AMATH 391 Midterm Review

1 Week 1

Lec 1 just for information.

1.1 Fourier series

The coefficients of Fourier expansion is given on **Midterm Examination FACT SHEET**.

The partial sums $S_N(x)$ are functions that will serve as **approximations** to the function $f(x)$.

$$\lim_{N\to\infty} S_N = f$$

1.2 Metric spaces

A metric space (X, d), is a set X with a "metric" d that assigns nonnegative "distances" between any two elements in X.

1. Positivity: $d(x, y) \geq 0, \quad d(x, x) = 0$
2. Strict positivity: $d(x, y) = 0 \implies x = y$
3. Symmetry: $d(x, y) = d(y, x)$
4. $\triangle$ inequality: $d(x, y) \leq d(x, z) + d(z, y)$

1.2.1 Metric spaces for functions

listed in fact sheet.

2 Week 2

2.1 Convergence

Cauchy sequence: for any $\varepsilon > 0$, there exists $N_\varepsilon > 0$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m > N_\varepsilon$$

Defn Complete metric space: if all every Cauchy sequence $\{x_n\}$ converges (to an element $x \in X$)

Convergence in d_∞ implies not only pointwise convergence but uniform convergence.

2.2 Normed Linear Spaces

Let X be a vector space. A real-valued function $\|x\|$ defined on X is a norm on X if the following properties are satisfied:

1. postivity
2. strict positivity $\|x\| = 0 \iff x = 0$
3. $\triangle$ inequality
4. homogeneity: $\|\alpha x\| = |\alpha|\|x\|$

The pair $(X, \|\cdot\|)$ is called a **normed linear space**. And it is a metric space.

If we consider a normed linear space X as a metric space d, then we may ask whether it is complete.

Defn A complete normed space is called **Banach space**.

2.3 Best approximation

Examples:

1. $X = C[a, b]$. The set of functions $\{1, x, x^2, \ldots\}$: linearly independent set.

$$\min_{c_0,\ldots,c_{n-1}} \|f - v_n\| = \min_{c_0,\ldots,c_{n-1}} \max_{x\in[a,b]} |f(x) - c_0 - c_1 x - \ldots - c_{n-1}x^{n-1}|$$

   Special case: $n = 1$. $d_\infty(f, c)$

2. $X = L^1[a, b]$. Special case $n = 1$.
3. $X = L^2[a, b]$
   (a) $n = 1$. Two methods:
      - expand the integrand and integrate to produce an expression for $\Delta_2^2(c)$ in terms of c.
      - use Lebiniz's Rule to differentiate the integral.

   (b) special case $n = 2$: $f(x) \approxeq c_0 + c_1 x$. Then minimize

$$h(c_0, c_1) = \Delta_2^2(c_0 c_1)$$

   Critical points (c_0, c_1) must satisfy the following stationarity conditions:

$$\frac{\partial h}{\partial c_0} = \frac{\partial h}{\partial c_1} = 0$$

4. We return to the approx. that yielded by partial sums of the Fourier series of a function $f(x)$ defined on the interval $[-\pi, \pi]$. We simply state that $S_N(x)$ is the best approximation to $f(x)$ in this $2N + 1$-dimensional space.

2.4 Inner product spaces

The inner product satisfies the following conditions:

1. $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
2. $\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$
3. $\langle x, y\rangle = \langle y, z\rangle$. If the field of scalars is $\mathbb{C}$, then this becomes $\langle x, y\rangle = \overline{\langle y, x\rangle}$
4. $\langle x, x\rangle \geq 0$

We then say that $(X, \langle , \rangle)$ is an inner product space.

An inner product defines a norm which, in tun, defines a metric.

A complete inner product space is called a **Hilbert space**.

2.4.1 Orthogonality in inner product spaces

convex, direct sum of two subspaces (algebraic complements). orthogonal complement: $S, S^{\perp}$.

2.5 Projection Theorem

Let H be a Hilbert space and $Y \subset H$ any closed subspace of H. Now let $Z = Y^{\perp}$. Then for any $x \in H$, there is a unique decomposition of the form

$$x = y + z, \qquad y \in Y,\ z \in Z = Y^{\perp}$$

The point y is called the (orthogonal) projection of x on Y.

mapping $P_Y : H \to Y$, the projection of H onto Y. This is idempotent operator: $P_Y^2 = P_Y$.

2.5.1 Best approximations in Hilbert spaces

Theorem Let $\{e_1, \ldots, e_n\}$ be an orthonormal set in a Hilbert space H. Define $Y = \text{span}\{e_i\}_{i=1}^{n}$. Y is a subspace of H. Then for any $x \in H$, the best approximation of x in Y is given by the unique element

$$y = P_Y(x) = \sum_{k=1}^{n} c_k e_k$$

where

$$c_k = \langle x, e_k\rangle$$

Furthermore, **Bessel's inequality**

$$\sum_{k=1}^{n} |c_k|^2 \leq \|x\|^2$$

3 Week 3

3.1 Complete orthonormal basis sets - "Generalized Fourier series"

Defn An orthonormal set $\{e_k\}_1^\infty$ is said to be complete or maximal if the following is true:

$$\text{If } \langle x, e_k \rangle = 0 \text{ for all } k \geq 1 \text{ then } x = 0$$

Here is the main result:

For any $x \in H$, (Generalized Fourier Series)

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle \, e_k$$

Parseval's equation:

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

3.2 Convergence of Fourier series expansions

pointwise/uniform convergence, convergence in mean.

uniform $\implies$ convergence in mean.

3.3 Higher variation means higher frequencies are needed

not be examined.

3.4 even and odd extensions

3.5 Discrete Fourier Transform

In the signal processing literature, the usual notation for such a **sampling** is as follows,

$$f[n] := f(nT), \qquad n \in \{0, 1, 2, \ldots\} \quad \text{or} \quad n \in \{\ldots, -1, 0, 1, \ldots\}$$

4 Week 4

4.1 An orthonormal periodic basis in $\mathbb{C}^N$

inner product: $\langle f, g \rangle = \sum_{n=0}^{N-1} f[n]\overline{g[n]}$

normalized:

$$u_k = (u_k[0], \ldots, u_k[N-1])$$

with components

$$u_k[n] = \frac{1}{\sqrt{N}} \exp\left(\frac{i2\pi kn}{N}\right), \qquad n = 0, 1, \ldots, N-1$$

index n plays the role of **time or spatial variable** and k is the index of the **frequency**.

4.2 DFT version 3

Given in fact sheet.

4.2.1 Some examples

constant signal: only frequency is zero frequency

linearity of DFT.

real-valued signal can have complex-valued DFT.

$$\|f\|^2 = \frac{1}{N}\|F\|^2$$

Important result: The N-point DFT of the sampled function $\exp(ik_0x), 0 \leq x \leq 2\pi$, is given by a single peak:

$$F[k] = \begin{cases} N, & k = k_0 \mod N \\ 0, & \text{otherwise} \end{cases}$$

4.3 Some properties

We should be able to derive them...

4.3.1 Linearity

$$\mathcal{F}(f+g) = \mathcal{F}f + \mathcal{F}g$$

by defn

4.3.2 Conjugate symmetry

$$F[k] = \overline{F[N-k]}$$

Note that $\exp(-i2\pi n) = 1$

4.3.3 Shift Theorem

See fact sheet

Proof By defn

$$
\begin{aligned}
G[k] &= \sum_{n=0}^{N-1} g[n] \exp\left(-\frac{i2\pi kn}{N}\right) \\
&= \sum_{n=0}^{N-1} f[n+1] \exp\left(-\frac{i2\pi kn}{N}\right) \\
&= \sum_{n=0}^{N-1} f[n+1] \exp\left(-\frac{i2\pi k(n+1)}{N}\right) \exp\left(\frac{i2\pi k}{N}\right) \\
&= \omega^{-k} F[k]
\end{aligned}
$$

4.3.4 Convolution Theorem

Proof by defn.

$$
\begin{aligned}
H[k] &= \sum_{n=0}^{N-1} h[n] \exp\left(-\frac{i2\pi kn}{N}\right) \\
&= \sum_{n=0}^{N-1} \left[\sum_{j=0}^{N-1} f[j] g[n-j]\right] \exp\left(-\frac{i2\pi kn}{N}\right) \\
&= \sum_{j=0}^{N-1} f[j] \exp\left(-\frac{i2\pi kj}{N}\right) \sum_{n=0}^{N-1} g[n-j] \exp\left(-\frac{i2\pi k(n-j)}{N}\right) \\
&= \left[\sum_{j=0}^{N-1} f[j] \exp\left(-\frac{i2\pi kj}{N}\right)\right] \left[\sum_{l=0}^{N-1} g[l] \exp\left(-\frac{i2\pi kl}{N}\right)\right] \\
&= F[k]G[k]
\end{aligned}
$$

The second-to-last line follows the fact from that the products $f[j]g[n-j]$ exhaust all possible pairs since the vectors are N-periodic.

$$|F[N-k]| = |F[k]|$$

5 Week 5

By "denosing" the signal f, we mean finding approximations to the noiseless signal f_0.

5.1 A closer look at Conv. Thm

$$h = f * g$$

In this way, we can view f as a signal, and g as a mask: the conv. thm produces a new signal h from f.

5.2 Averaging as a convolution

In the frequency domain, local averaging is shown to perform the greatest shrinkage of DFT coefficients in the high frequency region.

5.3 DCT

eliminate convergence problems due to discont. at the endpoints. True even extension.

6 Week 6

6.1 DFT of 2-d datasets

tensor product basis. In matlab, they are `fft2` and `ifft2`.

6.2 Fourier Transform

$$f(x) = \sum_{-\infty}^{\infty} a_n e^{in\pi x/L}$$

represents an expansion of the function $f(x)$ in terms of its frequency components.

- The Fourier series is a summation over discrete frequencies ω_n
- The Fourier transform is an integration over continuous frequencies ω.

6.2.1 Import Properties

1. Linearity
2. $\mathcal{F}(t^n f(t)) = i^n F^{(n)}(\omega)$
3. $\mathcal{F}^{-1}(\omega^n F(\omega)) = (-i)^n f^{(n)}(t)$
4. $\mathcal{F}(f^{(n)}(t)) = (i\omega)^n F(\omega)$
5. $\mathcal{F}^{-1}(F^{(n)}(\omega)) = (-it)^n f(t)$
6. $\mathcal{F}(f(t-a)) = e^{-\omega a} F(\omega)$
7. $\mathcal{F}(e^{i\omega_0 t} f(t)) = F(\omega - \omega_0)$

8. For a $b > 0$, $\mathcal{F}(f(bt)) = \frac{1}{b}F\left(\frac{\omega}{b}\right)$
9. Convolution Theorem

6.2.2 Frequency Shift Thm

number 7 of the properties above.

We may be interested in computing the FT of the product of either $\cos\omega_0 t$ or $\sin\omega_0 t$ with a function $f(t)$, which are known as **(amplitude) modulations** of $f(t)$.

6.2.3 Scaling Thm

number 8.

Suppose $b > 1$. $g(t) = f(bt)$ is obtained by contracting the latter horizontally toward y-axis by a factor of $\frac{1}{b}$.

G is obtained by stretching F outward.

6.3 Plancherel Formula

Using complex inner product:

$$\langle f, g\rangle = \langle F, G\rangle$$

norm-preserving. Can be viewed as the continuous version of **Parseval's equation**.

6.4 The FT of a Gaussian

The sdv of $f_\sigma(t)$ is σ, while $F_\sigma(\omega)$ is σ^{-1}. Consequence of the complementarity of time (or space) and frequency domains.

7 Week 7

Lecture 18 only

7.1 Convolution thm version 2

$$\mathcal{F}(f * g) = \sqrt{2\pi}FG$$

and version 2:

$$\mathcal{F}[f(t)g(t)] = \frac{1}{\sqrt{2\pi}}(F * G)(\omega)$$

A product of functions in one representation is equivalent to a convolution in the other.

7.2 Using Conv. Thm to reduce high freq

7.2.1 low-pass filter

$$H_{\omega_0}(\omega) = F(\omega)B_{\omega_0}(\omega)$$

$B_{\omega_0}(\omega)$ is a boxcar-type function.

7.2.2 Gaussian weighting

One may wish to employ smoother.

$$G_\kappa(\omega) = e^{-\frac{\omega^2}{2\kappa^2}}$$

(we have not normalized in order to ensure $G_\kappa(0) = 1$)

Gaussian-weighted FT:

$$H_\kappa(\omega) = F(\omega)G_\kappa(\omega)$$