General properties of Definite Integral

Prop. 3.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,\ a < c < b.$$

L.H.S. $= F(b) - F(a)$

R.H.S. $= F(c) - F(a) + F(b) - F(c) = F(b) - F(a).$

$\therefore$ L.H.S. = R.H.S.

We can generalise the property as follows :

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)\,dx,$$

where $a < c_1 < c_2 < \ldots < c_n < b.$

Prop. 4.

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Put $a - x = t$. Then $-dx = dt$.

Also when $x = 0, t = a$, and when $x = a, t = 0$.

$$\therefore \int_0^a f(a-x)\,dx = -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt = \int_0^a f(x)\,dx$$

Illustration :

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos^n x\,dx$$

Definition of Odd and Even Functions :

1. $f(x)$ is an odd function if $f(-x) = -f(x)$.
2. $f(x)$ is an even function if $f(-x) = f(x)$.

Examples: x^3 is odd function $\because (-x)^3 = -x^3$

$\sin x$ is an odd function $\because \sin(-x) = -\sin x$

x^2 is an even function $\because (-x)^2 = x^2$.

$\cos x$ is even function $\because \cos(-x) = \cos x$.

Prop. 5.

$$\int_{-a}^{a} f(x)\,dx = 0, \text{ when } f(x) \text{ is an odd function}$$

$$= 2\int_{0}^{a} f(x)\,dx\,, \text{ when } f(x) \text{ is an even function.}$$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx\,.$$

Put $x = -t$. Then $dx = -dt$.
Also when $x = -a$, $t = a$ and when $x = 0$, $t = 0$.

$$\therefore \int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-x)\,dx\,.$$

$$\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = 0, \text{ if } f(-x) = -f(x)$$

$$\text{and } = 2\int_{0}^{a} f(x)\,dx\,, \text{ if } f(-x) = f(x).$$

Illustration :

$$\int_{-\pi/2}^{\pi/2} \sin x\,dx = 0$$

$$\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\int_{0}^{\pi/2} \cos x\,dx\,.$$

Prop. 6.

$$\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx\,, \text{ when } f(2a-x) = f(x)$$

$$= 0, \text{ when } f(2a-x) = -f(x).$$

$$\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx.$$

Put $x = 2a - t$. Then $dx = -dt$.

Also when $x = a$, $t = a$ and when $x = 2a$, $t = 0$.

$$\therefore \int_a^{2a} f(x)\,dx = -\int_a^{0} f(2a-t)\,dt = \int_0^{a} f(2a-x)\,dx.$$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx$$

$$= 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

and $= 0$, if $f(2a - x) = -f(x)$.

Illustration :

$$\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx\,, \quad \because \sin^n(\pi - x) = \sin^n x.$$

$$\int_0^{\pi} \cos^3 x\,dx = 0, \quad \because \cos^3(\pi - x) = -\cos^3 x.$$

$$\int_0^{\pi} \cos^4 x\,dx = 2\int_0^{\pi/2} \cos^4 x\,dx\,, \quad \because \cos^4(\pi - x) = \cos^4 x$$

$$\int_0^{\pi} \cos^m x \sin^n x\,dx = 0 \text{ if } m \text{ is odd}$$

$$\text{and } = 2\int_0^{\pi/2} \cos^m x \sin^n x\,dx \text{ if } m \text{ is even.}$$

Prop. 7.

$$\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx, \text{ when } f(x) = f(a+x).$$

$$\int_0^{na} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx + \ldots + \int_{(n-1)a}^{na} f(x)\,dx$$

Put $x = a + t$. Then $dx = dt$.

Also when $x = a$, $t = 0$ and when $x = 2a$, $t = a$.

$$\therefore \int_a^{2a} f(x)\,dx = \int_0^a f(a+t)\,dt = \int_0^a f(a+x)\,dx = \int_0^a f(x)\,dx.$$

Similarly, we can prove that each of the integrals of the right side is equal to $\int_0^a f(x)\,dx$.

$$\therefore \int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx.$$

Examples Worked Out

1. Show that $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{4}$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}\,dx}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}$$ By - pr-4

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}} + \int_0^{\pi/2} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \int_0^{\pi/2} dx$$

$$= [x]_0^{\pi/2} = \frac{\pi}{2}. \qquad \therefore I = \frac{\pi}{4}.$$

2. Show that $\int_0^{\pi} \frac{x \sin x \, dx}{1 + \cos^2 x} = \frac{\pi^2}{4}$

$$I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x) \, dx}{1 + \cos^2 (\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x) \sin x \, dx}{1 + \cos^2 x} = \int_0^{\pi} \frac{\pi \sin x \, dx}{1 + \cos^2 x} - \int_0^{\pi} \frac{x \sin x \, dx}{1 + \cos^2 x}$$

$$= \pi \int_0^{\pi} \frac{\sin x \, dx}{1 + \cos^2 x} - I.$$

$$\therefore 2I = \pi \int_0^{\pi} \frac{\sin x \, dx}{1 + \cos^2 x}$$

Put $\cos x = t$. Then $-\sin x \, dx = dt$.
Also when $x = 0, t = 1$ and when $x = \pi, t = -1$.

$$\therefore 2I = \pi \int_1^{-1} - \frac{dt}{1 + t^2} = \pi \int_{-1}^{1} \frac{dt}{1 + t^2}$$

$$\therefore I = \frac{\pi}{2} [\tan^{-1} t]_{-1}^{1} = \frac{\pi}{2} [\tan^{-1} 1 - \tan^{-1} (-1)]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

3. Show that $\int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx = \frac{\pi}{2} \log \frac{1}{2}$.

$$I = \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \log \cos x \, dx.$$

$$\therefore 2I = \int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx$$

$$= \int_0^{\pi/2} (\log \sin x + \log \cos x)\, dx = \int_0^{\pi/2} \log (\sin x \cos x)\, dx$$

$$= \int_0^{\pi/2} \log \left(\frac{\sin 2x}{2}\right) dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$$

Put $2x = t$. Then $2dx = dt$.

Also when $x = 0, t = 0$ and when $x = \frac{\pi}{2}, t = \pi$.

$$\therefore \int_0^{\pi/2} \log \sin 2x\, dx = \frac{1}{2} \int_0^{\pi} \log \sin t\, dt = \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin t\, dt$$

$[\because \sin (\pi - t) = \sin t]$

$$= \int_0^{\pi/2} \log \sin x\, dx = I.$$

$$\therefore 2I = I - \log 2\, [x]_0^{\pi/2}$$

$$\therefore I = -\log 2 \,.\, \frac{\pi}{2} = \frac{\pi}{2} \log \frac{1}{2}\,.$$

4. Show that $\int_0^{\pi/4} \log (1 + \tan \theta)\, d\theta = \frac{\pi}{8} \log 2.$

$$I = \int_0^{\pi/4} \log (1 + \tan \theta)\, d\theta = \int_0^{\pi/4} \log \left[1 + \tan \left(\frac{\pi}{4} - \theta\right)\right] d\theta$$

$$= \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan \theta}{1 + \tan \theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan \theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log (1 + \tan \theta)\, d\theta$$

$$\therefore I = \log 2\,[\theta]_0^{\pi/4} - I \text{ or } 2I = \log 2 \cdot \frac{\pi}{4}$$

$$\therefore I = \frac{\pi}{8}\log 2.$$

5. Evaluate $\displaystyle\int_0^{\infty} \log\left(x + \frac{1}{x}\right)\frac{dx}{1+x^2}$

Put $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$.
Adjusting the limits,

$$I = \int_0^{\pi/2} \log\left(\tan\theta + \frac{1}{\tan\theta}\right)\frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\pi/2} \log\left(\frac{\sec^2\theta}{\tan\theta}\right) d\theta = \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$= -\int_0^{\pi/2} (\log\sin\theta + \log\cos\theta)\, d\theta$$

$$= -\frac{\pi}{2}\log\frac{1}{2} - \frac{\pi}{2}\log\frac{1}{2} \quad \text{(See Q. 3)}$$

$$= -\pi\log\frac{1}{2} = \pi\log 2.$$

Examples 4

Show that

1. $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x\cos x}\, dx = 0$

2. $\displaystyle\int_0^{\pi/2} \frac{\sin x\, dx}{\sin x + \cos x} = \frac{\pi}{4}$

3. $\displaystyle\int_0^{\pi/2} \log\tan x\, dx = 0$

4. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}\, dx}{\sqrt{\sin x} + \sqrt{\cos x}}$

5. $\displaystyle\int_0^{a} \frac{dx}{x + \sqrt{a^2 - x^2}} = \frac{\pi}{4}$

6. $\displaystyle\int_0^{a} \frac{\sqrt{x}\, dx}{\sqrt{x} + \sqrt{a - x}} = \frac{a}{2}$

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7. $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

8. $\int_0^{\pi/2} \sin 2x \log \tan x\,dx = 0$

9. $\int_0^{1} \frac{\log x}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\log\frac{1}{2}$ $\quad x = \sin\theta$

10. $\int_0^{1} \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$ $\quad x = \tan\theta$

11. $\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{4}$

12. $\int_0^{\pi/2} \frac{\sqrt{\cot x}\,dx}{1+\sqrt{\cot x}} = \frac{\pi}{4}$

13. $\int_0^{\pi} \frac{x\tan x\,dx}{\sec x+\tan x} = \frac{\pi}{2}(\pi-2)$

14. $\int_0^{\pi} \frac{x\tan x\,dx}{\sec x+\cos x} = \frac{\pi^2}{4}$

15. $\int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$

16. $\int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x} = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$

17. $\int_0^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx = \frac{\pi^2}{16}$

18. $\int_0^{1} \cot^{-1}(1-x+x^2)\,dx = \frac{\pi}{2} - \log 2$

$\left[\cot^{-1}(1-x+x^2) = \tan^{-1}\frac{x+(1-x)}{1-x(1-x)} = \tan^{-1}x + \tan^{-1}(1-x)\right]$

19. $\int_0^{\pi} \frac{x\,dx}{a^2\sin^2 x+b^2\cos^2 x}\ (a,b>0) = \frac{\pi^2}{2ab}$ $\quad a\tan x = bt$

20. $\int_0^{\pi} \frac{x\,dx}{(a^2\sin^2 x+b^2\cos^2 x)^2} = \frac{\pi^2(a^2+b^2)}{4a^3b^3}$

$2I = 2\int_0^{\pi/2} \phi \quad a\tan x = b\tan\theta$

$\tan^{-1}\frac{\pi}{2} = \infty$ & $\tan^{-1} 0 = 0$

$\int \operatorname{cosec}\theta\,d\theta = \log(\operatorname{cosec}\theta - \cot\theta)$