Definite Integra

CHAPTER 2
DEFINITE INTEGRALS

2.1 Definite Integral

If $F(x)$ is an integral of $f(x)$. Then we define $\int_a^b f(x)\,dx$ as $F(b) - F(a)$. $\int_a^b f(x)\,dx$ is called the definite integral of $f(x)$ between the limits a and b. b is called the upper limit and a the lower limit.

We can use the above definition to evaluate a definite integral. To evaluate $\int_a^b f(x)\,dx$, we first find the indefinite integral $F(x)$ of $\int_a^b f(x)\,dx$. Then we substitute b and a for x and subtract $F(a)$ from $F(b)$.

Examples worked out

1. $$I = \int_0^{\pi/2} \cos^4 x\,dx = \int_0^{\pi/2} \left[\frac{1 + \cos 2x}{2}\right]^2 dx$$

$$= \frac{1}{4}\int_0^{\pi/2} dx + \frac{1}{2}\int_0^{\pi/2} \cos 2x\,dx + \frac{1}{8}\int_0^{\pi/2} (1 + \cos 4x)\,dx$$

$$= \frac{1}{4}[x]_0^{\pi/2} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2} + \frac{1}{8}[x]_0^{\pi/2} + \frac{1}{8}\left[\frac{\sin 4x}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}(\sin\pi - \sin 0)$$

$$+ \frac{1}{8}\left(\frac{\pi}{2} - 0\right) + \frac{1}{32}(\sin 2\pi - \sin 0)$$

$$= \frac{\pi}{8} + \frac{\pi}{16} = \frac{3\pi}{16}.$$

2. $$I = \int_0^1 \frac{1 - x^2}{1 + x^2}\,dx = \int_0^1 \frac{2 - (1 + x^2)}{1 + x^2}\,dx$$

$$= \int_0^1 \left(\frac{2}{1+x^2} - 1\right) dx = [2\tan^{-1} x - x]_0^1$$

$$= (2\tan^{-1} 1 - 1) - (2\tan^{-1} 0 - 0) = 2\left(\frac{\pi}{4}\right) - 1 = \frac{\pi}{2} - 1.$$

When the variable in a definite integral is changed, we usual change the limits also. The method will be illustrated below.

3. $$I = \int_0^{\pi/4} \frac{\sin 2\theta}{\sin^4\theta + \cos^4\theta}\, d\theta$$

$$= \int_0^{\pi/4} \frac{2\sin\theta\cos\theta\, d\theta}{\sin^4\theta + \cos^4\theta} = \int_0^{\pi/4} \frac{2\tan\theta\sec\theta}{\tan^4\theta + 1}$$

Put $\tan^2\theta = T$. Then $2\tan\theta\sec^2\theta\, d\theta = dt$

When $\theta = 0, t = \tan^2 0 = 0.$

When $\theta = \frac{\pi}{4}, t = \tan^2\frac{\pi}{4} = 1.$

$$\therefore I = \int_0^1 \frac{dt}{t^2+1} = [\tan^{-1} t]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

4. $$I = \int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$$

Put $x + 1 = \frac{1}{t}$, Then $dx = -\frac{1}{t^2}\, dt.$

When $x = 1, t = \frac{1}{x+1} = \frac{1}{2}$

When $x = 2, t = \frac{1}{2+1} = \frac{1}{3}$

$$\therefore I = \int_{1/2}^{1/3} \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t} - 1\right)^2 - 1}}$$

$$= -\int_{1/2}^{1/3} \frac{dt}{\sqrt{(1-t)^2 - t^2}} = -\int_{1/2}^{1/3} \frac{dt}{\sqrt{1-2t}}$$

$$= -\left[\frac{(1-2t)^{1/2}}{\frac{1}{2}(-2)}\right]_{1/2}^{1/3} = \left[\sqrt{1-2t}\right]_{1/2}^{1/3}$$

$$= \sqrt{1-\frac{2}{3}} - \sqrt{1-1} = \frac{1}{\sqrt{3}}$$

5. $$I = \int_{\beta}^{\alpha} \sqrt{(x-\alpha)(\beta-x)}\ dx$$

$$= \int_{\beta}^{\alpha} \sqrt{-x^2 + (\alpha+\beta)x - \alpha\beta}\ dx$$

$$= \int_{\beta}^{\alpha} \sqrt{\left(\frac{\beta+\alpha}{2}\right)^2 - \alpha\beta - \left[x^2 - (\alpha+\beta)x + \left(\frac{\beta+\alpha}{2}\right)^2\right]}\ dx$$

$$= \int_{\beta}^{\alpha} \sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - \left(x - \frac{\beta+\alpha}{2}\right)^2}\ dx$$

Put $\frac{\beta-\alpha}{2} = a$ and $x - \frac{\beta+\alpha}{2} = y$

Then $dx = dy$

When $x = \alpha$, $y = \alpha - \frac{\beta+\alpha}{2} = \frac{\alpha-\beta}{2} = -a.$

When $x = \beta$, $y = \beta - \frac{\beta+\alpha}{2} = \frac{\beta-\alpha}{2} = a$

$$\therefore I = \int_{-a}^{a} \sqrt{a^2 - y^2}\ dy = \left[\frac{y\sqrt{a^2-y^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_{-a}^{a}$$

$$= \left(0 + \frac{a^2}{2}\sin^{-1} 1\right) - \left[0 + \frac{a^2}{2}\sin^{-1}(-1)\right]$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a^2}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}a^2 = \frac{\pi}{2}\left(\frac{\beta-\alpha}{2}\right)^2 = \frac{\pi}{8}(\beta-\alpha)^2.$$

Note: It can also be integrated if we put $x = \alpha\cos^2\theta + \beta\sin^2\theta$.

8. $I = \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}}$

Put $x = a \sin \theta$. Then $dx = a \cos \theta \, d\theta$.

When $x = a$, $\sin \theta = 1$ or $\theta = \frac{\pi}{2}$.

When $x = 0$, $\sin \theta = 0$ or $\theta = 0$.

$$\therefore I = \int_0^{\pi/2} \frac{a \cos \theta \, d\theta}{a \sin \theta + a \cos \theta}$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{(\cos \theta + \sin \theta) + (\cos \theta - \sin \theta)}{\sin \theta + \cos \theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} d\theta + \frac{1}{2} \int_0^{\pi/2} \frac{\cos \theta - \sin \theta}{\sin \theta + \cos \theta} d\theta$$

$$= \frac{1}{2} [\theta]_0^{\pi/2} + \frac{1}{2} [\log (\sin\theta + \cos \theta)]_0^{\pi/2}$$

$$= \frac{1}{2} \times \frac{\pi}{2} + \frac{1}{2} (\log 1 - \log 1) = \frac{\pi}{4}.$$

Examples 2

Evaluate :

1. $\int_0^1 \tan^{-1} x \, dx$

2. $\int_1^e \frac{dx}{x \sqrt{1 - (\log x)^2}}$ $\log e = 1$

3. $\int_0^{\pi/2} \sin^4 x \, dx$

4. $\int_0^{\pi/2} \sqrt{\cos \theta} \sin^3 \theta \, d\theta$

5. $\int_0^{1/2} \frac{dx}{(1 - 2x^2) \sqrt{1 - x^2}}$ $x = \sin \theta$

6. $\int_0^1 x \tan^{-1} x \, dx$

7. $\int_0^1 \frac{1 - x}{1 + x} dx$

8. $\int_1^2 \frac{dx}{x (1 + x^4)}$

47. $-\frac{1}{2}\log(1+\cos x)+\frac{1}{10}\log(1-\cos x)+\frac{2}{5}\log(3+2\cos x)$

48. $\frac{1}{2}\log\tan\frac{x}{2}+\frac{1}{4}\sec^2\frac{x}{2}+\tan\frac{x}{2}$ 49. $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}\,x}$

50. $\cos\alpha\cos^{-1}(\cos x\sec\alpha)-\sin\alpha\log\left(\sin x+\sqrt{\sin^2 x-\sin^2\alpha}\right)$

Examples 2

1. $\frac{\pi}{4}-\frac{1}{2}\log 2$ 2. $\frac{\pi}{2}$ 3. $\frac{3\pi}{16}$ 4. $\frac{8}{21}$

5. $\frac{1}{2}\log\left(2+\sqrt{3}\right)$ 6. $\frac{\pi-2}{4}$ 7. $-1+2\log 2$ 8. $\frac{1}{4}\log\frac{32}{17}$

9. $e^{\pi/2}$ 10. 0 11. $\frac{\pi}{4\sqrt{5}}$ 12. $\frac{1}{24}$

13. $2-\frac{\pi}{2}$ 14. $\frac{3a^4\pi}{16}$ 15. $\frac{\pi}{3\sqrt{3}}$

16. $\frac{1}{\sqrt{3}}\log\left(2+\sqrt{3}\right)$ 17. $\frac{1}{3}\log 2$ 18. $\frac{2}{3}\tan^{-1}\frac{1}{3}$

19. $\frac{\alpha}{\sin\alpha}$ 20. $\frac{\pi(a^2+b^2)}{4a^3b^3}$ 21. $\frac{\pi}{4}$

22. $\frac{\log 3}{20}$ 23. $\pi\sqrt{2}$ 24. $\frac{\pi}{4ab^2(a+b)}$ 25. $\frac{\pi}{3}$

26. $\frac{\pi}{6}$ 27. $\frac{\pi}{2\sqrt{2}}$ 28. $\log\frac{4}{3}$

Examples 3

1. $\frac{1}{3}$ 2. $e^{-a}-e^{-b}$ 3. $\frac{1}{m}(e^{mb}-e^{ma})$ 4. $\frac{27}{2}$

5. $\frac{1}{4}(b^4-a^4)$ 6. $\cos a-\cos b$ 7. 1 8. $\frac{1}{a}-\frac{1}{b}$

9. $\frac{2}{3}$ 10. 2 11. $\frac{1}{2}a+b$ 12. $\frac{\pi}{8}-\frac{1}{4}$

15. $\frac{3}{8}$ 16. $\frac{1}{2}\log 2+\frac{\pi}{4}$