Introduction to General Relativity

AMATH 475

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Preface

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Some of the notes (especially special relativity part) are projected to the screen instead of using blackboards. They can be found on https://sites.google.com/site/emmfis/teaching/gr.

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Pre-Math

0.1 Index notation

$$A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \qquad B = \begin{pmatrix} B^1{}_1 & B^1{}_2 \\ B^2{}_1 & B^2{}_2 \end{pmatrix}$$

$$(A \cdot B)^a{}_b = A^a{}_c B^c{}_b = B^c{}_b A^a{}_c \qquad \text{sum over all possible } c$$

Identify followings:

$$B_\kappa{}^\nu A_\mu{}^\kappa = A_\mu{}^\kappa B_\kappa{}^\nu = C_\mu{}^\nu = (A \cdot B)_\mu{}^\nu$$

$$A^\kappa{}_\mu B_\kappa{}^\nu = D_\mu{}^\nu = (A^T)_\mu{}^\kappa B_\kappa{}^\nu = (A^T \cdot B)_\mu{}^\kappa$$
$$A_\kappa{}^\nu B_\mu{}^\kappa = E_\mu{}^\nu = (B \cdot A)_\mu{}^\nu$$
$$A^\kappa{}_\mu B^\nu{}_\kappa = (A^T)_\mu{}^\kappa (B^T)_\kappa{}^\nu = \left((B \cdot A)^T\right)_\mu{}^\nu$$

$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 \qquad \{\mathbf{e}_1, \mathbf{e}_2\}$ Basis 1.

$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a \qquad \{\mathbf{e}'_1, \mathbf{e}'_2\}$ Basis 2.

Change of basis matrix Λ

$$\mathbf{e}'_a = \Lambda_a{}^b \mathbf{e}_b$$
$$v'^a = \tilde{\Lambda}^a{}_b v^b$$

$$\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a{}_b v^b \Lambda_a{}^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a{}_b \Lambda_a{}^c v^b \mathbf{e}_c \\
&= \underbrace{\left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c}_{\delta_b^c} v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b
\end{aligned}$$

$$\begin{aligned}
\implies \left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c &= \delta_b^c \\
\tilde{\Lambda}^T \cdot \Lambda &= \mathbb{1}
\end{aligned}$$

$\tilde{\Lambda}^T$ is the inverse transpose of Λ

covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis. Λ. (sub-indices)

A contravariant object transforms like components of vectors. ($\tilde{\Lambda} = (\Lambda^T)^{-1}$). (super-indices)

0.2 Vectors and one-forms

one-form

Let V be a vector space. A one-form is a linear map $\boldsymbol{\omega} : V \to \mathbb{R}$.

or we write: $(\boldsymbol{\omega}, \cdot) : V \to \mathbb{R}$ and $(\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}$.

dual vector space

The set of all one-forms on V (call V^*) is a vector space as well called the dual vector space to V.

dual basis

Let $\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2, \ldots\}$ (or $\{\boldsymbol{\Upsilon}_i\}$) be a basis of V so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = v^i \boldsymbol{\Upsilon}_i$.

We define the dual basis (of V^*) to $\{\boldsymbol{\Upsilon}_i\}$ as $\{\boldsymbol{\omega}^i\}$ such that $\boldsymbol{\omega}^i(\boldsymbol{\Upsilon}_j) = \delta^i_j$.

For a one form $\boldsymbol{\omega}$ we denote its "components of the basis $\boldsymbol{\Upsilon}$" as $(\boldsymbol{\omega}, \boldsymbol{\Upsilon}_m) = \omega_m$

Proposition 0.1

The dual basis of V^* is actually a basis of V^*.

The action of $\boldsymbol{\omega} \in V^*$ on a vector $\mathbf{v} = v^\mu \boldsymbol{\Upsilon} \in V$ is

$$(\boldsymbol{\omega}, \mathbf{v}) = (\boldsymbol{\omega}, v^\mu \boldsymbol{\Upsilon}_\mu) = v^\mu \omega_\mu$$

Let's prove $\{\boldsymbol{\Upsilon}^a\}$ is linear independent.

Proof:
A linear comb. $c_a \boldsymbol{\Upsilon}^a$ acts on a vector $\mathbf{v} = v^a \boldsymbol{\Upsilon}_a$

$$\begin{aligned}(c_a \boldsymbol{\Upsilon}^a, \mathbf{v}) &= c_a (\boldsymbol{\Upsilon}^a, \mathbf{v}) \\ &= c_a (\boldsymbol{\Upsilon}^a, v^b \boldsymbol{\Upsilon}_b) \\ &= c_a v^b \underbrace{(\boldsymbol{\Upsilon}^a, \boldsymbol{\Upsilon}_b)}_{\delta^a_b} \\ &= c_a v^b \delta^a_b = c_a v^a\end{aligned}$$

For LI,

$$c_a \boldsymbol{\Upsilon}^a = 0 \iff c_a = 0 \quad \forall a$$
$$c_a v^a = 0 \quad \forall \mathbf{v} \iff c_a = 0$$

$\square$

vectors: take one-forms $\to \mathbb{R}$ one-forms: take vectors $\to \mathbb{R}$

0.3 Tensor

type (m, n) tensor

A type (m, n) tensor is a multilinear map that

$$\mathbf{T} : V^n \otimes (V^*)^m \to \mathbb{R}$$

Components of $\mathbf{T}$:

$$\mathbf{T}(\boldsymbol{\Upsilon}_{a1}, \ldots, \boldsymbol{\Upsilon}_{an}, \boldsymbol{\Upsilon}^{b1}, \ldots, \boldsymbol{\Upsilon}^{bm}) = T_{a_1 \ldots a_n}{}^{b_1 \ldots b_m}$$

1. Tensor product takes $\binom{m}{n}$ and $\binom{m'}{n'} \to \binom{m+m'}{n+n'}$ tensor
2. Contraction takes $\binom{m}{n} \to \binom{m-1}{n-1}$

Example:

1. $T_a{}^b, S_c{}^d$.
$$(\mathbf{T}\otimes\mathbf{S})_a{}^b{}_c{}^d = T_a{}^d S_c{}^d = P_a{}^b{}_c{}^d$$

2. $T_a{}^{bc} \to c^b T_a{}^{ba}$
$$v^a, w_b \begin{cases} v^a\omega_b \\ v^a\omega_a \end{cases}$$

If you have a favorite type $(2,0)$ symmetric tensor $\mathbf{g}$

$$v_\mu = g_{\mu\nu}v^\nu$$

$g^{\mu\nu}$:= components of the inverse of $\mathbf{g}_{\mu\nu}$

$$v^\nu = g^{\mu\nu}$$

then

$$g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$$

$$g_{\mu\nu}v^\mu w^\nu = v_\mu w^\nu = \mathbf{v}\mathbf{w}$$
$$||\mathbf{v}||^2 = g_{\nu\mu}v^\mu v^\nu$$

Then we can define the angle

$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{w}||\ ||\mathbf{v}||} := \cos\theta$$
$$T_\mu{}^\nu = g^{\nu\sigma}T_{\mu\sigma}$$
$$T^{\mu\nu} = g^{\nu\sigma}g^{\mu\rho}T_{\sigma\rho}$$
$$g^\nu_\mu = g^{\nu\sigma}g_{\sigma\mu} = \sigma^\nu_\mu$$

0.4 Levi-Civita symbol

Levi-Civita symbol $\epsilon^{abc\ldots}, \epsilon_{abc\ldots}$

- is antisymmetric
- $\epsilon^{1234\ldots} = 1$, $\epsilon_{1234} = 1$

$\epsilon^{123} = 1, \quad \epsilon^{213} = -1, \quad \epsilon^{312} = 1, \quad \epsilon^{113} = 0$

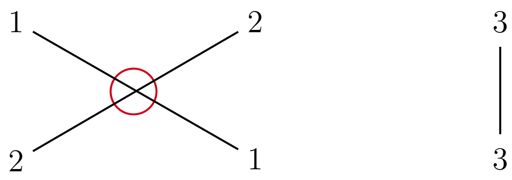
$\epsilon^{123456} = 1, \quad \epsilon^{612453} = -1$

Idea just see the permutations

Levi-Civita symbol

$$\varepsilon_{a_1 a_2 a_3 \ldots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an even permutation of } (1,2,3,\ldots,n) \\ -1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an odd permutation of } (1,2,3,\ldots,n) \\ 0 & \text{otherwise} \end{cases}$$

Here is a short-cut:



odd number crossings, so odd permutation.

Note that $\det(M) := \epsilon_{ijk\ldots} M^i{}_1 M^j{}_2 M^j{}_3 \ldots$

Exercise

prove $\epsilon^{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n} = n! i_j = 1, \ldots, n$

$$\epsilon^{ijk}\epsilon_{ilm} = \delta^j_l \delta^k_m - \delta^j_m \delta^k_l$$
$$\epsilon^{ijmn}\epsilon_{klmn} = 2(\delta^i_k \delta^j_l - \delta^j_k \delta^i_l)$$

Prove $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Proof:

Let $\vec{F} = \vec{A} \times (\vec{B} \times \vec{C})$ $\vec{D} = \vec{B} \times \vec{C}$

Then

$$D^k = \epsilon^k{}_{ij} B^i C_j$$
$$F^l = \epsilon^l{}_{mk} A^m D^k \implies F^l = \epsilon^l{}_{mk} \epsilon^k{}_{ij} A^m B^i C^j$$

Then

$$\begin{aligned} F^l &= (\delta^l_i \delta_{mj} - \delta^l_j \delta_{mi}) A^m B^i C^j \\ &= \delta^l_i \delta_{mj} A^m B^i C^j - \delta^l_j \delta_{mi} A^m B^i C^j \\ &= B^l (A_j C^j) - C^l (A_i B^i) \end{aligned}$$

where we use

$$\vec{A} \cdot \vec{B} = A^i B_i$$

$\square$

1

Special Relativity

1.1 Postulates of SR

Postulate 0

Newton's first law

Postulate 1: Principle of relativity

In the absence of gravity, all the laws of Physics are identical in all inertial reference frames.

Postulate 2

The speed of light in vacuum c is constant and the same from all inertial reference frames, regardless of their state of motion.

1.2 Lorentz Transformation

We define the spacetime interval Δs^2

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = -c^2 \left(t_2 - t_1\right)^2 + \left(\mathbf{x}_2 - \mathbf{x}_1\right)^2$$

Assuming the following:

1. The difference between the two frames is a constant speed $\succapprox$
2. The transformation has to be linear.

$$t' = \gamma \left(t - \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{c^2} \right), \quad \boldsymbol{x}' = \boldsymbol{x} + (\gamma - 1)(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - \gamma \boldsymbol{v} t$$

and index notation

$$t' = \gamma \left(t - \frac{v_i x^i}{c^2} \right), \quad x^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

1.3 Line element, proper time and spacelike, timelike and null separation

1.3.1 Classification of spacetime intervals

We can classify events according to the following criterion:

- Spacelike separated, $\Delta s^2 > 0$
- Timelike separated, $\Delta s^2 < 0$
- Lightlike (null) separated, $\Delta s^2 = 0$

Given the trajectory of a physical particle moving inertially, we will call co-moving frame (inertial) or proper frame (non-inertial) to the frame S_p where the particle is at rest.

1.3.2 Proper time and line element

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2$$

We will call ds^2 the spacetime line element.

$$\mathbf{v} := \frac{d\mathbf{x}}{dt}$$

P0, P1, P2 + linearity

$$\implies t' = t\left(t - \frac{v_i x^i}{c^2} \right) \tag{1}$$

$$x'^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

Particle trajectory in a given inertial (Lab) frame $\mathbf{x}(t)$

Particle trajectory in its proper frame $\boldsymbol{\xi}(t) = 0$

Comoving frame's trajectories at each t (from lab frame) $\mathbf{x} = \mathbf{v}(t)t$.

$$d\tau = dt' = \gamma(t)\left(1 - \frac{\mathbf{v}(t)^2}{c^2}\right) dt \tag{2}$$

$$ds^2 = -c^2 dt^2 \left(1 - \frac{1}{c^2}\underbrace{\left(\frac{d\mathbf{x}}{dt}\right)}_{\mathbf{v}(t)}^2\right) = -c^2 \underbrace{\gamma^{-2} dt^2}_{d\tau^2} \implies ds^2 = -c^2 d\tau^2 \tag{3}$$

Example:

Find $\tau(t)$ for the three following trajectories.

1. $x(t) = v(t)$

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x}^2 \implies d\tau = \gamma^{-1} dt \implies \Delta\tau = \gamma^{-1}\Delta t$$

2. $x(t) = \frac{c^2}{a}\left[\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right]$

Then $\frac{dx}{dt} = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2$$

$$\implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\frac{a^2t^2}{1 + \left(\frac{at}{c}\right)^2}}$$

$$\implies \tau(t) = \frac{c}{a}\operatorname{arcsinh}\left(\frac{at}{c}\right) \qquad \text{and} \qquad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

3. $x(t) = L\sin(\omega t) \implies \frac{dx}{dt} = Lw\cos(\omega t)$ with $L\omega < c$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 \implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2} \implies d\tau = \sqrt{1 - \frac{L^2\omega^2}{c^2}\omega t}\ dt$$

Then

$$\tau(t) = \frac{E\left(t\omega, \frac{1}{1-\frac{c^2}{L^2\omega^2}}\right)}{\omega\sqrt{\frac{1}{1-\frac{L^2\omega^2}{c^2}}}}$$

where

$$E(\phi|m) = \int_0^\phi (1 - m\sin^2\theta)^{1/2} d\theta$$

1.4 Lorentzian Tensors

See notes for details.

A_μ transposes with Λ and it's covariant.

A^μ transposes with $\tilde{\Lambda} = (\Lambda^{-1})^T$ and it's controvariant.

1.5 Poincare group

The derivations are in notes.

1.6 Relativistic dynamics

1.6.1 Hamilton's principle and Euler-Lagrange equations

There exists at least one function (called action) of the trajectories that the degrees of freedom of a system may take in phase space. The physical trajectories are obtained demanding stationarity of this functional under variations that keep the initial and final positions constant.

Usually, the action S of a system of n particles can be written in terms of a Lagrangian $L(s, \mathsf{x}, \mathring{\mathsf{x}})$ where $^\circ$ represents $\mathring{\mathsf{x}} = \frac{d\mathsf{x}}{ds}$ so that

$$S = \int_{s_1}^{s_2} ds\ L(s, \mathsf{x}, \mathring{\mathsf{x}})$$

$$\delta S = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} \delta x_n^\mu + \frac{\partial L}{\partial \mathring{x}_n^\mu} \delta \mathring{x}_n^\mu \right) = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \mathring{x}_n^\mu} \right) \delta x_n^\mu + \sum_n \left[\frac{\partial L}{\partial \mathring{x}_n^\mu} \delta x_n^\mu \right]_{s_1}^{s_2}$$

Impose Hamilton's Principle

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \mathring{x}_n^\mu} = 0$$

1.6.2 Conserved quantities and Noether's theorem

Noether's theorem

If the variation of the action around a physical trajectory under a continuous variation of the positions $\partial \mathsf{x}$ is zero, then the quantity

$$\delta Q = \sum_n \frac{\partial L}{\partial \mathring{x}_n^\mu} \delta x^\mu$$

is conserved. That is
$$\frac{d(\partial Q)}{ds} = 0.$$

Proof:
See notes. □

1.6.3 Four-momentum

Let S be invariant under $\partial \mathsf{x} = \mathsf{n}\delta\alpha$.
$$\implies \delta Q = \frac{\partial L}{\partial \overset{\circ}{x}^{\mu}} n^{\mu} \delta\alpha$$
is constant $\implies$ the projection $\mathsf{n} \cdot \mathsf{p} = n^{\mu} p_{\mu} = \eta_{\mu\nu} n^{\mu} p^{\nu}$ (where $p_{\mu} := \frac{\partial L}{\partial \overset{\circ}{x}^{\mu}}$) is conserved.

If the action is invariant under Lorentz transformation $\delta x^{\mu} = \delta\omega^{\mu}{}_{\nu} x^{\nu}$, then
$$J_{\mu\nu} := x_{\mu} p_{\nu} - x_{\nu} p_{\mu}$$
is conserved.

1.6.4 Angular momentum

The angular momentum $\boldsymbol{J}$ associated to spatial rotations and the vector $\boldsymbol{K}$ associated to boosts can be extracted directly from $J_{\mu\nu}$:
$$J^{i} = \frac{1}{2} \epsilon^{ijk} J_{jk}, \quad K_i = J_{i0}$$

1.6.5 Free particle dynamics

- S has to be a scalar (Invariant under Lorentz)
- Must coincide with the non-relativistic action in the limit $\frac{v}{c} \ll 1$.

$$S = mc \int ds = -mc^2 \int d\tau = -mc^2 \int dt \frac{d\tau}{dt} = -mc^2 \int \frac{dt}{\gamma} = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}}$$
$$= -mc^2 \int dt \left[1 - \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right)\right]$$
and
$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} = -mc^2 + \frac{1}{2} m v^2 + \mathcal{O}\left(\frac{v^4}{c^4}\right)$$

Euler-Lagrange $\frac{d}{dt}(\gamma m \boldsymbol{v}) = 0$

$$p_i = \frac{\delta S}{\delta v^i} = \frac{\partial L}{\partial v^i} = m\gamma v_i, \qquad \boldsymbol{p} = m\gamma \boldsymbol{v}$$

Hamiltonian

$$H = (\boldsymbol{p} \cdot \boldsymbol{v} - L)_{\boldsymbol{v} \to \boldsymbol{v}(\boldsymbol{p})} = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$$

Let's introduce Four-velocity.

$$\frac{dx^\mu}{d\tau} =: \dot{x}^\mu \equiv u^\mu$$

solid dot means derivative w.r.t proper time.

$$\dot{x}^\mu := \frac{dx^\mu}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d\boldsymbol{x}}{d\tau} \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d\boldsymbol{x}}{dt}\frac{dt}{d\tau} \end{pmatrix} = \gamma \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$$

If we choose action as (not four-velocity)

$$S = mv \int dt \sqrt{\eta_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}$$

Lagrangian

$$L = mc\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}$$

$$p_\mu = \frac{\delta S}{\delta \dot{x}^\mu} = m\dot{x}_\mu \implies p^\mu = m\dot{x}^\mu = m\gamma \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$$

p^0 in the proper frame: $p^0 = mc, \quad \boldsymbol{p} = \boldsymbol{0}$. so cp^0 is energy.

Let's compute

$$p^\mu p_\mu = m^2 \dot{x}^\mu \dot{x}_\mu = -m^2c^2$$

$$p^\mu p_\mu = -(p^0)^2 + \boldsymbol{p}^2$$

$$\implies -m^2c^2 = -(p^0)^2 + \boldsymbol{p}^2 \implies p^0 = \frac{1}{c}\sqrt{m^2c^4 + c^2\boldsymbol{p}^2} \implies E = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$$

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