AMATH 251

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Contents

Info

0.1 Schedule change

- lectures
  - Mon/Wed MC4063
  - Thur RCH 103 10:30
- Office Hours MC 6132
  - Wednesday, 4:30-5:30 p.m.
  - Thursday 4:00-5:00 p.m.
- TUT Fri 3:30
  - Friday Sept.14 Maple MC 3006
  - After next week TUT MC 4063
- Tutor: M. C.[1]
  - MC6501
  - Office Hour: after Friday tut.

0.2 Maple

Quick note: File → New → Worksheet

[1] too harsh...

1 Intro

1.1 Overview

In this course, we'll use differential equations as mathematical models of physical systems.

Definition 1.1.1. A differential equations (DE) is a equation for an unknown function involving one or more derivatives of the equation.

The unknown function is represented in terms of a dependent variable and the variables it depends on are called independent variable(s).

Simple Example

$$\frac{dy}{dx} = \cos(x)$$

unknown function $y(x)$

- $y \rightarrow$ dependent variable
- $x \rightarrow$ independent variable

independent variable.

$y(x)$

dependent variable

Simple mathematical model Newton's 2^{nd} law: "the rate of change of momentum of an object is equal to the sum of the forces acting on the object".

Write this as a DE:

$$\frac{d(mv)}{dt} = F \tag{1.1}$$

- t - time
- m - mass of object

- v - velocity of object
- F - Forces acting

If the mass of the object is constant, this becomes

$$m\frac{dv}{dt} = F \tag{1.2}$$

If we know m and F, this can be thought of as a DE for the unknown $v(t)$, where v is dependent variable, t is independent variable.

Example An object is thrown vertically upward with speed v_0. If the mass of object is m, and the only force acting is gravity, find the velocity of the object at any later time.

Solution Assumptions

1. only force acting is gravity
2. mass is constant
3. Assume object is close enough to the surface of earth that force of gravity is constant.
4. take upward as positive direction

Model:

$$m\frac{dv}{dt} = -mg \tag{1.3}$$

$$v(0) = v_0 \tag{1.4}$$

where g is the gravitational acceleration.

Units:

- m - kg
- t - s
- v - m/s
- v_0 - m/s
- $g = 9.8m/s^2$

Equations (3) - (4) can be solved to find $v(t) = v_0 - gt$.

Definition 1.1.2. An initial condition (IC) is an equation which gives the value of the dependent variable for a specific value of the independent variable

Definition 1.1.3. An initial value problem (IVP) (abbr. learned from Jordan) is a differential equation together with one or more initial conditions.

Key thing for us - how to solve a given IVP.

When formulating mathematical models, it is important to be aware what physical quantities the variables represent.

$$
\begin{aligned}
\text{Model:} \qquad m\frac{dv}{dt} &= -mg \\
kg \times m/s^2 &= kg \cdot m/s^2 \\
\text{Force} &= \text{Force}
\end{aligned}
$$

$$\underbrace{\underbrace{m}_{kg}\ \underbrace{\frac{dv}{dt}}_{m/s^2}}_{\text{Force}} = -\underbrace{\underbrace{m}_{kg}\ \underbrace{g}_{m/s^2}}_{\text{Force}}.$$

<u>Solution:</u>

$$
\begin{aligned}
\underline{v(t)} &= \underline{v_0} - \underline{gt} \\
m/s &= m/s - (m/s^2) \times (s)
\end{aligned}
$$

1.2 Fundamental Questions in the study of DEs

1. Does a solution to a given IVP exist? (existence)
2. Is there more than one solution? (uniqueness)
3. How do we find/approximate the solution? (Method)

<u>Classification of DEs:</u> See extra supplementary materials.

Most general, n^{th} order ODE:

$$H(x, y, y', y'', \cdots, y^{(n)}) = 0 \qquad (1)$$

where H is a given function.
To solve (1) means to find the unknown function $y(x)$.

More formally,

Definition 1.2.1. The function $\phi(x)$ is a <u>solution</u> of (1) on the open interval $I = \{x \in \mathbb{R} : a < x < b\}$ if $\phi, \phi', \cdots, \phi^{(n)}$ exist on I and

$$H(x, y, y', y'', \cdots, y^{(n)}) = 0 \quad \text{for all } x \in I.$$

The interval I is called the <u>interval of existence</u> (or domain of definition) of the solution.

Example $y' = y^2$ It is easy to check that $\phi(x) = \dfrac{1}{1-x}$ stisfies this DE where it is defined, i.e. on $(-\infty, 1)$ and $(1, \infty)$
There are two solutions of the DE

$$y = \frac{1}{1-x}, \quad x \in (-\infty, 1)$$
$$y = \frac{1}{1-x}, \quad x \in (1, \infty)$$

2 First Order ODEs

2.1 Intro

Most general, first order ODE: $H(x, y, y') = 0$
If H is linear as y' this can be written

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \qquad \text{(2) (quasi-linear form)}$$

Let $D \subset \mathbb{R}^2$ be an open, connected, nonempty set. If $N(x, y) \neq 0$ on D then (2) can be written on D as

$$\frac{dy}{dx} = f(x, y) \qquad \text{(3)(Normal form or standard form)}$$

Example Solve the DE $\frac{dy}{dx} = \cos(x)$

Solution Integrate both sides of DE with respect to x

$$\int \frac{dy}{dx} = \int \cos(x)dx$$

$$y(x) = \sin(x) + C$$

So for any constant C the function $y(x) = \sin(x) + C$, $x \in \mathbb{R}$ is a solution of the DE.

Definition 2.1.1. A general solution of a first order ODE is a solution containing one arbitrary constant that represents almost all the solutions of the DE.

In previous example $y(x) = \sin(x) + C$ is a general solution of the DE.

Definition 2.1.2. A particular solution of a first order ODE is a solution that contans no arbitrary constant

When solving an IVP, we will look for the particular solution that satisfies the given initial condition.

Example Solve the IVP $\frac{dy}{dx} = \cos(x)$, $y(\pi) = 1$

Solution
General solution: $y(x) = \sin(x) + C$
Initial condition: $y(\pi) = 1 \implies 1 = \sin(\pi) + C \implies C = 1$
Solution of the IVP is $y(x) = \sin(x) + 1, \quad x \in \mathbb{R}$

Example Consider the IVP $xy' = y, \quad y(0) = 5$
It can be shown that a general solution for the DE is $y(x) = Cx, \quad x \in \mathbb{R}$
Apply IC, $y(0) = 5$. $\rightarrow$ Not possible to satisfy. No choice of C works.

Example Consider the IVP $y' = 2x\sqrt{y}, \quad y(0) = 0$. Can verify (see review assignment) that this has two solutions.

$$y(x) = \frac{x^4}{2}, x \in \mathbb{R}$$

$$y(x) = 0, x \in \mathbb{R}$$

General solution for DE

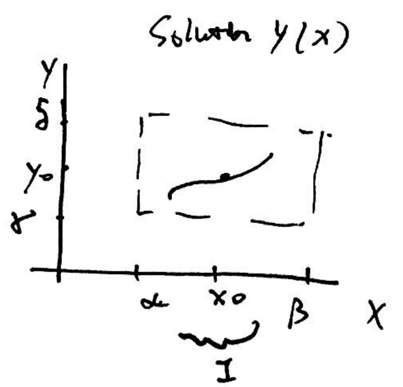
$$y(x) = (\frac{x^2}{2} + C)^2$$

(interval of existence depends on C)

How to predict if an IVP has a solution?

<u>**Theorem**</u> (Existence and Uniqueness Theorem)
Consider the IVP $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \quad (1)$
Suppose the functions f and $\frac{\partial f}{\partial y}$ are continuous on some rectangle R in the x-y plane containing the point (x_0, y_0), then for some open interval I containing x_0 the IVP(1) has one and only one solution defined on I.

<u>picture</u>



$R = \{(x,y) \mid \alpha < x < \beta;$
$\gamma < y < \delta\}$

Apply this to our examples.

1. $f(x, y) = \cos(x)$

   $\frac{\partial f}{\partial y} = 0$

   —— Both are continuous for all $(x, y) \in \mathbb{R}^2$

   The Existence and Uniqueness Theorem $\implies$ can find a solution to $\dfrac{dy}{dx} = \cos(x), \quad y(x_0) = y_0$ for any (x_0, y_0)

2. $x\dfrac{dy}{dx} = y \quad \Longrightarrow$ can't apply the Theorem in this form. Rewrite this in the standard/normal form

$$\frac{dy}{dx} = \frac{y}{x}$$

$\rightarrow$ only defined on $D = \{(x, y) \mid x > 0\}$ or $D = \{(x, y) \mid x < 0\}$
Initial condition $y(0) = 5$. Since$f$ is not defined when $x = 0$, Theorem doesn't apply. No conclusion.

3. $f(x, y) = 2x\sqrt{y}$
$\frac{\partial f}{\partial y} = 2x\dfrac{1}{2\sqrt{y}} = \dfrac{x}{\sqrt{y}}$
— f, $\frac{\partial f}{\partial y}$ are continuous on $D = \{(x, y) \mid y > 0\}$
Initial condition $y(0) = 0$ (problem)
Theorem doesn't apply. No conclusion.

DEs in examples 2 and 3 both have the solution $y = 0, x \in \mathbb{R}$

Such constant solutions are important in applications.

Definition 2.1.3. An equilibrium solution of an ODE is a solution where two dependent variable is constant that is $y(x) = K$, $x \in \mathbb{R}$ for some specific $K \in \mathbb{R}$
For the first order ODE $\dfrac{dy}{dx} = f(x, y) \quad y = k$ is an equilibrium solution if $f(x, k) = 0 \quad \forall x \in \mathbb{R}$

2.2 Slope Fields / Direction Fields

Can interpret $\dfrac{dy}{dx} = f(x, y)$ geometrically.
For any given point (x, y) (where $f(x, y)$ is defined), the DE tells us the slope of the solution passing through (x, y) is $f(x, y)$. Can use a computer to do this for many points and get an idea of what the solution looks like qualitatively.

2.3 Solving First Order ODEs - Exactness

Recall the quasilinear form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad (1)$$

Suppose there is a function $F(x, y)$ such that $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$
Then (1) can be written

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad (2)$$

From implicit differentiation, this is equivalent to

$$\frac{d}{dx}[F(x, y(x))] = 0$$

This can be solved using FTC.

$$F(x, y(x)) = C$$

$\rightarrow$ defines $y(x)$ implicitly as a function of x

From last class, $\quad M(x, y) + N(x, y)\frac{dy}{dx} = 0 \quad (1)$

If $\exists F(x,y)$, s.t. $\frac{\partial F}{\partial x} = M$ and $\frac{\partial F}{\partial y} = N$ then

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \quad (2)$

$\implies \frac{d}{dx}[F(x,y(x))] = 0$

$\implies F(x,y(x)) = C \quad (3) \to$ Defines $y(x)$ implicitly

Definition 2.3.1. Any equation (1) which can be written in the from (2) for some function $F(x,y)$ is called an exact differential equation.

Example $\underbrace{\cos(x)y^3}_{M(x,y)} + \underbrace{3\sin(x)y^2}_{N(x,y)}\frac{dy}{dx} = 0$

This equation is exact $\quad F(x,y) = \sin(x)y^3 \quad$ $\frac{\partial F}{\partial x} = \cos(x)y^3 = M$, $\quad \frac{\partial F}{\partial y} = 3\sin(x)y^3 = N$

DE is equivalent to $\frac{d}{dx}[\sin(x)y^3(x)] = 0$

$$\implies \sin(x)y^3(x) = C \qquad \implies \text{defines solutions implicitly}$$

This can be verified by the following Theorem.

Theorem (Criteria for Exactness)
Suppose that the functions $M(x,y)$ and $N(x,y)$ are continuous and have continuous first order partial derivatives in the open rectangle R: $\{(x,y)|\ a < x < b,\ c < y < d\}$. Then the DE (1) is exact in R if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (4) at each point of R.

Idea of Proof Suppose (1) is exact $\implies \exists F$ s. t. $\quad M = \frac{\partial F}{\partial x},\quad N = \frac{\partial F}{\partial y}$, $\quad \frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x},\quad \frac{\partial N}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}$

By hypothesis, $\frac{\partial^2 F}{\partial y\partial x}, \frac{\partial^2 F}{\partial x\partial y}$ are continuous on R $\implies \frac{\partial^2 F}{\partial y\partial x} = \frac{\partial^2 F}{\partial x\partial y} \implies$ (4) holds
Suppose that (4) holds. Can use (4) to construct F. (see text chapter (6))

Examples Determine if the following DEs are exact.

Test $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies$ exact

1. $\underbrace{\cos(x)y}_{M} + \underbrace{3\sin(x)}_{N}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \cos(x) \neq \frac{\partial N}{\partial x} = 3\cos(x) \qquad \implies$ Not exact

2. $\underbrace{4x^3y}_{M} + \underbrace{x^4}_{N}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4x^3 \qquad \implies$ Exact

   By inspection, $F(x,y) = x^4y$
   DE: $\frac{d}{dx}[x^4y] = 0$

3. $\underbrace{x^2}_{M} + \underbrace{y^2}_{N}\frac{dy}{dx} = 0 \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 0 \qquad \implies$ Exact

Last example illustrates a whole set of exact DEs. Any DE of the form.

$M(x) + N(y)\frac{dy}{dx} = 0$ is exact where M, N and first partials are continuous

Such DE's are called separable. In normal/standard form they can be written

$$\frac{dy}{dx} = -\frac{M(x)}{N(y)} = g(x)h(y) = f(x, y)$$

where $N(y) \neq 0$

2.4 Solving Separable Equations

Example 1 $x\frac{dy}{dx} = y$ Find a general solution in explicit from.

Solution Rewrite DE $\frac{dy}{dx} = \frac{y}{x}$ $(x \neq 0)$ DE is separable.
Separate: $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$ $(x, y \neq 0)$
Integrate:
$\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{x}dx$
$\int \frac{1}{y}dy = \int \frac{1}{x}dx \implies \ln|y| = \ln|x| + C$

Take exponential of both sides:

$e^{\ln|y|} = e^{\ln|x|+C}$
$|y| = e^C|x|$

Depending on signs of x and y there are two cases:

$$y = e^C x \quad \text{or} \quad y = -e^C x$$

Collect into one formula: $y = \bar{c}x$ where $\bar{c} = \pm e^C$
This satisfies the DE for all $x \in \mathbb{R}$
Check for equilibrium (constant) solutions: $y = 0$ $x \in \mathbb{R}$ (This is included in the general formula if we allow $C = 0$)

Example Find a general solution of

$$\frac{dy}{dx} = 2x\sqrt{y}$$

Solution DE is separable.
Separate: $\frac{1}{2\sqrt{y}}\frac{dy}{dx} = x$ $(y \neq 0)$
Integrate

$$\int \frac{1}{2\sqrt{y}}dy = \int x dx$$

$$\sqrt{y} = \frac{x^2}{2} + C$$

RHS should be ≥ 0 since LHS is.
If $C < 0$, only defined for $x > \sqrt{-2C}$ or $x < -\sqrt{-2C}$
Simlify

$$y = (\frac{x^2}{2} + C)^2$$

$$C \geq 0, x \in \mathbb{R}$$

$$C < 0, x > \sqrt{-2C} \text{ or } x < -\sqrt{-2C}$$

Check for equilibnom solutions: Constants k such that $2 \times \sqrt{k} = 0, \forall x \implies k = 0$
Also have the solution $y = 0, x \in \mathbb{R}$

Exercise: Verify that when $C < 0$ the function $y = (\frac{x^2}{2} + C)^2$ only satisfies the DE if $x > \sqrt{-2C}$ or $x < -\sqrt{-2C}$

Example A motor boat is travelling in a straight line with speed 10m/s when the motor is turned off. The water provides a resistive force which is proportional to the square of the velocity and acts in the opposite direction. The mass of the boat and passengers is 1000kg. If after 1 second the speed is 9m/s, when will the speed be 2m/s?

Solution Physical law - Newton's second law
Variables:

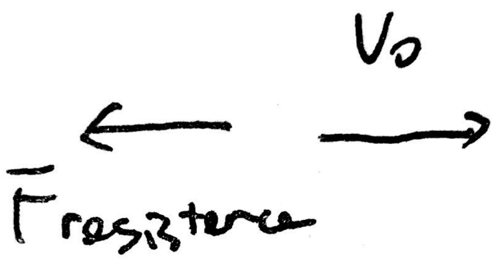
t - time (s)

v -velocity (m/s)

Constants: mass -1000 kg
Assumptions

- positive direction is direction of initial velocity
- $t = 0$ when the motor is turned off

Model: Newton's 2^{nd} Law:
$$m\frac{dv}{dt} = F_{\text{resistance}}$$
$$1000\frac{dv}{dt} = -kv^2$$



Note:

- minus means force acts in opposite direction of velocity
- $k > 0$ proportionally constant

Initial condition: $v(0) = 10$
Other condition: $v(1) = 9$
Solve the DE:

$$-\int v^2 dv = \int \frac{k}{1000} dt$$
$$\frac{1}{v} = \frac{kt}{1000} + C$$

Apply initial condition: $v(0) = 10$, $\frac{1}{10} = C$
Implicit solution of IVP $\frac{1}{v} = \frac{kt}{1000} + \frac{1}{10}$
Use $v(1) = 9$: $\frac{1}{9} = \frac{kt}{1000} + \frac{1}{10} \implies \frac{k}{1000} + \frac{1}{90}$
Put into solution $\frac{1}{v} = \frac{t}{90} + \frac{1}{10}$

Solve for v : $\quad v(t) = \frac{90}{t+9}, \qquad \underbrace{t \geq 0}_{\text{what makes sense for physical problem}}$

Let t^* be the time when $v = 2$m/s

$$v(t^*) = 2 \implies 2 = \frac{90}{t^*+9} \implies t^* = 36$$

The speed of the boat will be 2m/s 36 seconds after the motor is turned off.

2.5 Natural growth and decay

Many natural phenomena exhibit the following property

The time rate of change of some quantity is proportional to the quantity itself

Let $y(t)$ be the quantity, t be time, a model for this is $\dfrac{dy}{dt} = \alpha\ y$ (α - constant of proportionality)

Specific Examples

1. Let $p(t)$ be the size of a population

   t - time

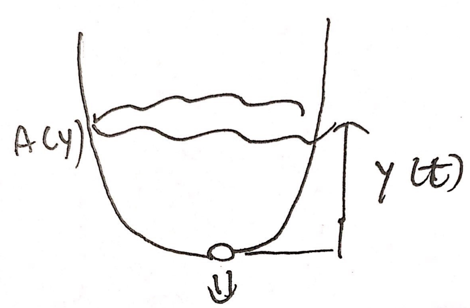
   $\frac{dP}{dt} = \alpha\ P$

   α positive or negative

2. Compound interest
   $A(t)$ - amount of money in investment
   t - time (year)
   Assuming - continuously compounded interest

   $$\frac{dA}{dt} = r\ A, \quad r > 0 \quad \text{- yearly interest rate}$$

3. Application Clepsydra (water clock)
   This is a device that measures time by the flow of water into or out of a container. We will develop a model for this.
   Consider a container with a hole at the bottom through which water flows.



Let $y(t)$ be the height of the water at time t and $V(t)$ the corresponding volume. It can only be shown that the speed of the water exiting the hole at time t is $\gamma\sqrt{2gy(t)}$ where

- g is gravitational acceleration
- $\sqrt{2gy}$ comes from Fluid Dynamics
- γ accounts for friction in the hole $\quad 0 < \gamma \leq 1$

Let a be the area of the hole at bottom.
By conservation of mass:

rate of change of volume in tank = rate water leaves tank

$$\frac{dV}{dt} = -a\gamma\sqrt{2gy}$$

or

$$\frac{dV}{dt} = -k\sqrt{y} \qquad (1) \qquad \text{Toricelli's Law}$$

Let $A(h)$ be the cross sectional area of the container a height h from the bottom.

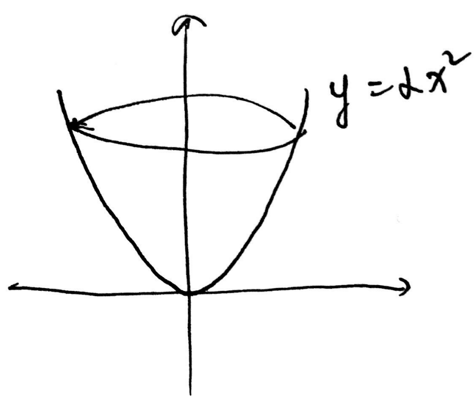
Then the volume of water is $\quad V(y) = \int\limits_0^y A(u)du$

$$\implies \frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}$$

So model (1) can be written

$$A(y)\frac{dy}{dt} = -k\sqrt{y} \qquad \rightarrow \text{ Nonlinear, separate DE for } y(t)$$

Example: Consider a container where the shape corresponds to the surface of revolution of a parabola, $y = \alpha x^2 (\alpha > 0)$ rotated around the y-axis. If the initial height is y_0, what time will it be empty?



$$A(y) = \pi x^2$$
$$= \frac{\pi}{2}y$$

The model becomes

$$\frac{\pi}{2}y\frac{dy}{dt} = -k\sqrt{y}, \qquad y(0) = y_0$$
$$y\frac{dy}{dt} = -\bar{k}\sqrt{y}, \qquad y(0) = y_0$$

Equilibrium solution $y(t) = 0, t \in \mathbb{R}$

Solve the DE

$$\sqrt{y}\frac{dy}{dt} = -\bar{k}$$
$$\int y^{1/2}dy = -\int \bar{k}dt$$
$$\frac{2}{3}y^{3/2} = -\bar{k}t + C$$

Apply IC: $\quad y(0) = y_0 \quad \Longrightarrow \quad \frac{2}{3}y_0^{3/2} = C$

Solution of IVP: $\quad \frac{2}{3}y^{3/2} = -\bar{k}t + \frac{2}{3}y_0^{3/2} \quad$ (implicit form)

$$y(t) = (y_0^{3/2} - \frac{2}{3}\bar{k}t)^{2/3}$$

Container will be empty when $y(t) = 0$

$$t = \frac{2}{3}\frac{y_0^{3/2}}{\bar{k}}$$

2.6 Integrating Factors

Recall some examples:

$\cos(x)y + 3\sin(x)\frac{dy}{dx} = 0 \quad \Longrightarrow \quad$ Not exact

$\cos(x)y^3 + 3y^2\sin(x)\frac{dy}{dx} = 0 \quad \Longrightarrow \quad$ Exact

Multiplying the first DE by y^2 makes it exact

The two DEs have the same general solution: $\quad y(x) = [\frac{C}{\sin(x)}]^{1/3}$

Suppose the DE $\quad M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad$ (1)

is not exact but $\quad v(x,y)M(x,y) + v(x,y)N(x,y)\frac{dy}{dx} = 0 \quad$ (2)

is exact. Then $v(x,y)$ is called an <u>integrating factor</u> for (1)

Note: Every solution of (1) is a solution of (2) <u>but</u> (2) may have solutions which are not solutions of (1). These are given by $v(x, y(x)) = 0$

There is no general method for finding $v(x,y)$.

Except in some special cases

(1) separable DEs

(2) DEs in the form: $\quad \underbrace{P(x)y}_{M} + \underbrace{1}_{N}\frac{dy}{dx} = 0$

$$P(x) \not\equiv 0$$

$\frac{\partial M}{\partial y} = P(x), \quad \frac{\partial N}{\partial x} = 0 \implies$ Not exact

Suppose $v(x)$ is an integrating factor.

Then

$$\underbrace{v(x)P(x)y}_{\bar{M}} + \underbrace{v(x)}_{\bar{N}}\frac{dy}{dx} = 0 \qquad \text{is exact}$$

$\Longrightarrow$

$$\frac{\partial \bar{M}}{\partial y} = \frac{\partial \bar{N}}{\partial x}$$

$v(x)P(x) = \frac{dv}{dx} \quad \Longrightarrow \quad$ A separable DE for $v(x)$

$$\int \frac{dv}{v} = \int P(x)dx$$

• • • $v(x) = Ke^{\int P(x)dx}, \quad K$ an arbitrary constant

usually we take $K = 1$

From last lecture $P(x)y + \dfrac{dy}{dx} = 0$ has integrating factor $e^{\int P(x)dx}$
We can find the solution of the DE as

$$e^{\int P(x)dx} P(x)y + e^{\int P(x)dx} P(x)\frac{dy}{dx} = 0$$
$$\frac{d}{dx}[e^{\int P(x)dx} y(x)] = 0$$

Implicit Differentiation

General Solution $e^{\int P(x)dx} P(x)y(x) = C$, C an arbitrary constant.

Recall our general first order ODE $H(x, y, \frac{dy}{dx}) = 0$

Definition A linear first order ODE is one where H is a linear function of y and $\frac{dy}{dx}$
Linear DE's can always can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (1)$$

Example Find the solution of $xy' + 2y = 4x^2$, $y(-1) = 2$

Solution Rewrite the DE in standard form

$$y' + \frac{2}{x}y = 4x, \quad x \neq 0$$

$P(x) = \dfrac{2}{x}$, Integrating factor

$$\begin{aligned} v(x) &= e^{\int \frac{2}{x} dx} \\ &= e^{2\ln|x|} \\ &= e^{\ln(x^2)} = x^2 \end{aligned}$$

Multiply through the DE by $v(x) = x^2$

$$x^2 y' + 2xy = 4x^3$$
$$\frac{d}{dx}[x^2 y(x)] = 4x^3$$

Product Rule + Implicit Differentiation

$$\int \frac{d}{dx}[x^2 y(x)]dx = \int 4x^3 dx$$
$$x^2 y(x) = x^4 + C$$

General Solution: $y(x) = x^2 + \dfrac{C}{x^2}$
Apply initial condition: $y(-1) = 2$

$$2 = 1 + \frac{C}{1} \implies C = 1$$

IVP has a unique solution: $y(x) = x^2 + \dfrac{1}{x^2}$, $x < 0$

Theorem (Existence and Uniqueness for Linear DEs)
Suppose the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing x_0. Then, for any y_0 the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0$$

Has a unique solution defined on I.

Proof Since $P(x), Q(x)$ are continuous on I, so are

$$\int P(x)dx, \quad v(x) = e^{\int P(x)dx}, \quad \int v(x)Q(x)dx$$

And $v(x) \neq 0$ on I
We showed general solution of the DE is

$$y(x) = \tfrac{C}{v(x)} + \tfrac{1}{v(x)} \int v(x)Q(x)dx$$

This is continuous on I, and satisfies DE.
For any $x_0 \in I$ rewrite this as

$$y(x) = \tfrac{\bar{C}}{v(x)} + \tfrac{1}{v(x)} \int\limits_{x_0}^{x} v(s)Q(s)ds$$

Apply the IC: $y(x_0) = y_0$

$$y_0 = \frac{\bar{C}}{v(x_0)} \implies \bar{C} = y_0 v(x_0)$$

Thus there is a unique value of C that satisfies the IC and hence a unique solution to the IVP. □

Applications Newton's Law of Cooling/Heating
The rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the ambient temperature (surroundings).

Model

T: temperature of object

t: time

$A(t)$: ambient temperature

$$\frac{dT}{dt} = K(A(t) - T) \qquad K > 0 \text{ constant}$$

2.6.1 Tutorial 2

To get 100% on assignment

1. Answer all questions.
2. Theory reasoning must be based on exact Definitions and Theorems (Title # 1 page source)

Thm 1 (Uniqueness + Existence) p.26 + addit.

chapter 1.4

 # 12 ODE $yy' = x(y^2+1)$

$$y\frac{dy}{dx} = x(y^2+1)$$

$$\int \frac{y}{y^2+1}dy = \int x\ dx$$

Substitute $u = y^2+1, \qquad du = 2y\ dy$

$$\implies \frac{1}{2}\int \frac{du}{u} = \int x\ dx$$

$$\ln|u| = x^2 + C$$

$$e^{\ln|u|} = e^{x^2+C}$$

$$u = e^{x^2}\cdot e^C$$

$$y^2+1 = Ce^{x^2}$$

Note that C is not the same in different equations...

$$y = \pm\sqrt{Ce^{x^2}-1}$$

However, we need $Ce^{x^2}-1 \geq 0$, so we must have restrictions on C. To make it easier to write, we have

$$y = \pm\sqrt{e^{x^2+C}-1}, \qquad C \geq 0$$

#21 $2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}}$
Immediately, we get $x \in (-\infty,-4)\cup(4,+\infty)$ Since we need $\exists f(x)$
This is IVP: ODE and IC $(y(5)=2)$
Solving it we get: $y^2 = \sqrt{x^2-16}+C$
Apply IC, we get: $C = 1$

$$y = \sqrt{\sqrt{x^2-16}+1}, \quad x > 4$$

 # 30 $(\frac{dy}{dx})^2 = 4y$
Taking square roots, $\frac{dy}{dx} = \pm 2y^{\frac{1}{2}}$
$y(x) = (x\pm C)^2,$ and we lose $y(x)\equiv 0$

Given (a,b)

(a) solution doesn't exist

(b) ∞ solutions (we can also have piecewise solution, but should be continuous)

(c) finite solutions

2.7 Application Cont'd

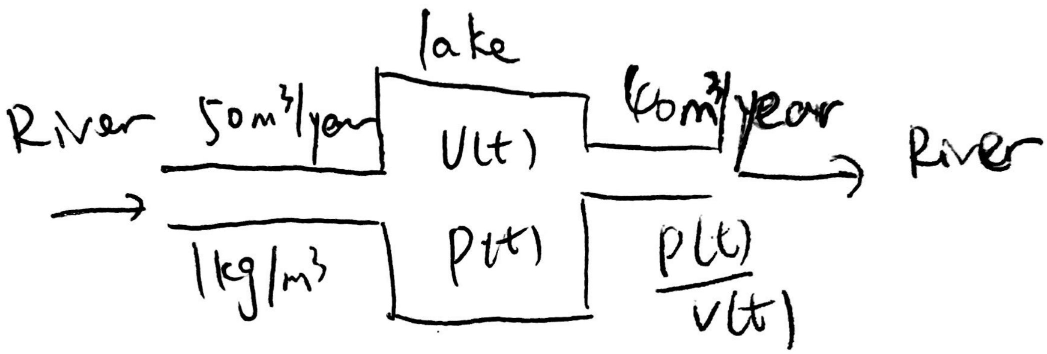
Application Mixture / compartment problems

Example A lake contains 10^4 m^3 of clean water when a factory starts dumping waste into a river that feeds the lake at 50 m^3/year. The concentration of pollution in the river is 1kg/m^3. Another river drains water from the lake at a rate 40 m^3/year.
Find the amount of pollution in the lake at any time after the dumping starts.

Model

- V: volume of the lake (m^3)
- p: mass of pollution in the lake (kg)
- t: time (years)

Physical principle - conservation of mass



Assume Lake is "well-mixed", so $p(t)$ same everywhere

$$\text{Rate of change of } V \text{ in time:} \quad \begin{array}{ccc} \textit{rate} & & \textit{rate} \\ \textit{water} & - & \textit{water} \\ \textit{in} & & \textit{out} \end{array}$$

$$\frac{dV}{dt} = 50 - 40, \qquad v(0) = 10^4 \tag{1}$$

$\Longrightarrow$ Easily solved: $\quad v(t) = 10t + 10^4$

$$\text{Rate of change of } p \text{ in time} = \begin{array}{ccc} \textit{rate} & & \textit{rate} \\ \textit{pollution} & - & \textit{pollution} \\ \textit{in} & & \textit{out} \end{array}$$

= (rate water in) (concentration pollutions in) $-$ (rate water out) (concentration pollution out)

$$\frac{dp}{dt} = 50 \cdot 1 - 40 \cdot \frac{p(t)}{V(t)}$$

Use the expression for $V(t)$, and rearrange

$$\frac{dp}{dt} + \frac{40}{10t + 10^4} p = 50 \qquad \text{(Linear DE for } p(t)\text{)}$$

Initial condition: $\quad p(0) = 0$ (no pollution at start)

Integrating factor: $\quad v(t) = e^{\int \frac{40}{10t+10^4} dt} = e^{\int \frac{4}{t+10^3} dt} = \ldots = (t + 10^3)^4$

Use $v(t)$ to solve DE:

$$(t + 10^3)^4 \frac{dp}{dt} + 4(t + 10^3)^3 p = 50(t + 10^3)^4$$

$$\frac{d}{dt}[(t + 10^3)^4 p] = 50(t + 10^3)^4$$

Integrate wrt t

$$(t + 10^3)^4 p = 10(t + 10^3)^5 + C$$

Apply IC, $p(0) = 0 \implies \cdots C = -10^{16}$
Put this into solution, and simplify, we get
Amount of pollution at time t:

$$p(t) = 10(t + 10^3) - \frac{10^{16}}{(t + 10^3)^4}$$

2.8 Solving DEs using a change of variables

Definition A homogeneous first order ODE is one that can be written in the form $\frac{dy}{dx} = F(\frac{y}{x})$ (1)
This kind of DE can be solved using the following change of variables.

$$\text{Let } v(x) = \frac{y(x)}{x}. \text{ Then } y = xv, \qquad \text{and } \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Putting this into (1)

$$v + x\frac{dv}{dx} = F(v)$$

$$\frac{dv}{dx} = \frac{F(v) - v}{x} \qquad \leftarrow \text{A separable DE}$$

Example $\frac{dy}{dx} = 2\frac{x}{y} + \frac{y}{x}$

Use the change of variables above: $y = xv$

$$v + x\frac{dv}{dx} = \frac{2}{v} + v$$

$$x\frac{dv}{dx} = \frac{2}{v}$$

Solve as a separable DE

$$\int v\ dv = \int \frac{2}{x} dx$$

$$\frac{v^2}{2} = 2\ln|x| + C = \ln\left(x^2\right) + C$$

Put $v(x) = \frac{y(x)}{x}$

$$\frac{1}{2} \cdot \frac{y^2}{x^2} = \ln\left(x^2\right) + C \implies y^2 = 2x^2\ln\left(x^2\right) + 2Cx^2$$

Definition A Bernoulli Equation is a first order ODE that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

If $n = 0$ or $n = 1$, then DE is linear.
If $n \neq 0, 1$ use the change of variables: $v(x) = [y(x)]^{1-n}$

$$\begin{aligned}
\frac{dv}{dx} &= (1-n)y^{-n}\frac{dy}{dx} \\
&= (1-n)y^{-n}[-P(x)y + Q(x)y^n] \\
&= -(1-n)y^{1-n}P(x) + (1-n)Q(x) \\
&= -(1-n)P(x)v + (1-n)Q(x)
\end{aligned}$$

This is a linear DE for $v(x)$

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$$

Example $\dfrac{dy}{dx} = 2\dfrac{x}{y} + \dfrac{y}{x}$
$\dfrac{dy}{dx} - \dfrac{1}{x}y = 2xy^{-1}$

Bernoulli DE with $n = -1$
Change of variables:

$$v = y^{1-(-1)} = y^2$$

$$\frac{dv}{dx} = 2y\frac{dy}{dx}$$

$$\begin{aligned}\frac{dv}{dx} &= 2y[\frac{1}{x} \cdot y + 2xy^{-1}] \\ &= \frac{2}{x}y^2 + 4x\end{aligned}$$

Then,

$$\frac{dv}{dx} - \frac{2}{x}v = 4x$$

Exercise: Solve and show you get the same answer as before.

Solution: We can find that integrating factor is $u(x) = \frac{1}{x^2}$
Then we get

$$\frac{d}{dx}[\frac{1}{x^2}v(x)] = \frac{4}{x}$$

$$v(x) = 4\ln(x)x^2 + Cx^2$$

Since $v(x) = y^2$, thus

$$y^2 = 4x^2\ln(x) + Cx^2$$

2.9 Second order ODEs reducible to first order

Most general second order ODE:

$$H(x, y, y', y'') = 0 \qquad (1)$$

If the dependent variable is missing in (1) we have

$$H(x, y', y'') = 0 \qquad (2)$$

Can solve as follow:

$$\text{Let } y' = v \qquad (3)$$
$$\text{then } y'' = v' \qquad (4)$$

Putting these into (2) gives

$$H(x, v, v') = 0 \qquad (5) \rightarrow \text{ first order ODE for} v$$

Can solve (5) for v and (3) for y

Example $xy'' = y'$

Note: $'$ means $\frac{d}{dx}$

Let $v = y', \quad v' = y''$

$$xv' = v$$
$$x\frac{dv}{dx} = v$$

We have solved this before: $v(x) = C_1 x$
Now solve for y

$$y' = v$$
$$\frac{dy}{dx} = C_1 x$$

Just integrate wrt x $\quad y(x) = C_1\frac{x^2}{2} + C_2, \quad x \in \mathbb{R}$ (two arbitrary constants)

If the independent variable is missing from (1) we have

$$H(y, y', y'') = 0 \qquad (6)$$

This may be solve via the change of varibles

$$y' = v(y) \qquad (7)$$
$$\frac{dy}{dx} = v(y(x))$$

Then

$$y'' = \frac{d^2y}{dx^2} = \frac{dv}{dy}\cdot\frac{dy}{dx} = \frac{dv}{dy}v \qquad (8)$$

Putting these into (6) gives

$$H(y, v, v\frac{dv}{dy}) = 0 \qquad (9)$$

First order ODE for $v(y)$

Solve (9) for $v(y)$ and (7) for $y(x)$

Example $yy'' = (y')^2$
Let $v(y) = y', \quad v\frac{dv}{dy} = y''$

$$yv\frac{dv}{dy} = v^2$$

$\rightarrow$ separable DE for $v(y)$

Note this has solution $v = 0$,
For $v \neq 0$ $\quad y\frac{dv}{dy} = v$

Solve this before: $v(y) = C_1 y$

...(exercise) $\implies y(x) = C_2 e^{C_1 x}$

3 Mathematical Models + Numerical Methods

3.1 Population Models

Let $P(t)$ be the number of individuals in a population as a function of time (t).

Basis model: $\dfrac{dP}{dt}$ = birth rate − death rate

Simplest assumption: birth rate and death rates are proportional to the population size.

$$\frac{dP}{dt} = \beta P - \delta P \qquad\qquad \beta,\ \delta > 0$$

$$\frac{dP}{dt} = (\beta - \delta)P \qquad \rightarrow \text{exponential growth or decay depending on whether } \beta > \delta \text{ or } \beta < \delta$$

More complicated model birth rate decreases with population size due to lack of resources

$$\frac{dP}{dt} = (\underbrace{\beta_0 - \beta_1 P}_{\beta} - \delta)P \qquad\qquad \beta_0,\ \beta_1,\ \delta > 0$$

Rewrite:

$$\frac{dP}{dt} = (\beta_0 - \delta)P - \beta_1 P^2$$

If $(\beta_0 - \delta) > 0$ this is called the Logistic Model and is usually written

$$\frac{dP}{dt} = kP(M - P) \qquad\qquad k,\ M > 0$$

Note:

(1) if you apply the Existence and Uniqueness Theorem to this DE → guarantee a unique solution for any IC $P(t_0) = P_0$

(2) The DE has two equilibrium solutions

- $P(t) = 0, \quad t \in \mathbb{R}$
- $P(t) = M, \quad t \in \mathbb{R}$

Solve the DE as a separable DE (it is also a Bernoulli DE)

$$\int \frac{dP}{P(M-P)} = \int k\ dt \qquad (P \neq 0,\ P \neq M)$$

Using Partial Fractions:

$$\frac{P_{(t)}}{M - P_{(t)}} = Ce^{kMt}$$

(C arbitrary constant)

Initial conditions: $P(0) = P_0$ $\implies C = \frac{P_0}{M-P_0}$
Putting this into the solution and rearranging (exercise)

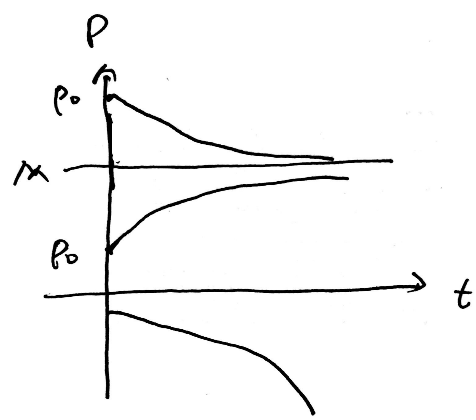
$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}}$$

$$\implies \begin{array}{ccc} P_0 = 0 & \to & P(t) = 0, t \in \mathbb{R} \\ P_0 = M & \to & P(t) = M, t \in \mathbb{R} \end{array}$$

Limiting behaviours:

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} = M$$

if $P_0 > 0$



$$\begin{cases} \frac{dP}{dt} > 0 & \text{if } 0 < P < M \\ \frac{dP}{dt} < 0 & \text{if } P > M \\ \frac{dP}{dt} < 0 & \text{if } P < 0 \end{cases}$$

Interpretation If $P_0 > 0$, then the population tends toward M as $t \to \infty$.

M is called the carrying capacity of the population

3.1.1 Equilibrium solutions and stability

Focus on DE's of the form $y' = f(y)$ (autonomous DEs)

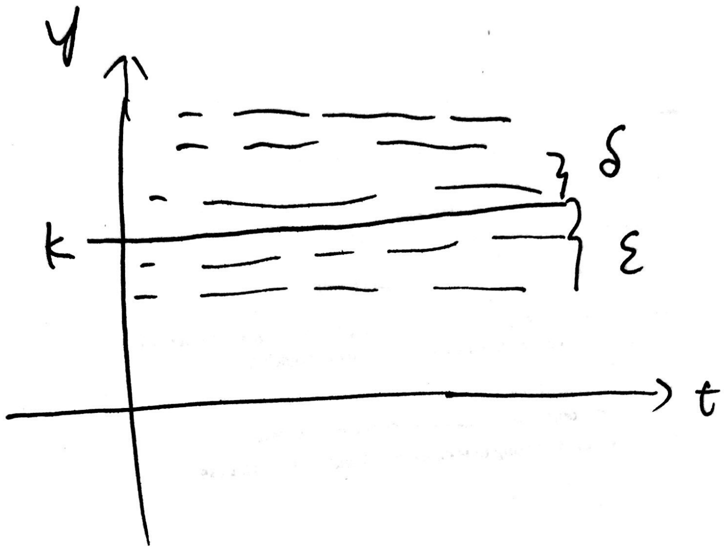
Equilibrium points: $y(t) = K, t \in \mathbb{R}$ $f(k) = 0$

Assume eg(1) with IC $y(0) = y_0$ has a unique solution.

Definition An equilibrium solution of (1) is stable if for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$|y_0 - k| < \delta \implies |y(t) - k| < \epsilon \quad \forall t \geq 0$$

otherwise it is unstable.



An equilibrium solution of (1) is asymptotically stable if it is stable and there is $\eta > 0$ such that

$$|y_0 - k| < \eta \implies \lim_{t\to\infty} y(t) = k$$

In logistic models:

- $P(t) = M$ is asymptotically stable
- $P(t) = 0$ is unstable

3.1.2 Logistic Growth with Harvesting

Example (Logistic Growth with Harvesting)
Consider a population of fish in a lake that obeys the logistic model. Suppose we wish to allow fishing at a constant rate h. How should we choose h so that the population doesn't go extinct?

Model

- $P(t)$ - population at time t
- k, M - as in logistic model
- $t = 0$ - when fishing start
- P_0 - population when fishing starts

$$\frac{dP}{dt} = \underbrace{kP(M-P)}_{\text{logistic growth}} \quad \underbrace{-\ h}_{\text{loss due to fishing}}$$

Equilibrium solutions

$$kP(M-P)-h=0$$

$$-k[P^2-MP+\frac{h}{k}]=0$$

$$P=\frac{M\pm\sqrt{M^2-4\frac{h}{k}}}{2}=M_\pm$$

Three cases:

- $M^2-4h/k>0$: two equilibrium solutions: $0<M_-<M_+<M$
- $M^2-4h/k=0$: one equilibrium solution: $P(t)=\frac{M}{2}$
- $M^2-4h/k<0$: no equilibrium solution

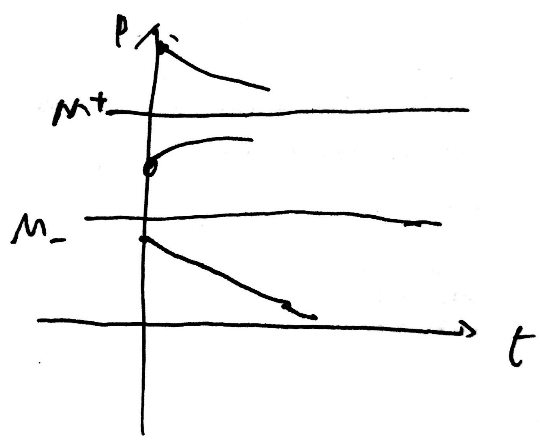
Case (1) Rewrite the model

$$\frac{dP}{dt}=k(P-M_-)(M_+-P),\quad P(0)=P_0$$

From the DE we can see:

$$\begin{cases}\frac{dP}{dt}<0 & \text{if } P<M_-\\ \frac{dP}{dt}>0 & \text{if } M_-<P<M_+\\ \frac{dP}{dt}<0 & \text{if } P>M_+\end{cases}$$

Can predict what solutions look like



Prediction: M_+ is (asymptotically) stable
M_- is unstable

Can show that the solution of the IVP is

$$P(t)=\frac{M_+(P_0-M_-)-M_-(P_0-M_+)e^{-k(M_+-M_-)t}}{(P_0-M_-)-(P_0-M_+)e^{-k(M_+-M_-)t}}$$

$$\lim_{t\to\infty}P(t)=M_+\qquad \text{if } P_0>M_-$$

Can show solutions look as we predicted

Interpretation

- $0 < P_0 < M_-$: population $\to 0$ in finite time
- $M_- < P_0 < M_+$: population increases to M_+
- $M_+ < P_0$: population decreases to M_+

To ensure population doesn't go extinct, choose h so that

- $M^2 > \frac{h}{k}$
- $M_- \leq P_0$

3.1.3 Tutorial 3

Step 1 Standard form: $y' + P(x)y = Q(x)$

Step 2 $\rho(x) = e^{\int P(x)dx}$

Step 3 $\rho(x)y(x) = \int \rho Q + C$

In Section 1.5 DE

#7 $2xy' + y = 10\sqrt{x}$ $\quad D : x \geq 0$
We are going to divide both sides by $2x, (x \neq 0)$
Check $y \equiv 0$. $y(0) = 0 \implies y \equiv 0$

$$y' + \underbrace{\frac{1}{2x}}_{P(x)} y = \underbrace{\frac{5}{\sqrt{x}}}_{Q(x)}$$

Find the integrating factor, $\rho(x) = e^{\int \frac{1}{2x} dx} = \sqrt{x}$

$$\sqrt{x}\, y(x) = 5x + C$$

$$\begin{cases} y(x) = 5\sqrt{x} + \frac{C}{\sqrt{x}}, & x_0 > 0 \\ y(x) \equiv 0, & x_0 = 0 \end{cases}$$

#? $y' + 2xy = x \quad y(0) = 2$
$\rho(x) = \ldots = e^{x^2}$
$\vdots$ done with some substitution in integral
$y(x) = \frac{1}{2} - Ce^{-x^2} \overset{\text{Apply IC}}{\Longrightarrow} y(x) = \frac{1}{2} - \frac{5}{2}e^{-x^2}$

Word Problem
Words $\overset{\text{translate}}{\Longrightarrow}$ Math $\overset{\text{translate}}{\Longrightarrow}$ Words (answer in words)

- denote all variables and parameters, units
- DE

#36

- t - time, min, ...
- $x(t)$ - amount of salt in tank at time t, 1b
- V_0 - amount of water in tank at time t

$0 \le t \le 60\,\text{min}$

$$\frac{dx}{dt} = r_i c_i - r_0 c_0 \quad F(x,y)$$

$$x' + \frac{3}{60-t}x = 2$$

$$x(t) = (60-t) + C(60-t)^3 \overset{x(0)=0}{\Longrightarrow} C = -\frac{1}{3600}$$

Find $x' = 0, x'' < 0$, and check endpoints

Newton's Law of Cooling

$$\frac{dT}{dt} = k(A - T) \qquad T(0) = T_0$$

$$A(t) = 25 - 20e^{0.1t}$$

$$\frac{dT}{dt} + kT = (25 - 20e^{0.1t})k$$

$$\frac{d}{dt}[e^{kt}T] = 25ke^{kt} - 20k^{(k-0.1)t}$$

(1) $k = 0.1$

$$T(t) = 25 - 20kte^{-kt}$$

(2) $k \neq 0.1$

$$T(t) = 25 - \frac{20}{k-0.1}(e^{kt} - e^{-0.1t})$$

3.2 Position and Velocity Problems

Air resistance A force that resists the motion of an object.

- Magnitude - proportional to a power of the speed
- Direction - opposite to the direction of the motion

If v is the velocity of an object

$$F_{\text{AR}} = \begin{cases} -k|v|^p & \text{if } v > 0 \\ k|v|^p & \text{if } v < 0 \end{cases}$$

where $k > 0$, $1 \le p \le 2$. k, p depends on the object.

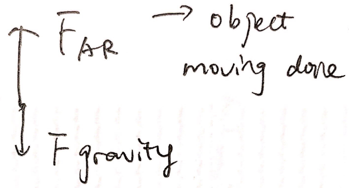
$$F_{\text{AR}} = -kv|v|^{p-1}$$

$p = 1 \quad F_{\text{AR}} = -kv$

$p = 2 \quad F_{\text{AR}} = -kv|v|$

Example An object is thrown downward form height y_0 with speed v_0. The object encounters air resistance proportional to the square of the speed. Find the velocity of the object.

Solution Diagram of forces



Assumptions

- F_{gravity} is constant
- Up is the positive direction

Model

- v - velocity
- t - time
- m - mass of object
- g - gravitational acceleration
- $k > 0$ - coefficient of A.R

Applying Newton's 2nd Law $m\frac{dv}{dt} = F_{\text{gravity}} + F_{\text{AR}}$

$$m\frac{dv}{dt} = -mg - kv|v|$$

$$v(0) = -v_0$$

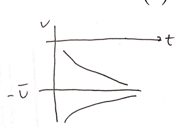
Since object is falling, $v < 0 \implies -v|v| = v^2$

$$m\frac{dv}{dt} = -mg + kv^2$$

Equilibrium solutions: $-g + \dfrac{k}{m}v^2 = 0 \implies v = \pm\sqrt{\dfrac{mg}{k}} = \pm\overline{v}$

Rewrite the DE: $\frac{dv}{dt} = \frac{k}{m}(v^2 - \overline{v}^2) = \frac{k}{m}(v - \overline{v})(v + \overline{v})$

Relevant equilibrium solution is $v(t) = -\overline{v}, t \in \mathbb{R}$



$$\begin{cases} \frac{dv}{dt} > 0 & \text{if } v < -\overline{v} \\ \frac{dv}{dt} < 0 & \text{if } v > -\overline{v} \end{cases}$$

Predict: $v(t) = -\overline{v}$ is asymptotically stable

Solving the DE using partial fractions

$$v(t) = \overline{v} \cdot \frac{\left[1 + Ce^{\frac{2k}{m}\overline{v}t}\right]}{\left[1 - Ce^{\frac{2k}{m}\overline{v}t}\right]}$$

where C is arbitrary

Apply IC: $v(0) = -v_0$ $\implies$ $C = \frac{v_0 + \overline{v}}{v_0 - \overline{v}}$

Solution for velocity:

$$v(t) = \overline{v} \cdot \frac{\left[v_0 - \overline{v} + (v_0 + \overline{v})e^{\frac{2k}{m}\overline{v}t}\right]}{\left[v_0 - \overline{v} - (v_0 + \overline{v})e^{\frac{2k}{m}\overline{v}t}\right]}$$

$$\lim_{t\to\infty} = -\overline{v} = -\sqrt{\frac{mg}{k}}$$

After a long time the velocity is effectively constant the value $-\sqrt{\frac{mg}{k}}$ is called the terminal velocity

3.2.1 Newton's Law of Gravitation

In previous examples, we used $F_{\text{gravity}} = mg$. This is an approximation of a more general law.

Two objects of masses m_1 and m_2, separated by distance r. For motion in one dimension:

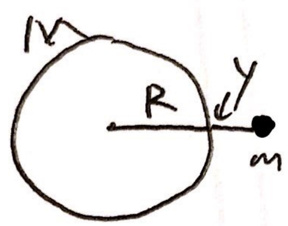
$$|F_{\text{gravity}}| = G\frac{m_1 m_2}{r^2}$$

where $G = 6.674 \times 10^{-11}\,\frac{\text{Nm}^2}{\text{kg}^2}$

For an object of mass m, a distance y above the surface of the earth

$$|F_{\text{gravity}}| = G\frac{m_1 m_2}{(R+y)^2}$$

where M = mass of earth = 5.972×10^{24}kg $\quad R$ = radius of earth = 6.371×10^6m



If y is small

$$G\frac{Mm}{(R+y)^2} \approx G\frac{Mm}{R^2} = m\underbrace{\frac{Gm}{R^2}}_{g}$$

Example A projectile is fixed vertically upward with velocity v_0 from the surface of the earth. Assuming gravity is the only force acting, but taking into account the variation of this force with distance, find the velocity.

Solution Diagram of forces $\downarrow F_{\text{gravity}}$

Assume up is positive direction

- v - velocity of object
- t - time
- y - position of object above surface of earth
- v_0 - initial velocity
- m - mass of object
- M - mass of earth
- R - radius of earth
- G - universal gravitational constant

Apply Newton's 2$^{\text{nd}}$ Law: $m\frac{dv}{dt} = F_{\text{gravity}}$

$$m\frac{dv}{dt} = -\frac{GMm}{(R+y)^2}$$

Initial conditions: $v(0) = v_0 \qquad y(0) = 0$
DE: $\frac{dv}{dt} = -\frac{GM}{(R+y)^2}$
Let $r(t) = R + y(t), \qquad \frac{dv}{dt} = \frac{d^2y}{dt^2} = \frac{d^2r}{dt^2}$

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

RHS doesn't depend on t: Think of v as a function of r
Let $v = \frac{dr}{dt}, \qquad \frac{d^2r}{dt^2} = \frac{dv}{dr} \cdot \frac{dr}{dt} = v\frac{dv}{dr}$

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$

Separable DE: $\frac{1}{2}v^2 = \frac{GM}{r} + C$
Initial condition: $t = 0 : v = v_0, y = 0 \quad \implies r = R \qquad v_{(r=R)} = v_0$

$$\frac{1}{2}v_0^2 = \frac{GM}{R} + C \implies C = \frac{1}{2}v_0^2 - \frac{GM}{R}$$

$$v^2(r) = v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R})$$

We take positive sign as object moving upward

$$v(r) = \sqrt{v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R})}$$

What is the interval of existence? Need $v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R}) > 0$
Two possibilities:

(1) $v_0^2 - \frac{2GM}{R} < 0$

$r < \frac{R}{\left(1-\frac{Rv_0^2}{2GM}\right)} = r_{\max} \implies v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R}) > 0$
Mathematically, the solution is defined for $0 < r < r_{\max}$
Physically, the solution is defined for $R \leq r < r_{\max}$

(2) $v_0^2 - \frac{2GM}{R} \geq 0$. Then $v_0^2 + 2GM(\frac{1}{r} - \frac{1}{R}) > 0$ for $r > 0$
Mathematically, the solution is defined for $0 < r$
Physically, the solution is defined for $R \leq r$

Interpretation

(1) $v(r) \to 0$ as $r \to r_{\max}$
Velocity goes to zero at maximum height
TO continue the solution, think about using solution with $-\sqrt{\cdots}$

(2) $v(r) > 0$ no matter how big r is.
Observe "escapes" from the earth's gravity. The minimum initial velocity needed to achieve this is
$v_0 = \sqrt{\frac{2GM}{R}}$ ——— called the escape velocity

In either case, we can use the solution to define IVP for the position

$$\frac{dr}{dt} = \underbrace{v(r)}_{\text{solution found above}}, \quad r(0) = R$$

3.3 Numerical Approximation of Solutions of DEs

So far - only considered DE's where we can write the solution in terms of elementary functions. (powers, trig functions, exp)
This is not always the case.

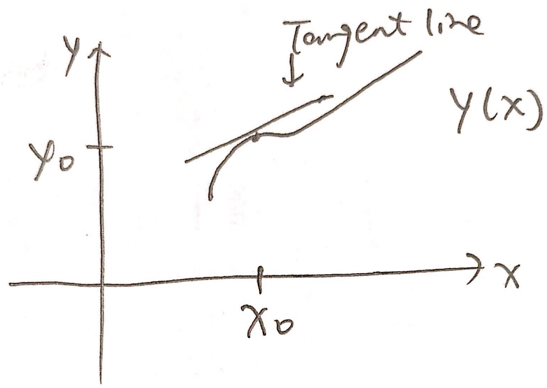
Example $\frac{dy}{dx} = e^{-x^2}$ - Linear, separable.
Has an unique solution for any IC $y(x_0) = y_0$. Can't express solution in terms of elementary function

$$y(x) = y_0 + \int_{x_0}^{x} e^{-t^2}\, dt$$

More generally, suppose we want to approximate the solution of

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0 \qquad (1)$$

Recall The tangent line (linear) approximation for $y(x)$ at the point (x_0, y_0)



$$y(x) \approx y(x_0) + y'(x_0)(x - x_0)$$

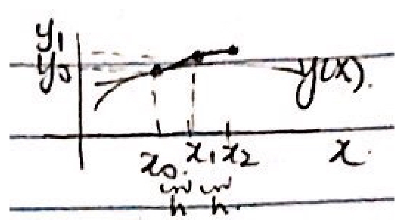
good approximation if x is close to x_0
Here $y(x)$ is unknown but $y'(x_0)$ is known from the DE.

Euler's Idea Use the tangent line approximation iteratively to approximate the solution of (1) at some particular values of x

From last class: tangent line approximation to $y(x)$ at (x_0, y_0)

$$y(x) \approx y(x_0) + y'(x_0)(x - x_0)$$

Euler's Method Use this to approximate solution of



$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0$$

Look at solution at set of values for $x : x_n = x_0 + nh, \quad n = 1, 2, \ldots \quad h$ is the step size.
For $x_0 \le x \le x_1$, use the tangent line approximation at (x_0, y_0)

$$y(x) \approx y_0 + f(x_0, y_0)(x - x_0)$$

At $x = x_1$, $\quad y(x_1) \approx y_0 + f(x_0, y_0)(x_1 - x_0) = y_0 + hf(x_0, y_0) = y_1$
Use this to generate new approximation for $x_1 \leq x \leq x_2$

$$y_1(x) \approx y_1 + f(x_1, y_1)(x - x_1)$$

Summary Euler Method Algorithm

1. (Discretization) Choose step size h. Let $x_n = x_0 + nh, \quad n = 0, 1, \ldots$ (grid points)
2. (Approximation) Approximate $y(t)$ at grid points
$$y(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n)$$
3. (Interpolation) Approximate $y(x)$ between grid points
$$y(x) \approx y_n + (x - x_n)f(x_n, y_n), \quad n = 0, 1, \ldots \qquad x_n \leq x \leq x_{n+1}$$

Example $\frac{dy}{dx} = x + \frac{1}{5}y, \quad y(0) = -3$
Find approximation of solution using Euler Method for $0 \leq x \leq 5$

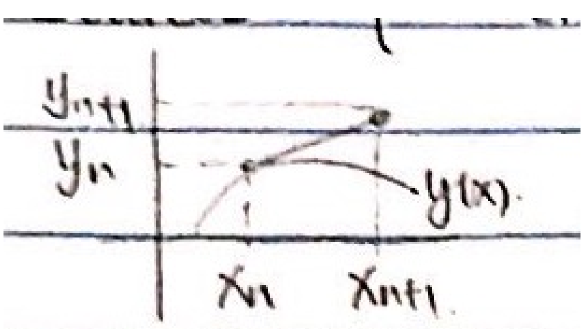
Solution Setup :

$$x_0 = 0, y_0 = -3$$
$$x_n = x_0 + nh = nh, \qquad n = 0, 1, \ldots$$
$$\begin{aligned} y_{n+1} &= y_n + hf(x_n, y_n) \\ &= y_n + h[x_n + \frac{1}{5}y_n] \end{aligned}$$

Take $h = 1, \qquad x_n = n, \quad n = 0, 1, 2, 3, 4, 5$

$$\begin{aligned}
x_0 = 0, \qquad & y_0 = -3 \\
x_1 = 1, \qquad & y_1 = y_0 + [x_0 + \frac{1}{5}y_0] \\
& = -3 + [0 - \frac{3}{5}] \\
& = -3.6 \\
x_2 = 2, \qquad & y_2 = y_1 + [x_1 + \frac{1}{5}y_1] \\
& = -3.6 + [0 - \frac{3.6}{5}] \\
& = -3.32
\end{aligned}$$

3.3.1 Sources of error in Euler's Method



$$y(x_{n+1}) \approx y_n + hf(x_n, y_n)$$

1 Error due to direct approximation

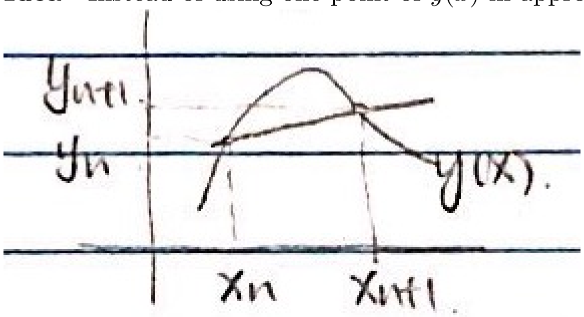
2 Error due to the fact that $y(x_n) = y_n$

Can show (see AMATH 332) the error in solving $y' = f(x, y); y(x_n) = y_n$, for $a \leq x \leq b$ satisfies $|y_n - y(x_n)| \leq ch$ for all n.

c depend on f, and $[a, b)$

One way to reduce error: make h smaller.
Another way: improve the approximation step.

Idea Instead of using one point of $y(x)$ in approximation use two points.



$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

Problem: y_{n+1} on both sides of equation

Solution approximate y_{n+1} on RHS using Euler's Method

3.3.2 Improved Euler Method

1. Discretization: As for Euler Method
2. Approximation:
   $y(x_{n+1}) \approx y_{n+1}$ where

$$k_1 = f(x_n, y_n)$$
$$u_{n+1} = y_n + hk_1$$
$$k_2 = f(x_{n+1}, u_{n+1})$$
$$y_{n+1} = y_n + \frac{h}{2}[k_1 + k_2]$$

   textbook P199

Can show error in Improved Euler satisfies

$$|y(x_n) - x_n| \leq \hat{c}h^2$$

$\hat{c}$ depend on f, and interval where solution approximates

3.3.3 Tutorial 4

1. IVP $\neq$ DE

$$\frac{dP}{dt} = kP(M - P), P(0) = P_0$$

Method 1 Separable

$$\int \frac{dP}{M(M-P)} = k\int dt$$
$$\frac{1}{M}\left(\int \frac{dP}{P} + \int \frac{dP}{M-P}\right) = k\int dt$$
$$\frac{1}{M}(\ln|P| - \ln|M-P|) = kt + C_0$$
$$e^{\ln|\frac{P}{M-P}|} = e^{kMt+MC_0}$$
$$\frac{P}{M-P} = e^{kMt}C \qquad C = \pm e^{MC_0}$$
$$\implies \text{general solution (with IC) } P(0) = P_0$$
$$\implies \boxed{P(t) = \frac{MP_0}{P_0 + (M-P)e^{-kMT}}} \quad \text{solution for IVP}$$

2. $\frac{dP}{dt} + k\overbrace{P^2}^{N} = kMP$
   New function:

$$v(t) = P^{1-N} = P^{-1}$$
$$\frac{dv}{dt} = -\frac{1}{P^2}\cdot\frac{dP}{dt} = -kMP^{-1} + k$$
$$\frac{dv}{dt} = -kMv + k$$
$$\mu(t) = e^{kMt}$$
$$\frac{d}{dt}\left[e^{kMt}v\right] = ke^{kMt}$$
$$v(t) = \frac{1}{M} + Ce^{-kMt}$$

3. $\frac{dx}{dt} = (x-2)^2$ §2.2 # 7, figure 2.2.9

4. $\frac{dv}{dt} = -g(1+\frac{\rho}{g})v^2 \qquad y(0) = y_0$ §2.3 #14

$$\frac{dv}{dt} = -g - \rho v^2$$

   Integrate to obtain

$$y(t) = y_0 + \frac{1}{\rho}\ln\left|\frac{\cos(C - \sqrt{\rho g})}{\cos C}\right|$$

5. Exact DE

$$\vec{D}F(x,y) = F_x dx + F_y dy$$

   $\boxed{\vec{D}F(x,y) = 0}$ Solution $\implies F(x,y) = C$
   check $F_x dx + F_y dy = 0 \iff F_{xy} = F_{yx} \iff N_y = M_x \implies \exists F(x,y)$

$$F(x,y) = \int F_x dx + f(y)$$

   To find $f(y)$, use $F_y = M$

3.4 (Advanced Topic)

Proof of Existence and Uniqueness Theorem: Text Reference - Appendix A

Existence and Uniqueness Theorem
Suppose $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on a rectangle $R = \{(x, y)|a < x < b, c < y < d\}$ such that $(x_0, y_0) \in \mathbb{R}$.

Then there is an open interval I with $x_0 \in I \subset (a, b)$ such that

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0 \quad (1)$$

has a unique solution defined on I.

Lemma Let $f(x, y)$ be continuous on R as in Existence and Uniqueness Theorem and I be an open interval containing x_0. $y(x)$ is a solution of (1) if and only if $y(x)$ is an continuous function satisfying

$$y(x) = y_0 + \int_x^{x_0} f(t, y(t))dt \qquad (2)$$

Proof
(1) $\Longrightarrow$ (2) Since $y(x)$ is a solution of (1) on I, it is continuous on I, hence $f(x, y(x))$ is a continuous function of x on I. Then by FTC the function $F(x) = \displaystyle\int_{x_0}^{x} f(t, y(t))dt$ satisfies $\frac{dF}{dx} = f(x, y(x))$

Thus F and y are both antiderivative of $f(x, y(x))$, so they must differ at most by a constant:

$$y(x) = F(x) + C$$

Using $y(x_0) = y_0$, and $F(x_0) = 0$ we have $C = y_0$.

$$y(x) = \int_x^{x_0} f(t, y(t))dt + y_0$$

(2) $\Longrightarrow$ (1) Suppose $y(x)$ is a continuous function satisfying (2) on I. Then by FTC we have

$$(a) \qquad \frac{dy}{dx} = f(x, y(x))\text{on } I$$

Also

$$\begin{aligned}(b) \qquad y(x_0) &= y_0 + \int_{x_0}^{x_0} f(t, y(t))dt \\ &= y_0\end{aligned}$$

These two (a) (b) implies $y(x)$ satisfies (1) on I. □

3.4.1 Method of Proof of Existence and Uniqueness Theorem due to Émile Picard (1856-1941)

1. Find a sequence of functions $\{y_n(x)\}_{n=0}^{\infty}$ that approximate a solution of (2)
2. Show the sequence converges
3. Show the limit of the sequence satisfies eq.(2)
4. Show this is the only solution of (2)

Outline of steps (1) - (3)

1. Use

$$y_0(x) = y_0, x \in (a, b)$$
$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt$$
$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))dt$$

In general

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt$$

This is called Picard Iteration or Method of Successive Approximations
Can show that all the $y_n(x)$ are continuous on some $I_0 \subset (a, b)$

2. Show that there is an interval $I_1 \subset I_0$ with $x_0 \in I_1$ such that $\lim_{n\to\infty} y_n(x) = \phi(x)$ uniquely on I_1
That is, given $\varepsilon > 0$ there is $N > 0$ such that

$$n > N \implies |y_n(x) - \phi(x)| < \varepsilon \qquad \forall x \in I_1$$

3. Show that there exists $I \subset I_1$ with $x_0 \in I$ such that

$$\int_{x_0}^{x} f(t, y_n(t))dt \qquad \text{converges uniformly to } \int_{x_0}^{x} f(t, \phi(t))dt \text{ on } I$$

Then we have

$$\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f(t, y_{n-1}(x))dt$$
$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(x))dt$$

Thus $\phi(x)$ is a solution of (1) by the Lemma.

Example $\frac{dy}{dx} = y \cos x, \quad y(0) = y_0$
$f(x, y) = y \cos x \quad f, f_y$ are continuous on $\mathbb{R}^2$
We will work through the steps of the proof for this problem

1. Method of Successive Approximations

$$y_n(x) = y_0 + \int_0^x y_{n-1}(t) \cos t \ dt$$

$$y_0(x) = y_0, x \in \mathbb{R}$$
$$y_1(x) = y_0 + \int_0^x y_0 \cos t \ dt = y_0(1 + \sin x), x \in \mathbb{R}$$
$$y_2(x) = y_0 + \int_0^x y_1 \cos t \ dt = y_0 + \int_0^x y_0(1 + \sin t) \cos t \ dt$$
$$y_2(x) = y_0(1 + \sin x + \frac{1}{2} \sin^2 x)$$

Can show by induction

$$y_n(x) = y_0 \sum_{k=0}^{n} \frac{\sin^k x}{k!}$$

2. Note that $\sum\limits_{k=0}^{n} \frac{\sin^k x_0}{k!}$ is the n^{th} partial sum of the series

$$\sum_{k=0}^{\infty} \frac{\sin^k x}{k!}$$

The terms in the series satisfy

$$\left|\frac{\sin^k x}{k!}\right| \leq \frac{1}{k!} \qquad \forall x \in \mathbb{R}$$

and $\sum_{k=0}^{\infty} \frac{1}{k!}$ is a convergent series of real numbers. It follows (by the Comparison Test) the series $\sum_{k=0}^{\infty} \frac{\sin^k x}{k!}$ converges uniformly $\implies \{y_n(x)\}_{n=0}^{\infty}$

3. Let $\lim\limits_{n\to\infty} y_n(x) = \phi(x)$

Show that $\phi(x)$ is a solution of the DE

- Show that $\displaystyle\int_0^x y_n(x) \cos t \; dt$ converges uniformly

Let $\alpha > 0$. Given $\varepsilon > 0$, there is $N > 0$ such that

$$n > N \implies |y_n(x) - \phi(x)| < \frac{\varepsilon}{\alpha}, \quad \forall x \in \mathbb{R}$$

Let $I = (-\alpha, \alpha)$ and assume $x \in I, x > 0, n > N$
Then

$$\begin{aligned}
\left|\int_0^x y_n(t) \cos t \; dt - \int_0^x \phi(t) \cos t \; dt\right| &= \left|\int_0^x [y_n(t) - \phi(t)] \cos t \; dt\right| \\
&\leq \int_0^x |y_n(t) - \phi(t)| \cdot |\cos t| dt \\
&< \int_0^x \frac{\varepsilon}{\alpha} \cdot 1 \; dt \\
&\leq \frac{\varepsilon}{\alpha} \int_0^{\alpha} dt \\
&= \varepsilon
\end{aligned}$$

Can do a similar analysis for $x < 0$
Conclusion

$$\int_0^x y_n(t) \cos t \; dt \to \int_0^x \phi(x) \cos t \; dt$$

uniformly on $I = (-\alpha, \alpha)$
Take limits in approximation scheme

$$\lim_{n\to\infty} y_n(x) = \lim_{n\to\infty} y_0 + \lim_{n\to\infty} \int_0^x y_{n-1}(t) \cos t \; dt$$

$$\phi(x) = y_0 + \int_0^x \phi(t) \cos t \; dt$$

$\implies \phi(x)$ is a solution of the IVP by Lemma from last class

3.4.2 Tutorial

Sec 2.5 #7 $\quad y' = -3x^2 y, y(0) = 3 \quad y = 3e^{-x^3}$ calculate error

Method Improved Euler

- predictor (1st Euler)
- corrector

Text: 499, Appendix Ex.2

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4 Dimensional Analysis

Not in textbook... See notes posted on Course webpage.

When formulating and solving models of physical systems, we need to know the dimensions of the physical quantities in the model, i.e. what the quantity represents physically.
(Linked to the units - an abstract way of representing this information)

Fundamental Physical Quantities

- mass - dimensions denoted $\mathfrak{M}$[1]
- length - dimensions denoted $\mathscr{L}$
- time - dimensions denoted τ
- [ ] - to mean the dimensions of

Let y be the position of an object and t be time

$$[y] = \mathscr{L} \quad [t] = \tau \quad \left[\frac{dy}{dt}\right] = \mathscr{L}\tau^{-1}$$

Fundamental Principles

1. One can only add, subtract or equate physical quantities with the same physical dimensions
2. Quantities with different dimensions may be combined by multiplication with dimensions given by

$$[AB] = [A][B], \quad \left[\frac{A}{B}\right] = \frac{[A]}{[B]}$$

Example Falling body with air resistance

$$m\frac{dv}{dt} = mg - kv$$

Find the dimensions of the constant k

Principle 1 $\left[m\frac{dv}{dt}\right] = [mg] = [kv]$

[1] I will match the fonts posted on learn later on, which is \mathcal

Principle 2 $[m]\left[\frac{dv}{dt}\right] = [m][g] = [k][v]$

$$\mathfrak{M}\mathscr{L}\tau^{-2} = \mathfrak{M}\mathscr{L}\tau^{-2} = [k]\mathscr{L}\tau^{-1}$$

$$\implies [k] = \mathfrak{M}\tau^{-1}$$

Dimensionless variables

Are variables formed by rescaling each physical variable by a combination of constants with the same dimensions

Procedure

1. Find dimensions of all variables and constants
2. For each variable find a combination of constants with the same dimensions
3. Define new variables by dividing the physical variables by the corresponding combination of constants

Example Logistic growth model

$$\frac{dP}{dt} = kP(M-P), \qquad P(0) = P_0$$

1. • Dimensions of variables
   $[P] = \mathcal{P}$ (population size)
   $[t] = \tau$
   • Dimensions of constants
   $[M] = \mathcal{P}$
   $[P_0] = \mathcal{P}$

   Use analysis as in previous example to find

$$\left[\frac{dP}{dt}\right] = [kPM] \implies \mathcal{P}\tau^{-1} = [k]\mathcal{P}^2 \implies [k] = \mathcal{P}^{-1}\tau^{-1}$$

2. Both M and P_0 have same dimensions as P

$$\left[\frac{1}{kM}\right] = \frac{1}{[k][M]} = \frac{1}{(\mathcal{P}^{-1}\tau^{-1})\mathcal{P}} = \tau$$

3. Let $y = \dfrac{P}{M}, \tau = \dfrac{t}{1/kM} = t\ kM$

$$[y] = \frac{[P]}{[M]} = \frac{\mathcal{P}}{\mathcal{P}} = 1 \implies \text{dimensionless}$$

   Make the change of variables in the model
   LHS:

$$\frac{dP}{dt} = \frac{d}{dt}(My) = M\frac{dy}{dt} = M\frac{d\tau}{dt}\frac{dy}{d\tau} = M^2k\frac{dy}{d\tau}$$

   RHS:

$$kP(M-P) = kMy(M-My) = kM^2y(1-y)$$

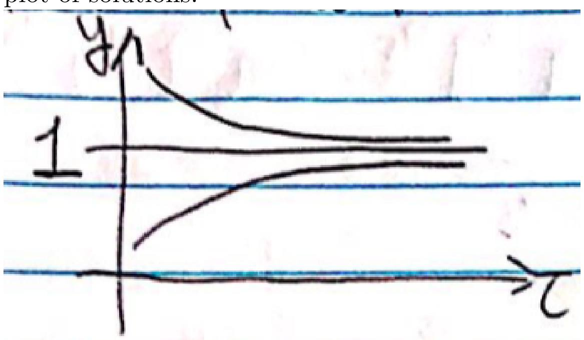
   Put together LHS = RHS

$$kM^2\frac{dy}{d\tau} = kM^2y(1-y) = \frac{dy}{d\tau} = y(1-y), \qquad y(0) = y_0$$

where $y_0 = \frac{P_0}{M}$

Solution for $y(\tau)$. $\quad y(\tau) = \dfrac{y_0}{y_0 + (1 - y_0)e^{-\tau}}$

plot of solutions:



Important constant for determining the behaviour is $y_0 = \frac{P_0}{M}$

4.1 Buckingham-π Theorem

Let $\{Q_1, Q_2, \ldots, Q_n\}$ be the set of all physical quantities (variables and constants) relevant to a particular problem. Suppose there is one and only one $\underbrace{\text{dimensionally homogeneous}}_{\text{satisfies property 1}}$ relationship between the Q_i:

$$Q_n = f(Q_1, Q_2, \ldots, Q_{n-1}) \tag{4.1}$$

where f is continuous and differentiable with respect to its variables.

Suppose there are r independent fundamental physical dimensions in the system. Then (4.1) is equivalent to

$$\pi_k = h(\pi_1, \pi_2, \ldots, \pi_{k-1}) \tag{4.2}$$

where $k = n - r$ and each π_j is a dimensionless quantity of the form

$$\pi_j = Q_1^{a_{1j}} Q_2^{a_{2j}} \ldots Q_n^{a_{nj}} \tag{4.3}$$

Proof see online notes. Relies on linear algebra □

Example Falling body with air resistance. (air resistance proportional to velocity)
Physical quantities: t, v, m, v_0, g, k
Fundamental dimensions: $\mathcal{T}, \mathcal{L}, \mathcal{M}$

Possible dimensionless quantities:

$$\pi = t^{a_1} v^{a_2} m^{a_3} v_0^{a_4} g^{a_5} k^{a_6}$$

Take dimensions of both sides, assume π is dimensionless

$$[\pi] = [t]^{a_1}[v]^{a_2}[m]^{a_3}[v_0]^{a_4}[g]^{a_5}[k]^{a_6}$$
$$1 = \mathcal{T}^{a_1}(\mathcal{L}\mathcal{T}^{-1})^{a_2}\mathcal{M}^{a_3}(\mathcal{L}\mathcal{T}^{-1})^{a_4}(\mathcal{L}\mathcal{T}^{-2})^{a_5}(\mathcal{M}\mathcal{T}^{-1})^{a_6}$$

Collet terms

$$\mathcal{T}^0\mathcal{L}^0\mathcal{M}^0 = \mathcal{T}^{a_1-a_2-a_4-2a_5-a_6}\mathcal{L}^{a_2+a_4+a_5}\mathcal{M}^{a_3+a_6}$$

Equate powers

$$\begin{aligned} a_1 - a_2 - a_4 - 2a_5 - a_6 &= 0 \\ a_2 + a_4 + a_5 &= 0 \\ a_3 + a_6 &= 0 \end{aligned}$$

Rewrite in matrix form

$$\begin{bmatrix} 1 & -1 & 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix has rank 3 (3 linearly independent columns). There are 4 linear independent solutions of the problem.
Solve for 3 of the A_j in terms of the others

$$\begin{aligned} a_3 &= -a_6 \\ a_2 &= -a_4 - a_5 \\ a_1 &= a_2 + a_4 + 2a_5 + a_6 = a_5 + a_6 \end{aligned}$$

Put this back into the expression for π

$$\begin{aligned} \pi &= t^{a_1} v^{a_2} m^{a_3} v_0^{a_4} g^{a_5} k^{a_6} \\ &= t^{a_5+a_6} v^{-a_4-a_5} m^{-a_6} v_0^{a_4} g^{a_5} k^{a_6} \\ &= \left(\frac{v_0}{v}\right)^{a_4} \left(\frac{gt}{v}\right)^{a_5} \left(\frac{kt}{m}\right)^{a_6} \end{aligned}$$

where a_4, a_5, a_6 are arbitrary
Can choose values of a_4, a_5, a_6 to get dimensionless quantities

1 $a_4 = -1, a_5 = 0, a_6 = 0$ $\quad \pi_1 = \frac{v}{v_0} \to$ dimensionless for v

2 $a_4 = 0, a_5 = 0, a_6 = 1$ $\quad \pi_2 = \frac{kt}{m} \to$ dimensionless for t

3 $a_4 = -1, a_5 = 1, a_6 = -1$ $\quad \pi_3 = \frac{mg}{v_0} \to$ dimensionless for constant

The $\pi-$Theorem tells us the relationship $v = f(t, m, g, k, v_0)$ is equivalent to

$$\pi_1 = h(\pi_2, \pi_3)$$

We can use the dimensionless variables to rewrite the model

$$m\frac{dv}{dt} = mg - kv, \qquad v(0) = v_0$$

$$\frac{dv}{dt} = g - \frac{k}{m}v, \qquad v(0) = v_0$$

Let $w = \frac{v}{v_0}, \tau = \frac{kt}{m}$. Make the change of variables: (Exercise)

$$\frac{dw}{d\tau} = \underbrace{\frac{mg}{kv_0}}_{\lambda} - w, \qquad w(0) = 1$$

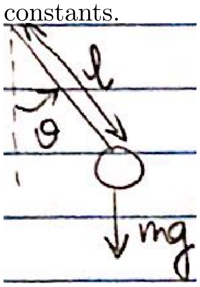
Solving the IVP for w gives

$$w(\tau) = \lambda + (1 - \lambda)e^{-\tau}$$

$w(\tau) \to \lambda$ as $\tau \to \infty$

4.2 Pendulum

Example Dimensional Analysis of a pendulum
Consider the motion of an object of mass m attached to a thin (massless) rod of length l. Let θ_0 be the initial angle the mass makes with the vertical. Assuming no friction at the pivot and no air resistances, the only force acting is gravity. In this (idealized) situation, the object will move back and forth with constant period, P. We will use the $\pi-$Theorem to deduce a relationship between P and the physical constants.



Assume that

$$P = f(\theta_0, m, l, g) \tag{$*$}$$

Note that θ measured in radius is dimensionless. $\theta = \frac{s}{r}$ $\quad [\theta] = \frac{[s]}{[r]} = \frac{\mathcal{L}}{\mathcal{L}} = 1$

Form a dimensionless quantity

$$\pi = P^{a_1}\theta_0^{a_2}m^{a_3}l^{a_4}g^{a_5}$$

Take dimensions:

$$[\pi] = [P]^{a_1}[\theta_0]^{a_2}[m]^{a_3}[l]^{a_4}[g]^{a_5}$$
$$1 = \mathcal{T}^{a_1}1^{a_2}\mathcal{M}^{a_3}\mathcal{L}^{a_4}(\mathcal{L}\mathcal{T}^{-2})^{a_5}$$

Equate powers

$$\mathcal{M}^0\mathcal{T}^0\mathcal{L}^0 = \mathcal{T}^{a_1-2a_5}\mathcal{M}^{a_3}\mathcal{L}^{a_4+a_5}$$
$$a_1 - 2a_5 = 0$$
$$a_3 = 0$$
$$a_4 + a_5 = 0$$

$$\implies \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{\text{rank 3}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix has rank 3 $\implies$ Expect $5 - 3 = 2$ linearly independent dimensionless quantities

Solving system $\begin{cases} a_3 = 0 \\ a_1 = 2a_5 \\ a_4 = -a_5 \end{cases}$ $\quad a_2$ is arbitrary

$$\pi = P^{2a_5}\theta_0^{a_2}m^0l^{-a_5}g^{a_5} = \theta_0^{a_2}\left(\frac{P^2g}{l}\right)^{a_5}$$

Solution

- $a_2 = 1, a_5 = 0 \qquad \pi_1 = \theta_0$
- $a_2 = 0, a_5 = \frac{1}{2} \qquad \pi_2 = P\sqrt{\frac{g}{l}}$

$\pi-$Theorem tells us $(*)$ is equivalent to $\pi_2 = h(\pi_1)$

$$P\sqrt{\frac{g}{l}} = h(\theta_0) \longrightarrow P = \sqrt{\frac{l}{g}}h(\theta_0)$$

Conclusions

- P doesn't depend on m
- P does depend on θ_0
- P is proportional to $\sqrt{\frac{l}{g}}$

4.2.1 Tutorial

1. The Schacfer model (fisheries):
   (a) Population obeys logistic, i.e. without fishing.
   (b) with fishing added, amount of caught $\sim$ (proportion to) P

$$\frac{dP}{dt} = k(M-P) - hP, \qquad P(0) = P_0$$

   $M, k, h > 0$

   $t, \tau \qquad \frac{dy}{d\tau} = (1-y) - \lambda y, \quad y(0) = y_0$

2. Moon: around Earth, period T.
   Assume T depends on M (earth), m (moon), r, G.

$$T = f(M, m, r, G)$$

$$\pi = T^{a_1} M^{a_2} m^{a_3} r^{a_4} G^{a_5}$$

$$\mathcal{T}^0 \mathcal{M}^0 \mathcal{L}^0 = \mathcal{T}^{a_1 - 2a_5} \mathcal{M}^{a_2 + a_3 - a_5} \mathcal{L}^{a_4 + 3a_5}$$

   Then ...

$$\pi_1 = T\sqrt{\frac{MG}{r^3}}(a_1 = 1, a_3 = 0) \qquad \pi_2 = \frac{m}{M}(a_1 = 0, a_3 = 1)$$

$$T = \sqrt{\frac{r^3}{MG}} h\left(\frac{m}{M}\right)$$

4.3 Supplementary notes

Notes on Dimensional Analysis for AMATH 251

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Chapter 3

Non-dimensionalization and Dimensional Analysis

In this course we will use *non-dimensionalization* to simply problems by introducing scaled dimensionless variables in order to make the problems look simple by reducing the number of parameters that appear in the problem. In more advanced courses, e.g., on perturbation theory, a further goal is to scale variables so that the dimensionless variables and their derivatives have typical sizes of about 1 in order to introduce small parameters that indicate which terms may be neglected.

Dimensional analysis is a useful method for deducing functional relationships between variables and simplifying problems.

In an equation modelling a physical process all of the terms separated by a '+' sign must have the same units. Otherwise we'd have something silly like 'oranges + avocados = houses', or $2\text{ kg} + 4\text{ m} = 10\text{ m s}^{-1}$. For example, in a mixing tank the equation for the mass of salt $A(t)$ in the tank is, based on the principle of conservation of mass,

$$\begin{aligned}\frac{dA}{dt} &= \text{rate mass enters} - \text{rate mass exits}, \\ &= c_{in}V_{f_{in}} - c_{out}V_{f_{out}}.\end{aligned} \tag{3.1}$$

Here

- $\frac{dA}{dt}$ has units of kg s^{-1}
- c_{in} has units of kg m^{-3}
- $V_{f_{in}}$ has units of m^3 s^{-1}

so $c_{in}V_{f_{in}}$ has units of kg s^{-1}, the same as those of $\frac{dA}{dt}$. Similarly for $c_{out}V_{f_{out}}$.

Checking the dimensions of terms in your equation can be helpful. It is always a good idea and often reveals simple mistakes.

We will use square brackets to denote the *dimensions* of a quantity. All scientific units can be written in terms of *primary dimensions* such as mass [M], length [L], time [T], electric current [I], temperature [θ] etc. The units for these quantities depend on the measurement system being used. The SI units for these quantities are kilograms (kg), meters (m), seconds (s), amperes (A) and Kelvin (K), although $^\circ C$ may be used for the latter. Other systems include the British foot-pound system and the cgs system (grams-centimeters-seconds).

Other *secondary* dimensions can be written in terms of primary dimensions. For example,

- force F = mass × acceleration has dimensions [F] = [M][L][T]$^{-2}$
- pressure P = force per unit area has dimensions [P] = [F][L]$^{-2}$ = [M][L]$^{-1}$[T]$^{-2}$
- energy E has dimensions [E] = [M][L]^2[T]$^{-2}$

etc.

Use of dimensionless parameters can reduce the number of parameters in a problem, simplify the presentation and analysis of results and improve understanding. For experimentalists or numerical modellers the use of dimensionless parameters can significantly reduce the number of parameters that need to be varied to explore the behaviour of a system in parameter space.

3.1 Non-dimensionalization

Problems are non-dimensionalized by scaling variables by a characteristic scale. We illustrate the process via a few examples.

Example: Consider the mixing tank problem above. We assume the volume fluxes into and out of the tank are equal and constant, i.e., $V_{f_{in}} = V_{f_{out}} = V_f$. Let V be the contant volume of the tank and $c(t) = A(t)/V$ be the concentration of salt in the tank. The fluid in the tank is well mixed and $c_{out} = c$. The governing equation is then

$$\frac{dc}{dt} = c_{in}\frac{V_f}{V} - c\frac{V_f}{V} \tag{3.2}$$

and the IVP is

$$\begin{aligned} \frac{dc}{dt} + \frac{V_f}{V}c &= c_{in}\frac{V_f}{V}, \\ c(0) &= c_0. \end{aligned} \tag{3.3}$$

The problem involves two variables c and t and three parameters V, V_f and c_{in}. We introduce a characteristic concentration C_c and time scale T_c and non-dimensional concentration $\tilde{c}$ and $\tilde{t}$ via

$$c = C_c\tilde{c} \qquad t = T_c\tilde{t} \tag{3.4}$$

where C_c and T_c are currently unknown. We will choose them to make the problem look relatively simple. Under our scaling time derivatives scale according to

$$\frac{d}{dt} = \frac{d\tilde{t}}{dt}\frac{d}{d\tilde{t}} = \frac{1}{T_c}\frac{d}{d\tilde{t}}. \tag{3.5}$$

Not that the factor $1/T_c$ carries the dimensions of the time derivative operator. Using this

$$\frac{dc}{dt} = \frac{1}{T_c}\frac{dc}{d\tilde{t}} = \frac{1}{T_c}\frac{d\big(C_c\tilde{c}\big)}{d\tilde{t}} = \frac{C_c}{T_c}\frac{d\tilde{c}}{d\tilde{t}}. \tag{3.6}$$

Note here that on the right-hand side C_c/T_c has dimensions of concentration over time, the dimensions of the left hand side. The term $\frac{d\tilde{c}}{d\tilde{t}}$ is dimensionless.

Substituting this, along with $c = C_c\tilde{c}$ into the IVP gives

$$\begin{aligned} \frac{C_c}{T_c}\frac{d\tilde{c}}{d\tilde{t}} + \frac{V_f}{V}C_c\tilde{c} &= c_{in}\frac{V_f}{V}, \\ C_c\tilde{c}(0) &= c_0. \end{aligned} \tag{3.7}$$

or

$$\begin{aligned} \frac{d\tilde{c}}{d\tilde{t}} + T_c\frac{V_f}{V}\tilde{c} &= \frac{T_c}{C_c}c_{in}\frac{V_f}{V}, \\ \tilde{c}(0) &= \frac{c_0}{C_c}, \end{aligned} \tag{3.8}$$

where now all the terms are dimensionless. **We now choose the scales C_c and T_c to put the problem in a simple form.** There are many ways to do this. The DE includes two 'messy' coefficients. We have two scalings to play around with so we can make both of them equal to one. That is, choose

$$T_c = \frac{V}{V_f} \tag{3.9}$$

so that the coefficient of $\tilde{c}$ in the DE is equal to one and then choose C_c to set the right hand side equal to one. That is, set

$$\frac{T_c}{C_c}c_{in}\frac{V_f}{V} = \frac{c_{in}}{C_c} = 1 \qquad \Longrightarrow \qquad C_c = c_{in} \tag{3.10}$$

(note we have used $T_c = V/V_f$). We can then define

$$\tilde{c}_0 = \frac{c_0}{C_c} \tag{3.11}$$

to get the final dimensionless form of the problem:

$$\begin{aligned} \frac{d\tilde{c}}{d\tilde{t}} + \tilde{c} &= 1, \\ \tilde{c}(0) &= \tilde{c}_0. \end{aligned} \tag{3.12}$$

This involves only one parameter, $\tilde{c}_{in}$. The solution of this IVP problem is

$$\tilde{c}(\tilde{t}) = 1 + (\tilde{c}_0 - 1)e^{-\tilde{t}}. \tag{3.13}$$

To get the solution of the dimensional problem you can now use $c = C_c\tilde{c}$ and $t = T_c\tilde{t}$ to get

$$\begin{aligned} c(t) &= C_c\Big(1 + (\tilde{c}_0 - 1)e^{-\tilde{t}}\Big) \\ &= C_c + \big(c_0 - C_c\big)e^{-t/T_c} \\ &= c_{in} + \big(c_0 - c_{in}\big)e^{-\frac{V_f}{V}t}. \end{aligned} \tag{3.14}$$

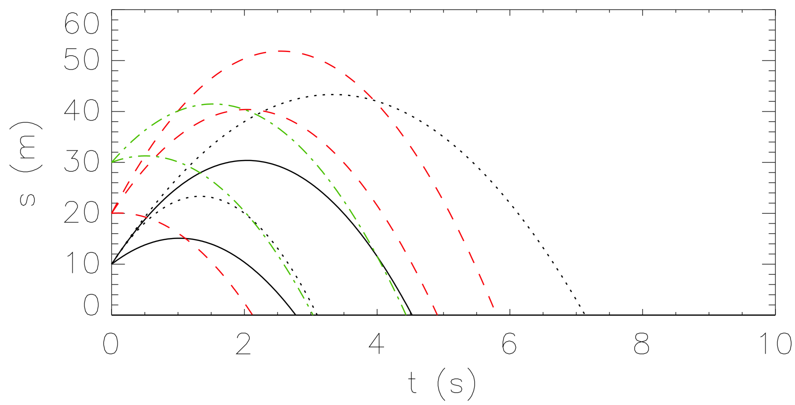


Figure 3.1: Height vs time for various initial heights s_0, initial velocities v_0 and different gravitational accelerations g.

Example: The height of an object thrown up from height $s_0 > 0$ with velocity $v_0 > 0$ at time t is governed by

$$\begin{aligned} \frac{d^2 s}{dt^2} &= -g \\ s(0) &= s_0 \\ \frac{ds}{dt}(0) &= v_0. \end{aligned} \tag{3.15}$$

which has the solution

$$s(t) = -\frac{1}{2} g t^2 + v_0 t + s_0. \tag{3.16}$$

This problem involves the two variables s and t, and three parameters g, v_0, and s_0. To plot solutions we need to vary g (for different planets), v_0 and s_0. Sample trajectories are shown in Figure 3.1.

We can make the problem dimensionless. Here we introduce characteristic length and time scales L_c and T_c along with dimensionless variables $\tilde{s}$ and $\tilde{t}$ given by $s = L_c \tilde{s}$ and $t = T_c \tilde{t}$. Using (3.5) we have

$$\frac{d^2}{dt^2} = \frac{d}{dt}\Big(\frac{d}{dt}\Big) = \frac{1}{T_c}\frac{d}{d\tilde{t}}\Big(\frac{1}{T_c}\frac{d}{d\tilde{t}}\Big) = \frac{1}{T_c^2}\frac{d^2}{d\tilde{t}^2}. \tag{3.17}$$

Thus

$$\frac{ds}{dt} = \frac{1}{T_c}\frac{d}{d\tilde{t}}\Big(L_c \tilde{s}\Big) = \frac{L_c}{T_c}\frac{d\tilde{s}}{d\tilde{t}}, \tag{3.18}$$

and

$$\frac{d^2 s}{dt^2} = \frac{1}{T_c^2}\frac{d^2}{d\tilde{t}^2}\Big(L_c \tilde{s}\Big) = \frac{L_c}{T_c^2}\frac{d^2 \tilde{s}}{d\tilde{t}^2}. \tag{3.19}$$

In terms of the dimensionless variables, the IVP is

$$
\begin{aligned}
\frac{L_c}{T_c^2}\frac{d^2\tilde{s}}{d\tilde{t}^2} &= -g \\
L_c\tilde{s}(0) &= s_0 \\
\frac{L_c}{T_c}\frac{d\tilde{s}}{d\tilde{t}}(0) &= v_0,
\end{aligned} \tag{3.20}
$$

or

$$
\begin{aligned}
\frac{d^2\tilde{s}}{d\tilde{t}^2} &= -g\frac{T_c^2}{L_c} \\
\tilde{s}(0) &= \frac{s_0}{L_c} \\
\frac{d\tilde{s}}{d\tilde{t}}(0) &= v_0\frac{T_c}{L_c},
\end{aligned} \tag{3.21}
$$

where now all the term are dimensionless.

There are many ways to choose the scalings T_c and L_c. If you are dropping an object from a height s_0 above the ground (at $s = 0$) then choosing $L_c = s_0$ is very sensible as then $\tilde{s}$ varies from an initial value of 1 to a final value of 0 when it hits the ground. If you fire a cannonball vertically upward from the ground at $s = 0$ (so $s_0 = 0$) then it will rise to a maximum height of $s_{max} = v_0^2/2g$ in which case using $L_c = v_0^2/2g$ is a good choice because then $\tilde{s}$ increases from its initial value of 0 to a maximum value of 1 before decreasing back to 0 as the cannonball returns to the ground. There are likewise many possible choices for T_c. As a general rule, the best choices for the scales will depend on your problem.

Here I will assume $s_0 \neq 0$ and $v_0 \neq 0$ and take $L_c = s_0$ and $T_c = v_0/g$. The latter is the time taken for the object to reach its maximum height. With these choices the problem becomes

$$
\begin{aligned}
\frac{d^2\tilde{s}}{d\tilde{t}^2} &= -g\frac{v_0^2}{g^2 s_0} = -2\lambda \\
\tilde{s}(0) &= 1 \\
\frac{d\tilde{s}}{d\tilde{t}}(0) &= v_0\frac{v_0}{g s_0} = 2\lambda,
\end{aligned} \tag{3.22}
$$

where

$$
\lambda = \frac{1}{2}\frac{v_0^2}{g s_0}. \tag{3.23}
$$

The solution in dimensionless terms is

$$
\tilde{s}(\tilde{t}) = -\lambda\tilde{t}^2 + 2\lambda\tilde{t} + 1. \tag{3.24}
$$

This involves three variables instead of five and to plot solutions we just need to vary the value of a single variable, λ, instead of three. There is now a 1-parameter family of solutions. All dimensionless solutions have the properties that

- the initial height is $\tilde{s}(0) = 1$.
- the initial velocity is $\frac{d\tilde{s}}{d\tilde{t}}(0) = 2\lambda$.

- $\frac{d\tilde{s}}{d\tilde{t}}(\tilde{t}) = 0$ when $\tilde{t} = 1$.
- the maximum height is $\tilde{s}(1) = 1 + \lambda$.
- the object reaches the ground ($\tilde{s}(\tilde{t}) = 0$) at time $\tilde{t}_f = 1 + \sqrt{\frac{1+\lambda}{\lambda}}$.

Some sample trajectories are plotted in Figure 3.2.

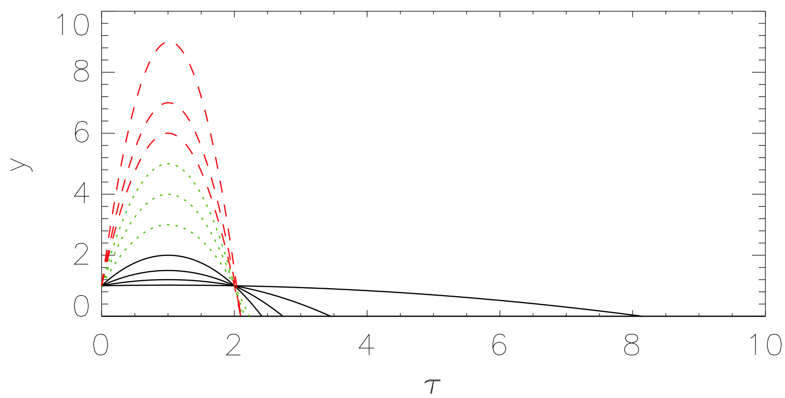


Figure 3.2: Nondimensional height vs time ($y = \tilde{s}$ and $\tau = \tilde{t}$) for several values of λ.

3.2 Dimensional Analysis: The Buckingham-π Theorem

The Buckingham-π Theorem is used to determine the number of dimensionless parameters in a problem and can simplify relationships among variables by reducing the number of variables. It is named after E. Buckingham who proved a general version of the theorem in 1914 however the theorem was first proved in 1878 by J. Bertrand and the technique was made widely known by Rayleigh in the 1890's.

Suppose there are n physical variables and parameters Q_1, Q_2, ..., Q_n and the solution of a mathematical model gives one in terms of the others:

$$Q_n = f(Q_1, Q_2, \ldots, Q_{n-1}). \tag{3.25}$$

Suppose that there are r independent basic physical dimensions [M], [L], [T], etc. Suppose that k dimensional quantities can be constructed via multiples of powers of the Q_j. We will see that there are at least $n - r$ of them. Then

$$\pi_k = h(\pi_1, \pi_2, \ldots, \pi_{k-1}). \tag{3.26}$$

To see how the Buckingham-π Theorem is used we consider a simple illustrative example: the simple nonlinear pendulum.

Example: Consider a simple pendulum of mass m in the absence of air damping. Assume the mass is attached to a frictionless pivot via an inextensible massless wire of length l. Assuming Newtonian mechanics applied (e.g., ignore realistic effects). The pendulum is released from rest at an angle α. How does the period τ of oscillation depend on the parameters α, m, l and the gravitational acceleration g?

The first step is to determine the dimensions of the various parameters. Here we have

- $[\tau] = [\mathrm{T}]$
- $[\mathrm{g}] = [\mathrm{L}][\mathrm{T}]^{-2}$
- $[\mathrm{l}] = [\mathrm{L}]$
- $[\alpha] = 1$ (the angle is dimensionless)
- $[\mathrm{m}] = [\mathrm{M}]$

There are $n = 5$ independent variables. These variables involve $r = 3$ dimensions: [L], [T], and [M]. In this case there are $k = n - r = 5 - 3 = 2$ independent dimensionless combinations of the parameters. Since α is dimensionless it can be taken as one of the dimensionless parameters. To find a second let

$$\pi = \tau^a m^b g^c l^d \tag{3.27}$$

(note: since the fifth parameter α is dimensionless there is no point in including it. Multiplying π by any power of α will yield another dimensionless parameter). The dimension of π is

$$\begin{aligned}
[\pi] &= [\tau]^a[m]^b[g]^c[l]^d, \\
&= [T]^a[M]^b\big([L][T]^{-2}\big)^c[L]^d, \\
&= [T]^{a-2c}[M]^b[L]^{c+d}.
\end{aligned} \tag{3.28}$$

This is dimensionless if

$$a - 2c = b = c + d = 0, \tag{3.29}$$

Table 3.1: The international system of units: fundamental (or primary, or basic) dimensions. Sometimes angles in radians is included as a dimension but not in the ISU.

Physical Quantity	unit symbol	unit name (MKS)
mass	kg	kilogram
length	m	meter
time	s	second
electric current	A	ampere
luminosity	C	candela
temperature	K	degree Kelvin
amount	mol	mole

i.e., if $c = a/2$, $b = 0$ and $d = -c = -a/2$. Thus, all dimensionless combinations of m, l, g and τ have the form

$$\pi = \left(\tau\sqrt{\frac{g}{l}}\right)^a. \tag{3.30}$$

We can choose a to have any non-zero value. For simplicity we take $a = 1$. Our two dimensionless parameters are $\pi_1 = \tau\sqrt{\frac{g}{l}}$ and $\pi_2 = \alpha$. Any other dimensionless parameter must have the form $\pi_1^\alpha \pi_2^\beta$ and hence is not independent of π_1 and π_2. According to the theorem

$$\pi_1 = h(\pi_2) \tag{3.31}$$

for some function h, or

$$\tau\sqrt{\frac{g}{l}} = h(\alpha), \tag{3.32}$$

which can be written as

$$\tau = \sqrt{\frac{l}{g}}h(\alpha). \tag{3.33}$$

Without even writing down any equations, let alone solving them, we have learnt two important things:

1. the period of oscillation does not depend on the mass m;
2. the period of oscillation is proportional to $\sqrt{l}$;
3. the period of oscillation is inversely proportional to $\sqrt{g}$;
4. $\tau\sqrt{\frac{g}{l}}$ depends only on the initial angle α.

Thus we know that doubling the length of the pendulum increases the period of oscillation by a factor of $\sqrt{2}$ and increasing the gravitational acceleration by a factor of two decreases the period of oscillation by a factor of $\sqrt{2}$. If one wanted to conduct a series of experiments to determine how the period τ depends on g, m, l and α, instead of doing a large set of experiments by varying all four parameters one only needs to vary α.

For small angles of oscillation, i.e., in the limit $\alpha \to 0$, the period of oscillation becomes

$$\tau_{lin} = \sqrt{\frac{l}{g}}h(0). \tag{3.34}$$

Mathematically the governing equation (a second-order ODE) is a nonlinear DE for arbitrary initial angles but in the limit $\alpha \to 0$ the equation becomes a linear DE hence we refer to the period of oscillation in this limit as the linear period τ_{lin}. As we will see later the period is $2\pi\sqrt{\frac{l}{g}}$, i.e., $h(0) = 2\pi$.

Comment: There are an infinite number of choices for the non dimensionless parameters. For example

$$\begin{aligned} \pi_1 &= \left(\tau\sqrt{\frac{g}{l}}\right)^{-2}, \\ \pi_2 &= \left(\tau\sqrt{\frac{g}{l}}\right)^{3}\alpha^2. \end{aligned} \tag{3.35}$$

A direct application of the theorem gives

$$\left(\tau\sqrt{\frac{g}{l}}\right)^{-2} = H\left(\left(\tau\sqrt{\frac{g}{l}}\right)^{3}\alpha^2\right) \tag{3.36}$$

for some function H. Inverting we have

$$\alpha^2 = H^{-1}\left(\left(\tau\sqrt{\frac{g}{l}}\right)^{-2}\right)/\left(\tau\sqrt{\frac{g}{l}}\right)^{3} \tag{3.37}$$

where H^{-1} is the inverse of H. Thus $\alpha = f\left(\tau\sqrt{\frac{g}{l}}\right)$. Inverting this leads to (3.33).

3.2.1 Proof of the Buckingham-π Theorem

The following is based on the discussion in [2]. See also [6] which is a very useful applied math text covering a broad range of subjects. A book all Applied Mathematicians should be familiar with.

Assumptions

Dimensional analysis is built upon the following assumptions:

(i) A quantity u is determined in terms of n measurable quantities (variables or parameters) $(W_1, W_2, \ldots, W_n)$ by a functional relationship of the form

$$u = f(W_1, W_2, \ldots, W_n) \tag{3.38}$$

(ii) The quantities $(u, W_1, W_2, \ldots, W_n)$ involve m fundamental dimensions which are labelled L_1, L_2, ..., L_m.

(iii) Let Z represent any of $(u, W_1, W_2, \ldots, W_n)$. Then the dimensions of Z, denoted by [Z], is a product of powers of the fundamental dimensions via

$$[Z] = L_1^{\alpha_1} L_2^{\alpha_2} \cdots L_m^{\alpha_m}. \tag{3.39}$$

The dimension vector of Z is the column vector

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \end{bmatrix} \tag{3.40}$$

A quantity Z is dimensionless if and only if all its dimension exponents are zero. Let

$$\vec{b}_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{mi} \end{bmatrix} \tag{3.41}$$

be the dimension vector of W_i, $i = 1, 2, 3, \ldots, n$, and let

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \tag{3.42}$$

be the $m \times n$ dimensional matrix of the problem.

Example: Before our final assumption consider the period of a simple pendulum τ which is released from rest at an initial angle α. The period is assumed to depend on the pendulum mass m, the length of the pendulum l, the gravitational acceleration g and the initial angle α. This gives $\tau = f(m, l, g, \alpha)$. The quantities $u = \tau$, $W_1 = m$, $W_2 = l$, $W_3 = g$ and $W_4 = \alpha$ involve three dimensions: $L_1 =$ [M], $L_2 =$ [L], and $L_3 =$ [T]. Hence $n = 4$ and $m = 3$. Now $[m] = [M] = L_1$, $l = [L] = L_2$, $[g] = [L][T[^{-2} = L_2 L_3^{-2}$ and $[\alpha] = 1$. Thus, for this problem

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \tag{3.43}$$

(iv) For any set of fundamental dimension one can choose a system of units for measuring the value of any quantity Z. For example, the SI system, the cgs system or the British foot-pound system. Changing from one system to another involves a positive scaling of each fundamental unit. E.g., 1 m = 100 cm, 1 cm = 2.54 inches, or 1 kg = 2.2 lbs. Secondary quantities are scaled accordingly. This results in a corresponding scaling of each quantity Z. *The final assumption is that the relationship between u and the variables* $(W_1, W_2, \ldots, W_n)$ *is invariant under any scaling of the fundamental units.* For example, the relationship between kinetic energy K and the mass m and velocity $\vec{v}$ of an object is $K = \frac{1}{2} m \vec{v} \cdot \vec{v}$ in all measurement systems, or Newton's second law of motion $\vec{F} = m\vec{a}$ is independent of the measurement system.

The invariance of the functional relationship means the following. Suppose under a scaling of the dimension L_j $L_j \to L_j^* = e^\epsilon L_j$ and under this scaling $(u, W_1, W_2, \ldots, W_n) \to (u^*, W_1*, W_2^*, \ldots, W_n^*)$. Then $u = f(W_1, W_2, \ldots, W_n)$ becomes $u^* = f(W_1^*, W_2^*, \ldots, W_n^*)$.

Illustration of the Buckingham-π theorem

Before proving the theorem we re-consider our simple pendulum which has period

$$\tau = f(m, l, g, \alpha). \tag{3.44}$$

This relationship is assumed to be independent of scalings of the fundamental dimensions. Suppose we scale the mass dimension by e^ϵ so $L_1 \to L_1^* = e^\epsilon L_1$. Under this scaling $m \to m^* = e^\epsilon m$ but as the other variables do not involve dimensions $L_1 = [M]$, $\tau \to \tau^* = \tau$, $g \to g^* = g$ and $l \to l^* = l$ (α, being dimensionless does not change). Thus, $\tau = f(m, l, g, \alpha)$ becomes $\tau^* = f(m^*, l^*, g^*, \alpha)$ according to assumption (iv). This means that

$$\tau = f(e^\epsilon m, l, g, \alpha). \tag{3.45}$$

This is true for all ϵ and the only way (3.44) and (3.45) can both hold is if τ does not depend on m, i.e., $\tau = f(l, g, \alpha) = f(W_2, W_3, W_4)$.

We now tackle the next dimension $[L] = L_2$. Scale L_2 by e^ϵ. Then $\tau^* = \tau$ is unchanged (it does not involve dimensions of length), $l^* = e^\epsilon l$, and $g* = e^\epsilon g$ so $\tau^* = f(l^*, g^*, \alpha)$ gives $\tau = f(e^\epsilon l, e^\epsilon g, \alpha)$. The appearance of e^ϵ in two term makes things slightly more complicated. To proceed we eliminate dimension L_2 from all but one variable by choosing two new independent variables. Let us remove it from the second variable, W_3, by defining new variables $X_2 = W_2$ and $X_3 = W_3/W_2 = g/l$. Here X_3 has been chosen so that $[X_3] = [\mathrm{T}]^{-2} = L_3^{-2}$ does not include the dimension L_2. In terms of the new variables $\tau = f(l, g) = \tilde{f}(X_2, X_3, \alpha) = \tilde{f}(l, g/l, \alpha)$. If we now scale l by e^ϵ we obtain

$$\tau = \tilde{f}(e^\epsilon X_2, X_3, \alpha) \tag{3.46}$$

which again is true for all ϵ. Hence $\tilde{f}$ must be independent of X_2, i.e.,

$$\tau = \tilde{f}(X_3, \alpha) = \tilde{f}\big(\frac{g}{l}, \alpha\big). \tag{3.47}$$

For the final step we note that $[\tau] = [T]$ and $[g/l] = [T]^{-2}$. Since $\tau = \tilde{f}(g/l, \alpha)$ it follows that $v = \tau\sqrt{\frac{g}{l}} = \tilde{f}((g/l, \alpha)\sqrt{\frac{g}{l}} \equiv h(g/l, \alpha) = h(X_3, \alpha)$ is dimensionless. Now $[v] = 1$ and $[X_3] = [T]^{-2}$. We now scale time by $L_3^* = e^\epsilon L_3$. Under this scaling $v \to v^* = v$ and $X_3 \to X_3^* = e^{-2\epsilon} X_3$. By the assumption $v^* = h(X_3^*, \alpha)$ or $v = h(e^{-2\epsilon} X_3, \alpha)$. The only way this can be true is if h is independent of X_3, i.e., $v = h(\alpha)$, or $\tau/\sqrt{g/l} = h(\alpha)$ recovering our previous result.

Statement of the Buckingham-π theorem

Consider

$$u = f(W_1, W_2, \ldots, W_n) \tag{3.48}$$

and let B be the dimension matrix for $W_1, W_2, \ldots, W_n$. Let

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \tag{3.49}$$

be the dimension vector of u. As in the simple pendulum example we want to introduce a new *dimensionless* dependent variable v by multiplying u by appropriate powers of the W_j. Thus, choose $(y_1, y_2, \ldots, y_n)$ so that

$$\pi = u W_1^{y_1} W_2^{y_2} \cdots W_n^{y_n} \tag{3.50}$$

is dimensionless. Recalling that the dimensions of W_i are $[W_i] = L_1^{b_{1i}} L_2^{b_{2i}} \cdots L_m^{b_{mi}}$ it follows that the dimensions of the expression on the right hand side is

$$\begin{aligned} &L_1^{a_1 + y_1 b_{11} + y_2 b_{12} + \cdots + y_n b_{1n}} L_2^{a_2 + y_1 b_{21} + y_2 b_{22} + \cdots + y_n b_{2n}} \cdots \\ &\quad \cdots L_j^{a_j + y_1 b_{j1} + y_2 b_{j2} + \cdots + y_n b_{jn}} \cdots L_m^{a_m + y_1 b_{m1} + y_2 b_{m2} + \cdots + y_n b_{mn}}. \end{aligned} \tag{3.51}$$

This is dimensionless if

$$a_j + y_1 b_{j1} + y_2 b_{j2} + \cdots + y_n b_{jn} = 0 \tag{3.52}$$

for all j, i.e., if $\vec{y}$ is a solution of

$$B\vec{y} = -\vec{a}. \tag{3.53}$$

Note that in B is often not invertible, e.g., it is not invertible if $m \neq n$ (for example if there are more variables W_k than dimensions L_i as in our simple pendulum example). There are in general many solutions. Indeed, if π is dimensionless and π_1 is any other dimensionless variables (e.g., $\tau/\sqrt{\frac{g}{l}}$ and α in our simple pendulum example) then $\pi^a\pi_1^b$ is an alternative dimensionless replacement for u for any nonzero a and b. It is also possible that there are no solutions which means the problem is poorly formulated. For example u may include a dimension that is not a dimension of any of the variables W_i. A scaling of this dimension leads to $e^{\epsilon}u = f(W_1, W_2, \ldots, W_n) = u$ which can't be true for all ϵ. For example, one may speculate that the period of revolution of the moon around the Earth, τ_r, depends on the mass of the Earth M_E, the mass of the moon M_m and the distance R_m of the moon from the centre of the Earth giving $\tau_r = f(M_E, M_m, R_m)$. M_E, M_m and R_m do not involve dimensions of time so the period τ_r can not be a function of these three variables only. Something is missing (in this case Newton's Universal Gravitational Constant). Henceforth we assume that (3.53) has a solution.

Next, we want to construct dimensionless variables by taking appropriate combinations of the W_j. Suppose

$$\pi_i = W_1^{x_{1i}} W_2^{x_{2i}} \cdots W_n^{x_{ni}} \tag{3.54}$$

is dimensionless. Then it follows that

$$B\vec{x_i} = 0 \tag{3.55}$$

which follows from (3.50) and (3.53) after replacing $\vec{a}$ by the zero vector. There are $k = n - r(B)$ linearly independent solutions of (3.55). Here $r(B)$ is the rank of the dimension matrix B (i.e., the number of linearly independent columns, or equivalently, the number of linearly independent rows). Let $\vec{x_i}$ $i = 1, 2, \cdots, k$ be any such set and let π_i given by (3.54) be the corresponding k dimensionless variables. Then the Buckingham-π theorem states that

$$\pi = g(\pi_1, \pi_2, \ldots, \pi_k). \tag{3.56}$$

Example: Consider again the simple pendulum. The dimension matrix (3.43) has rank $r = 3$. Hence there is a dimensionless dependent variable π, which we can take as $\tau\sqrt{g/l}$, and a single independent variable $\pi_1 = \alpha$. The theorem says that $\pi = h(\alpha)$ for some function h.

Proof of the Buckingham-π theorem

The proof of the Buckingham-π theorem follows procedures illustrated in the simple pendulum example. Suppose dimension L_1 is scaled by e^{ϵ}. Under this scale $u = f(W_1, W_2, \ldots, W_n)$ becomes $e^{\epsilon a_1}u = f(e^{\epsilon b_{11}}W_1, e^{\epsilon b_{12}}W_2, \ldots, e^{\epsilon b_{1n}}W_n)$. There are two cases to consider.

CASE I: If $b_{11} = b_{12} = \cdots = b_{1n} = a_1 = 0$ it follows the L_1 is not a fundamental dimension of the problem, i.e., the problem is independent of this dimension. We can assume this is not the case.

CASE II: If $b_{11} = b_{12} = \cdots = b_{1n} = 0$ and $a_1 \neq 0$ we have $e^{\epsilon a_1}u = f(W_1, W_2, \ldots, W_n)$ for all ϵ hence it follows that $u \equiv 0$, a situation that is not of interest.

CASE III: At least one of b_{11}, b_{12}, $\ldots$, b_{1n} is non-zero. Wlog assume $b_{11} \neq 0$. Define new quantities

$$X_i = W_i W_1^{-b_{1i}/b_{11}}, \quad i = 2, 3, \ldots, n \tag{3.57}$$

and a new unknown

$$v = uW_1^{-a_i/b_{11}}. \tag{3.58}$$

The formula $u = f(W_1, W_2, \ldots, W_n)$ is equivalent to $v = F(W_1, X_2, X_3, \ldots, X_n)$. By construction v and the X_i are independent of dimension L_1. Scaling L_1 by e^ϵ gives $v = F(e^{\epsilon b_{11}} W_1, X_2, X_3, \ldots, X_n)$. Since this is true for all ϵ it follows that F is independent of W_1. That is, $v = G(X_2, X_3, \ldots, X_n)$ where v and the quantities X_2, X_3, $\ldots$, X_n are independent of the dimension L_1.

Note that under the change of variables $W_j \to W_u W_1^{-b_{1i}/b_{11}}$ the dimension matrix is changed by subtracting $(b_{1i}/b_{11})\vec{b_1}$ from the ith column which makes the entries in row 1 zero apart from b_{11}:

$$
\begin{aligned}
B &= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \\
&= \begin{bmatrix} \vec{b_1} & \vec{b_2} & \cdots & \vec{b_n} \end{bmatrix} \\
&\to \begin{bmatrix} \vec{b_1} & \vec{b_2} - \frac{b_{12}}{b_{11}}\vec{b_1} & \cdots & \vec{b_n} - \frac{b_{1n}}{b_{11}}\vec{b_1} \end{bmatrix} \\
&= \begin{bmatrix} b_{11} & b_{12} - \frac{b_{12}}{b_{11}} b_{11} & \cdots & b_{1n} - \frac{b_{1n}}{b_{11}} b_{11} \\ b_{21} & b_{22} - \frac{b_{12}}{b_{11}} b_{21} & \cdots & b_{2n} - \frac{b_{1n}}{b_{11}} b_{21} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} - \frac{b_{12}}{b_{11}} b_{m1} & \cdots & b_{mn} - \frac{b_{1n}}{b_{11}} b_{m1} \end{bmatrix} \\
&= \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} - \frac{b_{12}}{b_{11}} b_{21} & \cdots & b_{2n} - \frac{b_{1n}}{b_{11}} b_{21} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} - \frac{b_{12}}{b_{11}} b_{m1} & \cdots & b_{mn} - \frac{b_{1n}}{b_{11}} b_{m1} \end{bmatrix} \\
&= \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & \tilde{b}_{22} & \cdots & \tilde{b}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & \tilde{b}_{m2} & \cdots & \tilde{b}_{mn} \end{bmatrix}.
\end{aligned}
\tag{3.59}
$$

We continue with the remaining $m-1$ dimensions. In turn we eliminate each dimension from all but one variable and reduce the number of independent variables. As discussed above, after the m dimensions have been eliminated we are left with $k+1 = n+1-r(B)$ dimensionless quantities π and π_1, π_2, $\ldots$, π_k where

$$\pi = g(\pi_1, \pi_2, \ldots, \pi_k). \tag{3.60}$$

or

$$u = W_1^{-y_1} W_2^{-y_2} \cdots W_n^{-y_n} g(\pi_1, \pi_2, \ldots, \pi_k). \tag{3.61}$$

Comments:

1. This proof makes no assumptions about the continuity of the functions f and g.

2. The assumed relationship $u = f(W_1, W_2, \ldots, W_n)$ may be incorrect. This can manifest itself in several ways. The non-dimensionalization procedure may fail (e..g, the orbital period of the moon example). Alternatively, the nondimensionalization procedure may work but the resulting $\pi = g(\pi_1, \pi_2, \ldots, \pi_n)$ may be incorrect because u depends on parameters or variables not included in the analysis. This may, however, still give a useful results in certain limiting cases. For example, if the period of the idealized simple pendulum is assumed to depend on l, g and m only the analysis follows through and results in $\tau = k\sqrt{\frac{l}{g}}$ for some constant k instead of the correct[1] result $\tau = f(\alpha)\sqrt{\frac{l}{g}}$ where f is an undetermined function to be found via experiments or mathematical analysis. The approximation $f(\alpha) = k$, a constant, is only valid in the small amplitude limit (which may be what you are interested in and hence OK).

3. The scaling $L_i^* = e^{\epsilon} L_i$ which induces the transformation $u \to u^* = e^{\epsilon a_i}$, and $W_j \to W_j^* = e^{\epsilon b_{ij}} W_j$ for $j = 1, 2, ..., n$ defines a one-parameter (ϵ) Lie group of scaling transformations of the $n+1$ quantities $(u, W_1, W_2, \ldots, W_n)$ with $\epsilon = 0$ corresponding to the identity transformation. Assumption (iv) behind dimensional analysis states that the relationships $u = f(W_1, W_2, \ldots, W_n)$ is invariant under this Lie group transformation. The Buckingham-π theorem is exploiting these symmetries to reduce the dimensionality of the problem. There is a whole field of study devoted to Lie group transformations which has given rise to many useful results in the study of differential equations.

3.3 Problems

1. Introduce suitable non-dimensional population $\tilde{P}$ and time $\tilde{t}$ to write the logistic equation in the form

$$\frac{d\tilde{P}}{dt} = \tilde{P}(1 - \tilde{P}). \tag{3.62}$$

What is the general solution $\tilde{P}(\tilde{t})$? Use your dimensionless solution to find the solution $P(t)$ of the dimensional problem.

[1] Even this is an approximation as we have ignored lots of other effects, e.g., relativistic effects, assuming them to be negligible.

5 Linear Equations of Higher Order

Most general n^{th} order ODE

$$G(x, y', y'', \ldots, y^{(n)}) = 0 \tag{5.1}$$

where $y' = \frac{dy}{dx}, \ldots, y^{(n)} = \frac{d^n y}{dx^n}$

Example $y'' = \sin x$
Solve by integrating twice

$$\frac{d^2y}{dx^2} = \sin x \implies \frac{dy}{dx} = -\cos x + C_1 \implies y = -\sin x + C_1 x + C_2$$

Solution has two arbitrary constants

Definition A general solution of (5.1) is a solution containing n arbitrary constants that represents almost all solutions of (5.1)

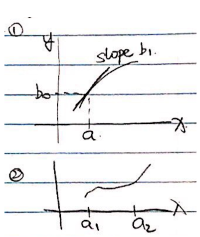
A particular solution is a solution of (5.1) with no arbitrary constants . One way to determine a particular solution from a general solution is to specify n conditions the solution must satisfy.

There can be many ways to do this.

Example $y'' = \sin x \implies y(x) = -\sin x + C_1 x + C_2$
specify conditions to determine C_1, C_2

1. Specify values of y and y' at one value of x

2. Specify values of y and/or y' at two values of x
   $y(a_0) = b_0, y(a_1) = b_1$ or $y'(a_0) = b_0, y'(a_1) = b_1$



Type (1) are called initial conditions
A DE together with conditions of type (1) is called an initial value problem

Type (2) are called boundary conditions
A DE together with conditions of type (2) is called a boundary value problem

We will focus on IVP

Definition An n^{th} order linear ODE is an equation of the form (5.1) where G is a linear function of $y, y', y'', \ldots, y^{(n)}$.
Any n^{th} order linear ODE can be written in the form

$$\mathscr{P}_0(x)y^{(n)} + \mathscr{P}_1(x)y^{(n-1)} + \ldots + \mathscr{P}_{n-1}(x)y' + \mathscr{P}_n(x)y = F(x) \tag{5.2}$$

If $\mathscr{P}_0(x) \neq 0$ for $x \in D$, then we can rewrite (5.2) in normal form

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_{n-1}(x)y' + P_n(x)y = f(x) \tag{5.3}$$

where $P_j(x) = \frac{\mathscr{P}_j(x)}{\mathscr{P}_0(x)}, f(x) = \frac{F(x)}{\mathscr{P}_0(x)}$

Definition If $f(x) = 0$ for all $x \in D$ then (5.3) is called homogeneous otherwise it is called non-homogeneous .

Theorem (Existence and Uniqueness) Suppose that $P_1(x), P_2(x), \ldots, P_n(x)$ and $f(x)$ are continuous on the open interval I containing the point a. Then, given real numbers $b_0, b_1, \ldots, b_{n-1}$ the DE (5.3) has a unique solution on I that satisfies

$$y(a) = b_0, y'(a) = b_1, y''(a) = b_2, \ldots, y^{(n-1)}(a) = b_{n-1} \tag{5.4}$$

The associated homogeneous equation of (5.3) is

$$y^{(n)} + P_1(x)y^{(n-1)} + \ldots + P_n(x)y = 0 \tag{5.5}$$

Note that this one always has solution $y(x) = 0, x \in I$

Theorem (Principle of Superposition)
Let $y_1, y_2, \ldots, y_n$ be solutions of (5.5) on I. Then for any $c_1, c_2, \ldots, c_n \in \mathbb{R}$, the linear combination $c_1y_1(x) + c_2y_2(x) + \ldots + c_ny_n(x)$ is also a solution of (5.5) on I.

Proof (For $n = 2$), let $y_1(x), y_2(x)$ be the solutions of

$$y'' + P_1(x)y' + P_2(x)y = 0$$

Consider $\phi(x) = c_1y_1(x) + c_2y_2(x)$ where $c_1, c_2 \in \mathbb{R}$

$$\phi'(x) = c_1y_1' + c_2y_2'$$
$$\phi''(x) = c_1y_1'' + c_2y_2''$$

$$\begin{aligned}\phi'' + P_1(x)\phi' + P_2(x)\phi &= c_1y_1'' + c_2y_2'' + P_1(x)[c_1y_1' + c_2y_2'] + P_2(x)[c_1y_1 + c_2y_2] \\ &= c_1[y_1'' + P_1(x)y_1' + P_2(x)y_1] + c_2[y_2'' + P_1(x)y_2' + P_2(x)y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0\end{aligned}$$

Example $y'' + y = 0$
Easy to check that $y_1(x) = \sin x, y_2(x) = 5\sin x$.
Question: Is $c_1 \sin x + c_2 5 \sin x$ a general solution?
No. Check that $y_3 = \cos x$ is also a solution of the DE. we can't write $y_3(x) = c_1 \sin x + c_2 5 \sin x$ for any values of c_1, c_2.

Definition (Linear Independence of Functions)
The functions $f_1, f_2, \ldots, f_n$ are said to be linearly dependent on the interval I provided there exist constants $c_1, c_2, \ldots, c_n$, which are not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0, \qquad \forall x \in I$$

The functions $f_1, \ldots, f_n$ are called linearly independent on the interval I if they are not linearly dependent.

Example 1 $f_1(x) = \sin x, f_2 = 5 \sin x$.
Since $5f_1(x) - f_2(x) = 0 \quad \forall x \in \mathbb{R}$, then they are linearly dependent.

Example 2 $f_1(x) = \sin x, f_2(x) = \cos x$
Let c_1, c_2 be such that

$$(*) \qquad c_1 \sin x + c_2 \cos x = 0, \qquad \forall x \in \mathbb{R}$$

$$x = 0: \quad 0 + c_2 = 0 \implies c_2 = 0$$

$$x = \frac{\pi}{2}: \quad c_1 + 0 = 0 \implies c_1 = 0$$

We want the same values of c_1, c_2 for all $x \in \mathbb{R}$. The only way for $(*)$ to be satisfied is if $c_1 = c_2 = 0$. Thus $\sin x, \cos x$ are linearly independent on $\mathbb{R}$

Definition Suppose the functions $f_1, f_2, \ldots, f_n$ are $(n-1)$ times differentiable on some interval I. The Wronskian of the functions is

$$W(f_1, f_2, \ldots, f_n) = \det \begin{bmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{bmatrix}$$

Note W is a scalar function of x, $\qquad W : I \to \mathbb{R}$.

Lemma Let $f_1, f_2, \ldots f_n$ be $(n-1)$ times differentiable on I. If $f_1, f_2, \ldots f_n$ are linearly dependent on I, the $W(f_1, f_2, \ldots, f_n) \equiv 0$ on I

Proof (For $n = 2$)
Since f_1, f_2 are linearly dependent on I there are $c_1, c_2 \in \mathbb{R}$ which are not both zero such that

$$c_1 f_1(x) + c_2 f_2(x) = 0, \qquad \forall x \in I$$

Differentiate:

$$c_1 f_1'(x) + c_2 f_2'(x) = 0, \qquad \forall x \in I$$

For any $x \in I$, these two equations form a homogeneous linear system for c_1, c_2:

$$\begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (*)$$

We know that c_1, c_2 are not both zero.Since the linear system $(*)$ has a nontrivial solution, we must have the determinant of the coefficient matrix is zero.

$$\det \begin{bmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{bmatrix} = 0 \qquad \forall x \in I$$

$W(f_1, f_2) \equiv 0$ on I $\square$

Theorem
Let $P_1(x), \ldots, P_n(x)$ be continuous on an open interval I. Suppose that $y_1, \ldots, y_n$ are solutions of the homogeneous n^{th} linear DE (5.5) on I. Then there are two possibilities

(a) $y_1(x), \ldots, y_n$ are linearly dependent on I, and $W(y_1, \ldots, y_n) \equiv 0$ on I

(b) $y_1(x), \ldots, y_n$ are linearly independent on I and $W(y_1, \ldots, y_n) \neq 0$ for any $x \in I$

Proof

(a) Follows from the lemma

(b) for $n = 2$. Let y_1, y_2 be two linearly independent solutions on I of (5.5). Assume for contradiction that $W(y_1, y_2) = 0$ at some point $a \in I$.
Consider the linear system:

$$(*) \qquad \begin{aligned} c_1 y_1(a) + c_2 y_2(a) = 0 \\ c_1 y_1'(a) + c_2 y_2'(a) = 0 \end{aligned} \implies \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $W(y_1, y_2) = 0$ at a, this system has a non-trivial solution for c_1, c_2. Let $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ using c_1, c_2 from the solution of $(*)$

- $Y(x)$ is a solution of (5.5) by the superposition principle
- $Y(x)$ satisfies the ICs: $\quad Y(a) = 0, Y'(a) = 0$

But the trivial (zero) solution also satisfies (5.5) and these ICs. By E/U Theorem

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) = 0 \qquad \forall x \in I$$

Since c_1, c_2 are not both zero, this means y_1, y_2 are linearly dependent on I. Contradiction. Thus $W(x) \neq 0 \qquad \forall x \in I$ $\square$

Example $\quad y'' + y = 0$
$y_1(x) = \sin x, y_2 = 2\sin x$ are solutions on $\mathbb{R}$.

$$W(y_1, y_2) = \det \begin{bmatrix} \sin x & 2\sin x \\ \cos x & 2\cos x \end{bmatrix} \equiv 0 \text{ on } \mathbb{R}$$

y_1, y_2 are linearly dependent on $\mathbb{R}$.
$y_1(x) = \sin x, y_3 = 2\cos x$ are solutions on $\mathbb{R}$.

$$W(y_1, y_2) = \det \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} = -1 \neq 0 \text{ on } \mathbb{R}$$

y_1, y_3 are linearly independent on $\mathbb{R}$.

Theorem (General Solution for a Linear Homogeneous Equation)
Let $P_1(x), \ldots, P_n(x)$ be continuous on the open interval I. Let $y_1(x), \ldots, y_n(x)$ be n linearly independent solutions on I of (5.5). If $\phi(x)$ is any solution on I, (5.5) then there are $c_1, \ldots, c_n$ such that

$$\phi(x) = c_1 y_1(x) + \ldots + c_n y_n(x), \quad \forall x \in I$$

Proof $(n = 2)$ Let $\phi(x)$ be a solution of (5.5) on I. Let $a \in I$. Consider the linear system.

$$(*) \qquad \begin{bmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \phi(a) \\ \phi'(a) \end{pmatrix}$$

Since y_1, y_2 are linearly independent on I , $W(y_1, y_2) \neq 0$ on I. Thus $\det(M) \neq 0$ and $(*)$ has a solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M^{-1} \begin{pmatrix} \phi(a) \\ \phi'(a) \end{pmatrix}$$

Using these values of c_1, c_2 define

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Then $y(x)$ satisfies the IVP on I consisting of (5.5) and $y(a) = \phi(a), y'(a) = \phi'(a)$. But $\phi(a)$ also satisfies this IVP on I. So by E/U we must have:

$$\phi(x) = y(x) = c_1 y_1(x) + c_2 y_2(x) \quad x \in I$$

In other words, given $y_1, \ldots, y_n$ linearly independent solutions of (5.5), and arbitrary constants $c_1, \ldots, c_n$

$$c_1 y_1(x) + \ldots + c_n y_n(x)$$

is a general solution of (5.5).

Theorem (General Solution for a linear non-homogeneous ODE)
Let $P_1(x), \ldots, P_n(x)$ and $f(x)$ be continuous on an open interval I. Let $y_1, \ldots, y_n$ be linearly independent solutions on I of the homogeneous DE (5.5) and $y_p(x)$ a particular solution on I of the non-homogeneous DE (5.3).
If $\phi(x)$ is any solution of (5.3) on I, then there are $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\phi(x) = c_1 y_1(x) + \ldots + c_n y_n(x) + y_p(x) \qquad \forall x \in I$$

Proof $n = 2$

$$(1)' \qquad y'' + P_1(x)y' + P_2(x)y = f(x)$$
$$(2)' \qquad y'' + P_1(x)y' + P_2(x)y = 0$$

Consider: $y_n(x) = \phi(x) - y_p(x), \qquad \phi, y_p$ satisfy $(1)'$

$$\begin{aligned} y_h'' + P_1(x)y_n' + P_2(x)y_n &= \phi'' + P_1(x)\phi' + P_2(x)\phi - (y_p'' + P_1(x)y_p' + P_2(x)y_p) \\ &= f(x) - f(x) = 0 \end{aligned}$$

Thus $y_n(x)$ satisfies $(2)'$
Form the previous theorem, there are constants $c_1, c_2 \in \mathbb{R}$ such that $y_n(x) = c_1 y_1(x) + c_2 y_2(x), \forall x \in I$.
(y_1, y_2 are linearly independent solutions of $(2)'$ on I)
Thus

$$\phi(x) = y_h(x) + y_p(x) = c_1 y_1 + c_2 y_2 + y_p, \qquad \forall x \in I$$

□

Summary To find a general solution of the non-homogeneous DE (5.3) we need

(1) n linearly independent solutions of the associated homogeneous equation (5.5)

(2) a particular solution of (5.3)

We will start with (5.3) for a special cases end of midterms...

5.1 Homogeneous, linear ODEs with constant coefficients

In this case $P_1(x), \ldots, P_n(x)$ are constants, so the DE can be written

$$a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \tag{5.6}$$

where $a_0, \ldots, a_n \in \mathbb{R}$.

What kinds of functions satisfy (5.6). There is a linear combination of y and its first n derivatives that is zero.
$y = e^{rx} \qquad y' = re^{rx} \qquad y'' = r^2 e^{rx}$

Look for solutions in the form $y = e^{rx}$, where r is TBD.
Substituting into (5.6)

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \ldots + a_1 r e^{rx} + a_0 e^{rx} = 0$$

$$(a_n r^n + \ldots + a_1 r + a_0) e^{rx} = 0$$

Since $e^{rx} \neq 0$ fot this equation to be satisfied, we need

$$\underbrace{a_n r^n + \ldots + a_1 r + a_0}_{p(r) \quad \text{polynomial in } r} = 0 \tag{5.7}$$

Equation (5.7) is called the <u>characteristic equation</u> for (5.6). $p(r)$ is called the <u>characteristic polynomial</u> for (5.6).

Summary If r is a root of (5.7) (a zero of $p(r)$) , e^{rx} is a solution of (5.6).

Special case: $n = 2$

$$a_2 y'' + a_1 y' + a_0 y = 0 \tag{5.8}$$

Characteristic equation

$$a_2 r^2 + a_1 r + a_0 = 0 \tag{5.9}$$

Roots of (5.9): $r \pm \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$

Three cases

(1) Two real distinct roots ($a_1^2 - 4a_0 a_2 > 0$), denote by $r_+, r_- \in \mathbb{R}$.
$\implies$ two solutions of (5.8): $\quad y_1(x) = e^{r_+ x},\ y_2(x) = e^{r_- x}, \qquad x \in \mathbb{R}$

(2) One real root ($a_1^2 - 4a_0 a_2 = 0$), $r_+ = r_- = r = -\frac{a_1}{2a_2} \in \mathbb{R}$
$\implies$ one solution of (5.8). $y_1(x) = e^{rx}, x \in \mathbb{R}$
$\implies$ Need a second, linearly independent solution

(3) Two complex roots ($a_1^2 - 4a_0 a_2 < 0$). $r_+, r_- \in \mathbb{C}$.
$r_\pm = \alpha \pm i\beta, \quad \alpha, \beta \in \mathbb{R}. \qquad \alpha = -\frac{a_1}{2a_2}, \beta = \frac{\sqrt{4a_0 a_2 - a_1^2}}{2a_2}$

$$e^{(\alpha \pm i\beta)x} = e^{\alpha x} e^{\pm i\beta x} = e^{\alpha x}(\cos \beta x \pm i \sin \beta x) \implies \text{complex valued function?}$$

Theorem (Distinct Real Roots)
If the roots $r_1, \ldots, r_n$ of the characteristic equation (5.7) are real and distinct and $c_1, \ldots, c_n$ are arbitrary constants then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x} \qquad x \in \mathbb{R} \tag{5.10}$$

is a general solution of (5.6)

Proof $(n = 2)$
If $r_1, r_2 \in \mathbb{R}$, $r - 1 \neq r_2$ are roots of (5.9), then from our analysis above $y_1(x) = e^{r_1 x}, y_2(x) = e^{r_2 x}$ are two solutions on $\mathbb{R}$ of (5.8).

$$W(y_1, y_2) = \det \begin{bmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{bmatrix} = (r_2 - r_1) e^{(r_1 + r_2)x} \neq 0, \forall x \in \mathbb{R}$$

$\implies y_1, y_2$ are linearly independent.
By the Theorem on general solution for homogeneous equations $c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad x \in \mathbb{R}$ us a general solution for (5.8) $\square$

To proceed to the other cases, we'll make a brief aside.

Linking DE's to Linear Operations

Operator a mapping from function to function

$$T[f(x)] = g(x)$$

Definition The differential operator, D, is the operator defined by $D[y] = y'$. y gets mapped to its derivative.

Definition The identity operator, g, is the operator that maps a function to itself: $g[y] = y$.
Compositions of operators are defined as

$$(T_1 \circ T_2)[f] = T_1[T_2[f]]$$

Examples
$(D \circ D)[y] = D^2[y] = D[D[y]] = D[y'] = y''$
$D^k[y] = D[D[\ldots D[y] \ldots]] = y^{(k)}$

Definition An operator T is linear if for any $c_1, c_2 \in \mathbb{R}$ and any f_1, f_2 in the domain of T, $c_1 f_1 + c_2 f_2$ is in the domain of T and

$$T[c_1 f_1 + c_2 f_2] = c_1 T[f_1] + c_2 T[f_2]$$

Exercise Show D^k is a linear operator for any k.

Given $a_1, a_2 \in \mathbb{R}$ and operators T_1, T_2, we define

$$(a_1 T_1 + a_2 T_2)[y] = a_1 T_1[y] + a_2 T_2[y]$$

Example (A polynomial differential operator)
Let $a_1, a_2 \in \mathbb{R}$. For any differential function y

$$(a_1 D + a_0 g) = a_1 D[y] + a_0 g[y] = a_1 y' + a_0 y$$

Examples
Commutativity

$$(D - ag) \circ (D - bg)[y] = (D - bg) \circ (D - ag)[y]$$

Factoring

$$(D^2 - (a + b)D - abg)[y] = (D - ag) \circ (D - bg)[y]$$

Often we will write $ag[y] = a[y]$ where $a \in \mathbb{R}$

Using these ideas we may rewrite the homogeneous DE (5.6) as an operator equation

$$\begin{aligned} &a_n y^{(n)} + \ldots + a_1 y' + a_0 y = 0 \\ &a_n D^n[y] + \ldots + a_1 D[y] + a_0 g[y] = 0 \to \text{ function} \\ &\underbrace{(a_n D^n + \ldots + a_1 D + a_0 g)}_{P(D)}[y] = 0 \end{aligned} \tag{5.11}$$

Then solving (5.6) is equivalent to finding the functions y that are mapped to 0 by $P(D)$.
Note that $P(D)$ is a linear operator.

Example For $n = 2$, we have $a_2 y'' + a_1 y' + a_0 y = 0$.
Operator form:

$$\begin{aligned} &(a_2 D^2 + a_1 D + a_0)[y] = 0 \\ &(a_2 g) \circ (D^2 + b_1 D + b_0 g)[y] = 0 \qquad b_1 = \frac{a_1}{a_2}, b_2 = \frac{a_0}{a_2} \\ &(a_2 g) \circ (D - r_+) \circ (D - r_-)[y] = 0 \end{aligned}$$

where $r_\pm = \dfrac{-b_1 \pm \sqrt{b_1^2 - 4b_0}}{2} = \dfrac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}$

Repeated Roots $r_+ = r_- = r = -\frac{a_1}{2a_2}$
Operator form of DE: $(*)$

$$a_2 (D - r)^2[y] = 0$$

$$a_2 (D - r)[(D - r)[y]] = 0$$

One solution is $y = e^{rx}$ since $(D - r)[e^{rx}] = re^{rx} - re^{rx} = 0$
Look for a second solution in the form $u(x)e^{rx}$.
Substitute this into $(*)$

$$\begin{aligned} a_2 (D - r)[(D - r)[u(x)e^{rx}]] &= a_2 (D - r)[u' e^{rx} + rue^{rx} - rue^{rx}] \\ &= a_2 (D - r)[u' e^{rx}] \\ &= a_2 [u'' e^{rx} + u' re^{rx} - u' re^{rx}] \\ &= a_2 u'' e^{rx} \end{aligned}$$

In order for $u(x)e^{rx}$ to satisfy $(*)$ we need

$$a_2 u'' e^{rx} = 0 \iff u''(x) = 0 \iff u(x) = c_1 + c_2 x, \qquad c_1, c_2 \text{ constants}$$

$\implies$ Any solution of $(*)$ is in this form $(c_1 + c_2 x)e^{rx}$, $x \in \mathbb{R}$.
Can check that e^{rx}, xe^{rx} are linearly independent functions.

Theorem (Repeated Roots)
If the characteristic equation (5.7) has a repeated root $\overline{r} \in \mathbb{R}$ of multiplicity k, then the following are k linearly independent solutions of (5.6) on $\mathbb{R}$.

$$e^{\overline{r}x}, xe^{\overline{r}x}, x^2e^{\overline{r}x}, \ldots, x^{k-1}e^{\overline{r}x}$$

Proof Since the characteristic equation as a root $\overline{r}$ of multiplicity k it may be written.

$$p(r) = q(r)(r - \overline{r})^k = 0 \qquad \text{where } q(r) \text{ is of degree } n - k$$

Thus (5.6) in operator form can be written

$$q(D)(D - \overline{r})^k[y] = 0$$

We know $e^{\overline{r}x}$ is a solution of the DE and y will be a solution if $(D - \overline{r})^k[y] = 0$
Consider $u(x)e^{rx}$. This will be a solution if

$$(D - \overline{r})^k[ue^{\overline{r}x}] = 0$$

Can show by induction that

$$(D - \overline{r})^k[ue^{\overline{r}x}] = u^{(k)}e^{\overline{r}x}$$

$ue^{\overline{r}x}$ is a solution of (5.6) if and only if $u^{(k)}e^{\overline{r}x} = 0$ if and only if $u^{(k)} = 0$ if and only if

$$u(x) = c_1 + c_2x + \ldots + c_kx^{k-1}, \qquad c_1, \ldots, c_k \in \mathbb{R}$$

Thus $c_1e^{\overline{r}x} + \ldots + c_kx^{k-1}e^{\overline{r}x}$ is a solution of (5.6) for any $c_1, \ldots, c_k \in \mathbb{R}$
$\implies e^{\overline{r}x}, xe^{\overline{r}x}, \ldots, x^{k-1}e^{\overline{r}x}$ are solutions on $\mathbb{R}$ of (5.6).
Can show that these are linearly independent. □
(See problem 29 in section 3.2 of text)

Complex roots
Recall for $n = 2$

$$a_2y'' + a_1y' + a_0y = 0$$

$$p(r) = a_2r^2 + a_1r + a_0 = 0$$

If $a_1^2 - 4a_0a_2 < 0$, the roots are complex.
$r_\pm = \alpha \pm i\beta$ where $\alpha = -\frac{a_1}{2a_2}, \beta = \frac{1}{2a_2}\sqrt{4a_0a_2 - a_1^2}$.
Note that $p(r)$ may be rewritten:

$$\begin{aligned} p(r) &= a_2(r^2 + b_1r + b_0r) \\ &= a_2(r - (\alpha + i\beta))(r - (\alpha - i\beta)) \\ &= a_2(r^2 - 2\alpha r + \alpha^2 + \beta^2) \\ &= a_2((r - \alpha)^2 + \beta^2) \end{aligned}$$

Theorem (Complex Roots)
If the characteristic equation (5.7) has a pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity 1. Then two linearly independent solutions of (5.6) on $\mathbb{R}$ are

$$y_1(x) = e^{\alpha x}\cos(\beta x), \qquad y_2(x) = e^{\alpha x}\sin(\beta x)$$

Proof In this situation the characteristic polynomial can be written $p(r) = q(r)((r-\alpha)^2+\beta^2)$ where $q(r)$ is of degree $n-2$.
Thus the DE (5.6) in operator form is

$$q(D)((D-\alpha)^2+\beta^2)[y] = 0$$

Consider

$$(D-\alpha)[e^{\alpha x}\cos\beta x] = \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x - \alpha e^{\alpha x}\cos\beta x = -\beta e^{\alpha x}\sin\beta x$$

Similarly $(D-\alpha)[e^{\alpha x}\sin\beta x] = \beta e^{\alpha x}\cos\beta x$

Thus

$$\begin{aligned}[(D-\alpha)^2+\beta^2][e^{\alpha x}\cos\beta x] &= (D-\alpha)^2[e^{\alpha x}\cos\beta x] + \beta^2 e^{\alpha x}\cos\beta x\\ &= (D-\alpha)[-\beta e^{\alpha x}\sin\beta x + \beta^2 e^{\alpha x}\cos\beta x]\\ &= -\beta^2 e^{\alpha x}\cos\beta x + \beta^2 e^{\alpha x}\cos\beta x = 0\end{aligned}$$

Similarly, we can show $((D-\alpha)^2+\beta^2)[e^{\alpha x}\sin\beta x] = 0$
So $e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x$ are solutions of (5.6)

Consider

$$W = \det\begin{bmatrix} e^{\alpha x}\cos\beta x & e^{\alpha x}\sin\beta x\\ \alpha e^{\alpha x}\cos\beta x - \beta e^{\alpha x}\sin\beta x & \alpha e^{\alpha x}\sin\beta x + \beta e^{\alpha x}\cos\beta x\end{bmatrix}$$

Thus $W(e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x) = \beta e^{2\alpha x} \neq 0$ on $\mathbb{R}$.
So $e^{\alpha x}\cos\beta x,\ e^{\alpha x}\sin\beta x$ are linearly independent on $\mathbb{R}$. □

Theorem (Repeated Complex Roots)
If the characteristic equation (5.7) has a pair of complex conjugate roots $\alpha \pm i\beta$ of multiplicity k, then the following are $2k$ linearly independent solutions of (5.6)

$$e^{\alpha x}\cos\beta x,\quad e^{\alpha x}\sin\beta x,\quad xe^{\alpha x}\cos\beta x,\ldots,x^{k-1}e^{\alpha x}\cos\beta x,\quad x^{k-1}e^{\alpha x}\sin\beta x$$

Proof Puts together the ideas from previous two theorems. □

Example Find the solution of the IVP $y'' - 2y' + 5y = 0,\quad y(0) = 1,\quad y'(0) = 2$

Solution The characteristic equation is: $r^2 - 2r + 5 = 0, \qquad (D^2 - 2D - 5g)(y) = 0.$
Roots $r = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$
From the theorem last class, two linearly independent solutions are $y_1(x) = e^x\cos 2x,\quad y_2(x) = e^x\sin 2x$
General solutions: $y(x) = c_1e^x\cos 2x + c_2e^x\sin 2x.$ where $c_1, c-2$ are arbitrary constants.

$$y'(x) = c_1e^x\cos 2x - 2c_1e^x\sin 2x + c_2e^x\sin 2x + 2c_2e^x\cos 2x$$

Apply IC's
$y(0) = 1 \implies 1 = c_1$
$y'(0) = 2 \implies 2 = c_1 + 2c_2$
$\implies c_1 = 1, c_2 = \frac{1}{2}$

Solution of the IVP

$$y(x) = e^x \cos 2x + \frac{1}{2} e^x \sin 2x, \quad x \in \mathbb{R}$$

Example Find a general solution of $y''' - 3y'' + 3y' - y = 0$.
Characteristic equation $r^3 - 3r^2 + 3r - 1 = 0 \implies (r-1)^3 = 0$
Roots $r = 1$, with multiplicity 3.
From Theorem last class, 3 linearly independent solution of the DE are e^x, xe^x, x^2e^x
$\implies$ General solution: $y(x) = c_1e^x + c_2xe^x + c_3x^2e^x$ where c_1, c_2, c_3 are arbitrary.

Example Find a general solution of

$$y^{(4)} - 3y'' - 4y = 0$$

Characteristic equation: $r^4 - 3r^2 - 4 = 0$
$(r^2+1)(r^2-4) = 0$
$r^2 = 4 \implies r = \pm 2$
$r^2 = -1 \implies \implies r = \pm i \implies \alpha = 0, \beta = 1$

From our Theorem from last class, 4 solutions of the DE are:

$$y_1(x) = e^{2x}, y_2(x) = e^{-2x}, y_3(x) = \cos x, y_4(x) = \sin x$$

Need to check these are linearly independent.
From Maple, $W(y_1, y_2, y_3, y_4) = -100 \neq 0 \implies$ linear independent on $\mathbb{R}$.
General solution:

$$y(x) = c_1e^{2x} + c_2e^{-2x} + c_3 \cos x + c_4 \sin x, \qquad x \in \mathbb{R}$$

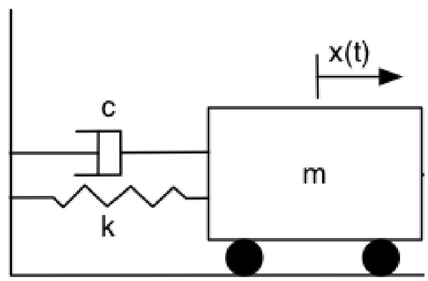
where c_1, c_2, c_3, c_4 are arbitrary.

5.2 Applications

Slides on learn. (and appended here)

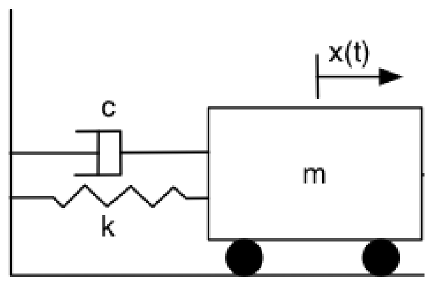
Applications of Second Order, Constant Coefficient, Homogeneous ODEs

Mass Spring System



Consider an object of mass *m* attached to a spring which is attached to wall. The object moves on a frictionless surface but is a attached to a damper that provides resistance proportional to the speed of the object.

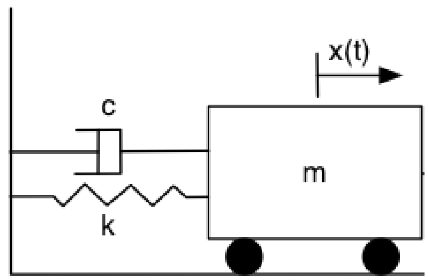
Mass Spring System



Consider an object of mass m attached to a spring which is attached to wall. The object moves on a frictionless surface but is a attached to a damper that provides resistance proportional to the speed of the object.
Let $x(t)$ be the displacement of the object from its resting position.
Take x positive to the right as in the figure. Then

$$
\begin{aligned}
v(t) &= \frac{dx}{dt} \text{ is the velocity of the object} \\
a(t) &= \frac{dv}{dt} = \frac{d^2x}{dt^2} \text{ is the acceleration of the object}
\end{aligned}
$$

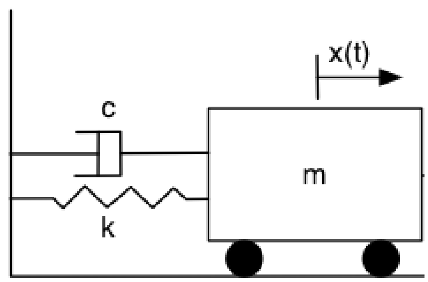
Forces



Spring Force: Hooke's law

- magnitude of force is proportional to amount spring is stretched/compressed
- force acts in direction to restore spring to original state.

Forces



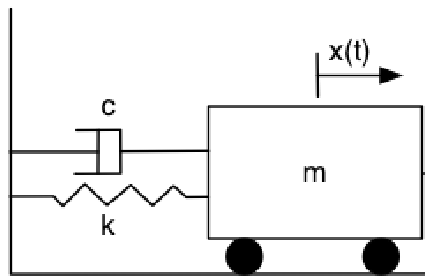
Spring Force: Hooke's law

- magnitude of force is proportional to amount spring is stretched/compressed
- force acts in direction to restore spring to original state.

$$F_{\text{spring}} = -kx, \quad \text{where } k > 0$$

k is called the spring constant

Forces



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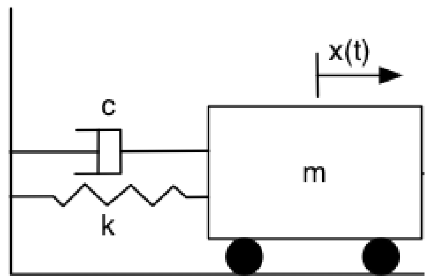
Resistance/Damping:

- magnitude of force is proportional to speed of object
- force acts in opposite direction of velocity

Forces



Spring Force: Hooke's law

- magnitude of force is proportional to amount spring is stretched/compressed
- force acts in direction to restore spring to original state.

$$F_{\text{spring}} = -kx, \quad \text{where } k > 0$$

k is called the spring constant

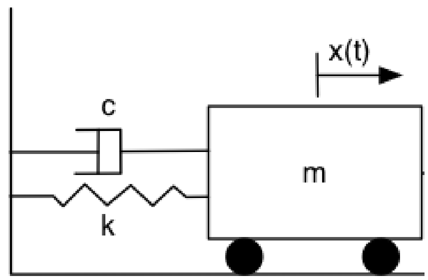
Resistance/Damping:

- magnitude of force is proportional to speed of object
- force acts in opposite direction of velocity

$$F_{\text{damping}} = -c\frac{dx}{dt}, \quad \text{where } c > 0$$

c is called the damping constant

Model



Applying Newton's second law

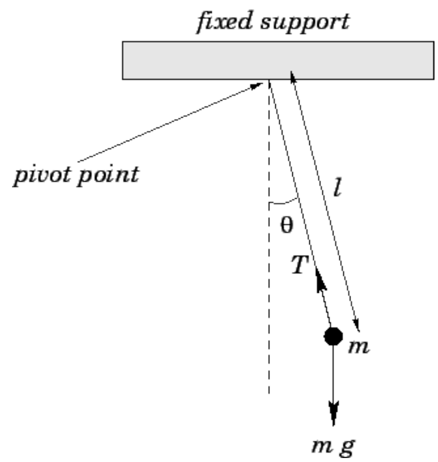
$$\text{mass} \times \text{acceleration} = F_{\text{spring}} + F_{\text{damping}}$$

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx$$

or

$$mx'' + cx' + kx = 0$$

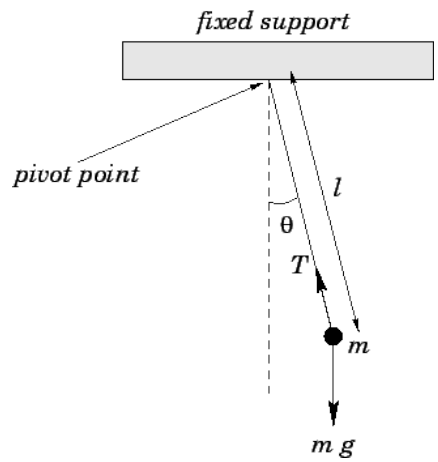
Simple Pendulum



Consider an object of mass *m* swinging on a string (or massless rod) of length *l*.
Assume there is no friction or air resistance so the only forces acting are gravity *mg* and the tension in the string *T*.

Simple Pendulum

Consider an object of mass m swinging on a string (or massless rod) of length l.
Assume there is no friction or air resistance so the only forces acting are gravity mg and the tension in the string T.



Let $\theta(t)$ be the angle the string makes with the vertical at time t.

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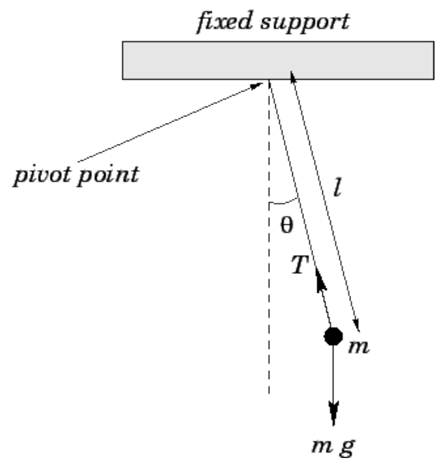
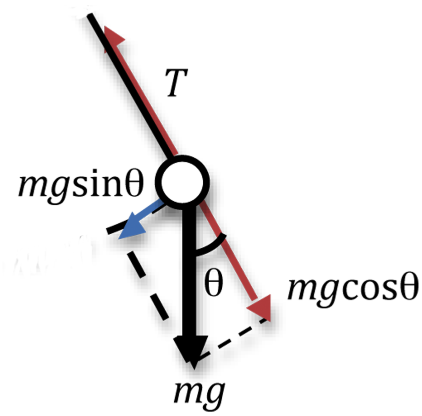
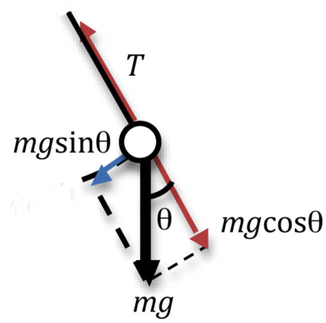


Let $\theta(t)$ be the angle the string makes with the vertical at time t.

Resolve forces into radial (along string) and tangential (perpendicular to string) coordinates.



Applying Newton's second law - radial direction

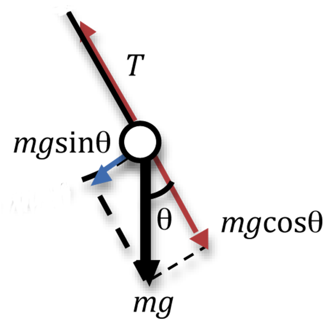


$$\text{mass} \times (\text{radial acceleration}) \quad = \quad \text{sum of radial forces}$$

Assuming the string doesn't stretch or bend, the radial displacement is fixed. Thus the radial velocity and acceleration are zero giving

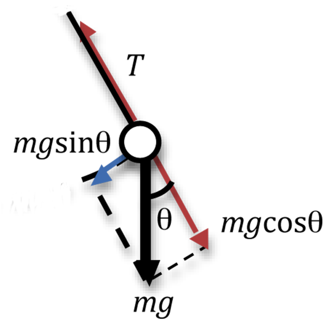
$$0 = T - mg\cos(\theta) \quad \Rightarrow \quad T = mg\cos(\theta)$$

Applying Newton's second law - tangential direction



mass $\times$ (tangential acceleration) $=$ sum of tangential forces

Applying Newton's second law - tangential direction



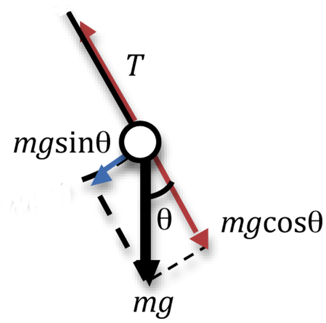
mass $\times$ (tangential acceleration) $=$ sum of tangential forces

Let s be the tangential displacement from the bottom.

Then $s = l\theta$, $l\dfrac{d\theta}{dt}$ is the tangential velocity of the object

and $l\dfrac{d^2\theta}{dt^2}$ is the tangential acceleration of the object

Applying Newton's second law - tangential direction



mass $\times$ (tangential acceleration) $=$ sum of tangential forces

Let s be the tangential displacement from the bottom.

Then $s = l\theta$, $l\dfrac{d\theta}{dt}$ is the tangential velocity of the object

and $l\dfrac{d^2\theta}{dt^2}$ is the tangential acceleration of the object

Then we have

$$ml\frac{d^2\theta}{dt^2} = -mg\sin(\theta) \quad \text{or} \quad l\theta'' + g\sin(\theta) = 0$$

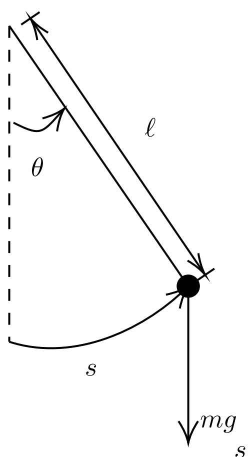
5.2.1 Mass-Spring System Model

$mx'' + cx' + kx = 0$ (constant coefficient, 2nd order)

- x - displacement of object from rest position
- t - time
- $m > 0$ - mass
- $c > 0$ - damping constant
- $k > 0$ - spring constant

Initial condition $x(0) = x_0$ (initial position) $x'(0) = v_0$ (initial velocity)

5.2.2 Pendulum Model



$s = \ell\theta$ $\quad\theta = \dfrac{s}{\ell}$

$$m\ell\theta'' + mg\sin(\theta) = 0$$

$$\theta'' + \frac{g}{l}\sin(\theta) = 0 \qquad \text{not linear}$$

g - gravitational acceleration
ℓ - length of a string
Initial conditions: $\theta(0) = \theta_0$ (initial angle) $\quad\theta'(0) = v_0$ (initial angular velocity)

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin(\theta) = 0 \qquad \theta(0) = \theta_0, \quad \theta'(0) = v_0$$

$\theta(t)$ - angle pendulum makes with vertical
This is a non-linear DE due to $\sin\theta$

Note this has the equilibrium solution: $\theta(t) = 0, t \in \mathbb{R}$
Corresponds to the pendulum hanging straight down.
If $|\theta|$ is small, then we can use the linear approximation. $\sin\theta \approx \theta$
If we put this in the model we get a linear, constant coefficient DE:

$$\theta'' + \frac{g}{\ell}\theta = 0, \qquad \theta(0) = \theta_0, \theta'(0) = v_0$$

The two applications have models in the form

$$y'' + b_1 y + b_0 y = 0$$

with mass spring: $b_1 = \frac{c}{m} > 0, \quad b_0 = \frac{k}{m} > 0$
linear pendulum: $b_1 = 0, \quad b_2 = \frac{g}{l} > 0$

Depending on the values of b_1, b_0 the systems will have different behaviour.

Free, undamped case $b_1 = 0$ (Pendulum or mass-spring with $c = 0$)
Sine $b_0 > 0$, let $b_0 = \omega_0^2, (\omega_0 > 0)$
Characteristic equation: $r^2 + \omega_0^2 = 0 \implies$ roots: $r = \pm i\omega_0$
General solution: $y(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t \qquad c_1, c_2$ arbitrary.
Rewrite (Using trig identity): $y(t) = A\cos(\omega_0 t - B) \qquad A, B$ arbitrary.
Those arbitrary constants are determined by IC's
The motion is periodic with amplitude A and period $\frac{2\pi}{\omega_0}$
This is called simple harmonic motion

Free, damped motion $(b_1, b_0 > 0)$
Characteristic equation: $r^2 + b_1 r + b_0 = 0$

- case 1 $b_1 - 4b_0 < 0$
  Complex conjugate roots $= r_\pm = \alpha \pm i\beta = -\frac{b_1}{2} + \frac{i}{2}\sqrt{b_1^2 - 4b_0}$
  General solution:
  $$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t = Ae^{\alpha t}\cos(\beta t - B)$$
  Since $b_1 > 0, \alpha < 0$, so $\lim_{t\to\infty} y(t) = 0$
  Motion is oscillatory with amplitude that decays in time
  This is called the underdamped case.

- case 2 $b_1^2 = 4b_0 \qquad$ One real, repeated root $r = -\frac{b_1}{2} < 0$
  General solution $y(t) = c_1 e^{rt} + c_2 t e^{rt}$
  $\lim_{t\to\infty} y(t) = 0$ Motion is not oscillatory.
  This is called the critically damped case

- case 3 $b_1^2 > 4b_0 \qquad$ Two real, distinct roots
  $$r_\pm = -\frac{b_1}{2} \pm \frac{1}{2}\sqrt{b_1^2 - 4b_0}$$
  Since $b_1 > 0$ and $b_0 > 0 \implies r_- < r_+ < 0$
  General solution: $y(t) = c_1 e^{r_+ t} + c_2 e^{r_- t} R$
  $\lim_{t\to\infty} y(t) = 0$ Motion is not oscillatory.
  This is called the overdamped case

5.3 Non-homogeneous DE

$$y^{(n)} + P_1(x) y^{(n-1)} + \ldots + P_n(x) y = f(x) \tag{5.12}$$

Associated homogeneous DE

$$y^{(n)} + P_1(x) y^{(n-1)} + \ldots + P_n(x) y = 0 \tag{5.13}$$

General solution of (5.12):

$$y(x) = \underbrace{y_h(x)}_{\text{general solution of (5.13)}} + \underbrace{y_p(x)}_{\text{particular solution of (5.12)}}$$

We'll see two methods for finding $y_p(x)$

5.3.1 Method of Undetermined Coefficients

Applies if $P_1(x), \ldots, P_n(x)$ are constants.
In this case the DE can be written in operator form

$$\underbrace{P(D)}_{\text{characteristic polynomial}} = f(x) \qquad \longrightarrow \text{Look for } y_p(x) \text{ that is mapped from } P(D) \text{ to } f(x)$$

Idea If $f(x)$ has a finite number of derivative look for $y_p(x)$ as a linear combination of derivatives of f.
If $y_p(x)$ is a solution of (5.13) $\quad P(D)[y_p(x)] = 0$
In this case if we consider $xy_p(x)$

$$P(D)[xy_p(x)] = a_1 y_p(x)$$

Example:

Example: Find a general solution of

$$y'' - 2y' + 5y = x^2 + \cos(2x) \tag{1}$$

Solution: The associated homogeneous equation is

$$y'' - 2y' + 5y = 0.$$

From last lecture we know a general solution of this is

$$y_h(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

Thus we just need to find a particular solution of (1).
The RHS of (1) is $f(x) = x^2 + \cos(2x) = f_1(x) + f_2(x)$.
Thus we choose $y_p(x) = y_{p1}(x) + y_{p2}(x)$ where

$$\begin{aligned} y_{p1}(x) &= Ax^2 + Bx + C \\ y_{p2}(x) &= M_1 \cos(2x) + M_2 \sin(2x) \end{aligned}$$

No part of either of these is in $y_h(x)$, thus

$$y_p(x) = Ax^2 + Bx + C + M_1 \cos(2x) + M_2 \sin(2x)$$

Take derivatives:

$$\begin{aligned}
y_p(x) &= Ax^2 + Bx + C + M_1\cos(2x) + M_2\sin(2x) \\
y_p'(x) &= 2Ax + B - 2M_1\sin(2x) + 2M_2\cos(2x) \\
y_p''(x) &= 2A - 4M_1\cos(2x) - 4M_2\sin(2x)
\end{aligned}$$

Substitute into the DE, $y'' - 2y' + 5y = x^2 + \cos(2x)$:

$$\begin{aligned}
2A - 4M_1\cos(2x) - 4M_2\sin(2x)& \\
-4Ax - 2B + 4M_1\sin(2x) - 4M_2\cos(2x)& \\
+5Ax^2 + 5Bx + 5C + 5M_1\cos(2x) + 5M_2\sin(2x) &= x^2 + \cos(2x)
\end{aligned}$$

Collect terms

$$\begin{aligned}
5Ax^2 + (5B - 4A)x + (2A - 2B + 5C)& \\
+(M_1 - 4M_2)\cos(2x) + (4M_1 + M_2)\sin(2x) &= x^2 + \cos(2x)
\end{aligned}$$

Form linear system by equating coefficients of like terms

$$\begin{aligned} 5A &= 1 \\ -4A+5B &= 0 \\ 2A-2B+5C &= 0 \end{aligned} \quad \Rightarrow A=\frac{1}{5},\ B=\frac{4}{25},\ C=-\frac{2}{125}$$

$$\begin{aligned} M_1-4M_2 &= 1 \\ 4M_1+M_2 &= 0 \end{aligned} \Rightarrow M_1=\frac{1}{17},\ M_2=-\frac{4}{17}$$

Thus

$$y_p(x)=\frac{1}{5}x^2+\frac{4}{25}x+-\frac{2}{125}+\frac{1}{17}\cos(2x)+-\frac{4}{17}\sin(2x)$$

General solution of the DE $y''-2y'+5y=x^2+\cos(2x)$ is

$$y(x)=c_1e^x\cos(2x)+c_2e^x\sin(2x)+y_p(x), \quad x\in\mathbb{R}.$$

Example Find the general solution of

$$y^{(4)} - 3y'' - 4y = e^{2x}\sin x + 2\cos x$$

Solution Associated homogeneous equation: $y^{(4)} - 3y'' - 4y = 0$
has general solution: $y_h(x) = c_1e^{2x} + c_2e^{-2x} + c_3\cos x + c_4\sin x$
Form for $y_p(x)$:

$$f(x) = e^{2x}\sin x + 2\cos x = f_1(x) + f_2(x)$$

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$$

$$y_{p_1}(x) = M_1e^{2x}\cos x + M_2e^{2x}\sin x$$

$$y_{p_2}(x) = A\cos x + B\sin x$$

Since $y_{p_2}(x)$ is a solution of associated homogeneous equation.
Multiply $y_{p_2}(x)$ by x:

$$y_{p_2}(x) = Ax\cos x + Bx\sin x$$

5.3.2 Method of Variation of Parameters

Method of Variation of Parameters (Joseph Lagrange, 1774)

Consider the second order, nonhomogeneous ordinary differential equation

$$y'' + p_1(x)y' + p_2(x)y = f(x) \tag{1}$$

Suppose that $p_1(x), p_2(x), f(x)$ are continous on some open interval $I \subset \mathbb{R}$. Suppose that the associated homogeneous equation

$$y'' + p_1(x)y' + p_2(x)y = 0 \tag{2}$$

has linearly independent solutions on I, $y_1(x),\ y_2(x)$, giving the general solution

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),\ x \in I \tag{3}$$

Lagrange's Idea: Look for a solution of (1) by "varying the parameters" in (3), i.e., by replacing c_1, c_2 by functions of x which are to be determined:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),\ x \in I \tag{4}$$

Substituting (4) into (1) will give one condition on u_1, u_2, we will impose another to get two equations to solve for u_1, u_2

Differentiate expression for $y_p(x)$

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ y_p' &= u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2' \\ y_p'' &= u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2'' \end{aligned}$$

Impose the condition:

$$u_1' y_1 + u_2' y_2 = 0$$

so that the derivatives simplify

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ y_p' &= u_1 y_1' + u_2 y_2' \\ y_p'' &= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' \end{aligned}$$

Substitute expressions for y_p, y_p', y_p'' into the DE (1)

$$\underbrace{u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''}_{y_p''} + p_1(x)[\underbrace{u_1y_1' + u_2y_2'}_{y_p'}] + p_2(x)[\underbrace{u_1y_1 + u_2y_2}_{y_p}] = f(x)$$

Rearrange

$$u_1[\underbrace{y_1'' + p_1(x)y_1' + p_2(x)y_1}_{=0 \text{ since } y_1 \text{ solution of } (2)}] + u_2[\underbrace{y_2'' + p_1(x)y_2' + p_2(x)y_2}_{=0 \text{ since } y_2 \text{ solution of } (2)}] + u_1'y_1' + u_2'y_2' = f(x)$$

Thus we have

$$u_1'y_1' + u_2'y_2' = f(x)$$

Summary: Look for a solution of the nonhomogeneous equation (1) in the form $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$, where u_1, u_2 satisfy

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 && (5) \\ u_1'y_1' + u_2'y_2' &= f(x) && (6) \end{aligned}$$

The two conditions (5) and (6) form a linear system for u_1', u_2'

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(x) \end{pmatrix}$$

Since y_1, y_2 are linearly independent solutions of (2) on I, $W(y_1, y_2) \neq 0$ on I.
So this linear system can be solved for u_1', u_2' (on I).
The expressions obtained can be integrated to find $u_1(x), u_2(x)$.

Note: This approach may be extended to higher order equations, we just need to impose more conditions

Example Find a general solution of

$$y'' + 4y = \sec 2x \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}$$

Solution Associated Homogeneous equation

$$y'' + 4y = 0$$

Characteristic equation: $r^2 + 4 = 0 \implies r = \pm 2i$
General solution: $y_h(x) = c_1 \cos 2x + c_2 \sin 2x$
Can't use Method of Undetermined coefficients due to $\sec 2x$ on RHS.

Variation of Parameters:
Assume $y_p(x) = u_1(x) \cos 2x + u_2 \sin 2x$
Using the conditions we derived:

$$u_1' \cos 2x + u_2' \sin 2x = 0$$

$$u_1'(-2) \sin 2x + u_2' 2 \sin 2x = \sec 2x$$

Solve for u_1' and u_2'

$$u_1' = -\frac{1}{2} \tan 2x \implies u_1(x) = \frac{1}{4} \ln(\cos 2x)$$

$$u_2' = \frac{1}{2} \implies u_2 = \frac{x}{2}$$

So $y_p(x) = \frac{1}{4} \ln(\cos 2x) \quad \cos 2x + \frac{x}{2} \sin 2x$
General solution: $y(x) = y_h(x) + y_p(x)$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4} \ln(\cos 2x) \quad \cos 2x + \frac{x}{2} \sin 2x$$

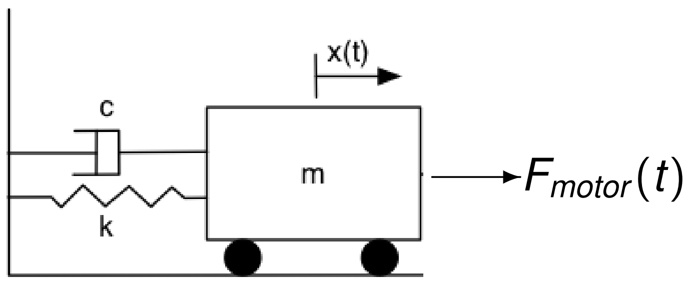
5.4 Applications

see slides

5.4.1 Undamped Forced Motion

Forced, Undamped Motion: Resonance and Beating

Mass Spring System



Consider an object of mass m attached to a spring with spring constant $k > 0$. Suppose there is no damping ($c = 0$), but the mass is attached to a motor that drives it with a force that varies periodically, given by $F_{motor}(t) = F_0 \cos(\omega t)$ where $\omega > 0$. Assume the system starts with the mass at rest in its equilibrium position.

Applying Newton's second law gives

$$\text{mass} \times \text{acceleration} = F_{spring} + F_{motor}$$

which becomes

$$mx'' + kx = F_0 \cos(\omega t), \quad x(0) = 0, \; x'(0) = 0$$

The DE may be rewritten

$$x'' + \omega_0^2 x = A_0 \cos(\omega t), \tag{1}$$

where $\omega_0 = \sqrt{k/m}$, $A_0 = F_0/m$.
We have seen before the solution of the associated homogeneous equation

$$x'' + \omega_0^2 x = 0$$

is $x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$.

We will find a particular solution of (1) using Method of Undetermined Coefficients.
The form of the particular solution is

$$x_p(t) = M_1 \cos(\omega t) + M_2 \sin(\omega t) \qquad \text{if } \omega \neq \omega_0.$$

The form of the particular solution is

$$x_p(t) = M_1 t \cos(\omega_0 t) + M_2 t \sin(\omega_0 t) \qquad \text{if } \omega = \omega_0.$$

Case I: $x_p(t) = M_1 \cos(\omega t) + M_2 \sin(\omega t)$ $(\omega \neq \omega_0)$

Substituting into the DE gives (exercise)

$$-\omega^2(M_1 \cos(\omega t) + M_2 \sin(\omega t)) + \omega_0^2(M_1 \cos(\omega t) + M_2 \sin(\omega t)) = A_0 \cos(\omega t)$$

Thus $\quad x_p(t) = \dfrac{A_0}{\omega_0^2 - \omega^2} \cos(\omega t).$

General solution

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Applying the initial conditions $x(0) = 0,\ x'(0) = 0$ yields (exercise)

$$C_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)},\ C_2 = 0$$

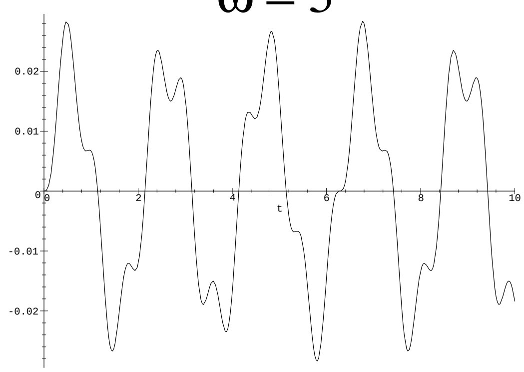
Thus the solution of the IVP is

$$x(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos(\omega t) - \cos(\omega_0 t))$$

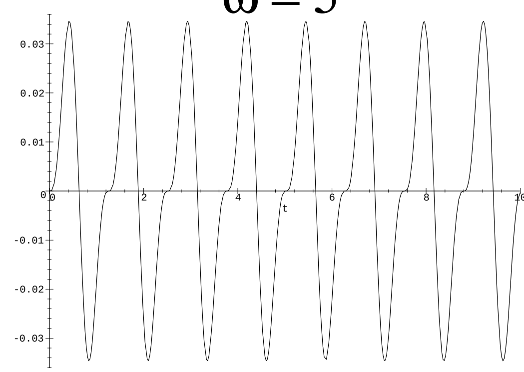
Solution has two frequencies: ω_0 (natural frequency) and ω (forcing frequency).

Example: $\omega_0 = 10$

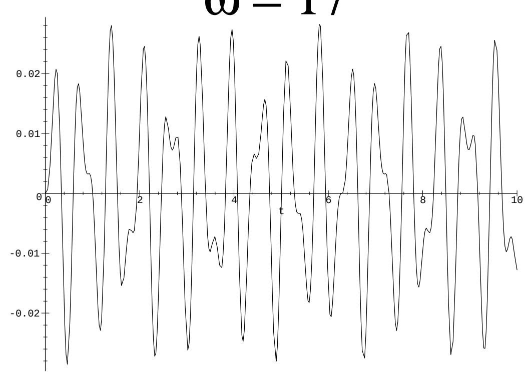
$\omega = 3$



$\omega = 5$

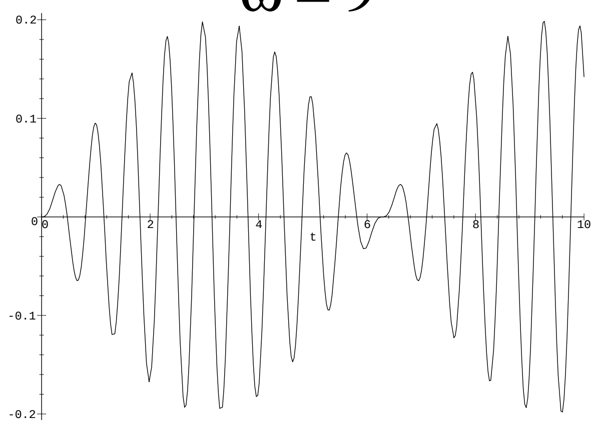


$\omega = 17$

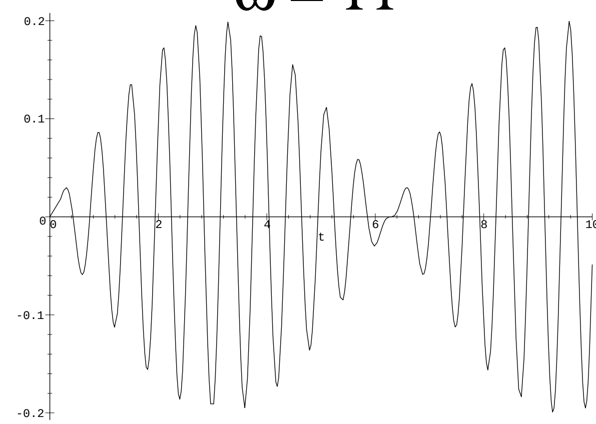


Example: $\omega_0 = 10$

$\omega = 9$



$\omega = 11$



To understand the last two figures, use the trig identity

$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$$

two rewrite the solution (exercise)

$$x(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right) \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$$

If ω is close to ω_0 then $\omega_0 - \omega$ is small
and $\omega_0 + \omega \approx 2\omega_0$ is much larger.

$$x(t) = \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)}_{\text{slowly varying amplitude}} \underbrace{\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)}_{\text{fast oscillation}}$$

This is called **beating**.

Case II: $x_p(t) = M_1 t\cos(\omega_0 t) + M_2 t\sin(\omega_0 t)$ $(\omega = \omega_0)$

Substituting into the differential equation gives (exercise)

$$M_1 = 0, M_2 = \frac{F_0}{2m\omega_0} \qquad \Rightarrow \qquad x_p(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$

General solution is

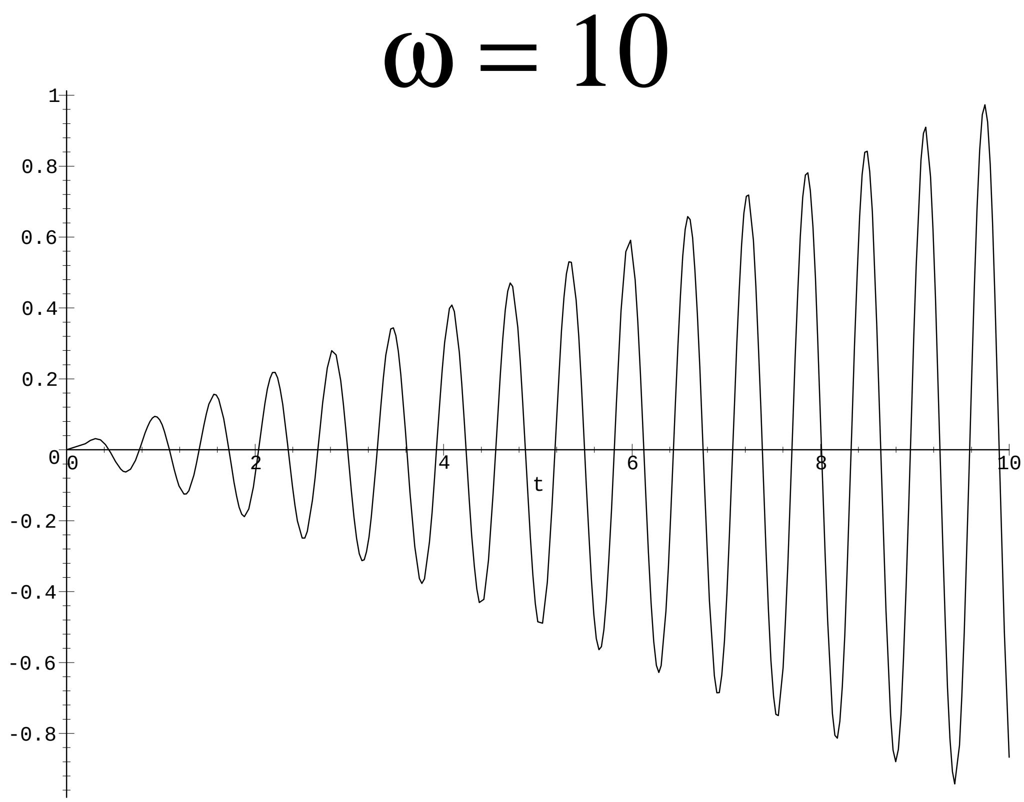
$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$

Applying the initial conditions $x(0) = 0,\ x'(0) = 0$ yields (exercise)
$C_1 = C_2 = 0$.
Solution of initial value problem.

$$x(t) = \frac{F_0}{2m\omega_0} t\sin(\omega_0 t)$$

Note that the amplitude of the solution grows without bound as t increases.
This is called **pure resonance**

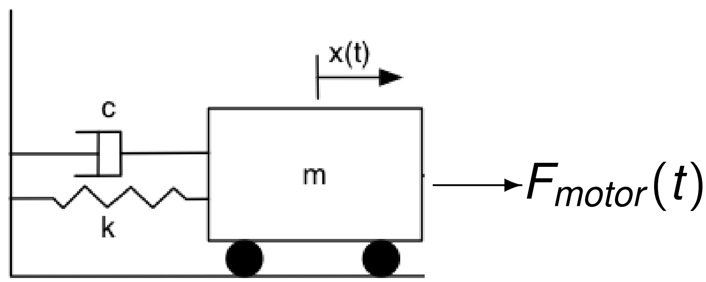
Example: $\omega_0 = 10$



5.4.2 Damped Forced Motion

Forced, Damped Motion:
Practical Resonance

Mass Spring System



Consider an object of mass m attached to a spring with spring constant $k > 0$. There is damping with damping constant $c > 0$, and the mass is attached to a motor that drives it with a force that varies periodically, given by $F_{motor} = F_0 \cos(\omega t)$ where $\omega > 0$. Assume the system starts with the mass at rest in its equilibrium position. Applying Newton's second law gives

$$\text{mass} \times \text{acceleration} = F_{spring} + F_{damping} + F_{motor}$$

which becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \quad x(0) = 0, \; x'(0) = 0 \tag{1}$$

We have seen before the solution of the associated homogeneous equation

$$mx'' + cx' + kx = 0$$

has three cases, depending on the relative sizes of m, c, k.
In all cases, the solution satisfies

$$\lim_{t\to\infty} x_h(t) = 0$$

So $x_h(t)$ will not affect the **longterm** behaviour of the solution.
We thus focus on $x_p(t)$.
We will find a particular solution of (1) using Method of Undetermined Coefficients.
The form of the particular solution is

$$x_p(t) = M_1 \cos(\omega t) + M_2 \sin(\omega t).$$

Substituting into the DE gives (exercise)

$$[(k-m\omega^2)M_1+c\omega M_2]\cos(\omega t)+[(k-m\omega^2)M_2-c\omega M_1]\sin(\omega t)=F_0\cos(\omega t)$$

Solving gives (exercise)

$$M_1=\frac{(k-m\omega^2)F_0}{(k-m\omega^2)^2+c^2\omega^2},\quad M_2=\frac{c\omega F_0}{(k-m\omega^2)^2+c^2\omega^2} \tag{2}$$

Rewrite $x_p(t)$ as

$$x_p(t)=M_1\cos(\omega t)+M_2\sin(\omega t)=A\cos(\omega t-\alpha)$$

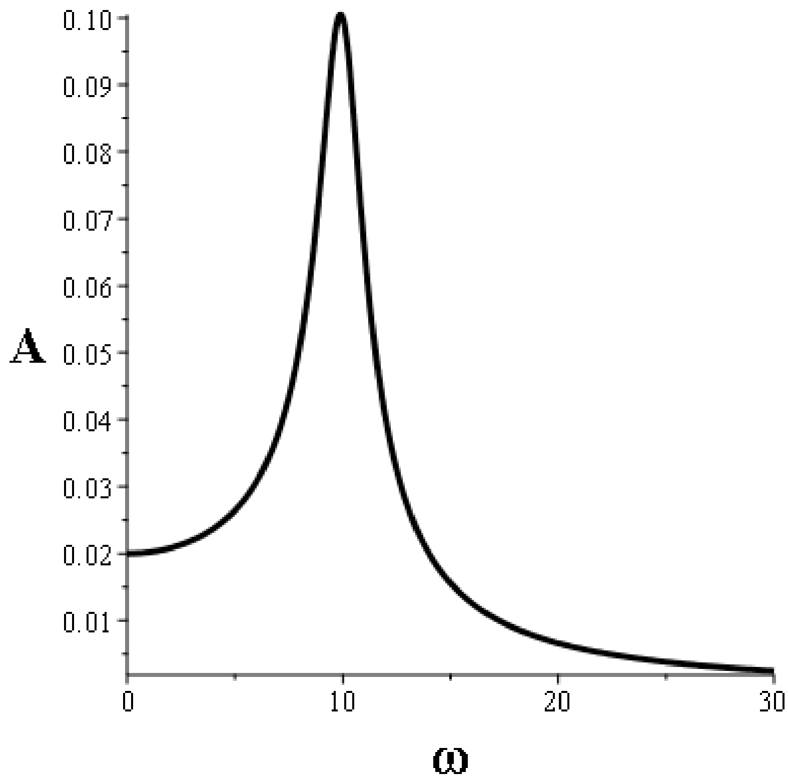
where

$$A=\sqrt{M_1^2+M_2^2},\quad \cos(\alpha)=\frac{M_1}{\sqrt{M_1^2+M_2^2}},\quad \sin(\alpha)=\frac{M_2}{\sqrt{M_1^2+M_2^2}}.$$

A is the **amplitude** of the oscillation, α is the **phase shift**.
Using (2) we see that the amplitude of the oscillations is

$$A=\frac{F_0}{\sqrt{(k-m\omega^2)^2+c^2\omega^2}}.$$

Variation of the amplitude with the forcing frequency, ω.
$(F_0 = 10,\ m = 5,\ c = 10,\ k = 500)$



In particular, one can show that, depending on the values of m, k, c,

$A(\omega)$ may have a maximum at $\omega = \sqrt{\dfrac{k}{m} - \dfrac{c^2}{2m}}$

This is called **practical resonance**.

6 Linear Systems of DEs

Consider the mass spring model

$$mx'' + cx' + kx = F(t) \tag{$*$}$$

Introduce the variables: $\begin{array}{l} x_1 = x \\ x_2 = x' \end{array} \implies \begin{array}{l} x_1' = x' = x_2 \\ x_2' = x'' = x_1'' \end{array}$

Then an equivalent way of writing $(*)$ is

$$\begin{cases} x_1' = x_2 \\ x_2' = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + F(t) \end{cases}$$

This is a system of two first order ODEs for $x_1(t), x_2(t)$.
In general, any n^{th} order ODE can be written as a system of n first order ODEs.

6.1 Matrices and Linear Systems

Definition A first order n-dimensional linear system of ODEs is a set of n equations involving n unknown functions $x_1, x_2, \ldots, x_n$ and their first derivatives. which can be written in the form

$$\begin{aligned} x_1' &= P_{11}(t)x_1 + \ldots + P_{1n}x_n + f_1(t) \\ &\vdots \\ x_n' &= P_{n1}(t)x_1 + \ldots + P_{nn}x_n + f_n(t) \end{aligned} \tag{6.1}$$

Alternatively, we can write this:

$$\vec{x}' = P(t)\vec{x} + \vec{f}(t) \tag{6.2}$$

where

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, (\vec{x} : \mathbb{R} \to \mathbb{R}^n) \qquad P(t) = \underbrace{\begin{bmatrix} P_{11}(t) & \ldots & P_{1n}(t) \\ \vdots & & \\ P_{n1}(t) & \ldots & P_{nn}(t) \end{bmatrix}}_{\text{matrix valued function}}, (P : \mathbb{R} \to \mathbb{R}^{n\times n}) \qquad \vec{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$P(t)$ called the coefficient matrix.

Definition A solution of (6.2) on the open interval I is a function $\vec{x}$ which is differentiable on I and satisfies (6.2) on I.

Definition An initial condition for (6.2) is a specification of $\vec{x}(t)$ for a given value of t

$$\vec{x}(t_0) = x_0 \tag{6.3}$$

Theorem (Existence and Uniqueness)
Suppose the functions $P(t)$ and $\vec{f}(t)$ are continuous on the open interval I containing the point t_0. Then there is a unique solution on I to the IVP consisting of (6.2) and (6.3), for any $\vec{x}_0 \in \mathbb{R}^n$

Definition The associated homogeneous equation for (6.2) is

$$\vec{x}' = P(t)\vec{x} \tag{6.4}$$

Theorem (Principle of Superposition)
Let $\vec{x}_1, \ldots, \vec{x}_n$ be solutions of (6.4) on the open interval I. Then for any $c_1, \ldots, c_n \in \mathbb{R}$

$$\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)$$

is also a solution on I of (6.4)

Proof

$$\begin{aligned}
\vec{x}' &= c_1\vec{x}_1' + \ldots + c_n\vec{x}_n' \\
&= c_1 P(t)\vec{x}_1 + \ldots + c_n P(t)\vec{x}_n \\
&= P(t)[c_1\vec{x}_1 + \ldots + c_n\vec{x}_n] \\
&= P(t)\vec{x}
\end{aligned}$$

□

Definition Let $I \subset \mathbb{R}$ and $\vec{f}_j : I \to \mathbb{R}^n$, $j = 1, \ldots, n$
The functions $\vec{f}_1, \ldots, \vec{f}_n$ are <u>linear dependent</u> on I if there are $c_1, \ldots, c_n \in \mathbb{R}$, not all of which are zero, such that $c_1\vec{f}_1(t) + \ldots + c_n\vec{f}_n(t) = 0 \ \forall t \in I$. Otherwise they are <u>linear independent</u>.

Definition Let $\vec{x}_1, \ldots, \vec{x}_n$ be solutions of (6.4). Let $M = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}$ be the matrix with j^{th} column is $\vec{x}_j$.
The wronskian of $\vec{x}_1, \ldots, \vec{x}_n$ is

$$W(\vec{x}_1, \ldots, \vec{x}_n) = \det(M) = \det \begin{bmatrix} x_{11}(t) & \ldots & x_{n1}(t) \\ \vdots & \ddots & \vdots \\ \underbrace{x_{1n}(t)}_{\vec{x}_1(t)} & \ldots & \underbrace{x_{nn}(t)}_{\vec{x}_n(t)} \end{bmatrix}$$

Theorem (Wronskian of Solutions)
Suppose $\vec{x}_1, \ldots, \vec{x}_n$ are n solutions of (6.4) on an open interval I where $P(t)$ is continuous, then there are two possibilities:

(1) $\vec{x}_1, \ldots, \vec{x}_n$ are linearly dependent on I and $W(\vec{x}_1, \ldots, \vec{x}_n) \equiv 0$ on I.

(2) $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent on I and $W(\vec{x}_1, \ldots, \vec{x}_n) \neq 0$ on $\forall t \in I$.

Proof (Analogous to case for n^{th} order ODEs) □

Theorem (General Solution of Homogeneous Linear Systems)
Let $\vec{x}_1, \ldots, \vec{x}_n$ be n linearly independent solutions of (6.4) on an open interval I where $P(t)$ is continuous. If $\vec{x}(t)$ is any solution on I of (6.4) then there exist $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) \qquad \forall t \in I$$

Proof Let $\vec{x}(t)$ be any solution on I of (6.4). Let $t_0 \in I$, and $M(t)$ be as in the definition of the Wronskian. Since $\vec{x}_1, \ldots, \vec{x}_n$ are linearly independent on I,

$$\det(M(t_0)) = W(\vec{x}_1(t_0), \ldots, \vec{x}_n(t_0)) \neq 0$$

Thus the linear system $M(t_0)\vec{c} = \vec{x}(t_0)$ $(*)$
has a unique solution

$$\vec{c} = M^{-1}(t_0)\vec{x}(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Define $\vec{y}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t)$. This is a solution of (6.4) by the Superposition Principle and satisfies the initial condition $\vec{y}(t_0) = \vec{x}(t_0)$. But $\vec{x}(t)$ is also a solution of (6.4) satisfying the same IC. By the E/U Theorem we must have

$$\vec{x}(t) = \vec{y}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) \qquad \forall t \in I$$

$\square$

Theorem (General Solution of Non-homogeneous Linear Systems)
Let $\vec{x}_p$ be a particular solution of (6.2) on I where $P(t)$ and $\vec{f}(t)$ are continuous. Let $\vec{x}_1, \ldots, \vec{x}_n$ be n linearly independent solution on I of (6.4).
If $\vec{x}(t)$ is any solution of (6.2) on I then there are $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{x}(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) + \vec{x}_p(t), \qquad \forall t \in I$$

Proof Let $\vec{x}(t)$ be any solution on I of (6.2).
Let $\vec{y}(t) = \vec{x}(t) - \vec{x}_p(t)$, then

$$\vec{y}' = \vec{x}' - \vec{x}_p' = [P(t)\vec{x} + \vec{f}] - [P\vec{x}_p + \vec{f}] = P(\vec{x} - \vec{x}_p) = P\vec{y} \quad \forall t \in I$$

So $\vec{y}(t)$ is a solution on I of (6.4). By Theorem from last class, there are $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\vec{y}(t) = c_1\vec{x}_1(t) + \ldots + \vec{c}_n\vec{x}_n(t) \qquad \forall t \in I$$

Thus

$$\vec{x}(t) = \vec{y}(t) + \vec{x}_p(t) = c_1\vec{x}_1(t) + \ldots + c_n\vec{x}_n(t) + \vec{x}_p(t) \qquad \forall t \in I$$

$\square$

6.2 Constant Coefficient Homogeneous Systems

6.2.1 The Eigenvalue Method

See slides

Constant Coefficient Homogeneous Systems
The Eigenvalue Method

Consider a homogeneous system:

$$\mathbf{x}' = P(t)\mathbf{x}$$

If the coefficient matrix is constant, $P(t) = A$, with A an $n \times n$ matrix, we will call this a **constant coefficient, homogeneous system** and write it

$$\mathbf{x}' = A\mathbf{x} \tag{1}$$

Motivated by the solution for n^{th} order constant coefficient equations, we will look for solutions of this in the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v} = \begin{pmatrix} e^{\lambda t} v_1 \\ e^{\lambda t} v_2 \\ \vdots \\ e^{\lambda t} v_n \end{pmatrix} \tag{2}$$

where the constant λ and the vector $\mathbf{v}$ are to be determined.

Substituting $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ into the DE (1) we find

$$\lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v}$$

Since $e^{\lambda t} \neq 0$ we require

$$\lambda \mathbf{v} = A\mathbf{v} \tag{3}$$

Thus $e^{\lambda t}\mathbf{v}$ is a solution of the DE (1) if λ is an eigenvalue of A with eigenvector $\mathbf{v}$.
Eq. (3) will have nontrivial solutions if and only if

$$\det(A - \lambda I) = 0 \tag{4}$$

This is the **characteristic equation** of the DE (1).

Since A is an $n \times n$ matrix, we know from Linear Algebra that it will have n eigenvalues, which may be real or complex and may not be all distinct.

We will use these eigenvalues to construct n linearly independent solutions of the DE (1)

For simplicity we set $n = 2$ to illustrate the different cases.
In this situation there are three cases.

Case 1. There are two real eigenvalues, $\lambda_1,\ \lambda_2$, and the corresponding eigenvectors, $\mathbf{v}_1,\ \mathbf{v}_2$ are linearly independent. (Note: This will always be the case if $\lambda_1 \neq \lambda_2$. This *may* be the case if $\lambda_1 = \lambda_2$.)

The corresponding solutions are $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1,\ \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$. Consider

$$\begin{aligned} W(\mathbf{x}_1(t), \mathbf{x}_1(t)) &= \det \begin{pmatrix} e^{\lambda_1 t} v_{11} & e^{\lambda_2 t} v_{21} \\ e^{\lambda_1 t} v_{12} & e^{\lambda_2 t} v_{22} \end{pmatrix} = e^{(\lambda_1 + \lambda_2)t}(v_{11}v_{22} - v_{12}v_{21}) \\ &= e^{(\lambda_1+\lambda_2)t} \det \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} \end{aligned}$$

$W(\mathbf{x}_1(t), \mathbf{x}_2(t)) \neq 0,\ t \in \mathbb{R}$ since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent vectors. Thus $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are linearly independent solutions on $\mathbb{R}$.

General solution of the DE: $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2,\ t \in \mathbb{R}$.

Case 2. There is one repeated real eigenvalue, λ, with corresponding eigenvector, $\mathbf{v}$, and every other eigenvector of λ is a scalar multiple of $\mathbf{v}$.

One solution of the DE is $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}$.

Second idea. Look for a solution of the form $e^{\lambda t}\mathbf{u} + te^{\lambda t}\mathbf{w}$, where $\mathbf{u},\ \mathbf{w}$ are to be determined. Differentiating and substituting into the DE gives

$$\lambda e^{\lambda t}\mathbf{u} + e^{\lambda t}\mathbf{w} + \lambda te^{\lambda t}\mathbf{w} = A(e^{\lambda t}\mathbf{u} + te^{\lambda t}\mathbf{w}) \Rightarrow \lambda\mathbf{u} + \mathbf{w} + \lambda t\mathbf{w} = A\mathbf{u} + At\mathbf{w}$$

Equating like powers of t

$$\begin{aligned} \lambda\mathbf{w} &= A\mathbf{w} \\ \lambda\mathbf{u} + \mathbf{w} &= A\mathbf{u} \end{aligned}$$

First equation implies $\mathbf{w}$ is an eigenvector of λ. Take $\mathbf{w} = \mathbf{v}$.
Then $\mathbf{u}$ must satisfy

$$(A - \lambda I)\mathbf{u} = \mathbf{v} \tag{5}$$

It can be shown that (5) can always be solved for $\mathbf{u} \neq \mathbf{0}$ with $\mathbf{u},\ \mathbf{v}$ linearly independent vectors.
(Note: $\mathbf{u}$ satisfying (5) is called a **generalized eigenvector** of λ).

Case 2. (Continued)
A second solution solution of the DE in this case is $\mathbf{x}_2(t) = e^{\lambda t}\mathbf{u} + te^{\lambda t}\mathbf{v}$ where $\mathbf{u}$ satisfies (5).

It can be shown (exercise) that $W(\mathbf{x}_1(t), \mathbf{x}_2(t) = e^{2\lambda t}(v_1 u_2 - v_2 u_1)$.
Since $\mathbf{u}, \mathbf{v}$ are linearly independent vectors, $W(\mathbf{x}_1(t), \mathbf{x}_2(t)) \neq 0, \ \forall t \in \mathbb{R}$.
So $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are linearly independent solutions on $\mathbb{R}$.

General solution of the DE: $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2(e^{\lambda t}\mathbf{u} + te^{\lambda t}\mathbf{v}), \ t \in \mathbb{R}$ where $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue λ and $\mathbf{u}$ satisfies

$$(A - \lambda I)\mathbf{u} = \mathbf{v}$$

Case 3. Complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm i\beta,\ \alpha, \beta \in \mathbb{R}$
with corresponding eigenvectors $\mathbf{v}_{1,2} = \mathbf{u} + i\mathbf{w},\ \mathbf{u}, \mathbf{w} \in \mathbb{R}^n$.
Form the complex solution:

$$\begin{aligned}
\mathbf{z}(t) &= e^{\lambda t}\mathbf{v} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\mathbf{u} + i\mathbf{w}) \\
&= e^{\alpha t}(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}) + ie^{\alpha t}(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}) \\
&= \mathbf{x}_1(t) + i\mathbf{x}_2(t)
\end{aligned}$$

Can show by substitution that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two solutions of the DE on $\mathbb{R}$. Can show using the Wronskian that they linearly independent on $\mathbb{R}$.

General solution of the DE:
$\mathbf{x}(t) = c_1 e^{\alpha t}(\cos(\beta t)\mathbf{u} - \sin(\beta t)\mathbf{w}) + c_2 e^{\alpha t}(\sin(\beta t)\mathbf{u} + \cos(\beta t)\mathbf{w}),\ t \in \mathbb{R}$

Example 1 Find a solution of the IVP

$$\vec{x}' = A\vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Solution Find the eigenvalues and eigenvectors of A.
Characteristic equation: $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - 4 = 0 \implies \lambda = 3, -1$

Eigenvector for $\lambda = 3 : \vec{v} = \binom{1}{2} \to \vec{x}_1(t) = e^{3t}\binom{1}{2}$.
$\lambda = -1, \vec{v} = \binom{1}{-2} \to \vec{x}_2(t) = e^{-t}\binom{1}{-2}$
General

$$\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{pmatrix}$$

Initial condition: $\vec{x}(0) = \binom{1}{3}$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{pmatrix} \implies c_1 = \frac{5}{4}, c_2 = -\frac{1}{4}$$

Solution of IVP: $\vec{x}(t) = \begin{pmatrix} 5/4e^{3t} - 1/4e^{-4} \\ 5/2e^{3t} + 1/2e^{-t} \end{pmatrix}$

Example 2 Find a general solution of $\vec{x}'(t) = A\vec{x}$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

Solution Characteristic equation: $(\lambda - 2)^2 = 0, \lambda = 2, 2$
Solve for eigenvector $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector $\vec{v} = \binom{1}{-1} \to \vec{x}_1(t) = e^{2t}\binom{1}{-1}$
2^{nd} solution $\vec{x}_2(t) = e^{2t}\vec{u} + te^{2t}\vec{v}$ where $\vec{u}$ satisfies $(A - \lambda I)\vec{u} = \vec{v}$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Choose one solution: $u = \binom{0}{-1}$
$x_2(t) = e^{2t}\binom{0}{-1} + te^{2t}\binom{1}{-1}$
General solution: $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t), \qquad t \in \mathbb{R}$

Example Find a general solution of $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} -1 & -1 \\ 5 & -3 \end{pmatrix}$

Solution Characteristic equation: $\det(A - \lambda I) = 0 \implies \lambda = -2 \pm 2i = \alpha \pm i\beta$
Find an eigenvector, $\lambda = -2 + 2i$

$$(A - \lambda I)\vec{v} = 0 \qquad \begin{pmatrix} 1-2i & -1 \\ 5 & -1-2i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies v_2 = (1-2i)v_1$$

Choose $\vec{v} = \begin{pmatrix} 1 \\ 1-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \vec{u} + i\vec{w}$
General solution (See derivation last class)

$$\begin{aligned}\vec{x}(t) &= c_1 e^{\alpha t}(\vec{u}\cos(\beta t) - \vec{w}\sin(\beta t)) + c_2 e^{\alpha t}(\vec{u}\sin(\beta t) + \vec{w}\cos(\beta t)) \\ &= c_1 e^{-2t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix}\sin(2t)\right] + c_2 e^{-2t}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin(2t) + \begin{pmatrix} 0 \\ -2 \end{pmatrix}\cos(2t)\right]\end{aligned}$$

6.3 Sketching Solutions for 2-D Systems

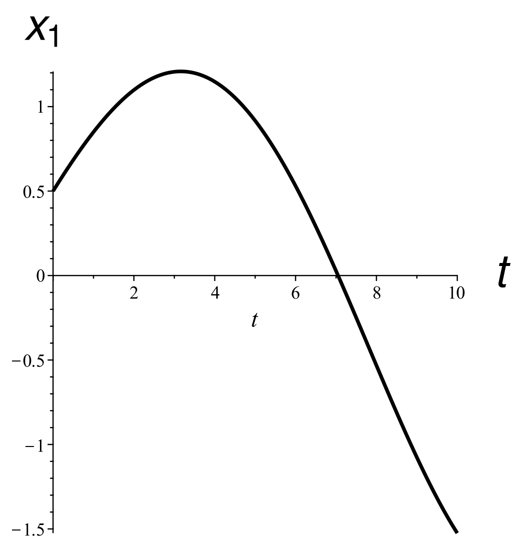
Sketching Solution Curves for 2D Systems

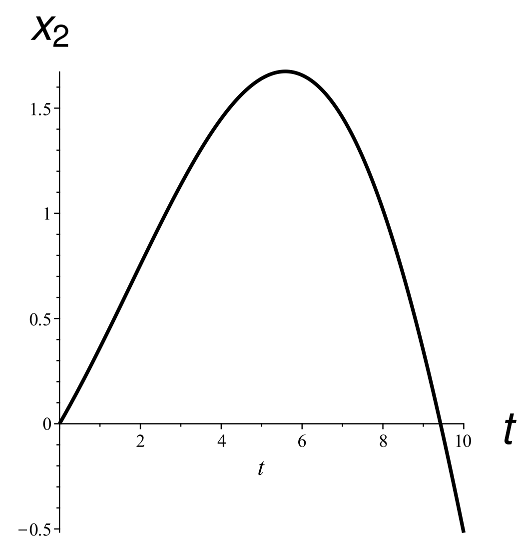
Consider the constant coefficient, homogeneous system

$$\mathbf{x}' = A\mathbf{x} \tag{1}$$

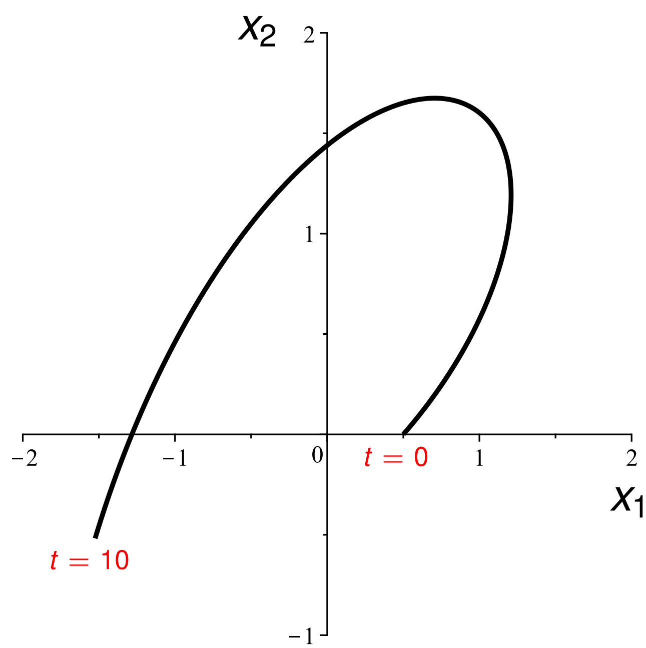
where A is 2×2 matrix. A solution is a two dimensional vector valued function $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$.

One way to visualize a solution is by plotting the component functions vs t





Another way to visualize a solution is to think of $\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ as defining a curve **parametrically** in $\mathbb{R}^2$, by plotting the points $(x_1(t), x_2(t))$ in $\mathbb{R}^2$ as t is varied.



Physical Analogy. If $x_1(t)$ and $x_2(t)$ represent the position of an object in two dimensions, the curve represents the path of the object in space.

The second approach may be more useful if we wish to compare different solutions of the system.

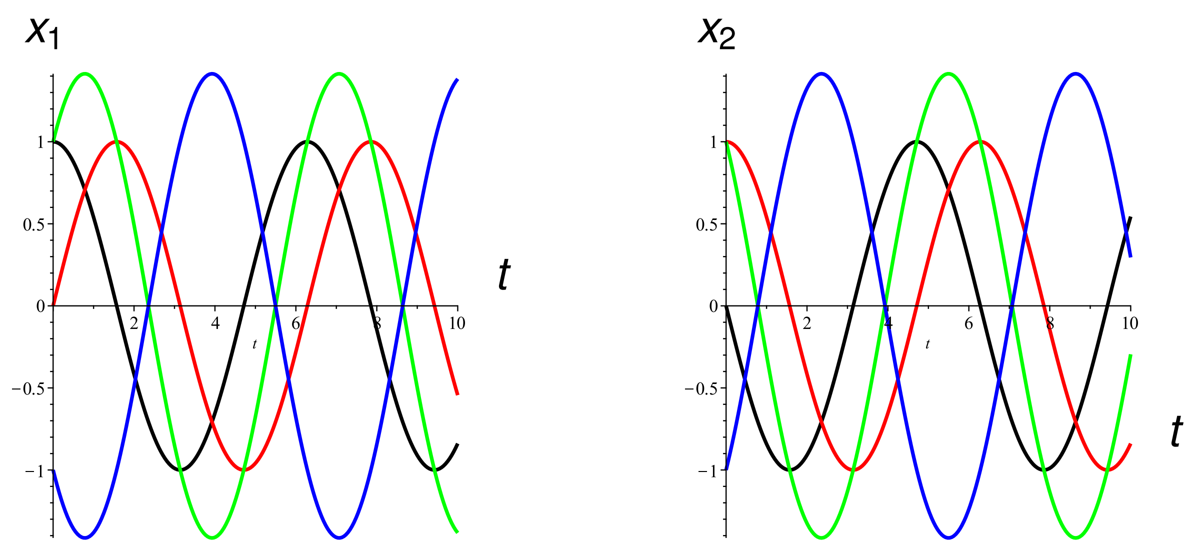
Example. Consider the initial value problem consisting of system (1) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It can be shown (exercise) a general solution of this system is

$$\mathbf{x}(t) = c_1 \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}.$$

Solution curves plotted separately for four values of c_1, c_2.



As we saw in the example, solutions with different initial conditions can lie on the same curve when plotted in $\mathbb{R}^2$.

The curve in $\mathbb{R}^2$ associated with a solution $\mathbf{x}(t)$ of the system (1) is called a **trajectory** of the system.

A plot of the qualitatively different trajectories of system (1) is called the **phase portrait** of the system.

Tools for sketching phase portraits

1. **Trajectories never cross.** It follows from the Existence and Uniqueness Theorem that two different solutions of (1) either lie on the same trajectory or on two different trajectories that never intersect.
2. **Special Solutions.**
   - The system (1) always has the constant solution $\mathbf{x}(t) = \mathbf{0},\ t \in \mathbb{R}$. This is an equilibrium solution for the system. The corresponding trajectory is the point $(0, 0)$.
   - For some systems there are trajectories that lie on straight lines, i.e., $x_2(t) = kx_1(t),\ t \in \mathbb{R}$ where $k \in \mathbb{R}$.
3. **Asymptotic Behaviour.**
   Consider the behaviour of $\mathbf{x}(t)$ as $t \to \pm\infty$.
   Does it approach one of the exceptional solutions?

Tools for sketching phase portraits

4. **Nullclines.**
   Rewrite system (1) in component form

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2\\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2\end{aligned}$$

From chain rule

$$\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{a_{21}x_1 + a_{22}x_2}{a_{11}x_1 + a_{12}x_2}$$

Thus we can use the system to tell us about the slope of the tangent to a trajectory at a given point (x_1, x_2).

The trajectory will have a *horizontal tangent* at points where $\frac{dx_2}{dt} = 0$. These points form a line called the **horizontal nullcline**.

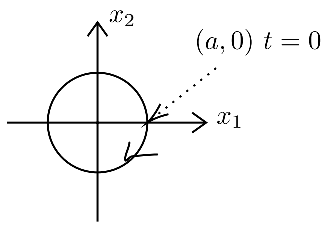
The trajectory will have a *vertical tangent* at points where $\frac{dx_1}{dt} = 0$. These points form a line called the **vertical nullcline**.

Example $\vec{x}' = A\vec{x}, A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Can show general solution is

$$\vec{x}(t) = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad t \in \mathbb{R}$$

Plot of component functions separately is messy.
Suppose: $c_1 = a, \quad c_2 = 0$, then $x_1(t) = a\cos t, x_2(t) = -a\sin t, t \in \mathbb{R}$.
Think of this as parametric equation for a curve in $\mathbb{R}^2$

$$x_1^2 + x_2^2 = a^2\cos^2 t + a^2 \sin^2 t = a^2 \rightarrow \text{ a circle with radius } a$$



$t = 0 \quad x_1 = a, \ x_2 = 0$
$t = \pi/2 \quad x_1 = 0, x_2 = -a$

Put arrow to indicate how curve traversed as t increased

Suppose $c_1 = 0, c_2 = a$, similar. Can show for any values of c_1, c_2 the solution corresponding to a circle.

Example 1 Sketch the phase portrait of $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

Solution Form last lecture, A has eigenvalues $3, -1$ with eigenvectors $\binom{1}{2}, \binom{1}{-2}$.
General solution: $\vec{x}(t) = c_1 e^{3t}\binom{1}{2} + c_2 e^{-t}\binom{1}{-2}, \quad t \in \mathbb{R}$.
Special solutions: $c_1 = 0, c_2 = 0 \rightarrow \vec{x}(t) = \binom{0}{0}, t \in \mathbb{R}$.

$$c_1 \neq 0, c_2 = 0, \quad \vec{x}(t) = \begin{pmatrix} c_1 e^{3t} \\ 2c_1 e^{3t} \end{pmatrix}, t \in \mathbb{R} \implies x_2(t) = 2x_1(t), t \in \mathbb{R}$$

$$c_1 = 0, c_2 \neq 0, \quad \vec{x}(t) = \begin{pmatrix} c_2 e^{-t} \\ -2c_2 e^{-t} \end{pmatrix}, t \in \mathbb{R} \implies x_2(t) = -x_1(t), t \in \mathbb{R}$$

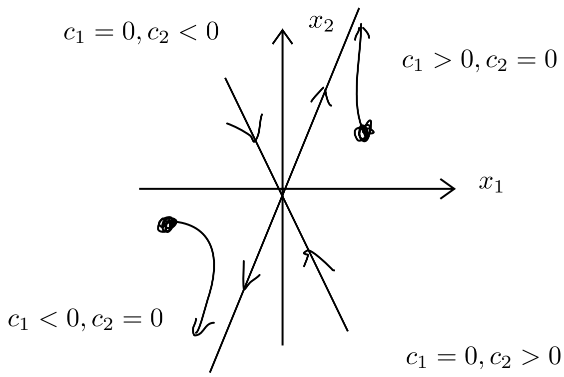
Asymptotic Behaviour $\vec{x}(t) = c_1 e^{3t}\binom{1}{2} + c_2 e^{-t}\binom{1}{-2}, \quad t \in \mathbb{R}$

- $c_1 = 0$: $\lim_{t\to\infty} \vec{x}(t) = \binom{0}{0}$ On the line $x_2 = -2x_1$ solutions move toward origin.
- $c_1 > 0, c_2 \neq 0$: $\lim_{t\to\infty} \vec{x}_1(t) = \infty, \lim_{t\to\infty} \vec{x}_2(t) = \infty$
- $c_1 < 0, c_2 \neq 0$: $\lim_{t\to\infty} \vec{x}_1(t) = -\infty, \lim_{t\to\infty} \vec{x}_2(t) = -\infty$

How do solutions behave as $t \to \infty$?

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = \lim_{t\to\infty} \frac{2c_1 e^{3t} - 2c_1 e^{-t}}{c_1 e^{3t} + c_2 e^{-t}} = 2$$

As $t \to \infty, x_2(t) \to 2x_1(t)$, $\vec{x}(t)$ has the line $x_2 = 2x_1$ as an asymptote as $t \to \infty$

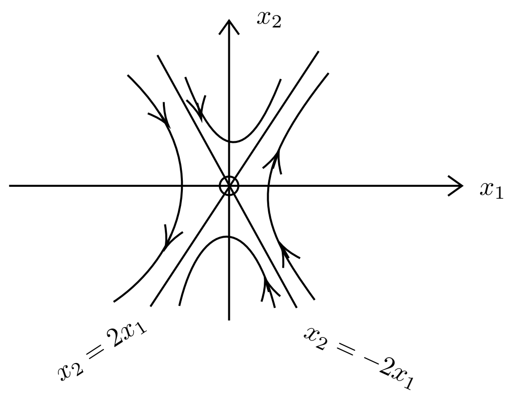


Cont'd

$$\lim_{t\to-\infty} = \frac{x_2(t)}{x_1(t)} = \lim_{t\to\infty} \frac{2c_1e^{3t} - 2c_1e^{-t}}{c_1e^{3t} + c_2e^{-t}} = -2$$

Key Points

- equilibrium point $\binom{0}{0}$ is unstable (saddle point)
- eigenvectors determine special solutions of $x_2(x_1)$
- sign of corresponding eigenvalue determines direction on special solution



Example Sketch phase potrait of $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}$.

Solution A has eigenvalues 2,2, eigenvector $\binom{1}{-1}$, generalized eigenvector $\binom{0}{-1}$.
General solution

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} \right], \quad t \in \mathbb{R}$$

Special solutions:

c_1	c_2	
0	0	$\vec{x}(t) = \binom{0}{0}$
$\neq 0$	0	$x_2(t) = -x_1(t)$

$$c_2 = 0 \implies \lim_{t\to\infty} \vec{x}(t) = \begin{cases} \binom{+\infty}{-\infty} & c_1 > 0 \\ \binom{-\infty}{+\infty} & c_1 < 0 \end{cases}$$

$$\lim_{t\to-\infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 \neq 0, c_2 \neq 0 \qquad \lim_{t\to-\infty} \frac{x_2(t)}{x_1(t)} = -1$$

as $t \to -\infty$, $\vec{x}(t)$ has asymptote $x_2 = -x_1$.
Nullclines

- horizontal: $\dfrac{x_2}{t} = 0 \qquad x_1 + 3x_2 = 0 \implies x_2 = -\frac{x_1}{3}$
- vertical: $\dfrac{x_1}{t} = 0 \qquad x_1 - x_2 = 0 \implies x_1 = x_2$

equilibrium point $\binom{0}{0}$ is unstable.

Example Sketch phase potrait of $\vec{x}'(t) = \begin{pmatrix} -1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$
Eigenvalues: $-2 \pm 2i$.
General solution

$$\vec{x}(t) = c_1 e^{-2t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(2t) - \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin(2t) \right] + c_2 e^{-2t} \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(2t) + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \cos(2t) \right]$$

Special solutions: $\vec{x}(t) = \binom{0}{0}$
Rewrite solution:

$$\vec{x}(t) = \kappa e^{-2t} \begin{pmatrix} \cos(2t - \delta) \\ \cos(2t - \delta) + 2\sin(2t - \delta) \end{pmatrix} \qquad \kappa = \sqrt{c_1^2 + c_2^2}, \quad \tan(\delta) = \frac{c_2}{c_1}$$

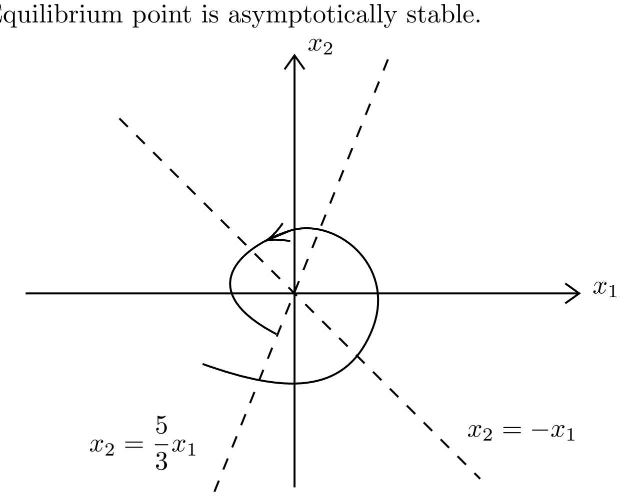
$$\lim_{t\to\infty} \vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad x_1(t), x_2(t) \text{ are oscillating with amplitude} \to 0 \text{ as } t \to \infty.$$

The trajectories will be spirals.

Nullclines

- horizontal: $\frac{dx_2}{dt} = 0 \implies x_2 = \frac{5}{3} x_1$
- vertical: $\frac{dx_1}{dt} = 0 \implies x_2 = -x_1$

Equilibrium point is asymptotically stable.



More details and examples in the textbook.

6.4 Fundamental Matrix

Fundamental Matrices for Linear Systems

Consider a n-dimensional, homogeneous linear system

$$\mathbf{x}' = P(t)\mathbf{x} \tag{1}$$

Assume that $P(t)$ is continous on an open interval I.

Definition. Let $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ be n, linearly independent solutions of (1) on I. The $n \times n$ matrix valued function, $\Phi : \mathbb{R} \to \mathbb{R}^{n\times n}$ formed by taking $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ as the columns is called a **fundamental matrix** for (1). That is,

$$\Phi(t) = \begin{pmatrix} x_{11}(t) & x_{21}(t) & \cdots & x_{n1}(t) \\ x_{12}(t) & x_{22}(t) & \cdots & x_{n2}(t) \\ \vdots & \vdots & \vdots & \vdots \\ x_{1n}(t) & x_{2n}(t) & \cdots & x_{nn}(t) \end{pmatrix}$$

Properties of a Fundamental Matrix

Proposition 1

Every solution $\mathbf{x}(t)$ of (1) can be written in the form $\mathbf{x}(t) = \Phi(t)\mathbf{c}$ for $t \in I$ and some $\mathbf{c} \in \mathbb{R}^n$.

Proof. Let $\mathbf{x}(t)$ be a solution of (1). By the theorem on general solutions for (1) there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that, for $t \in I$

$$
\begin{aligned}
\mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n \\
&= \begin{pmatrix} c_1 x_{11}(t) + \cdots + c_n x_{n1}(t) \\ \vdots \\ c_1 x_{1n}(t) + \cdots + c_n x_{nn}(t) \end{pmatrix} \\
&= \begin{pmatrix} x_{11}(t) & \cdots & x_{n1}(t) \\ & \vdots & \\ x_{1n}(t) & \cdots & x_{nn}(t) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\
&= \Phi(t)\mathbf{c}
\end{aligned}
$$

□

Properties of a Fundamental Matrix

Proposition 2

$\Phi(t)$ is an invertible matrix for each $t \in I$.

Proof. Since $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent solutions of (1) on I, $W(\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)) = \det(\Phi(t)) \neq 0$ on I. Thus $\Phi(t)$ is invertible matrix on I. $\square$

Proposition 3

$\Phi(t)$ satisfies the matrix differential equation

$$\Phi'(t) = P(t)\Phi(t) \tag{2}$$

for $t \in I$.

Proof. Let $\mathbf{c} \in \mathbb{R}^n$ and define $\mathbf{x}(t) = \Phi(t)\mathbf{c} = \sum_{j=k}^{N} c_k \mathbf{x}_k(t)$. Then $\mathbf{x}(t)$ is a solution of (1) on I, that is for all $t \in I$

$$\begin{aligned} \mathbf{x}'(t) &= P(t)\mathbf{x}(t) \\ \Phi'(t)\mathbf{c} &= P(t)\Phi(t)\mathbf{c} \\ (\Phi'(t) - P(t)\Phi(t))\mathbf{c} &= 0 \end{aligned}$$

Since this is true for any $\mathbf{c} \in \mathbb{R}^n$ we must have $\Phi'(t) = P(t)\Phi(t)$ for all $t \in I$. $\square$

Theorem (Fundamental Matrix Solution)
Let $P(t)$ be continuous on the open interval I. Let $\Phi(t)$ be a fundamental matrix for the system (1). For any $t_0 \in I$ and $\mathbf{x}_0 \in \mathbb{R}^n$, the unique solution of the initial value problem

$$\mathbf{x}' = P(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(t_0)\mathbf{x}_0, \; t \in I$$

Proof. By Proposition 1, a general solution for the differential equation is $\mathbf{x}(t) = \Phi(t)\mathbf{c}, \; t \in I$. Applying the initial condition we have

$$\mathbf{x}_0 = \Phi(t_0)\mathbf{c}$$

Since $\Phi(t_0)$ is invertible this implies

$$\mathbf{c} = \Phi^{-1}(t_0)\mathbf{x_0}$$

The result follows. $\square$

Definition. Let A be an $n \times n$ matrix. The **exponential of A** is defined by the matrix series

$$e^A = I + A + \frac{1}{2}A^2 + \cdots = \sum_{k=1}^{\infty} \frac{A^k}{k!}$$

It can be shown that the series converges.

Theorem (Matrix Exponential Solution)
If A is an $n \times n$ matrix then the unique solution of the initial value problem

$$\mathbf{x}' = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0,\ t \in \mathbb{R}.$$

Idea of Proof. Show that e^{At} is a fundamental matrix for the differential equation and then apply the previous theorem.

6.5 Nonhomogeneous Linear Systems

Nonhomogeneous system

$$\vec{x}' = P(t)\vec{x} + \vec{f}(t) \tag{1}$$

Associated homogeneous system

$$\vec{x}' = P(t)\vec{x} \tag{2}$$

Recall General solution of (1) is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t) \tag{3}$$

where $\vec{x}_h(t)$ is a general solution of (2), $\vec{x}_p(t)$ is a particular solution of (1). From last class

$$\vec{x}_h(t) = \Phi(t)\vec{c} \tag{4}$$

where $\Phi(t)$ is a fundamental matrix for (2), $\vec{c} \in \mathbb{R}^n$ is arbitrary .

To find $\vec{x}_p(t)$ we will use the following.

Theorem (Variation of Parameters for Linear Systems)
If $\phi(t)$ is a fundamental matrix for (2) on some open interval I where $P(t), \vec{f}(t)$ are continuous then a particular solution of (1) is

$$\vec{x}_p(t) = \Phi(t) \int \Phi^{-1}\vec{f}(t)\ dt, \quad t \in I$$

Proof Assume a solution of (1) in the form $\vec{x}_p(t) = \Phi(t)\vec{u}(t)$ where $\vec{u}(t)$ is TBD.
Then for $t \in I$

$$\vec{x}_p'(t) = \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) = P(t)\Phi(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) \quad \text{(prop3 from last class)}$$

Substitute $\vec{x}_p, \vec{x}_p'$ into (1)

$$P(t)\Phi(t)\vec{u}(t) + \Phi(t)\cdot\vec{u}'(t) = P(t)\Phi(t)\vec{u}(t) + \vec{f}(t)$$
$$\Phi(t)\cdot\vec{u}'(t) = \vec{f}(t)$$

Since Φ is invertible on I we can solve for $\vec{u}'$

$$\vec{u}'(t) = \Phi^{-1}(t)\vec{f}(t)$$

Thus we need $\vec{u}(t)$ to be an antiderivative of $\Phi^{-1}(t)\vec{f}(t)$ we write this as

$$\vec{u}(t) = \int \Phi^{-1}(t)\vec{f}(t)\ dt$$

Thus

$$\vec{x}_p(t) = \Phi(t)\vec{u}(t) = \Phi(t)\int \Phi^{-1}(t)\vec{f}(t)\ dt$$

is a solution of (1)

It follows from this Theorem and (3) that a general solution for (1) can be written as

$$\vec{x}(t) = \Phi(t)\vec{c} + \Phi(t)\int \Phi^{-1}(t)\vec{f}(t)\ dt, \quad t \in I$$

This is called the variation of parameters formula.
Sometimes this formula is written

$$\vec{x}(t) = \underbrace{\Phi(t)\vec{c}}_{\vec{x}_h(t)} + \underbrace{\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\vec{f}(s)\ ds}_{\vec{x}_p(t)}, \quad t \in I$$

where $t_0 \in I$.
In this case we've chosen $\vec{x}_p(t)$ so that $\vec{x}_p(t_0) = 0$

Example Find a general solution of $\vec{x}'(t) = A\vec{x} + \vec{f}(t)$ where $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$ and $\vec{f}(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}$.

Solution We showed in a previous lecture that

$$\vec{x}_h(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right], \quad t \in \mathbb{R}$$

A fundamental matrix for $\vec{x}' = A\vec{x}$ is

$$\Phi(t) = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -(1+t)e^{2t} \end{pmatrix}$$

To find $\Phi^{-1}(t)$ we can use the formula. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus

$$\Phi^{-1}(t) = \begin{bmatrix} (1+t)e^{-2t} & te^{-2t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix} \implies \Phi'(t)\vec{f}(t) = \begin{pmatrix} 1+t \\ -1 \end{pmatrix}$$

$$\int \Phi'(t)\vec{f}(t)dt = \begin{pmatrix} t + \frac{t^2}{2} \\ -t \end{pmatrix} \qquad \text{leave out constants of integration}$$

$$\vec{x}_p(t) = \Phi(t) \int \Phi'(t)\vec{f}(t)dt = matrix\ multiplication = \begin{pmatrix} e^{-2t}(t - \frac{t^2}{2}) \\ e^{2t}(\frac{t^2}{2}) \end{pmatrix}, \qquad t \in \mathbb{R}$$

General solution: $\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$

7

Laplace Transforms

7.1 Laplace Transforms and Inverse Transforms

Definition Given a function $f(t)$ defined for all $t \geq 0$, the Laplace Transform of f is the function F defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

for all values of s for which the improper integral converges.

Recall We say the improper integral above converges if

$$\lim_{b\to\infty} \int_0^b e^{-st} f(t) dt$$

exists, otherwise we say it diverges.

Example $f(t) = 1$

$$(\text{if } s \neq 0) \quad \lim_{b\to\infty} \int_0^b e^{-st} f(t) dt = \lim_{b\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^b = \lim_{b\to\infty} \frac{1-e^{-sb}}{s} = \begin{cases} \frac{1}{s} & \text{if } s > 0 \\ DNE \end{cases}$$

$$\text{if } s = 0 \lim_{b\to\infty} t|_0^b = \lim_{b\to\infty} b \qquad DNE$$

Summary

$$\int_0^\infty e^{-st} dt \begin{cases} \text{converges with value } \frac{1}{s} \text{ if } s > 0 \\ \text{diverges otherwise} \end{cases}$$

Implication: $\mathcal{L}\{1\} = \frac{1}{s}$, for $s > 0$.

Example 2 Let $f(t) = e^{at}, a \in \mathbb{R}$. Assume $a \neq s$

$$\lim_{b\to\infty} \int_0^b e^{at} e^{-st} dt = \lim_{b\to\infty} \int_0^b e^{(a-s)t} dt = \lim_{b\to\infty} \frac{e^{(a-s)b} - 1}{a-s} = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ DNE & \text{if } s < a \end{cases}$$

The case $s = a$ is the same as the case $s = 0$ in the previous example.
Conclusion: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ if $s > a$.

Example 3 Let $a \in \mathbb{C}$, i.e. $a = \alpha + i\beta$

$$e^{au} = e^{\alpha u} e^{i\beta u} = e^{\alpha u}(\cos \beta u + i \sin \beta u)$$

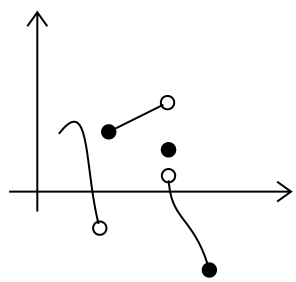
Evaluation of the integral proceeds as in the last example until the last step

$$\lim_{b\to\infty} \int_0^\infty e^{(\alpha+i\beta)t} e^{-st} dt = \lim_{b\to\infty} \frac{e^{(\alpha-s)b}[\cos \beta b + i \sin \beta b] - 1}{a - s} \begin{cases} \frac{1}{a-s} & \text{if } s > \alpha \\ DNE & \text{if } s \le \alpha \end{cases}$$

To proceed further, we define conditions to guarantee that $\mathcal{L}\{f(t)\}$ exists.

Definition The function $f(t)$ is said to be piecewise continuous for $a \le t \le b$ provided that $[a, b]$ can be subdivided into finitely many abutting subintervals, such that

1. f is continuous in the interior of each subinterval
2. f has finite limit as t approaches each end point of each subinterval from its interior



Definition The function $f(t)$ is said to be piecewise continuous on $t \ge 0$ if it is piecewise continuous on each bounded subinterval of $[0, +\infty)$.

Example The unit step function, $u(t)$ is defined by

$$\begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \ge 0 \end{cases}$$

The unite step function at a is defined by

$$u_a(t) = u(t - a) = \begin{cases} 0 & t < a \\ 1 & t \ge a \end{cases}$$

Definition The function $f(t)$ is said to be of exponential order as $t \to \infty$ if there are constants $M \ge 0, c \ge 0, T \ge 0$ such that

$$|f(t)| \le Me^{ct}, \qquad \text{for } t \ge T \qquad (2)$$

Theorem (Existence of the Laplace Transform)
If f is piecewise continuous on $t \ge 0$ and of exponential order as $t \to \infty$ with constant c in eq(2), then $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c$.

Proof Since f is piecewise continuous on $t \geq 0$, we can find $M \geq 0$ such that (2) is satisfied with $T = 0$. i.e.

$$|f(t)| \leq Me^{ct}, \qquad \text{for } t \geq 0$$

From example 2 above $\int_0^\infty Me^{ct}e^{-st}dt$ converges if $s > c$. Thus using a comparison theorem $\int_0^\infty |f(t)e^{-st}|dt$ converges for $s > c$.

It follows that $\int_0^\infty f(t)e^{-st}dt$ converges. $\square$

Theorem (Linearity of the Laplace Transform)
If α and β are constants and $\mathcal{L}\{f(t)\}, \mathcal{L}\{g(t)\}$, exist for $s > c$ then

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\}$$

Proof

$$\begin{aligned}
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \lim_{b\to\infty} \int_0^b \left[\alpha f(t) + \beta g(t)\right] e^{-st}dt \\
&= \lim_{b\to\infty} \left[\alpha \int_0^b f(t)e^{-st}dt + \beta \int_0^b g(t)e^{-st}dt\right] \\
&= \lim_{b\to\infty} \alpha \int_0^b f(t)e^{-st}dt + \lim_{b\to\infty} \beta \int_0^b g(t)e^{-st}dt \qquad \text{since these limits exist} \\
&= \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} \text{ for } s > c
\end{aligned}$$

$\square$

Example Find $\mathcal{L}\{\cos(kt)\}$

Solution $\cos(kt) = \frac{e^{ikt}+e^{-ikt}}{2}$

$$\mathcal{L}\{\cos(kt)\} = \mathcal{L}\left\{\frac{e^{ikt}+e^{-ikt}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{ikt}\} + \frac{1}{2}\mathcal{L}\{e^{-ikt}\} = \frac{1}{2}\left[\frac{1}{s-ik} + \frac{1}{s+ik}\right] = \frac{s}{s^2+k^2} \quad \text{if } s > 0$$

Using Example 3, $\alpha = 0, \beta = k$

Similarly, we can show:

$$\mathcal{L}\left\{\sin(kt)\right\} = \frac{k}{s^2+k^2}, s > 0, \quad \mathcal{L}\left\{\cosh(kt)\right\} = \frac{s}{s^2-k^2}, s > k > 0, \quad \mathcal{L}\left\{\sinh(kt)\right\} = \frac{k}{s^2-k^2}, s > k > 0$$

Definition The gamma function is defined by $\Gamma(x) = \displaystyle\int_0^\infty e^{-t}t^{x-1}dt$ for $x > 0$

Properties $\Gamma(1) = 1$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(n+1) = n!$ (n a positive integer)

Example Let $a \in \mathbb{R}$, $a > -1$. Find $\mathcal{L}\left\{t^a\right\}$.

$$\begin{aligned}
&\int_0^\infty e^{-st}t^a dt \qquad \text{Let } u = st,\ t = \frac{u}{s},\ dt = \frac{1}{s}du \\
=&\int_0^\infty e^{-u}\left(\frac{u}{s}\right)^a \frac{1}{s}du \\
=&\frac{1}{s^{a+1}}\int_0^\infty e^{-u}u^0 du \qquad \text{since } t \geq 0, u > 0 \text{ if } s > 0 \\
=&\frac{1}{s^{a+1}}\Gamma(a+1) \qquad \text{for } s > 0
\end{aligned}$$

$$\implies \mathcal{L}\left\{t^a\right\} = \frac{\Gamma(a+1)}{s^{a+1}} \quad \text{for } s > 0$$

n is a positive integer $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, s > 0$

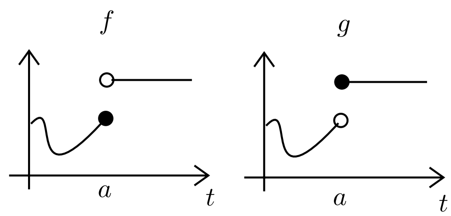
$a = -\frac{1}{2} \qquad \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \mathcal{L}\left\{t^{\frac{-1}{2}}\right\} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}} = \sqrt{\frac{\pi}{s}}, \quad s > 0$

Suppose we are given $F(s)$, how can we find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$?

Theorem (Uniqueness of the Inverse Laplace Transform)
Suppose that f and g satisfy the hypothesis of the Existence Theorem for the Laplace Transform, with $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. If for some c, $F(s) = G(s)$ $\forall s > c$, then $f(t) = g(t)$ wherever on $[0, \infty)$ both functions are continuous.

Proof Requires concepts form Complex Analysis. □

Implication $f(t)$ and $g(t)$ may differ only at points of discontinuity.



$f = g$ except at a
$\implies \mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$

Suppose $g(t) = \begin{cases} f(t), & t \geq 0 \\ h(t), & t < 0 \end{cases}$ and $\mathcal{L}\{f(t)\}$ is defined for $s > c$.

$$\mathcal{L}\{g(t)\} = \int_0^\infty g(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt = \mathcal{L}\{f(t)\} \qquad s > c$$

Convention If $F(s) = \mathcal{L}\{f(t)\}$ where $f(t)$ is continuous on $t \geq 0$ we call $f(t)$ the inverse Laplace Transform of $F(s)$ and write $\mathcal{L}^{-1}\{F(s)\} = f(t)$.

Theorem (Linearity of Inverse Laplace Transform)
If $H(s) = \alpha F(s) + \beta G(s)$ for $s > c$ where $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$, and f, g are continuous on $t \geq 0$m then

$$\mathcal{L}^{-1}\{H(s)\} = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}, \qquad \text{where } \alpha, \beta \text{ are constants}$$

Proof Let $h(t) = \alpha f(t) + \beta g(t)$, hence $h(t)$ is continuous on $t \geq 0$.

$$\mathcal{L}\{h(t)\} = \alpha\mathcal{L}\{f(t)\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s) = H(s) \qquad \text{by linearity of } \mathcal{L}$$

Then by uniqueness,

$$\mathcal{L}^{-1}\{H(s)\} = h(t) = \alpha f(t) + \beta g(t) = \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\}, \qquad s > c$$

□

Example Find $\mathcal{L}^{-1}\left\{\dfrac{2s-3}{s^2-s-6}\right\} = \mathcal{L}^{-1}\{F(s)\}$.

Solution Apply partial fractions to $F(s)$

$$\frac{2s-3}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2} \implies A = \frac{3}{5}, B = \frac{7}{5}$$

Now we take inverse Laplace Transform using linearity

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}\right\} = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$$

Example Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s^3+4s}\right\}$.

Solution Using partial Fractions

$$\frac{1}{s^3+4s} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \implies A = \frac{1}{4}, B = -\frac{1}{4}, C = 0$$

Now we take inverse Laplace Transform using linearity

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+4s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4s} - \frac{s}{4(s^2+4)}\right\} = \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \frac{1}{4} - \frac{1}{4}\cos(2t)$$

7.2 Laplace Transforms & IVPs

Theorem (Laplace Transform of Derivatives)
Suppose $f(t)$ is continuous on $t \geq 0$ and $f'(t)$ is piecewise continuous on $t \geq 0$ and $f(t)$ is of exponential order as $t \to \infty$ (with constants c, M, T). Then $\mathcal{L}\{f'(t)\}$ exists for $s > c$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Proof (for f' continuous) General Proof on p455-466 of text
Using IBP

$$\lim_{b\to\infty}\int_0^b f'(t)e^{-st}dt = \lim_{b\to\infty}\left[\left[f(t)e^{-st}\right]_0^b + s\int_0^b f(t)e^{-st}dt\right] = \lim_{b\to\infty}\left[\underbrace{f(b)e^{-sb}}_{\substack{\text{can show}\\ \to 0 \text{ for } s>c\\ \text{since}\\ |f(t)|\leq Me^{ct}}} - f(0) + s\underbrace{\int_0^b f(t)e^{-st}dt}_{\substack{\text{limit exists since}\\ \mathcal{L}\{f(t)\} \text{ defined for } s>c}}\right]$$

□

Corollary (Laplace Transform of Higher Derivatives)
Suppose

- $f^{(h)}$ is piecewise continuous on $t \geq 0$
- $f, f', \ldots, f^{(n-1)}$ are continuous on $t \geq 0$ and of exponential order as $t \to \infty$ (constants, M, c, T)

Then $\mathcal{L}\left\{f^{(n)}(t)\right\}$ exists for $s > c$ and

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) = F(s)$$

Example Use Laplace Transforms to find the solution of the IVP

$$x'' - x' - 6x = 0, \qquad x(0) = 2, x'(0) = -1$$

Solution By E/U Theorem we know a unique solution, $x(t)$, to the IVP on $\mathbb{R}$ which is twice differentiable. We assume that $\mathcal{L}\{x(t)\}$ exists and let $X(s) = \mathcal{L}\{x(t)\}$

$$\mathcal{L}\{x'' - x' - 6x\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 6\mathcal{L}\{x\} = 0 \qquad \text{linearity}$$

$$s^2X(s) - sx(0) - x'(0) - [sX(s) - x(0)] - 6X(s) = 0 \quad \text{(Derivative Theorems)}$$

$$(s^2 - s - 6)X(s) - 2s + 3 = 0 \qquad \text{(using IC's)}$$
$$X(s) = \frac{2s-3}{s^2 - s - 6}$$

Since $x(t)$ is continuous, we use the uniqueness of the inverse.
Laplace transform to conclude.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-s-6}\right\} = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t} \quad \text{from previous lec}$$

Solution of IVP is $x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}, t \in \mathbb{R}$

Theorem (Laplace Transform of Integrals)
If $f(t)$ is piecewise continuous on $t \geq 0$ and is of exponential order as $t \to \infty$ (with constants c, T, M) then

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \quad \text{for } s > c$$

equivalently:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau$$

Proof Since f is piecewise continuous, on $t \geq 0$. $g(t) = \int_0^t f(t)dt$ is continuous on $t \geq 0$, g' is piecewise continuous on $t \geq 0$.
Further,

$$|g(t)| = \left|\int_0^t f(\tau)d\tau\right| \leq \int_0^t |f(\tau)|d\tau \leq \int_0^t Me^{c\tau}d\tau = \frac{M}{c}(e^{ct} - 1) \leq \frac{M}{c}e^{ct} \qquad t \geq 0$$

So $g(t)$ is of exponential order as $t \to \infty$ and we can apply the Theorem on Laplace Transform of Derivatives.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\} \qquad \text{for } s > c$$
$$\implies \mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\} \qquad \text{for } s > c$$

note the notation is :

$$\int_0^t f(\tau)d\tau$$

□

Example Solve the IVP $\quad x''' + 4x' = 0, x(0) = x'(0) = 0, x''(0) = 1$

Solution Let $X(s) = \mathcal{L}\{x(t)\}$
Apply L.T to DE

$$\mathcal{L}\{x''' + 4x'\} = \mathcal{L}\{0\}$$
$$\mathcal{L}\{x'''\} + 4\mathcal{L}\{x'\} = 0$$
$$s^3X(s) - s^2x(0) - sx'(0) - x''(0) + 4[sX(s) - x(0)] = 0$$
$$(s^3 + 4s)X(s) = (s^2 + 4)x(0) + sx'(0) + x''(0) = 1$$
$$X(s) = \frac{1}{s^3 + 4s} = \frac{1}{s(s^2 + 4)}$$

To find $x(t) = \mathcal{L}^{-1}\{X(s)\}$ could use partial fractions.
Instead we'll use previous Theorem.

$$\frac{1}{s(s^2+4)} = \frac{F(s)}{s} \qquad \text{where } F(s) = \frac{1}{s^2+4}$$

Then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+1}\right\} = \frac{1}{2}\sin 2t$$

By Laplace Transform of Integrals Theorem

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)d\tau = \frac{1}{2}\int_0^t \sin(2\tau)d\tau = \frac{1}{4}(1-\cos(2t))$$

7.3 Translation and Partial Fractions

Note List of partial fractions "rules" is on p. 458 - 459 in textbook.

Theorem (Translation in s)
If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > c$, then $\mathcal{L}\{e^{at}f(t)\}$ exists for $s > a + c$ and $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.
Equivalently, $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Proof Since $F(s)$ exists for $s > c$

$$\int_0^\infty e^{-st}f(t)dt \qquad \text{converges for } s > c$$

Replacing s by $s - a$

$$\int_0^\infty e^{-(s-a)t}f(t)dt \qquad \text{converges for } s - a > c$$

$$\implies \underbrace{\int_0^\infty e^{-st}e^{at}f(t)dt}_{\mathcal{L}\{e^{at}f(t)\}} \qquad \text{converges for } s - a > c$$

Thus $F(s-a) = \mathcal{L}\{e^{at}f(t)\}$ for $s > a + c$. □

Immediate consequence

$$\mathcal{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}, s > a \qquad \mathcal{L}\left\{e^{at}\sin(kt)\right\} = \frac{k}{(s-a)^2+k^2}, s > a$$

$$\mathcal{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}, s > a \qquad n \text{ is postive integer}$$

Example Solve the IVP $x'' + 2x' + 5x = e^t, \qquad x(0) = 0, x'(0) = 1$

Solution Let $X(s) = \mathcal{L}\{x(t)\}$. Take Laplace Transform of DE.

$$s^2X(s) - sx(0) - x'(0) + 2[sX(s) - x(0)] + 5X(s) = \frac{1}{s-1}$$

$$(s^2 + 2s + 5)[X(s)] = \frac{1}{s-1} + 1 \qquad \text{using ICs}$$

$$X(s) = \frac{1}{(s-1)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

Using Partial Fractions on first term

$$\frac{1}{(s-1)(s^2+2s+5)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5} \implies A = 1/8, B = -1/8, C = -3/8$$

$$\begin{aligned} X(s) &= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{s+3}{s^2+2s+5} + \frac{1}{s^2+2s+5} \\ &= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{s}{[(s+1)^2+4]} + \frac{1}{8}\frac{5}{[(s+1)^2+4]} \\ &= \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{s+1}{[(s+1)^2+4]} + \frac{3}{8}\frac{2}{[(s+1)^2+4]} \end{aligned}$$

Take the inverse transform

$$x(t) = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{s+1}{[(s+1)^2+4]}\right\} + \frac{3}{8}\mathcal{L}^{-1}\left\{\frac{2}{[(s+1)^2+4]}\right\} = \frac{1}{8}e^t - \frac{1}{8}e^{-t}\cos(2t) + \frac{3}{8}e^{-t}\sin(2t)$$

7.4 Derivatives, Integrals and Products of Laplace Transforms

Theorem (Differentiation of Laplace Transforms)
Suppose that $f(t)$ satisfies the conditions of the Existence Theorem and $\mathcal{L}\{f(t)\} = F(s)$ for $s > c$. Then

$$\mathcal{L}\{-tf(t)\} = \frac{dF}{ds} \qquad \text{for } s > c$$

Equivalently,

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}$$

Examples Find $\mathcal{L}\{t\sin(kt)\}$

Solution $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2} = F(s)$ for $s > 0$.
Apply Theorem:

$$\mathcal{L}\{t\sin kt\} = -\frac{dF}{ds} = \frac{2ks}{(s^2+k^2)^2}$$

Application Forced, undamped mass-spring system.
$x(t)$: position of object at time t. $\omega_0 = \sqrt{\frac{k}{m}}$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t, \qquad x(0) = x'(0) = 0$$

Apply the Laplace Transform to the DE ($\mathcal{L}\{x(t)\} = X(s)$)

$$(s^2+\omega_0^2)X(s) = \frac{F_0}{m}\frac{s}{s^2+\omega_0^2}$$

$$X(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2}$$

Take inverse transform:

$$x(t) = \frac{F_0}{2m\omega_0}t\sin(\omega_0 t)$$

Corollary $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n}$ for $s > c$.

Theorem (Integration of Laplace Transforms)
Suppose $f(t)$ satisfies the conditions of the Existence theorem with $\mathcal{L}\{f(t)\} = F(s)$ for $s > c$, and $\lim_{t\to\infty} \dfrac{f(t)}{t}$ exists. Then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)d\sigma \qquad \text{for } s > c$$

Equivalently

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)d\sigma\right\}$$

Example Find $\mathcal{L}^{-1}\left\{\dfrac{2s}{(s^2-1)^2}\right\}$

Solution We'll apply the previous theorem. Let $F(s) = \frac{2s}{(s^2-1)^2}$.
Consider

$$\lim_{b\to\infty}\int_s^b F(\sigma)d\sigma = \lim_{b\to\infty}\int_s^b \frac{2\sigma}{(\sigma^2-1)^2}d\sigma = \lim_{b\to\infty}\left[\frac{1}{s^2-1} - \frac{1}{b^2-1}\right] = \frac{1}{s^2-1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s}{(s^2-1)^2}\right\} = t\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2\sigma}{(\sigma^2-1)^2}d\sigma\right\} = t\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = t\sinh(t)$$

7.4.1 Products of Laplace Transforms

Consider the IVP $x'' + x = g(t), \qquad x(0) = x'(0) = 0$
Suppose L.T. of $g(t)$ exists, $\mathcal{L}\{g(t)\} = G(s)$
Let $\mathcal{L}\{x(t)\} = X(s)$ and apply L.T. to DE. $\qquad \mathcal{L}\{x''\} + \mathcal{L}\{x\} = \mathcal{L}\{g(t)\}$

$$(s^2+1)X(s) = G(s) \implies X(s) = \frac{1}{s^2+1}G(s)$$

Note that $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

Might be useful to have a way to represent $\mathcal{L}^{-1}\{X(s)\}$ in terms of $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$ and $\mathcal{L}^{-1}\{G(s)\}$.

Definition The convolution of the piecewise continuous functions f and g is defined for ≥ 0 by

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(t)g(t-\tau)d\tau$$

Exercise: show $f * g = g * f$.

Theorem (Convolution)
Suppose that $f(t)$ and $g(t)$ satisfy the conditions of the L.T. Existence Theorem and $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ for $s > c$. Then for $s > c$

$$\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} \qquad \text{and} \qquad \mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)$$

Proof See text □

Back to the example $x'' + x = g(t)$, $x(0) = x'(0) = 0$

$$X(s) = \frac{1}{s^2+1}G(s), \qquad \mathcal{L}\{x(t)\} = X(s),\ \mathcal{L}\{g(t)\} = G(s),\ \mathcal{L}\{\sin(t)\} = \frac{1}{s^2+1}$$

Solution of IVP: $x(t) = \mathcal{L}^{-1}\{X(s)\} = \int_0^t \sin(\tau)g(t-\tau)d\tau = \sin(t) * g(t)$
gives a representation of the solution for any $g(t)$ which satisfies conditions of L.T. existence theorem.

Suppose $g(t) = t$, $\mathcal{L}\{g(t)\} = \frac{1}{s^2}$
Solution:

$$x(t) = \int_0^t \sin(\tau)(t-\tau)d\tau = t\int_0^t \sin(\tau)d\tau - \int_0^t \tau\sin(\tau)d\tau = t\sin(t)$$

7.5 Piecewise Continuous Input Functions

Recall unit step function (Heaviside step function) at $a \geq 0$.

$$u_a(t) = u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$

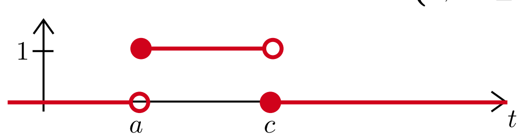
To find $\mathcal{L}\{u(t-a)\}$ use the definition

$$\lim_{b\to\infty}\int_0^b u(t-a)e^{-st}dt = \lim_{b\to\infty}\int_a^b e^{-st}dt = \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_a^b = \lim_{b\to\infty}\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right] = \frac{e^{-as}}{s} \quad \text{for } s > 0$$

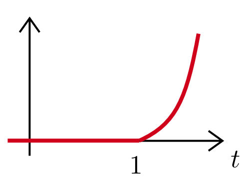
Can use $u(t-a)$ to represent piecewise defined functions.

Examples

1. $a < c$ $f(t) = u(t-a) - u(t-c) = \begin{cases} 0, & t - a \\ 1, & a \leq t < c \\ 0, & t \geq c \end{cases}$



2. $f(t) = (t-1)^2 u(t-1) = \begin{cases} 0, & t < 1 \\ (t-1)^2, & t \geq 1 \end{cases}$



Theorem (Translation in t)
Suppose $f(t)$ satisfies the conditions of the L.T existence theorem and $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \qquad \text{for } s > c$$ [1]

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = u(t-a)f(t-a)$$

[1] where textbook is incorrect

Proof Consider

$$\begin{aligned}
\lim_{b\to\infty}\int_0^b e^{-st}u(t-a)f(t-a)\,dt &= \lim_{b\to\infty}\int_a^b e^{-st}f(t-a)\,dt \qquad \text{Let } v=t-a, dv=dt \\
&= \lim_{b\to\infty}\int_0^{b-a} e^{-s(v+a)}f(v)\,dv \\
&= e^{-sa}\left[\lim_{b\to\infty}\int_0^{b-a} e^{-sv}f(v)\,dv\right] = e^{-sa}F(s) \qquad \text{for } s>c
\end{aligned}$$

$\lim_{b\to\infty}\int_0^b e^{-st}f(t)dt = \mathcal{L}\{f(t)\} \qquad s>c$ □

Alternate form: $\mathcal{L}\{u(t-a)g(t)\} = e^{-as}\mathcal{L}\{g(t+a)\}$

Example Find $\mathcal{L}\{g(t)\}$ where $g(t)=\begin{cases}0, & t<2\\ t^2, & t\ge 2\end{cases}$.

Here $g(t)=u(t-2)t^2$

$$\begin{aligned}
\mathcal{L}\{g(t)\} = \mathcal{L}\left\{t^2u(t-2)\right\} &= e^{-2s}\mathcal{L}\left\{(t+2)^2\right\} \quad \text{using alternate form of Theorem}\\
&= e^{-2s}\mathcal{L}\left\{t^2+4t+4\right\}\\
&= e^{-2s}\left(\mathcal{L}\left\{t^2\right\}+4\mathcal{L}\{t\}+4\mathcal{L}\{1\}\right)\\
&= e^{-2s}\left(\frac{1}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right), \quad s>0
\end{aligned}$$

Example Solve the IVP $\qquad x''+4x=u(t-1), \qquad x(0)=x'(0)=0$

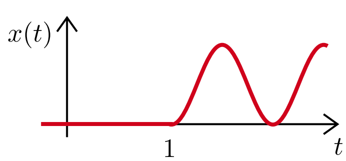
Solution Let $X(s)=\mathcal{L}\{x(t)\}$. Take L.T. of DE

$$\mathcal{L}\{x''\}+4\mathcal{L}\{x\}=\mathcal{L}\{u(t-1)\} \implies (s^2+4)X(s)=\frac{e^{-s}}{s} \longrightarrow X(s)=\frac{e^{-s}}{s(s^2+4)}$$

We saw before $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\}=\frac{1}{4}(1-\cos(2t))$

Apply the previous theorem with $a=1, F(s)=\frac{1}{s(s^2+4)}$. Thus

$$x(t)=\mathcal{L}^{-1}\left\{e^{-s}F(s)\right\}=u(t-1)f(t-1)=\frac{1}{4}u(t-1)(1-\cos[2(t-1)])=\begin{cases}0, & t<1\\ \frac{1}{4}(1-\cos[2(t-1)]), & t\ge 1\end{cases}$$



7.6 Applications: Models with discontinuous forcing

- Systems with an on/off switch
  - → circuits
  - → forced mass/spring system
- Growth of population with harvesting (fishing, hunting)
  - → typically harvesting only allowed during limited time hunting season.

Example Consider an object of mass 1kg attached to a spring with spring constant 4 N/m. There is no damping but the object is attached to a motor that provides a force $F(t) = \cos(2t)$ N. If the object is at rest in its equilibrium position, when the motor is turned on for 2πseconds, find the position of the object as a function of time.

Solution

- x - position of the object (metres)
- t - time (seconds) $t = 0$ when motor is turned on.
- Model form Newton's 2$^{\text{nd}}$ law: $mx'' + kx = F(t)$

Here m (mass) =1 kg, $k = 4$N/m (spring constant)

$$F(t) = \begin{cases} \cos(2t) & 0 \le t \le 2\pi \\ 0 & t \ge 2\pi \end{cases} = \cos(2t)\ [u(t) - u(t-2\pi)] = \cos(2t)\, u(t) - \cos(2(t-2\pi))\, u(t-2\pi)$$

Model: $x'' + 4x = \cos(2t)\, u(t) - \cos(2(t-2\pi))\, u(t-2\pi), \quad x(0) = x'(0) = 0$
Let $X(s) = \mathcal{L}\{x(t)\}$. Take Laplace Transform of DE using Derivative Theorem and initial conditions

$$s^2 X(s) + 4X(s) = \mathcal{L}\{\cos(2t)\, u(t)\} - \mathcal{L}\{\cos(2(t-2\pi))\, u(t-2\pi)\}$$

Shift in t: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

$$(s^2+4)X(s) = \mathcal{L}\{\cos(2t)\} - e^{-2\pi s}\mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4} - \frac{e^{-2\pi s}s}{s^2+4}$$
$$X(s) = \frac{s}{(s^2+4)^2} - e^{-2\pi s}\frac{s}{(s^2+4)^2}$$

Recall from previous lecture: $\mathcal{L}\{t\sin(kt)\} = \dfrac{2ks}{(s^2+k^2)^2}$

$$\begin{aligned} x(t) = \mathcal{L}^{-1}\{X(s)\} &= \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} - \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{s}{(s^2+4)^2}\right\} \\ &= \frac{1}{4}t\sin 2t - \frac{1}{4}(t-2\pi)\sin(2(t-2\pi))u(t-2\pi) \\ &= \frac{1}{4}t\sin 2t - \frac{1}{4}(t-2\pi)\sin(2t)u(t-2\pi) \\ &= \frac{1}{4}\sin(2t)(1-u(t-2\pi)) + \frac{\pi}{2}\sin(2t)u(t-2\pi) \\ &= \begin{cases} \frac{t}{4}\sin 2t & 0 \le t < 2\pi \\ \frac{\pi}{2}\sin 2t & t \ge 2\pi \end{cases} \end{aligned}$$