EN 312 - LIST OF IMPORTANT FORMULAE

Laplace Transform:

$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}.dt$ $(s = \sigma + j\omega) \rightarrow$ complex frequency

properties:

i) Linearity : $L[f_1(t) + f_2(t) \ldots\ldots + f_n(t)] = F_1(s) + F_2(s) \ldots\ldots + F_n(s)$

ii) Scaling : $L[kf(t)] = kF(s)$

iii) Real differentiation : $L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$

$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) \ldots - f^{(n-1)}(0^-)$

iv) Real integration : $L\left\{\int_0^t f(t)\,dt\right\} = F(s)/s$

$L\left\{\int_0^{t_1}\int_0^{t_2} \ldots \int_0^{t_n} f(t)\,dt_1\,dt_2 \ldots dt_n\right\} = F(s)/s^n$

v) Differention by s : $L\{tf(t)\} = -dF(s)/ds$

$L\{t^n\} = n!/s^{n+1}$

vi) Complex translation : $F(s-a) = L[e^{at}f(t)]$

and $F(s+a) = L[e^{-at}f(t)]$

vii) Real translation (shifting theorem)

$L\{f(t-T)\} = e^{-Ts}F(s)$

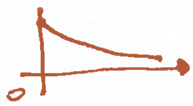
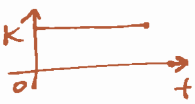
viii) Initial value theorem : $f(0^+) = \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

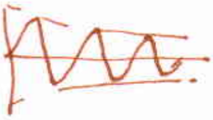
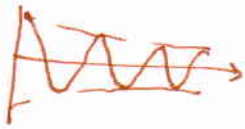
< f(t) to be continuous or step discontinuity @ $t=0$ > = validity

ix) Final value theorem : $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

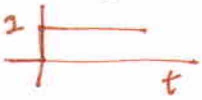
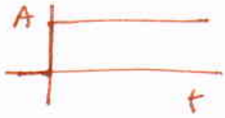
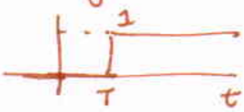
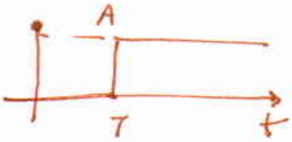
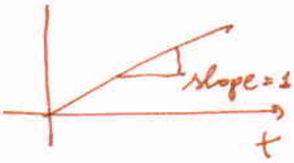
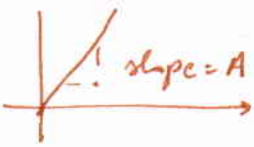
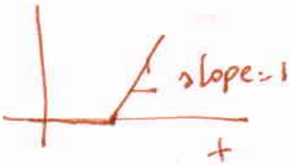
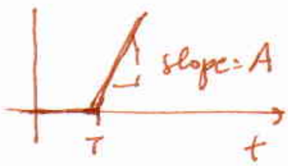
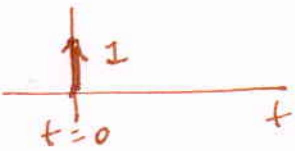
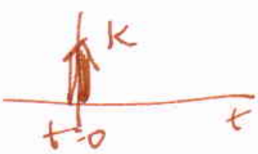
< roots of denominator polynomial of $F(s)$ i.e poles to have negative real parts > = validity condition.

Table of Laplace Transforms

$f(t)$		$F(s)$	$f(t)$		$F(s)$
1		$1/s$	e^{-at}		$\frac{1}{s+a}$
k		k/s	e^{at}		$\frac{1}{s-a}$
$k\,f(t)$		$kF(s)$	$e^{-at}t^n$	$\longrightarrow$	$n!/(s+a)^{n+1}$
t		$1/s^2$	$\sin\omega t$		$\frac{\omega}{s^2+\omega^2}$
t^n		$n!/s^{n+1}$	$\cos\omega t$		$\frac{s}{s^2+\omega^2}$

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$e^{-at}\sin\omega t$ 	$\rightarrow \dfrac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$ 	$\rightarrow \dfrac{(s+a)}{(s+a)^2+\omega^2}$
$\sinh\omega t$	$\rightarrow \dfrac{\omega}{s^2-\omega^2}$	$\cosh\omega t$	$\rightarrow \dfrac{s}{s^2-\omega^2}$

Standard Laplace transforms

$f(t)$	$F(s)$	$f(t)$	$F(s)$
unit step $= u(t)$ 	$\rightarrow 1/s$	$A u(t)$ 	$\rightarrow A/s$
delayed unit step  $u(t-T)$	$\rightarrow e^{-Ts}/s$	$A u(t-T)$ 	$\rightarrow \dfrac{Ae^{-Ts}}{s}$
unit ramp $r(t) = t u(t)$ 	$\rightarrow 1/s^2$	$A t u(t)$ 	$\rightarrow A/s^2$
Delayed ramp. $r(t-T) = (t-T)u(t-T)$ 	$\rightarrow \dfrac{e^{-Ts}}{s^2}$	$A(t-T)u(t-T)$ 	$\rightarrow \dfrac{Ae^{-Ts}}{s}$
Unit impulse $= \delta(t)$ 	$\rightarrow 1$	Delayed impulse $\delta(t-T)$ 	$\rightarrow e^{-Ts}$
impulse of strength K. 	$\rightarrow K$	Delayed impulse amplitude K $K\delta(t-T)$ 	$\rightarrow Ke^{-Ts}$.

Relationship between the standard inputs

$$\boxed{r(t)} \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} \boxed{u(t)} \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} \boxed{\delta(t)}$$

ramp function — unit step function — impulse function.

Inverse Laplace transform:

Decompose using partial fraction method and use the properties and L.T of standard function tables.

- Simple/real roots
- Repeated/ Multiple roots
- Complex conjugate roots

Transfer function (T.F)

$$\frac{\text{output response}}{\text{input excitation/ driving function}}$$

Cond: only linear time invariant systems.
- neglecting all initial conditions.
- Single o/p single i/p system.

Electrical

$R \longrightarrow R$

$L \longrightarrow sL \quad (\sim j\omega L) \rightarrow X_L.$

$C \longrightarrow 1/sC \quad (\sim \frac{1}{j\omega C}) \rightarrow X_C$

* $\boxed{\text{Transfer function (T.F) of a system} = \text{unit impulse (i/p) response}.}$

$$T.F. = \frac{P(s)}{Q(s)} \xrightarrow{\text{factorized}} \frac{K(s-s_a)(s-s_b)\cdots(s-s_m)}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

* K = system gain factor.
* put $s=0$. overall value of T.f. = DC gain
* equate denominator to zero → roots = poles (×)
  (i.e $s_1, s_2 \cdots s_n$ are poles)

polynomial equation obtained in s ⟶ CHARACTERISTIC EQUATION.
i.e $Q(s)=0$

* equate numerator to zero ⟶ roots are zeros (o)
  i.e $P(s)=0$

* highest power of 's' present in the characteristic equation (denominator polynomial in s equated to zero) of a CLOSED LOOP T.F. of system = "ORDER" of the system

* if OPEN LOOP T.F of the system expressed in time constant form. i.e

$$T.F = \frac{K(1+T_a s)(1+T_b s)\cdots}{s^j (1+T_1 s)(1+T_2 s)\cdots}$$

in denominator power of $s = j$ = "TYPE" of system

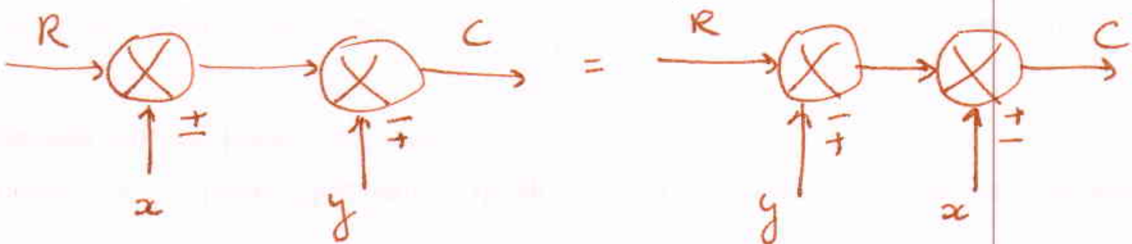
Note: Order based on closed loop system (& charac. equation) while type based on open loop (& time constant form)

Block diagram Reduction (BDR)

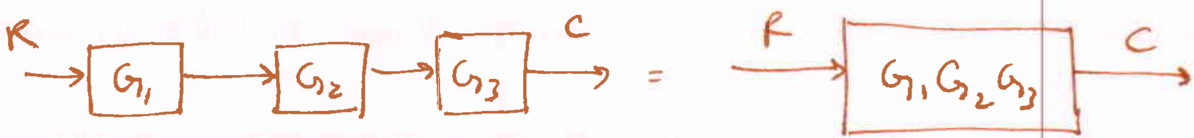
* Hint: As far as possible shift take off points to right and summing points to the left.

BDR rules <confirm using algebra after every step>

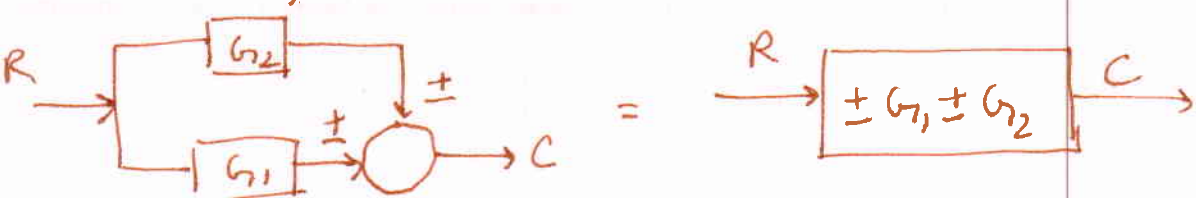
1. Associative law



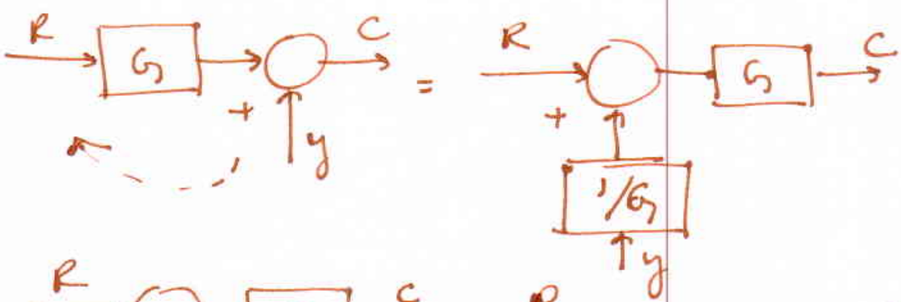
2. Blocks in series



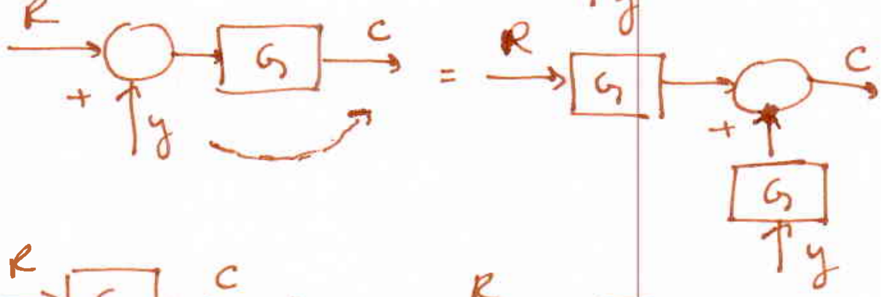
3. Blocks in parallel



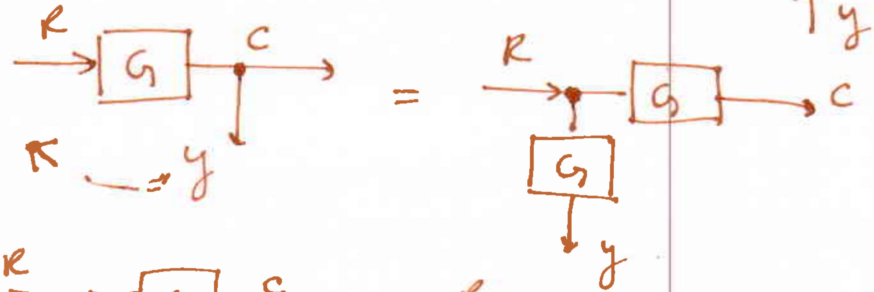
4. Summing point before block



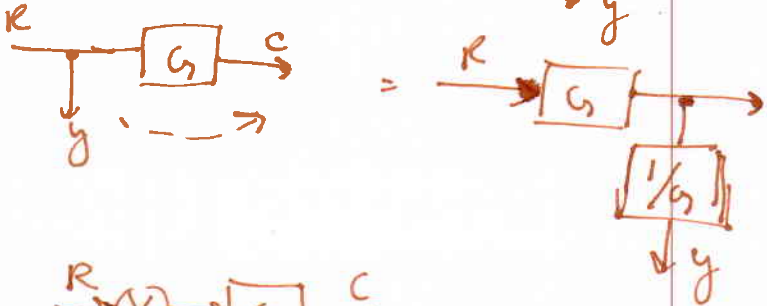
5. Summing point after block



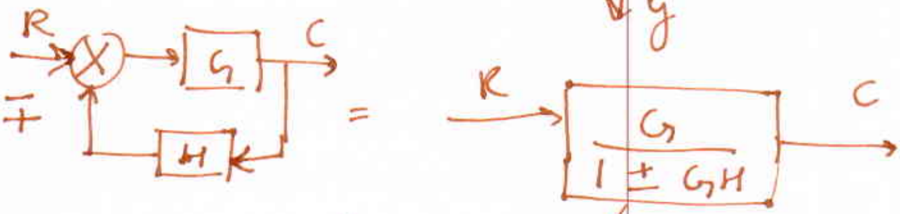
6. Takeoff point before block



7. Take off point after block

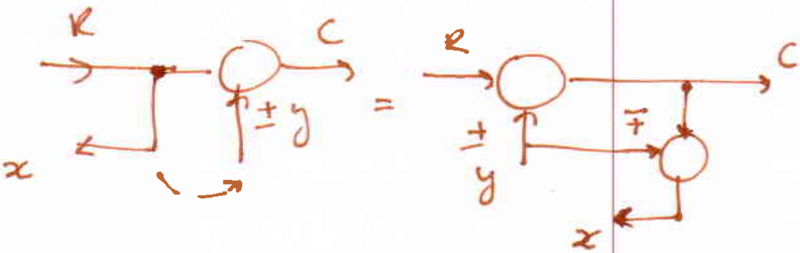


8. Minor feedback loop



\+ for −ve feedback
− for +ve feedback

9. Takeoff point after summing junction



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CRITICAL RULES

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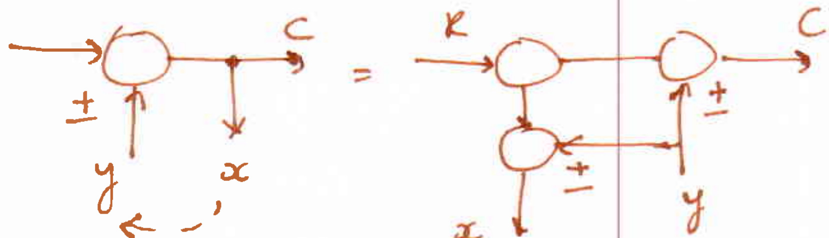
10. Take off point before summing junction



Mason's Gain formula (for signal flow graphs)

Overall T.F $= \dfrac{\sum T_k \Delta_k}{\Delta} = \dfrac{T_1\Delta_1 + T_2\Delta_2 \ldots\ldots}{\Delta}$

where k = no: of forward paths.

T_k = gain of k^{th} forward path.

Δ = system determinant

$= 1 - [\sum$ all individual feedback loop gains including self loops$] + [\sum$ gain products of all possible combinations of non-touching loops$]$

$- [\sum$ gain products of combinations of three non-touching loops$] + \ldots\ldots$

Δ_k = value of above Δ by eliminating all loop gains and associated products which are touching the k^{th} forward path.

Time response and system design:

- Total time response $c(t) = c_{ss} + c_t(t)$ = steady state response + transient response
- Mathematically for stable operating systems: transient response $\rightarrow 0$ i.e $\lim_{t\to\infty} c_t(t) = 0$.
- Steady state error $e_{ss}(t) = r(t) - c_{ss}$, where $r(t)$ = reference input.
- Types of inputs: (standard test inputs)

$r(t)$	Symbol	$R(s)$	
* unit step / position function	$u(t)$	$1/s$	1 or A (A/s)
* unit ramp / velocity function	$r(t)$	$1/s^2$	t or At (A/s^2)
* Unit parabolic / acceleration function	$p(t)$	$1/s^3$	$t^2/2$ or $\frac{At^2}{2}$ (A/s^3)
* Unit impulse	$\delta(t)$	1	(1 or A)

$E(s)$ = steady state error = $\dfrac{R(s)}{1+G(s)H(s)}$

$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$ $\therefore$ $\boxed{e_{ss} = \lim_{s\to 0} \dfrac{sR(s)}{1+G(s)H(s)}}$

Static error coefficient	Corresponding S.S error
- positional error coefficient $k_p = \lim_{s\to 0} G(s)H(s)$	$\dfrac{A}{1+K_p}$
- velocity error coefficient $K_v = \lim_{s\to 0} sG(s)H(s)$	$\dfrac{A}{K_v}$
- Acceleration error coefficient $K_a = \lim_{s\to 0} s^2G(s)H(s)$	$\dfrac{A}{K_a}$

Type of system	Error coefficients			Error e_{ss} for		
	K_p	K_v	K_a	Step input	Ramp input	Parabolic i/p
0	K	0	0	$\frac{A}{1+K}$	∞	∞
1	∞	K	0	0	$\frac{A}{K}$	∞
2	∞	∞	K	0	0	$\frac{A}{K}$

First order system: general T.F = $\dfrac{1}{1+s\tau}$

τ = time constant of the system.

$1+s\tau = 0$ = charc. equation

$\hookrightarrow$ 1 pole $s = -1/\tau$

- Standard second order system : $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

ζ = damping ratio/coefficient (unit less) and ω_n = natural frequency of oscillations (rad/s)

**

↳ ⟨Note: above expression in denominator coefficient of s^2 is $\underline{\underline{1}}$⟩

- damped frequency of oscillations $\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$ rad/s.

No.	ζ Range	Type of closed loop poles	Nature of response	System Classification
1.	$\zeta = 0$	purely imaginary	Oscillation with constant frequency & amplitude	Undamped
2.	$0 < \zeta < 1$	complex conjugate with neg. real parts	Damped oscillations	Underdamped
3.	$\zeta = 1$	real, equal & negative	Critical and pure exponential	Critically damped
4	$1 < \zeta < \infty$	real, unequal	purely exponential slow/sluggish response	Overdamped.

Result for standard second order systems which is underdamped ($0<\zeta<1$) and excited by unit step

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ rad/s and $\theta = \cos^{-1}\zeta$ or $\tan^{-1}\left\{\frac{\sqrt{1-\zeta^2}}{\zeta}\right\}$ rad

↳ if i/p is step function with input 'A' amplitude

then: $c(t) = A\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)\right]$

If system is not in the standard form:

i.e $\dfrac{C(s)}{R(s)} = \dfrac{K}{s^2+2\zeta\omega_n s+\omega_n^2}$ K = constant then,

the result can be used after expressing the transfer function as:

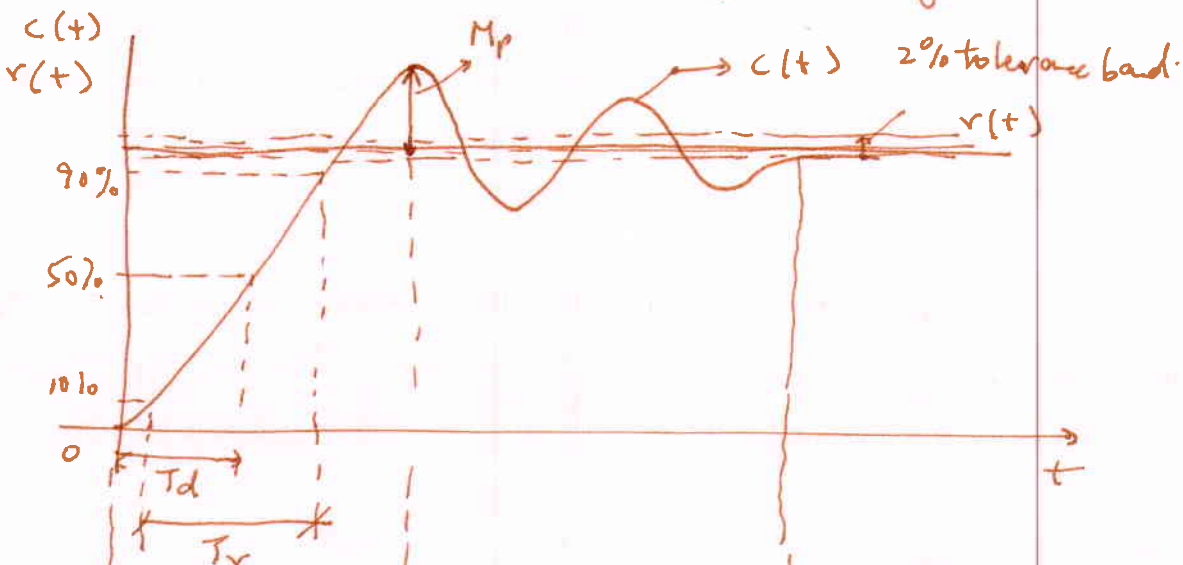
$$\frac{C(s)}{R(s)} = \frac{K}{\omega_n^2}\left[\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right]$$

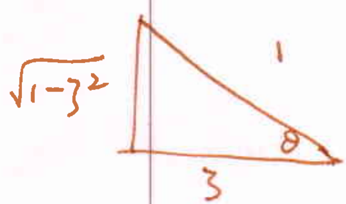
and hence $c(t) = \dfrac{K}{\omega_n^2}\left[1-\dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)\right]$

↳ Above result not applicable for case if:

a) Numerator is a polynomial in s and

b) input is not a step (position) function.





* Delay time $T_d = \dfrac{1+0.7\zeta}{\omega_n}$ sec

* Rise time $T_r = \dfrac{\pi-\theta}{\omega_d}$ sec   θ in radians. $\theta = \cos^{-1}\zeta$ or $= \tan^{-1}\dfrac{\sqrt{1-\zeta^2}}{\zeta}$.

* Peak time $T_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ sec.

* Peak overshoot $M_p = \left\{c(t)\big|_{t=T_p}\right\} - 1$ for unit step.

$$\% M_p = \left(e^{\left\{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right\}}\right)\times 100\,\%.$$

* Settling time $T_s = \frac{4}{\zeta\omega_n}$ ... for a tolerance of $\pm 2\%$ of steady state.

↳ Time constant: time required by system o/p to reach 63.2% of the final value during first attempt.

Other formulae:

(1) time for nth overshoot $T_{p_n} = \frac{n\pi}{\omega_d} = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$ sec

$n = 2, 3 \ldots$ so on.

(2) oscillation period $T_\omega = \underline{T_{p(n+2)} - T_{p(n)}} = \frac{2\pi}{\omega_d}$

(1^{st} is peak $n=1$; $n=2$ is ~~crest~~ valley; $n=3$ again peak etc.).

(3) Time constant of the system $\tau = \frac{1}{\zeta\omega_n}$ sec.

(4) Settling time for $\pm 5\%$ tolerance band:

$c(t)|_{t=T_s} = 0.95$ or $0.95 = 1 - e^{-\zeta\omega_n T_s}$. ⟨2% ideally $T_s = \frac{3.912}{\zeta\omega_n}$⟩

$T_s \simeq \frac{3}{\zeta\omega_n}$ ⟨$T_s = 2.995/\zeta\omega_n$⟩

(5) Similarly settling time for $\pm x\%$ band

$\Rightarrow (1-x) = 1 - e^{-\zeta\omega_n T_s} \rightarrow$ solve for T_s.

or in other words $T_s = \frac{1}{\zeta\omega_n} \times \ln\left(\frac{x}{100}\right) = \tau \times \ln\left(\frac{x}{100}\right)$

(6) Extending for third order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(1+s\tau)(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$= \frac{\omega_n^2\alpha}{(s+\alpha)(s^2+2\zeta\omega_n s+\omega_n^2)} \quad \text{where } \alpha = 1/\tau$$

$$c(t) = 1 - Ae^{-\alpha t} + Be^{-\zeta\omega_n t}\sin\left[\omega_n\sqrt{1-\zeta^2}\,t - \phi\right]$$

where,

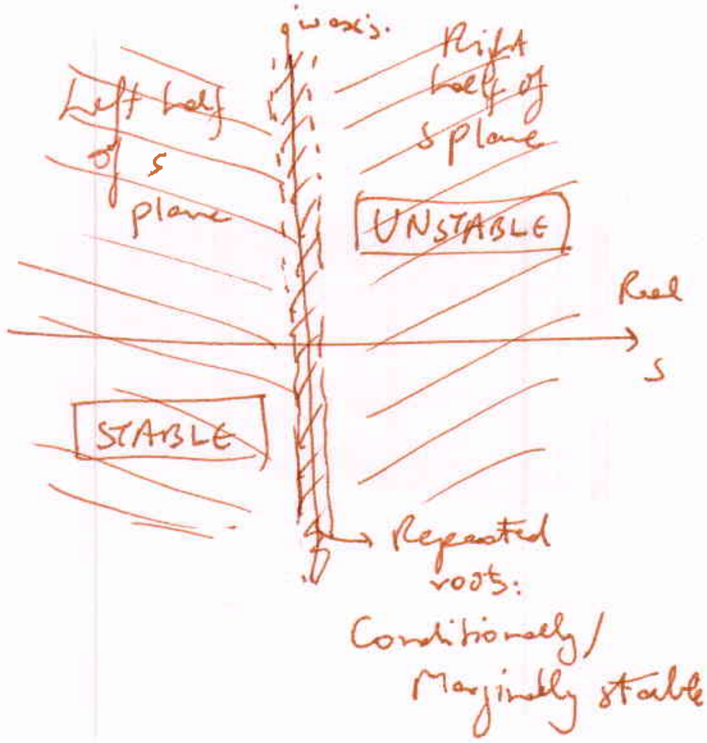
$$A = \frac{\omega_n^2}{\omega_n^2 - 2\zeta\omega_n\alpha + \alpha^2}, \quad B = \frac{1}{\sqrt{1-\zeta^2\left(1 - \frac{2\zeta\omega_n}{\alpha} + \frac{\omega_n^2}{\alpha^2}\right)}}$$

and $$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} + \tan^{-1}\frac{\omega_n\sqrt{1-\zeta^2}}{\alpha - \zeta\omega_n}$$

Controller Types:

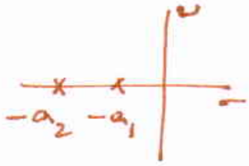
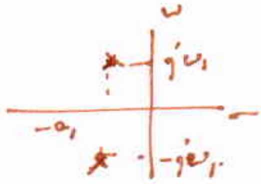
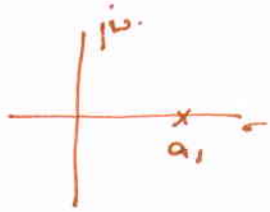
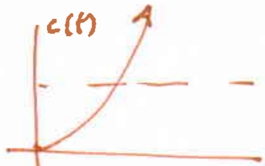
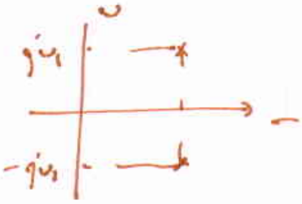
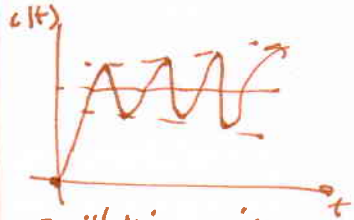
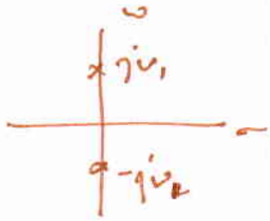
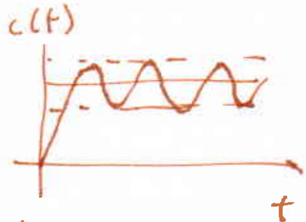
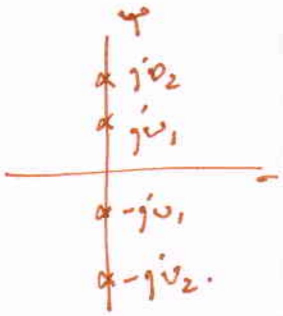
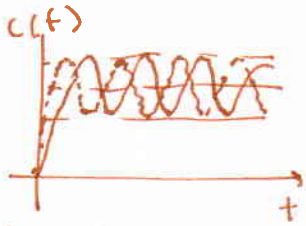
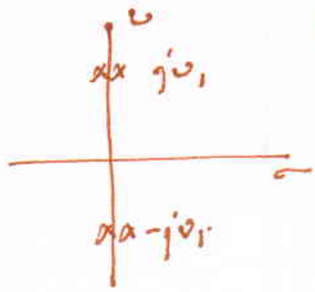
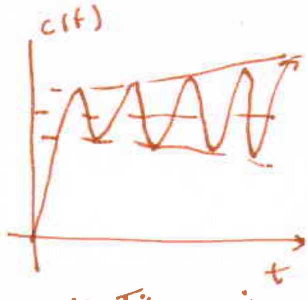
- Proportional (P) $\rightarrow$ output $= Ke(t)$ — $e(t)$ = error signal $= r(t) - c(t)$.
- proportional derivative (PD) $\rightarrow$ O/P $= Ke(t) + T_d \frac{de(t)}{dt}$
- proportional integral (PI) $\rightarrow$ O/P $= Ke(t) + K_i \int e(t)\,dt$
- proportional integral-derivative (PID) $\rightarrow$ O/P $= Ke(t) + T_d \frac{de(t)}{dt} + K_i \int e(t)\,dt$.
- Rate feedback controller $\rightarrow$ O/P $= Ke(t) - K_t \frac{dc(t)}{dt}$.

Stability & R-H Criterion:



Types of stability

- Absolutely stable
- Critically stable
- Conditionally stable
- BIBO stable
- zero input stable
- Asymptotic stable

Nature of closed loop poles	Location of closed loop poles.	Step Response	Stability condition
- Real, -ve in LHS plane		 pure exponential	Absolutely stable.
- Complex conjugate with -ve real part		 Damped oscillations	Absolutely stable.
- Real, +ve in RHS of s plane. (even one pole on RHS)		 exponential	Unstable
- Complex conjugate with +ve real part		 Oscillations with increasing amplitude	Unstable.
Non repeated roots on imaginary axis.		 freq of oscln = ω_1	Marginally / Critically stable
		 freq of oscillations = ω_1 & ω_2	Marginally / Critically stable
Repeated pair on imaginary axis.		 oscillations with increasing amplitude	Unstable

Routh's stability condition

Routh's array

general characteristic equation:

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$$

$$\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & \cdots & \\
s^{n-3} & c_1 & c_2 & c_3 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s^0 & a_n & & & &
\end{array}$$

Coefficients for first 2 rows are written directly from charc. eqn.

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \quad (\nearrow f\!-\!\searrow)$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \quad \text{\& so on.}$$

Necessary and sufficient condition for system to be stable is "All the terms in the first column of the Routh Array should have same sign → no sign change in the first column."

if sign changes are there: no. of sign of changes = no: of poles of CL system in RHS of s plane

& system is unstable.

Problems.

- find stability or # of roots on RHS.
- # of roots bet $s = -\sigma_1$ & $s = -\sigma_2$
- find range of parameter K for stability
- find freq of oscillation. K_{mar} & ω_{mar}.

→ Special case 1: first element of any of the rows of R array is 0.

→ Special case 2: All the elements of any row in the array are zero

Special case 1: first element in Routh array zero. $\Rightarrow$ 2 methods to solve

Complete array by substituting the zero with small $\underline{+ve}$ no: ε & look for sign change taking $\lim_{\varepsilon \to 0}$

Replace 's' by '$1/z$' in the original equation. Taking LCM rearrange the charc eqn in descending powers of 'z' and complete the array.

<u>Special case 2:</u> All elements in a row are zero.

$$
\begin{array}{c|ccc}
s^5 & a & b & c \\
s^4 & d & e & f \\
s^3 & 0 & 0 & 0 \\
 & 4d & e & 0.
\end{array}
$$

$\rightarrow$ get auxillary equation $A(s) = ds^4 + es^2 + f$

for s^3: Complete array by taking derivative of $A(s)$ w.r.t s.

$$\frac{dA(s)}{ds} = 4ds^3 + 2es.$$

$\hookrightarrow$ Check for sign chnge in the array $\boxed{\text{and}}$ roots of the <u>auxillary equation</u> as well. $\rightarrow$ these roots (of the auxillary equation) are the <u>most dominant</u> roots of the original characteristic equation from the stability point of view.

** for finding no: of roots bet $s = \sigma_1$ and $s = \sigma_2$ put $s = s' + \sigma_1$ $\rightarrow$ Compute array in s' $\rightarrow$ # of signchanges = # of roots to RHS of $s = \sigma_1$

and then $s = s' + \sigma_2$ in the charc. equation. $\rightarrow$ Compute the array. $\hookrightarrow$ # of sign chngs = # of roots of to RHS of $s = \sigma_2$.

$\hookrightarrow$ the difference = # of roots bet $s = \sigma_1$ & $s = \sigma_2$

Root Locus Technique

- introduced by W.R. Evans in 1948.
- Variation of closed loop poles of a system on the s-plane as a gain / parameter is varied. ⟨K⟩.

K variation

0 to ∞ (i.e K +ve)
direct root locus
↳ what we use!

0 to −∞ (i.e K −ve)
complimentary or inverse root locus
⇒ ⟨NOT CONSIDERED⟩

→ Angle condition

if a point has to lie on the root locus of a system whose characteristic equation is given by $1+G(s)H(s)=0$; then

$$\angle G(s)H(s)\Big|_{@\ s=\text{point of interest}} = \pm(2q+1)180^\circ = \pm(\underline{\underline{\text{odd}}}\ \text{multiple of } 180^\circ)$$

$q = 0,1,2$

→ Angle condition used to find if the point lies on the root locus
eg: validity of breakaway or breakin points

→ Magnitude condition

If a point is already known to lie on the root locus, then it should satisfy

$$|G(s)H(s)|\Big|_{@\ s=\text{point of interest (on the root locus)}} = 1$$

↳ Magnitude condition is used to find the value of parameter/gain K for a given point on the root locus.

graphical method to determine K:

$$K = \frac{\text{product of phasor lengths drawn from open loop } \boxed{\text{poles}} \text{ upto a point on the root locus.}}{\text{product of phasor lengths drawn from open loop } \boxed{\text{zeros}} \text{ upto a point on the root locus}}$$

↳ if no open loop zeros exist denominator to be taken as unity.

Rules for plotting root locus.

Rule 1: Root locus is always symmetrical about the real axis.

Rule 2: The no: of branches in a root locus = no: of OL poles OR no: of OL zeros ⟨whichever is higher⟩

No. of branches of root locus approaching infinity

$= P - Z$

P = # of openloop (OL) poles.

Z = # of open loop (OL) zeros.

Rule 3: A point on the real axis lies on the root locus if the sum of the OL poles ~~(P)~~ and OL zeros on the right hand side of the point ON THE REAL AXIS is $\boxed{\text{ODD}}$.

Note: No complex poles or zeros are considered in this rule.

Rule 4: Branches approach to or from infinity along straight lines called Asymptotes. Angles of these asymptotes are given by:

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z} \quad \text{where } q = 0, 1, 2 \ldots \underset{\text{upto.}}{(P-Z-1)}$$

Rule 5: The asymptotes intersect at a common point given by σ (known as the centroid)

$$\Rightarrow \sigma = \frac{\sum \text{Real part of poles of } G(s)H(s) - \sum \text{Real parts of zeros of } G(s)H(s)}{P - Z}$$

note: Only real parts of poles & zeros need to be used in the above expression.

Rule 6: Breakaway / Breakin points: these are points on the root locus where multiple roots of the characteristic equation occur for a particular value of K.

Occurence: i) Two adjacently placed poles on real axis with the axis being part of the root locus.

ii) Two adjacently placed zeros on the real axis with the space in between being part of the root locus.

iii) if there is a zero on the real axis to the left of which there is no pole or zero but the left of this is part of the root locus.

Steps to determine breakaway / breakin point:

a) Construct charc. eqn $1+G(s)H(s)=0$

b) Separate terms involving k & $K = F(s)$

c) differentiate w.r.t s and equate to zero $\frac{dk}{ds}=0$ $F'(s)=0$

d) Roots of the above eqn (step (c)) give the breakaway point.

e) check for validity by substituting in step (b) & ensuring that the value of K at the roots is +ve.

Rule 7: Intersection of the root locus with the imaginary axis:

a) Construct RH array using K and $1+G(s)H(s)=0$.

b) Determine Kmarginal which makes one of the rows zero (eg: s^1 row) < not s^0 row >

c) Construct the auxiliary equation $A(s)=0$ by using coefficients of a row just above the row of zeros.

d) Roots of the auxiliary equation using Kmarginal are the intersection points of the root locus with the imaginary axis. Note: Only if Kmar is +ve the root locus will intersect with the imaginary axis; else entire root locus lies in the left half plane

Rule 8: Angles of departure / arrival. complex poles or complex zeros respectively.

$\phi_d = 180° - \phi$ = $\angle$ of departure $\phi_a = 180° + \phi$ = $\angle$ of arrival.

where ϕ (for both cases) $= \Sigma\phi_p - \Sigma\phi_z$.

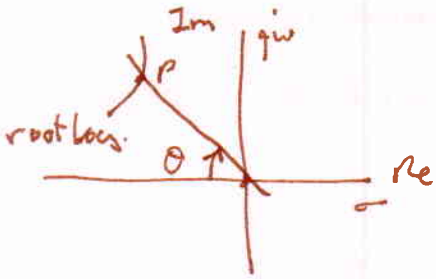
$\Sigma\phi_p$ = contributions by angles made by rem. open loop poles @ the pole/zero of interest

$\Sigma\phi_z$ = contribution by angles made by remaining OL zeros @ the pole/zero of interest.

Draw the root locus: Remember to mark clearly the axes, all poles and zeros, centroid & asymptote lines / angles, the breakaway / breakin point (with corresponding K values), intersection points & angles of departure / arrival (if any).

⇒ predict stability performance depending on value of Kmer / K.

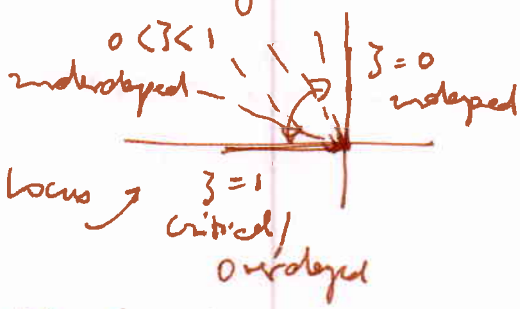
<u>Graphical determination of K for a specified damping ratio ζ</u> (given.)



$\theta = \cos^{-1}\zeta$.

- Draw the root locus on a graph sheet using <u>same</u> scale on X & Y axis.
- Draw a line passing through the origin with a slope of ($\theta = \cos^{-1}\zeta$) <u>measured from −ve real axis</u> as shown in fig.
- Determine point (P) of intersection with root locus & the line from graph = $(a+jb)$.
- Apply magnitude condition at this point to $G(s)H(s)$ to determine the value of K for which the system will have the given damping ratio (ζ).

⇒ lines with constant slope correspond to lines having constant ζ on the root locus



<u>Drawing root locus if characteristic equation is given.</u>

Remember root locus is for OLTF $G(s)H(s)$

if eg: $s^3+7s^2+12s+Ks+10K=0$ given as charc eqn → get $G(s)H(s)$ as follows:

$$\underbrace{(s^3+7s^2+12s)}_{\text{only } s}+\underbrace{K(s+10)}_{\text{terms having } K}=0$$

÷ by polynomial in s without K

i.e $$\frac{(s^3+7s^2+12s)}{(s^3+7s^2+12s)}+\frac{K(s+10)}{(s^3+7s^2+12s)}$$

$$1+\left\{\frac{K(s+10)}{s^3+7s^2+12s}\right\} = G(s)H(s)$$

FREQUENCY DOMAIN ANALYSIS:

for a standard second order system given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

* resonant frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ rad/s

* resonant peak $M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ dB

* Bandwidth $(BW) = \omega_n\sqrt{1-2\zeta^2+\sqrt{2-4\zeta^2+4\zeta^4}}$ rad/s

Note: if $\zeta > 1$ in time domain peak overshoot M_p vanishes.
Similarly if $\zeta \geq 0.707$, resonant peak M_r vanishes.

Other results for standard second order system:

* Gain crossover frequency $(\omega_{gc}) = \omega_n\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}$ rad/s.

* Phase margin $PM = +\tan^{-1}\left\{\dfrac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{4\zeta^4+1}}}\right\}$

where gain crossover frequency = frequency at which
$|G(j\omega)H(j\omega)| = 1$.

and phase margin $= 180° + \angle G(j\omega)H(j\omega)\big|_{@\,\omega=\omega_{gc}}$.

POLAR AND NYQUIST PLOTS

Complex systems – Nyquist plot can be used effectively to determine the stability of the closed loop system.

To note: poles of $1+G(s)H(s)$ = open loop poles of a system
zeros of $1+G(s)H(s)$ = closed loop poles of a system.

Specification of frequency domain plots:

– Gain cross over frequency (ω_{gc}) is the ω at which $|G(j\omega)H(j\omega)| = 0\,dB$ or
(substitute s by $j\omega$ in O.L.T.F) ∴ $|G(j\omega)H(j\omega)| = 1$

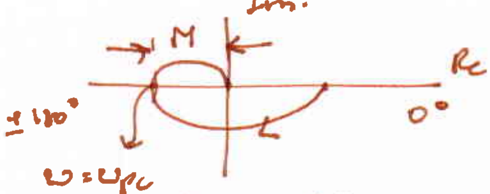
– phase cross over frequency (ω_{pc}) ω at which $\angle G(j\omega)H(j\omega) = -180^\circ$

→ Gain margin (GM) $= \dfrac{1}{|G(j\omega)H(j\omega)|}\Big|_{@\,\omega=\omega_{pc}} = -20\log_{10}|G(j\omega)H(j\omega)|\,dB\Big|_{@\,\omega=\omega_{pc}}$

$= K_{marginal} / K$ of system

→ Phase margin (PM) $= 180^\circ + \angle G(j\omega)H(j\omega)\Big|_{@\,\omega=\omega_{gc}}$

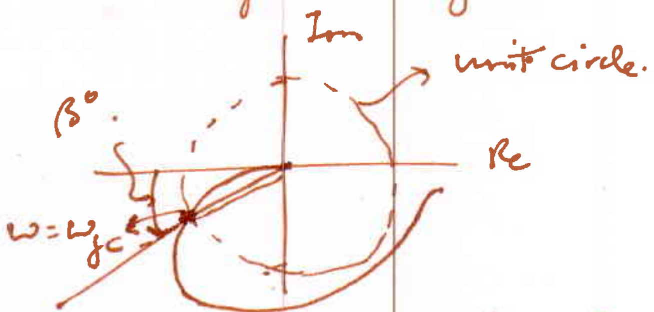
↳ In polar plots



$GM = 1/M$
or $-20\log_{10}|M|\,dB$.

$\langle GM = 1/|G(j\omega)H(j\omega)|_{@\,\omega=\omega_{pc}} \rangle$

graphically drawn & superimposed a circle of unit ~~ra~~ radius.



$\langle PM = 180^\circ + \angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}} \rangle$
$= \underline{\beta^\circ}$.

Stability criteria:

$\omega_{gc} < \omega_{pc}$ & $GM > 0$ & $PM > 0$ ⇒ STABLE

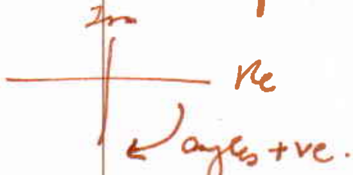
$\omega_{gc} = \omega_{pc}$ & $GM = PM = 0$ ⇒ CRITICALLY STABLE

$\omega_{gc} < \omega_{pc}$ & $GM < 0$ & $PM < 0$ ⇒ UNSTABLE

↳ if $GM = \infty$ means gain K needs to become ∞ for instability to set in ⇒ always stable system.

Polar plots : every pole at origin shifts the starting point by $+90°$.

every $(1+Ts)$ factor in the denominator – make the polar plot end by $+90°$



**(Use sign convention thought in class)

Steps involved in Nyquist plot problems:

1) get pole - zero plot

2) Consider appropriate Nyquist contour on entire Right half of s plane ⟨ singularities = poles on imaginary axis ⟩

3) Map corresponding polar plot of the Nyquist contour

4) Check Nyquist stability criterion and judge the stability.

5) Determine GM / PM or value of K for stability depending on the question.

Nyquist stability criterion : $\boxed{Z = P + N}$

Z = no. of roots of the characteristic equation on the Right hand side of the s plane

P = no: of ~~roots~~ poles of $G(s)H(s)$ in the right hand side of the s plane.

N = no. of encirclements of the polar plot of the point $(-1+j0)$.

⟨ Use sign convention as explained in the class ⟩

Clockwise encirclements →

Counter clockwise encirclements ⟶

If the equation is satisfied then the $\boxed{\text{closed loop}}$ system is stable.

Special points on gain & phase margin.

→ The cross over (gain & phase) frequencies can be obtained mathematically and can be plugged in the formulae to calculate the gain and phase margins.

→ The gain of the system generally needs to be kept as high as possible to reduce the steady state error and obtain accurate & fast system response. and yet maintain adequate GM & PM. Typically as a rule of thumb a gain margin of 2 (6 dB) and phase margin of +30° is generally considered good.

→ To obtain ω_{pc} mathematically, separate real and imaginary parts of $G(j\omega)H(j\omega)$ and equate the imaginary part to zero → solve for ω → result is ω_{pc}.

$\boxed{\text{Bode plot}}$

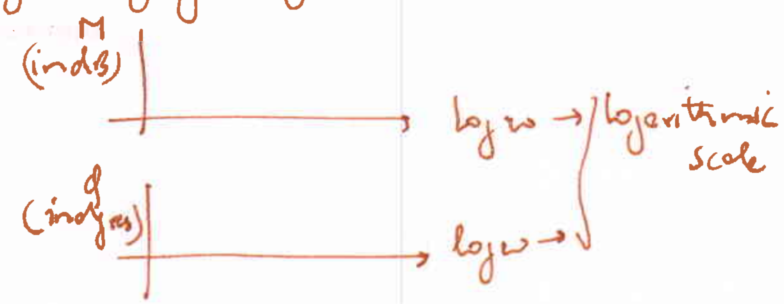
Steps: get O.L.T.F. into time constant form

$$G(s)H(s) = \frac{K(1+T_a s)(1+T_b s)\cdots}{s^l(1+T_1 s)(1+T_2 s)\cdots}$$

- replace s by $j\omega$ to convert into frequency domain.
- find magnitude as a function of ω & express in dB $20\log_{10}|G(j\omega)H(j\omega)|$

↳ find phase angle using $\tan^{-1}\left(\frac{\text{imaginary part}}{\text{real part}}\right)$ in degrees.

↳ with required approximations plot magnitude (in dB) & phase ∠ in degrees v.s. ($\log\omega$) in x axis. on the same semi-log sheet by varying ω from 0 to ∞.



points

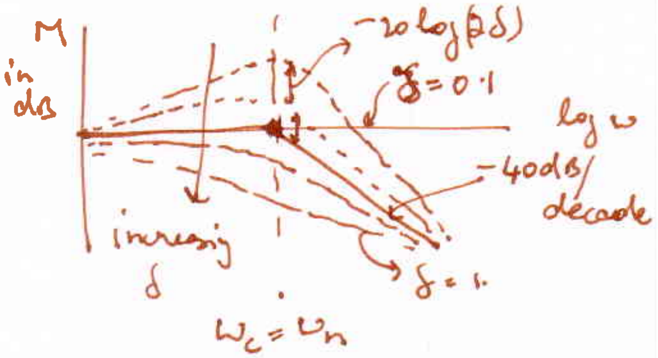
a) 6dB/octave = 20dB/decade
12dB/octave = 40dB/decade.

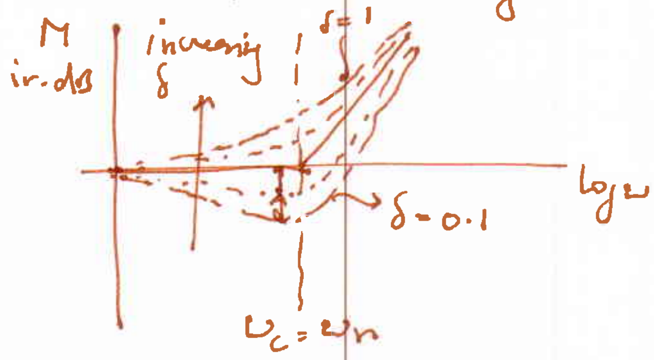
b) corner frequencies $\omega_{ci} = 1/\tau_i$ (from time constant form) of $G(j\omega)H(j\omega)$.

c) if quadratic poles exist (complex conjugate pairs) or zeros
then compare to $s^2 + 2\delta\omega_n s + \omega_n^2 \longrightarrow$ corner frequency $= \omega_n$.

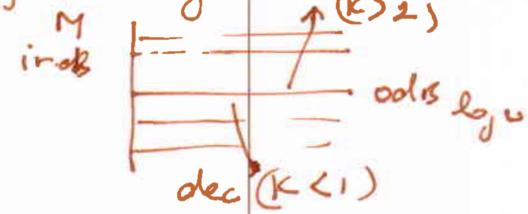
d) We confine ourselves to asymptotic plots. (even for quadratic factors). In reality a correction of ~~$-20\log_{10}(2\delta)$~~ $-20\log_{10}(2\delta)$ need to be applied at the corner frequency as shown:

quadratic factor at denominator (poles) — quadratic factor at numerator (zeros)





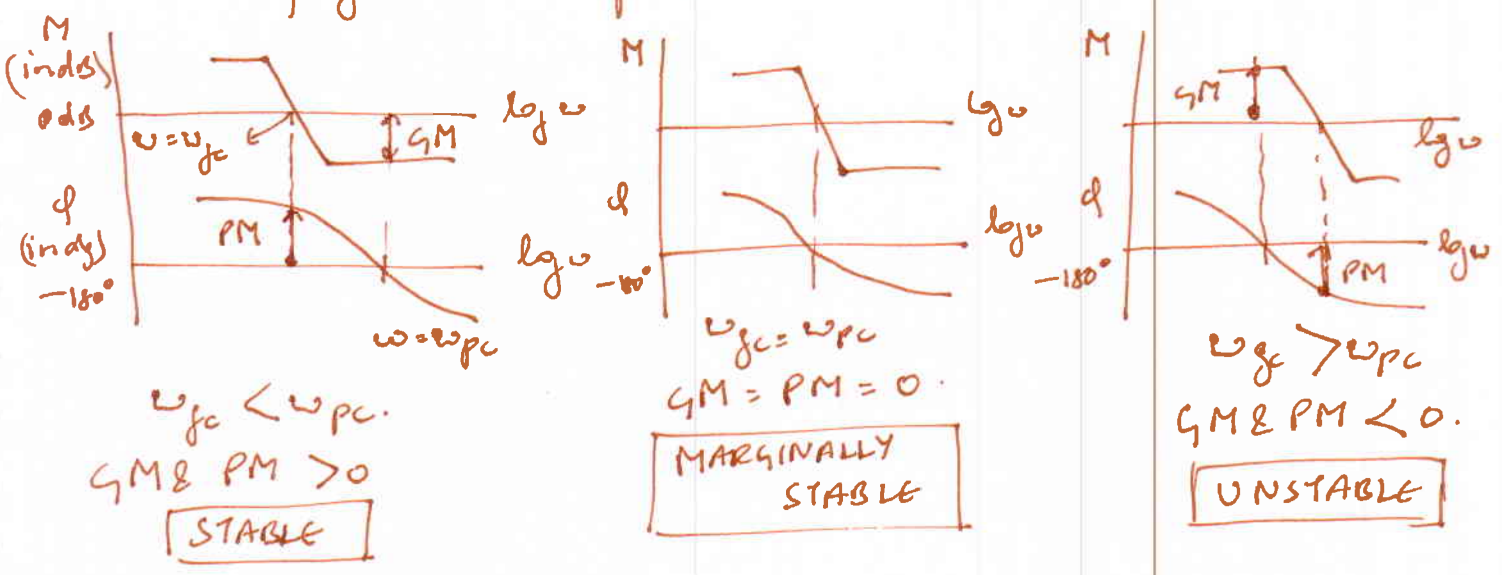
e) gain 'K' shifts the magnitude plot of $|G(j\omega)H(j\omega)|$ by a distance of $20\log K$ dB upwards if $K > 1$ and downwards if $K < 1$. $\longrightarrow$ phase plots are unaffected by K.



f) Each zero @ the origin increases the magnitude at a rate of +20dB/decade and each pole at the origin decreases the magnitude at a rate of −20dB/decade

g) Each zero @ the origin contributes +90° angle in the overall phase angle plot and each pole @ the origin contributes −90° angle to the ~~polar~~ phase plot irrespective of the value of ω.

h) Stability from Bode plot.



Sketching the Bode plot

- Draw a line of $20 \log_{10} K$ dB.
- Draw a line of appropriate slope representing poles or zeros at the origin passing through the intersection point of $\omega = 1$ and 0dB.
- Shift this intersection point on the $20 \log K$ line and draw a parallel line to the line drawn above. This is addition of constant K and no: of poles/zeros at the origin.
- Change the slope of this line at various corner frequencies by the appropriate value depending on whether it is a pole (−20dB/dec) or zero (+20dB/dec) ⟨for simple poles & zeros⟩ & ±40dB/dec for quadratic factors. Keep changing the slope at each increasing corner frequency. ⇒ MAGNITUDE PLOT IS ASYMPTOTIC (~~dis~~continuous straight lines)
- Tabulate phase angle vs. ω & plot this below the magnitude plot and draw a smooth curve obtaining the necessary phase angle plot.
- Obtain the necessary specifications (GM/PM/ω_{gc}/ω_{pc}) & comment on the stability.

Types of problems (BODE plot)

Draw bode plot & judge stability & get specifications GM/PM/ω_{gc}/ω_{pc}

Design problem.
- Compensators
- Determine K for a given GM or PM

given the Bode plot (obtain the transfer function.

Note.

$(s-1)$ factor $\rightarrow = j\omega - 1 = \sqrt{\omega^2+1}\ \angle \left(180^\circ - \tan^{-1}(\omega)\right)^\circ$

$(1-s)$ factor $\rightarrow = 1 - j\omega = \sqrt{\omega^2+1}\ \angle -\tan^{-1}(\omega)^\circ$

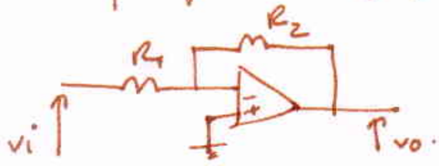
$e^{-Ts} = e^{-Tj\omega} = 1\angle T\omega$ rad or $= 1\angle -\frac{T\omega \times 180^\circ}{\pi}{}^\circ$.

↳ non linear (but approximated by Taylor series).

↳ Typically the transfer function of a delay block (of T seconds) in control systems is represented by $\boxed{e^{-Ts}}$

Controllers

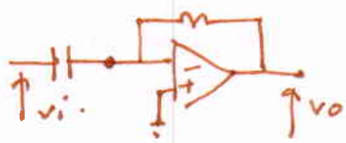
proportional (k)



$v_o/v_i = K = R_2/R_1$

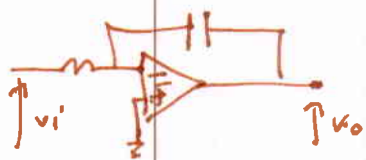
- possibility of oscillation if $K > K_{marginal}$

proportional derivative $(K_d s)$



- high pass filter action
- improves transient performance
- BW ↑ t_s ↓

proportional integral (K_i/s)



- low pass filter action
- δ ↑
- improves steady state performance (reduces e_{ss})

$\boxed{\text{Compensators}}$

- indirect correlation between the transient response & frequency response ⇒ design specifications can be met using the BODE plot

→ PM/GM/M_r → rough estimate of δ (damping coefficient)

→ ω_{gc}/ω_r → estimate of speed of response

→ static error coefficients → estimate steady state accuracy.

(can be obtained from Bode plot)

↳ GAIN INCREMENT alone cannot be used to satisfy both steady state & transient behaviour specifications → or even if satisfied – done at the expense of stability.

⇓

Need for compensators $\xrightarrow{\text{types.}}$ a) Cascade compensation (Series with plant)

b) feed back compensation (series with the feed bak path)

c) Output compensation d) input compensation.

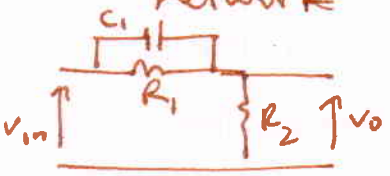
Compensators

Lead compensator

used for better/fast transient response & BW increment

⇓

high pass filter network



$$T(s) = \frac{V_0(s)}{V_{in}(s)} = \frac{s + 1/\tau}{s + 1/\alpha\tau}$$

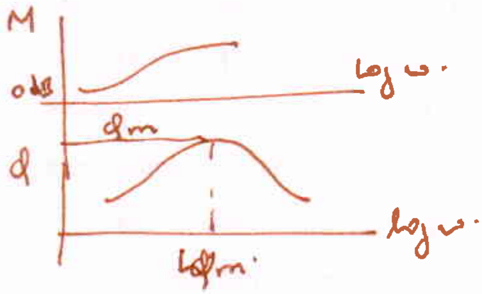
$$\alpha = \frac{R_2}{R_1+R_2} \ (<1) \quad \tau = R_1 C$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

also $T(s) = s + \omega_1 / s + \omega_2$

$\omega_1 = 1/\tau$ & $\omega_2 \ 1/\alpha\tau$

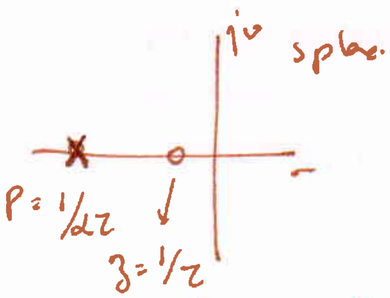
where $\alpha < 1$.



$\omega_m = \sqrt{\omega_1\omega_2}$

‖

differentiator



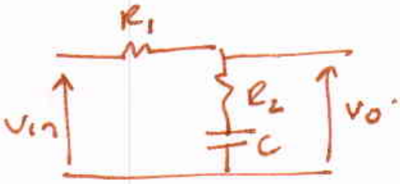
($\alpha > 0.07$) for good signal to noise ratio

Lag compensator

used for reducing the steady state error & decrementing the BW.

⇓

low pass filter network.



$$T(s) = \frac{V_0(s)}{V_{in}(s)} = \frac{1+s\tau}{1+s\beta\tau}$$

$\beta = 1/\alpha \ (>1) \quad \tau = R_2 C$

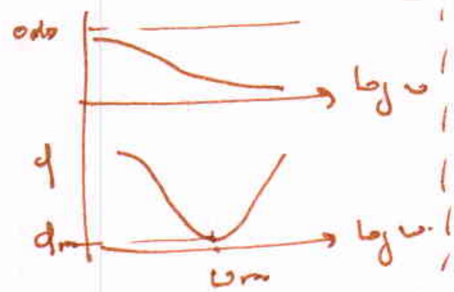
also $T(s) = \dfrac{1+s/\omega_2}{1+s/\omega_1}$

$\omega_1 = 1/\beta\tau$ & $\omega_2 = 1/\tau$

$(\beta > 1)$

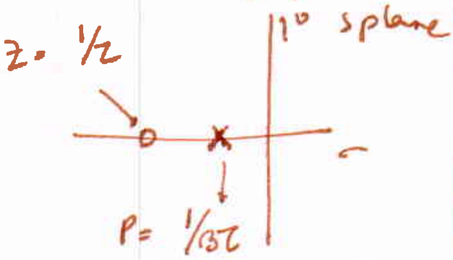
$$\beta = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

$$= R_1 + R_2 / R_2$$



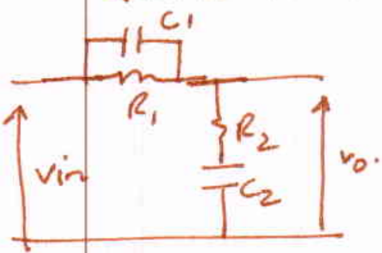
‖

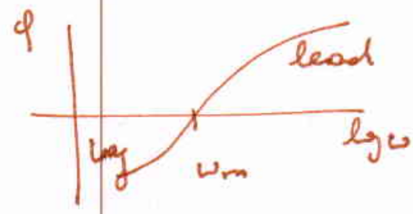
integrator



Lead-Lag compensator

<COMBINATION>





$$T(s) = \frac{V_0(s)}{V_{in}(s)}$$

$$\underbrace{\left(\frac{s + 1/\tau_1}{s + \beta/\tau_1}\right)}_{\text{lead}} \times \underbrace{\left(\frac{s + 1/\tau_2}{s + 1/\beta\tau_2}\right)}_{\text{lag}}$$

$\tau_1 = R_1 C_1$ & $\tau_2 = R_2 C_2$

$\omega_m = 1/\sqrt{\tau_1 \tau_2}$