic form does not exceed 4. And this value is attained for $x_1 = (0, 1, 0)$, which is an eigenvector for the second greatest eigenvalue of the matrix of the quadratic form. $\square$

The next theorem generalizes the previous theorem and gives a useful characterization of all the eigenvalues of A. The proof is omitted.

Theorem

Let A be a symmetric $n \times n$ matrix with an orthogonal diagonalization $A = P\Lambda P^{-1}$, where the entries on the diagonal of Λ are arranged so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and where the columns of P are corresponding unit eigenvectors $u_1, u_2, \cdots, u_n$. Then for $k = 2, \cdot, n$, the maximum value of $x^T Ax$ subject to the constraints

$$x^T x = 1,\ x^T u_1 = 0, \cdots, x^T u_{k-1} = 0$$

is the eigenvalue λ_k, and this maximum is attained at $x = u_k$.