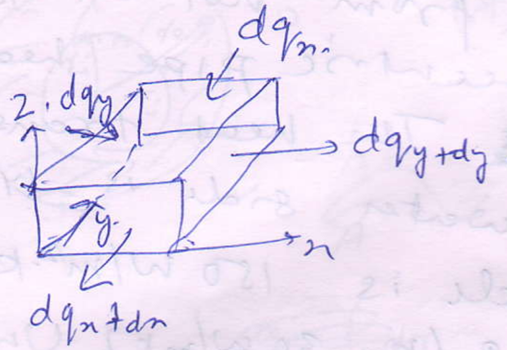
20/9/18.

Differential eqn. of heat conduction



$\bar{q} = f(x, y, z, t)$.

rate of heat generated in given volume

From Fourier's law, heat conducted in x-dir

$$dq_x = -K\frac{\partial T}{\partial x}\, dA = -K\frac{\partial T}{\partial x}\, dy\, dz$$

$$dq_{x+dx} = dq_x + \frac{\partial}{\partial x}(dq_x)\, dx$$

$$= \cancel{dq_x} - K\left(\frac{\partial T}{\partial x}\right) dy\, dz - \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) dx\cdot dy\cdot dz$$

$$= -\left[K\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) dx\right] dy\, dz.$$

Similarly,

$$dq_{y+dy} = -\left[K\left(\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) dy\right] dx\, dz$$

$$dq_{z+dz} = -\left[K\left(\frac{\partial T}{\partial z}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right) dz\right] dx\, dy.$$

∴ Net amount of heat conducted in element per unit time

$$= (dq_x + dq_y + dq_z) - (dq_{x+dx} + dq_{y+dy} + dq_{z+dz})$$

$$= \frac{\partial}{\partial x}\left[\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial T}{\partial z}\right)\right] dx\, dy\, dz$$

$$= K(\nabla^2 T)$$

Now, heat generated/absorbed in given volume per unit time,

$$= \bar{q}\, dx\, dy\, dz.$$

Now,

Rate of change of internal energy to the element,

$$= \rho c_p \frac{\partial T}{\partial t}\, dx\, dy \cdot dz$$

$$\therefore \cancel{\nabla^2 T + \frac{\bar{q}}{K} = \rho c_p}$$

$$K \nabla^2 T + \bar{q} = \rho c_p \frac{\partial T}{\partial t}.$$

$$\therefore \nabla^2 T + \frac{\bar{q}}{K} = \frac{\rho c_p}{K} \frac{\partial T}{\partial t}.$$

$$\Rightarrow \boxed{\nabla^2 T + \frac{\bar{q}}{K} = \frac{1}{\alpha} \frac{\partial T}{\partial t}}$$

$\alpha = \frac{K}{\rho c_p} \rightarrow$ thermal diffusivity. (m^2/s)

for steady state $\left(\frac{\partial}{\partial t} = 0\right)$ & no heat gene.

$$\boxed{\nabla^2 T = 0.} \longrightarrow \text{Laplace's eqn.}$$

Initial & Boundary Condⁿ

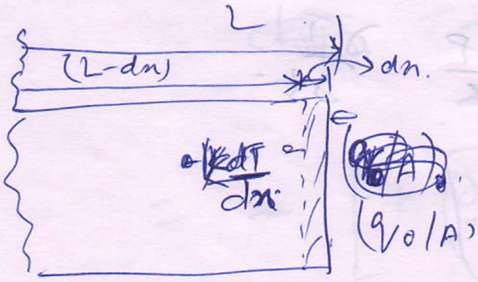
at $t = 0$, $T = f(x, y, z)$

@ $\underbrace{T = 0, \ T = T_0}_{\text{uniform temp at } T=0}$ → Initial condition needed for un-steady problem

Boundary Condⁿ

i) prescribed surf. temp.
@ $x = L$, $T = f(x, y, z)$ → known.
or $T = T_0$.

ii) prescribed heat flux incident on the surface.



applying 1st law or energy cons. at volume of thickness 'dx'.

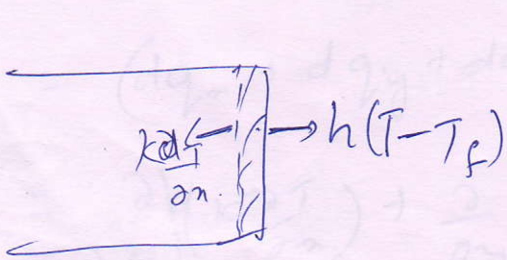
Heat in − Heat out = Heat accumulated

$$\Rightarrow A\left(\frac{q_0}{A}\right) - KA\frac{\partial T}{\partial x}\bigg|_{L=(x-dx)} = \rho C_p (A\,dx)\frac{\partial T}{\partial t}$$

~~when~~ limit $dx \to 0$,

$$\frac{q_0}{A} - K\frac{\partial T}{\partial x} = 0 \quad \Rightarrow K\left(\frac{\partial T}{\partial x}\right)_{x=L} = \frac{q_0}{A}.$$

iii) Prescribed heat flux incident on the surface.

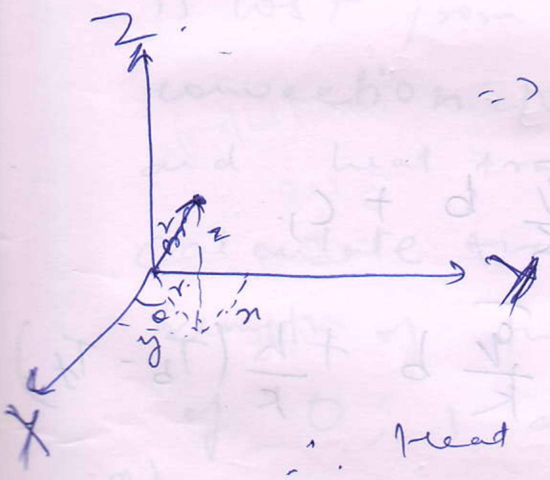


$$-KA\left(\frac{\partial T}{\partial x}\right)_{x=(L-dx)} - hA\left(T_{x=L} - T_f\right)$$
$$= \rho C_p (A\,dx)\left(\frac{\partial T}{\partial t}\right)_{x=L}$$

at limit $dx \to 0$.

$$-K\left(\frac{\partial T}{\partial x}\right)_{x=L} = h\left(T_{x=L} - T_f\right).$$

Cylindrical coordinates.

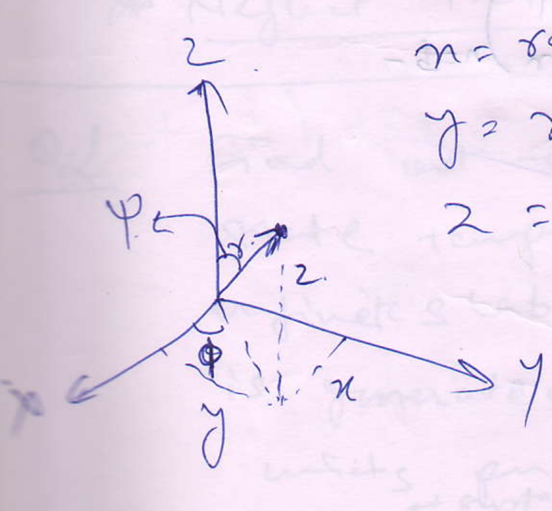


$\Rightarrow$ $x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$\therefore$ Heat conduction eqn. becomes.

$$K\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right\} + \bar{q} = \rho c_p \frac{\partial T}{\partial t}$$

Spherical coordinates.



$x = r\sin\psi\cos\phi$

$y = r\sin\psi\sin\phi$.

$z = r\cos\psi$.

$$\Rightarrow K\left\{\frac{1}{r}\frac{\partial^2(rT)}{\partial r^2} + \frac{1}{r^2\sin\psi}\frac{\partial}{\partial\psi}\left(\sin\psi\frac{\partial T}{\partial\psi}\right) + \frac{1}{r^2\sin^2\psi}\frac{\partial^2 T}{\partial\phi^2}\right.$$

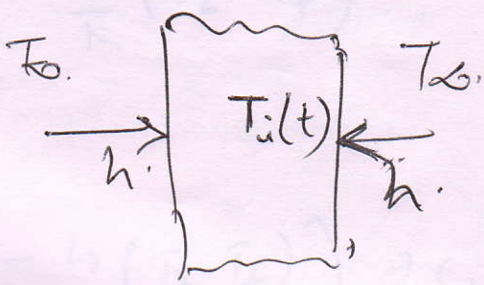
$$+ \bar{q} = \rho c_p \frac{\partial T}{\partial t}$$

Transient

~~Unsteady~~ heat conduction — 1/10/18

1) Lumped system analysis - Ignoring the temperature variation within the system.

↳ Thermally thin



$$\begin{pmatrix}\text{Heat transfer into the} \\ \text{body during time} \\ \text{'dt'}\end{pmatrix} = \begin{pmatrix}\text{Incr in enthalpy} \\ \text{of the body during} \\ \text{'dt'}\end{pmatrix}$$

$$h A_s (T_\infty - T)\, dt = + m\, c_p \cdot dT$$

Taking ~~Now~~, $dT = d(T - T_\infty)$ as T_∞ = const

$$\therefore \frac{d(T - T_\infty)}{T - T_\infty} = -\frac{h A_s}{m c_p}\, dt$$

BC

$\Rightarrow$ ~~$\ln \left| \frac{T(t) - T_\infty}{T_i - T_\infty} \right|$~~

$$\Rightarrow \ln |T - T_\infty| = -\frac{h A_s}{m c_p} t$$

$\Rightarrow$ B.C's $\rightarrow$ @ $t=0 \rightarrow T=T_i$

$t=t \rightarrow T=T(t)$

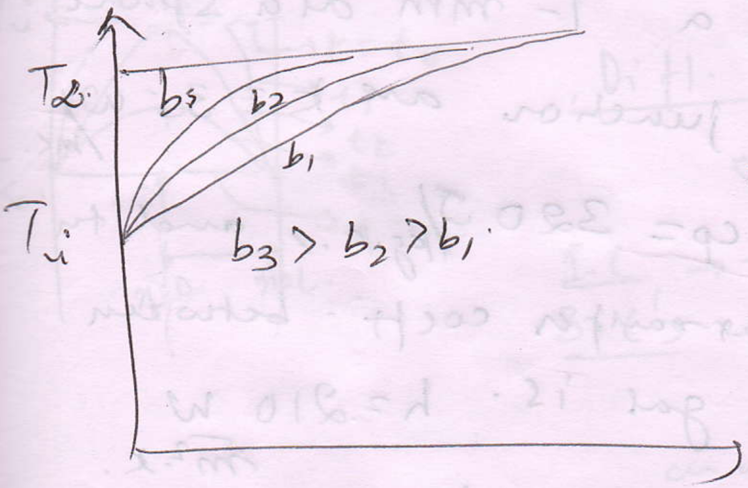
$$\Rightarrow \ln\left(\frac{T(t)-T_\infty}{T_i-T_\infty}\right) = -\frac{hA_s}{mC_p}\,t$$

$$\Rightarrow \frac{T(t)-T_\infty}{T_i-T_\infty} = e^{-\frac{hA_s}{mC_p}t} = e^{-bt}$$

where, $b = -\frac{hA_s}{mC_p} = -\frac{hA_s}{\rho V C_p}$

$= \frac{-h}{\rho C_p (V/A_s)} = \frac{-h}{\rho C_p L_c}$

characteristic length.



* Large 'b' indicates the body achieves ambient/outside temp. faster.

Criteria for lumped sys. analysis

Biot number $\rightarrow$ Dimension less number.

$= \frac{hL_c}{K} = \frac{h}{(K/L_c)} = \frac{\text{conv. at the Surface of body}}{\text{conduction within the body}}$

or $= \frac{(L_c/K)}{1/h} = \frac{\text{conductive resistance in body}}{\text{conv. resis...}}$

∴ Lumped ~~bod~~ sys. analysis is good approx ⟶ when 'Bi' is low. or object is thermally thin

Generally accepted number;

$Bi \leq 0.1$ ⟶ thermally thin.

↳ Temp. gradient within the body will be negligible.

L_c ⟶.

slab ⟶ $L/2$.

cyl. ⟶ $r_0/2$

sphere ⟶ $r_0/3$

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = 0.01 \; (1\%)$$

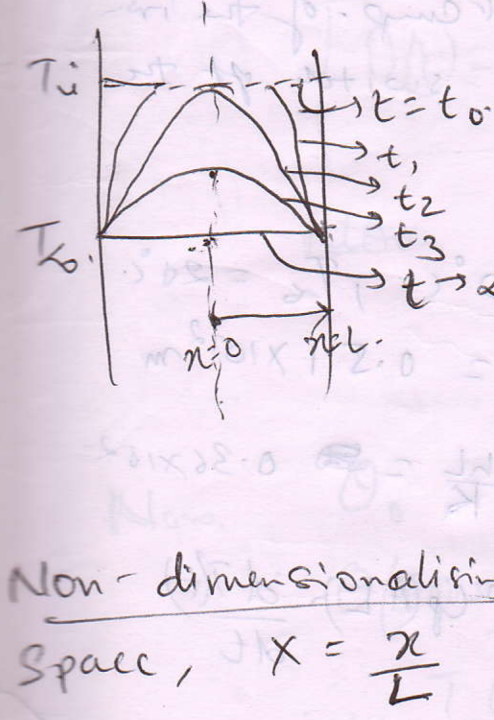
$$\Rightarrow \quad e^{-bt} = 0.01$$

and $\quad b = \frac{h A_s}{\rho c_p V} = \frac{h}{\rho c_p L_c} = 0.462 \, s^{-1}$

$$\Rightarrow \quad e^{-bt} = 0.01$$

$$\Rightarrow \quad \underline{t = 10 \, s}$$

<u>Transient heat conduction</u> – when internal temp gradients can't be neglected.



<u>Difl. eqn.</u>

$$\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \qquad \left(\alpha = \frac{k}{\rho c_p}\right)$$

<u>I.C</u> $\rightarrow T(x,0) = T_i$

<u>B.C's</u> $\rightarrow \frac{\partial T}{\partial x}(0,t) = 0$

and $\quad -k \frac{\partial T}{\partial x}(L,t) = h\left[T(L,t) - T_\infty\right]$

<u>Non-dimensionalising</u>

Space, $X = \frac{x}{L}$, ~~Time~~ Temp, $\theta(x,t) = \frac{(T(x,t) - T_\infty)}{T_i - T_\infty}$

$0 \le X \le 1$, Now, $\frac{\partial \theta}{\partial X} =$

$$= \frac{\partial \theta}{\partial (x/L)} = \frac{L}{(T_i - T_\infty)} \frac{\partial T}{\partial x}$$

Later

Trasien heat cond. (Cont.) 8/10/18

$$\frac{\partial \theta}{\partial X} = \frac{L}{(T_i - T_\infty)} \frac{\partial T}{\partial x} \Rightarrow \qquad \frac{\partial \theta}{\partial t} = \frac{1}{(T_i - T_\infty)} \frac{\partial T}{\partial t}$$

$$\Rightarrow \frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{(T_i - T_\infty)} \frac{\partial^2 T}{\partial x^2}$$

Governing eqn.

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \Leftrightarrow \frac{\partial^2 \theta}{\partial X^2} = \frac{L^2}{\alpha} \frac{\partial \theta}{\partial t}$$

and B.C's, $\frac{\partial \theta}{\partial X}(0, t) = 0$,

$$\frac{\partial \theta}{\partial X}(1, t) = -\left(\frac{hL}{K}\right) \theta(1, t)$$

$\left(\frac{hL}{K}\right) \to Bi$

I.C $\to \theta(X, 0) = 1$.

$$= -Bi \cdot \theta(1, t)$$

$\left(\frac{hL}{K}\right) = Bi$

Another non-dimensional number $\to$ Time

$$\tau = \frac{\alpha t}{L^2} \to \text{Fourier Numb. (Fo)}$$

$\therefore$ Governing eqn $\to \frac{\partial^2 T}{\partial X^2} = \frac{\partial \theta}{\partial \tau}$

B.C's $\to \frac{\partial \theta}{\partial X}(0, \tau) = 0$, $\frac{\partial \theta(1, \tau)}{\partial X} = -Bi \cdot \theta(1, \tau)$.

$\underline{I.C} \rightarrow \Theta(X,0) = 1.$

when,

$\Theta \Rightarrow$ Dimentionless temp.

$X \rightarrow$ Dimensionless dist. from center.

Bi ~~Cht~~ $\rightarrow$ Dimensionless heat transfer coeff.

$\tau \rightarrow$ Dimensionless time.

Orig. prob.

$$T = f(x, L, t, k, \alpha, h, T_i, T_\infty).$$

Non-dimensional.

$$\Theta = f(X, \tau, Bi)$$

Special case – Thermally thin. $Bi \le 0.1$ (Ignore 'X').

$$\Theta = \frac{T(t) - T_\infty}{T_i - T_\infty} = e^{-bt} = e^{-\frac{hAt}{\rho V c_p}}$$

$$= e^{-\left(\frac{hL}{k}\right)\left(\frac{k}{\rho c_p \cdot L^2} \cdot t\right)} = e^{-Bi \cdot \cancel{\tau}}$$

$$\therefore \Theta = f(Bi, \tau)$$

Exact Sol. of 1-D transient Cond. problem.

Separation of Variables. → ~~as $\theta = f(X, Bi, \tau)$~~

→ PDE to ODE.

$\therefore \theta(X,\tau) = F(X)G(\tau)$.

$\therefore \dfrac{\partial^2\theta}{\partial X^2} = \dfrac{\partial\theta}{\partial\tau} \iff \dfrac{1}{F}\dfrac{d^2F}{dX^2} = \dfrac{1}{G}\dfrac{dG}{d\tau}$.

$\dfrac{d^2F}{dX^2} + \lambda^2 F = 0$, $\dfrac{dG}{d\tau} + \lambda^2 G = 0$.

↓ Sol^n ↓ Sol^n

$F = C_1\cos(\lambda X) + C_2\sin(\lambda X)$, $G = C_3 e^{-\lambda^2\tau}$

$\therefore \theta = FG = C_3 e^{-\lambda^2\tau}\left[C_1\cos(\lambda X) + C_2\sin(\lambda X)\right]$

$= e^{-\lambda^2\tau}\left[A\cos(\lambda X) + B\sin(\lambda X)\right]$

When, $A = C_1C_3$
$B = C_2C_3$

Applying B.C , $\dfrac{\partial\theta(0,\tau)}{\partial X} = 0$.

$\Rightarrow e^{-\lambda^2\tau}(A\lambda\sin 0 + B\cos 0) = 0$.

$\Rightarrow \boxed{B = 0}$