# Unit 2 H: Dot Products and Angles

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# Summary

Now, we'll discuss geometric properties of $\mathbb{R}^n$. This includes angles and perpendicularity; linear combinations, spans, and linear dependence/independence; basis vectors, including orthogonal/orthonormal basis vectors; and projections onto basis vectors. Your are familiar with many of these ideas in $\mathbb{R}^2$ and $\mathbb{R}^3$. At present, we'll generalize them to $\mathbb{R}^n$.

## Dot product

Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$. The dot product $x \cdot y$ is defined as

$$x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i.$$

## Properties

1. $x \cdot y = y \cdot x$.
2. $(x + y) \cdot z = x \cdot z + y \cdot z$.
3. $(cx) \cdot y = x \cdot (cy) = c(x \cdot y)$.
4. $x \cdot x \geq 0$.
5. $x \cdot x = 0 \Leftrightarrow x = 0$.

As we will see, the dot (inner) product is important for many reasons. One reason is that it has close connections with a particular vector norm.

What is that connection ?

## Euclidean norm

The length or norm or Euclidean norm of a vector $x \in \mathbb{R}^n$ is given by

$$\|x\|_2 = (x \cdot x)^{1/2} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$$

## Problem

Let $x, y \in \mathbb{R}^5$ with $x \cdot y = -1$, $\|x\| = 2$ and $\|y\| = 3$. Find each of the following.

1. $x \cdot 2y$
2. $(x + y) \cdot y$
3. $(2x + 4y) \cdot (x - 7y)$.

If we view these two (column) vectors as matrices, then they are $n \times 1$ matrices. Then the dot product can be expressed in terms of matrix multiplication by taking transposes.

$$x^T y = \sum_{i=1}^{n} x_i y_i = y^T x.$$

So, we have

$$x \cdot y = x^T y = y^T x.$$

This is a very special case of a matrix multiplication.

**Theorem (Cauchy-Schwartz Inequality)**

If $x$ and $y$ are vectors in $\mathbb{R}^n$, then

$$|x \cdot y| \leq \|x\|_2 \|y\|_2.$$

**Proof.** For $u_i,\ v_i \leq 0$, we have

$$\sum_{i=1}^{n} \sqrt{u_i v_i} \leq \sum_{i=1}^{n} \frac{u_i + v_i}{2}$$

**Proof.** For $u_i, v_i \leq 0$, we have

$$\sum_{i=1}^{n} \sqrt{u_i v_i} \leq \sum_{i=1}^{n} \frac{u_i + v_i}{2}$$

By choosing

$$u_i = \frac{x_i^2}{\|x\|^2} \text{ and } v_i = \frac{y_i^2}{\|y\|^2}$$

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in the above inequality, we get

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|\|y\|} \leq \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{\|x\|^2} + \frac{1}{2} \sum_{i=1}^{n} \frac{y_i^2}{\|y\|^2}$$

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$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|\|y\|} \leq \frac{1}{2} \sum_{i=1}^{n} \frac{x_i^2}{\|x\|^2} + \frac{1}{2} \sum_{i=1}^{n} \frac{y_i^2}{\|y\|^2} = 1.$$

**Proof...**

This implies

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\| \|y\|$$

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We know that

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Therefore,

$$|x \cdot y| \leq \|x\| \|y\|.$$

$\square$

## Theorem

If $\theta$ is the angle between two non-zero vectors $x, y \in \mathbb{R}^n$, then

$$x \cdot y = \|x\|\|y\| \cos\theta.$$

**Case I:**

Let the two vectors $x$ and $y$ not be scalar multiples of each other. By the Law of Cosines:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta.$$

Now, observe that

$$\begin{aligned} \|x - y\|^2 &= (x - y) \cdot (x - y) \\ &= x \cdot x + (y \cdot y) - 2(x \cdot y) \\ &= \|x\|^2 + \|y\|^2 - 2(x \cdot y) \end{aligned}$$

Therefore,

$$x \cdot y = \|x\|\|y\| \cos\theta.$$

**Case II:**

Let $y = cx$. Then

$$c > 0 \Rightarrow \theta = 0 \Rightarrow \cos\theta = 1,$$
$$c < 0 \Rightarrow \theta = \pi \Rightarrow \cos\theta = -1.$$

Now,

$$x \cdot y = x \cdot cx = c\|x\|^2 = \|x\|(c\|x\|)$$

If $c > 0$, then we have

$$x \cdot y = \|x\|\|cx\| = \|x\|\|y\|\cos\theta$$

If $c < 0$, then we have

$$x \cdot y = -\|x\|\|cx\| = \|x\|\|y\|\cos\theta.$$

$\square$