Applied Real Analysis

AMATH 331

Prof. Henry Shum

LaTeXed by Sibelius Peng

Preface

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Sibelius Peng

Contents

1 Real Numbers

Refs 1 for review. 2.1-2.2, 2.9

1.1 Decimal expansions and the real number line

finite decimal expansion

A finite decimal expansion has the form

$$x = a_0{}_\circ a_1 a_2 a_3 \ldots a_N$$

where a_0 is an integer (positive, negative or zero) for $1 \leq n \leq N$ $a_n \in \{0, 1, \ldots, 9\}$

Example:

$$1_\circ 45$$

$$-38_\circ 298743$$

You can think of this as

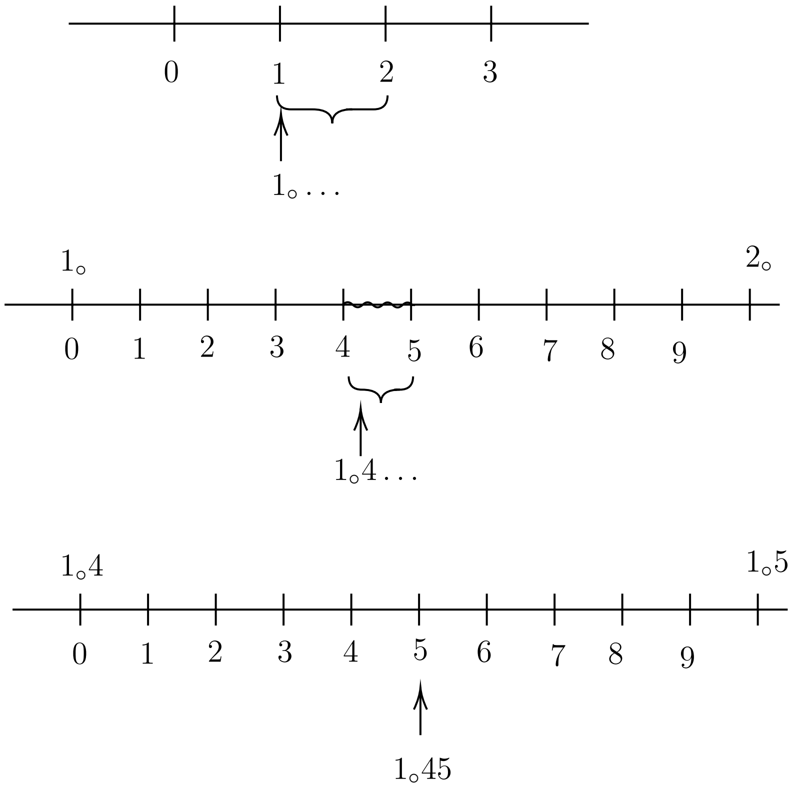
$$x = a_0 + a_1 \left(\frac{1}{10}\right) + \ldots + a_N \left(\frac{1}{10^N}\right)$$

Warning This looks like the usual decimal representation but it is not the same for negative numbers.

Any finite decimal expansion can be replaced on the real number line.

Example:

Where is $1_{\circ}45$?



We can similarly define infinite decimal expansions

infinite decimal expansions

$$x = a_{0\circ}a_1a_2\ldots$$

Example:

$$1_{\circ}450000000\ldots$$

$$\pi = 3_{\circ}1415926535\ldots$$

Assuming the real number line has no gaps, every infinite decimal expansion x corresponds to a point on the line.

Given any positive integer k, let $y = a_{0\circ}a_1a_2\ldots a_k$ be the finite decimal expansion of x to the k-th decimal space. Then, x lies in the interval from y to $(y + 10^{-k})$. So, y approximates x to an accuracy of $1/10^k$. As we increase k, we improve the accuracy; in fact, the error can be made arbitrarily small.

The converse direction: given a point on the real number line, can we find its decimal expansion?

Yes!

It is possible for two decimal expansions to represent the same point. This happens precisely when one ends in an infinite string of 0's.

Example:

$$1.000\ldots \qquad \text{and} \qquad 0.999\ldots$$

$$25.300\ldots \qquad \text{and} \qquad 25.2999\ldots$$

We define the real numbers $\mathbb{R}$ as the set of all infinite decimal expansions.

1.2 Ordering of real numbers

Suppose

$$x = x_{0\circ}x_1x_2x_3\ldots, \qquad y = y_{0\circ}y_1y_2y_3\ldots$$

We say that x and y are equal and write $x = y$ if infinite decimal expansions are identical or equivalent, as discussed previously.

If x and y are not equal, then we say that x are not equal, then x is *less than* y and write $x < y$ if there exists integer $k \geq 0$ such that $x_k < y_k$ and $x_i = y_i$ for $i < k$. x is *greater than* y $(x > y)$ if ...

For any two real numbers x, y, exactly one of the following holds:

$$x = y \qquad x < y \qquad x > y$$

2

Bounds and Limits

2.1 Bounded sets of real numbers

upper bound

A set $S \subseteq \mathbb{R}$ is *bounded above* if there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. M is an *upper bound* of S.

lower bound

A set $S \subseteq \mathbb{R}$ is *bounded below* if there exists $m \in \mathbb{R}$ such that $s \geq m$ for all $s \in S$. m is an *lower bound* of S.

bounded

A set is *bounded* if it is both bounded above and bounded below.

supremum

The *supremum* or *least upper bound* of a nonempty set S that is bounded above is the upper bound L satisfies $L \leq M$ for all upper bounds M of S is written as $\sup S$.

infimum

The *infimum* or *greatest lower bound* of a nonempty set S is the lower bound ℓ satisfying $\ell \geq m$ for all lower bounds m of S. The infimum is denoted $\inf S$.

max

If there exists $M \in S$ such that $s \leq M$ for all $s \in S$, then M is called the *maximum* of S, $\max S$.

min

Analogous defn for $\min S$.

2.2 Examples

0. $S_0 = \varnothing$. Bounded above and below. No supremum or infimum.
1. $S_1 = \{n \in \mathbb{Z}^+\} = \{1, 2, 3, \dots\}$ not bounded above, bounded below.

   1 is infimum and minimum
2. $S_2 = \{-3, -2, 0.5, 1.423\}$. Bounded above and below. Bounded. Has max, min.
3. $S_3 = \left\{1 - \frac{1}{n} : n \in \mathbb{Z}^+\right\} = \{0, \frac{1}{2}, \frac{2}{3}, \dots\}$

   Bounded above by 1. Bounded below by 0.

   Supremum is 1, but there is no max.

2.3 Least Upper Bound Principle

Theorem 2.1: Least Upper Bound Principle

Every nonempty set S of $\mathbb{R}$ that is bounded above has a supremum. Every nonempty set that is bounded below has an infimum.

Sketch of proof for "infimum". There are only finitely many integers from m_0 to $s_0 + 2$. Choose the greatest integer lower bound $\to$ call it a_0.

$a_0 + 1$ is not a lower bound. Divide $[a_0, a_0 + 1]$ into 10, find a_1 such that $a_{0\circ}a_1$ is lower bound of S, but $a_{0\circ}a_1 + 1/10$ is not. Repeat infinitely many times to construct $L = a_{0\circ}a_1a_2a_3\dots$.

Now, show that L is infimum.[1] $\square$

[1] See details in textbook.

3

Limits of Sequences

3.1 Sequences

An *infinite sequence of real* numbers is an infinite, enumerated list of real numbers, denoted by

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, \ldots)$$

Each $a_n \in \mathbb{R}$ is an *element* of the sequence.

We will just refer to them as sequences, and often write (a_n). Formally, a sequence is a function that maps positive integers to $\mathbb{R}$.

We say that a sequence is [bounded above/bounded below/bounded] if the set $A = \{a_n\}$ is respectively [bounded above/bounded below/bounded].

3.2 Examples

1. $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^n$ for $n \geq 1$.
2. $a_n = \frac{1}{n}$, for $n \geq 1$.
3. $(a_n) = (1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \ldots)$

3.3 Limits of Sequences

limit

Let $(a_n)_{n=1}^{\infty}$ be a sequence. We call $L \in \mathbb{R}$ the *limit* of the sequence if for all $\epsilon > 0$, there exists an integer N such that

$$|a_n - L| < \epsilon$$

for all $n \geq N$.

If such L exists, then we say that (a_n) is convergent, and converges to L and we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$.

If a sequence does not have such a limit, then we say it *diverges*, or is *divergent*.

A sequence (a_n) *diverges to* ∞ if for all $M > 0$, there exists N such that $a_n > M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = \infty$.

A sequence (a_n) *diverges to* $-\infty$ if for all $M < 0$, there exists N such that $a_n < M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = -\infty$.

Note

$\lim_{n\to\infty} a_n = \pm\infty$ does not mean limit exists.

3.4 Examples

1. $a_n = 1/n, \quad \lim_{n\to\infty} a_n = 0$

   For any $\epsilon > 0$, we need to show that there exists N such that $|a_n - 0| < \epsilon$ for all $n \geq N$.

   Choose N to be any integer greater than $1/\epsilon$. ($N > \frac{1}{\epsilon}$)

   For any $n \geq N$, $a_n = 1/n \leq \frac{1}{N} < \epsilon$. We also have $a_n \geq 0$

   $$\implies |a_n| < \epsilon$$

   for all $n \geq N$ as required.

3.5 Some basic properties of limits

Theorem 3.1: Squeeze Theorem

Let $(a_n), (b_n), (c_n)$ be sequences.

If $a_n \leq b_n \leq c_n$ for all $n \geq 1$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = 1$$

Proof:

We want to show that for all $\epsilon > 0$, there exists N such that $|b_n - L| < \epsilon$ for all $n \geq N$.

Let $\epsilon > 0$. Since $a_n \to L$, we can find N_1 such that $|a_n - L| < \epsilon$ for all $n \geq N_1$.

Similarly, there exists N_2 s.t. $|c_n - L| < \epsilon$ for all $n \geq N_2$.

Define $N := \max\{N_1, N_2\}$. Then, for $n \geq N$, $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$.

Equivalently,

$$L - \epsilon < a_n < L + \epsilon \qquad L - \epsilon < c_n < L + \epsilon$$

Since $a_n \leq b_n \leq c_n$. $L - \epsilon < b_n < L + \epsilon$, or

$$|b_n - L| < \epsilon$$

as required. □

Proposition 3.2

If a sequence converges to a limit L, then this limit is unique.

Proof:

See PDF. □

Proposition 3.3

If a sequence (a_n) converges, then the set $A := \{a_n : n \geq 1\}$ is bounded.

Proof:

Exercises. □

Theorem 3.4

Let (a_n) and (b_n) be two convergent sequences. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. $\lim_{n\to\infty}(a_n + b_n) = L + M$
2. for any $\alpha \in \mathbb{R}$, $\lim_{n\to\infty}(\alpha a_n) = \alpha L$
3. $\lim_{n\to\infty}(a_n b_n) = LM$, and
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$ and $b_n \neq 0$ for all n.

4

Monotone Sequence and Applications

4.1 Monotone Sequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. it is

1. monotone increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.
2. strictly monotone increasing if $a_{n+1} > a_n$ for all $n \geq 1$.
3. monotone decreasing if $a_{n+1} \leq a_n$
4. strictly monotone decreasing if $a_{n+1} < a_n$

monotone

A sequence is monotone is *monotone* if it is either (monotone) increasing or (monotone) decreasing.

Theorem 4.1: Monotone Convergence Theorem

Monotone Convergence Theorem:

(i) Every monotone increasing sequence that is bounded above converges

(ii) Every monotone decreasing sequence that is bounded below converges

Proof:

We will first show that (i) $\implies$ (ii).

Let (a_n) be a monotone decreasing sequence that is bounded below by m.

The sequence $(-a_n)_{n=1}^{\infty}$ is monotone increasing and is bounded above by $-m$. By part (i), $(-a_n)$ must converge. Call the limit $L = \lim_{n\to\infty}(-a_n)$.

By Theorem 3.4 Part 2,

$$\lim_{n\to\infty} = \lim_{n\to\infty}[(-1)(-a_n)] = (-1)\lim_{n\to\infty}(-a_n) = -L$$

To prove Part(i) of this theorem, suppose (a_n) is monotone increasing and bounded above.

The set $A = \{a_n | n \in \mathbb{Z}^+\}$ is bounded above, and nonempty.

By LUBP(Theorem 2.1), A has a supremum, which we call $L = \sup A$. We show that L is the limit of (a_n).

Given $\epsilon > 0$, we know that $L - \epsilon$ cannot be an upper bound of A.

So there exists N such that $a_n > L - \epsilon$.

Since (a_n) is increasing, $a_n > L - \epsilon$ for all $n \geq N$. Since L is an upper bound of A, $a_n \leq L$ for all $n \geq N$.

$$\implies L - \epsilon < a_n \leq L < L + \epsilon$$

That is $|a_n - L| \leq \epsilon$ for all $n \geq N$. □

4.2 Applications: Calculate Square Roots

The square root of a real number $a > 0$ can be obtained as the limit of the sequence defined recursively by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right), \qquad \text{for } n \geq 1$$

where the starting point x_0 is any positive number.

Moreover, for any $n \geq 1$, the error in approximating $\sqrt{a}$ by x_n satisfies the bound

$$0 \leq x_n - \sqrt{a} < x_n - \frac{a}{x_n}$$

Proof:

Strategy:

1. Prove that (x_n) is bounded below.
2. Prove that (x_n) is monotone decreasing.
3. Prove that (x_n) is monotone decreasing.

4. Use MCT to prove that (x_n) converges.

5. Use properties of limits to determine that $\sqrt{a}$ is the limit.

6. Look for upper and lower bounds for error.

See PDF for full proof. $\square$

4.3 Warning about computing limits that don't exist

Example:

$a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$ for $n \geq 1$.

If we assume (a_n) has a limit L, then we can get nonsense.

$$a_{n+1} = \frac{1}{2}(a_n^2 + 1)$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n^2 + 1)$$
$$\implies L = \frac{1}{2}\left(\lim_{n\to\infty} a_n\right)^2 + \frac{1}{2} = \frac{1}{2}L^2 + \frac{1}{2}$$
$$L^2 - 2L + 1 = 0 \implies L = 1 \text{ is a solution}$$

However, it can be shown that (a_n) is monotone increasing. Since $a_1 = 2$, (a_n) cannot possibly converge to 1.

(In fact, it does not converge.)

5 Subsequences

5.1 Definitions of subsequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence. The sequence $(b_k)_{k=1}^{\infty}$ is a *subsequence* of (a_n) of there exist integers n_k with $1 \leq n_1 < n_2 < n_3 < \ldots$ such that $b_k = a_{n_k}$ for each $k \geq 1$.

Example:

$(a_1, a_2, a_3, a_4, a_5, \ldots)$

$(b_1, b_2, b_3, b_4, b_5, \ldots)$

cannot do the following:

$(a_1, a_2, a_3, a_4, a_5, \ldots)$

$(b_1, b_2, b_3, b_4, b_5, \ldots)$

not allowed to change order

Example:

$$(a_n)_{n=1}^{\infty} = \left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \ldots\right)$$

The sequence (b_k) with $b_k = a_k$ for all $k \geq 1$ is a subsequence of (a_n).

The sequence $\left(-1, -\frac{1}{3}, -\frac{1}{5}, \ldots\right)$ is a subsequence.

The sequence $\left(\frac{1}{2}, \frac{1}{4}, \ldots\right)$ is another subsequence.

5.2 Some properties of Subsequences

Lemma 5.1

Let n_k be integers satisfying $n_1 \geq 1$ and $n_k < n_{k+1}$ for all $k \geq 1$. Then $n_k \geq k$ for all $k \geq 1$.

Theorem 5.2

Suppose the sequence $(a_n)_{n=1}^{\infty}$ converges to the limit L. Then every subsequence of (a_n) also converges to L.

Proof:

By definition of limit, for every $\epsilon > 0$, there exists N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Let $(b_k)_{k=1}^{\infty}$ be any subsequence of (a_n), where $b_k = a_{n_k}$ for each $k \geq 1$.

From Lemma 5.1, we know that $n_k \geq k$ for each k. Given $\epsilon > 0$, chose N as in definition of $\lim\limits_{n\to\infty} a_n = L$. For every $k \geq N$,

$$n_k \geq k \geq N \implies |b_k - L| = |a_{n_k} - L| < \epsilon$$

$\square$

Example:

1. From 5.1, the theorem holds just as it is.
2. Converse is not true. If a subsequence converges, we cannot conclude that the original sequence converges.

5.3 Bolzano-Weierstrass

If for every integer $n \geq 1$, we have a nonempty, closed interval $I_n = [a_n, b_n]$ such that $I_{n+1} \subseteq I_n$, then we say that (I_n) is a *nested sequence of closed, bounded intervals.*

Lemma 5.3: Nested Intervals Lemma

If (I_n) is a nested sequence of closed bounded intervals, then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof:

Exercise. □

Theorem 5.4: Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Proof:

Outline.

1. Given a bounded sequence (a_n), construct a nested sequence of closed, bounded intervals I_n with lengths decreasing to zero, and such that each I_n contains infinitely many elements of the sequence (a_n).
2. Construct a subsequence (b_k) such that $b_k \in I_k$ for each $k \geq 1$.
3. Show that (b_k) converges.

□

Proof:

Step 1: Suppose $(a_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers. Let m_1 be a lower bound and M_1 be an upper-bound for $A = \{a_n : n \geq 1\}$.

Define an interval $I_1 = [m_1, M_1]$. Define the point $c_1 = \frac{1}{2}(m_1 + M_1)$. Choose one smaller interval either $[m_1, c_1]$ or $[c_1, M_1]$ that contains an infinite member of elements of $(a_n) \to$ call this interval $I_2 = [m_2, M_2]$.

We repeat this process for all $k \geq 2$. This gives a sequence of intervals $(I_k)_{k=1}^{\infty}$ such that $I_{n+1} \subseteq I_n$ for all $n \geq 1$, and lengths of I_n converges to zero. Also each I_k contains an infinite number of elements of (a_n).

Step 2: Let $n_1 = 2$ so $b_1 = a_1$. Suppose we have our subsequence (b_j) up to element k. Then we have $n_i \geq 1$ for all $i = 1, 2, \ldots, k$ and $n_i < n_{i+1}$ for all $i = 1, 2, \ldots, k-1$.

Since there are an infinite number of elements of (a_n) contained in I_{k+1}, we can choose n_{k+1} such that $n_{k+1} > n_k$ and $a_{n_{k+1}} \in I_{k+1}$, i.e. $b_{k+1} \in I_{k+1}$. In this way, we inductively define (b_j) as a subsequence of (a_n).

Step 3: By Nested Intervals Lemma (Lemma 5.3), $\displaystyle\bigcap_{k=1}^{\infty} I_k \neq \varnothing$, so there must exist a point $L \in \cap_{k=1}^{\infty} I_k$. The length of interval I_j is $\dfrac{(M_1 - m_1)}{2^{j-1}}$. For any $k \geq 1$, we have $L \in I_k$ and $b_k \in I_k$. Hence $|b_k - L| \leq \dfrac{(M_1 - m_1)}{2^{k-1}}$.

Consider sequence $(|b_k - L|)_{k=1}^{\infty}$. We can use Squeeze Theorem to show that $\lim_{n\to\infty} |b_k - L| = 0$ since

$$0 \leq |b_k - L| \leq \frac{(M_1 - m_1)}{2^{k-1}}.$$

Hence $\lim\limits_{k\to\infty} b_k = L$.

$\square$

6 Cauchy Sequences

6.1 Definition

A sequence (a_n) is *Cauchy* if for any $\epsilon > 0$, there exists an integer N such that

$$|a_n - a_m| < \epsilon$$

for all $n, m \geq N$.

Example:

$$(a_n)_{n=1}^{\infty} = (3, 3.1, 3.14, 3.141, \ldots)$$

More generally, if x is any real number with infinite decimal expression $x_{0\circ}x_1x_2x_3\ldots$, then the sequence of finite truncations, i.e., a_k is the truncation of x to k decimal places, is Cauchy.

$$a_k = x_{0\circ}x_1 \ldots x_k000\ldots$$

Given $\epsilon > 0$, we can find N such that $10^{-N} < \epsilon$.

For any $n \geq 1$, we have

$$a_n \leq x \leq a_n + 10^{-n}$$

In particular,

$$a_N \leq x \leq a_N + 10^{-N}$$

Note that (a_n) is monotone increasing, so $a_N \leq a_n, a_m \leq x \leq a_N + 10^{-N}$ for any $n, m \geq N$.

So

$$|a_n - a_m| \leq \text{length of interval} = 10^{-N} < \epsilon$$

$\implies (a_n)_{n=1}^{\infty}$ is Cauchy.

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