5 Beta gama function

11. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx \, dx$, prove that

$$I_{m,n} = \left[\frac{m(m-1)}{m^2 - n^2}\right] I_{m-2,n}.$$

3.6 Beta and Gamma Functions.

In the application of Integral Calculus to various problems, we often use Beta and Gamma functions. We give here their properties and uses.

Definitions :

1. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ (m > 0, n > 0)$ is known as the First Eulerian Integral or Beta function and is written as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,\ (m > 0, n > 0).$$

2. The integral $\int_0^\infty e^{-x} x^{n-1}\,dx,\ (n > 0)$ is known as the Second Eulerian Integral or Gamma function and is written as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx,\ (n > 0).$$

3.7 Some Important Results.

1. $\beta(m, n) = \beta(n, m).$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^1 (1-x)^{m-1}\{1-(1-x)\}^{n-1}dx$$

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\,dx = \beta(n, m).$$

2. $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$

Putting $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$.

Also when $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \frac{\pi}{2}$.

$$\therefore \beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta.$$

3. $\Gamma(1) = 1$

We have $\Gamma(1) = \int_0^\infty e^{-x}\, dx = \underset{b\to\infty}{Lt} \int_0^b e^{-x}\, dx$

$$= \underset{b\to\infty}{Lt} [-e^{-x}]_0^b = \underset{b\to\infty}{Lt} (-e^{-b} + e^0) = 0 + 1 = 1.$$

4. $\Gamma(n+1) = n\,\Gamma(n).$

We have $\Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx.$

Integrating by parts,

$$\Gamma(n+1) = [x^n(-e^{-x})]_0^\infty + \int_0^\infty n\, x^{n-1} e^{-x}\, dx.$$

But $\underset{x\to\infty}{Lt} \frac{x^n}{e^x} = 0.$ $= \frac{x^n}{1+x+\frac{x^2}{2!}+ \dots + \frac{x^n}{n!} + \dots}$

$= \frac{x^n}{x^n\left(\frac{1}{x^n} + \dots + \frac{1}{n!} + \frac{x}{(n+1)!}\right)}$

$$\therefore \Gamma(n+1) = n\int_0^\infty x^{n-1} e^{-x}\, dx = n\,\Gamma(n).$$

$= \frac{1}{0+\infty} = 0$

Cor. When n is a positive integer, $\Gamma(n+1) = n!$

$\Gamma(n) = (n-1)\,\Gamma(n-1),\ \Gamma(n-1) = (n-2)\,\Gamma(n-2)$ and so on.

Proceeding in this way,

$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1) = n(n-1)(n-2)\dots 1\,\Gamma(1)$

$= n!$ $\qquad (\because \Gamma(1) = 1).$

5. We give below three important results without proof. The ... should remember them.

(i) $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ show it

(ii) $\beta(m, n) = \displaystyle\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}}$

(iii) $\Gamma(m)\,\Gamma(1-m) = \dfrac{\pi}{\sin m\pi} \quad (0 < m < 1)$

6. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. $\therefore \beta(m,n) = \dfrac{\Gamma m\,\Gamma n}{\Gamma m+n}$

If we put $m = n = \dfrac{1}{2}$ in (i) above,

$$\frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1/2 + 1/2)} = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

or $\dfrac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(1)} = \displaystyle\int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx.$

Put $x = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\,d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \dfrac{\pi}{2}$.

$$\therefore \Gamma(1/2)\,\Gamma(1/2) = \int_0^{\pi/2} \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\,2\sin\theta\cos\theta\,d\theta$$

$$= [2\theta]_0^{\pi/2} = \pi.$$

$\therefore \Gamma(1/2) = \sqrt{\pi}$

The value of $\Gamma(1/2)$ can also be deduced by putting

$m = \dfrac{1}{2}$ in $\Gamma(m)\,\Gamma(1-m) = \dfrac{\pi}{\sin m\pi}$ so that

$$\Gamma(1/2)\,\Gamma(1/2) = \frac{\pi}{\sin \pi/2} = \pi$$

$\therefore \Gamma(1/2) = \sqrt{\pi}$.

3.8 The Integrals $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$, $\int_0^{\pi/2} \sin^n x \, dx$ and $\int_0^{\pi/2} \cos^n x \, dx$.

(1) $\int_0^{\pi/2} \sin^m x \cos^n x \, dx, \ (m > -1, n > -1)$

$$= \int_0^{\pi/2} (\sin^2 x)^{m/2} (1 - \sin^2 x)^{n/2} \, dx$$

Put $\sin^2 x = t$. Then $2 \sin x \cos x \, dx = dt$

or $dx = \dfrac{dt}{2\sqrt{\sin^2 x}\sqrt{1 - \sin^2 x}} = \dfrac{dt}{2\sqrt{t}\sqrt{1-t}}$

$$\therefore I = \int_0^1 \frac{t^{m/2}(1-t)^{n/2} \, dt}{2\sqrt{t}\sqrt{1-t}}$$

$$= \frac{1}{2}\int_0^1 t^{(m+1)/2 \, - 1} (1-t)^{(n+1)/2 \, -1} \, dt$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}.$$

(2) If we put $n = 0$ in the above result,

$$\int_0^{\pi/2} \sin^m x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{m+2}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{m+1}{2}\right)}{2\Gamma\left(\frac{m+2}{2}\right)}$$

Replacing m by n, $\int_0^{\pi/2} \sin^n x \, dx = \dfrac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}$

Similarly by putting $m = 0$, we get

$$\int_0^{\pi/2} \cos^n x \, dx = \frac{\sqrt{\pi}\, \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}$$

3.9 The Integral $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$.

Put $x^2 = z$. Then $2x\,dx = dz$ or $dx = \frac{dz}{2\sqrt{z}}$ ·

$$\therefore I = \int_0^\infty \frac{1}{2} e^{-z} z^{-1/2} dz = \frac{1}{2}\int_0^\infty e^{-z} z^{1/2-1} dz = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}$$

Examples worked out

1. Find the value of (i) $\Gamma(8)$, (ii) $\Gamma\left(\frac{7}{2}\right)$ and (iii) $\frac{\Gamma\left(\frac{3}{2}\right) . \Gamma\left(\frac{5}{2}\right)}{\Gamma(5)}$

(i) We have $\Gamma(n+1) = n!$, when n is a positive integer

$\therefore \Gamma(8) = \Gamma(7+1) = 7!$.

(ii) We have $\Gamma(n+1) = n\,\Gamma(n)$

$$\therefore \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}$$

(iii) $$\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma(5)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{4\ .\ 3\ .\ 2\ .\ 1} = \frac{\sqrt{\pi}\ .\ \sqrt{\pi}}{64} = \frac{\pi}{64}$$

2. Evaluate (i) $\int_0^a x^3 (a^2 - x^2)^{5/2} dx$ (ii) $\int_0^{\pi/4} (\cos 2\theta)^{3/2} \cos\theta\, d\theta$

(i) Put $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2} a^3 \sin^3\theta\, (a^2\cos^2\theta)^{5/2}\, a\cos\theta\, d\theta$$

$$= a^9 \int_0^{\pi/2} \sin^3\theta \cos^6\theta\, d\theta$$

$$= a^9 \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\,\Gamma\left(\frac{3+6+2}{2}\right)} = a^9 \frac{\Gamma(2)\,\Gamma\left(\frac{7}{2}\right)}{2\,\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{a^9}{2} \cdot \frac{1 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{2a^9}{63}.$$

(ii) $I = \int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta$

Put $\sqrt{2}\sin\theta = \sin x$. Then $\sqrt{2}\cos\theta\, d\theta = \cos x\, dx$. Also when $\theta = 0, x = 0$ and when $\theta = \frac{\pi}{4}, x = \frac{\pi}{2}$.

$$\therefore I = \int_0^{\pi/2} (1 - \sin^2 x)^{3/2} \frac{\cos x\, dx}{\sqrt{2}} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \cos^4 x\, dx$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{6}{2}\right)} = \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\,.\,1}$$

$$= \frac{3}{16\sqrt{2}} \cdot \sqrt{\pi} \cdot \sqrt{\pi} = \frac{3\pi}{16\sqrt{2}}.$$

3. Show that $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$.

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1 - \frac{1}{4}\right)$$

$$= \frac{\pi}{\sin\frac{1}{4}\pi} \quad \left[\because \Gamma(m)\,\Gamma(1-m)\right] = \frac{\pi}{\sin m\pi} = \sqrt{2}\,\pi.$$

4. Show that $\frac{\beta(m, n+1)}{n} = \frac{\beta(m+1)}{m} = \frac{\beta(m, n)}{m+n}$

$$\frac{\beta(m, n+1)}{n} = \frac{\Gamma(m)\,\Gamma(n+1)}{n\,\Gamma(m+n+1)} = \frac{\Gamma(m)\, n\,\Gamma(n)}{n\,(m+n)\,\Gamma(m+n)}$$

$$= \frac{\beta(m, n)}{m+n}$$

$$\frac{\beta(m+1,n)}{m} = \frac{\Gamma(m+1)\,\Gamma(n)}{m\,\Gamma(m+n+1)} = \frac{m\,\Gamma(m)\,\Gamma(n)}{m\,(m+n)\,\Gamma(m+n)}$$

$$= \frac{\beta(m,n)}{m+n}$$

5. Show that $\int_{-1}^{1} (1+x)^p (1-x)^q \, dx = 2^{p+q+1} \frac{\Gamma(p+1)\,\Gamma(q}{\Gamma(p+q+}$

$(p > -1, q >$

Put $1 + x = 2t$. Then $dx = 2dt$. Also when $x = -1, t = 0$ and $x = 1, t = 1$.

$$\therefore I = \int_0^1 (2t)^p (2-2t)^q \, 2dt = \int_0^1 2^{p+q+1} t^p (1-t)^q \, dt$$

$$= 2^{p+q+1} \int_0^1 t^{(p+1)-1} (1-t)^{(q+1)-1} \, dt$$

$$= 2^{p+q+1} \beta(p+1, q+1) \quad (\because p+1 > 0, q+1 > 0)$$

$$= 2^{p+q+1} \frac{1(p+1)\,\Gamma(q+1)}{\Gamma(p+q+2)}$$

6. Show that $\int_0^\infty e^{-x^4} x^2 \, dx \times \int_0^\infty e^{-x^4} \, dx = \frac{\pi}{8\sqrt{2}}$.

Put $x^4 = t$. Then $4x^3 \, dx = dt$ or $dx = \frac{dt}{4x^3} = \frac{dt}{4t^{3/4}}$

$$\therefore I = \int_0^\infty e^{-t} \frac{t^{1/2}\,dt}{4\,t^{3/4}} \times \int_0^\infty e^{-t} \frac{dt}{4\,t^{3/4}}$$

$$= \frac{1}{4} \int_0^\infty e^{-t} t^{-1/4} \, dt \times \frac{1}{4} \int_0^\infty e^{-t} t^{-3/4} \, dt$$

$$= \frac{1}{16} \int_0^\infty t^{3/4-1} e^{-t} \, dt \times \int_0^\infty t^{1/4-1} e^{-t} \, dt$$

$$= \frac{1}{16} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) = \frac{1}{16} \sqrt{2}\,\pi$$ (from Q.3. abo

$$= \frac{\pi}{8\sqrt{2}}.$$

7. Prove that $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx = \pi.$

$$\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\int_0^{\pi/2} \sqrt{\sin x}\, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+2}{2}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{3}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}$$

$$\therefore \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$$

$$= \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \times \frac{\sqrt{\pi}}{2} \frac{4\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} = \pi.$$

8. Prove that $\sqrt{\pi}\, \Gamma(2n) = 2^{2n-1}\, \Gamma(n)\, \Gamma\left(n + \frac{1}{2}\right).$

We have $\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)}$ ...... (1)

If we put $m = n$,

$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= 2\,\frac{1}{2^{2n-1}} \int_0^{\pi/2} (\sin 2\theta)^{2n-1}\, d\theta.$$

Put $2\theta = t$. Then $2\, d\theta = dt$.

When $\theta = 0$, $t = 0$ and when $\theta = \frac{\pi}{2}$, $t = \pi$.

$$\therefore \frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \frac{1}{2^{2n-1}} \int_0^{\pi} (\sin t)^{2n-1}\, t\, dt$$

$\Rightarrow \frac{(\Gamma n)^2}{\Gamma 2n}$

$$= \frac{1}{2^{2n-1}} \cdot 2\int_0^{\pi/2} \sin^{2n-1} t\, dt \;\ldots\ldots (2)$$

Again putting $n = \frac{1}{2}$ in (1), we have

$$2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta = \frac{\Gamma(m)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}.$$

Replacing m by n and θ by t

$$\int_0^{\pi/2} \sin t^{2n-1}\, t\, dt = \frac{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(n+\frac{1}{2}\right)} \;\ldots\ldots\ldots\ldots (3)$$

From (2) and (3)

$$\frac{\Gamma(n)\,\Gamma(n)}{\Gamma(2n)} = \frac{2}{2^{2n-1}} \cdot \frac{\Gamma(n)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(n+\frac{1}{2}\right)} = \frac{1}{2^{2n-1}} \frac{\Gamma(n)\sqrt{\pi}}{\Gamma\left(n+\frac{1}{2}\right)}$$

by cross multiplication

$$\therefore \sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right). \quad \text{— ①}$$

This important result is known as 'Duplication Formula.'

Q. $B(n,n) = \frac{(\Gamma n)^2}{\Gamma 2n}$ Formula $\Gamma 2n = \frac{2^{2n-1}\,\Gamma n\,\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}$

$\therefore B(n,n) = (\Gamma n)^2 \cdot \frac{\sqrt{\pi}}{2^{2n-1}\,\Gamma n\,\Gamma(n+\frac{1}{2})}$

Examples 7

1. Show that

(i) $\Gamma\left(\frac{9}{2}\right) = \frac{105}{16}\sqrt{\pi}$ (ii) $\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(6)} = \frac{\pi}{128}$

2. Show that

(i) $\int_0^{\pi/2} \sin^3 x \cos^5 x\, dx = \frac{1}{24}$ (ii) $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx =$

3. Use Gamma function to evaluate

(i) $\int_0^1 x^6 \sqrt{1-x^2}\, dx$ (ii) $\int_0^a x^2 (a^2-x^2)^{3/2}\, dx$

(iii) $\int_0^a x^4 \sqrt{a^2-x^2}\, dx$ (iv) $\int_0^1 x^{3/2} (1-x)^{3/2}\, dx$

4. Evaluate by using Gamma function :

(i) $\int_0^a \frac{x^4\, dx}{\sqrt{a^2-x^2}}$ (ii) $\int_0^1 \frac{x^6\, dx}{\sqrt{1-x^2}}$

(iii) $\int_0^{2a} x^5 \sqrt{2ax-x^2}\, dx$ $x = 2a \sin^2\theta$

(iv) $\int_0^{2a} x^{9/2} (2a-x)^{-1/2}\, dx$ (Put $x = 2a \sin^2 \theta$

5. Use Gamma function to prove

$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$

(i) $\int_0^{\pi/8} \cos^3 4x\, dx = \frac{1}{6}$

(ii) $\int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2}\, dx = \frac{5\pi}{2^{11}}$

(iii) $\int_0^{\pi/4} \sin^4 x \cos^2 x\, dx = \frac{3\pi-4}{192}$ $\left(\frac{1-\cos 2x}{2}\right)^2 \left(\frac{1+\cos 2x}{2}\right)$

Take $2x = \theta$

(iv) $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta\, d\theta = \frac{5\pi}{192}$

$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$

(v) $\int_0^{\pi/6} \cos^2 6\theta \sin^4 3\theta\, d\theta = \frac{7\pi}{192}$

6. Prove that $\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$.

7. Show that $\beta(m, n)\, \beta(m+n, l) = \beta(n, l)\, \beta(n+l, m)$.

8. Prove that $\frac{2^n}{\sqrt{\pi}} \Gamma\left(\frac{2n+1}{2}\right) = 1.3.5. \ldots\ldots\ldots\, (2n-3)(2n-1)$.

Sol $\Gamma\left(\frac{2n+1}{2}\right) = \Gamma\left(\frac{2n-1}{2}+1\right) = \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\cdots \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}$

59 If $\phi(n) = \int_0^\infty e^{-x} x^{n-1} \log x \, dx$ $n>0$, show that $\phi(n+2) - (2n+1)\phi(n+1) + n^2\phi(n) = 0$

9. Show that $\int_0^\infty e^{-x^2} x^\alpha \, dx = \frac{1}{2}\Gamma\left(\frac{\alpha+1}{2}\right)$. (Put $x^2 = t$)

10. Prove that $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$. $\left[\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$

11. Show that $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right) = \frac{16}{3}\pi^4$.

[Find $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)$, $\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{7}{9}\right)$ and so on.]

12. Use Gamma function to prove that $\int_0^1 \frac{dx}{(1-x^6)^{1/6}} = \frac{\pi}{3}$.

$x^6 = t$

$\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 60^\circ \cdot \sin 80^\circ = \frac{16}{3}$

$2 \sin A \sin B = \cos(B-A) - \cos(A+B)$
$2 \sin A \cdot \cos B = \sin(A+B) + \sin(A-B)$

58) $\int_a^b \frac{(x-a)^m (b-x)^n}{m+n+1} dx = \frac{m!\, n!\, (b-a)^{m+n}}{(m+n)!}$

Let $x = a + (b-a)y \Rightarrow dx = (b-a)\,dy$
when $x = a \Rightarrow y = 0$ & $x = b \Rightarrow y = 1$

$$I = \int_0^1 \frac{[(b-a)y]^m \, [(b-a)-(b-a)y]^n \, (b-a)\,dy}{m+n+1}$$

$$= \frac{(b-a)^{m+n+1}}{m+n+1} \int_0^1 y^m (1-y)^n \, dy$$

put $m = p-1$
$n = q-1$

$$= \frac{(b-a)^{p+q-1}}{p+q-1} \int y^{p-1} (1-y)^{q-1} \, dy$$