Introduction to Optimization

CO 255

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LATEXed by Sibelius Peng

Preface

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Sibelius Peng

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Info

Ricardo: MC 5036. OH: M 1:30 - 3pm
TA: Adam Brown: MC 5462. OH: F 10-11am

Books (not required)

- Intro to Linear Opt. Bertsimas
- Int Programming. Conforti

Grading

- assns: 20% (≈ 5)
- mid: 30% (Feb 11 in class)
- final: 50%

1 Introduction

Given a set S, and a function $f : S \to \mathbb{R}$. An optimization problem is:

$$\begin{array}{ll} \max & f(x) \\ \underbrace{\text{s.t.}}_{\text{subject to}} & x \in S \end{array} \tag{OPT}$$

- S **feasible region**
- A point $\overline{x} \in S$ is a **feasible solution**
- $f(x)$ is **objective function**

(OPT) means: "Find a feasible solution x^* such that $f(x) \leq f(x^*), \forall x \in S$"

- Such x^* is an **optimal solution**
- $f(x^*)$ is **optimal value**

Other ways to write (OPT):

$$\max\{f(x), x \in S\}$$

$$\max_{x \in S} f(x)$$

Analogous problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}$$

Note

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & x \in S \end{array} = -1 \begin{pmatrix} \min & -f(x) \\ \text{s.t.} & x \in S \end{pmatrix}$$

Problem x^* may not exist

a) Problem is unbounded:

$$\forall M \in \mathbb{R}, \exists \overline{x} \in S, \text{ s.t. } f(\overline{x}) > M$$

b) $S = \varnothing$, i.e. (OPT) is **INFEASIBLE**

c) There may not exist x^* achieving supremum.

> Example:
>
> $$\begin{array}{ll} \max & x \\ \text{s.t.} & x < 1 \end{array}$$

supremum

$$\sup\{f(x) : x \in S\} = \begin{cases} +\infty & \text{if OPT unbounded} \\ -\infty & \text{if } S = \varnothing \\ \min\{x : x \geq f(x), \forall x \in S\} & \text{otherwise} \end{cases}$$

always exist and are well-defined

infimum

$$\inf\{f(x) : x \in S\} = -1 \cdot \sup\{-f(x) : x \in S\}$$

From this point on, we will abuse notation and say $\max\{f(x) : x \in S\}$ is $\sup\{f(x) : x \in S\}$.

One way to specify that I want an opt. sol. (if exists) is

$$x^* \in \operatorname{argmax}\{f(x) : x \in S\}$$

2 Linear Optimization (Programming) (LP)

$$S = \{x \in \mathbb{R}^n : Ax \le b\}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $f(x) = c^T x, c \in \mathbb{R}^n$.

$$\downarrow$$

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \end{array} \tag{LP}$$

Note

$$A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix} \qquad A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

Clarifying

$$u, v \in \mathbb{R}^n, \qquad u \le v \iff u_j \le v_j, \forall j \in 1, \dots, n$$

Note

$u \not\le v$ is not the same as $u > v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \not\le \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example:

$$\begin{array}{lrrl} \max & 2x_1 + & 0.5x_2 & \\ \text{s.t.} & x_1 & & \le 2 \\ & x_1 + & x_2 & \le 2 \\ & x & & \ge 0 \end{array}$$

- Strict ineq. not allowed

halfspace, hyperplane, polyhedron

Let $h \in \mathbb{R}^n, h_0 \in \mathbb{R}$.

$\{x \in \mathbb{R}^n : h^T x \leq h_0\}$ is a **halfspace**.

$\{x \in \mathbb{R}^n : h^T x = h_0\}$ is a **hyperplane**.

$Ax \leq b$ is a **polyhedron** (i.e. intersection of finitely many halfspaces).

Example:

n products, m resources. Producing $j \in \{1, \ldots, n\}$ given c_j profit/unit and consumes a_{ij} units of resource i, $\forall i \in \{1, \ldots, m\}$. There are b_i units available $\forall i \in \{1, \ldots, m\}$.

$$\begin{aligned} \max & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad \forall i = 1, \ldots, m \\ & x \geq 0 \end{aligned}$$

which is an LP.

2.1 Determining Feasibility

Given a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

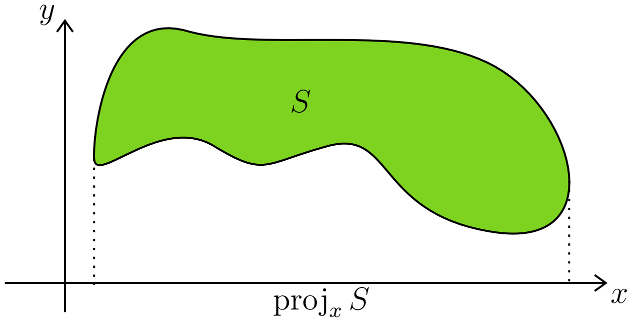
either find $\overline{x} \in P$ or show $P = \varnothing$.

Idea In 1-d, easy. $\rightarrow$ Reduce problem in dimension n to one in dimension $n-1$.

Notation Let $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \leq b\}$, then

$$\operatorname{proj}_x S := \{x \in \mathbb{R}^n : \exists y \text{ so that } (x, y) \in S\}$$

is the (orthogonal) *projection* if S onto x.



We will find if $P = \varnothing$ by looking at $\text{proj}_{x_1,\dots,x_{n-1}}$ (P)

2.2 Fourier-Motzkin Elimination

Call a_{ij} entries of A. Let

$$\begin{aligned} M &:= \{1, 2, \dots, m\} \\ M^+ &:= \{i \in M : a_{in} > 0\} \\ M^- &:= \{i \in M : a_{in} < 0\} \\ M^0 &:= \{i \in M : a_{in} = 0\} \end{aligned}$$

For $i \in M^+$:

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n} a_{ij} x_j \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j + x_n \leq \frac{b_i}{a_{in}}, \qquad \forall i \in M^+ \tag{1}$$

For $i \in M^-$

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j - x_n \leq \frac{b_i}{-a_{in}}, \qquad \forall i \in M^- \tag{2}$$

For $i \in M^0$

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} a_{ij} x_j \leq b_i, \qquad \forall i \in M^0 \tag{3}$$

$$P = \{x \in \mathbb{R}^n : (1)(2)(3)\}$$

Define

$$\sum_{j=1}^{n-1} \left(\frac{a_{ij}}{a_{in}} - \frac{a_{kj}}{a_{kn}} \right) x_j \leq \frac{b_i}{a_{in}} - \frac{b_i}{a_{kn}}, \qquad \forall i \in M^+, \forall k \in M^- \tag{4}$$

Theorem 2.1

$$(\overline{x}_1, \dots, \overline{x}_{n-1}) \text{ satisfies } (3), (4) \iff \exists \overline{x}_n : (\overline{x}_1, \dots, \overline{x}_n) \in P$$

Proof:

$\Longleftarrow$ If $(\overline{x}_1, \dots, \overline{x}_n)$ satisfies (1), (2), (3) then $(\overline{x}_1, \dots, \overline{x}_{n-1})$ satisfies (3) and adding (1), (2) $\implies$ $(\overline{x}_1, \dots, \overline{x}_{n-1})$ satisfies (4)

$\implies$ If $(\overline{x}_1, \dots, \overline{x}_{n-1})$ satisfies (4)

$$\sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \overline{x}_j - \frac{b_i}{a_{in}} \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \overline{x}_j - \frac{b_k}{a_{kn}}, \qquad \forall i \in M^+, k \in M^-$$

Let

$$\overline{x}_n := \max_{i \in M^+} \left\{ \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \overline{x}_j - \frac{b_i}{a_{in}} \right\}$$

$$\implies \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \overline{x}_j - \frac{b_i}{a_{in}} \leq -\overline{x}_n, \quad \forall i \in M^+$$

and

$$-\overline{x}_n \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \overline{x}_j - \frac{b_k}{a_{kn}}, \qquad \forall k \in M^-$$

$$\implies (\overline{x}_1, \ldots, \overline{x}_n) \in P$$

□

Note

Proof assumes M^+, M^- are nonempty. But statement holds regardless.

(if M^+ or $M^- = \varnothing$ then (4) yields no constraints)

Algorithm 1: Fourier-Motzkin

---

1 $A^n = A, b^n = b$

2 given A^i, b^i obtain A^{i-1}, b^{i-1} (A^{i-1} has one less column than A^i column than A^i) by applying the steps described

$$P_i := \{x \in \mathbb{R}^i : A^i x \leq b^i\}$$

then

$$P_{i-1} = \operatorname{proj}_{x_1, \ldots, x_{i-1}} P_i$$

3 Keep applying projection until $i = 1$.

$$P_0 = \varnothing \iff P_n = P = \varnothing$$

Let

$$P_i^n = P_i \times \mathbb{R}^{n-i} = \{x \in \mathbb{R}^n (A^i, 0)x \leq b^i\}$$

not hard to see $P_i^n = \varnothing \iff P_i = \varnothing$

Notice that

$$P_0 = \varnothing \iff P_0^n = \varnothing, P_0^n = \{0 \leq b^0\}$$

Example:

$$P_2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{rrr} x_1 & +2x_2 & \leq 1 \\ -x_1 & & \leq 0 \\ & -x_2 & \leq -2 \\ -3x_1 & -3x_2 & \leq -6 \end{array} \right\}$$

draw the graph, clearly empty

M^+: $\frac{1}{2}x_1 + x_2 \leq \frac{1}{2}$

M^-: $-x_2 \leq -2 \qquad -x_1 - x_2 \leq -2$

M^0: $-x_1 \leq 0$

$$P_1 = \left\{ x_1 \in \mathbb{R} : \begin{array}{ll} -x_1 & \leq 0 \\ \frac{1}{2}x_1 & \leq -\frac{3}{2} \\ -\frac{1}{2}x_1 & \leq -\frac{3}{2} \end{array} \right\}$$

M^+: $x_1 \leq -3$

M^-: $-x_1 \leq 0$ and $-x_1 \leq -3$

$$P_0^2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{l} 0 \leq -3 \\ 0 \leq -6 \end{array} \right\} = \varnothing$$

Here $b^0 = \binom{-3}{-6}$

Remark:
Inequality in P_i^n:

- All inequalities are obtained by a nonnegative combination of inequality in P_{i+1}^n
  $\implies$ all nonnegative combination of inequalities in P.
- If all A, b are rational then so are all A^i, b^i
- If $b = 0, b_i = 0, \forall i$

Theorem 2.2: Farkas' Lemma

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = \varnothing \iff \exists u \in \mathbb{R}^m : \begin{array}{l} u^T A = 0 \\ u^T b < 0 \\ u \geq 0 \end{array}$$

Proof:
($\Longleftarrow$) Suppose $\overline{x}$ satisfies $A\overline{x} \leq b$.

$$0 = u^T A\overline{x} \leq u^T b < 0$$

which is impossible.

($\Longrightarrow$) If $P = \varnothing$. Apply Fourier-Motzkin until we get

$$P_0^n = \varnothing = \{x \in \mathbb{R}^n : 0x \leq b^0\}$$

i.e. there exists j for which $b_j^0 < 0$.

If we look at corresponding constraint in P_0^n is

$$0^T x \leq b_j^0$$

which can be obtained by a vector u such that $u^T A = 0, u^T b = b_j^0, u \geq 0$.

□

Farkas' Lemma (alternate statement)

Exactly one of the following has a solution:

a) $Ax \leq b$

b) $\begin{aligned} u^T A &= 0 \\ u^T b &< 0 \\ u &\geq 0 \end{aligned}$

Farkas' Lemma (Different Form)

Exactly one of the following has a solution:

a) $\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$

b) $\begin{aligned} u^T A &\geq 0 \\ u^T b &< 0 \end{aligned}$

Proof:
(Sketch)

$$P = \left\{ x : \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \right\} = \left\{ x : \underbrace{\begin{pmatrix} A \\ -A \\ -I \end{pmatrix}}_{A'} x \leq \underbrace{\begin{pmatrix} b \\ -b \\ -0 \end{pmatrix}}_{b'} \right\}$$

Apply original Farkas' Lemma to get $P = \varnothing \iff \exists u_1 \in \mathbb{R}^m, u_2 \in \mathbb{R}^m, v \in \mathbb{R}^n$:

$$\begin{aligned} u_1^T A - u_2^T A - v &= 0 \\ u_1^T b - u_2^T b &< 0 \\ u_1, u_2, v &\geq 0 \end{aligned}$$

Let $u = (u_2 - u_2)$

$$u^T A - v = 0 \implies u^T A \geq 0, \quad u^T b < 0$$

□

Consider a linear programming (LP):

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \tag{LP}$$

Theorem 2.3: Fundamental Theorem of Linear Programming

(LP) has exactly one of 3 outcomes:

a) Infeasible

b) Unbounded

c) There exists an optimal solution.

Proof:

Let's assume a), b) don't hold.

If $n = 1$, then (LP) has an optimal solution. (Why?)

Else, define

$$\begin{array}{ll} \max & z \\ \text{s.t.} & \begin{array}{l} z - c^T x \leq 0 \\ Ax \leq b \end{array} \end{array} \tag{LP'}$$

(LP') is also not in case a) or b). (Why?)

Also if (x^*, z^*) is an optimal solution to (LP'), then x^* is an optimal solution to (LP). (Why?)

Apply Fourier-Motzkin to

$$\left\{ (x, z) : \begin{array}{r} z - c^T x \leq 0 \\ Ax \leq b \end{array} \right\}$$

Until we are left with a polyhedron

$$\{z \in \mathbb{R} : A'z \leq b'\}$$

Now $\begin{array}{ll} \max & z \\ \text{s.t.} & A'z \leq b' \end{array}$ is not cases a) or b). (Why?)

$\rightarrow$ can get an optimal solution z^* to such problem. Apply Fourier-Motzkin back to get (x^*, z^*) optimal solution to (LP'). (Why?) $\square$

2.3 Certifying Optimality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \tag{LP}$$

and let $\overline{x} \in P = \{x : Ax \leq b\}$

Question Can we certify that $\overline{x}$ is optimal?

Example:

$$\begin{aligned} \max\ & 2x_1 + x_2 \\ \text{s.t.}\ & x_1 + 2x_2 \leq 2 \\ & x_1 + x_2 \leq 2 \\ & x_1 - x_2 \leq 0.5 \end{aligned}$$

Consider $\overline{x} = (0,1)^T$ is clearly NOT optimal.

$x^* = (1, 0.5)^T$ and $c^T x^* = 2.5$. Any feasible solution satisfies

$$\begin{array}{rll} x_1 + 2x_2 & \leq 2 & \times 1/3 \\ x_1 + x_2 & \leq 2 & \times 1 \\ +\quad x_1 - x_2 & \leq 0.5 & \times 2/3 \\ \hline 2x_1 + x_2 & \leq 3 & \end{array}$$

Instead do $1 \times 1st$ constraint $+ 1 \times 3rd$ constraint $\implies 2x_1 + x_2 \leq 2.5$

In general:

$$\begin{array}{rll} x_1 + 2x_2 & \leq 2 & \times y_1 \\ x_1 + x_2 & \leq 2 & \times y_2 \\ +\quad x_1 - x_2 & \leq 0.5 & \times y_3 \\ \hline \end{array}$$
$$(y_1 + y_2 + y_3)x_1 + (2y_1 + y_2 - y_3)x_2 \leq 2y_1 + 2y_2 + 0.5y_3$$

As long as $y_1, y_2, y_3 \geq 0$ and

$$\begin{aligned} y_1 + y_2 + y_3 &= 2 \\ 2y_1 + y_2 - y_3 &= 1 \end{aligned}$$

This leads to the following linear program:

$$\begin{aligned} \min\ & 2y_1 + 2y_2 + 0.5y_3 \\ s.t.\ & y_1 + y_2 + y_3 = 2 \\ & 2y_1 + y_2 - y_3 = 1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

This is called the dual LP.

In general:

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \leq b \end{aligned} \tag{P}$$

Dual of (P)

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & y^T A = c^T \\ & y \geq 0 \end{aligned} \tag{D}$$

Remark:
We call (P) primal LP.

Theorem 2.4: Weak Duality

Let $\overline{x}$ feasible for (P), $\overline{y}$ feasible for (D). Then $c^T x \leq b^T y$.

Proof:

$$c^T \overline{x} = \overline{y}^T (A\overline{x}) \leq \overline{y}^T b$$

where we used $A\overline{x} \leq b$ and $\overline{y} \geq 0$. □

Corrollary 2.5

Several results:

- If (P) is unbounded then (D) is infeasible.
- If (D) is unbounded then (P) is infeasible.

Note
(P) and (D) can both be infeasible.

- If $\overline{x}$ is feasible for (P) $\overline{y}$ feasible for (D) $c^T \overline{x} = b^T \overline{y}$, then $\overline{x}$ optimal for (P), $\overline{y}$ optimal for (D).

Theorem 2.6: Strong Duality

x^* is optimal for (P) $\iff$ $\exists y^*$ feasible for (D) such that $c^T x^* = b^T y^*$.

Proof:
$(\impliedby)$ ✓

$(\implies)$ Is (D) infeasible?

Suppose $\left\{ y \in \mathbb{R}^n : \begin{array}{r} A^T y = c \\ y \geq 0 \end{array} \right\} = \varnothing$

(Alternate version of Farkas' Lemma) $\exists u : \begin{array}{r} u^T A^T \geq 0 \\ u^T c < 0 \end{array} \iff \exists d : \begin{array}{r} Ad \leq 0 \\ c^T d > 0 \end{array}$

Take look at $x' = x^* + d$, then

$$\begin{aligned} Ax' &= Ax^* + Ad \leq b \\ c^T x' &= c^T x^* + c^T d > c^T x^* \end{aligned}$$

Contradiction. Thus (D) has an optimal solution y^*.

Now let $\gamma = b^T y^*$, and let $\theta := \left\{ x \in \mathbb{R}^n : \begin{array}{r} Ax \leq b \\ -c^T x \leq -\gamma \end{array} \right\}$.

If $\theta = \varnothing$, by Farkas'

$$\exists \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix} : \begin{cases} \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix}^T \begin{pmatrix} A \\ -c^T \end{pmatrix} = 0 \\ \\ \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix}^T \begin{pmatrix} b \\ -\gamma \end{pmatrix} < 0 \\ \\ \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix} \geq 0 \end{cases} \iff \begin{array}{l} A^T \overline{y} = c\overline{\lambda} \\ b^T \overline{y} < \gamma \overline{\lambda} \\ \overline{y} \geq 0 \\ \overline{\lambda} \geq 0 \end{array}$$

<u>Case 1:</u> $\overline{\lambda} > 0$.

Let $y' = \dfrac{\overline{y}}{\overline{\lambda}}$. Then we have

$$A^T y' = A^T \frac{\overline{y}}{\overline{\lambda}} = c \quad \text{and} \quad b^T y' = b^T \frac{\overline{y}}{\overline{\lambda}} < \gamma \quad \text{and} \quad y' = \frac{\overline{y}}{\overline{\lambda}} \geq 0$$

Contradicts optimality of y^*.

<u>Case 2:</u> $\overline{\lambda} = 0$. Then $\begin{array}{l} A^T y = 0 \\ b^T y < 0 \\ \overline{y} \geq 0 \end{array}$

Now we can do the same thing previously. Let $y' = y^* + \overline{y}$, then

$$A^T y' = A^T y^* + A^T \overline{y} = c$$

and

$$y' = y^* + \overline{y} \geq 0$$
$$b^T y' = b^T y^* + b^T \overline{y} < b^T y^*$$

Contradicts optimality of y^*.

Thus $\theta \neq \varnothing$.

Let $\overline{x} \in \theta$,

$$c^T x^* \underbrace{\leq}_{\text{weak duality}} b^T y^* = \gamma \underbrace{\leq}_{\overline{x} \in \theta} c^T \overline{x} \leq c^T x^*$$

where the last inequality is because $\overline{x}$ feasible for (P), x^* optimal for (P).

$\square$

2.4 Possible Outcomes

See here.

2.5 Duals of generic LPs

$$
\begin{array}{lllll}
\max & 2x_1 + 3x_2 - 4x_3 \\
\text{s.t.} & x_1 & & +7x_3 & \leq 5 \\
 & & 2x_2 & -x_3 & \geq 3 \\
 & x_1 & & +x_3 & = 8 \\
 & & x_2 & & \leq 6 \\
 & x_1 & & & \geq 0 \\
 & & x_2 & & \leq 0
\end{array}
$$

$$
\begin{array}{ll}
\max & (2,3,-4)x \\
\text{s.t.} & \begin{pmatrix} 1 & 0 & 7 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 5 \\ -3 \\ 8 \\ -8 \\ 6 \\ 0 \\ 0 \end{pmatrix}
\end{array}
$$

and dual

$$
\begin{array}{ll}
\min & (5,-3,8,-8,6,0,0)y \\
\text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0
\end{array} \qquad (D_1)
$$

$$
\begin{array}{ll}
\min & (5,-3,8,-8,6)y \\
\text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0
\end{array} \qquad (D_2)
$$

Claim $(y_1^*, \ldots, y_5^*)$ is optimal for (D_2) $\iff$ $(y_1^*, \ldots, y_5^*, y_6^*, y_7^*)$ optimal for (D_1) with

$$
\begin{aligned}
y_6^* &= y_1^* + y_3^* - y_4^* - 2 \\
y_7^* &= 3 - (-2y_2^* + y_5^*)
\end{aligned}
$$

$$\begin{array}{ll} \min & (5,3,8,6)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 7 & -1 & 1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y_1 \geq 0, y_2 \leq 0 \qquad y_4 \geq 0 \end{array} \tag{D_3}$$

Claim Opt value of (D_2) and (D_3) are same.

In general

$$\begin{array}{lll} \max & c^T x & \\ \text{s.t.} & \begin{array}{r} Ax?b \\ x?0 \end{array} & (P) \end{array} \Bigg| \begin{array}{lll} \min & b^T y & \\ \text{s.t.} & \begin{array}{r} A^T y?c \\ y?0 \end{array} & (D) \end{array}$$

2.5.1 Cheat Sheet

Here or

Primal (max)		Dual (min)	
Constraint	$\leq$	≥ 0	Variable
	$\geq$	≤ 0	
	$=$	free	
Variable	$\geq$	≥ 0	Constraint
	$\leq$	≤ 0	
	free	$=$	

Remark:
This is not symmetric... The way you can remember it is by thinking natural variables in real life, like you cannot have negative number of cars and so on...

Q What if you start with a minimization LP as primal?

Example:

$$\begin{array}{ll} \min & x_1 - x_2 \\ \text{s.t.} & \begin{array}{l} 2x_1 + 3x_2 \leq 5 \\ x_1 - x_2 \geq 3 \\ x_1 + 5x_2 = 7 \\ x_1 \geq 0, x_2 \leq 0 \end{array} \end{array} \tag{P}$$

Rewrite as:

$$-1 \times \begin{pmatrix} \max & -x_1 + x_2 \\ \downarrow & \\ \text{s.t.} & \dots \end{pmatrix}$$

Will lead to finding dual:

$$\begin{array}{ll} \max & 5y_1 + 3y_2 + 7y_3 \\ \downarrow & \\ & 2y_1 + y_2 \leq 1 \\ \text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\ & y_1 \leq 0, y_2 \geq 0, y_3 \text{ free} \end{array}$$

Also

- Weak duality holds.
  If $\overline{x}$ feasible for (P), $\overline{y}$ feasible for (D), then $c^T\overline{x} \geq b^T\overline{y}$.
- Strong duality holds

Note
The dual of the dual of (P) is (P).

Example:
Given a simple undirected graph $G = (V, E)$. $M \subseteq E$ is a *matching* if every vertex $v \in V$ is incident to ≤ 1 edge in M.

See examples of matching in CO 342 or MATH 249.

Max cardinality matching

Find matching M with largest $|M|$.

Define $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{otherwise} \end{cases}$.

$$\begin{array}{ll} \max & \sum_{e \in E} x_e \\ \downarrow & \\ \text{s.t.} & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\ & 0 \leq x_e, \quad \forall e \in E \end{array}$$

where $\delta(v) =$ set of edges in E incident to v.

$$\begin{array}{lll} \min & \sum_{v \in V} y_v & \\ \downarrow & & \\ \text{s.t.} & y_u + y_v \geq 1, & \forall e = uv \in E \\ & y \geq 0 & \end{array}$$

2.6 Other interpretations of dual

Example:

			Resources	
		Per unit Profit	Per unit consumption A	B
Product	1	5	2	3
	2	3	4	1
Available Resources			15	10

$$\begin{array}{ll} \max & 5x_1 + 3x_2 \\ \downarrow & \\ & 2x_1 + 4x_2 \le 15 \\ \text{s.t.} & 3x_1 + x_2 \le 10 \\ & x \ge 0 \end{array}$$

Suppose somebody wants to buy A, B from me. What is the lowest price I should ask?

Let y_A, y_B be prices:

$$\begin{array}{ll} \min & 15y_A + 10y_B \\ \downarrow & \\ & 2y_A + 3y_B \ge 5 \\ \text{s.t.} & 4y_A + y_B \ge 3 \\ & y \ge 0 \end{array}$$

Example: Zero-Sum

Alice, Bob play game. A: m choices. B: n choices. Alice play i, Bob plays j, Bob pays Alice M_{ij} dollars.

		Alice R	P	S
	R	0	1	-1
Bob	P	-1	0	1
	S	1	-1	0

Zero-sum: Amount won by Alice - Amount won by Bob = 0

Let $y \in \mathbb{R}^m_+$, Alice's probability distribution.
Let $x \in \mathbb{R}^n_+$, Bob's probability distribution.

Expected Amount Bob pays Alice:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i M_{ij} x_j = y^T M_x$$

$$P = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \sum x_j = 1 \\ x \ge 0 \end{array} \right\}$$

$$Q = \left\{ y \in \mathbb{R}^m : \begin{array}{l} \sum y_i = 1 \\ y \geq 0 \end{array} \right\}$$

Alice wants $\max\limits_{y \in Q} \left\{ \min\limits_{x \in P} \ y^T M_x \right\}$. Bob wants $\min\limits_{x \in P} \left\{ \max\limits_{y \in Q} \ y^T M_x \right\}$.

Suppose $\overline{y} \in Q$ is fixed. Bob's problem is

$$\min_{x \in P} \ \overline{y}^T M_x \ = \ \begin{array}{ll} \min & \sum\limits_{j=1}^{n} \left(\sum\limits_{i=1}^{m} M_{ij} \overline{y}_i \right) x_j \\ \downarrow & \\ \text{s.t.} & \sum\limits_{j=1}^{n} x_j = 1 \\ & x \geq 0 \end{array}$$

This is equivalent to picking smallest number in

$$\left\{ \sum_{i=1}^{m} M_{ij} \overline{y}_i \right\}_{j=1}^{n}$$

$$\implies \max_{y \in Q} \min_{x \in P} y^T M_x = \max_{y \in Q} \left\{ \begin{array}{lll} \max & u & \\ \downarrow & & \\ \text{s.t.} & u \leq y^T M e_j, & \forall j = 1, \ldots, n \end{array} \right\}$$

$$= \begin{array}{lll} \max & u & \\ \downarrow & & \\ & u \leq y^T M e_j, & \forall j = 1, \ldots, n \\ \text{s.t.} & y^T = 1 & \\ & y \geq 0 & \end{array}$$

Similarly Bob's problem:

$$\begin{array}{lll} \min & v & \\ \downarrow & & \\ & v \geq e_i^T M_x, & \forall i = 1, \ldots, m \\ \text{s.t.} & x^T = 1 & \\ & x \geq 0 & \end{array}$$

There are x^*, y^* for which strategy values match $\rightarrow$ Nash's Equilibrium.

Now get back to Farkas' Lemma Theorem 2.2. [1]

Proof:

$$\begin{array}{ll} \max & 0^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \tag{P}$$

[1] Rephrase it a little bit: Exactly one of the two has a solution (i) $Ax \leq b$ (ii) $u^T \ldots$.

$$\begin{array}{ll} \min & b^T u \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} u^T A = 0 \\ u \geq 0 \end{array} \end{array} \tag{D}$$

(D) is always feasible ($u = 0$).

If $\exists \overline{x} : A\overline{x} \leq b$, $\overline{x}$ optimal for (P) $\implies$ optimal for (D) has value 0.
$\implies \nexists u$ satisfying (ii).

And the converse is also true. □

2.7 Complementary Slackness (C.S.)

Let x^*, y^* be feasible for primal and dual respectively.

Complementary Slackness

Abbreviated as C.S.

i) Either $x_j^* = 0$ or corresponding dual constraint is tight at y^*, $\forall j = 1, \ldots, n$.

ii) Either $y_i^* = 0$ or corresponding primal constraint is tight at x^*, $\forall i = 1, \ldots, m$.

Example:

$$\begin{array}{ll} \min & x_1 - x_2 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} 2x_1 + 3x_2 \leq 5 \\ x_1 - x_2 \geq 3 \\ x_1 + 5x_2 = 7 \\ x_1 \geq 0, x_2 \leq 0 \end{array} \end{array} \tag{P}$$

$$\begin{array}{ll} \max & 5y_1 + 3y_2 + 7y_3 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} 2y_1 + y_2 + y_3 \leq 1 \\ 3y_1 - y_2 + 5y_3 \geq -1 \\ y_1 \leq 0, y_2 \geq 0 \end{array} \end{array} \tag{D}$$

i) $x_1^* = 0$ OR $2y_1^* + y_2^* + y_3^* = 1$
$x_2^* = 0$ OR $3y_1^* - y_2^* + 5y_3^* = -1$

ii) $y_1^* = 0$ OR $2x_1^* + 3x_2^* = 5$
$y_2^* = 0$ OR $x_1^* - x_2^* = 3$
$y_3^* = 0$ OR $x_1^* + 5x_2^* = 7$

Theorem 2.7

Let x^*, y^* be feasible for primal/dual respectively. TFAE[a]

a) x^* opt for primal AND y^* opt. for dual

b) Obj. value of x^* = Obj. value of y^*

c) x^*, y^* satisfy C.S.

[a] the following are equivalent

Proof:

a) $\iff$ b) done.

b) $\iff$ c) Proof for

$$\begin{array}{lll} \max & c^T x & \\ \downarrow & & \\ \text{s.t.} & \begin{array}{l} Ax \le b \\ x \ge 0 \end{array} \end{array} \qquad \begin{array}{ll} \min & b^T y \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} A^T y \ge c \\ y \ge 0 \end{array} \end{array}$$

Note

$$A^T y \ge c \iff \sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad \forall j = 1, \dots, n$$

$$\begin{aligned} c^T x^* &= \sum_{j=1}^{n} c_j x^* \\ &\le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^* \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_i^* \right) y_i^* \\ &\le \sum_{i=1}^{m} b_i y_i^* = b^T y^* \end{aligned}$$

where first and second inequalities come from $x \ge 0, y \ge 0$ respectively.

(b) $c^T x^* = b^T y^* \iff$ C.S. holds. (Just play with some strict inequality conditions)

□

Example:

$$\begin{array}{ll} \max & x_1 + x_2 \\ \downarrow & \\ \text{s.t.} & x_1 + x_2 \le 1 \end{array} \qquad \begin{array}{ll} \min & y \\ \downarrow & \\ & y = 1 \\ \text{s.t.} & y = 1 \\ & y \ge 0 \end{array}$$

Consider a pair $x^* = (0, 0), y^* = 1$ which violates CS.

2.7.1 Geometric Interpretation of C.S.

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \le b \end{array} \qquad \begin{array}{ll} \min & c^T y \\ \downarrow & \\ \text{s.t.} & A^T y = c \\ & y \ge 0 \end{array}$$

$$A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

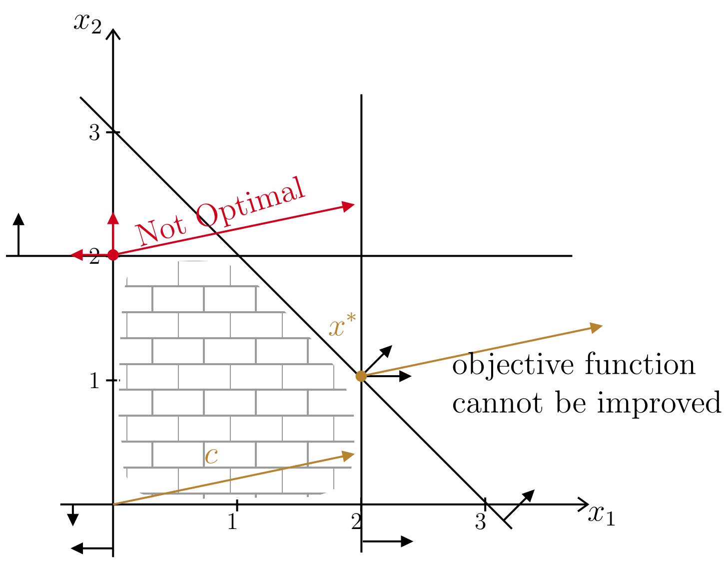
C.S says $a_i^T x^* = b_i$ or $y_i^* = 0$.

$$A^T y = c \implies \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{pmatrix} y = c \implies \sum_{i=1}^m a_i y_i = c$$

C.S. says c is a nonnegative combination of tight constraint at x^*.

Example:

$$\begin{array}{ll} \max & 2x_1 + 0.5x_2 \\ \downarrow & \\ & x_1 \le 2 \\ \text{s.t.} & x_2 \le 2 \\ & x_1 + x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{array}$$



Theorem 2.8

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \tag{P}$$

is unbounded iff (P) is feasible and $\exists d \in \mathbb{R}^n : \begin{array}{l} c^T d > 0 \\ Ad \leq 0 \end{array}$.

Proof:

$\Longrightarrow$) Let $\overline{x}$ feasible for (P), $\overline{x} + \lambda d$ is also feasible for (P) $\forall \lambda \geq 0$.

$c^T(\overline{x} + \lambda d)$ can be made arbitrary large.

$\Longleftarrow$) Hard exercise but doable.

□

2.8 Geometry of Polyhedra

line segment

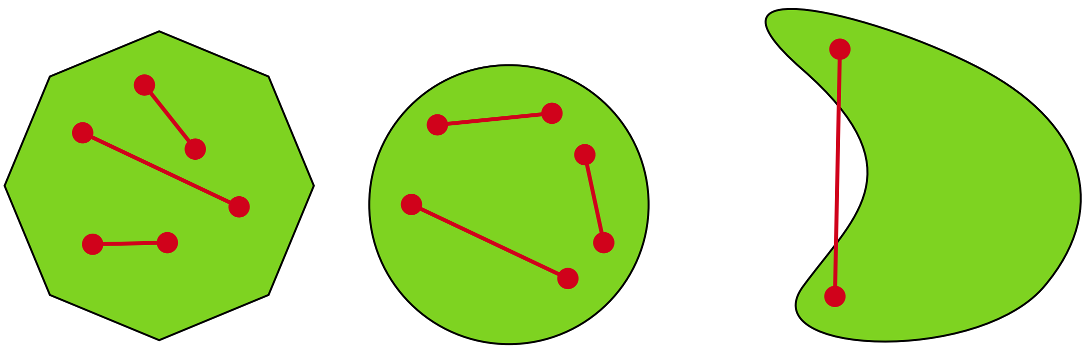
$\overline{x}, \overline{y} \in \mathbb{R}^n$ the line segment between $\overline{x}, \overline{y}$ is

$$\left\{ x \in \mathbb{R}^n : \begin{array}{l} x = \lambda \overline{x} + (1-\lambda)\overline{y} \\ \text{for some } \lambda \in [0,1] \end{array} \right\}$$

convex set

S is a convex set if $\forall x, y \in S$, line segment between x, y is contained in S.

Example:



NOT a convex set

Polyhedra are convex sets. $P = \{x : Ax \leq b\}$. $\overline{x}, \overline{y} \in P$ then

$$A(\underbrace{\lambda}_{\geq 0} \overline{x} + \underbrace{(1-\lambda)}_{\geq 0} \overline{y}) \leq \lambda b + (1-\lambda) b = b$$

convex combination

Given $x^1, \ldots, x^k \in \mathbb{R}^n$. We say $\overline{x}$ is a convex combination of $x^1, \ldots, x^k$ if $\exists \lambda$:

$$\overline{x} = \sum_{i=1}^{k} \lambda_i x^i$$
$$1 = \sum_{i=1}^{k} \lambda_i$$
$$\lambda \geq 0$$

Optimal solution seems to be happen at "corners".

Let P be a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

vertex

$\overline{x}$ is a vertex of P if $\exists c$: $\overline{x}$ is unique optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

extreme point

$\overline{x}$ is an extreme point of P if $\nexists u, v \in P \setminus \{\overline{x}\}$ such that $\overline{x}$ is in line segment between u, v.

basic feasible solution

$\overline{x} \in P$ is a basic feasible solution of P if there are n linearly independent tight constraints at $\overline{x}$.

Note

Constraints

$$a_i^T x \leq b_i, \quad \forall i = 1, \ldots, m$$

are linearly independent if $\{a_i\}_{i=1}^m$ are linearly independent.

Theorem 2.9

Let $\overline{x} \in P$. TFAE:

a) $\overline{x}$ is a vertex of P.

b) $\overline{x}$ is a basic feasible solution of P.

c) $\overline{x}$ is a extreme point of P.

Proof:

a) $\implies$ c) Suppose $\exists u, v \in P \setminus \{\overline{x}\}$ such that

$$\overline{x} = \lambda u + (1-\lambda)v$$

for some $\lambda \in (0,1)$. Consider c for which $\overline{x}$ is an optimal solution to

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & x \in P \end{aligned}$$

$$\implies \begin{aligned} c^T\overline{x} \geq c^T u \\ c^T\overline{x} \geq c^T v \end{aligned}$$

and

$$c^T\overline{x} = \underbrace{\lambda}_{\geq 0} c^T u + \underbrace{(1-\lambda)}_{\geq 0} c^T v \leq \lambda c^T\overline{x} + (1-\lambda)c^T\overline{x} = c^T\overline{x}$$

$$\implies c^T u = c^T v = c^T \overline{x}$$

$\implies$ $\overline{x}$ NOT a vertex.

c) $\implies$ b) Suppose $\overline{x}$ is not a BFS. Let $I \subseteq \{1, \dots, m\}$ be the index set of tight constraint at $\overline{x}$. Consider

$$a_i^T d = 0, \quad \forall i \in I \tag{$*$}$$

But since $\overline{x}$ not BFS, $\exists \overline{d} \neq 0$ satisfying $(*)$.[a]

$$x(\epsilon) = \overline{x} + \epsilon \overline{d}$$

$$\begin{aligned} a_i^T x(\epsilon) &= a_i^T \overline{x} \leq b_i, \quad \forall i \in I \\ a_i^T x(\epsilon) &= \underbrace{a_i^T \overline{x}}_{<b_i} + \epsilon a_i^T d \leq b_i, \quad \forall i \notin I \end{aligned}$$

which is satisfied if $|\epsilon|$ is small enough.

$x(\epsilon) \in P$ if $|\epsilon|$ is small enough.

But then

$$\overline{x} = \frac{1}{2}x(\epsilon) + \frac{1}{2}x(-\epsilon)$$

b) $\implies$ a) Let $I \subseteq \{1, \dots, m\}$ index set of tight constraint at $\overline{x}$.

Define

$$c := \sum_{i \in I} a_i$$

Then $\forall x \in P$

$$c^T x = \sum_{i \in I} a_i^T x \leq \sum_{i \in I} b_i$$

And

$$c^T \overline{x} = \sum_{i \in I} a_i^T \overline{x} = \sum_{i \in I} b_i$$

$\implies \overline{x}$ is optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array} \tag{**}$$

If $x' \in P$ is optimal solution to (**), then

$$a_i^T x' = b_i, \qquad \forall i \in I \tag{***}$$

But since there are n linear independent constraints in I, $\overline{x}$ is unique solution to (***). $\implies x' = \overline{x}$.

□

[a]by Rank-Nullity Theorem.

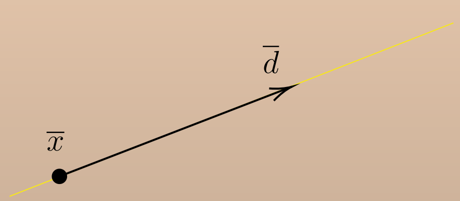
Q When does P have extreme points?

line

Let $\overline{x}, \overline{d} \in \mathbb{R}^n$, $\overline{d} \neq 0$. The set

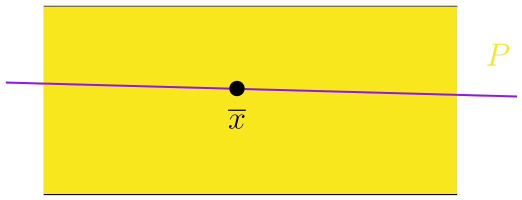
$$\{x \in \mathbb{R}^n : x = \overline{x} + \lambda d \text{ for some } \lambda \in \mathbb{R}\}$$

is called a line.



We say a polyhedron P has a line if $\exists \overline{x}, \overline{d}$ has a line if $\exists \overline{x}, \overline{d}$ s.t. $\overline{x} \in P, \overline{d} \neq 0$ and

$$\{x \in \mathbb{R} : x = \overline{x} + \lambda \overline{d} \text{ for some } \lambda \in \mathbb{R}\} \subseteq P$$



Proposition 2.10

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has a line iff $P \neq \varnothing$ and $\exists \overline{d} \neq 0$ such that $A\overline{d} = 0$

$\iff P \neq \varnothing$ and $\operatorname{rank}(A) < n$

Proof:
Exercise. □

Theorem 2.11

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has an extreme point

$\iff P \neq \varnothing$ and P has no lines.

Proof:
Exercise. $\square$

pointed polyhedron

A non-empty polyhedron is called pointed if it has no lines.

Note
not pointed does not imply bounded. For example, in $\mathbb{R}^2$, $x \geq 0$ and $y \geq 0$.

Theorem 2.12

Let $P \neq \varnothing$ pointed polyhedron. If $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$ (LP) has an optimal solution, it has an optimal solution that is an extreme point.

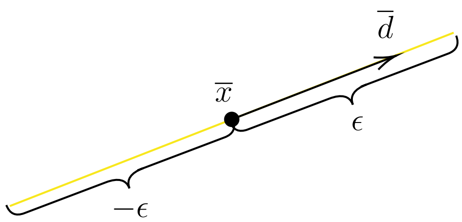
Proof:
Let $\overline{x}$ be an optimal solution to (LP) with largest number of linear independent tight constraints.

Suppose there are $\leq n-1$ linear independent tight constraints at $\overline{x}$.

Pick $\overline{d} \neq 0$ such that $a_i^T \overline{d} = 0, \forall i \in I$, where I is the index set of tight constraints. By the exact same argument as before, $\overline{x} \pm \epsilon \overline{d} \in P$ for ϵ small enough. But

$$c^T(\overline{x} \pm \epsilon \overline{d}) = c^T \overline{x} \pm \epsilon c^T \overline{d}$$

$\implies c^T \overline{d} = 0$
$\implies c^T d(\overline{x} \pm \epsilon d) = c^T \overline{x}$



Since P is pointed, $\exists \overline{\epsilon}$ for which

$$\overline{x} \pm \overline{\epsilon} \overline{d} \in P$$

and one of them not in P if $|\epsilon| > \overline{\epsilon}$. That can only happen if

$$a_k^T(\overline{x} + \overline{\epsilon} \overline{d}) = b_k \qquad \text{or} \qquad a_k^T(\overline{x} - \overline{\epsilon} \overline{d}) = b_k$$

for some $k \notin I$.

$\implies a_k^T \overline{d} \neq 0, \implies a_k$ is linear independent from $\{a_i\}_{i \in I}$ since non-zero cannot be linear combination of zeros. Contradiction to choice of $\overline{x}$. $\square$

2.9 Simplex Algorithm

Standard Equality Form

A linear program is in Standard Equality Form (SEF) if it is of the form

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \end{array}$$

Proposition 2.13

Given any linear program, there exists an equivalent LP in SEF.

Example:

$$\begin{array}{ll} \max & x_1 + 2x_2 + x_3 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} 3x_1 + x_2 \leq 5 \\ -x_1 + x_3 \geq 6 \\ x_1 \leq 0, x_3 \geq 0 \end{array} \end{array} \tag{P1}$$

$x_1' = -x_1 \geq 0$ and
$x_2 = x_2^+ - x_2^-$ where $x_2^+ \geq 0, x_2^- \geq 0$

We introduce

$$s_1 = 5 - 3x_1 - x_2 \geq 0, \qquad s_2 = -x_1 + x_3 - 6 \geq 0$$

Then

$$\begin{array}{ll} \max & -x_1' + 2x_2^+ - 2x_2^- + x_3 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} -3x_1' + 2x_2^+ - x_2^- + s_1 = 5 \\ x_1' + x_3 - s_2 = 6 \\ x_1', x_2^+, x_2^-, x_3, s_1, s_2 \geq 0 \end{array} \end{array} \tag{P2}$$

x feasible for (P1) $\iff (x_1', x_2^+, x_2^-, x_3, s_1, s_2)$ feasible for (P2) and they have same cost.

Assumption $A \in \mathbb{R}^{m \times n} \to \operatorname{rank}(A) = m$. This is WLOG. Since if

$$a_i = \sum_{k \neq i} \lambda_k a_k$$

Either

$$b_i \neq \sum_{k \neq i} \lambda_k b_k$$

in which case (SEF) is infeasible. Or $a_i^T x = b_i$ is redundant. So it can be removed from (SEF).

Note

$\{x : Ax = b, x \geq 0\}$ is *pointed* polyhedron (if nonempty).

Structure of BFS Any feasible solution has m linear independent tight constraints $(n - m)$ extra tight constraint must come from $x_j \geq 0$.

Let $B \subseteq \{1, \ldots, n\}$ such that $|B| = m$ and A_B[2] is invertible.

$N = \{1, \ldots, n\} \setminus B$. $x_N = 0$, i.e. $x_j = 0, \forall j \in N$.

Feasible solutions obtained this way are precisely BFS.

Example:

$$\begin{array}{ll} \max & \begin{pmatrix} 3 & 2 & 1 & 4 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x \geq 0 \end{array}$$

If we pick

$$\begin{aligned} B = \{1, 2\} \qquad & A_B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ N = \{3, 4\} \qquad & A_N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ C_B = (3 \quad 2)^T \qquad & C_N = (1 \quad 4)^T \end{aligned}$$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$B = \{1, 3\}, B = \{2, 4\}, A_B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, A_N = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$C_B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, C_N = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

If we set $x_N = 0$ (for $B = \{1, 3\}$) we are left with

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

This has a unique solution $x_1 = 3.5, x_3 = -1.5$, but not feasible.

[2] A_B is submatrix obtained by picking columns of A indexed by B. Such B is called a basis.

If we pick $B = \{1, 2\}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$\underbrace{x_3 = x_4}_{x_N} = 0,\ x_1 = 3, x_2 = 1$, which is feasible.

In general,

$$Ax = b \iff A_B x_B + \cancelto{0}{A_N x_N} = b$$

has unique solution $x_b = A_B^{-1} b$.

For any basis B, the corresponding *basic solution* is

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}$$

If $A_B^{-1} b \geq 0$, then it is a *BFS*.

2.9.1 Canonical Form

Let B be a feasible basis (i.e. corresponding basis solution is feasible).

$$\begin{aligned} Ax = b &\iff A_B x_B + A_N x_N = b \\ &\iff x_B + A_B^{-1} A_N x_N = A_B^{-1} b \end{aligned}$$

Now let's take a look at objective.

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N - c_B^T (x_B + A_B^{-1} A_N x_N - A_B^{-1} b) \\ &= (c_N^T - c_B^T A_B^{-1} A_N) x_N + c_B^T A_B^{-1} b \end{aligned}$$

Thus (SEF) is said to be in canonical form for B if it is written as

$$\begin{aligned} \max \quad & \overbrace{(c_N^T - c_B^T A_B^{-1} A_N)}^{\bar{c}_N^T \to \text{Reduced costs}} x_N + c_B^T A_B^{-1} b \\ \downarrow \quad & \\ \text{s.t.} \quad & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\ & x_B, x_N \geq 0 \end{aligned}$$

Example:
Back to our previous example...

$B = \{1, 2\}$. Rewriting in canonical form for B:

$$A_B^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

$$A_B A = \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix}$$

$$c_B^T A_B^{-1} A_N = \begin{pmatrix}3 & 2\end{pmatrix} \begin{pmatrix}1/3 & -2/3 \\ 2/3 & -1/3\end{pmatrix} = \begin{pmatrix}7/3 & -8/3\end{pmatrix}$$

$$c_N^T - c_B^T A_B^{-1} A_N = \begin{pmatrix}-4/3 & 4/3\end{pmatrix}$$

Then

$$\begin{aligned} \max \quad & \begin{pmatrix}0 & 0 & -4/3 & 4/3\end{pmatrix} x + 11 \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix}1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3\end{pmatrix} x = \begin{pmatrix}3 \\ 1\end{pmatrix} \\ & x \geq 0 \end{aligned}$$

is in canonical form for $B = \{1, 2\}$.

Example:

$$\begin{aligned} \max \quad & \begin{pmatrix}1 & 3 & -2 & 0 & 0\end{pmatrix} x \underbrace{+0}_{\text{obj. value}} \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix}1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1\end{pmatrix} x = \begin{pmatrix}4 \\ 1\end{pmatrix} \\ & x \geq 0 \end{aligned} \tag{LP}$$

Canonical form for $B = \{4, 5\}$.

Corresponding BFS $\begin{matrix} x_4 = 4 \\ x_5 = 1 \end{matrix}, \quad x_j = 0, \forall j \in N$

$x = \begin{pmatrix}0 & 0 & 0 & 4 & 1\end{pmatrix}^T$

Objective value $= 0$

If increase x_1 or x_2. Objective function increases.

Let's try to increase x_1 from $0 \to \theta$. (Keep $x_2 = x_3 = 0$)

$$\begin{aligned} \theta + x_4 = 4 &\iff x_4 = 4 - \theta \\ \theta + x_5 = 1 &\iff x_5 = 1 - \theta \end{aligned}$$

New objective: $0 + \theta$. However, we have

$$\begin{matrix} x_4 \geq 0 \implies \theta \leq 4 \\ x_5 \geq 0 \implies \theta \leq 1 \end{matrix} \implies \text{Increase } x_1 \text{ by } 1$$

x_5 will be $0 \to \begin{matrix} x_1 \text{ enters basis} \\ x_5 \text{ leaves basis} \end{matrix}$. Then new basis $B = \{1, 4\}$.

Rewriting (LP) in canonical form for $B = \{1, 4\}$.

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 4 & -5 & 0 & -1\end{pmatrix} x + \underbrace{1}_{\text{obj. value}} \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix}1 & -1 & 3 & 0 & 1 \\ 0 & 2 & -2 & 1 & -1\end{pmatrix} x = \begin{pmatrix}1 \\ 3\end{pmatrix} \\ & x \geq 0 \end{array}$$

Corresponding BFS:

$$x = \begin{pmatrix}1 & 0 & 0 & 3 & 0\end{pmatrix}^T$$

Obj. value $= 1$

Pick $j \in N$: $\bar{c}_j > 0$ $(j = 2)$

Increase x_2 to θ, keep $x_3 = x_5 = 0$

$$\begin{aligned} x_1 - \theta = 1 &\iff x_1 = 1 + \theta \\ x_4 + 2\theta = 3 &\iff x_4 = 3 - 2\theta \end{aligned}$$

and

$$\begin{aligned} x_1 \geq 0 &\implies \theta \geq -1 \\ x_4 \geq 0 &\implies \theta \leq \frac{3}{2} \end{aligned}$$

Set $\theta \leftarrow \frac{3}{2} \rightarrow \begin{array}{l} x_2 \text{ enters basis} \\ x_4 \text{ leaves basis} \end{array}$

New basis $B = \{1, 2\}$.

(LP) in canonical form for $B = \{1, 2\}$.

$$\begin{array}{ll} \max & \begin{pmatrix}0 & 0 & -1 & -2 & 1\end{pmatrix} x + 7 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix}1 & 0 & 2 & 0.5 & 0.5 \\ 0 & 1 & -1 & 0.5 & -0.5\end{pmatrix} x = \begin{pmatrix}2.5 \\ 1.5\end{pmatrix} \\ & x \geq 0 \end{array}$$

Corresponding BFS:

$$x = \begin{pmatrix}2.5 & 1.5 & 0 & 0 & 0\end{pmatrix}^T$$

Obj. value $= 7$

Find $j \in N$, $\bar{c}_j > 0$ $(j = 5)$

$$\begin{array}{l} x_1 = 2.5 - 0.5\theta \geq 0 \\ x_2 = 1.5 + 0.5\theta \geq 0 \end{array} \implies \begin{array}{l} \theta \leq 5 \\ \theta \geq -3 \end{array} \rightarrow \begin{array}{l} x_1 \text{ leaves basis} \\ x_5 \text{ enters basis} \end{array}$$

New basis $B = \{2, 5\}$

(LP) in canonical form for $B = \{2, 5\}$

$$
\begin{array}{ll}
\max & \begin{pmatrix} -2 & 0 & -5 & -3 & 0 \end{pmatrix} x + 12 \\
\downarrow & \\
\text{s.t.} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\
& x \geq 0
\end{array}
$$

$$
\left.\begin{array}{l}
\text{BFS } x = \begin{pmatrix} 0 & 4 & 0 & 0 & 5 \end{pmatrix}^T \\
\text{Obj. value} = 12.
\end{array}\right\} \text{Optimal Solution}
$$

2.9.2 Iteration of simplex

Algorithm 2: Iteration of simplex

1. Start with feasible basis B
2. Rewrite LP in canonical form for B
3. Pick $j \in N : \bar{c}_j > 0$ (x_j enters basis)
4. Let $\bar{b} = A_B^{-1} b$, $\overline{A}_N = A_B^{-1} A_N$
   Find largest θ so that $\bar{b} - \theta \overline{A}_j \geq 0$.
   Corresponding basic variable that becomes 0 (say x_k) leaves basis.
5. $B \leftarrow B \setminus \{k\} \cup \{j\}$. Iterate.

If problem has optimal solution AND θ is always > 0, simplex finishes.

Note
If at current BFS we have a basic variable $= 0$, we may have $\theta = 0$. $\rightarrow$ May lead to cycling. (i.e. return to current basis in future iteration)

Bland's Rule

If there are multiple choices of entering or leaving variables, always pick lowest index variable.

Using Bland's Rule avoids cycling

Observations If $\bar{c}_N \leq 0$, then the (LP) obj. value in canonical form is

$$
\underbrace{\bar{c}_N^T}_{\leq 0} \underbrace{x_N}_{\geq 0} + c_B^T A_B^{-1} b \leq c_B^T A_B^{-1} b
$$

For any feasible solution $\implies$ Current BFS is optimal

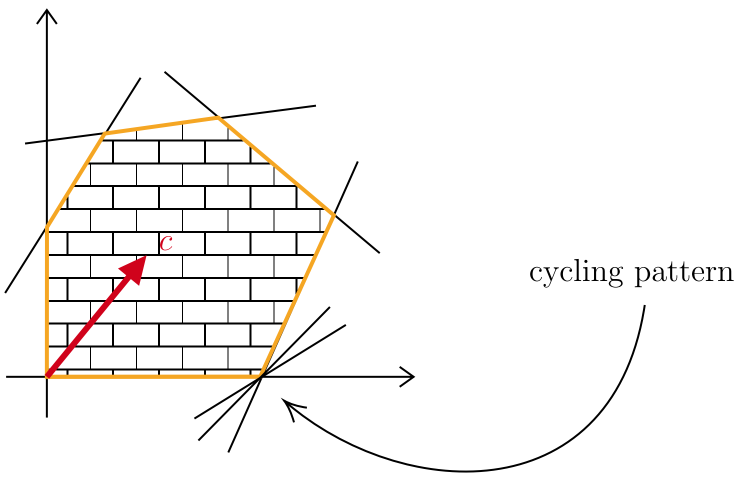


Figure 2.1: Simplex method

Original LP

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \end{array}$$

Dual

$$\begin{array}{ll} \min & b^T y \\ \downarrow & \\ \text{s.t.} & A^T y \geq c \end{array} \iff \begin{array}{ll} \min & y^T b \\ \downarrow & \\ \text{s.t.} & y^T A \geq c^T \end{array}$$

$$\iff \begin{array}{ll} \min & y^T b \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} y^T A_B \geq c_B^T \\ y^T A_N \geq c_N^T \end{array} \end{array}$$

If satisfies C.S with BFS corresponding to B

$$\implies \begin{array}{r} y^T A_B = c_B^T \\ y^T = c_B^T A_B^{-1} \\ y_T A_N \geq c_N^T \end{array} \iff c_B^T A_B^{-1} A_N \geq c_N^T \iff \bar{c}_N \leq 0$$

2.9.3 Mechanics of Simplex

Example: 1

$$\begin{array}{ll} \max & \begin{array}{c} \begin{array}{ccccc} \text{enters basis} & & & j & \\ \uparrow & & & \uparrow & \end{array} \\ \begin{pmatrix} 1 & 3 & -2 & 0 & 0 \end{pmatrix} x \end{array} \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} \overset{\text{pivot}}{} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{row } \ell \\ x \geq 0 \end{array} \end{array}$$

For θ

$$\theta \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and we have

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4-\theta \\ 1-\theta \end{pmatrix} \geq 0 \implies \begin{array}{c} \theta \leq 4 \\ \boxed{\theta \leq 1} \end{array}$$

We are actually picking $\min \left\{ \frac{4}{1}, \frac{1}{1} \right\}$

Pick, out of all rows $\min \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$ where j is entering variable.

Then now in row ℓ (second row here). Make row operations so that pivot element become 1, all others in col j becomes 0.

→ Row 2 ×1

→ Subtract tow 2 from row 1

→ subtract row 2 from objective function (with RHS multiplied by -1)

$$\begin{array}{ll} \max & \begin{array}{c} j \\ \uparrow \\ \begin{pmatrix} 0 & 4 & -5 & 0 & -1 \end{pmatrix} x + 1 \end{array} \\ \downarrow & \quad \text{pivot} \\ \text{s.t.} & \begin{pmatrix} 0 & 2 & -2 & 1 & -1 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{row } \ell \\ & x \geq 0 \end{array}$$

$$\begin{aligned} 2\theta + x_4 = 3 &\iff x_4 = 3 - 2\theta \geq 0 \implies \theta \leq \frac{3}{2} \\ -\theta + x_1 = 1 &\iff x_1 = \theta + 1 \geq 0 \implies \theta \geq -1 \end{aligned}$$

where we are finding $\min\limits_{\bar{a}_{ij}>0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$. Now follow the similar procedure, we have

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -2 & 1 \end{pmatrix} x + 7 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 0 & 1 & -1 & 0.5 & -0.5 \\ 1 & 0 & 2 & 0.5 & 0.5 \end{pmatrix} x = \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} \end{array}$$

In general Pick $j \in N : \bar{c}_j > 0$.

Let $\ell = \operatorname*{argmin}\limits_{\bar{a}_{ij}>0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$ (Ratio Test)

- Multiply row ℓ by $\frac{1}{\bar{a}_{\ell j}}$
- Add $-\frac{\bar{a}_{ij}}{\bar{a}_{\ell j}}$ times row ℓ to row $i \neq \ell$.

- Add $-\dfrac{\overline{c}_j \cdot \overline{a}_{\ell k}}{\overline{a}_{\ell j}}$ to variable coeff in objective. $\forall k \in 1, \ldots, n$
- Add $\dfrac{\overline{b}_\ell \cdot \overline{c}_j}{\overline{a}_{ij}}$ to objective value in objective function

Example: 2

$$
\begin{array}{ll}
\max & \begin{array}{c} \begin{array}{ccccc} j & & & & \end{array} \\ \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \end{pmatrix} x \end{array} \\
\downarrow & \\
\text{s.t.} & \text{pivot} \begin{pmatrix} 1 & 2 & -1 & 1 & 0 \\ 2 & -2 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \begin{array}{c} \\ \text{row } \ell \end{array} \\
& x \geq 0
\end{array}
$$

Ratio Test $\min\left\{\dfrac{2}{1}, \dfrac{3}{2}\right\} = 1.5$. $\ell = 2$. (x_2 enters, x_5 leaves)

$$
\begin{array}{ll}
\max & \begin{pmatrix} 0 & 3 & 2_j & 0 & -1 \end{pmatrix} x + 3 \\
\downarrow & \\
\text{s.t.} & \begin{pmatrix} 0 & 3 & -0.5 & 1 & -0.5 \\ 1 & -1 & -0.5 & 0 & 0.5 \end{pmatrix} x = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} \\
& x \geq 0
\end{array}
$$

If we increase $x_3 \to \theta$ and keep $x_2 = x_5 = 0$

$$
\begin{aligned} -0.5\theta + x_4 &= 0.5 \\ -0.5\theta + x_1 &= 1.5 \end{aligned} \implies \begin{aligned} x_1 &= 1.5 + 0.5\theta \\ x_4 &= 0.5 + 0.5\theta \end{aligned} \to \text{ Problem is unbounded!}
$$

In general Let B be a basis

$$
\begin{array}{ll}
\max & \overline{c}_N^T x_N \\
\downarrow & \\
\text{s.t.} & x_B + \overline{A}_N x_N = \overline{b} \\
& x_B, x_N \geq 0
\end{array}
$$

Found $j : \overline{c}_j > 0$ AND $\overline{A}_j \leq 0$.

Construct $d \in \mathbb{R}^n$ to reflect what we are trying to do when we increase $x_j \to \theta$.

Right now, we are at BFS:

$$
\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}
$$

We want:

$$
\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix} + \theta \begin{pmatrix} d_B \\ d_N \end{pmatrix}
$$

where $d_N = \begin{array}{cc} \begin{pmatrix} 0\\ 0\\ \vdots\\ 1\\ \vdots\\ 0\\ 0 \end{pmatrix} & \begin{matrix} \\ \\ \\ j \\ \\ \\ \\ \end{matrix} \end{array} = e_j$ and $d_B = -\overline{A}_j = -A_B^{-1}A_j$.

Found d: $d \geq 0$, then

$$Ad = A_Bd_B + A_Nd_N = -A_BA_B^{-1}A_j + A_j = 0$$

and

$$c^Td = c_B^Td_B + c_N^Td_N = -c_B^TA_B^{-1}A_j + c_j = \overline{c}_j > 0$$

i.e.,

$$\left.\begin{aligned} c^Td &> 0\\ Ad &= 0\\ d &\geq 0 \end{aligned}\right. \implies \text{Problem is unbounded}$$

But wait, how to find an initial BFS?

Given

$$\begin{array}{ll} \max & c^Tx \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \end{array} \tag{LP}$$

where $b \geq 0$.

Construct auxiliary

$$\begin{array}{ll} \max & -e^Tw \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax + Iw = b \\ x, w \geq 0 \end{array} \end{array} \tag{AUX}$$

Note

- (AUX) is feasible $(x = 0, w = b)$
- (AUX) is bounded $-e^Tw \leq 0$

So (AUX) has an optimal solution.

Proposition 2.14

(AUX) has optimal value 0 iff (LP) is feasible.

Proof:

If optimal solution (x^*, w^*) has value 0, then $w^* = 0$ so $Ax^* + I0 = b$

$\implies$ x^* is feasible for (LP)

If x is feasible for (LP) then $(x, 0)$ has value 0 in (AUX).

Moreover, if optimal value of (AUX) is < 0, then we can use the dual for a certificate.

$$\begin{array}{ll} \min & y^T b \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} y^T A \geq 0 \\ y \geq -e \end{array} \end{array} \tag{DAUX}$$

y^* optimal ${y^*}^T b < 0$ and ${y^*}^T A \geq 0$

$\implies$ y^* satisfies $\{x : Ax = b,\ x \geq 0\} = \varnothing$ $\square$

2.9.4 Two Stage Simplex

Phase 1

- write (AUX)
- solve (AUX) with BFS corresponding to w
- if opt value < 0, get certificate y^* (LP) is infeasible
- opt value 0, BFS x where $w = 0$

Phase 2

- simplex with x as initial BFS

Example: 1

$$\begin{array}{ll} \max & \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} x \begin{array}{cc} \leq & -1 \\ \geq & 3 \end{array} \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 2 & 1 & 3 & 0 & 0 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 \\ 1 & 1 & 2 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array} \tag{SEF}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array} \tag{AUX}$$

canonical form: $B = \{6, 7\}$

$$\begin{array}{ll} \max & \begin{pmatrix} -1 & 0 & 2 & -1 & -1 & 0 & 0 \end{pmatrix} x - 4 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array}$$

add 3 to the basis

$\min \left(\frac{b_i}{a_{i3}} \right) = \frac{3}{2}$

7 leaves the basis.

canonical form for $B = \{3, 6\}$

$$\begin{array}{ll} \max & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix} x - 1 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1/2 & 1/2 & 1 & 0 & -1/2 & 0 & 1/2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} \end{array}$$

$x^* = \begin{pmatrix} 0 & 0 & \frac{3}{2} & 0 & 0 & 1 & 0 \end{pmatrix}$

certificate of infeasibility

$$\begin{aligned} y^T &= c_B^T A_B^{-1} \\ &= \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \end{pmatrix} \end{aligned}$$

Example: 2

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix} x = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

in SEF.

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \end{array}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \end{array} \tag{AUX}$$

canonical form $B = \{4, 5\}$

$$\begin{array}{ll} \max & \begin{pmatrix} 3 & 2 & 3 & 0 & 0 \end{pmatrix} x - 12 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

1 enters basis $x + \theta d \quad d = \begin{pmatrix} 1 & 0 & 0 & -2 & -1 \end{pmatrix}^T$

$\min \left(\frac{b_i}{a_{i1}} \right) = \frac{7}{2}$

4 leaves the basis

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1/2 & 3/2 & -3/2 & 0 \end{pmatrix} x - 3/2 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 1 \end{pmatrix} x = \begin{pmatrix} 7/2 \\ 3/2 \end{pmatrix} \\ & x \geq 0 \end{array}$$

2 enters the basis

$\min \left(\frac{b_i}{a_{i2}} \right) = \frac{3/2}{1/2}$

5 leaves the basis

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \end{pmatrix} x + 0 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 3 & -1 & 2 \end{pmatrix} x = \begin{pmatrix} 2 & 3 \end{pmatrix} \\ & x \geq 0 \end{array}$$

Thus $x = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 \end{pmatrix}$ is optimal for (AUX)

Forget (AUX). Start Simplex with $x = \begin{pmatrix} 2 & 3 & 0 \end{pmatrix}$ as initial BFS.

Now return to SEF.

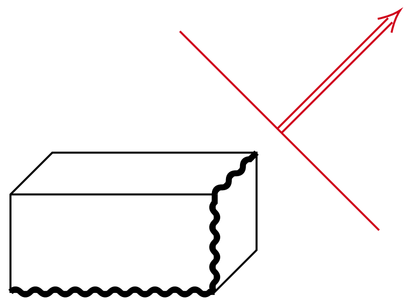
$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array} \tag{SEF}$$

canonical form for $B = \{1, 2\}$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 3 \end{pmatrix} x + 2 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array}$$

How long does simplex take?

At each pivot, we move from an extreme point to another.



Every pivot rule has a bad example.

Sprelman & Teng (2001): bad examples are pathological. Small changes become good examples.

Polynomial Hirsch Conjecture

~~Polynomially many vertex for bounded Polyhedral.~~

Let G be the graph of a d-polytope with n facets. Then the diameter of G is bounded above by a polynomial of d and n.

or

The (combinatorial) diameter of a polytope of dimension d with n facets cannot be greater than $n - d$.

Remark:
Here we call combinatorial diameter of a polytope the maximum number of steps needed to go from one vertex to another, where a step consists in traversing an edge.

What this conjecture tells us is that it will take only finitely many edges from initial BFS to optimal one.

There's one counterexample: 43-dimensional polytope with 86 facets and diameter (at least) 44.

2.10 Ellipsoid Algorithm

Feasibility Given polyhedron P, find $\overline{x} \in P$ or show $P = \varnothing$.

Fourier-Motzkin & simplex solve this problem.

Aside Given an algorithm an input I to it,

$$\text{size}(I) = \#\text{ of bits needed to represent } I.$$

> Example:
>
> $$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array}$$
>
> Assume $c \in \mathbb{Q}^n, A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^n$.
>
> By scaling, we may assume $c \in \mathbb{Z}^n, A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$.
>
> Let $\alpha = \max\{\|c\|_\infty, \|A\|_\infty, \|b\|_\infty\}$.
>
> Size of input to LP $\approx (n + n, m + m)\log(\alpha)$

Efficient Algorithm # of operations to solve an instance of size k are bounded by a polynomial on k.

Thus Simplex & FM NOT Efficient.

Goal Derive an efficient alg.

If you have an efficient algorithm to solve feasibility for any polyhedron P, can be used to solve LP.

Option 1

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

Assume I know $L \leq \text{OPT} \leq U$.

Algorithm 3: Option 1

```
while Repeat do
    V = (L + U)/2
    P' = { x : Ax ≤ b, c^T x ≥ V }
    if P' == ∅ then
        U ← V
    else
        L ← V
    end
end
```

Option 2

Is the following nonempty?

$$\left\{ x, y : \begin{array}{l} Ax \leq b \\ y^T A = c^T \\ y \geq 0 \\ c^T x = b^T y \end{array} \right\}$$

2.10.1 Ellipsoid

Ball $B(z, R) := \{x \in \mathbb{R}^n : \|x - z\| \leq R\}$

Unit Ball $B := B(0, 1)$

Apply an affine map to B.

$f(x) = A(x - b)$ where $b \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$ invertible

$$f(B) := \{x \in \mathbb{R}^n : \|f(x)\| \leq 1\} = \{x \in \mathbb{R}^n : \|A(x - b)\| \leq 1\}$$

Sets of this form are **Ellipsoid**. Denoted $E(A, b)$.

Idea

- Suppose I know $P \subseteq B(0, R)$
- Also, suppose either $P = \varnothing$ OR $\operatorname{Vol} P \geq \epsilon > 0$.

Algorithm 4: Ellipsoid Algorithm

$E \leftarrow E(M, z)$, where $P \subseteq E(M, z)$.
while $\operatorname{Vol}(E) \geq \epsilon$ **do**
  if $z \in P$ **then**
    STOP
  else
    • Find $\alpha^T x \leq \alpha_0$ so that $\alpha^T x \leq \alpha_0, \forall x \in P$ and $\alpha^T z > \alpha_0$
    • Find $E(M', z')$ such that $E \cap \{x : \alpha^T x \leq \alpha_0\} \subseteq E(M', z')$ and volume of $E(M', z')$ is much lower than E
    • $E \leftarrow E(M', z')$
  end
end

> **Note**
> At any point $P \subseteq E$.
>
> The reason why we choose ellipsoid instead of ball is that it can actually shrink "thinner" than ball.

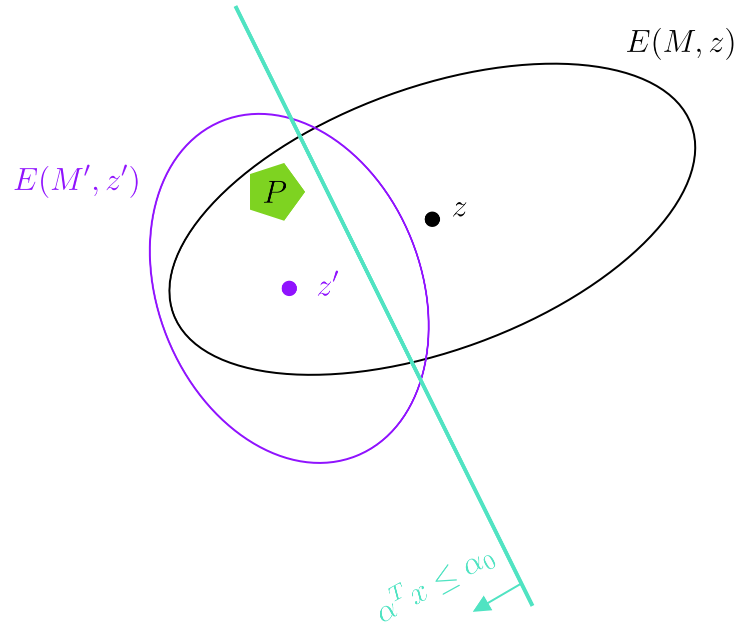


Figure 2.2: Ellipsoid Algorithm

Lemma 2.15

There exists $E(M', z')$ that can be computed in polynomial time such that

$$\frac{\text{Vol}(E(M', z'))}{\text{Vol}(E(M, z))} \leq e^{-\frac{1}{2n+2}}$$

Number of While Loop Iterations

If $B(0, R)$ initial ellipsoid, then $\text{Vol}(B(0, R)) \leq (2R)^n$. After $k(2n + 2)$ iterations, $\text{Vol}(E) \leq e^{-k}(2R)^n$.

We want

$$e^{-k}(2R)^n < \epsilon \implies -k + n\ln(2R) < \ln(\epsilon) \implies k \geq \lceil n\ln(2R) - \ln(\epsilon)\rceil$$

Alg stops after $\lceil n\ln(2R) - \ln(\epsilon)\rceil(2n + 2)$ iterations.

We only used that

$$z \notin P \iff \begin{array}{l} \exists \alpha^T x \leq \alpha_0 \text{ such that} \\ \quad \alpha^T \overline{x} \leq \alpha_0, \forall \overline{x} \in P \\ \quad \alpha^T z > \alpha_0 \end{array}$$

Theorem 2.16: Separating Hyperplane

Let C be a closed, convex set, $z \in \mathbb{R}^n$. Then $z \notin C \iff \exists$ a hyperplane $\alpha^T x \leq \alpha_0$ separating z and C.

Is runtime polynomial?

- $\ln(R)$ is polynomial in input size $\to$ NOT a problem
- Finding a separating hyperplane: can be done in polynomial time.

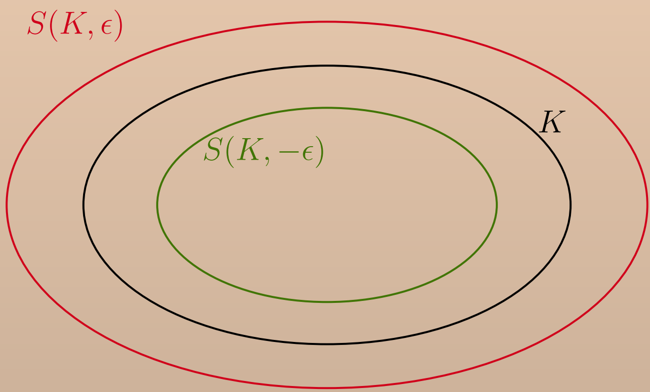
2.11 Grötchel-Lovász-Schrijver (GLS)

$S(K, \pm\epsilon)$

Let $K \subseteq \mathbb{R}^n$ be closed bounded convex set.

$$S(K,\epsilon) := \{x : \|x - y\| \leq \epsilon, \text{ for some } y \in K\}$$

$$S(K,-\epsilon) := \{x : S(x,\epsilon) \subseteq K\}$$



3 problems

- Optimization

  Given $K \subseteq \mathbb{R}^n$, $c \in \mathbb{Q}^n$.

  Find $x^* \in K$ such that
  $$c^T x^* \geq c^T x, \forall x \in K$$
  or determine $K = \varnothing$.

- Separation

  Given $K \subseteq \mathbb{R}^n$, $w \in \mathbb{R}^n$.

  Determine if $w \in K$ or find α:
  $$\|\alpha\|_\infty = 1 \qquad \alpha^T x < \alpha^T w, \forall x \in K$$

- FEASIBILITY

  Given $K \subseteq \mathbb{R}^n$.

  Find $\overline{x} \in K$ or determine $K = \varnothing$.

Feas $\leq_p$ Opt. (i.e. if we can solve opt efficiently, we can solve feas efficiently)

Weaker version...

- WEAK OPTIMIZATION

  Give $K \subseteq \mathbb{R}^n, c \in \mathbb{Q}^n, \epsilon > 0$

  Find $x^* \in S(K, \epsilon)$ such that

  $$c^T x \leq c^T x^* + \epsilon, \qquad \forall x \in S(K, -\epsilon)$$

  or determine $S(K, -\epsilon) = \varnothing$

- WEAK SEPARATION

  Given $K \subseteq \mathbb{R}^n, w \in \mathbb{R}^n, \epsilon > 0$.

  Determine if $w \in S(K, \epsilon)$ or find α:

  $$\|\alpha\|_\infty = 1 \qquad \alpha^T x < \alpha^T w + \epsilon, \forall x \in S(K, -\epsilon)$$

- WEAK FEASIBILITY

  Given $K \subseteq \mathbb{R}^n$.

  Determine $S(K, -\epsilon) = \epsilon$ or find $\overline{x} \in S(K, \epsilon)$

W-Feas $\leq_p$ W-Opt.

Ellipsoid gives us: W-Feas $\leq_p$ W-Sep.

- Grötchel-Lovász-Schrijver (GLS) have shown that

  W-SEP, W-Feas, W-OPT are polynomially equivalent.

In particular, for rational polyhedra[3] (even unbounded) then OPT, FEAS, SEP are polynomially equivalent.

Khachiyan ('80) used ellipsoid to give polytime algorithm for LPs.

2.11.1 Consequence of GLS

Example TSP: **complete** graph $G = (V, E)$

[3] $\{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$

Edge costs $c_e, \forall e \in E$.

Find a tour visiting every vertex exactly once of min cost.

IP formulation $x_e = \begin{cases} 1, & \text{if } e \text{ is in tour} \\ 0, & \text{otherwise} \end{cases}$

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \downarrow & \\ \text{s.t.} & \sum_{e \in \delta(v)} x_e = 2, \quad \forall v \in V \end{array}$$

In general, $\delta(S) = \left\{ uv \in E : \begin{array}{c} u \in S \\ v \notin S \end{array} \right\}$ where $S \subseteq V$.

Subtour elimination $\displaystyle\sum_{e \in \delta(S)} x_e \geq 2, \quad \forall \varnothing \subsetneq S \subsetneq V$

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ \downarrow & & \\ & \sum_{e \in \delta(v)} x_e = 2, & \forall v \in V \\ \text{s.t.} & \sum_{e \in \delta(S)} x_e \geq 2, & \forall \varnothing \subsetneq S \subsetneq V \\ & x_e \in \{0,1\}, & \forall e \in E \end{array}$$

LP-relaxation Replace $x_e \in \{0,1\}$ by $0 \leq x_e \leq 1, \forall e \in E$.

Can I solve the LP in polynomial time on # vertices/edges?

Separation/Feasibility Given $\overline{x}_e, \forall e \in E$. Can I know if $\overline{x}_e$ if feasible for LP in time polynomial in # vertices?

If YES, GLS tells we can also solve OPT.

In polytime (in # vertices) I can check $\begin{cases} \sum_{e \in \delta(v)} \overline{x}_e = 2, & \forall v \in V \\ 0 \leq \overline{x}_e \leq 1, & \forall e \in E \end{cases}$

Min-Cut problem Given $G = (V, E), w_e \geq 0$. Find $\displaystyle\sum_{e \in \delta(S)} w_e$

Problem can be solved in polytime in # vertices.

Then we solve mincut with $w_e = \overline{x}_e$. If optimal value is ≥ 2, then $\overline{x}$ feasible for LP. Otherwise found $S : \displaystyle\sum_{e \in \delta(S)} \overline{x}_e < 2$.

3 Integer Programming

An integer program is a problem of the form:

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax \leq b \\ x_i \in \mathbb{Z}, \forall j \in I \end{array} \end{array}$$

where $\varnothing \neq I \subseteq \{1, \ldots, n\}$.

If $I = \{1, \ldots, n\}$, it's pure IP. Otherwise, Mixed IP (MIP).

If all variables are constrained to be in $\{0, 1\}$, it's a Binary IP.

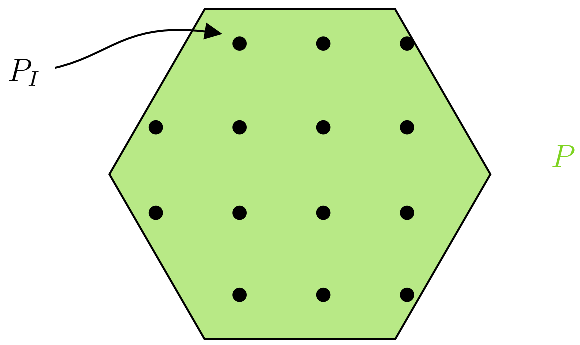
Key Assumption: All data is rational ($A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$) i.e, $Ax \leq b$ is a rational polyhedron.

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, $P_I = P \cap \{x_j \in \mathbb{Z} : j \in I\}$.

Theorem 3.1

$conv(P_I)$ is a polyhedron.

From now on, assume we have a pure IP.



recession cone

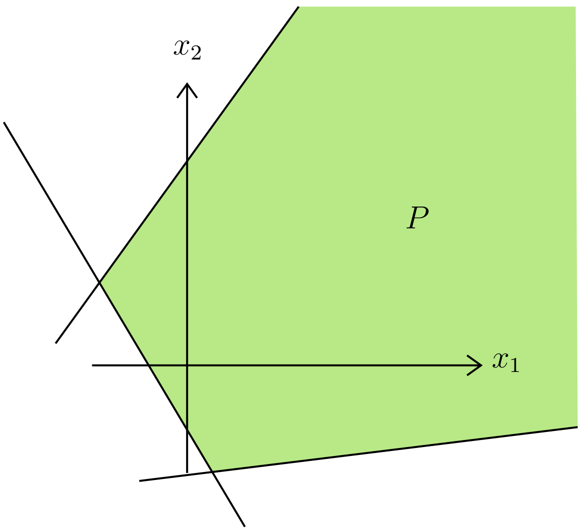
Let P be a polyhedron. Its recession cone is

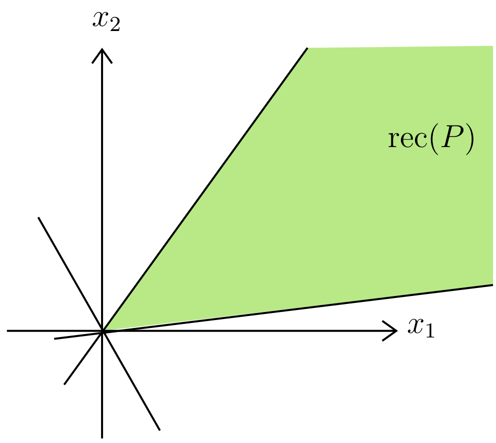
$$\operatorname{rec}(P) := \left\{ r \in \mathbb{R}^n : \begin{array}{l} \forall \overline{x} \in P \\ \forall \lambda \geq 0 \\ \overline{x} + \lambda r \in P \end{array} \right\}$$

Lemma 3.2

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \varnothing$ then

$$\underbrace{\operatorname{rec}(P)}_{R_1} = \underbrace{r \in \mathbb{R}^n : Ar \leq 0}_{R_2}$$





Proof:

$R_2 \subseteq R_1$) Let $\overline{x} \in P, \lambda \geq 0, r \in R_2$

$$A(\overline{x} + \lambda r) = A\overline{x} + \lambda A r \leq b \implies \overline{x} + \lambda r \in P \implies r \in R_1$$

$R_1 \subseteq R_2$) Let $r \notin R_2$, i.e., $\exists i : a_i^T r > 0$

Let $\overline{x} \in P$, it is clear $\exists \lambda > 0 : a_i^T(\overline{x} + \lambda r) > b_i \implies r \notin R_1$.

$\square$

Theorem 3.3

$P \neq \varnothing$ is a bounded polyhedron

$\iff P = conv(x^1, \ldots, x^k)$ for some vectors $x^1, \ldots, x^k \in \mathbb{R}^n$.

$conv(x^1, \ldots, x^k)$ is smallest convex set containing $x^1, \ldots, x^k \iff$ set of all finite combinations of $x^1, \ldots, x^k$.

Proof:

$\Leftarrow$) $P = \left\{ x \in \mathbb{R}^n : \begin{array}{r} x = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \geq 0 \end{array} \right\}$

$P' = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^k : \begin{array}{r} x = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \geq 0 \end{array} \right\}$ is a bounded polyhedron.

$P = \text{proj}_x P'$ which is a bounded polyhedron.

$\Rightarrow$) P bounded $\implies$ P has no lines.

Let $x^1, \ldots, x^k$ be extreme points. Want to show $P = conv(x^1, \ldots, x^k)$

$P \supseteq conv(x^1, \ldots, x^k)$ follows since P is a convex set containing $x^1, \ldots, x^k$.

Suppose $\exists \overline{x} \in P \setminus conv(x^1, \ldots, x^k)$

Consider

$$\begin{array}{llll} \min & 0^T \lambda & & \\ \downarrow & & & \\ & \sum_{i=1}^k \lambda_i x^i & = \overline{x} & \alpha \in \mathbb{R}^n \\ \text{s.t.} & \sum_{i=1}^k \lambda_i & = 1 & \alpha_0 \in \mathbb{R} \\ & \lambda & \geq 0 & \end{array} \tag{1}$$

and its dual

$$\begin{array}{ll} \max & \alpha^T \overline{x} + \alpha_0 \\ \text{s.t.} & \alpha^T x^i + \alpha_0 \leq 0, \quad \forall i = 1, \ldots, k \end{array} \tag{2}$$

$(\alpha, \alpha_0) = (0, 0)$ feasible for (2). By assumption, (1) is infeasible.

Let $(\overline{\alpha}, \overline{\alpha}_0)$ be such that $\overline{\alpha}^T \overline{x} + \overline{\alpha}_0 > 0$

Now consider

$$\begin{array}{ll} \max & \overline{\alpha}^T x + \overline{\alpha}_0 \\ \text{s.t.} & x \in P \end{array} \tag{3}$$

(3) has optimal solution since $P \neq \varnothing$ bounded and its has an optimal extreme point, i.e., $\overline{\alpha}^T x^i + \overline{\alpha}_0$ is optimal value. But by (2)

$$\overline{\alpha}^T x^i + \overline{\alpha}_0 \leq 0 < \overline{\alpha}^T \overline{x} + \overline{\alpha}_0$$

Contradiction.

$\square$

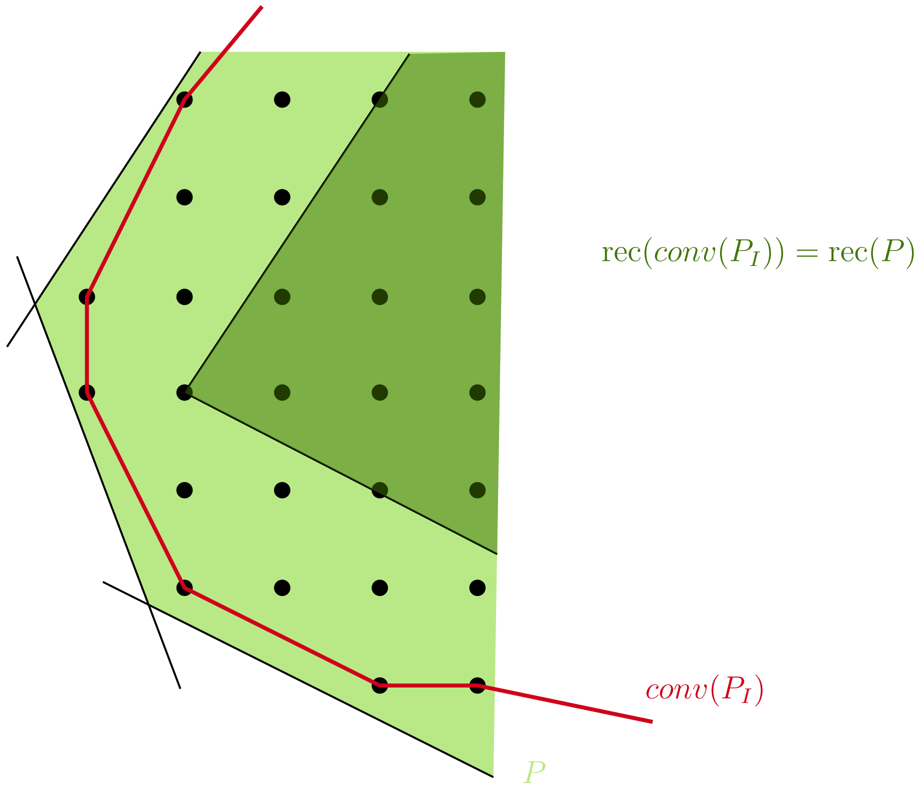
Back to IP...

Theorem 3.4

If P is a rational polyhedron, then $conv(P_I)$ is also a rational polyhedron ($P_I = P \cap \mathbb{Z}^n$). Moreover, if $P_I \neq \varnothing$, $\text{rec}(conv(P_I)) = \text{rec}(P)$.

Proof:
Done if P is bounded ($\{0\}$).

Skipped for unbounded P.



□

Theorem 3.5

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I \end{array} = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & conv(P_I) \end{array}$$

Note

1. Using Fund Thm of LP. I know IP is either infeas., unbounded, or $\exists$ opt. sol.

2. If $P_I \neq \varnothing$, then unboundedness can be detected by checking if $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$ is unbounded. Since $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$ unbounded iff $P \neq \varnothing$ and $\exists r : \begin{array}{l} c^T r > 0 \\ Ar \leq 0 \end{array}$.

$P_I \neq \varnothing \implies P \neq \varnothing$. But then this implies $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in conv(P_I) \end{array}$ unbounded.

Proof:

WMA $P_I \neq \varnothing$.

Let $z_1 = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I \end{array}$, $z_2 = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in conv(P_I) \end{array}$.

Since $P_I \subseteq conv(P_I) \implies z_1 \leq z_2$.

Now let $x^* \in conv(P_I) \implies \begin{array}{r} x^* = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \geq 0 \end{array}$ for $x^1, \dots, x^k \in P_I$.

$\implies \exists i : c^T x^i \geq c^T x^*$ since otherwise

$$c^T x^* = \sum_{i=1}^k \lambda_i (c^T x^*) > \sum_{i=1}^k \lambda_i (c^T x^i) = c^T \left(\sum_{i=1}^k \lambda_i x^i \right) = c^T x^*$$

contradiction $\implies z_1 \geq z_2$. □

Corrollary 3.6

If $P \neq \varnothing$ and pointed. Then $conv(P_I)$ is pointed and any extreme point of $conv(P_I)$ is integral.

Proof:

$\text{rec}(P) = \text{rec}(conv(P_I))$ implies $conv(P_I)$ pointed.

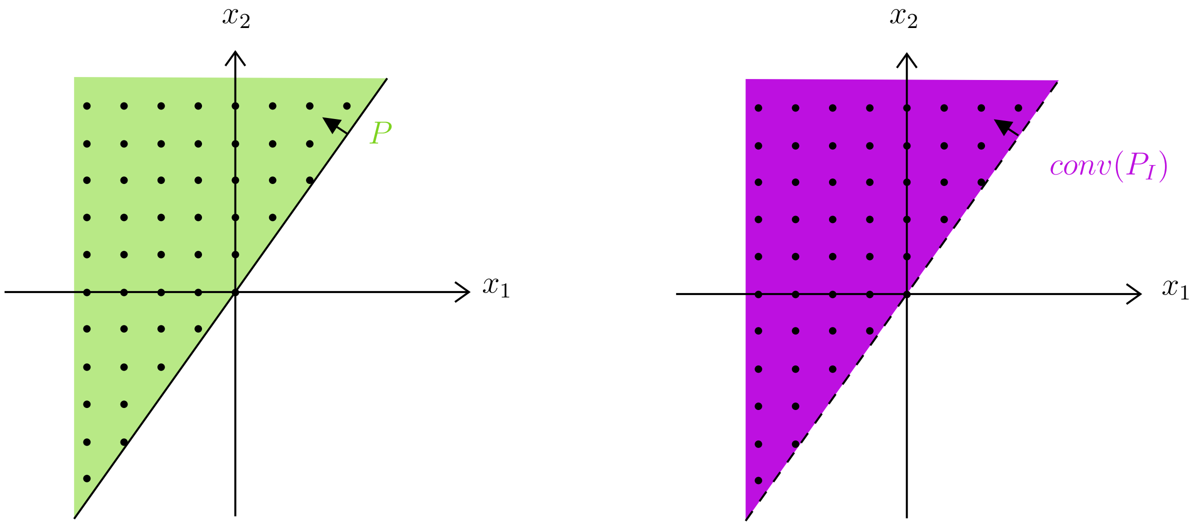
Let x^* be extreme point of $conv(P_I)$. Let c be such that x^* is unique optimal solution to $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in conv(P_I) \end{array}$.

By theorem, $\exists \overline{x} \in P_I : c^T \overline{x} = c^T x^*$.

By uniqueness of x^*, $\overline{x} = x^*$, then x^* is integral. □

Note

$P = \{x \in \mathbb{R}^2 : x_2 \geq \sqrt{2} x_1\}$



$conv(P_I)$ is not even closed (dotted line plus $(0,0)$), NOT a polyhedron.