Unit 2F: Linear transformations

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Summary

Transformations and matrices

1. Inverse matrices

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Relatedly, what does it "mean"?

Relatedly, why is it so useful?

We will consider the answers to these questions.

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Thus, we can consider the dot product to be a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

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From this discussion, we can say that matrix multiplication is a generalization of dot product in some sense.

Link between a dot product and a linear function:

We have viewed a dot product

as a function that fixes y and takes x as an input. In this case,

$$f(x) = f_y(x) = y^T x,$$

and this is a function that takes as input a vector $x \in \mathbb{R}^n$ and returns as output a number that is an element of $\mathbb{R}$ that equals $\sum_{i=1}^{n} x_i y_i$, where recall y is assumed to be fixed.

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This enables us **to view a matrix as a linear transformation**.

For example, we have noted earlier that if we consider a **random walk transition matrix**

$$W = \begin{pmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{pmatrix}$$

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and define

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then we take one vector in $\mathbb{R}^4$ and we apply W to get another vector in $\mathbb{R}^4$.

This perspective as viewing a matrix in terms of transformations or as a function is true more generally.

Linear transformations as matrices

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This representation also explains the definition of matrix multiplication:

Applying a function twice (as in iterating the random walk matrix) corresponds exactly to doing a multiplication of the two matrices.

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$$w = Av = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \in \mathbb{R}^2.$$

Thus, the $m \times n$ matrix A transforms vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$ according to the rule

$$w_i = \sum_{j=1}^{n} A_{ij} v_j \qquad \text{for } i = 1, 2, ..., m.$$

This is a linear function.

More abstractly, here is the definition of a linear transformation.

Linear transformation

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if

1. $\forall\, x, y \in \mathbb{R}^n \quad T(x + y) = T(x) + T(y)$,
2. $\forall\, x \in \mathbb{R}^n,\ \forall\, \alpha \in \mathbb{R} \quad T(\alpha x) = \alpha T(x)$.

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In particular, this means that the function would be a linear transformation if it were modified by taking the image of the origin and transforming it back to the origin.

In $\mathbb{R}$, this is just another way of saying that

$$y_1 = ax_1 + b, \quad \text{for } b \neq 0,$$

is not a linear function, but $y_1 = ax_1$ is a linear function. This holds true more generally.

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i.e., $x \mapsto Ax + b$, where the first term is the linear transformation and the second term is the affine offset. Here the constant offset $b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ is a vector offset and not a scalar offset.

Examples of nonlinear functions.

- A function given by the equation $y_1 = (x_1 - 2)^2 + 3$ is not linear.
- A function given by the equation $y_1 = \sin(x_1)$ is not a linear.

Example (linear)

1. A function given by the equation $y_1 = ax_1$ is a linear function.
2. The function that takes as input the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and returns as output the vector
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}.$$
Note that by writing the RHS of this equation as a matrix-vector product, we obtain
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Example

The composition of two (and thus more than two) linear functions is a linear function.

For example, if $f(x) = ax$ and $g(x) = bx$ (where $x, a, b, f(x), g(x)$ are all real numbers in $\mathbb{R}$), then

$$f(g(x)) = a(bx) = (ab)x,$$

which is a linear function.

Consider two linear functions that take as input vectors and returns as output vectors defined by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \Rightarrow \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

Clearly, the last equation defines a linear function.

Notice that a linear function can be represented as a matrix. Let us return to the equation considered above:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1}$$

Put

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$$f = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

We now observe that Equation (1) can be rewritten as follows:

$$y = f(x).$$

In general, if we have a linear function $f : \mathbb{R}^m \to \mathbb{R}^n$, then it can be fully described by an $m \times n$ matrix.

Recall that the multiplicative inverse of a number such as 5 is 1/5 or 5^{-1}. This inverse satisfies the equations

$$5^{-1} \cdot 5 = 1 \text{ and } 5 \cdot 5^{-1} = 1.$$

The matrix generalization requires both equations and avoids the slanted-line notation (for division) because matrix multiplication is not commutative. Furthermore, a full generalization is possible only if the matrices involved are square matrices.

We need the identity matrix in order to define and explain the inverse matrix. So, we begin with the identity matrix.
What does the term identity matrix mean?

Identity matrix

The **identity matrix** is a matrix denoted by I such that

$$AI = A \text{ for any matrix } A$$

What does the identity matrix look like?

An identity matrix is a square matrix defined by

$$I = (I_{kj}) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}$$

This means, all the diagonal elements of a matrix I are 1:

$$i_{11} = i_{22} = i_{33} = ... = 1$$

and all the other entries are zero.

Example.

For a 2×2 matrix we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c and d are real numbers.

Invertible

A square matrix A is said to be **invertible** or **non-singular** if there is a matrix B of the same size such that

$$AB = BA = I.$$

Matrix B is called the (multiplicative) **inverse** of A and is denoted by A^{-1}.

Here is a basic result about inverses.

Theorem

If A and B are invertible, then AB is invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Proof.

... $\square$

From a linear equation perspective, inverse matrices are important. For example, a system of linear equations can generally be written as $Ax = b$ where x is the vector of unknowns that we need to find. If we multiply both sides of this $Ax = b$ by the inverse matrix A^{-1} we obtain:

$$\begin{aligned} & A^{-1}Ax = A^{-1}b \\ \Rightarrow \qquad & x = A^{-1}b. \end{aligned}$$

Hence we can find the unknowns by finding the inverse matrix.

Question. When does such an inverse matrix exist, i.e., when is a matrix invertible?

Answer. Here is the partial answer.

<u>Case $m = n = 1$</u>: We can write a 1×1 matrix as $A = (a)$. In this case, $A^{-1} = 1/a$, which is defined for all $a \neq 0$. So, for 1×1 matrices, i.e., real numbers, such a number exists for every $a \in \mathbb{R}$ such that $a \neq 0$. This is particularly simple, and the situation is considerably more subtle for matrices of larger size.

Case $m = n = 2$: We can write a 2×2 matrix as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which is defined only when $ad - bc \neq 0$, and otherwise the inverse is not defined.

Check that the above matrix A^{-1} is the inverse of the matrix A, and vice versa

A 2×2 matrix A doesn't have an inverse in two cases:

A 2×2 matrix A doesn't have an inverse in two cases:

1. When it is the all-zeros matrix, in which case it doesn't have any non-trivial information and it sends all input vectors to the zero vector; and
2. when it's two columns are the same, up to scaling, in which case you might imagine that it is missing some information. Note that this degeneracy corresponds to multiplication by a scalar, i.e., one column is a scalar multiple of the other column. Note also that if $\alpha = 0$, then the second column is the all-zeros column, and it still holds that the matrix is not invertible.

The quantity $ad - bc$ is called the **determinant** of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we write

$$\det A = ad - bc.$$

We also write

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

For the case of 3×3 matrices, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

the determinant of this 3×3 matrix is defined by

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

In this way, we can also define the determinant of an $n \times n$ matrix with $n \geq 4$.

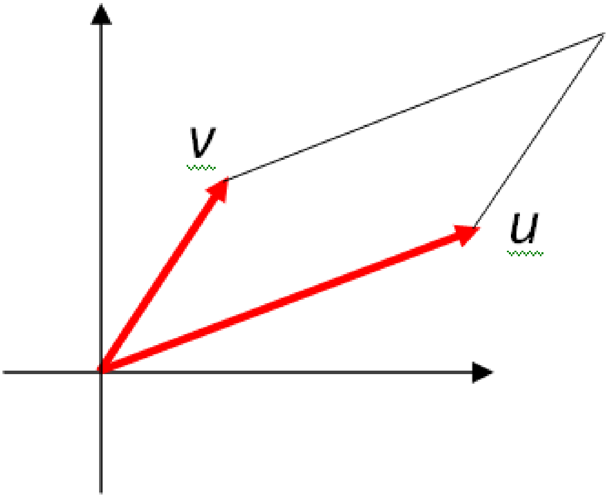
From the above discussion, we observe that

Determinant as a function

$\det(\cdot)$ is a real-valued function defined on the set of all square matrices.

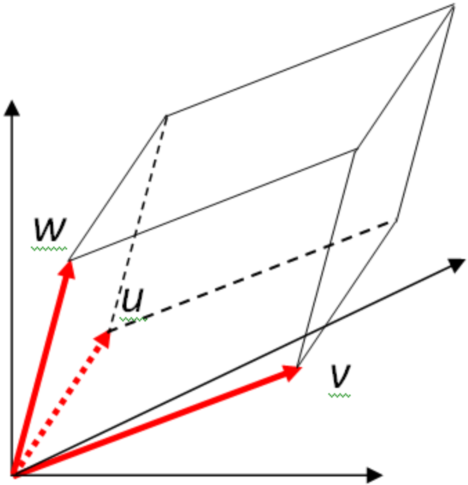
Geometrical interpretation of determinants.

For a 2×2 matrix, if we consider the parallelogram defined by the two columns (or rows) of the matrix, for example, $x = \begin{pmatrix} a \\ c \end{pmatrix}$ and $y = \begin{pmatrix} b \\ d \end{pmatrix}$, then the determinant equals the area of the parallelogram.



Clearly, this equals zero if the two columns are linearly dependent, i.e., if one is a scalar multiple

For a 3×3 matrix, if we consider the parallelepiped defined by the three columns (or rows) of the matrix, then the determinant equals the volume of the parallelepiped.



Clearly, this equals zero if the three columns are linearly dependent, i.e., if one is a scalar multiple