



# *Introduction to Optimization*

CO 255



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# Preface

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# Info

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Ricardo: MC 5036. OH: M 1:30 - 3pm  
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## Books (not required)

- Intro to Linear Opt. Bertsimas
- Int Programming. Conforti

## Grading

- assns: 20% ( $\approx 5$ )
- mid: 30% (Feb 11 in class)
- final: 50%

# Introduction

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Given a set  $S$ , and a function  $f : S \rightarrow \mathbb{R}$ . An optimization problem is:

$$\begin{array}{ll} \max & f(x) \\ \underbrace{\text{s.t.}}_{\text{subject to}} & x \in S \end{array} \quad (\text{OPT})$$

- $S$  **feasible region**
- A point  $\bar{x} \in S$  is a **feasible solution**
- $f(x)$  is **objective function**

(OPT) means: “Find a feasible solution  $x^*$  such that  $f(x) \leq f(x^*), \forall x \in S$ ”

- Such  $x^*$  is an **optimal solution**
- $f(x^*)$  is **optimal value**

Other ways to write (OPT):

$$\begin{aligned} \max \{ & f(x), x \in S \} \\ \max_{x \in S} & f(x) \end{aligned}$$

Analogous problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}$$

## Note

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & x \in S \end{array} = -1 \left( \begin{array}{ll} \min & -f(x) \\ \text{s.t.} & x \in S \end{array} \right)$$

**Problem**  $x^*$  may not exist

a) Problem is unbounded:

$$\forall M \in \mathbb{R}, \exists \bar{x} \in S, \text{ s.t. } f(\bar{x}) > M$$

b)  $S = \emptyset$ , i.e. (OPT) is **INFEASIBLE**

c) There may not exist  $x^*$  achieving supremum.

Example:

$$\begin{array}{ll} \max & x \\ \text{s.t.} & x < 1 \end{array}$$

### supremum

$$\sup\{f(x) : x \in S\} = \begin{cases} +\infty & \text{if OPT unbounded} \\ -\infty & \text{if } S = \emptyset \\ \min\{x : x \geq f(x), \forall x \in S\} & \text{otherwise} \end{cases}$$

always exist and are well-defined

### infimum

$$\inf\{f(x) : x \in S\} = -1 \cdot \sup\{-f(x) : x \in S\}$$

From this point on, we will abuse notation and say  $\max\{f(x) : x \in S\}$  is  $\sup\{f(x) : x \in S\}$ .

One way to specify that I want an opt. sol. (if exists) is

$$x^* \in \operatorname{argmax}\{f(x) : x \in S\}$$

# 2

## Linear Optimization (Programming) (LP)

---

$$S = \{x \in \mathbb{R}^n : Ax \leq b\}$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$  and  $f(x) = c^T x$ ,  $c \in \mathbb{R}^n$ .

↓

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \quad (\text{LP})$$

### Note

$$A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix} \quad A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

### Clarifying

$$u, v \in \mathbb{R}^n, \quad u \leq v \iff u_j \leq v_j, \forall j \in 1, \dots, n$$

### Note

$u \not\leq v$  is not the same as  $u > v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \not\leq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

### Example:

$$\begin{array}{ll} \max & 2x_1 + 0.5x_2 \\ \text{s.t.} & x_1 \leq 2 \\ & x_1 + x_2 \leq 2 \\ & x \geq 0 \end{array}$$

- Strict ineq. not allowed

**halfspace, hyperplane, polyhedron**

Let  $h \in \mathbb{R}^n, h_0 \in \mathbb{R}$ .

$\{x \in \mathbb{R}^n : h^T x \leq h_0\}$  is a **halfspace**.

$\{x \in \mathbb{R}^n : h^T x = h_0\}$  is a **hyperplane**.

$Ax \leq b$  is a **polyhedron** (i.e. intersection of finitely many halfspaces).

**Example:**

$n$  products,  $m$  resources. Producing  $j \in \{1, \dots, n\}$  given  $c_j$  profit/unit and consumes  $a_{ij}$  units of resource  $i$ ,  $\forall i \in \{1, \dots, m\}$ . There are  $b_i$  units available  $\forall i \in \{1, \dots, m\}$ .

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

which is an LP.

## 2.1 Determining Feasibility

Given a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

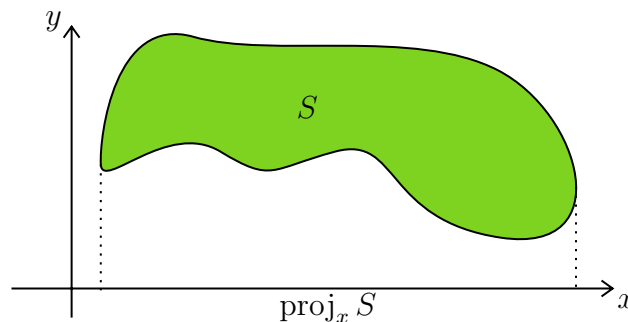
either find  $\bar{x} \in P$  or show  $P = \emptyset$ .

**Idea** In 1-d, easy.  $\rightarrow$  Reduce problem in dimension  $n$  to one in dimension  $n - 1$ .

**Notation** Let  $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \leq b\}$ , then

$$\text{proj}_x S := \{x \in \mathbb{R}^n : \exists y \text{ so that } (x, y) \in S\}$$

is the (orthogonal) *projection* of  $S$  onto  $x$ .





We will find if  $P = \emptyset$  by looking at  $\text{proj}_{x_1, \dots, x_{n-1}} \quad (P)$

## 2.2 Fourier-Motzkin Elimination

Call  $a_{ij}$  entries of  $A$ . Let

$$\begin{aligned} M &:= \{1, 2, \dots, m\} \\ M^+ &:= \{i \in M : a_{in} > 0\} \\ M^- &:= \{i \in M : a_{in} < 0\} \\ M^0 &:= \{i \in M : a_{in} = 0\} \end{aligned}$$

For  $i \in M^+$ :

$$a_i^T x \leq b_i \iff \sum_{j=1}^n a_{ij} x_j \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j + x_n \leq \frac{b_i}{a_{in}}, \quad \forall i \in M^+ \quad (1)$$

For  $i \in M^-$

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j - x_n \leq \frac{b_i}{-a_{in}}, \quad \forall i \in M^- \quad (2)$$

For  $i \in M^0$

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} a_{ij} x_j \leq b_i, \quad \forall i \in M^0 \quad (3)$$

$$P = \{x \in \mathbb{R}^n : (1)(2)(3)\}$$

Define

$$\sum_{j=1}^{n-1} \left( \frac{a_{ij}}{a_{in}} - \frac{a_{kj}}{a_{kn}} \right) x_j \leq \frac{b_i}{a_{in}} - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, \forall k \in M^- \quad (4)$$

### Theorem 2.1

$$(\bar{x}_1, \dots, \bar{x}_{n-1}) \text{ satisfies (3), (4)} \iff \exists \bar{x}_n : (\bar{x}_1, \dots, \bar{x}_n) \in P$$

**Proof:**

$\Leftarrow$  If  $(\bar{x}_1, \dots, \bar{x}_n)$  satisfies (1), (2), (3) then  $(\bar{x}_1, \dots, \bar{x}_{n-1})$  satisfies (3) and adding (1), (2)  $\Rightarrow$   $(\bar{x}_1, \dots, \bar{x}_{n-1})$  satisfies (4)

$\Rightarrow$  If  $(\bar{x}_1, \dots, \bar{x}_{n-1})$  satisfies (4)

$$\sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, k \in M^-$$

Let

$$\bar{x}_n := \max_{i \in M^+} \left\{ \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \right\}$$

$$\implies \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq -\bar{x}_n, \quad \forall i \in M^+$$

and

$$\begin{aligned} -\bar{x}_n &\leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall k \in M^- \\ &\implies (\bar{x}_1, \dots, \bar{x}_n) \in P \end{aligned}$$

□

### Note

Proof assumes  $M^+, M^-$  are nonempty. But statement holds regardless.

(if  $M^+$  or  $M^- = \emptyset$  then (4) yields no constraints)

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### Algorithm 1: Fourier-Motzkin

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- 1  $A^n = A, b^n = b$
- 2 given  $A^i, b^i$  obtain  $A^{i-1}, b^{i-1}$  ( $A^{i-1}$  has one less column than  $A^i$  column than  $A^i$ ) by applying the steps described

$$P_i := \{x \in \mathbb{R}^i : A^i x \leq b^i\}$$

then

$$P_{i-1} = \text{proj}_{x_1, \dots, x_{i-1}} P_i$$

- 3 Keep applying projection until  $i = 1$ .

$$P_0 = \emptyset \iff P_n = P = \emptyset$$

---

Let

$$P_i^n = P_i \times \mathbb{R}^{n-i} = \{x \in \mathbb{R}^n : (A^i, 0)x \leq b^i\}$$

not hard to see  $P_i^n = \emptyset \iff P_i = \emptyset$

Notice that

$$P_0 = \emptyset \iff P_0^n = \emptyset, P_0^n = \{0 \leq b^0\}$$

**Example:**

$$P_2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{rrcl} x_1 & +2x_2 & \leq & 1 \\ -x_1 & & \leq & 0 \\ & -x_2 & \leq & -2 \\ -3x_1 & -3x_2 & \leq & -6 \end{array} \right\}$$

draw the graph, clearly empty

$$M^+: \frac{1}{2}x_1 + x_2 \leq \frac{1}{2}$$

$$M^-: -x_2 \leq -2 \quad -x_1 - x_2 \leq -2$$

$$M^0: -x_1 \leq 0$$

$$P_1 = \left\{ x_1 \in \mathbb{R} : \begin{array}{ll} -x_1 & \leq 0 \\ \frac{1}{2}x_1 & \leq -\frac{3}{2} \\ -\frac{1}{2}x_1 & \leq -\frac{3}{2} \end{array} \right\}$$

$$M^+: x_1 \leq -3$$

$$M^-: -x_1 \leq 0 \text{ and } -x_1 \leq -3$$

$$P_0^2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{l} 0 \leq -3 \\ 0 \leq -6 \end{array} \right\} = \emptyset$$

$$\text{Here } b^0 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$$

**Remark:**

Inequality in  $P_i^n$ :

- All inequalities are obtained by a nonnegative combination of inequality in  $P_{i+1}^n$   
 $\implies$  all nonnegative combination of inequalities in  $P$ .
- If all  $A, b$  are rational then so are all  $A^i, b^i$
- If  $b = 0, b_i = 0, \forall i$

### Theorem 2.2: Farkas' Lemma

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = \emptyset \iff \begin{array}{l} u^T A = 0 \\ \exists u \in \mathbb{R}^m : u^T b < 0 \\ u \geq 0 \end{array}$$

**Proof:**

( $\Leftarrow$ ) Suppose  $\bar{x}$  satisfies  $A\bar{x} \leq b$ .

$$0 = u^T A\bar{x} \leq u^T b < 0$$

which is impossible.

( $\Rightarrow$ ) If  $P = \emptyset$ . Apply Fourier-Motzkin until we get

$$P_0^n = \emptyset = \{x \in \mathbb{R}^n : 0x \leq b^0\}$$

i.e. there exists  $j$  for which  $b_j^0 < 0$ .

If we look at corresponding constraint in  $P_0^n$  is

$$0^T x \leq b_j^0$$

which can be obtained by a vector  $u$  such that  $u^T A = 0, u^T b = b_j^0, u \geq 0$ .

□

#### Farkas' Lemma (alternate statement)

Exactly one of the following has a solution:

- a)  $Ax \leq b$
- $u^T A = 0$
- b)  $u^T b < 0$
- $u \geq 0$

#### Farkas' Lemma (Different Form)

Exactly one of the following has a solution:

- a)  $Ax = b$
- $x \geq 0$
- b)  $u^T A \geq 0$
- $u^T b < 0$

**Proof:**

(Sketch)

$$P = \left\{ x : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} = \left\{ x : \underbrace{\begin{pmatrix} A \\ -A \\ -I \end{pmatrix}}_{A'} x \leq \underbrace{\begin{pmatrix} b \\ -b \\ -0 \end{pmatrix}}_{b'} \right\}$$

Apply original Farkas' Lemma to get  $P = \emptyset \iff \exists u_1 \in \mathbb{R}^m, u_2 \in \mathbb{R}^m, v \in \mathbb{R}^n$ :

$$\begin{aligned} u_1^T A - u_2^T A - v &= 0 \\ u_1^T b - u_2^T b &< 0 \\ u_1, u_2, v &\geq 0 \end{aligned}$$

Let  $u = (u_2 - u_1)$

$$u^T A - v = 0 \implies u^T A \geq 0, \quad u^T b < 0$$

□

Consider a linear programming (LP):

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \tag{LP}$$

**Theorem 2.3: Fundamental Theorem of Linear Programming**

(LP) has exactly one of 3 outcomes:

- a) Infeasible
- b) Unbounded
- c) There exists an optimal solution.

**Proof:**

Let's assume a), b) don't hold.

If  $n = 1$ , then (LP) has an optimal solution. (Why?)

Else, define

$$\begin{array}{ll} \max & z \\ \text{s.t.} & z - c^T x \leq 0 \\ & Ax \leq b \end{array} \quad (\text{LP}')$$

(LP') is also not in case a) or b). (Why?)

Also if  $(x^*, z^*)$  is an optimal solution to (LP'), then  $x^*$  is an optimal solution to (LP). (Why?)

Apply Fourier-Motzkin to

$$\left\{ (x, z) : \begin{array}{l} z - c^T x \leq 0 \\ Ax \leq b \end{array} \right\}$$

Until we are left with a polyhedron

$$\{z \in \mathbb{R} : A'z \leq b'\}$$

Now  $\max_{\text{s.t.}} z$   $A'z \leq b'$  is not cases a) or b). (Why?)

→ can get an optimal solution  $z^*$  to such problem. Apply Fourier-Motzkin back to get  $(x^*, z^*)$  optimal solution to (LP'). (Why?)  $\square$

## 2.3 Certifying Optimality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \quad (\text{LP})$$

and let  $\bar{x} \in P = \{x : Ax \leq b\}$

**Question** Can we certify that  $\bar{x}$  is optimal?

Example:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ & x_1 + 2x_2 \leq 2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & x_1 - x_2 \leq 0.5 \end{aligned}$$

Consider  $\bar{x} = (0, 1)^T$  is clearly NOT optimal.

$x^* = (1, 0.5)^T$  and  $c^T x^* = 2.5$ . Any feasible solution satisfies

$$\begin{array}{rcl} x_1 + 2x_2 & \leq 2 & \times 1/3 \\ x_1 + x_2 & \leq 2 & \times 1 \\ + \quad x_1 - x_2 & \leq 0.5 & \times 2/3 \\ \hline 2x_1 + x_2 & \leq 3 & \end{array}$$

Instead do  $1 \times 1st$  constraint +  $1 \times 3rd$  constraint  $\implies 2x_1 + x_2 \leq 2.5$

In general:

$$\begin{array}{rcl} x_1 + 2x_2 & \leq 2 & \times y_1 \\ x_1 + x_2 & \leq 2 & \times y_2 \\ + \quad x_1 - x_2 & \leq 0.5 & \times y_3 \\ \hline (y_1 + y_2 + y_3)x_1 + (2y_1 + y_2 - y_3)x_2 & \leq 2y_1 + 2y_2 + 0.5y_3 \end{array}$$

As long as  $y_1, y_2, y_3 \geq 0$  and

$$\begin{aligned} y_1 + y_2 + y_3 &= 2 \\ 2y_1 + y_2 - y_3 &= 1 \end{aligned}$$

This leads to the following linear program:

$$\begin{aligned} \min \quad & 2y_1 + 2y_2 + 0.5y_3 \\ & y_1 + y_2 + y_3 = 2 \\ \text{s.t.} \quad & 2y_1 + y_2 - y_3 = 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

This is called the dual LP.

In general:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \tag{P}$$

Dual of (P)

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & y^T A = c^T \\ & y \geq 0 \end{aligned} \tag{D}$$

**Remark:**

We call (P) primal LP.

**Theorem 2.4: Weak Duality**

Let  $\bar{x}$  feasible for (P),  $\bar{y}$  feasible for (D). Then  $c^T \bar{x} \leq b^T \bar{y}$ .

**Proof:**

$$c^T \bar{x} = \bar{y}^T (A\bar{x}) \leq \bar{y}^T b$$

where we used  $A\bar{x} \leq b$  and  $\bar{y} \geq 0$ . □

**Corollary 2.5**

Several results:

- If (P) is unbounded then (D) is infeasible.
- If (D) is unbounded then (P) is infeasible.

**Note**

(P) and (D) can both be infeasible.

- If  $\bar{x}$  is feasible for (P)  $\bar{y}$  feasible for (D)  $c^T \bar{x} = b^T \bar{y}$ , then  $\bar{x}$  optimal for (P),  $\bar{y}$  optimal for (D).

**Theorem 2.6: Strong Duality**

$x^*$  is optimal for (P)  $\iff \exists y^*$  feasible for (D) such that  $c^T x^* = b^T y^*$ .

**Proof:**

( $\Leftarrow$ ) ✓

( $\Rightarrow$ ) Is (D) infeasible?

$$\text{Suppose } \left\{ y \in \mathbb{R}^n : \begin{array}{l} A^T y = c \\ y \geq 0 \end{array} \right\} = \emptyset$$

$$(\text{Alternate version of Farkas' Lemma}) \exists u : \begin{array}{l} u^T A^T \geq 0 \\ u^T c < 0 \end{array} \iff \exists d : \begin{array}{l} Ad \leq 0 \\ c^T d > 0 \end{array}$$

Take look at  $x' = x^* + d$ , then

$$\begin{aligned} Ax' &= Ax^* + Ad \leq b \\ c^T x' &= c^T x^* + c^T d > c^T x^* \end{aligned}$$

Contradiction. Thus (D) has an optimal solution  $y^*$ .

$$\text{Now let } \gamma = b^T y^*, \text{ and let } \theta := \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax \leq b \\ -c^T x \leq -\gamma \end{array} \right\}.$$

If  $\theta = \emptyset$ , by Farkas'

$$\exists \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} : \begin{cases} \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix}^T \begin{pmatrix} A \\ -c^T \end{pmatrix} = 0 \\ \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix}^T \begin{pmatrix} b \\ -\gamma \end{pmatrix} < 0 \\ \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} \geq 0 \end{cases} \iff \begin{cases} A^T \bar{y} = c \bar{\lambda} \\ b^T \bar{y} < \gamma \bar{\lambda} \\ \bar{y} \geq 0 \\ \bar{\lambda} \geq 0 \end{cases}$$

Case 1:  $\bar{\lambda} > 0$ .

Let  $y' = \frac{\bar{y}}{\bar{\lambda}}$ . Then we have

$$A^T y' = A^T \frac{\bar{y}}{\bar{\lambda}} = c \quad \text{and} \quad b^T y' = b^T \frac{\bar{y}}{\bar{\lambda}} < \gamma \quad \text{and} \quad y' = \frac{\bar{y}}{\bar{\lambda}} \geq 0$$

Contradicts optimality of  $y^*$ .

$$A^T y = 0$$

Case 2:  $\bar{\lambda} = 0$ . Then  $b^T y < 0$

$$\bar{y} \geq 0$$

Now we can do the same thing previously. Let  $y' = y^* + \bar{y}$ , then

$$A^T y' = A^T y^* + A^T \bar{y} = c$$

and

$$y' = y^* + \bar{y} \geq 0$$

$$b^T y' = b^T y^* + b^T \bar{y} < b^T y^*$$

Contradicts optimality of  $y^*$ .

Thus  $\theta \neq \emptyset$ .

Let  $\bar{x} \in \theta$ ,

$$c^T x^* \underbrace{\leq}_{\text{weak duality}} b^T y^* = \gamma \underbrace{\leq}_{\bar{x} \in \theta} c^T \bar{x} \leq c^T x^*$$

where the last inequality is because  $\bar{x}$  feasible for (P),  $x^*$  optimal for (P).

□

## 2.4 Possible Outcomes

See [here](#).



## 2.5 Duals of generic LPs

$$\begin{array}{rcll}
 \max & 2x_1 + 3x_2 - 4x_3 & & \\
 & x_1 & +7x_3 & \leq 5 \\
 & & 2x_2 - x_3 & \geq 3 \\
 \text{s.t.} & x_1 & +x_3 & = 8 \\
 & & x_2 & \leq 6 \\
 & x_1 & & \geq 0 \\
 & x_2 & & \leq 0
 \end{array}$$

$$\begin{array}{rcl}
 \max & (2, 3, -4)x & \\
 \text{s.t.} & \begin{pmatrix} 1 & 0 & 7 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 5 \\ -3 \\ 8 \\ -8 \\ 6 \\ 0 \\ 0 \end{pmatrix} &
 \end{array}$$

and dual

$$\begin{array}{rcl}
 \min & (5, -3, 8, -8, 6, 0, 0)y & \\
 \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and } y \geq 0 & (D_1)
 \end{array}$$

$$\begin{array}{rcl}
 \min & (5, -3, 8, -8, 6)y & \\
 \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and } y \geq 0 & (D_2)
 \end{array}$$

**Claim**  $(y_1^*, \dots, y_5^*)$  is optimal for  $(D_2) \iff (y_1^*, \dots, y_5^*, y_6^*, y_7^*)$  optimal for  $(D_1)$  with

$$\begin{aligned}
 y_6^* &= y_1^* + y_3^* - y_4^* - 2 \\
 y_7^* &= 3 - (-2y_2^* + y_5^*)
 \end{aligned}$$

$$\begin{array}{rcl}
 \min & (5, 3, 8, 6)y & \\
 \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 7 & -1 & 1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and } y_1 \geq 0, y_2 \leq 0 \quad y_4 \geq 0 & (D_3)
 \end{array}$$

**Claim** Opt value of  $(D_2)$  and  $(D_3)$  are same.

**In general**

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \quad (P) \quad \left| \quad \begin{array}{ll} \min & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array} \quad (D)$$

### 2.5.1 Cheat Sheet

Here or

Primal (max)		Dual (min)	
Constraint	$\leq$	$\geq 0$	Variable
	$\geq$	$\leq 0$	
	$=$	free	
Variable	$\geq$	$\geq 0$	Constraint
	$\leq$	$\leq 0$	
	free	$=$	

**Remark:**

This is not symmetric... The way you can remember it is by thinking natural variables in real life, like you cannot have negative number of cars and so on...

**Q** What if you start with a minimization LP as primal?

**Example:**

$$\begin{array}{ll} \min & x_1 - x_2 \\ & 2x_1 + 3x_2 \leq 5 \\ \text{s.t.} & x_1 - x_2 \geq 3 \\ & x_1 + 5x_2 = 7 \\ & x_1 \geq 0, x_2 \leq 0 \end{array} \quad (P)$$

Rewrite as:

$$-1 \times \begin{pmatrix} \max & -x_1 + x_2 \\ \downarrow & \\ \text{s.t.} & \dots \end{pmatrix}$$

Will lead to finding dual:

$$\begin{array}{ll} \max & 5y_1 + 3y_2 + 7y_3 \\ \downarrow & \\ & 2y_1 + y_2 \leq 1 \\ \text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\ & y_1 \leq 0, y_2 \geq 0, y_3 \text{ free} \end{array}$$

**Also**

- Weak duality holds.

If  $\bar{x}$  feasible for (P),  $\bar{y}$  feasible for (D), then  $c^T \bar{x} \geq b^T \bar{y}$ .

- Strong duality holds

### Note

The dual of the dual of (P) is (P).

### Example:

Given a simple undirected graph  $G = (V, E)$ .  $M \subseteq E$  is a *matching* if every vertex  $v \in V$  is incident to  $\leq 1$  edge in  $M$ .

See examples of matching in [CO 342](#) or [MATH 249](#).

## Max cardinality matching

Find matching  $M$  with largest  $|M|$ .

Define  $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{otherwise} \end{cases}$ .

$$\begin{aligned} & \max \quad \sum_{e \in E} x_e \\ & \quad \downarrow \\ & \text{s.t.} \quad \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\ & \quad \quad 0 \leq x_e, \quad \forall e \in E \end{aligned}$$

where  $\delta(v)$  = set of edges in  $E$  incident to  $v$ .

$$\begin{aligned} & \min \quad \sum_{v \in V} y_v \\ & \quad \downarrow \\ & \text{s.t.} \quad y_u + y_v \geq 1, \quad \forall e = uv \in E \\ & \quad \quad y \geq 0 \end{aligned}$$

## 2.6 Other interpretations of dual

### Example:

		Per unit Profit	Resources	
			Per unit consumption A	B
Product	1	5	2	3
	2	3	4	1
Available Resources			15	10

$$\begin{array}{ll}
\max & 5x_1 + 3x_2 \\
\downarrow & \\
& 2x_1 + 4x_2 \leq 15 \\
\text{s.t.} & 3x_1 + x_2 \leq 10 \\
& x \geq 0
\end{array}$$

Suppose somebody wants to buy  $A, B$  from me. What is the lowest price I should ask?

Let  $y_A, y_B$  be prices:

$$\begin{array}{ll}
\min & 15y_A + 10y_B \\
\downarrow & \\
& 2y_A + 3y_B \geq 5 \\
\text{s.t.} & 4y_A + y_B \geq 3 \\
& y \geq 0
\end{array}$$

### Example: Zero-Sum

Alice, Bob play game. A:  $m$  choices. B:  $n$  choices. Alice play  $i$ , Bob plays  $j$ , Bob pays Alice  $M_{ij}$  dollars.

		Alice		
		R	P	S
Bob	R	0	1	-1
	P	-1	0	1
	S	1	-1	0

Zero-sum: Amount won by Alice - Amount won by Bob = 0

Let  $y \in \mathbb{R}_+^m$ , Alice's probability distribution.

Let  $x \in \mathbb{R}_+^n$ , Bob's probability distribution.

Expected Amount Bob pays Alice:

$$\sum_{i=1}^m \sum_{j=1}^n y_i M_{ij} x_j = y^T M x$$

$$P = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \sum x_j = 1 \\ x \geq 0 \end{array} \right\}$$

$$Q = \left\{ y \in \mathbb{R}^m : \begin{array}{l} \sum y_i = 1 \\ y \geq 0 \end{array} \right\}$$

Alice wants  $\max_{y \in Q} \left\{ \min_{x \in P} y^T M x \right\}$ . Bob wants  $\min_{x \in P} \left\{ \max_{y \in Q} y^T M x \right\}$ .

Suppose  $\bar{y} \in Q$  is fixed. Bob's problem is

$$\begin{aligned} \min_{x \in P} \bar{y}^T M_x &= \min \sum_{j=1}^n \left( \sum_{i=1}^m M_{ij} \bar{y}_i \right) x_j \\ &\downarrow \\ \text{s.t.} \quad &\sum_{j=1}^n x_j = 1 \\ &x \geq 0 \end{aligned}$$

This is equivalent to picking smallest number in

$$\begin{aligned} &\left\{ \sum_{i=1}^m M_{ij} \bar{y}_i \right\}_{j=1}^n \\ \Rightarrow \max_{y \in Q} \min_{x \in P} y^T M_x &= \max_{y \in Q} \left\{ \begin{array}{l} \max u \\ \downarrow \\ \text{s.t.} \quad u \leq y^T M e_j, \quad \forall j = 1, \dots, n \end{array} \right\} \\ &\downarrow \\ &= \begin{array}{l} \max u \\ \downarrow \\ \text{s.t.} \quad u \leq y^T M e_j, \quad \forall j = 1, \dots, n \\ y^T = 1 \\ y \geq 0 \end{array} \end{aligned}$$

Similarly Bob's problem:

$$\begin{aligned} \min \quad &v \\ \downarrow \\ &v \geq e_i^T M x, \quad \forall i = 1, \dots, m \\ \text{s.t.} \quad &x^T = 1 \\ &x \geq 0 \end{aligned}$$

There are  $x^*, y^*$  for which strategy values match  $\rightarrow$  Nash's Equilibrium.

Now get back to Farkas' Lemma Theorem 2.2. <sup>1</sup>

**Proof:**

$$\begin{aligned} \max \quad &0^T x \\ \downarrow \\ \text{s.t.} \quad &Ax \leq b \end{aligned} \tag{P}$$

$$\begin{aligned} \min \quad &b^T u \\ \downarrow \\ \text{s.t.} \quad &u^T A = 0 \\ &u \geq 0 \end{aligned} \tag{D}$$

(D) is always feasible ( $u = 0$ ).

<sup>1</sup>Rephrase it a little bit: Exactly one of the two has a solution (i)  $Ax \leq b$  (ii)  $u^T \dots$

If  $\exists \bar{x} : A\bar{x} \leq b$ ,  $\bar{x}$  optimal for (P)  $\implies$  optimal for (D) has value 0.  
 $\implies \nexists u$  satisfying (ii).

And the converse is also true. □

## 2.7 Complementary Slackness (C.S.)

Let  $x^*, y^*$  be feasible for primal and dual respectively.

### Complementary Slackness

Abbreviated as C.S.

- i) Either  $x_j^* = 0$  or corresponding dual constraint is tight at  $y^*$ ,  $\forall j = 1, \dots, n$ .
- ii) Either  $y_i^* = 0$  or corresponding primal constraint is tight at  $x^*$ ,  $\forall i = 1, \dots, m$ .

Example:

$$\begin{array}{ll}
 \min & x_1 - x_2 \\
 \downarrow & \\
 & 2x_1 + 3x_2 \leq 5 \\
 \text{s.t.} & x_1 - x_2 \geq 3 \\
 & x_1 + 5x_2 = 7 \\
 & x_1 \geq 0, x_2 \leq 0
 \end{array} \tag{P}$$

$$\begin{array}{ll}
 \max & 5y_1 + 3y_2 + 7y_3 \\
 \downarrow & \\
 & 2y_1 + y_2 + y_3 \leq 1 \\
 \text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\
 & y_1 \leq 0, y_2 \geq 0
 \end{array} \tag{D}$$

- i)  $x_1^* = 0$  OR  $2y_1^* + y_2^* + y_3^* = 1$   
 $x_2^* = 0$  OR  $3y_1^* - y_2^* + 5y_3^* = -1$
- ii)  $y_1^* = 0$  OR  $2x_1^* + 3x_2^* = 5$   
 $y_2^* = 0$  OR  $x_1^* - x_2^* = 3$   
 $y_3^* = 0$  OR  $x_1^* + 5x_2^* = 7$

### Theorem 2.7

Let  $x^*, y^*$  be feasible for primal/dual respectively. TFAE<sup>a</sup>

- a)  $x^*$  opt for primal AND  $y^*$  opt. for dual
- b) Obj. value of  $x^* =$  Obj. value of  $y^*$

c)  $x^*, y^*$  satisfy C.S.

<sup>a</sup>the following are equivalent

**Proof:**

a)  $\iff$  b) done.

b)  $\iff$  c) Proof for

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \min & b^T y \\ \downarrow & \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

**Note**

$$A^T y \geq c \iff \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \forall j = 1, \dots, n$$

$$\begin{aligned} c^T x^* &= \sum_{j=1}^n c_j x_j^* \\ &\leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} y_i^* \right) x_j^* \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \\ &\leq \sum_{i=1}^m b_i y_i^* = b^T y^* \end{aligned}$$

where first and second inequalities come from  $x \geq 0, y \geq 0$  respectively.

(b)  $c^T x^* = b^T y^* \iff$  C.S. holds. (Just play with some strict inequality conditions)

□

**Example:**

$$\begin{array}{ll} \max & x_1 + x_2 \\ \downarrow & \\ \text{s.t.} & x_1 + x_2 \leq 1 \end{array} \qquad \begin{array}{ll} \min & y \\ \downarrow & \\ & y = 1 \\ \text{s.t.} & y = 1 \\ & y \geq 0 \end{array}$$

Consider a pair  $x^* = (0, 0), y^* = 1$  which violates CS.

## 2.7.1 Geometric Interpretation of C.S.

$$\begin{array}{ll}
 \max & c^T x \\
 \downarrow & \\
 \text{s.t.} & Ax \leq b
 \end{array}
 \qquad
 \begin{array}{ll}
 \min & c^T y \\
 \downarrow & \\
 \text{s.t.} & A^T y = c \\
 & y \geq 0
 \end{array}$$

$$A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

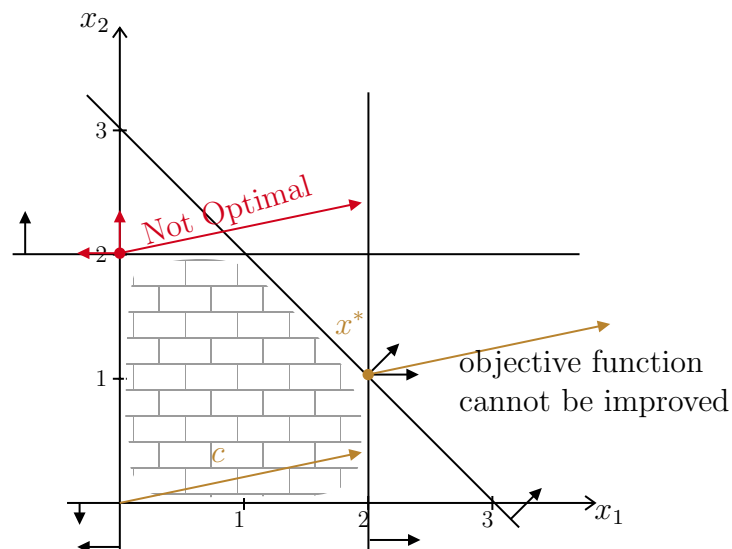
C.S. says  $a_i^T x^* = b_i$  or  $y_i^* = 0$ .

$$A^T y = c \implies \begin{pmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{pmatrix} y = c \implies \sum_{i=1}^m a_i y_i = c$$

C.S. says  $c$  is a nonnegative combination of tight constraint at  $x^*$ .

Example:

$$\begin{array}{ll}
 \max & 2x_1 + 0.5x_2 \\
 \downarrow & \\
 & x_1 \leq 2 \\
 & x_2 \leq 2 \\
 \text{s.t.} & x_1 + x_2 \leq 3 \\
 & x_1, x_2 \geq 0
 \end{array}$$





**Theorem 2.8**

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \quad (\text{P})$$

is unbounded iff (P) is feasible and  $\exists d \in \mathbb{R}^n : \begin{array}{l} c^T d > 0 \\ Ad \leq 0 \end{array}$ .

**Proof:**

$\Rightarrow$ ) Let  $\bar{x}$  feasible for (P),  $\bar{x} + \lambda d$  is also feasible for (P)  $\forall \lambda \geq 0$ .

$c^T(\bar{x} + \lambda d)$  can be made arbitrary large.

$\Leftarrow$ ) Hard exercise but doable.

□

## 2.8 Geometry of Polyhedra

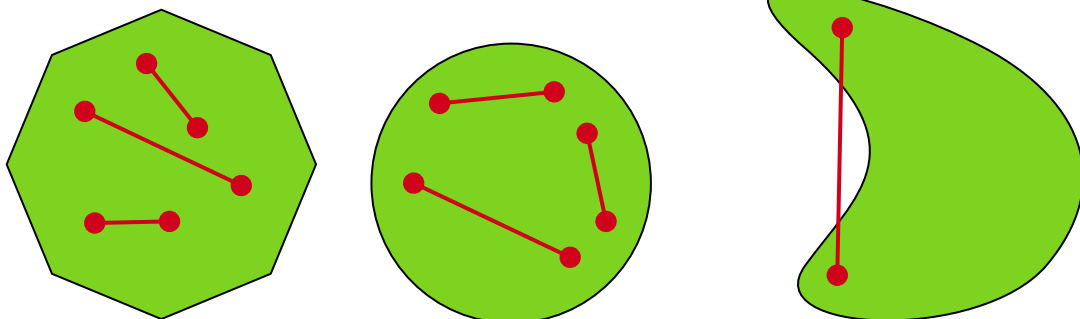
**line segment**

$\bar{x}, \bar{y} \in \mathbb{R}^n$  the line segment between  $\bar{x}, \bar{y}$  is

$$\left\{ x \in \mathbb{R}^n : \begin{array}{l} x = \lambda \bar{x} + (1 - \lambda) \bar{y} \\ \text{for some } \lambda \in [0, 1] \end{array} \right\}$$

**convex set**

$S$  is a convex set if  $\forall x, y \in S$ , line segment between  $x, y$  is contained in  $S$ .

**Example:**

NOT a convex set

Polyhedra are convex sets.  $P = \{x : Ax \leq b\}$ .  $\bar{x}, \bar{y} \in P$  then

$$A(\underbrace{\lambda}_{\geq 0} \bar{x} + \underbrace{(1 - \lambda)}_{\geq 0} \bar{y}) \leq \lambda b + (1 - \lambda)b = b$$

**convex combination**

Given  $x^1, \dots, x^k \in \mathbb{R}^n$ . We say  $\bar{x}$  is a convex combination of  $x^1, \dots, x^k$  if  $\exists \lambda$ :

$$\begin{aligned}\bar{x} &= \sum_{i=1}^k \lambda_i x^i \\ 1 &= \sum_{i=1}^k \lambda_i \\ \lambda &\geq 0\end{aligned}$$

Optimal solution seems to be happen at “corners”.

Let  $P$  be a polyhedron  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ .

**vertex**

$\bar{x}$  is a vertex of  $P$  if  $\exists c$ :  $\bar{x}$  is unique optimal solution to

$$\begin{aligned}\max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b\end{aligned}$$

**extreme point**

$\bar{x}$  is an extreme point of  $P$  if  $\nexists u, v \in P \setminus \{\bar{x}\}$  such that  $\bar{x}$  is in line segment between  $u, v$ .

**basic feasible solution**

$\bar{x} \in P$  is a basic feasible solution of  $P$  if there are  $n$  linearly independent tight constraints at  $\bar{x}$ .

**Note**

Constraints

$$a_i^T x \leq b_i, \quad \forall i = 1, \dots, m$$

are linearly independent if  $\{a_i\}_{i=1}^m$  are linearly independent.

**Theorem 2.9**

Let  $\bar{x} \in P$ . TFAE:

- a)  $\bar{x}$  is a vertex of  $P$ .
- b)  $\bar{x}$  is a basic feasible solution of  $P$ .
- c)  $\bar{x}$  is a extreme point of  $P$ .

**Proof:**a)  $\implies$  c) Suppose  $\exists u, v \in P \setminus \{\bar{x}\}$  such that

$$\bar{x} = \lambda u + (1 - \lambda)v$$

for some  $\lambda \in (0, 1)$ . Consider  $c$  for which  $\bar{x}$  is an optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$$

$$\implies \begin{array}{l} c^T \bar{x} \geq c^T u \\ c^T \bar{x} \geq c^T v \end{array}$$

and

$$c^T \bar{x} = \underbrace{\lambda}_{\geq 0} c^T u + \underbrace{(1 - \lambda)}_{\geq 0} c^T v \leq \lambda c^T \bar{x} + (1 - \lambda) c^T \bar{x} = c^T \bar{x}$$

$$\implies c^T u = c^T v = c^T \bar{x}$$

 $\implies \bar{x}$  NOT a vertex.c)  $\implies$  b) Suppose  $\bar{x}$  is not a BFS. Let  $I \subseteq \{1, \dots, m\}$  be the index set of tight constraint at  $\bar{x}$ . Consider

$$a_i^T d = 0, \quad \forall i \in I \tag{*}$$

But since  $\bar{x}$  not BFS,  $\exists \bar{d} \neq 0$  satisfying  $(*)$ .<sup>a</sup>

$$x(\epsilon) = \bar{x} + \epsilon \bar{d}$$

$$a_i^T x(\epsilon) = a_i^T \bar{x} \leq b_i, \quad \forall i \in I$$

$$a_i^T x(\epsilon) = \underbrace{a_i^T \bar{x}}_{< b_i} + \epsilon a_i^T \bar{d} \leq b_i, \quad \forall i \notin I$$

which is satisfied if  $|\epsilon|$  is small enough. $x(\epsilon) \in P$  if  $|\epsilon|$  is small enough.

But then

$$\bar{x} = \frac{1}{2}x(\epsilon) + \frac{1}{2}x(-\epsilon)$$

b)  $\implies$  a) Let  $I \subseteq \{1, \dots, m\}$  index set of tight constraint at  $\bar{x}$ .

Define

$$c := \sum_{i \in I} a_i$$

Then  $\forall x \in P$ 

$$c^T x = \sum_{i \in I} a_i^T x \leq \sum_{i \in I} b_i$$

And

$$c^T \bar{x} = \sum_{i \in I} a_i^T \bar{x} = \sum_{i \in I} b_i$$

$\implies \bar{x}$  is optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array} \quad (**)$$

If  $x' \in P$  is optimal solution to  $(**)$ , then

$$a_i^T x' = b_i, \quad \forall i \in I \quad (***)$$

But since there are  $n$  linear independent constraints in  $I$ ,  $\bar{x}$  is unique solution to  $(***)$ .  $\implies x' = \bar{x}$ .

□

---

<sup>a</sup>by Rank-Nullity Theorem.

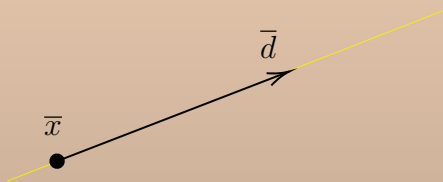
**Q** When does  $P$  have extreme points?

#### line

Let  $\bar{x}, \bar{d} \in \mathbb{R}^n$ ,  $\bar{d} \neq 0$ . The set

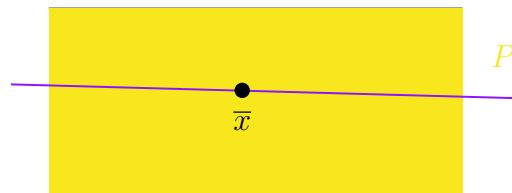
$$\{x \in \mathbb{R}^n : x = \bar{x} + \lambda \bar{d} \text{ for some } \lambda \in \mathbb{R}\}$$

is called a line.



We say a polyhedron  $P$  has a line if  $\exists \bar{x}, \bar{d}$  has a line if  $\exists \bar{x}, \bar{d}$  s.t.  $\bar{x} \in P, \bar{d} \neq 0$  and

$$\{x \in \mathbb{R}^n : x = \bar{x} + \lambda \bar{d} \text{ for some } \lambda \in \mathbb{R}\} \subseteq P$$



#### Proposition 2.10

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$  has a line iff  $P \neq \emptyset$  and  $\exists \bar{d} \neq 0$  such that  $A\bar{d} = 0$

$$\iff P \neq \emptyset \text{ and } \text{rank}(A) < n$$

**Proof:**  
Exercise.

□

**Theorem 2.11**

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$  has an extreme point

$\iff P \neq \emptyset$  and  $P$  has no lines.

**Proof:**

Exercise. □

**pointed polyhedron**

A non-empty polyhedron is called pointed if it has no lines.

**Note**

not pointed does not imply bounded. For example, in  $\mathbb{R}^2$ ,  $x \geq 0$  and  $y \geq 0$ .

**Theorem 2.12**

Let  $P \neq \emptyset$  pointed polyhedron. If  $\max_{x \in P} c^T x$  (LP) has an optimal solution, it has an optimal solution that is an extreme point.

**Proof:**

Let  $\bar{x}$  be an optimal solution to (LP) with largest number of linear independent tight constraints.

Suppose there are  $\leq n - 1$  linear independent tight constraints at  $\bar{x}$ .

Pick  $\bar{d} \neq 0$  such that  $a_i^T \bar{d} = 0, \forall i \in I$ , where  $I$  is the index set of tight constraints. By the exact same argument as before,  $\bar{x} \pm \epsilon \bar{d} \in P$  for  $\epsilon$  small enough. But

$$c^T(\bar{x} \pm \epsilon \bar{d}) = c^T \bar{x} \pm \epsilon c^T \bar{d}$$

$$\implies c^T \bar{d} = 0$$

$$\implies c^T d(\bar{x} \pm \epsilon d) = c^T \bar{x}$$



Since  $P$  is pointed,  $\exists \bar{\epsilon}$  for which

$$\bar{x} \pm \bar{\epsilon} \bar{d} \in P$$

and one of them not in  $P$  if  $|\epsilon| > \bar{\epsilon}$ . That can only happen if

$$a_k^T(\bar{x} + \bar{\epsilon} \bar{d}) = b_k \quad \text{or} \quad a_k^T(\bar{x} - \bar{\epsilon} \bar{d}) = b_k$$

for some  $k \notin I$ .

$\implies a_k^T \bar{d} \neq 0, \implies a_k$  is linear independent from  $\{a_i\}_{i \in I}$  since non-zero cannot be linear combination of zeros. Contradiction to choice of  $\bar{x}$ .  $\square$

## 2.9 Simplex Algorithm

### Standard Equality Form

A linear program is in Standard Equality Form (SEF) if it is of the form

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

### Proposition 2.13

Given any linear program, there exists an equivalent LP in SEF.

**Example:**

$$\begin{array}{ll} \max & x_1 + 2x_2 + x_3 \\ \downarrow & \\ & 3x_1 + x_2 \leq 5 \\ \text{s.t.} & -x_1 + x_3 \geq 6 \\ & x_1 \leq 0, x_3 \geq 0 \end{array} \quad (\text{P1})$$

$$\begin{aligned} x'_1 &= -x_1 \geq 0 \text{ and} \\ x_2 &= x_2^+ - x_2^- \text{ where } x_2^+ \geq 0, x_2^- \geq 0 \end{aligned}$$

We introduce

$$s_1 = 5 - 3x_1 - x_2 \geq 0, \quad s_2 = -x_1 + x_3 - 6 \geq 0$$

Then

$$\begin{array}{ll} \max & -x'_1 + 2x_2^+ - 2x_2^- + x_3 \\ \downarrow & \\ & -3x'_1 + 2x_2^+ - x_2^- + s_1 = 5 \\ \text{s.t.} & x'_1 + x_3 - s_2 = 6 \\ & x'_1, x_2^+, x_2^-, x_3, s_1, s_2 \geq 0 \end{array} \quad (\text{P2})$$

$x$  feasible for (P1)  $\iff (x'_1, x_2^+, x_2^-, x_3, s_1, s_2)$  feasible for (P2) and they have same cost.

**Assumption**  $A \in \mathbb{R}^{m \times n} \rightarrow \text{rank}(A) = m$ . This is WLOG. Since if

$$a_i = \sum_{k \neq i} \lambda_k a_k$$

Either

$$b_i \neq \sum_{k \neq i} \lambda_k b_k$$

in which case (SEF) is infeasible. Or  $a_i^T x = b_i$  is redundant. So it can be removed from (SEF).

**Note**

$\{x : Ax = b, x \geq 0\}$  is *pointed* polyhedron (if nonempty).

**Structure of BFS** Any feasible solution has  $m$  linear independent tight constraints ( $n - m$ ) extra tight constraint must come from  $x_j \geq 0$ .

Let  $B \subseteq \{1, \dots, n\}$  such that  $|B| = m$  and  $A_B$ <sup>2</sup> is invertible.

$N = \{1, \dots, n\} \setminus B$ .  $x_N = 0$ , i.e.  $x_j = 0, \forall j \in N$ .

Feasible solutions obtained this way are precisely BFS.

**Example:**

$$\begin{array}{ll} \max & (3 \ 2 \ 1 \ 4) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x \geq 0 \end{array}$$

If we pick

$$\begin{array}{ll} B = \{1, 2\} & A_B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ N = \{3, 4\} & A_N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ C_B = (3 \ 2)^T & C_N = (1 \ 4)^T \end{array}$$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

---


$$B = \{1, 3\}, B = \{2, 4\}, A_B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, A_N = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$C_B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, C_N = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

If we set  $x_N = 0$  (for  $B = \{1, 3\}$ ) we are left with

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

This has a unique solution  $x_1 = 3.5, x_3 = -1.5$ , but not feasible.

---

<sup>2</sup> $A_B$  is submatrix obtained by picking columns of  $A$  indexed by  $B$ . Such  $B$  is called a basis.

If we pick  $B = \{1, 2\}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$\underbrace{x_3 = x_4}_{x_N} = 0, x_1 = 3, x_2 = 1$ , which is feasible.

In general,

$$Ax = b \iff A_B x_B + \cancel{A_N x_N}^0 = b$$

has unique solution  $x_b = A_B^{-1}b$ .

For any basis  $B$ , the corresponding *basic solution* is

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1}b \\ 0 \end{pmatrix}$$

If  $A_B^{-1}b \geq 0$ , then it is a *BFS*.

### 2.9.1 Canonical Form

Let  $B$  be a feasible basis (i.e. corresponding basis solution is feasible).

$$\begin{aligned} Ax = b &\iff A_B x_B + A_N x_N = b \\ &\iff x_B + A_B^{-1} A_N x_N = A_B^{-1}b \end{aligned}$$

Now let's take a look at objective.

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N - \cancel{c_B^T (x_B + A_B^{-1} A_N x_N - A_B^{-1}b)} \\ &= (c_N^T - c_B^T A_B^{-1} A_N) x_N + c_B^T A_B^{-1}b \end{aligned}$$

Thus (SEF) is said to be in canonical form for  $B$  if it is written as

$$\begin{aligned} \max \quad & \overbrace{(c_N^T - c_B^T A_B^{-1} A_N) x_N}^{\text{Reduced costs}} + c_B^T A_B^{-1}b \\ \downarrow \\ \text{s.t.} \quad & x_B + A_B^{-1} A_N x_N = A_B^{-1}b \\ & x_B, x_N \geq 0 \end{aligned}$$

**Example:**

Back to our previous example...

$B = \{1, 2\}$ . Rewriting in canonical form for  $B$ :

$$A_B^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

$$A_B A = \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix}$$



$$c_B^T A_B^{-1} A_N = (3 \quad 2) \begin{pmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{pmatrix} = (7/3 \quad -8/3)$$

$$c_N^T - c_B^T A_B^{-1} A_N = (-4/3 \quad 4/3)$$

Then

$$\begin{aligned} \max \quad & (0 \quad 0 \quad -4/3 \quad 4/3)x + 11 \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

is in canonical form for  $B = \{1, 2\}$ .

Example:

$$\begin{aligned} \max \quad & (1 \quad 3 \quad -2 \quad 0 \quad 0)x \quad \underbrace{+0}_{\text{obj. value}} \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned} \quad (\text{LP})$$

Canonical form for  $B = \{4, 5\}$ .

Corresponding BFS  $\begin{matrix} x_4 = 4 \\ x_5 = 1 \end{matrix}, \quad x_j = 0, \forall j \in N$

$$x = (0 \quad 0 \quad 0 \quad 4 \quad 1)^T$$

Objective value = 0

If increase  $x_1$  or  $x_2$ . Objective function increases.

Let's try to increase  $x_1$  from  $0 \rightarrow \theta$ . (Keep  $x_2 = x_3 = 0$ )

$$\begin{aligned} \theta + x_4 = 4 &\iff x_4 = 4 - \theta \\ \theta + x_5 = 1 &\iff x_5 = 1 - \theta \end{aligned}$$

New objective:  $0 + \theta$ . However, we have

$$\begin{aligned} x_4 \geq 0 &\implies \theta \leq 4 \\ x_5 \geq 0 &\implies \theta \leq 1 \end{aligned} \implies \text{Increase } x_1 \text{ by } 1$$

$x_5$  will be 0  $\rightarrow$   $\begin{matrix} x_1 \text{ enters basis} \\ x_5 \text{ leaves basis} \end{matrix}$ . Then new basis  $B = \{1, 4\}$ .

---

Rewriting (LP) in canonical form for  $B = \{1, 4\}$ .

$$\begin{array}{ll}
\max & (0 \ 4 \ -5 \ 0 \ -1) x + \underbrace{1}_{\text{obj. value}} \\
\downarrow & \\
\text{s.t.} & \begin{pmatrix} 1 & -1 & 3 & 0 & 1 \\ 0 & 2 & -2 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
& x \geq 0
\end{array}$$

Corresponding BFS:

$$x = (1 \ 0 \ 0 \ 3 \ 0)^T$$

Obj. value = 1

---

Pick  $j \in N$ :  $\bar{c}_j > 0$  ( $j = 2$ )

Increase  $x_2$  to  $\theta$ , keep  $x_3 = x_5 = 0$

$$\begin{aligned}
x_1 - \theta &= 1 \iff x_1 = 1 + \theta \\
x_4 + 2\theta &= 3 \iff x_4 = 3 - 2\theta
\end{aligned}$$

and

$$\begin{aligned}
x_1 \geq 0 &\implies \theta \geq -1 \\
x_4 \geq 0 &\implies \theta \leq \frac{3}{2}
\end{aligned}$$

Set  $\theta \leftarrow \frac{3}{2} \rightarrow$   $x_2$  enters basis  
 $x_4$  leaves basis

New basis  $B = \{1, 2\}$ .

---

(LP) in canonical form for  $B = \{1, 2\}$ .

$$\begin{array}{ll}
\max & (0 \ 0 \ -1 \ -2 \ 1) x + 7 \\
\downarrow & \\
\text{s.t.} & \begin{pmatrix} 1 & 0 & 2 & 0.5 & 0.5 \\ 0 & 1 & -1 & 0.5 & -0.5 \end{pmatrix} x = \begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix} \\
& x \geq 0
\end{array}$$

Corresponding BFS:

$$x = (2.5 \ 1.5 \ 0 \ 0 \ 0)^T$$

Obj. value = 7

---

Find  $j \in N$ ,  $\bar{c}_j > 0$  ( $j = 5$ )

$$\begin{aligned}
x_1 = 2.5 - 0.5\theta \geq 0 &\implies \theta \leq 5 \rightarrow x_1 \text{ leaves basis} \\
x_2 = 1.5 + 0.5\theta \geq 0 &\implies \theta \geq -3 \rightarrow x_5 \text{ enters basis}
\end{aligned}$$

New basis  $B = \{2, 5\}$

---

(LP) in canonical form for  $B = \{2, 5\}$

$$\begin{aligned} \max \quad & (-2 \ 0 \ -5 \ -3 \ 0) x + 12 \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

BFS  $x = (0 \ 4 \ 0 \ 0 \ 5)^T$  } Optimal Solution  
 Obj. value = 12.

## 2.9.2 Iteration of simplex

---

**Algorithm 2:** Iteration of simplex

---

- 1 Start with feasible basis  $B$
  - 2 Rewrite LP in canonical form for  $B$
  - 3 Pick  $j \in N : \bar{c}_j > 0$  ( $x_j$  enters basis)
  - 4 Let  $\bar{b} = A_B^{-1}b$ ,  $\bar{A}_N = A_B^{-1}A_N$   
 Find largest  $\theta$  so that  $\bar{b} - \theta\bar{A}_j \geq 0$ .  
 Corresponding basic variable that becomes 0 (say  $x_k$ ) leaves basis.
  - 5  $B \leftarrow B \setminus \{k\} \cup \{j\}$ . Iterate.
- 

If problem has optimal solution AND  $\theta$  is always  $> 0$ , simplex finishes.

### Note

If at current BFS we have a basic variable = 0, we may have  $\theta = 0$ .  $\rightarrow$  May lead to cycling. (i.e. return to current basis in future iteration)

### Bland's Rule

If there are multiple choices of entering or leaving variables, always pick lowest index variable.

Using Bland's Rule avoids cycling

**Observations** If  $\bar{c}_N \leq 0$ , then the (LP) obj. value in canonical form is

$$\underbrace{\bar{c}_N^T}_{\leq 0} \underbrace{x_N}_{\geq 0} + c_B^T A_B^{-1} b \leq c_B^T A_B^{-1} b$$

For any feasible solution  $\implies$  Current BFS is optimal

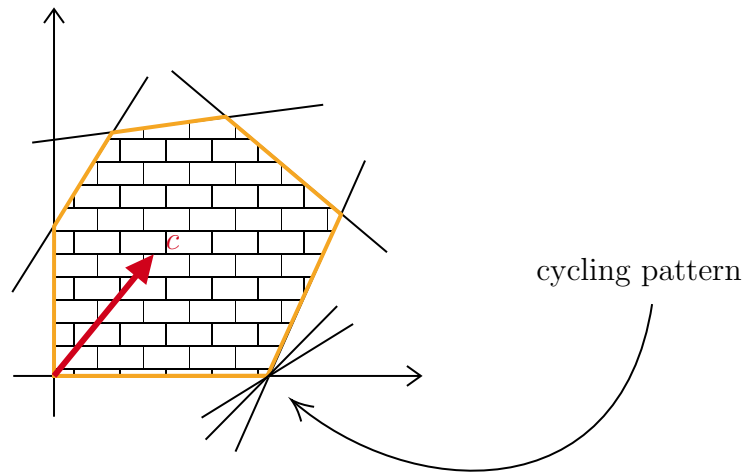


Figure 2.1: Simplex method

Original LP

$$\begin{aligned} \max \quad & c^T x \\ \downarrow \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \min \quad & b^T y \\ \downarrow \\ \text{s.t.} \quad & A^T y \geq c \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min \quad & y^T b \\ \downarrow \\ \text{s.t.} \quad & y^T A \geq c^T \end{aligned}$$

$$\begin{aligned} \min \quad & y^T b \\ \downarrow \\ \text{s.t.} \quad & y^T A_B \geq c_B^T \\ & y^T A_N \geq c_N^T \end{aligned}$$

If satisfies C.S with BFS corresponding to  $B$

$$\begin{aligned} y^T A_B &= c_B^T \\ \implies y^T &= c_B^T A_B^{-1} \Longleftrightarrow c_B^T A_B^{-1} A_N \geq c_N^T \Longleftrightarrow \bar{c}_N \leq 0 \\ y^T A_N &\geq c_N^T \end{aligned}$$

### 2.9.3 Mechanics of Simplex

Example: 1

$$\begin{aligned} \max \quad & \overset{\text{enters basis}}{\uparrow} (1 \quad 3 \quad -2 \quad \overset{j}{\uparrow} 0 \quad 0) x \\ \downarrow \\ \text{s.t.} \quad & \overset{\text{pivot}}{\uparrow} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{row } \ell \\ & x \geq 0 \end{aligned}$$

For  $\theta$

$$\theta \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and we have

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 - \theta \\ 1 - \theta \end{pmatrix} \geq 0 \implies \boxed{\theta \leq 4}$$

We are actually picking  $\min \left\{ \frac{4}{1}, \frac{1}{1} \right\}$

Pick, out of all rows  $\min \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$  where  $j$  is entering variable.

Then now in row  $\ell$  (second row here). Make row operations so that pivot element become 1, all others in col  $j$  becomes 0.

→ Row  $2 \times 1$

→ Subtract row 2 from row 1

→ subtract row 2 from objective function (with RHS multiplied by  $-1$ )

$$\begin{array}{ll} \max & (0 \quad 4 \quad -5 \quad 0 \quad -1)x + 1 \\ \downarrow & \text{pivot} \leftarrow \begin{matrix} j \\ \uparrow \end{matrix} \\ \text{s.t.} & \begin{pmatrix} 0 & 2 & -2 & 1 & -1 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{row } \ell \\ & x \geq 0 \end{array}$$

$$2\theta + x_4 = 3 \iff x_4 = 3 - 2\theta \geq 0 \implies \theta \leq \frac{3}{2}$$

$$-\theta + x_1 = 1 \iff x_1 = \theta + 1 \geq 0 \implies \theta \geq -1$$

where we are finding  $\min_{\bar{a}_{ij} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$ . Now follow the similar procedure, we have

$$\begin{array}{ll} \max & (0 \quad 0 \quad -1 \quad -2 \quad 1)x + 7 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 0 & 1 & -1 & 0.5 & -0.5 \\ 1 & 0 & 2 & 0.5 & 0.5 \end{pmatrix} x = \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} \end{array}$$

**In general** Pick  $j \in N : \bar{c}_j > 0$ .

Let  $\ell = \operatorname{argmin}_{\bar{a}_{ij} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$  (Ratio Test)

- Multiply row  $\ell$  by  $\frac{1}{\bar{a}_{\ell j}}$
- Add  $-\frac{\bar{a}_{ij}}{\bar{a}_{\ell j}}$  times row  $\ell$  to row  $i \neq \ell$ .

- Add  $-\frac{\bar{c}_j \cdot \bar{a}_{\ell k}}{\bar{a}_{\ell j}}$  to variable coeff in objective.  $\forall k \in 1, \dots, n$
- Add  $\frac{\bar{b}_\ell \cdot \bar{c}_j}{\bar{a}_{\ell j}}$  to objective value in objective function

**Example: 2**

$$\begin{array}{ll}
 \max & (2 \quad 1 \quad 1 \quad 0 \quad 0) x \\
 \downarrow & \\
 \text{s.t.} & \begin{array}{c} \text{pivot} \uparrow \\ \left( \begin{array}{ccccc} 1 & 2 & -1 & 1 & 0 \\ 2 & -2 & -1 & 0 & 1 \end{array} \right) x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array} \quad \text{row } \ell \\
 & x \geq 0
 \end{array}$$

**Ratio Test**  $\min \left\{ \frac{2}{1}, \frac{3}{2} \right\} = 1.5. \ell = 2. (x_2 \text{ enters, } x_5 \text{ leaves})$

$$\begin{array}{ll}
 \max & (0 \quad 3 \quad 2_j \quad 0 \quad -1) x + 3 \\
 \downarrow & \\
 \text{s.t.} & \begin{pmatrix} 0 & 3 & -0.5 & 1 & -0.5 \\ 1 & -1 & -0.5 & 0 & 0.5 \end{pmatrix} x = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} \\
 & x \geq 0
 \end{array}$$

If we increase  $x_3 \rightarrow \theta$  and keep  $x_2 = x_5 = 0$

$$\begin{array}{ll}
 -0.5\theta + x_4 = 0.5 & \Rightarrow x_1 = 1.5 + 0.5\theta \\
 -0.5\theta + x_1 = 1.5 & \Rightarrow x_4 = 0.5 + 0.5\theta \rightarrow \text{Problem is unbounded!}
 \end{array}$$

**In general** Let  $B$  be a basis

$$\begin{array}{ll}
 \max & \bar{c}_N^T x_N \\
 \downarrow & \\
 \text{s.t.} & x_B + \bar{A}_N x_N = \bar{b} \\
 & x_B, x_N \geq 0
 \end{array}$$

Found  $j : \bar{c}_j > 0$  AND  $\bar{A}_j \leq 0$ .

Construct  $d \in \mathbb{R}^n$  to reflect what we are trying to do when we increase  $x_j \rightarrow \theta$ .

Right now, we are at BFS:

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}$$

We want:

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix} + \theta \begin{pmatrix} d_B \\ d_N \end{pmatrix}$$

where  $d_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$   $\xrightarrow{j} = e_j$  and  $d_B = -\bar{A}_j = -A_B^{-1}A_j$ .

Found  $d$ :  $d \geq 0$ , then

$$Ad = A_B d_B + A_N d_N = -A_B A_B^{-1} A_j + A_j = 0$$

and

$$c^T d = c_B^T d_B + c_N^T d_N = -c_B^T A_B^{-1} A_j + c_j = \bar{c}_j > 0$$

i.e.,

$$c^T d > 0$$

$$Ad = 0 \implies \text{Problem is unbounded}$$

$$d \geq 0$$

*But wait, how to find an initial BFS?*

Given

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \quad (\text{LP})$$

where  $b \geq 0$ .

Construct auxiliary

$$\begin{array}{ll} \max & -e^T w \\ \downarrow & \\ \text{s.t.} & Ax + Iw = b \\ & x, w \geq 0 \end{array} \quad (\text{AUX})$$

#### Note

- (AUX) is feasible ( $x = 0, w = b$ )
- (AUX) is bounded  $-e^T w \leq 0$

So (AUX) has an optimal solution.

#### Proposition 2.14

(AUX) has optimal value 0 iff (LP) is feasible.

#### Proof:

If optimal solution  $(x^*, w^*)$  has value 0, then  $w^* = 0$  so  $Ax^* + I0 = b$

$\implies x^*$  is feasible for (LP)

If  $x$  is feasible for (LP) then  $(x, 0)$  has value 0 in (AUX).

Moreover, if optimal value of (AUX) is  $< 0$ , then we can use the dual for a certificate.

$$\begin{array}{ll} \min & y^T b \\ \downarrow & \\ \text{s.t.} & y^T A \geq 0 \\ & y \geq -e \end{array} \quad (\text{DAUX})$$

$y^*$  optimal  $y^{*T} b < 0$  and  $y^{*T} A \geq 0$

$\implies y^*$  satisfies  $\{x : Ax = b, x \geq 0\} = \emptyset$

□

## 2.9.4 Two Stage Simplex

### Phase 1

- write (AUX)
- solve (AUX) with BFS corresponding to  $w$
- if opt value  $< 0$ , get certificate  $y^*$  (LP) is infeasible
- opt value 0, BFS  $x$  where  $w = 0$

### Phase 2

- simplex with  $x$  as initial BFS

Example: 1

$$\begin{array}{ll} \max & (2 \ 1 \ 3) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} x \begin{array}{l} \leq -1 \\ \geq 3 \end{array} \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \max & (2 \ 1 \ 3 \ 0 \ 0) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 \\ 1 & 1 & 2 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array} \quad (\text{SEF})$$

$$\begin{array}{ll} \max & (0 \ 0 \ 0 \ 0 \ 0 \ -1 \ -1) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array} \quad (\text{AUX})$$



canonical form:  $B = \{6, 7\}$

$$\begin{array}{ll} \max & (-1 \ 0 \ 2 \ -1 \ -1 \ 0 \ 0) x - 4 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array}$$

add 3 to the basis

$$\min \left( \frac{b_i}{a_{i3}} \right) = \frac{3}{2}$$

7 leaves the basis.

canonical form for  $B = \{3, 6\}$

$$\begin{array}{ll} \max & (-2 \ -1 \ 0 \ -1 \ 0 \ 0 \ -1) x - 1 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1/2 & 1/2 & 1 & 0 & -1/2 & 0 & 1/2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} \end{array}$$

$$x^* = (0 \ 0 \ \frac{3}{2} \ 0 \ 0 \ 1 \ 0)$$

certificate of infeasibility

$$\begin{aligned} y^T &= c_B^T A_B^{-1} \\ &= (0 \ -1) \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \\ &= (0 \ -1) \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} \\ &= (-1 \ 0) \end{aligned}$$

Example: 2

$$\begin{array}{ll} \max & (1 \ 0 \ 2) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix} x = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

in SEF.

$$\begin{array}{ll} \max & (1 \ 0 \ 2) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\ \\ \max & (0 \ 0 \ 0 \ -1 \ -1) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \end{array}$$

(AUX)

canonical form  $B = \{4, 5\}$

$$\begin{array}{ll} \max & (3 \ 2 \ 3 \ 0 \ 0) x - 12 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

1 enters basis  $x + \theta d \quad d = (1 \ 0 \ 0 \ -2 \ -1)^T$

$$\min \left( \frac{b_i}{a_{i1}} \right) = \frac{7}{2}$$

4 leaves the basis

$$\begin{array}{ll} \max & (0 \ 1/2 \ 3/2 \ -3/2 \ 0) x - 3/2 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 1 \end{pmatrix} x = \begin{pmatrix} 7/2 \\ 3/2 \end{pmatrix} \\ & x \geq 0 \end{array}$$

2 enters the basis

$$\min \left( \frac{b_i}{a_{i2}} \right) = \frac{3/2}{1/2}$$

5 leaves the basis

$$\begin{array}{ll} \max & (0 \ 0 \ 0 \ -1 \ -1) x + 0 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 3 & -1 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array}$$

Thus  $x = (2 \ 3 \ 0 \ 0 \ 0)$  is optimal for (AUX)

Forget (AUX). Start Simplex with  $x = (2 \ 3 \ 0)$  as initial BFS.

Now return to SEF.

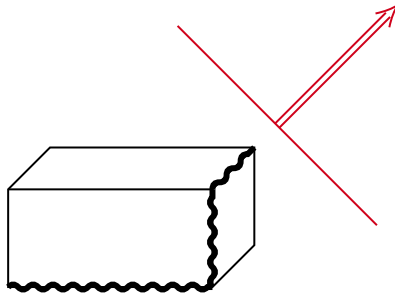
$$\begin{array}{ll} \max & (1 \ 0 \ 2) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array} \quad (\text{SEF})$$

canonical form for  $B = \{1, 2\}$

$$\begin{array}{ll} \max & (0 \ 0 \ 3) x + 2 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array}$$

*How long does simplex take?*

At each pivot, we move from an extreme point to another.



Every pivot rule has a bad example.

Sprelman & Teng (2001): bad examples are pathological. Small changes become good examples.

#### Polynomial Hirsch Conjecture

Polynomially many vertex for bounded Polyhedral.

Let  $G$  be the graph of a  $d$ -polytope with  $n$  facets. Then the diameter of  $G$  is bounded above by a polynomial of  $d$  and  $n$ .

or

The (combinatorial) diameter of a polytope of dimension  $d$  with  $n$  facets cannot be greater than  $n - d$ .

#### Remark:

Here we call combinatorial diameter of a polytope the maximum number of steps needed to go from one vertex to another, where a step consists in traversing an edge.

What this conjecture tells us is that it will take only finitely many edges from initial BFS to optimal one.

There's one **counterexample**: 43-dimensional polytope with 86 facets and diameter (at least) 44.

## 2.10 Ellipsoid Algorithm

**Feasibility** Given polyhedron  $P$ , find  $\bar{x} \in P$  or show  $P = \emptyset$ .

Fourier-Motzkin & simplex solve this problem.

**Aside** Given an algorithm an input  $I$  to it,

$$\text{size}(I) = \# \text{ of bits needed to represent } I.$$

**Example:**

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array}$$

Assume  $c \in \mathbb{Q}^n, A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$ .

By scaling, we may assume  $c \in \mathbb{Z}^n, A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ .

Let  $\alpha = \max\{\|c\|_\infty, \|A\|_\infty, \|b\|_\infty\}$ .

Size of input to LP  $\approx (n + m) \log(\alpha)$

**Efficient Algorithm** # of operations to solve an instance of size  $k$  are bounded by a polynomial on  $k$ .

Thus Simplex & FM NOT Efficient.

**Goal** Derive an efficient alg.

If you have an efficient algorithm to solve feasibility for any polyhedron  $P$ , can be used to solve LP.

## Option 1

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

Assume I know  $L \leq \text{OPT} \leq U$ .

---

### Algorithm 3: Option 1

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```

1 while Repeat do
2    $V = \frac{L + U}{2}$ 
3    $P' = \left\{ x : \begin{array}{l} Ax \leq b \\ c^T x \geq V \end{array} \right\}$ 
4   if  $P' == \emptyset$  then
5      $U \leftarrow V$ 
6   else
7      $L \leftarrow V$ 
8   end
9 end

```

---

## Option 2

Is the following nonempty?

$$\left\{ x, y : \begin{array}{l} Ax \leq b \\ y^T A = c^T \\ y \geq 0 \\ c^T x = b^T y \end{array} \right\}$$

### 2.10.1 Ellipsoid

**Ball**  $B(z, R) := \{x \in \mathbb{R}^n : \|x - z\| \leq R\}$

**Unit Ball**  $B := B(0, 1)$

Apply an affine map to  $B$ .

$f(x) = A(x - b)$  where  $b \in \mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$  invertible

$$f(B) := \{x \in \mathbb{R}^n : \|f(x)\| \leq 1\} = \{x \in \mathbb{R}^n : \|A(x - b)\| \leq 1\}$$

Sets of this form are **Ellipsoid**. Denoted  $E(A, b)$ .

#### Idea

- Suppose I know  $P \subseteq B(0, R)$
- Also, suppose either  $P = \emptyset$  OR  $\text{Vol } P \geq \epsilon > 0$ .

---

#### Algorithm 4: Ellipsoid Algorithm

---

```

1  $E \leftarrow E(M, z)$ , where  $P \subseteq E(M, z)$ .
2 while  $\text{Vol}(E) \geq \epsilon$  do
3   if  $z \in P$  then
4     | STOP
5   else
6     • Find  $\alpha^T x \leq \alpha_0$  so that  $\alpha^T x \leq \alpha_0, \forall x \in P$  and  $\alpha^T z > \alpha_0$ 
7     • Find  $E(M', z')$  such that  $E \cap \{x : \alpha^T x \leq \alpha_0\} \subseteq E(M', z')$  and volume
        of  $E(M', z')$  is much lower than  $E$ 
8     •  $E \leftarrow E(M', z')$ 
9   end
10 end
```

---

#### Note

At any point  $P \subseteq E$ .

The reason why we choose ellipsoid instead of ball is that it can actually shrink “thinner” than ball.

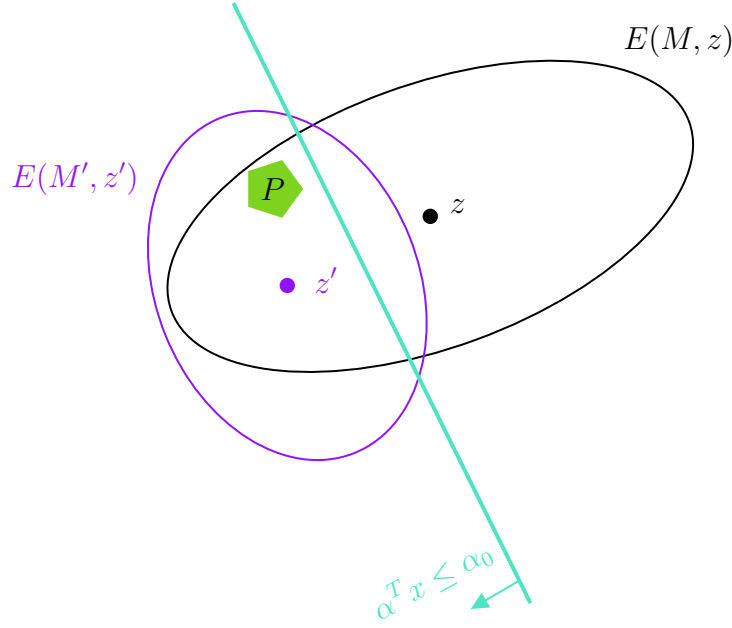


Figure 2.2: Ellipsoid Algorithm

**Lemma 2.15**

There exists  $E(M', z')$  that can be computed in polynomial time such that

$$\frac{\text{Vol}(E(M', z'))}{\text{Vol}(E(M, z))} \leq e^{-\frac{1}{2n+2}}$$

**Number of While Loop Iterations**

If  $B(0, R)$  initial ellipsoid, then  $\text{Vol}(B(0, R)) \leq (2R)^n$ . After  $k(2n + 2)$  iterations,  $\text{Vol}(E) \leq e^{-k}(2R)^n$ .

We want

$$e^{-k}(2R)^n < \epsilon \implies -k + n \ln(2R) < \ln(\epsilon) \implies k \geq \lceil n \ln(2R) - \ln(\epsilon) \rceil$$

Alg stops after  $\lceil n \ln(2R) - \ln(\epsilon) \rceil (2n + 2)$  iterations.

We only used that

$$z \notin P \iff \begin{array}{l} \exists \alpha^T x \leq \alpha_0 \text{ such that} \\ \alpha^T \bar{x} \leq \alpha_0, \forall \bar{x} \in P \\ \alpha^T z > \alpha_0 \end{array}$$

**Theorem 2.16: Separating Hyperplane**

Let  $C$  be a closed, convex set,  $z \in \mathbb{R}^n$ . Then  $z \notin C \iff \exists$  a hyperplane  $\alpha^T x \leq \alpha_0$  separating  $z$  and  $C$ .

Is runtime polynomial?

- $\ln(R)$  is polynomial in input size  $\rightarrow$  NOT a problem
- Finding a separating hyperplane: can be done in polynomial time.

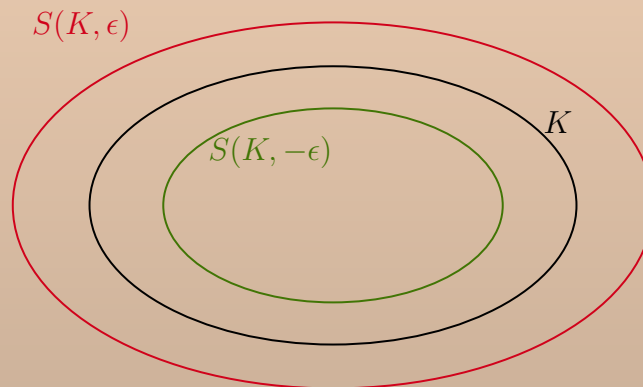
## 2.11 Grötschel-Lovász-Schrijver (GLS)

$S(K, \pm\epsilon)$

Let  $K \subseteq \mathbb{R}^n$  be closed bounded convex set.

$$S(K, \epsilon) := \{x : \|x - y\| \leq \epsilon, \text{ for some } y \in K\}$$

$$S(K, -\epsilon) := \{x : S(x, \epsilon) \subseteq K\}$$



### 2.11.1 3 problems

- OPTIMIZATION

Given  $K \subseteq \mathbb{R}^n$ ,  $c \in \mathbb{Q}^n$ .

Find  $x^* \in K$  such that

$$c^T x^* \geq c^T x, \forall x \in K$$

or determine  $K = \emptyset$ .

- SEPARATION

Given  $K \subseteq \mathbb{R}^n$ ,  $w \in \mathbb{R}^n$ .

Determine if  $w \in K$  or find  $\alpha$ :

$$\|\alpha\|_\infty = 1 \quad \alpha^T x < \alpha^T w, \forall x \in K$$

- **FEASIBILITY**

Given  $K \subseteq \mathbb{R}^n$ .

Find  $\bar{x} \in K$  or determine  $K = \emptyset$ .

Feas  $\leq_p$  Opt. (i.e. if we can solve opt efficiently, we can solve feas efficiently)

Weaker version...

- **WEAK OPTIMIZATION**

Give  $K \subseteq \mathbb{R}^n, c \in \mathbb{Q}^n, \epsilon > 0$

Find  $x^* \in S(K, \epsilon)$  such that

$$c^T x \leq c^T x^* + \epsilon, \quad \forall x \in S(K, -\epsilon)$$

or determine  $S(K, -\epsilon) = \emptyset$

- **WEAK SEPARATION**

Given  $K \subseteq \mathbb{R}^n, w \in \mathbb{R}^n, \epsilon > 0$ .

Determine if  $w \in S(K, \epsilon)$  or find  $\alpha$ :

$$\|\alpha\|_\infty = 1 \quad \alpha^T x < \alpha^T w + \epsilon, \forall x \in S(K, -\epsilon)$$

- **WEAK FEASIBILITY**

Given  $K \subseteq \mathbb{R}^n$ .

Determine  $S(K, -\epsilon) = \emptyset$  or find  $\bar{x} \in S(K, \epsilon)$

W-Feas  $\leq_p$  W-Opt.

Ellipsoid gives us: W-Feas  $\leq_p$  W-Sep.

- Grötschel-Lovász-Schrijver (GLS) have shown that

W-SEP, W-Feas, W-OPT are polynomially equivalent.

In particular, for rational polyhedra<sup>3</sup> (even unbounded) then OPT, FEAS, SEP are polynomially equivalent.

Khachiyan ('80) used ellipsoid to give polytime algorithm for LPs.

## 2.11.2 Consequence of GLS

**Example** TSP: **complete** graph  $G = (V, E)$

---

<sup>3</sup> $\{x \in \mathbb{R}^n : Ax \leq b\}$  where  $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$



Edge costs  $c_e, \forall e \in E$ .

Find a tour visiting every vertex exactly once of min cost.

$$\begin{aligned} \text{IP formulation} \quad x_e &= \begin{cases} 1, & \text{if } e \text{ is in tour} \\ 0, & \text{otherwise} \end{cases} \\ \min \quad & \sum_{e \in E} c_e x_e \\ \downarrow \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e = 2, \quad \forall v \in V \end{aligned}$$

In general,  $\delta(S) = \left\{ uv \in E : \begin{matrix} u \in S \\ v \notin S \end{matrix} \right\}$  where  $S \subseteq V$ .

$$\begin{aligned} \text{Subtour elimination} \quad & \sum_{e \in \delta(S)} x_e \geq 2, \quad \forall \emptyset \subsetneq S \subsetneq V \\ \min \quad & \sum_{e \in E} c_e x_e \\ \downarrow \\ \text{s.t.} \quad & \begin{aligned} & \sum_{e \in \delta(v)} x_e = 2, \quad \forall v \in V \\ & \sum_{e \in \delta(S)} x_e \geq 2, \quad \forall \emptyset \subsetneq S \subsetneq V \\ & x_e \in \{0, 1\}, \quad \forall e \in E \end{aligned} \end{aligned}$$

**LP-relaxation** Replace  $x_e \in \{0, 1\}$  by  $0 \leq x_e \leq 1, \forall e \in E$ .

Can I solve the LP in polynomial time on # vertices/edges?

**Separation/Feasibility** Given  $\bar{x}_e, \forall e \in E$ . Can I know if  $\bar{x}_e$  is feasible for LP in time polynomial in # vertices?

If YES, GLS tells we can also solve OPT.

In polytime (in # vertices) I can check  $\begin{cases} \sum_{e \in \delta(v)} \bar{x}_e = 2, & \forall v \in V \\ 0 \leq \bar{x}_e \leq 1, & \forall e \in E \end{cases}$

**Min-Cut problem** Given  $G = (V, E), w_e \geq 0$ . Find  $\sum_{e \in \delta(S)} w_e$

Problem can be solved in polytime in # vertices.

Then we solve mincut with  $w_e = \bar{x}_e$ . If optimal value is  $\geq 2$ , then  $\bar{x}$  feasible for LP. Otherwise found  $S : \sum_{e \in \delta(S)} \bar{x}_e < 2$ .

# Integer Programming

---

An integer program is a problem of the form:

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \\ & x_i \in \mathbb{Z}, \forall j \in I \end{array}$$

where  $\emptyset \neq I \subseteq \{1, \dots, n\}$ .

If  $I = \{1, \dots, n\}$ , it's pure IP. Otherwise, Mixed IP (MIP).

If all variables are constrained to be in  $\{0, 1\}$ , it's a Binary IP.

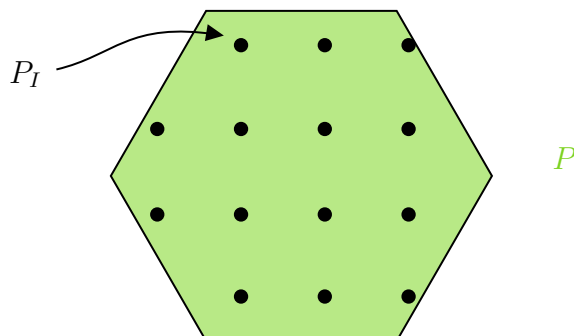
**Key Assumption:** All data is rational ( $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$ ) i.e,  $Ax \leq b$  is a rational polyhedron.

Let  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ ,  $P_I = P \cap \{x_j \in \mathbb{Z} : j \in I\}$ .

## Theorem 3.1

$\text{conv}(P_I)$  is a polyhedron.

From now on, assume we have a pure IP.



**recession cone**

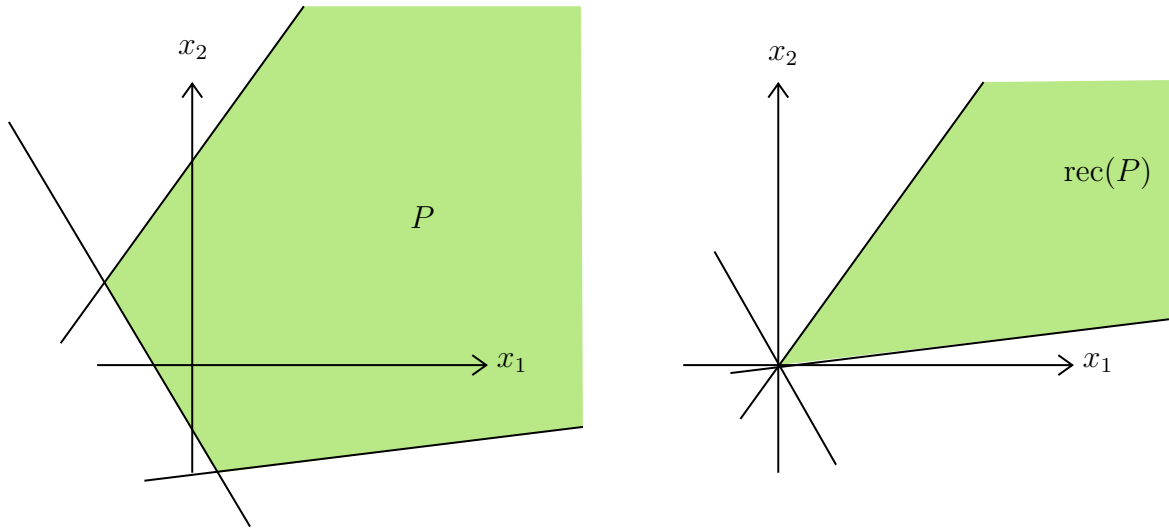
Let  $P$  be a polyhedron. Its recession cone is

$$\text{rec}(P) := \left\{ r \in \mathbb{R}^n : \begin{array}{l} \forall \bar{x} \in P \\ \forall \lambda \geq 0 \\ \bar{x} + \lambda r \in P \end{array} \right\}$$

**Lemma 3.2**

Let  $P = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$  then

$$\underbrace{\text{rec}(P)}_{R_1} = \underbrace{r \in \mathbb{R}^n : Ar \leq 0}_{R_2}$$

**Proof:**

$R_2 \subseteq R_1$ ) Let  $\bar{x} \in P, \lambda \geq 0, r \in R_2$

$$A(\bar{x} + \lambda r) = A\bar{x} + \lambda Ar \leq b \implies \bar{x} + \lambda r \in P \implies r \in R_1$$

$R_1 \subseteq R_2$ ) Let  $r \notin R_2$ , i.e.,  $\exists i : a_i^T r > 0$

Let  $\bar{x} \in P$ , it is clear  $\exists \lambda > 0 : a_i^T(\bar{x} + \lambda r) > b_i \implies r \notin R_1$ .

□

**Theorem 3.3**

$P \neq \emptyset$  is a bounded polyhedron

$$\iff P = \text{conv}(x^1, \dots, x^k) \text{ for some vectors } x^1, \dots, x^k \in \mathbb{R}^n.$$

$\text{conv}(x^1, \dots, x^k)$  is smallest convex set containing  $x^1, \dots, x^k \iff$  set of all finite

combinations of  $x^1, \dots, x^k$ .

**Proof:**

$$\Leftrightarrow) P = \left\{ x \in \mathbb{R}^n : \begin{array}{l} x = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \geq 0 \end{array} \right\}$$

$$P' = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^k : \begin{array}{l} x = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \geq 0 \end{array} \right\} \text{ is a bounded polyhedron.}$$

$P = \text{proj}_x P'$  which is a bounded polyhedron.

$\Rightarrow) P \text{ bounded} \implies P \text{ has no lines.}$

Let  $x^1, \dots, x^k$  be extreme points. Want to show  $P = \text{conv}(x^1, \dots, x^k)$

$P \supseteq \text{conv}(x^1, \dots, x^k)$  follows since  $P$  is a convex set containing  $x^1, \dots, x^k$ .

Suppose  $\exists \bar{x} \in P \setminus \text{conv}(x^1, \dots, x^k)$

Consider

$$\begin{array}{ll} \min & 0^T \lambda \\ \downarrow & \\ \text{s.t.} & \begin{array}{ll} \sum_{i=1}^k \lambda_i x^i & = \bar{x} & \alpha \in \mathbb{R}^n \\ \sum_{i=1}^k \lambda_i & = 1 & \alpha_0 \in \mathbb{R} \\ \lambda & \geq 0 \end{array} \end{array} \quad (1)$$

and its dual

$$\begin{array}{ll} \max & \alpha^T \bar{x} + \alpha_0 \\ \text{s.t.} & \alpha^T x^i + \alpha_0 \leq 0, \quad \forall i = 1, \dots, k \end{array} \quad (2)$$

$(\alpha, \alpha_0) = (0, 0)$  feasible for (2). By assumption, (1) is infeasible.

Let  $(\bar{\alpha}, \bar{\alpha}_0)$  be such that  $\bar{\alpha}^T \bar{x} + \bar{\alpha}_0 > 0$

Now consider

$$\begin{array}{ll} \max & \bar{\alpha}^T x + \bar{\alpha}_0 \\ \text{s.t.} & x \in P \end{array} \quad (3)$$

(3) has optimal solution since  $P \neq \emptyset$  bounded and its has an optimal extreme point, i.e.,  $\bar{\alpha}^T x^i + \bar{\alpha}_0$  is optimal value. But by (2)

$$\bar{\alpha}^T x^i + \bar{\alpha}_0 \leq 0 < \bar{\alpha}^T \bar{x} + \bar{\alpha}_0$$

Contradiction.

□

Back to IP...

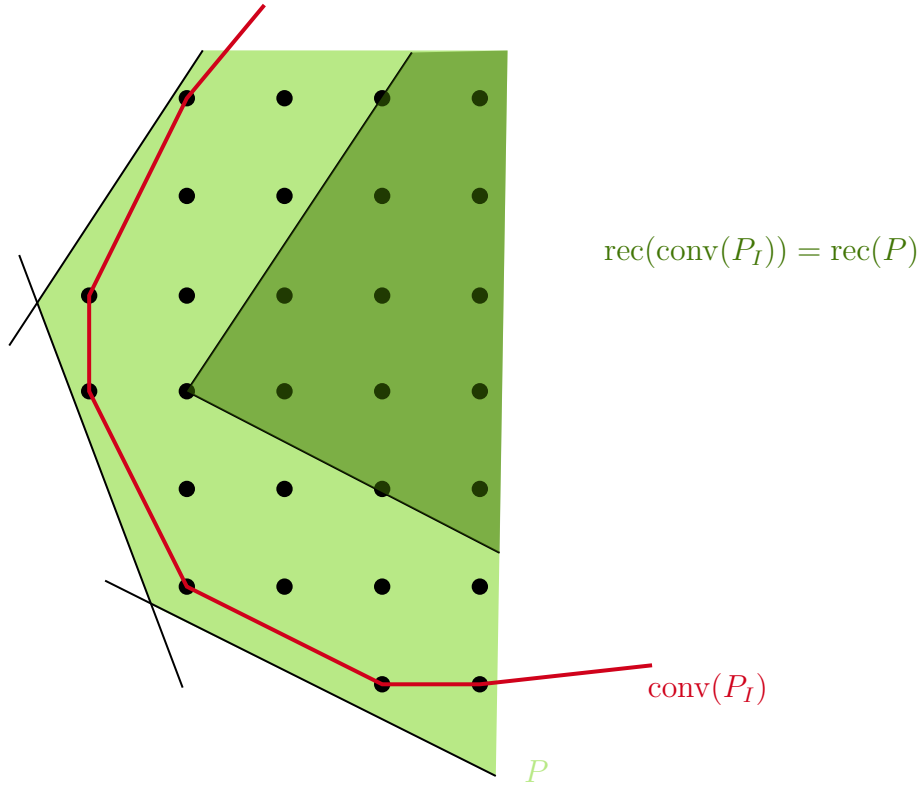
**Theorem 3.4**

If  $P$  is a rational polyhedron, then  $\text{conv}(P_I)$  is also a rational polyhedron ( $P_I = P \cap \mathbb{Z}^n$ ). Moreover, if  $P_I \neq \emptyset$ ,  $\text{rec}(\text{conv}(P_I)) = \text{rec}(P)$ .

**Proof:**

Done if  $P$  is bounded ( $\{0\}$ ).

Skipped for unbounded  $P$ .



□

**Theorem 3.5**

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I \end{array} = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in \text{conv}(P_I) \end{array}$$

**Note**

1. Using Fund Thm of LP. I know IP is either infeas., unbounded, or  $\exists$  opt. sol.
2. If  $P_I \neq \emptyset$ , then unboundedness can be detected by checking if  $\max_{x \in P} c^T x$  is unbounded. Since  $\max_{x \in P} c^T x$  is unbounded iff  $P \neq \emptyset$  and  $\exists r : \begin{array}{l} c^T r > 0 \\ Ar \leq 0 \end{array}$ .

$P_I \neq \emptyset \implies P \neq \emptyset$ . But then this implies  $\max_{\text{s.t. } x \in \text{conv}(P_I)} c^T x$  unbounded.

**Proof:**

WMA (we may assume)  $P_I \neq \emptyset$ .

Let  $z_1 = \max_{\text{s.t. } x \in P_I} c^T x$ ,  $z_2 = \max_{\text{s.t. } x \in \text{conv}(P_I)} c^T x$ .

Since  $P_I \subseteq \text{conv}(P_I) \implies z_1 \leq z_2$ .

Now let  $x^* \in \text{conv}(P_I) \implies \begin{matrix} x^* = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \text{ for } x^1, \dots, x^k \in P_I. \\ \lambda \geq 0 \end{matrix}$

$\implies \exists i : c^T x^i \geq c^T x^*$  since otherwise

$$c^T x^* = \sum_{i=1}^k \lambda_i (c^T x^*) > \sum_{i=1}^k \lambda_i (c^T x^i) = c^T \left( \sum_{i=1}^k \lambda_i x^i \right) = c^T x^*$$

contradiction  $\implies z_1 \geq z_2$ . □

### Corollary 3.6

If  $P \neq \emptyset$  and pointed. Then  $\text{conv}(P_I)$  is pointed and any extreme point of  $\text{conv}(P_I)$  is integral.

**Proof:**

$\text{rec}(P) = \text{rec}(\text{conv}(P_I))$  implies  $\text{conv}(P_I)$  pointed.

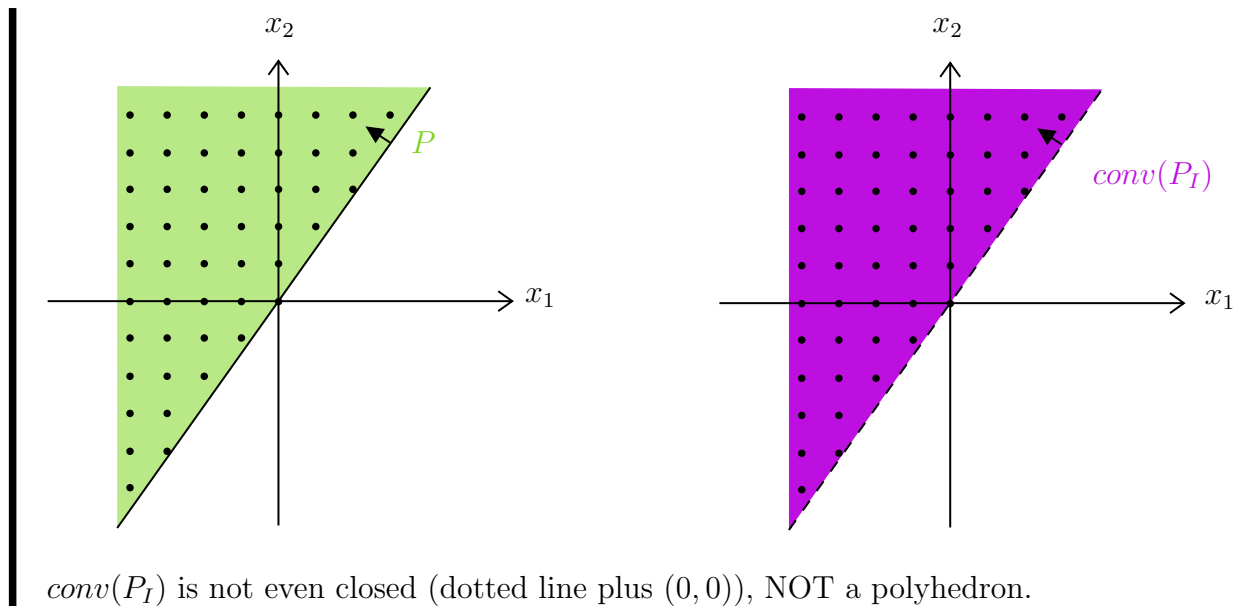
Let  $x^*$  be extreme point of  $\text{conv}(P_I)$ . Let  $c$  be such that  $x^*$  is unique optimal solution to  $\max_{\text{s.t. } x \in \text{conv}(P_I)} c^T x$ .

By theorem,  $\exists \bar{x} \in P_I : c^T \bar{x} = c^T x^*$ .

By uniqueness of  $x^*$ ,  $\bar{x} = x^*$ , then  $x^*$  is integral. □

**Note**

$$P = \{x \in \mathbb{R}^2 : x_2 \geq \sqrt{2}x_1\}$$



### 3.1 Cutting Plane Algorithm

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I := P \cap \mathbb{Z}^n \end{array} \quad (\text{IP})$$

where  $P$  is rational polyhedron.

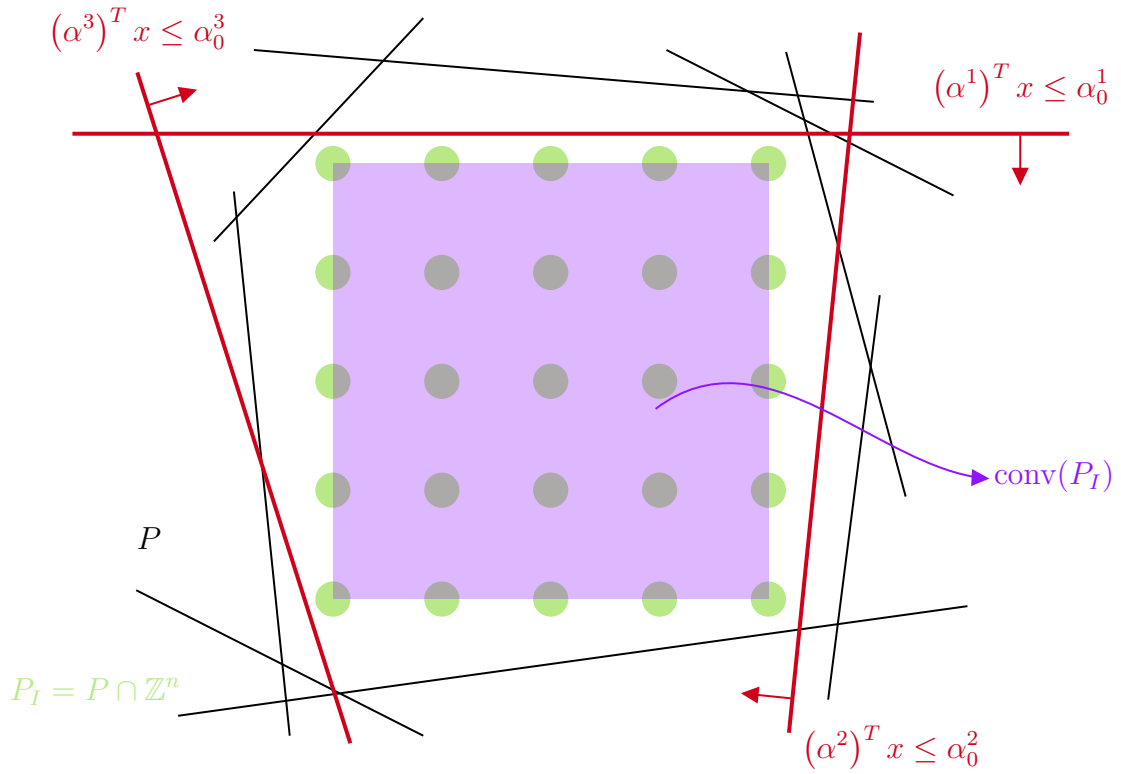
We know it can be solved by solving 
$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & \text{conv}(P_I) \end{array}$$

**Problem** Hard to compute  $\text{conv}(P_I)$ .

$\text{conv}(P_I)$  is smallest convex set containing  $P_I$ .  $P$  is a convex set containing  $P_I$ .

**Idea**

- Start with  $P$
- Iteratively make  $P$  “closer” to  $\text{conv}(P_I)$



**Idea 2** Want to know only part of  $\text{conv}(P_I)$  that is in the “direction I am optimizing”.

#### LP relaxation

The LP you obtain from (IP) after dropping integrality, i.e.,

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$$

#### valid ineq

An ineq  $\alpha^T x \leq \alpha_0$  is valid for  $S \subseteq \mathbb{R}^n$  if  $\forall \bar{x} \in S: \alpha^T \bar{x} \leq \alpha_0$ .

**Assumption** LP relaxation has an optimal solution.

If  $P = \emptyset$ , then  $P_I = \emptyset$ . If LP relaxation is unbounded, either  $P_I = \emptyset$  or (IP) is



unbounded.

---

**Algorithm 5:** Cutting Plane Algorithm

---

```

1  $R \leftarrow P$ 
2 do
3   Let  $x^*$  be optimal solution to  $\max_{x \in R} c^T x$ 
4   if  $x^*$  is integral then
5     STOP //  $x^*$  is opt sol for (IP)
6   else
7     Find valid ineq  $\alpha^T x \leq \alpha_0$  for  $\text{conv}(P_I)$  s.t.  $\alpha^T x^* > \alpha_0$ 
8      $R \leftarrow R \cap \{x : \alpha^T x \leq \alpha_0\}$ 
9   end
10 while  $R \neq \emptyset$ ;
11 Declare (IP) infeasible

```

---

Issues...

1.  $\alpha, \alpha_0$  must be rational
2. Finiteness?
3. How to find  $\alpha, \alpha_0$ ?

**Note**

Any any point  $P_I \subseteq \text{conv}(P_I) \subseteq R \subseteq P$ .

$$\max_{x \in P_I} c^T x \leq \max_{x \in R} c^T x$$

If  $x^* \in \mathbb{Z}^n$ , then  $x^* \in P_I$ .

$$\implies \max_{x \in P_I} c^T x \geq c^T x^* \implies x^* \text{ is optimal for } P_I$$

To solve the issues, impose  $x^*$  being an opt. BFS of  $\max_{x \in R} c^T x$

**Proposition 3.7**

Let  $R$  be a pointed rational polyhedron such that  $R \cap \mathbb{Z}^n = P_I$ . Let  $x^*$  be a BFS of  $R$ .

Then  $x^*$  is integral  $\iff x^* \in \text{conv}(P_I)$

**Proof:**

Exercise. □

How to find valid ineq for  $\text{conv}(P_I)$   $\alpha^T x \leq \alpha_0$  s.t.  $\alpha^T x^* > \alpha_0$ ?

Call such ineq. a **CUTTING PLANE** or a **CUT** separating  $\text{conv}(P_I)$  and  $x^*$ .

**Assumption**  $R = \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\}.$

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \end{array} \quad (1)$$

Let  $B$  be opt. basis.

$$\begin{array}{ll} \max & \bar{c}_N^T x_N + c_B^T A_B^{-1} b \\ \downarrow & \\ (1) \quad \iff & \begin{array}{l} \text{s.t.} \quad x_B + \overbrace{A_B^{-1} A_N}^{\bar{A}_N} x_N = \overbrace{A_B^{-1} b}^{\bar{b}} \\ x \geq 0 \end{array} \end{array}$$

$$x^* \text{ is integral} \iff A_B^{-1} b \in \mathbb{Z}^m$$

If  $x^*$  is not integral, then  $\exists i \in \{1, \dots, m\} : (A_B^{-1} b)_i \notin \mathbb{Z}.$

Look at constraint

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i$$

is valid for  $P_I$  since it is valid for  $R$ .

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \bar{b}_i$$

is valid for  $P_I$  since it is valid for  $R$ .

Since  $\lfloor \bar{a}_{ij} \rfloor \leq \bar{a}_{ij}$  and  $x_j \geq 0 \implies \lfloor \bar{a}_{ij} \rfloor x_j \leq \bar{a}_{ij} x_j.$

Since LHS is integer  $\forall x \in P_I,$

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor \quad (\star)$$

is valid for  $P_I$ .

**Note**

For  $x^*, \quad x_j^* = 0, \forall j \in N \quad x_i^* = \bar{b}_i.$

Thus

$$x_i^* + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j^* = \bar{b}_i > \lfloor \bar{b}_i \rfloor$$

( $\star$ ) is the cut we wanted. Called a Chvátal-Gomory (CG) cut.

---

**Algorithm 6:** Cutting Plane Algorithm (**Correct**)
 

---

```

1  $R \leftarrow P$  // ( $P$  pointed)
2 do
3   Let  $x^*$  be optimal BFS solution to  $\max_{x \in R} c^T x$ 
4   if  $x^*$  is integral then
5     STOP //  $x^*$  is opt sol for (IP)
6   else
7     Find valid ineq  $\alpha^T x \leq \alpha_0$  for  $\text{conv}(P_I)$  s.t.  $\alpha^T x^* > \alpha_0$ 
8      $R \leftarrow R \cap \{x : \alpha^T x \leq \alpha_0\}$ 
9   end
10 while  $R \neq \emptyset$ ;
11 Declare (IP) infeasible
  
```

---

**Theorem 3.8**

The cutting plane algorithm using CG cuts terminates in finitely many iterations (for pure IPs).

**Proof:**  
SKIPPED. □

**Example:**

$$\begin{aligned}
 & \max \quad (1 \ 3 \ -2 \ 0 \ 0) x \\
 & \downarrow \\
 & \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
 & \text{s.t.} \\
 & \quad x \geq 0, \quad x \in \mathbb{Z}^5
 \end{aligned}$$

Opt basis for LP relaxation:  $B = \{2, 5\}$ .

In canonical form:

$$\begin{aligned}
 & \max \quad (-0.5 \ 0 \ -3.5 \ -1.5 \ 0) x + 4.5 \\
 & \downarrow \\
 & \begin{pmatrix} 0.5 & 1 & 0.5 & 0.5 & 0 \\ 1.5 & 0 & 3.5 & 0.5 & 1 \end{pmatrix} x = \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} \\
 & \text{s.t.} \\
 & \quad x \geq 0
 \end{aligned}$$

and  $x^* = (0 \ 1.5 \ 0 \ 0 \ 2.5)^T$

**CG-cut:**

$$\begin{aligned}
 0x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 &\leq 1 \iff x_2 \leq 1 && \text{From 1st constraint} \\
 x_1 + 3x_3 + x_5 &\leq 2 && \text{CG-cut from 2nd constraint}
 \end{aligned}$$

Can add both to  $R$ .

**New LP**

$$\begin{array}{ll}
\max & (1 \ 3 \ -2 \ 0 \ 0) x \\
\downarrow & \\
\text{s.t.} & \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} \\
& x \geq 0
\end{array}$$

Add  $x_6, x_7 \geq 0$  convert to SEF, where

$$x_2 + x_6 = 1, \quad x_1 + 3x_3 + x_5 + x_7 = 2$$

If  $x_1, \dots, x_5 \in \mathbb{Z}$ , then  $x_6, x_7 \in \mathbb{Z}$ .

New Opt for LP:

$$x^T = (1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0)$$

So opt sol to original LP is  $(1 \ 1 \ 0 \ 0 \ 1)$ .

## 3.2 Total Unimodularity

### totally unimodular

A matrix  $U$  is called totally unimodular (TU) if all its square submatrices have determinant in  $\{-1, 0, 1\}$ .

#### Example:

$\begin{pmatrix} \boxed{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  is not TU.

$\begin{pmatrix} \boxed{1} & 1 & \boxed{-1} & 0 \\ 0 & 0 & 0 & 0 \\ \boxed{1} & 0 & \boxed{1} & 1 \end{pmatrix}$  is NOT TU.

#### Note

Square submatrices are obtained by deleting rows/columns.

$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$  is TU.

### Theorem 3.9

If  $A \in \mathbb{Z}^{m \times n}$  is TU and  $b \in \mathbb{Z}^m$  then every BFS of  $P = \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\}$  is integral.

Recall

**Cramer's Rule**

If  $D$  is  $n \times n$  invertible, then unique solution to  $Dx = b$  is given by

$$x_i = \frac{\det D(i)}{\det D}$$

where  $D(i)$  is  $D$  replacing  $i$ -th column with  $b$ .

**Example:**

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solution

$$x_1 = \frac{\det \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}} = \frac{7}{3}, \quad x_2 = \frac{\det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}} = \frac{1}{3}$$

**Proof:**

Let  $x^*$  be a BFS of  $\left\{ x : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\}$ ,  $B$  corresponding basis.

Then  $x_B^* = A_B^{-1}b, x_N^* = 0$

**Note**  $x_B^*$  is unique solution to  $A_B x_B = b$

$\Rightarrow$  By Cramer's rule,

$$x_i^* = \frac{\det A_B(i)}{\det A_B} \in \mathbb{Z}$$

since  $\det A_B(i) \in \mathbb{Z}$  and by TU,  $\det A_B \in \{1, -1\}$  which cannot be 0 since invertible.  $\square$

**Note**

Result remains true if  $P = \{x : Ax \leq b\}$  or  $P = \left\{ x : \begin{array}{l} Ax \leq b \\ x \geq 0 \end{array} \right\}$

**integral**

We say a polyhedron is integral if all its extreme points are integral.

**Lemma 3.10**

$P$  is an integral polyhedron iff  $P = \text{conv}(P \cap \mathbb{Z}^n)$ .

**Proof:**

Exercise.  $\square$

**Lemma 3.11**

Let  $A \in \mathbb{Z}^{m \times n}$  TU.

Then applying any of the following operations on  $A$  yields a TU matrix.

- a) Delete row/column
- b) Multiply row/column by  $-1$
- c) Permute rows/columns
- d) Transpose
- e) Duplicate row/column
- f) Add a row/column with at most one nonzero entry, which is in  $\{+1, -1\}$ .

**Proof:**

a) ✓

b)-d) Potentially changes signs of det.

e) Only can create new submatrices if row and its duplicate are in it. But that has  $\det = 0$ .

f) Recall

**Laplace formula**

$D$  square:

$$D = \begin{pmatrix} & | & \\ -- & d_{ij} & -- \\ & | & \end{pmatrix}$$

Let  $M_{ij}$  be the matrix obtained by deleting row  $i$ , column  $j$ .

Then for any row  $i$  of  $D$ :

$$\det(D) = \sum_j (-1)^{i+j} d_{ij} \det(M_{ij})$$

For any column  $j$ :

$$\det(D) = \sum_i (-1)^{i+j} d_{ij} \det(M_{ij})$$

$$A' = \left( \begin{array}{c|c} \begin{matrix} 1 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{matrix} & A \end{array} \right)$$

Let  $D$  be square submatrix of  $A'$ . If  $D$  does not contain first col, then  $\det(D) \in \{\pm 1, 0\}$  since  $A$  is TU.

If  $D$  does not contain first row, but contains first column, then  $\det(D) = 0$ .

Else,

$$D = \left( \begin{array}{c|cccccc} 1 & \times & \times & \times & \times & \times \\ \hline 0 & & & & & \\ \vdots & & & \overline{D} & & \\ 0 & & & & & \\ 0 & & & & & \end{array} \right)$$

By Laplace formula:  $|\det(D)| = |\det(\overline{D})| \in \{0, 1\}$ .

□

**Application 1** Suppose  $A$  is TU  $\in \mathbb{Z}^{m \times n}$ . If  $b \in \mathbb{Z}^m$  and  $\ell, u \in \mathbb{Z}^n$ , then

$$P = \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax \leq b \\ \ell \leq x \leq u \end{array} \right\}$$

is integer polyhedron.

$$P = \left\{ x \in \mathbb{R}^n : \underbrace{\begin{pmatrix} A \\ I \\ -I \end{pmatrix} x}_{A'} \leq \underbrace{\begin{pmatrix} b \\ u \\ -\ell \end{pmatrix}}_{b'} \right\}$$

$b'$  integral,  $A'$  TU  $\implies P$  is integral

**Application 2**  $A \in \mathbb{Z}^{m \times n}$  TU,  $b \in \mathbb{Z}^m$ ,  $c \in \mathbb{Z}^n$ , then

$$\begin{array}{c|c} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \quad \left| \quad \begin{array}{c|c} \min & b^T y \\ \downarrow & \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array} \right.$$

have integral opt solutions (if both are feasible).

### 3.3 Sufficient condition for TU

#### Lemma 3.12

Let  $A \in \mathbb{Z}^{m \times n}$  with entries  $\{-1, 0, 1\}$ . If  $A$  has:

- At most two nonzeros per column, AND
- There exists a partition  $I_1, I_2$  of its rows such that, for every column:
  - i) Nonzero entries of same sign lie in different partitions
  - ii) Nonzero entries of opposite signs lie in same partition.

Then  $A$  is TU.

**Example:**

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

above the line:  $I_1$ ; below:  $I_2$ .  $A$  is TU.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Line 1 and line 3:  $I_1$ ; Line 2 and 4:  $I_2$ .  $A$  is TU.

**Proof:**

Suppose Lemma is False. Let  $M$  be a minimal counterexample, i.e.,

- $M$  is not TU,
- $M$  satisfies conditions of Lemma,
- Any submatrix of  $M$  is TU.

Then  $M$  itself is a square matrix with  $\det(M) \notin \{-1, 0, 1\}$  and all its submatrix have  $\det \in \{-1, 0, 1\}$ .

If  $M$  has  $\leq 1$  nonzero in a column, then  $M$  is obtained by adding a column with at most 1 nonzero to a TU matrix  $\implies M$  is TU (By Lemma 3.11).



Thus, we may assume all columns of  $M$  has exactly two nonzero elements.

$$M = \begin{pmatrix} - & M_1^T & - \\ & \vdots & \\ - & M_m^T & - \end{pmatrix}$$

Consider:

$$\sum_{i \in I_1} M_i - \sum_{i \in I_2} M_i = 0$$

since i) and ii) hold. Then this means  $\{M_i\}_{i=1}^m$  are **not** linearly independent, which implies  $\det(M) = 0$ .  $\square$

**Example:**

Given  $G = (V, E)$  undirected simple graph.

$G$  is bipartite if  $V = \underbrace{V_1 \dot{\cup} V_2}_{\text{disjoint union}}$  and  $\forall u, v \in E$  has  $u \in V_1, v \in V_2$ .

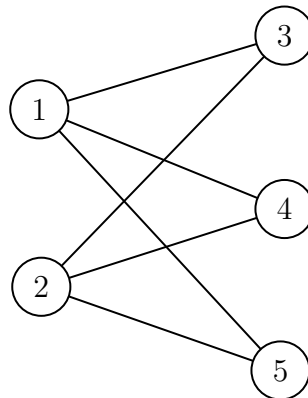
$M \subseteq E$  is a matching if  $|M \cap \delta(v)| \leq 1, \forall v \in V$  where  $\delta(v) := \{e \in E : v \text{ is an endpoint of } e\}$ .

Given  $G$  bipartite. **Goal:** Find max cardinality matching.

Let  $x_e \in \{0, 1\}$  and  $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{if } e \notin M \end{cases}$ .

$$\begin{aligned} & \max \quad \sum_{e \in E} x_e \\ & \downarrow \\ & \text{s.t.} \quad \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\ & \quad \quad x \in \{0, 1\}^E \end{aligned} \tag{1}$$

Let's now take a look at example.



$$\begin{aligned}
 x &= (x_{13} \ x_{14} \ x_{15} \ x_{23} \ x_{24} \ x_{25})^T \\
 \max \quad & (1 \ 1 \ 1 \ 1 \ 1 \ 1) x \\
 \downarrow \\
 \text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \\
 & x \in \{0, 1\}^E
 \end{aligned}$$

vertex

In general:

- $I_1 \rightarrow$  constraints correspond to  $V_1$
- $I_2 \rightarrow$  constraints correspond to  $V_2$

If we look at a column  $x_{uv}$ , it will have a 1 in row of  $u$  a 1 in row of  $v$ , 0 everywhere else.

$\rightarrow$  Bipartite  $\implies$  Lemma is satisfied  $\implies$  (1) can be solved via LP.

Let (2) be LP relaxation of (1) without  $x_e \leq 1, \forall e \in E$ , otherwise the first constraint is violated.

$$\begin{aligned}
 \max \quad & \sum_{e \in E} x_e \\
 \downarrow \\
 \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\
 & x \geq 0
 \end{aligned} \tag{2}$$

Let us write the dual of (2)

$$\begin{aligned}
 \min \quad & \sum_{v \in V} y_v \\
 \downarrow \\
 \text{s.t.} \quad & y_u + y_v \geq 1, \quad \forall uv \in E \\
 & y \geq 0
 \end{aligned} \tag{3}$$

and add integral constraints,

$$\begin{aligned}
 \min \quad & \sum_{v \in V} y_v \\
 \downarrow \\
 \text{s.t.} \quad & y_u + y_v \geq 1, \quad \forall uv \in E \\
 & y \in \{0, 1\}^V
 \end{aligned} \tag{4}$$

Let  $z_i$  be the optimal value for (i) then

$$z_1 \leq z_2 = z_3 \leq z_4$$

$$\begin{aligned}
 G \text{ bipartite} \implies & z_1 = z_2 \\
 & z_3 = z_4
 \end{aligned}$$

**Vertex Cover:** such that  $\forall e \in E, |e \cap U| \geq 1$ . **Problem:** Finding smallest vertex cover.

### König's Theorem

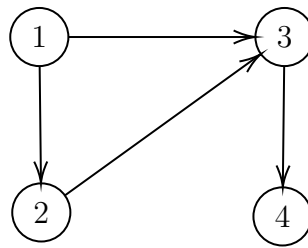
In bipartite graph  $G$ , size of largest matching = size of smallest vertex cover.

### Example:

Consider a directed graph  $D = (V, A)$ .

Incidence matrix of  $D$  has one row per vertex, one column per arc.

For  $v \in V$ ,  $(w, y) \in A$ , then  $a_{ve} = \begin{cases} -1, & \text{if } v = w \\ 1, & \text{if } v = y \\ 0, & \text{otherwise} \end{cases}$



$$\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

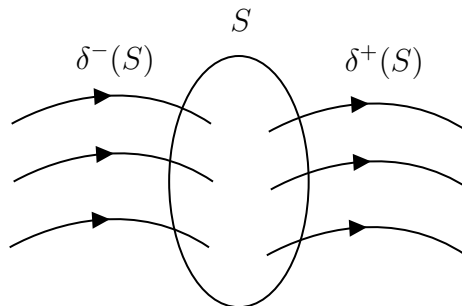
$I_1 = \text{everything}$ ,  $I_2 = \emptyset \implies$  Matrix is TU

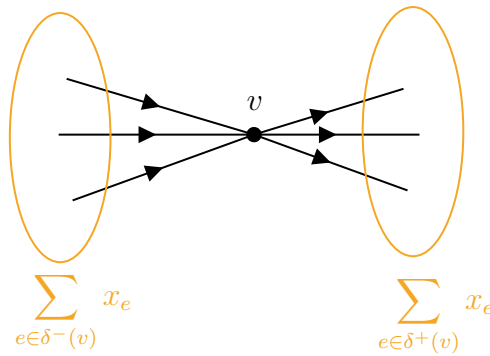
**Max Flow:** Given  $D = (V, A)$ ,  $s, t \in V (s \neq t)$ . An  $s$ - $t$  flow is a nonnegative vector  $x \in \mathbb{R}^A$ , where

$$\sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = 0, \quad \forall v \in V \setminus \{s, t\}$$

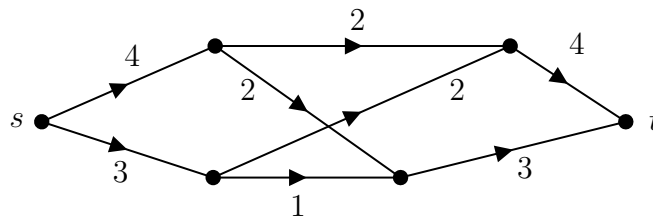
where

$$\delta^-(S) = \left\{ (u, v) \in A : \begin{matrix} u \notin S \\ v \in S \end{matrix} \right\} \quad \text{and} \quad \delta^+(S) = \left\{ (u, v) \in A : \begin{matrix} u \in S \\ v \notin S \end{matrix} \right\}$$





**Goal:** Find a flow maximizing  $\sum_{e \in \delta^+(S)} x_e$



also  $0 \leq x_e \leq c_e, \forall e \in A$  where  $c_e$  is some capacity constraint.

TU  $\implies$  max flow is integral if  $c_e \in \mathbb{Z}, \forall e \in A$ .

### Theorem 3.13

An  $m \times n$  integral matrix  $A$  is TU iff for every subset  $R \subseteq \{1, \dots, m\}$ , there exists a partition of  $R$  into  $R_1, R_2$  (that is,  $R_1 \cup R_2 = R$  and  $R_1 \cap R_2 = \emptyset$ ) such that

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\}, \forall j = 1, \dots, n$$

### Note

Careful that in the previous result that we had seen, we just needed to partition the original rows into two such sets.

This result says that if I pick ANY SUBSET of rows, I must be able to do the same.

Skipped branch-and-bound, Minimum Cost Perfect Matching in Bipartite Graphs...  
due to one week suspension

# Nonlinear Programming

---

The general form: Let  $f, g_1, \dots, g_m : \mathbb{R}^m \rightarrow \mathbb{R}$ .

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{array} \quad (\text{NLP})$$

Note that this is minimization problem with “ $\leq$ ” constraints.

## Example: Linear Programs

$f(x) := c^T x$  and  $g_i(x) := a_i^T x - b_i$ . These give us

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \leq b_i, \quad \forall i = 1, \dots, m \end{array}$$

## Example: Binary integer program

Let  $f(x) := c^T x$ ,  $g_1(x) := x_1(1 - x_1)$  and  $g_2(x) := -x_1(1 - x_1)$ . These give us

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x_1(1 - x_1) = 0 \end{array}$$

where the constraint is equivalent to  $x_1 \in \{0, 1\}$ . Extend it to

$$\begin{array}{ll} \min & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \\ & x \in \{0, 1\}^n \end{array}$$

## 4.1 Convex functions

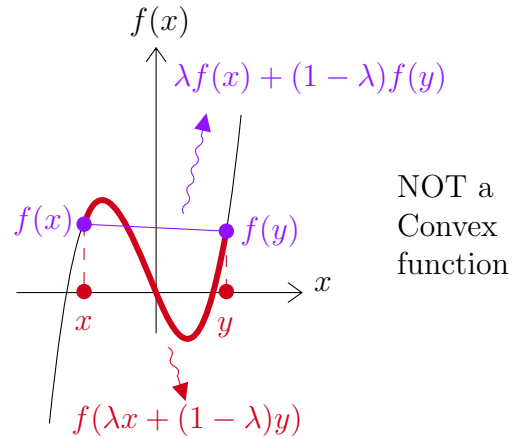
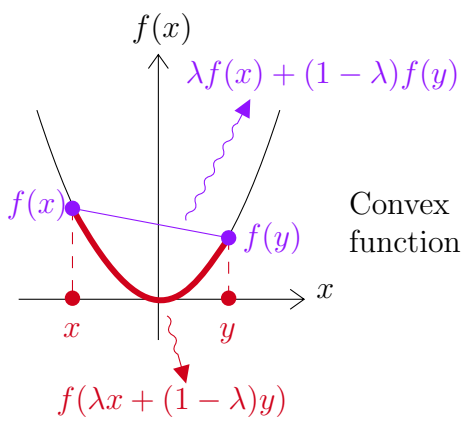
### convex functions

Let  $S \subseteq \mathbb{R}^n$  be a convex set. The function  $f : S \rightarrow \mathbb{R}^n$  is a convex function if  $\forall x, y \in S, \forall \lambda \in [0, 1]$ ,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

#### Example:

Here we let  $S = \mathbb{R}$ .



A **convex NLP** is one of the form:

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{aligned} \quad (\text{CVX})$$

where  $f, g_1, \dots, g_m: \mathbb{R}^n \rightarrow \mathbb{R}$  are convex functions.

#### Note

It is important that constraints are  $\leq$  and that the objective is a minimization problem.

**Proposition 4.1**

If  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function, then  $S = \{x \in \mathbb{R}^n : g(x) \leq 0\}$  is a convex set.

**Proof:**

Let  $x, y \in S$ , i.e.,  $g(x) \leq 0, g(y) \leq 0$ . Now we want to prove  $\lambda x + (1 - \lambda)y \in S$ .

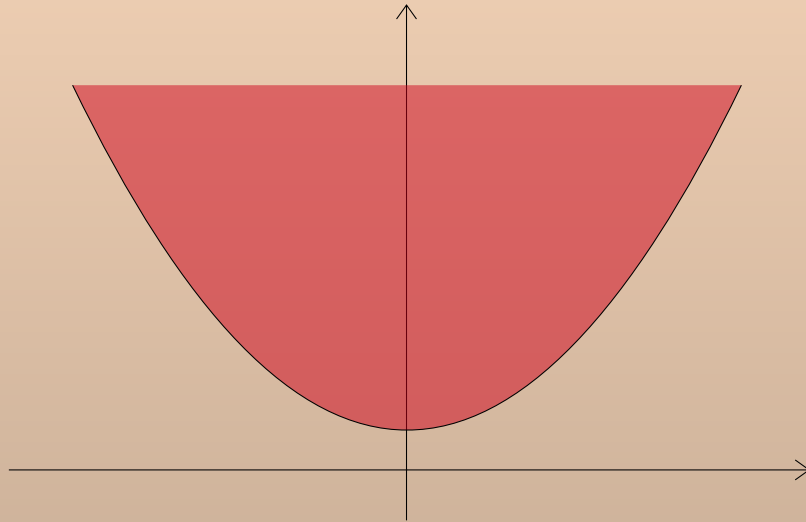
$$\begin{aligned} g(\lambda x + (1 - \lambda)y) &\leq \lambda g(x) + (1 - \lambda)g(y) \quad \text{since } g \text{ is a convex function} \\ &\leq 0 \end{aligned}$$

where the last ineq is from  $\begin{matrix} g(x) \leq 0, \lambda \geq 0 \\ g(y) \leq 0, (1 - \lambda) \geq 0 \end{matrix}$ .

This implies  $\lambda x + (1 - \lambda)y \in S, \quad \forall \lambda \in [0, 1]$ . □

**epigraph**

$$\text{epi}(f) = \{(x, y) : y \geq f(x)\}$$



$f$  is convex  $\iff$   $\text{epi}(f)$  is convex.

## 4.2 Gradients & Hessian

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a twice differentiable function.

The **gradient** of  $f$  at  $\bar{x}$  is the vector

$$\nabla f(\bar{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

The **Hessian** of  $f$  at  $\bar{x}$  is the  $n \times n$  symmetric matrix

$$\nabla^2 f(\bar{x})$$

where the element is defined as

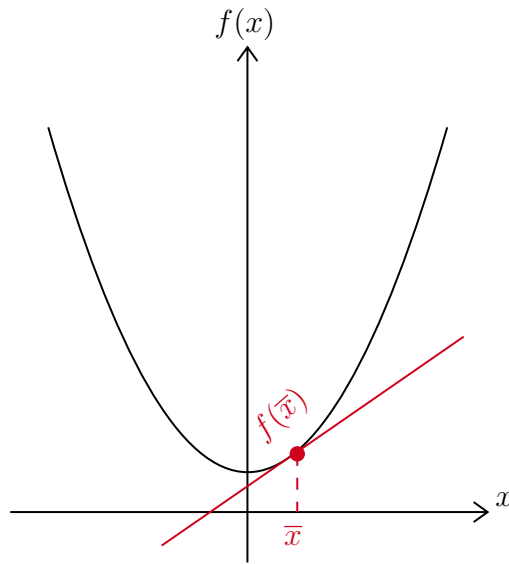
$$[\nabla^2 f(\bar{x})]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

**Example:**

$f(x) = x_1^2 x_2 + 2x_1 + 3$ . Then

$$\nabla f(x) = \begin{pmatrix} 2x_1 x_2 + 2 \\ x_1^2 \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x) = \begin{pmatrix} 2x_2 & 2x_1 \\ 2x_1 & 0 \end{pmatrix}$$

Now looking at 1-D convex functions, two key properties stand out:



- second derivative is  $\geq 0$  (at any point  $\bar{x}$ )
- value of  $f$  is above tangent line at  $\bar{x}$

Translating:

- $f''(x) \geq 0, \forall x$
- $f(x) \geq f(\bar{x}) + f'(\bar{x})(x - \bar{x}), \forall x, \bar{x}$

#### Theorem 4.2

Let  $S \subseteq \mathbb{R}$  be a convex set. Let  $S \rightarrow \mathbb{R}$  be twice differentiable. TFAE:

- $f$  is convex on  $S$
- $f(x) \geq f(\bar{x}) + f'(\bar{x})(x - \bar{x}), \forall x, \bar{x} \in S$
- $(f'(x) - f'(\bar{x}))(x - \bar{x}) \geq 0, \forall x, \bar{x} \in S$
- $f''(x) \geq 0, \forall x \in S$ .

What is the generalization of b), c), d) to  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ?



- b):  $f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T(x - \bar{x}), \quad \forall x, \bar{x} \in S.$   
 c):  $(\nabla f(x) - \nabla f(\bar{x}))^T(x - \bar{x}) \geq 0, \quad \forall x, \bar{x} \in S.$   
 d):  $\nabla^2 f(x)$  is Positive Semidefinite (PSD),  $\forall x \in S.$

**Note**

A symmetric  $n \times n$  matrix  $Q$  is said to be **positive semidefinite** if  $\forall y \in \mathbb{R}^n$ ,

$$y^T Q y \geq 0$$

Denoted as  $Q \succeq 0$ .

$Q$  is said to be **positive definite** (PD) if  $\forall y \in \mathbb{R}^n, y \neq 0$ ,

$$y^T Q y > 0$$

Denoted as  $Q \succ 0$ .

**Theorem 4.3**

Let  $S \subseteq \mathbb{R}^n$  be a convex set. Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a continuous twice differentiable function. TFAE:

- a)  $f$  is convex on  $S$   
 b)  $f(x) \geq f(\bar{x}) + \nabla f(\bar{x})^T(x - \bar{x}), \quad \forall x, \bar{x} \in S$   
 c)  $(\nabla f(x) - \nabla f(\bar{x}))^T(x - \bar{x}) \geq 0, \quad \forall x, \bar{x} \in S$   
 d)  $\nabla^2 f(x) \succeq 0, \forall x \in S.$

**Example:**

$$f(x) = \|x\|^2 = \sum_{j=1}^n x_j^2$$

$$\nabla f(x) = \begin{pmatrix} 2x_1 \\ \vdots \\ 2x_n \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x) = 2I$$

Now

$$y^T \nabla^2 f(x) y = 2y^T I y = 2y^T y = 2\|y\|^2 \geq 0$$

$$\implies \nabla^2 f(x) \succeq 0, \forall x \implies f(x) \text{ is convex.}$$

**Example:**

$f(x) = \frac{1}{2}x^T Q x + d^T x + p$  where  $Q$  is PSD.

$$f(x) = \sum_{j=1}^n \frac{x_j^2}{2} g_{jj} + \frac{1}{2} \sum_{i=1}^n \sum_{j>i}^n 2x_i x_j q_{ij} + \sum_{j=1}^n x_j d_j + p$$

$$\nabla f(x) = \begin{pmatrix} \frac{2x_1}{2}q_{11} + \sum_{j=2}^n x_j q_{1j} + d_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n x_j q_{1j} + d_1 \\ \vdots \end{pmatrix} = Qx + d$$

$\nabla^2 f(x) = Q \succeq 0 \implies f$  is convex.

### 4.3 Local vs. Global optimality