4-Reduction Formula

REDUCTION FORMULA & BETA AND GAMMA FUNCTIONS

3.1 Reduction Formula

A reduction formula connects an integral with another integral of the same type but of a lower order. Usually, a reduction formula is obtained by the method of integration by parts. The reduction formula is applied repeatedly to express a given integral in terms of a much simpler integral which can be easily evaluated. We will get some reduction formulae below.

3.2 Reduction formula for $\int x^n e^{ax}\,dx$.

Let $I_n = \int x^n e^{ax}\,dx$

Integrating by parts,

$$\int x^n e^{ax}\,dx = x^n \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx$$

$$\therefore I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}.$$

Example : Denoting $\int x^3 e^{ax}\,dx$ by I_3 and putting $n = 3, 2, 1$ successively in the above formula, we get

$$I_3 = \frac{x^3 e^{ax}}{a} - \frac{3}{a} I_2 = \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left(\frac{x^2 e^{ax}}{a} - \frac{2}{a} I_1\right)$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^2}\left(\frac{x\,e^{ax}}{a} - \frac{1}{a}\int e^{ax}\,dx\right)$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6x\,e^{ax}}{a^3} - \frac{6}{a^3}\cdot\frac{e^{ax}}{a}$$

$$= \frac{e^{ax}}{a^4}\left(a^3x^3 - 3a^2x^2 + 6ax - 6\right).$$

$\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx.$

3.3 Reduction formula for $\int \sin^n x \, dx$, $\int_0^{\pi/2} \sin^n x \, dx$, $\int \cos^n x \, dx$ and $\int_0^{\pi/2} \cos^n x \, dx$.

Let $I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$ where n is a **positive** integer.

Integrating by parts,

$$I_n = \sin^{n-1} x (-\cos x) + \int (n-1) \sin^{n-2} x \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$$

By transposition,

$$n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$$

$$\therefore I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2} \quad \text{.........(1)}$$

which is the reduction formula for $\int \sin^n x \, dx$.

If $J_n = \int_0^{\pi/2} \sin^n x \, dx$, we have

$$J_n = -\left[\frac{\sin^{n-1} x \cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} J_{n-2} \qquad \text{from (1).}$$

$$= 0 + \frac{n-1}{n} J_{n-2} = \frac{n-1}{n} J_{n-2} \quad \text{......(2)}$$

This is the reduction formula for $\int_0^{\pi/2} \sin^n x \, dx$. Replacing n by $n-2, n-4, \ldots$ successively, we have from (2),

$$J_{n-2} = \frac{n-3}{n-2} J_{n-4}, \qquad J_{n-4} = \frac{n-5}{n-4} J_{n-6} \qquad \text{and so on.}$$

$$\therefore J_n = \frac{n-1}{n} J_{n-2} = \frac{(n-1)(n-3)}{n(n-2)} J_{n-4}$$

$$= \frac{(n-1)(n-3)(n-5)}{n(n-2)(n-4)} J_{n-6}$$

Proceeding in this way,

$$J_n = \frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3} J_1 \text{, where n is odd.}$$

$$J_n = \frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 2} J_0 \text{, where n is even.}$$

Now $J_1 = \int_0^{\pi/2} \sin x\, dx = [-\cos x]_0^{\pi/2} = 1$

and $J_0 = \int_0^{\pi/2} \sin^0 x\, dx = \int_0^{\pi/2} dx = \frac{\pi}{2}$.

$$\therefore \quad J_n = \frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3} \text{, when } n \text{ is odd}$$

and $\frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 2} \cdot \frac{\pi}{2}$, when n is even ... (

\## If $I_n = \int \cos^n x\, dx$, we can similarly prove that

$$I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

If $J_n = \int_0^{\pi/2} \cos^n x\, dx$,

$$J_n = \frac{(n-1)(n-3)(n-5)\ldots 2}{n(n-2)(n-4)\ldots 3} \text{, when } n \text{ is odd}$$

and $\frac{(n-1)(n-3)(n-5)\ldots 1}{n(n-2)(n-4)\ldots 2} \cdot \frac{\pi}{2}$, when n is even ... (

Thus $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx.$

Formula (3) and (4) above are called Walli's Formula.

It may be noted that the factors in the numerator begin with $(n -$ go on decreasing by 2 and end with 2 or 1. The factors in the denomina begin with n, go on decreasing by 2 and end with 3 or 2. When n is eve we multiply by $\frac{\pi}{2}$.

Examples :

1. $\int_0^{\pi/2} \sin^6 x\, dx = \frac{(6-1)(6-3)(6-5)}{6(6-2)(6-4)} \cdot \frac{\pi}{2} = \frac{5.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{5\pi}{32}$

2. $\int_0^{\pi/2} \cos^7 x\, dx = \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)} = \frac{6.4.2}{7.5.3} = \frac{16}{35}$.

3.4 Reduction formula for $\int \cos^m x \sin nx\, dx$.

Let $I_{m,n} = \int \cos^m x \sin nx\, dx$.

Integrating by parts,

$$I_{mn} = \cos^m x \left(-\frac{\cos nx}{n}\right) - \int m \cos^{m-1} x\,(-\sin x)\left(\frac{-\cos nx}{n}\right)$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x\,(\cos nx \sin x)\, dx \quad \text{......... (1)}$$

Now $\sin(nx - x) = \sin nx \cos x - \cos nx \sin x$

$\therefore \cos nx \sin x = \sin nx \cos x - \sin(n-1)x$

Substituting in (1), we have

$$I_{m,n} = -\frac{\cos^m x \cos nx}{n}$$

$$- \frac{m}{n}\int \cos^{m-1} x\,[\sin nx \cos x - \sin(n-1)x]\, dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

or $\left(1 + \frac{m}{n}\right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}.$

$$\therefore I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}.$$

This is the required reduction formula.

From the above formula

$$\int_0^{\pi/2} \cos^m x \sin nx\, dx$$

$$= \frac{1}{m+n} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x\, dx.$$

3.5 Reduction formula for $\int \cos^m x \cos nx \, dx$

Let $I_{m,n} = \int \cos^m x \cos nx \, dx$.

Integrating by parts,

$$I_{m,n} = \cos^m x \frac{\sin nx}{n} - \int -m \cos^{m-1} x \sin x \frac{\sin nx}{n} \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x (\sin nx \sin x) \, dx$$

Now $\cos (n-1) x = \cos nx \cos x + \sin nx \sin x$ .... (1)

$\therefore \sin nx \sin x = \cos (n-1) x - \cos nx \cos x$

Substituting in (1), we have

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x [\cos (n-1) x - \cos nx \cos x] \, dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

or $\frac{m+n}{n} I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}.$$

and $\int_0^{\pi/2} \cos^m x \cos nx \, dx$

$$= \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \cos (n-1) x \, dx.$$

Examples Worked Out

1. Obtain a reduction formula for $\int \tan^n x \, dx$ and hence find $\int \tan^6 x \, dx$

Let $I_n = \int \tan^n x \, dx$

Then $I_n = \int \tan^{n-2} x \tan^2 x \, dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$, which is the required reduction formula

3.

$\therefore \int \tan^6 x\,dx = I_6 = \dfrac{\tan^5 x}{5} - I_4$

$= \dfrac{\tan^5 x}{5} - \left(\dfrac{\tan^3 x}{3} - I_2\right) = \dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + I_2$

$= \dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \dfrac{\tan x}{1} - I_0$

But $I_0 = \int dx = x.$

$\therefore \int \tan^6 x\,dx = \dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x - x.$

2. Obtain a reduction formula for $\int \operatorname{cosec}^n x\,dx$ and find $\int \operatorname{cosec}^5 x\,dx.$

$I_n = \int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\,dx$

Integrating by parts,

$$I_n = \operatorname{cosec}^{n-2} x\,(-\cot x) - \int (n-2) \operatorname{cosec}^{n-3} x\,(-\operatorname{cosec} x \cot x)\,(-\cot x)\,dx$$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x \cot^2 x\,dx$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^{n-2} x\,(\operatorname{cosec}^2 x - 1)\,dx$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) \int \operatorname{cosec}^n x\,dx + (n-2) \int \operatorname{cosec}^{n-2} x\,dx$$

$= -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$

$\therefore (n-1) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$

$\therefore I_n = -\dfrac{\operatorname{cosec}^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} I_{n-2}.$

Applying this reduction formula,

$\int \operatorname{cosec}^5 x\,dx = I_5 = -\dfrac{\operatorname{cosec}^3 x \cot x}{4} + \dfrac{3}{4} I_3$

$= -\dfrac{\operatorname{cosec}^3 x \cot x}{4} + \dfrac{3}{4}\left(-\dfrac{\operatorname{cosec} x \cot x}{2} + \dfrac{1}{2} I_1\right)$

$= -\dfrac{\operatorname{cosec}^3 x \cot x}{4} - \dfrac{3}{8} \operatorname{cosec} x \cot x + \dfrac{3}{8} \operatorname{cosec} x\,dx$

$= -\dfrac{\operatorname{cosec}^3 x \cot x}{4} - \dfrac{3}{8} \operatorname{cosec} x \cot x + \dfrac{3}{8} \log \tan \dfrac{x}{2}.$

3. From the reduction formula for $\int \cos^m x \sin nx\,dx$, find

$$\int \cos^2 x \sin 3x\,dx.$$

We know that if $I_{m,n} = \int \cos^m x \sin nx\,dx$,

$$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}.$$

$$\therefore \int \cos^2 x \sin 3x \, dx = I_{2,3}$$

$$= -\frac{\cos^2 x \cos 3x}{5} + \frac{2}{5} I_{1,2}$$

$$= -\frac{\cos^2 x \cos 3x}{5} + \frac{2}{5}\left[\frac{-\cos x \cos 2x}{3} + \frac{1}{3} I_{0,1}\right]$$

Now $I_{0,1} = \int \sin x \, dx = -\cos x$

$$\therefore I_{2,3} = -\frac{\cos^2 x \cos 3x}{5} - \frac{2\cos x \cos 2x}{15} - \frac{2\cos x}{15}$$

4. Use the reduction formula for $\int \cos^m x \cos nx \, dx$ to prove th

$$\int_0^{\pi/2} \cos^n x \cos nx \, dx = \frac{\pi}{2^{n+1}}$$

We have $\int \cos^m x \cos nx \, dx$

$$= \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} \int \cos^{m-1} x \cos (n-1)$$

$$\therefore \int_0^{\pi/2} \cos^m x \cos nx \, dx$$

$$= \left[\frac{\cos^m x \sin nx}{m+n}\right]_0^{\pi/2} + \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \cos (n-1)$$

$$= 0 + \frac{m}{m+n} \int_0^{\pi/2} \cos^{m-1} x \cos (n-1) x \, dx$$

Putting $m = n$ and denoting $\int_0^{\pi/2} \cos^n x \cos nx \, dx$ by I_n,

$$I_n = \frac{1}{2} I_{n-1}$$

Similarly, $I_{n-1} = \frac{1}{2} I_{n-2}$, $I_{n-2} = \frac{1}{2} I_{n-3}$ and so on.

$$\therefore I_n = \frac{1}{2} I_{n-1} = \frac{1}{2^2} I_{n-2} = \frac{1}{2^3} I_{n-3} = \dots = \frac{1}{2^n} I_{n-n}.$$

Now $I_{n-n} = I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$ $\quad \therefore I_n = \frac{1}{2^n} \cdot \frac{\pi}{2} = \frac{\pi}{2^{n+1}}$

Examples 6

1. Obtain a reduction formula for $\int \sin^n x\, dx$ and find $\int \sin^6 x\, dx$.

2. Obtain a reduction formula for $\int_0^{\pi/2} \sin^n x\, dx$ and hence evaluate

$$\int_0^{\pi/2} \sin^6 x\, dx.$$

3. Obtain a reduction formula for $\int_0^{\pi/2} \cos^n x\, dx$ and hence evaluate

$$\int_0^{\pi/2} \cos^{10} x\, dx.$$

4. Obtain a reduction formula for $\int \cot^n x\, dx$ and hence integrate $\int \cot^7 x\, dx$.

5. Obtain a reduction formula for $\int \sec^n x\, dx$ and hence find
(i) $\int \sec^5 x\, dx$ (ii) $\int \sec^6 x\, dx$

6. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, show that $I_n + I_{n-2} = \frac{1}{n-1}$ and deduce the value of I_5.

7. From the reduction formula for $\int \cos^m x \cos nx\, dx$ find $\int \cos^3 x \cos 5x\, dx$.

8. Prove that $\int_0^{\pi/2} \cos^5 x \sin 3x\, dx = \frac{1}{3}$.

9. Prove that

$$\int_0^{\pi/2} \cos^m x \sin mx\, dx = \frac{1}{2^{m+1}}\left(2 + \frac{2^2}{2} + \frac{2^3}{3} + \frac{2^4}{4} + \dots + \frac{2^m}{m}\right).$$

10. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\, dx$, prove that

$$I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}.$$