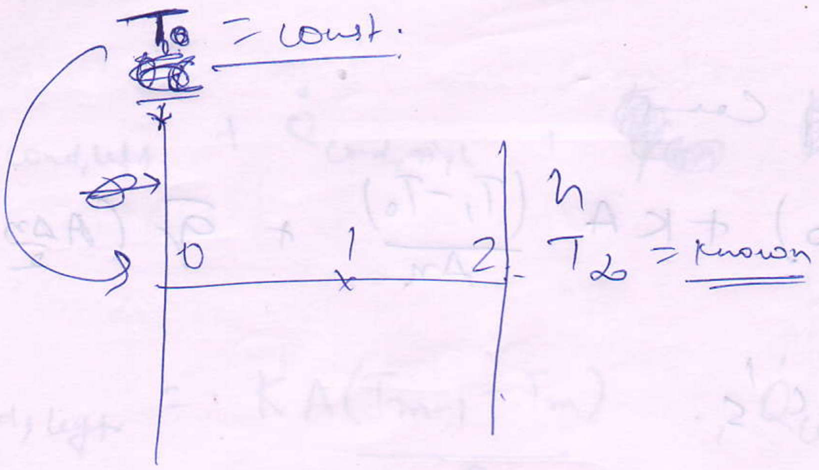


$$\left(\frac{T_0 - 2T_1 + T_2}{\Delta x^2}\right) + \frac{\bar{q}}{k} = 0 \Rightarrow 2T_1 - T_0 - T_2 = \frac{\bar{q}\Delta x^2}{k}$$

$$hA(T_\infty - T_2) + kA\frac{(T_1 - T_2)}{\Delta x} + \bar{q}\left(\frac{A\Delta x}{2}\right) = 0.$$

$$\Rightarrow T_1 - \left(1 + \frac{h\Delta x}{k}\right)T_2 = -\frac{h\Delta x}{k}T_\infty - \frac{\bar{q}\Delta x^2}{2k}.$$

T_0 = const ~~T_∞~~, T_∞ = const.

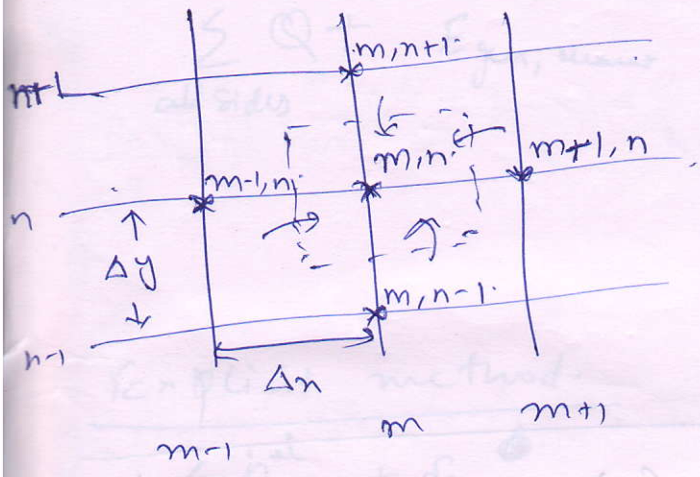
~~3 eqn. 3 vari~~ → 2 eqn. 2 variable.

Solution methodology

1. Direct method → elimination / matrix sol. etc.

2. Indirect method → ~~App~~ Initial Approx. & then iteratively refining solution to required degree of precision.
   ↳ Gauss-Seidel method

2-D steady state heat conduction:



Energy Balance

$\begin{pmatrix}\text{Rate of heat conduction at the top, left, right bottom}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat generation}\end{pmatrix} = \begin{pmatrix}\text{Rate of change of energy in element}\end{pmatrix}$

$$\dot{Q}_{cond,left} + \dot{Q}_{cond,right} + \dot{Q}_{cond,top} + \dot{Q}_{cond,bottom} + \dot{E}_{gen,element} = \frac{\Delta E_{element}}{\Delta t} \to 0 \longrightarrow \text{steady state}$$

$$\Rightarrow k(\Delta y \Delta z)\frac{(T_{m-1,n} - T_{m,n})}{\Delta x} + k(\Delta y \Delta z)\frac{(T_{m+1,n} - T_{m,n})}{\Delta x}$$

$$+ k(\Delta x \Delta z)\frac{(T_{m,n+1} - T_{m,n})}{\Delta y} + k(\Delta x \Delta z)\frac{(T_{m,n-1} - T_{m,n})}{\Delta y}$$

$$+ \bar{q}(\Delta x \Delta y \Delta z) = 0.$$

$$\Rightarrow \boxed{\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\bar{q}}{k} = 0}$$

$$\hookrightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\bar{q}}{k} = 0$$

12/8/18

Transient heat conduction

$$\sum_{\text{all sides}} \dot{Q} + E_{gen,\,element} = \frac{\Delta E_{element}}{\Delta t} = m c_p \frac{\Delta T}{\Delta t}$$

$$= \rho V_{element} c_p \frac{(T_m^{i+1} - T_m)}{\Delta t}$$

Explicit method

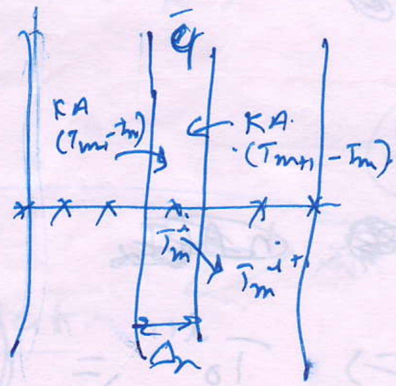
$$\sum \dot{Q}^{i} + E_{gen,\,element} = \rho V_{ele} c_p \frac{(T_m^{i+1} - T_m)}{\Delta t}$$

Implicit method

$$\sum Q^{i+1} + E_{gen,\,elem} = \rho V_{ele} c_p \frac{(T_m^{i+1} - T_m)}{\Delta t}$$

$$\frac{KA(T_{m-1} - T_m)}{\Delta x} + KA\frac{(T_{m+1} - T_m)}{\Delta x} + \bar{q}(A\Delta x)$$

$$= \rho(A\Delta x) c_p \frac{(T_m^{i+1} - T_m)}{\Delta t}$$



$$\Rightarrow T_{m-1} - 2T_m + T_{m+1} + \frac{\bar{q}}{k}\Delta x^2 = \frac{\Delta x^2}{\Delta t}\left(\frac{\rho c_p}{k}\right)(T_m^{i+1} - T_m^{i})$$

$$= \frac{1}{\alpha}\frac{\Delta x^2}{\Delta t}(T_m^{i+1} - T_m^{i})$$

$$= \frac{(T_m^{i+1} - T_m^{i})}{\tau}$$

$\tau \rightarrow$ Mesh Fourier No.

∴ **Explicit**

$$T_m^{i+1} = \tau\left(T_{m-1}^{i} + T_{m+1}^{i}\right) + (1-2\tau)T_m^{i} + \tau\left(\frac{\dot{q}_m^{i}\Delta x^2}{k}\right)$$

Implicit

$$\tau T_{m-1}^{i+1} - (1+2\tau)T_m^{i+1} + \tau T_{m+1}^{i+1} + \tau\left(\frac{\dot{q}_m^{i+1}\Delta x^2}{k}\right) + T_m^{i} =$$

B.C.

$$hA(T_\infty - T_0^{i}) + kA\left(\frac{T_1^{i} - T_0^{i}}{\Delta x}\right) + \dot{q}_0^{i}\left(A\frac{\Delta x}{2}\right) = \rho A\left(\frac{\Delta x}{2}\right)c_p\frac{\left(T_0^{i+1} - T_0^{i}\right)}{\Delta t}$$

$$\Rightarrow T_0^{i+1} = \left(1 - 2\tau - 2\tau\left(\frac{h\Delta x}{k}\right)\right)T_0^{i} + 2\tau T_1^{i} + 2\tau\left(\frac{h\Delta x}{k}\right)T_\infty + \tau\dot{q}_0^{i}\left(\frac{\Delta x^2}{k}\right).$$

$$\Rightarrow h\Delta x\,(T_\infty - T_0^i) + k(T_1^i - T_0^i) + \bar{q}_0^i\left(\frac{\Delta x^2}{2}\right) = \rho\left(\frac{\Delta x^2}{2}\right) c_p \frac{(T_0^{i+1} - T_0^i)}{\Delta t}$$

$$\Rightarrow \frac{h\Delta x}{k}(T_\infty - T_0^i) + (T_1^i - T_0^i) + \frac{\bar{q}_0^i}{k}\left(\frac{\Delta x^2}{2}\right) = \left(\frac{\rho c_p}{k}\frac{\Delta x^2}{\Delta t}\right)\frac{(T_0^{i+1} - T_0^i)}{2}$$

$$\Rightarrow \left(\frac{h\Delta x}{k}\right)(T_\infty - T_0^i) + (T_1^i - T_0^i) + \frac{\bar{q}_0}{k}\left(\frac{\Delta x^2}{2}\right) = \frac{1}{2\tau}(T_0^{i+1} - T_0^i)$$

$$\Rightarrow T_0^{i+1} = 2\tau\left(\frac{h\Delta x}{k}\right)(T_\infty - T_0^i) + 2\tau(T_1^i - T_0^i) + \frac{\bar{q}_0^i}{k}\left(\frac{\Delta x^2}{\cancel{2}}\right) + T_0^i$$

$$\boxed{\Rightarrow T_0^{i+1} = T_0^i\left(1 - 2\tau - 2\tau\left(\frac{h\Delta x}{k}\right)\right) + T_\infty\left(\frac{h\Delta x}{k}\right)2\tau + \tau\frac{\bar{q}_0^i}{k}(\Delta x^2)} \quad (2)$$

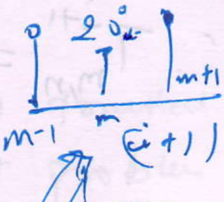
Stability criteria for explicit

From (1) → coeff. of all T_m^i in the T_m^{i+1} expression ≥ 0.

Primary coeff. ≥ 0 → +ve values.

↓

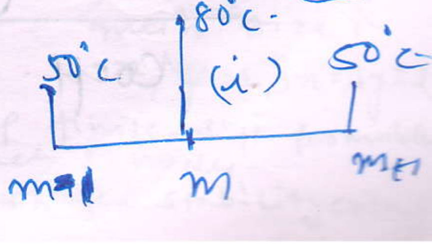
last time step coeff. →



$$(1 - 2\tau) \geq 0 \Rightarrow \tau \leq 1/2$$

$\tau = 1$

$$\Rightarrow \frac{\alpha\Delta t}{\Delta x^2} \leq \frac{1}{2}$$



From B.C. eqn. (2)

Primary coeff.

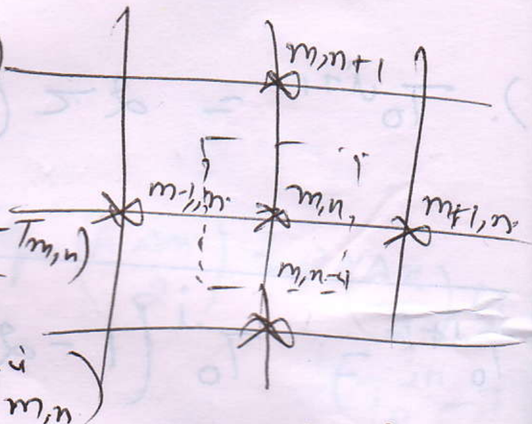
$$\left(1-2\tau-2\tau\left(\frac{h\Delta x}{k}\right)\right) \geq 0 \;\Rightarrow\; \tau \leq \frac{1}{2\left(1+\frac{h\Delta x}{k}\right)}$$

2-D transient heat conduction

$$k(\Delta y\Delta z)\frac{(T_{m-1,n}-T_{m,n})}{\Delta x} + k(\Delta y\Delta z)\frac{(T_{m+1,n}-T_{m,n})}{\Delta x}$$

$$+\,k(\Delta x\Delta z)\frac{(T_{m,n+1}-T_{m,n})}{\Delta x} + k(\Delta x\Delta z)\frac{(T_{m,n-1}-T_{m,n})}{\Delta x}$$

$$+\,\bar{q}(\Delta x\Delta y\Delta z) = \rho(\Delta x\Delta y\Delta z)c_p\frac{(T^{i+1}_{m,n}-T^{i}_{m,n})}{\Delta t}$$



m-s

~~Assuming~~ Taking $\Delta x = \Delta y = l$

$$T^i_{m-1,n}+T^i_{m+1,n}+T^i_{m,n-1}+T^i_{m,n+1}-4T^i_{m,n}+\frac{\bar{q}\,l^2}{k} = \left(\frac{\rho c_p}{k}\right)\frac{l^2}{\Delta t}(T^{i+1}_{m,n}-T^{i}_{m,n})$$

$$= \frac{T^{i+1}_{m,n}-T^{i}_{m,n}}{\tau}$$

$$\therefore\; T^{i+1}_{m,n} = \tau\left(T^i_{m-1,n}+T^i_{m+1,n}+T^i_{m,n-1}+T^i_{m,n+1}\right)+(1-4\tau)T^i_{m,n}+\tau\frac{\bar{q}l^2}{k}$$

Stability criteria.

Coeff. of $T^i_{m,n} \geq 0 \Rightarrow (1-4\tau) \geq 0$

$\therefore\; \tau \leq 1/4.$

Tutorial 8

Q.1

When diff. boundary condn & diff. ~~heat~~ energy bal. eqn. for diff. geometry

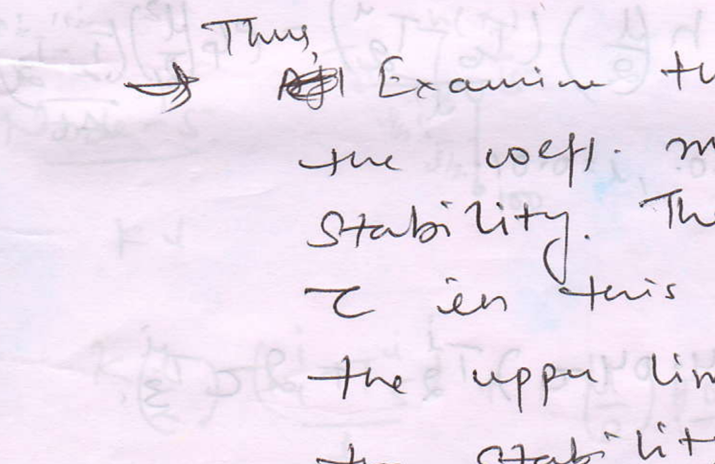
Then,

→ Examine each and every difference eqn. and the coeff. of all the primary elements (Nodes under consideration at earlier time stamp, $T^{i}_{m,n}$) should be ≥ 0.

Thus,
→ Examine the diagonal elements in the coeff. matrix (of $T_{m,n}$) for stability. The smallest coeff. of τ in this diagonal elements gives the upper limit for τ that satisfies the stability criteria.

Tutorial 8

Prob 3 (Pg#200) Ozisik.



convection

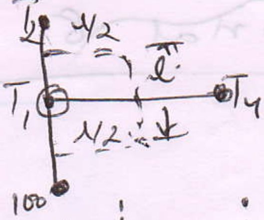
$T_\infty = 0°C$

$h = 60 \, W/(m^2.K)$

$l = 0.01 \, m$.

- 2-D transient heat problem
- take equal mesh size in x/y ⇒ $\Delta x = \Delta y = l$
- Develop finite-diff. formulation for each 5 nodes.
- establish the stability criteria

Node 1

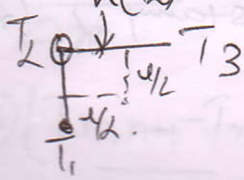


$$K\left(\frac{l}{2}\right)\left(\frac{T_2^i - T_1^i}{l}\right) + Kl\left(\frac{T_4^i - T_1^i}{l}\right) + K\left(\frac{l}{2}\right)\left(\frac{100 - T_1^i}{l}\right)$$

$$= \rho C_p\left(\frac{l^2}{2}\right)\left(\frac{T_1^{i+1} - T_1^i}{\Delta t}\right)$$

$$\therefore T_1^{i+1} = (1-4\tau)T_1^i + \tau T_2^i + 2\tau T_4^i + 100\tau$$

Node 2



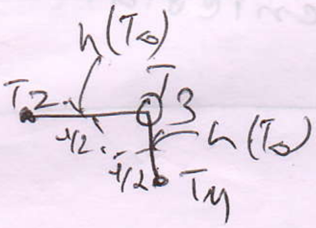
$$\Rightarrow K\left(\frac{l}{2}\right)\left(\frac{T_1^i - T_2^i}{l}\right) + K\left(\frac{l}{2}\right)\left(\frac{T_3^i - T_2^i}{l}\right)$$

$$+ h\left(\frac{l}{2}\right)\left(\frac{T_\infty - T_2^i}{l}\right) = \rho C_p\left(\frac{l^2}{4}\right)\left(\frac{T_2^{i+1} - T_2^i}{\Delta t}\right)$$

$T_\infty = 0$, $h = 60$, $K = 30$, $l = 0.01$.

$$\Rightarrow T_2^{i+1} = 2\tau T_1^i + (1 - 4.04\tau)T_2^i + 2\tau T_3^i.$$

Node-3

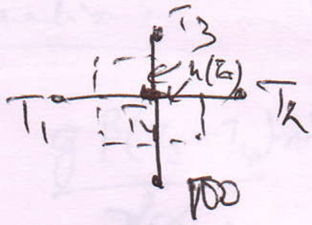


$$\Rightarrow K\left(\frac{l}{2}\right)\left(\frac{T_2^i - T_3^i}{l}\right) + K\left(\frac{l}{2}\right)\left(\frac{T_4^i - T_3^i}{l}\right) + 2\left(h\left(\frac{l}{2}\right)(T_\infty - T_3^i)\right)$$

$$= \rho C_p\left(\frac{l^2}{4}\right)\left(\frac{T_3^{i+1} - T_3^i}{\Delta t}\right)$$

$$\Rightarrow T_3^{i+1} = 2\tau T_2^i + (1 - 4.08\tau)T_3^i + 2\tau T_4^i$$

Mode-4

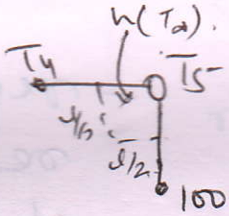


$$\Rightarrow k\left(\frac{d}{\alpha}\right)\frac{(T_3^i - T_4^i)}{d} + k\left(\frac{d}{\alpha}\right)(T_1^i - T_4^i) + k\frac{d}{\alpha}\frac{(100 - T_4^i)}{d}$$

$$+ k\left(\frac{d}{\alpha}\right)\frac{(T_5^i - T_4^i)}{d} + 2\left(h\left(\frac{d}{\alpha}\right)\right)(T_\infty - T_4^i) = \rho C_p\left(\frac{3d^2}{4}\right)\frac{(T_4^{i+1} - T_4^i)}{\Delta t}$$

$$\Rightarrow T_4^{i+1} = 1.33\,\tau T_1^i + 0.67\,\tau T_3^i + (1 - 4.027\tau)T_4^i + 0.67\,\tau T_5^i + 133.3\,\tau$$

Node-5.



$k\,d$

$$k\left(\frac{d}{\alpha}\right)\frac{(T_4^i - T_5^i)}{d} + k\left(\frac{d}{2}\right)\frac{(100 - T_5^i)}{d} + h\left(\frac{d}{2}\right)(T_\infty - T_5^i) = \rho C_p\left(\frac{d^2}{4}\right)\left(\frac{T_5^{i+1} - T_5^i}{\Delta t}\right)$$

$$\Rightarrow T_5^{i+1} = 2\tau T_4^i + (1 - 4.04\tau)T_5^i + 200\tau$$

$$\begin{bmatrix} T_1^{i+1} \\ T_2^{i+1} \\ T_3^{i+1} \\ T_4^{i+1} \\ T_5^{i+1} \end{bmatrix} = \begin{bmatrix} 1-4\tau & \tau & 0 & 2\tau & 0 \\ 2\tau & 1-4.04\tau & 2\tau & 0 & 0 \\ \cancel{0} & 2\tau & 1-4.08\tau & 2\tau & 0 \\ 1.33\tau & 0 & 0.67\tau & 1-4.027\tau & 0.67\tau \\ 0 & 0 & 0 & 2\tau & 1-4.04\tau \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} + \begin{bmatrix} 100\tau \\ 0 \\ 0 \\ 133.3\tau \\ 200\tau \end{bmatrix}$$

16/10/18

Convection

Free Convection

- Density diff. driven
- Buoyancy force driven
- Grashof No.

= ~~Buoyan~~ Buoyancy force / Viscous force

$= \frac{g \beta (T_s - T_\infty) x^3}{\nu^2}$

coeff. of thermal expansion

→ analogous to Re No. in forced convection.

→ transition from laminar to turbulent flow is governed by Gr. No.

Vertical plates → Laminar → $Gr < 10^8$
Turbulent → $Gr > 10^9$

- $\frac{Gr}{Re^2}$ → measure of relative importance of free over forced convection

- $\frac{Gr}{Re^2} \simeq 1$ → Both free & forced conv. play role

$\ll 1$ → primarily forced convection

$\gg 1$ – " free convection.

- $Nu = f(Re, Pr, Gr)$

~~Prandtl~~ Rayleigh No.

= for cases, $Pr \simeq 1$.

So $Nu = f(Gr)$ → free

Forced Convection

- forced mixing of fluid

- Prandtl Number

= momentum diffusion / heat diffusivity

$= \frac{\nu}{\alpha}$

$= \frac{\mu/\rho}{k/\rho C_p} = \frac{\mu C_p}{K}$

- represents the relative importance of momentum and energy transport.

- represents the relative thickness of velocity/momentum boundary layer thick and thermal B.L.T.

Free Convection,

- Rayleigh No.

$= Gr \cdot Pr = \frac{g\beta(T_s - T_\infty)x^3}{\nu\alpha}$

- Indicator for heat transfer as conduction or convection
- Below critical value of Rayleigh No — heat transfer is ~~better or~~ through conduction.
- Above cri value — heat transfer is through convection

Nusselt No. — ratio of $\frac{\text{heat transfer by convection}}{\text{heat tran by conduction}}$

$$Nu = \frac{hL}{K} = \frac{(h\Delta T)}{\left(\frac{K\Delta T}{L}\right)}$$

$$= f(Re, Pr, Gr).$$

For forced conv., $Nu = f(Re, Pr)$.

free " $Nu = f(Gr, Pr)$.

" " in gases ($Pr \approx 1$), $Nu = f(Gr)$.

External flow

Pr.	Pr.
Liq. metals	0.004 - 0.03
Gases	0.7 - 1.0
Water	1.7 - 13.7
light organic fluid	5 - 50
Oils	50 - 100,000.

1) Flow over flat plates

ⓐ Uniform temp.

Laminar → $Nu = \frac{hL}{k} = 0.664\, Re_L^{0.5} Pr^{1/3}$,

$Re_L < 5 \times 10^5$

for $Pr > 0.6$.

Turbulent → $Nu = 0.037\, Re_L^{0.8} Pr^{1/3}$.

$Re_L \geq 5 \times 10^5,\ \leq 10^7$

$0.6 \leq Pr \leq 60$.

Liq. metall –

$Nu_x = 0.565\, (\underbrace{Re_x Pr}_{P_e})^{1/2}$, $Pr \leq 0.05$

P_e → Peclet no.

ⓑ uniform heat flux.

$Nu_x = 0.453\, Re_x^{0.5} Pr^{1/3}$ → lamina.

$= 0.0308\, Re_x^{0.8} Pr^{1/3}$ → turbulent.

Flow over Sphere & cylinder.

cylinder

$$Nu_{cy.} = \frac{hD}{k} = 0.3 + \left[\frac{0.62\, Re^{1/2} Pr^{1/3}}{\left(1 + (0.4/Pr)^{2/3}\right)^{1/4}}\right]\left[1 + \left(\frac{Re}{282000}\right)^{5/8}\right]^{4/5}.$$

↳ $Re \cdot Pr > 0.2$

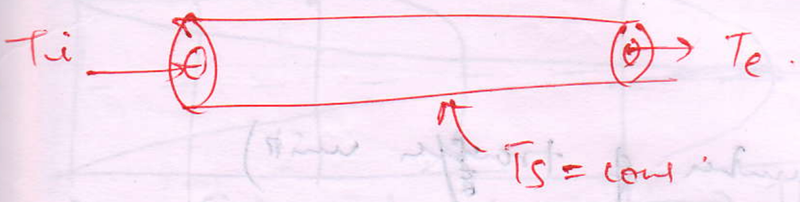
Sphere

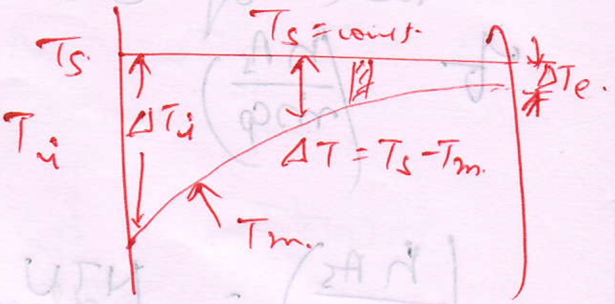
$$Nu_{sph} = \frac{hD}{k} = 2 + \left[0.4\, Re^{1/2} + 0.06\, Re^{2/3}\right] Pr^{0.4} \left(\frac{\mu_\infty}{\mu_s}\right)^{1/4}.$$

↳ $3.5 \leq Re \leq 8 \times 10^4$, $0.7 \leq Pr \leq 380$.

$\mu_\infty / \mu_s \geq 1$

Log mean Temp differen





$$\dot{Q} = h A_s \Delta T_{avg} = h \cdot A_s (T_s - T_m)_{avg}$$

$$= \dot{m} c_p \Delta T_{avg}$$

Now,

$$\dot{m} c_p \, dT_m = h (T_s - T_m) \, dA_s$$

$$A_s = \text{Perimeter} \times dx = p\,dx$$

$$dT_m = -d(T_s - T_m)$$

$$\Rightarrow \frac{d(T_s - T_m)}{(T_s - T_m)} = \frac{-hp}{\dot{m} c_p} dx$$

$$\cancel{\Rightarrow \ln T_s}$$

$$\cancel{\Rightarrow \ln \left(\frac{T_s - T_e}{T_s - T_i} \right)}$$

$$\Rightarrow \ln |T_s - T_m| \Big|_{x=0}^{x=L} = -\frac{hp}{\dot{m} c_p} L$$

A_s

$$\Rightarrow \boxed{\ln \left(\frac{T_s - T_e}{T_s - T_i} \right) = -\frac{h A_s}{\dot{m} c_p}}$$

at $x = 0$, $T_m = T_i$
at $x = L$, $T_m = T_e$.

$$\Rightarrow \boxed{T_e = T_s - (T_s - T_i) \exp\left(-\frac{h A_s}{\dot{m} c_p} \right)}$$

- Temp. diff. between fluid and surface decays exponentially in the flow direction and the