Fourier series and waveform symmetry

Consider a periodic function $f(t)$ of period 2π rad (corresponds to T sec and $\omega = \dfrac{2\pi}{T}$). The Fourier series expansion of $f(t)$ is given below.

$$
\begin{aligned}
f(t) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \\
a_0 &= \frac{1}{T}\int_0^T f(t)\ dt \ = \ \frac{1}{2\pi}\int_0^{2\pi} f(t)\ d\omega t \\
a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\ dt \ = \ \frac{2}{2\pi}\int_0^{2\pi} f(t)\cos(n\omega t)\ d\omega t, \qquad n = 1, 2, 3, ... \\
b_n &= \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\ dt \ = \ \frac{2}{2\pi}\int_0^{2\pi} f(t)\sin(n\omega t)\ d\omega t, \qquad n = 1, 2, 3, ...
\end{aligned}
$$

Simplified calculation of Fourier coefficients based on certain symmetries of the waveform

Symmetry	**Conditions**	**Fourier coefficients**
Even	$f(-t) = f(t)$	$b_n = 0$
Odd	$f(-t) = -f(t)$	$a_n = 0$
Half-wave symmetry	$f\left(\dfrac{T}{2} + t\right) = -f(t)$	$a_n = \dfrac{2}{(T/2)}\displaystyle\int_0^{(T/2)} f(t)\cos(n\omega t)\ dt$ $b_n = \dfrac{2}{(T/2)}\displaystyle\int_0^{(T/2)} f(t)\sin(n\omega t)\ dt$
Half-wave symmetry **and** Quarter-wave symmetry	$f\left(\dfrac{T}{2} + t\right) = -f(t)$ **and** $f\left(\dfrac{T}{4} + t\right) = f\left(\dfrac{T}{4} - t\right)$	$a_n = \dfrac{2}{(T/4)}\displaystyle\int_0^{(T/4)} f(t)\cos(n\omega t)\ dt$ $b_n = \dfrac{2}{(T/4)}\displaystyle\int_0^{(T/4)} f(t)\sin(n\omega t)\ dt$

Expressions in the table above are rewritten in the next page, with the variable of integration being ωt.

Symmetry	**Conditions**	**Fourier coefficients**
Even	$f(-\omega t) = f(\omega t)$	$b_n = 0$
Odd	$f(-\omega t) = -f(\omega t)$	$a_n = 0$
Half-wave symmetry	$f\left(\pi + \omega t\right) = -f(\omega t)$	$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos(n\omega t)\ d\omega t$ $b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin(n\omega t)\ d\omega t$
Half-wave symmetry **and** Quarter-wave symmetry	$f\left(\pi + \omega t\right) = -f(\omega t)$ **and** $f\left(\frac{\pi}{2} + \omega t\right) = f\left(\frac{\pi}{2} - \omega t\right)$	$a_n = \frac{2}{(\pi/2)}\int_0^{(\pi/2)} f(t)\cos(n\omega t)\ d\omega t$ $b_n = \frac{2}{(\pi/2)}\int_0^{(\pi/2)} f(t)\sin(n\omega t)\ d\omega t$