



Introduction to Optimization

CO 255



Ricardo Fukasawa

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 255 during Winter 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

For any questions, send me an email via <https://notes.sibeliusp.com/contact/>.

You can find my notes for other courses on <https://notes.sibeliusp.com/>.

Sibeliusp Peng

Contents

| | |
|--|----------|
| Preface | 1 |
| 0 Info | 3 |
| 1 Introduction | 4 |
| 2 Linear Optimization | 6 |
| 2.1 Determining Feasibility | 7 |
| 2.2 Fourier-Motzkin Elimination | 8 |
| 2.3 Certifying Optimality | 13 |
| 2.4 Possible Outcomes | 16 |
| 2.5 Duals of generic LPs | 16 |
| 2.5.1 Cheat Sheet | 17 |
| 2.6 Other interpretations of dual | 19 |
| 2.7 Complementary Slackness | 21 |
| 2.7.1 Geometric Interpretation of C.S. | 23 |
| 2.8 Geometry of Polyhedra | 24 |
| 2.9 Simplex Algorithm | 29 |
| 2.9.1 CANONICAL FORM | 31 |

Info

Ricardo: MC 5036. OH: M 1:30 - 3pm
TA: Adam Brown: MC 5462. OH: F 10-11am

Books (not required)

- Intro to Linear Opt. Bertsimas
- Int Programming. Conforti

Grading

- assns: 20% (≈ 5)
- mid: 30% (Feb 11 in class)
- final: 50%

Introduction

Given a set S , and a function $f : S \rightarrow \mathbb{R}$. An optimization problem is:

$$\begin{array}{ll} \max f(x) \\ \underbrace{s.t.}_{\text{subject to}} x \in S & (\text{OPT}) \end{array}$$

- S **feasible region**
- A point $\bar{x} \in S$ is a **feasible solution**
- $f(x)$ is **objective function**

(OPT) means: “Find a feasible solution x^* such that $f(x) \leq f(x^*), \forall x \in S$ ”

- Such x^* is an **optimal solution**
- $f(x^*)$ is **optimal value**

Other ways to write (OPT):

$$\max\{f(x), x \in S\}$$

$$\max_{x \in S} f(x)$$

Analogous problem

$$\begin{array}{ll} \min f(x) \\ s.t. \quad x \in S \end{array}$$

Note

$$\max_{s.t. \quad x \in S} f(x) = -1 \left(\min_{s.t. \quad x \in S} -f(x) \right)$$

Problem x^* may not exist

a) Problem is unbounded:

$$\forall M \in \mathbb{R}, \exists \bar{x} \in S, \text{ s.t. } f(\bar{x}) > M$$

b) $S = \emptyset$, i.e. (OPT) is **INFEASIBLE**

c) There may not exist x^* achieving supremum.

Example:

$$\begin{array}{ll} \max & x \\ \text{s.t.} & x < 1 \end{array}$$

supremum

$$\sup\{f(x) : x \in S\} = \begin{cases} +\infty & \text{if OPT unbounded} \\ -\infty & \text{if } S = \emptyset \\ \min\{x : x \geq f(x), \forall x \in S\} & \text{otherwise} \end{cases}$$

always exist and are well-defined

infimum

$$\inf\{f(x) : x \in S\} = -1 \cdot \sup\{-f(x) : x \in S\}$$

From this point on, we will abuse notation and say $\max\{f(x) : x \in S\}$ is $\sup\{f(x) : x \in S\}$.

One way to specify that I want an opt. sol. (if exists) is

$$x^* \in \operatorname{argmax}\{f(x) : x \in S\}$$

2

Linear Optimization (Programming) (LP)

$$S = \{x \in \mathbb{R}^n : Ax \leq b\}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $f(x) = c^T x$, $c \in \mathbb{R}^n$.

$$\begin{array}{ll} \downarrow \\ \max c^T x \\ \text{s.t. } Ax \leq b \end{array} \quad (LP)$$

Note

$$A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix} \quad A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

Clarifying

$$u, v \in \mathbb{R}^n, \quad u \leq v \iff u_j \leq v_j, \forall j \in 1, \dots, n$$

Note

$u \not\leq v$ is not the same as $u > v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \not\leq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example:

$$\begin{array}{ll} \max & 2x_1 + 0.5x_2 \\ \text{s.t.} & x_1 \leq 2 \\ & x_1 + x_2 \leq 2 \\ & x \geq 0 \end{array}$$

- Strict ineq. not allowed

halfspace, hyperplane, polyhedron

Let $h \in \mathbb{R}^n, h_0 \in \mathbb{R}$.

$\{x \in \mathbb{R}^n : h^T x \leq h_0\}$ is a **halfspace**.

$\{x \in \mathbb{R}^n : h^T x = h_0\}$ is a **hyperplane**.

$Ax \leq b$ is a **polyhedron** (i.e. intersection of finitely many halfspaces).

Example:

n products, m resources. Producing $j \in \{1, \dots, n\}$ given c_j profit/unit and consumes a_{ij} units of resource i , $\forall i \in \{1, \dots, m\}$. There are b_i units available $\forall i \in \{1, \dots, m\}$.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

which is an LP.

2.1 Determining Feasibility

Given a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

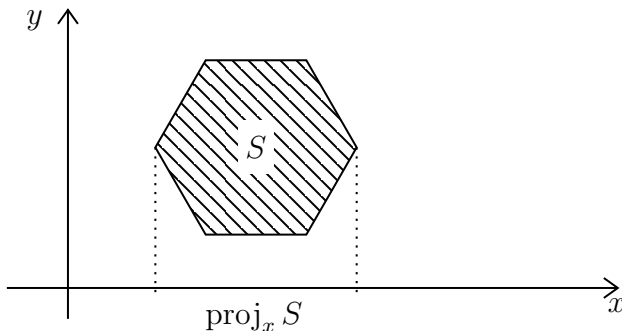
either find $\bar{x} \in P$ or show $P = \emptyset$.

Idea In 1-d, easy. \rightarrow Reduce problem in dimension n to one in dimension $n - 1$.

Notation Let $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \leq b\}$, then

$$\text{proj}_x S := \{x \in \mathbb{R}^n : \exists y \text{ so that } (x, y) \in S\}$$

is the (orthogonal) *projection* of S onto x .



We will find if $P = \emptyset$ by looking at $\text{proj}_{x_1, \dots, x_{n-1}} \quad (P)$

2.2 Fourier-Motzkin Elimination

Call a_{ij} entries of A . Let

$$\begin{aligned} M &:= \{1, 2, \dots, m\} \\ M^+ &:= \{i \in M : a_{in} > 0\} \\ M^- &:= \{i \in M : a_{in} < 0\} \\ M^0 &:= \{i \in M : a_{in} = 0\} \end{aligned}$$

For $i \in M^+$ (1):

$$a_i^T x \leq b_i \iff \sum_{j=1}^n a_{ij} x_j \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j + x_n \leq \frac{b_i}{a_{in}}, \quad \forall i \in M^+$$

For $i \in M^-$ (2):

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j - x_n \leq \frac{b_i}{-a_{in}}, \quad \forall i \in M^-$$

For $i \in M^0$ (3):

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} a_{ij} x_j \leq b_i, \quad \forall i \in M^0$$

$$P = \{x \in \mathbb{R}^n : (1)(2)(3)\}$$

Define (4):

$$\sum_{j=1}^{n-1} \left(\frac{a_{ij}}{a_{in}} - \frac{a_{kj}}{a_{kn}} \right) x_j \leq \frac{b_i}{a_{in}} - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, \forall k \in M^-$$

Theorem 2.1

$$(\bar{x}_1, \dots, \bar{x}_{n-1}) \text{ satisfies (3), (4)} \iff \exists \bar{x}_n : (\bar{x}_1, \dots, \bar{x}_n) \in P$$

Proof:

\Leftarrow If $(\bar{x}_1, \dots, \bar{x}_n)$ satisfies (1), (2), (3) then $(\bar{x}_1, \dots, \bar{x}_{n-1})$ satisfies (3) and adding (1), (2) $\implies (\bar{x}_1, \dots, \bar{x}_{n-1})$ satisfies (4)

\implies If $(\bar{x}_1, \dots, \bar{x}_{n-1})$ satisfies (4)

$$\sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, k \in M^-$$

Let

$$\bar{x}_n := \max_{i \in M^+} \left\{ \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \right\}$$

$$\implies \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq -\bar{x}_n, \quad \forall i \in M^+$$

and

$$-\bar{x}_n \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall k \in M^-$$

$$\implies (\bar{x}_1, \dots, \bar{x}_n) \in P$$

□

Note

Proof assumes M^+, M^- are nonempty. But statement holds regardless.

(if M^+ or $M^- = \emptyset$ then (4) yields no constraints)

Fourier-MotzKin

- $A^n = A, b^n = b$
- given A^i, b^i obtain A^{i-1}, b^{i-1} (A^{i-1} has one less column than A^i column than A^i) by applying the steps described

$$P_i := \{x \in \mathbb{R}^i : A^i x \leq b^i\}$$

then

$$P_{i-1} = \text{proj}_{x_1, \dots, x_{i-1}} P_i$$

$$\text{and } P_{i-1} = \emptyset \iff P_i = \emptyset.$$

- Keep applying projection until $i = 1$.

$$P_0 = \emptyset \iff P_n = P = \emptyset$$

Let

$$P_i^n = P_i \times \mathbb{R}^{n-i} = \{x \in \mathbb{R}^n (A^i, 0)x \leq b^i\}$$

$$\text{not hard to see } P_i^n = \emptyset \iff P_i = \emptyset$$

Notice that

$$P_0 = \emptyset \iff P_0^n = \emptyset, P_0^n = \{0 \leq b^0\}$$

Example:

$$P_2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{rcl} x_1 & +x_2 & \leq 1 \\ -x_1 & & \leq 0 \\ & -x_2 & \leq -2 \\ -3x_1 & -3x_2 & \leq -6 \end{array} \right\}$$

draw the graph, clearly empty

$$M^+: \frac{1}{2}x_1 + x_2 \leq \frac{1}{2}$$

$$M^-: -x_2 \leq -2 \quad -x_1 - x_2 \leq -2$$

$$M^0: -x_1 \leq 0$$

$$P_1 = \left\{ x_1 \in \mathbb{R} : \begin{array}{rcl} & -x_1 & \leq 0 \\ \frac{1}{2}x_1 & & \leq -\frac{3}{2} \\ & -\frac{1}{2}x_1 & \leq -\frac{3}{2} \end{array} \right\}$$

$$M^+: x_1 \leq -3$$

$$M^-: -x_1 \leq 0 \text{ and } -x_1 \leq -3$$

$$P_0^2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{rcl} 0 & \leq & -3 \\ 0 & \leq & -6 \end{array} \right\} = \emptyset$$

Here $b^0 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

Remark:

Inequality in P_i^n :

- All inequalities are obtained by a nonnegative combination of inequality in P_{i+1}^n
 \implies all nonnegative combination of inequalities in P .
- If all A, b are rational then so are all A^i, b^i
- If $b = 0, b_i = 0, \forall i$

Theorem 2.2: Farkas' Lemma

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = \emptyset \iff \begin{array}{l} \exists u \in \mathbb{R}^m : u^T A = 0 \\ u^T b < 0 \\ u \geq 0 \end{array}$$

Proof: (\Leftarrow) Suppose \bar{x} satisfies $A\bar{x} \leq b$.

$$0 = u^T A\bar{x} \leq u^T b < 0$$

which is impossible.

 (\Rightarrow) If $P = \emptyset$. Apply Fourier-Motzkin until we get

$$P_0^n = \emptyset = \{x \in \mathbb{R}^n : 0x \leq b^0\}$$

i.e. there exists j for which $b_j^0 < 0$.If we look at corresponding constraint in P_0^n is

$$0^T x \leq b_j^0$$

which can be obtained by a vector u such that $u^T A = 0, u^T b = b_j^0, u \geq 0$.

□

Farkas' Lemma (alternate statement)

Exactly one of the following has a solution:

a) $Ax \leq b$

$u^T A = 0$

b) $u^T b < 0$

$u \geq 0$

Farkas' Lemma (Different Form)

Exactly one of the following has a solution:

a) $Ax = b$

$x \geq 0$

b) $u^T A \geq 0$

$u^T b < 0$

Proof:

(Sketch)

$$P = \left\{ x : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} = \left\{ x : \underbrace{\begin{pmatrix} A \\ -A \\ -I \end{pmatrix}}_{A'} x \leq \underbrace{\begin{pmatrix} b \\ -b \\ -0 \end{pmatrix}}_{b'} \right\}$$

Apply original Farkas' Lemma to get $P = \emptyset \iff \exists u_1 \in \mathbb{R}^m, u_2 \in \mathbb{R}^m, v \in \mathbb{R}^n$:

$$\begin{aligned} u_1^T A - u_2^T A - v &= 0 \\ u_1^T b - u_2^T b &< 0 \\ u_1, u_2, v &\geq 0 \end{aligned}$$

Let $u = (u_1 - u_2)$

$$u^T A - v = 0 \implies u^T A \geq 0, \quad u^T b < 0$$

□

Consider a linear programming (LP):

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \quad (LP)$$

Theorem 2.3: Fundamental Theorem of Linear Programming

(LP) has exactly one of 3 outcomes:

- a) Infeasible
- b) Unbounded
- c) There exists an optimal solution.

Proof:

Let's assume a), b) don't hold.

If $n = 1$, then (LP) has an optimal solution. (Why?)

Else, define

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c^T x \leq 0 \\ & Ax \leq b \end{aligned} \quad (LP')$$

(LP') is also not in case a) or b). (Why?)

Also if (x^*, z^*) is an optimal solution to (LP'), then x^* is an optimal solution to (LP). (Why?)

Apply Fourier-Motzkin to

$$\left\{ (x, z) : \begin{aligned} z - c^T x &\leq 0 \\ Ax &\leq b \end{aligned} \right\}$$

Until we are left with a polyhedron

$$\{z \in \mathbb{R} : A'z \leq b'\}$$

Now $\max z \quad \text{s.t.} \quad A'z \leq b'$ is not cases a) or b). (Why?)

→ can get an optimal solution z^* to such problem. Apply Fourier-Motzkin back to get (x^*, z^*) optimal solution to (LP'). (Why?) \square

2.3 Certifying Optimality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \quad (LP)$$

and let $\bar{x} \in P = \{x : Ax \leq b\}$

Question Can we certify that \bar{x} is optimal?

Example:

$$\begin{array}{ll} \max & 2x_1 + x_2 \\ & x_1 + 2x_2 \leq 2 \\ \text{s.t.} & x_1 + x_2 \leq 2 \\ & x_1 - x_2 \leq 0.5 \end{array}$$

Consider $\bar{x} = (0, 1)^T$ is clearly NOT optimal.

$x^* = (1, 0.5)^T$ and $c^T x^* = 2.5$. Any feasible solution satisfies

$$\begin{array}{rcl} x_1 + 2x_2 & \leq 2 & \times 1/3 \\ x_1 + x_2 & \leq 2 & \times 1 \\ + \quad x_1 - x_2 & \leq 0.5 & \times 2/3 \\ \hline 2x_1 + x_2 & \leq 3 & \end{array}$$

Instead do $1 \times 1st$ constraint $+ 1 \times 3rd$ constraint $\implies 2x_1 + x_2 \leq 2.5$

In general:

$$\begin{array}{rcl} x_1 + 2x_2 & \leq 2 & \times y_1 \\ x_1 + x_2 & \leq 2 & \times y_2 \\ + \quad x_1 - x_2 & \leq 0.5 & \times y_3 \\ \hline (y_1 + y_2 + y_3)x_1 + (2y_1 + y_2 - y_3)x_2 & \leq & 2y_1 + 2y_2 + 0.5y_3 \end{array}$$

As long as $y_1, y_2, y_3 \geq 0$ and

$$\begin{array}{l} y_1 + y_2 + y_3 = 2 \\ 2y_1 + y_2 - y_3 = 1 \end{array}$$

This leads to the following linear program:

$$\begin{array}{ll} \min & 2y_1 + 2y_2 + 0.5y_3 \\ & y_1 + y_2 + y_3 = 2 \\ \text{s.t.} & 2y_1 + y_2 - y_3 = 1 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

This is called the dual LP.

In general:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \quad (P)$$

Dual of (P)

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & y^T A = c^T \\ & y \geq 0 \end{aligned} \quad (D)$$

Remark:

We call (P) primal LP.

Theorem 2.4: Weak Duality

Let \bar{x} feasible for (P), \bar{y} feasible for (D). Then $c^T \bar{x} \leq b^T \bar{y}$.

Proof:

$$c^T \bar{x} = \bar{y}^T (A\bar{x}) \leq \bar{y}^T b$$

where we used $A\bar{x} \leq b$ and $\bar{y} \geq 0$. □

Corollary 2.5

Several results:

- If (P) is unbounded then (D) is infeasible.
- If (D) is unbounded then (P) is infeasible.

Note

(P) and (D) can both be infeasible.

- If \bar{x} is feasible for (P) \bar{y} feasible for (D) $c^T \bar{x} = b^T \bar{y}$, then \bar{x} optimal for (P), \bar{y} optimal for (D).

Theorem 2.6: Strong Duality

x^* is optimal for (P) $\iff \exists y^*$ feasible for (D) such that $c^T x^* = b^T y^*$.

Proof:

(\Leftarrow) ✓

(\Rightarrow) Is (D) infeasible?

$$\text{Suppose } \left\{ y \in \mathbb{R}^n : \begin{array}{l} A^T y = c \\ y \geq 0 \end{array} \right\} = \emptyset$$

(Alternate version of Farkas' Lemma) $\exists u : \begin{matrix} u^T A \geq 0 \\ u^T c < 0 \end{matrix} \iff \exists d : \begin{matrix} Ad \leq 0 \\ c^T d > 0 \end{matrix}$

Take look at $x' = x^* + d$, then

$$\begin{aligned} Ax' &= Ax^* + Ad \leq b \\ c^T x' &= c^T x^* + c^T d > c^T x^* \end{aligned}$$

Contradiction. Thus (D) has an optimal solution y^* .

Now let $\gamma = b^T y^*$, and let $\theta := \left\{ x \in \mathbb{R}^n : \begin{matrix} Ax \leq b \\ -c^T x \leq -\gamma \end{matrix} \right\}$.

If $\theta = \emptyset$, by Farkas'

$$\exists \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} : \begin{cases} \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix}^T \begin{pmatrix} A \\ -c^T \end{pmatrix} = 0 \\ \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix}^T \begin{pmatrix} b \\ -\gamma \end{pmatrix} < 0 \\ \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} \geq 0 \end{cases} \iff \begin{aligned} A^T \bar{y} &= c \bar{\lambda} \\ b^T \bar{y} &< \gamma \bar{\lambda} \\ \bar{y} &\geq 0 \\ \bar{\lambda} &\geq 0 \end{aligned}$$

Case 1: $\bar{\lambda} > 0$.

Let $y' = \frac{\bar{y}}{\bar{\lambda}}$. Then we have

$$A^T y' = A^T \frac{\bar{y}}{\bar{\lambda}} = c \quad \text{and} \quad b^T y' = b^T \frac{\bar{y}}{\bar{\lambda}} < \gamma \quad \text{and} \quad y' = \frac{\bar{y}}{\bar{\lambda}} \geq 0$$

Contradicts optimality of y^* .

$$A^T y = 0$$

Case 2: $\bar{\lambda} = 0$. Then $b^T y < 0$

$$\bar{y} \geq 0$$

Now we can do the same thing previously. Let $y' = y^* + \bar{y}$, then

$$A^T y' = A^T y^* + A^T \bar{y} = c$$

and

$$\begin{aligned} y' &= y^* + \bar{y} \geq 0 \\ b^T y' &= b^T y^* + b^T \bar{y} < b^T y^* \end{aligned}$$

Contradicts optimality of y^* .

Thus $\theta \neq \emptyset$.

Let $\bar{x} \in \theta$,

$$c^T x^* \underbrace{\leq}_{\text{weak duality}} b^T y^* = \gamma \underbrace{\leq}_{\bar{x} \in \theta} c^T \bar{x} \leq c^T x^*$$

where the last inequality is because \bar{x} feasible for (P), x^* optimal for (P).

□

2.4 Possible Outcomes

See [here](#).

2.5 Duals of generic LPs

$$\begin{array}{ll} \max & 2x_1 + 3x_2 - 4x_3 \\ & x_1 \quad \quad + 7x_3 \leq 5 \\ & \quad 2x_2 - x_3 \geq 3 \\ \text{s.t.} & x_1 \quad \quad + x_3 = 8 \\ & \quad x_2 \leq 6 \\ & x_1 \geq 0 \\ & \quad x_2 \leq 0 \end{array}$$

$$\begin{array}{ll} \max & (2, 3, -4)x \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 7 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 5 \\ -3 \\ 8 \\ -8 \\ 6 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

and dual

$$\begin{array}{ll} \min & (5, -3, 8, -8, 6, 0, 0)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0 \end{array} \quad (D_1)$$

$$\begin{array}{ll} \min & (5, -3, 8, -8, 6)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 \end{pmatrix} y \begin{array}{l} \geq \\ \leq \\ = \end{array} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0 \end{array} \quad (D_2)$$

Claim (y_1^*, \dots, y_5^*) is optimal for $(D_2) \iff (y_1^*, \dots, y_5^*, y_6^*, y_7^*)$ optimal for (D_1) with

$$\begin{aligned} y_6^* &= y_1^* + y_3^* - y_4^* - 2 \\ y_7^* &= 3 - (-2y_2^* + y_5^*) \end{aligned}$$

$$\begin{aligned} \min \quad & (5, 3, 8, 6)y \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 7 & -1 & 1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y_1 \geq 0, y_2 \leq 0 \quad y_4 \geq 0 \end{aligned} \quad (D_3)$$

Claim Opt value of (D_2) and (D_3) are same.

In general

$$\begin{array}{l|l} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \quad (P) \quad \left| \quad \begin{array}{l|l} \min & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array} \quad (D)$$

2.5.1 Cheat Sheet

Here or

| Primal (max) | | Dual (min) | |
|--------------|--------|------------|------------|
| Constraint | \leq | ≥ 0 | Variable |
| | \geq | ≤ 0 | |
| | $=$ | free | |
| Variable | \geq | ≥ 0 | Constraint |
| | \leq | ≤ 0 | |
| | free | $=$ | |

Remark:

This is not symmetric... The way you can remember it is by thinking natural variables in real life, like you cannot have negative number of cars and so on...

Q What if you start with a minimization LP as primal?

Example:

$$\begin{aligned} \min \quad & x_1 - x_2 \\ & 2x_1 + 3x_2 \leq 5 \\ \text{s.t.} \quad & x_1 - x_2 \geq 3 \\ & x_1 + 5x_2 = 7 \\ & x_1 \geq 0, x_2 \leq 0 \end{aligned} \quad (P)$$

Rewrite as:

$$-1 \times \begin{pmatrix} \max & -x_1 + x_2 \\ \downarrow \\ \text{s.t.} & \dots \end{pmatrix}$$

Will lead to finding dual:

$$\begin{array}{ll} \max & 5y_1 + 3y_2 + 7y_3 \\ \downarrow & \\ & 2y_1 + y_2 \leq 1 \\ \text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\ & y_1 \leq 0, y_2 \geq 0, y_3 \text{ free} \end{array}$$

Also

- Weak duality holds.

If \bar{x} feasible for (P), \bar{y} feasible for (D), then $c^T \bar{x} \geq b^T \bar{y}$.

- Strong duality holds

Note

The dual of the dual of (P) is (P).

Example:

Given a simple undirected graph $G = (V, E)$. $M \subseteq E$ is a *matching* if every vertex $v \in V$ is incident to ≤ 1 edge in M .

See examples of matching in [CO 342](#) or [MATH 249](#).

Max cardinality matching

Find matching M with largest $|M|$.

Define $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{otherwise} \end{cases}$.

$$\begin{array}{ll} \max & \sum_{e \in E} x_e \\ \downarrow & \\ & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\ \text{s.t.} & \\ & 0 \leq x_e, \quad \forall e \in E \end{array}$$

where $\delta(v)$ = set of edges in E incident to v .

$$\begin{array}{ll} \min & \sum_{v \in V} y_v \\ \downarrow & \\ \text{s.t.} & y_u + y_v \geq 1, \quad \forall e = uv \in E \\ & y \geq 0 \end{array}$$

2.6 Other interpretations of dual

Example:

| | | | Resources | |
|---------------------|---|-----------------|----------------------|----|
| | | Per unit Profit | Per unit consumption | |
| | | | A | B |
| Product | 1 | 5 | 2 | 3 |
| | 2 | 3 | 4 | 1 |
| Available Resources | | | 15 | 10 |

$$\begin{aligned}
 & \max \quad 5x_1 + 3x_2 \\
 & \downarrow \\
 & \quad 2x_1 + 4x_2 \leq 15 \\
 & \text{s.t.} \quad 3x_1 + x_2 \leq 10 \\
 & \quad x \geq 0
 \end{aligned}$$

Suppose somebody wants to buy A, B from me. What is the lowest price I should ask?

Let y_A, y_B be prices:

$$\begin{aligned}
 & \min \quad 15y_A + 10y_B \\
 & \downarrow \\
 & \quad 2y_A + 3y_B \geq 5 \\
 & \text{s.t.} \quad 4y_A + y_B \geq 3 \\
 & \quad y \geq 0
 \end{aligned}$$

Example: Zero-Sum

Alice, Bob play game. A: m choices. B: n choices. Alice play i , Bob plays j , Bob pays Alice M_{ij} dollars.

| | | Alice | | |
|-----|---|-------|----|----|
| | | R | P | S |
| Bob | R | 0 | 1 | -1 |
| | P | -1 | 0 | 1 |
| | S | 1 | -1 | 0 |

Zero-sum: Amount won by Alice - Amount won by Bob = 0

Let $y \in \mathbb{R}_+^m$, Alice's probability distribution.

Let $x \in \mathbb{R}_+^n$, Bob's probability distribution.

Expected Amount Bob pays Alice:

$$\sum_{i=1}^m \sum_{j=1}^n y_i M_{ij} x_j = y^T M x$$

$$P = \left\{ x \in \mathbb{R}^n : \sum x_j = 1, x \geq 0 \right\}$$

$$Q = \left\{ y \in \mathbb{R}^m : \begin{array}{l} \sum y_i = 1 \\ y \geq 0 \end{array} \right\}$$

Alice wants $\max_{y \in Q} \left\{ \min_{x \in P} y^T M_x \right\}$. Bob wants $\min_{x \in P} \left\{ \max_{y \in Q} y^T M_x \right\}$.

Suppose $\bar{y} \in Q$ is fixed. Bob's problem is

$$\begin{aligned} \min_{x \in P} \bar{y}^T M_x &= \min \sum_{j=1}^n \left(\sum_{i=1}^m M_{ij} \bar{y}_i \right) x_j \\ &\downarrow \\ \text{s.t.} \quad &\sum_{j=1}^n x_j = 1 \\ &x \geq 0 \end{aligned}$$

This is equivalent to picking smallest number in

$$\begin{aligned} &\left\{ \sum_{i=1}^m M_{ij} \bar{y}_i \right\}_{j=1}^n \\ \Rightarrow \max_{y \in Q} \min_{x \in P} y^T M_x &= \max_{y \in Q} \left\{ \begin{array}{l} \max u \\ \downarrow \\ \text{s.t.} \quad u \leq y^T M e_j, \quad \forall j = 1, \dots, n \end{array} \right\} \\ &= \begin{array}{l} \max u \\ \downarrow \\ \text{s.t.} \quad u \leq y^T M e_j, \quad \forall j = 1, \dots, n \\ y^T = 1 \\ y \geq 0 \end{array} \end{aligned}$$

Similarly Bob's problem:

$$\begin{aligned} \min v & \\ \downarrow & \\ \text{s.t.} \quad &v \geq e_i^T M x, \quad \forall i = 1, \dots, m \\ &x^T = 1 \\ &x \geq 0 \end{aligned}$$

There are x^*, y^* for which strategy values match \rightarrow Nash's Equilibrium.

Now get back to Farkas' Lemma Theorem 2.2. ¹

Proof:

$$\begin{aligned} \max \quad &0^T x \\ \downarrow & \\ \text{s.t.} \quad &Ax \leq b \end{aligned} \tag{P}$$

¹Rephrase it a little bit: Exactly one of the two has a solution (i) $Ax \leq b$ (ii) $u^T \dots$

$$\begin{array}{ll}
\min & b^T u \\
\downarrow & \\
\text{s.t.} & u^T A = 0 \\
& u \geq 0
\end{array} \tag{D}$$

(D) is always feasible ($u = 0$).

If $\exists \bar{x} : A\bar{x} \leq b$, \bar{x} optimal for (P) \implies optimal for (D) has value 0.
 $\implies \nexists u$ satisfying (i).

And the converse is also true. □

2.7 Complementary Slackness (C.S.)

Let x^*, y^* be feasible for primal and dual respectively.

C.S.

Complementary Slackness:

- i) Either $x_j^* = 0$ or corresponding dual constraint is tight at y^* , $\forall j = 1, \dots, n$.
- ii) Either $y_i^* = 0$ or corresponding primal constraint is tight at x^* , $\forall i = 1, \dots, m$.

Example:

$$\begin{array}{ll}
\min & x_1 - x_2 \\
\downarrow & \\
& 2x_1 + 3x_2 \leq 5 \\
\text{s.t.} & x_1 - x_2 \geq 3 \\
& x_1 + 5x_2 = 7 \\
& x_1 \geq 0, x_2 \leq 0
\end{array} \tag{P}$$

$$\begin{array}{ll}
\max & 5y_1 + 3y_2 + 7y_3 \\
\downarrow & \\
& 2y_1 + y_2 + y_3 \leq 1 \\
\text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\
& y_1 \leq 0, y_2 \geq 0
\end{array} \tag{D}$$

- i) $x_1^* = 0$ OR $2y_1^* + y_2^* + y_3^* = 1$
 $x_2^* = 0$ OR $3y_1^* - y_2^* + 5y_3^* = -1$
- ii) $y_1^* = 0$ OR $2x_1^* + 3x_2^* = 5$
 $y_2^* = 0$ OR $x_1^* - x_2^* = 3$
 $y_3^* = 0$ OR $x_1^* + 5x_2^* = 7$

Theorem 2.7

Let x^*, y^* be feasible for primal/dual respectively. TFAE^a

- a) x^* opt for primal AND y^* opt. for dual
- b) Obj. value of $x^* =$ Obj. value of y^*
- c) x^*, y^* satisfy C.S.

^athe following are equivalent

Proof:

a) \iff b) done.

b) \iff c) Proof for

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \min & b^T y \\ \downarrow & \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

Note

$$A^T y \geq c \iff \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \forall j = 1, \dots, n$$

$$\begin{aligned} c^T x^* &= \sum_{j=1}^n c_j x_j^* \\ &\leq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i^* \right) x_j^* \\ &= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \\ &\leq \sum_{i=1}^m b_i y_i^* = b^T y^* \end{aligned}$$

where first and second inequalities come from $x \geq 0, y \geq 0$ respectively.

(b) $c^T x^* = b^T y^* \iff$ C.S. holds. (Just play with some strict inequality conditions)

□

Example:

$$\begin{array}{ll} \max & x_1 + x_2 \\ \downarrow & \\ \text{s.t.} & x_1 + x_2 \leq 1 \end{array} \qquad \begin{array}{ll} \min & y \\ \downarrow & \\ & y = 1 \\ \text{s.t.} & y = 1 \\ & y \geq 0 \end{array}$$

Consider a pair $x^* = (0, 0), y^* = 1$ which violates CS.

2.7.1 Geometric Interpretation of C.S.

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \qquad \begin{array}{ll} \min & c^T y \\ \downarrow & \\ \text{s.t.} & A^T y = c \\ & y \geq 0 \end{array}$$

$$A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

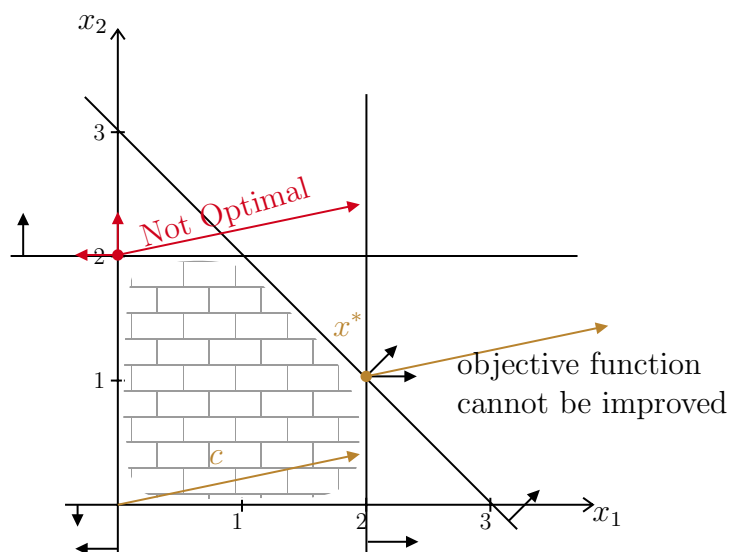
C.S. says $a_i^T x^* = b_i$ or $y_i^* = 0$.

$$A^T y = c \implies \begin{pmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & \cdots & | \end{pmatrix} y = c \implies \sum_{i=1}^m a_i y_i = c$$

C.S. says c is a nonnegative combination of tight constraint at x^* .

Example:

$$\begin{array}{ll} \max & 2x_1 + 0.5x_2 \\ \downarrow & \\ & x_1 \leq 2 \\ & x_2 \leq 2 \\ \text{s.t.} & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$



Theorem 2.8

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \quad (\text{P})$$

is unbounded iff (P) is feasible and $\exists d \in \mathbb{R}^n : \begin{array}{l} c^T d > 0 \\ Ad \leq 0 \end{array}$.

Proof:

\Rightarrow) Let \bar{x} feasible for (P), $\bar{x} + \lambda d$ is also feasible for (P) $\forall \lambda \geq 0$.

$c^T(\bar{x} + \lambda d)$ can be made arbitrary large.

\Leftarrow) Hard exercise but doable.

□

2.8 Geometry of Polyhedra

line segment

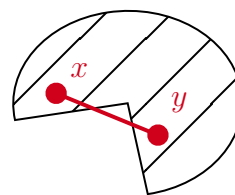
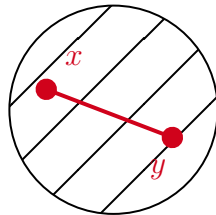
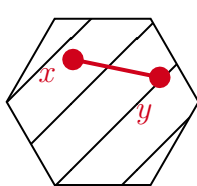
$\bar{x}, \bar{y} \in \mathbb{R}^n$ the line segment between \bar{x}, \bar{y} is

$$\left\{ x \in \mathbb{R}^n : \begin{array}{l} x = \lambda \bar{x} + (1 - \lambda) \bar{y} \\ \text{for some } \lambda \in [0, 1] \end{array} \right\}$$

convex set

S is a convex set if $\forall x, y \in S$, line segment between x, y is contained in S .

Example:



NOT a convex set

Polyhedra are convex sets. $P = \{x : Ax \leq b\}$. $\bar{x}, \bar{y} \in P$ then

$$A(\underbrace{\lambda}_{\geq 0} \bar{x} + \underbrace{(1 - \lambda)}_{\geq 0} \bar{y}) \leq \lambda b + (1 - \lambda)b = b$$

convex combination

Given $x^1, \dots, x^k \in \mathbb{R}^n$. We say \bar{x} is a convex combination of x^1, \dots, x^k if $\exists \lambda$:

$$\begin{aligned}\bar{x} &= \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i &= 1 \\ \lambda &\geq 0\end{aligned}$$

Optimal solution seems to be happen at “corners”.

Let P be a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

vertex

\bar{x} is a vertex of P if $\exists c$: \bar{x} is unique optimal solution to

$$\begin{aligned}\max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b\end{aligned}$$

extreme point

\bar{x} is an extreme point of P if $\nexists u, v \in P \setminus \{\bar{x}\}$ such that \bar{x} is in line segment between u, v .

basic feasible solution

$\bar{x} \in P$ is a basic feasible solution of P if there are n linearly independent tight constraints at \bar{x} .

Note

Constraints

$$a_i^T x \leq b_i, \quad \forall i = 1, \dots, m$$

are linearly independent if $\{a_i\}_{i=1}^m$ are linearly independent.

Theorem 2.9

Let $\bar{x} \in P$. TFAE:

- a) \bar{x} is a vertex of P .
- b) \bar{x} is a basic feasible solution of P .
- c) \bar{x} is a extreme point of P .

Proof:a) \implies c) Suppose $\exists u, v \in P \setminus \{\bar{x}\}$ such that

$$\bar{x} = \lambda u + (1 - \lambda)v$$

for some $\lambda \in (0, 1)$. Consider c for which \bar{x} is an optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$$

$$\implies \begin{array}{l} c^T \bar{x} \geq c^T u \\ c^T \bar{x} \geq c^T v \end{array}$$

and

$$c^T \bar{x} = \underbrace{\lambda}_{\geq 0} c^T u + \underbrace{(1 - \lambda)}_{\geq 0} c^T v \leq \lambda c^T \bar{x} + (1 - \lambda) c^T \bar{x} = c^T \bar{x}$$

$$\implies c^T u = c^T v = c^T \bar{x}$$

 $\implies \bar{x}$ NOT a vertex.c) \implies b) Suppose \bar{x} is not a BFS. Let $i \subseteq \{1, \dots, m\}$ be the index set of tight constraint at \bar{x} . Consider

$$a_i^T d = 0, \quad \forall i \in I \quad (*)$$

But since \bar{x} not BFS, $\exists \bar{d} \neq 0$ satisfying $(*)$.^a

$$x(\epsilon) = \bar{x} + \epsilon \bar{d}$$

$$a_i^T x(\epsilon) = a_i^T \bar{x} \leq b_i, \quad \forall i \in I$$

$$a_i^T x(\epsilon) = \underbrace{a_i^T \bar{x}}_{< b_i} + \epsilon a_i^T \bar{d} \leq b_i, \quad \forall i \notin I$$

which is satisfied if $|\epsilon|$ is small enough. $x(\epsilon) \in P$ if $|\epsilon|$ is small enough.

But then

$$\bar{x} = \frac{1}{2}x(\epsilon) + \frac{1}{2}x(-\epsilon)$$

b) \implies a) Let $I \subseteq \{1, \dots, m\}$ index set of tight constraint at \bar{x} .

Define

$$c := \sum_{i \in I} a_i$$

Then $\forall x \in P$

$$c^T x = \sum_{i \in I} a_i^T x \leq \sum_{i \in I} b_i$$

And

$$c^T \bar{x} = \sum_{i \in I} a_i^T \bar{x} = \sum_{i \in I} b_i$$

$\implies \bar{x}$ is optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array} \quad (**)$$

If $x' \in P$ is optimal solution to $(**)$, then

$$a_i^T x' = b_i, \quad \forall i \in I \quad (***)$$

But since there are n linear independent constraints in I , \bar{x} is unique solution to $(***)$. $\implies x' = \bar{x}$.

□

^aby Rank-Nullity Theorem.

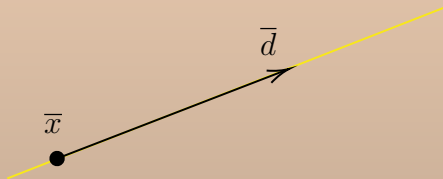
Q When does P have extreme points?

line

Let $\bar{x}, \bar{d} \in \mathbb{R}^n$, $\bar{d} \neq 0$. The set

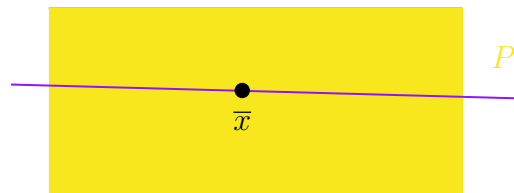
$$\{x \in \mathbb{R}^n : x = \bar{x} + \lambda \bar{d} \text{ for some } \lambda \in \mathbb{R}\}$$

is called a line.



We say a polyhedron P has a line if $\exists \bar{x}, \bar{d}$ has a line if $\exists \bar{x}, \bar{d}$ s.t. $\bar{x} \in P, \bar{d} \neq 0$ and

$$\{x \in \mathbb{R}^n : x = \bar{x} + \lambda \bar{d} \text{ for some } \lambda \in \mathbb{R}\} \subseteq P$$



Proposition 2.10

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has a line iff $P \neq \emptyset$ and $\exists \bar{d} \neq 0$ such that $A\bar{d} = 0$

$\iff P \neq \emptyset$ and $\text{rank}(A) < n$

Proof:

Exercise.

□

Theorem 2.11

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has an extreme point

$\iff P \neq \emptyset$ and P has no lines.

Proof:

Exercise. □

pointed polyhedron

A non-empty polyhedron is called pointed if it has no lines.

Note

not pointed does not imply bounded. For example, in \mathbb{R}^2 , $x \geq 0$ and $y \geq 0$.

Theorem 2.12

Let $P \neq \emptyset$ pointed polyhedron. If $\max_{x \in P} c^T x$ (LP) has an optimal solution, it has an optimal solution that is an extreme point.

Proof:

Let \bar{x} be an optimal solution to (LP) with largest number of linear independent tight constraints.

Suppose there are $\leq n - 1$ linear independent tight constraints at \bar{x} .

Pick $\bar{d} \neq 0$ such that $a_i^T \bar{d} = 0, \forall i \in I$, where I is the index set of tight constraints. By the exact same argument as before, $\bar{x} \pm \epsilon \bar{d} \in P$ for ϵ small enough. But

$$c^T(\bar{x} \pm \epsilon \bar{d}) = c^T \bar{x} \pm \epsilon c^T \bar{d}$$

$$\implies c^T \bar{d} = 0$$

$$\implies c^T d(\bar{x} \pm \epsilon d) = c^T \bar{x}$$



Since P is pointed, $\exists \bar{\epsilon}$ for which

$$\bar{x} \pm \bar{\epsilon} \in P$$

and one of them not in P if $|\epsilon| > \bar{\epsilon}$. That can only happen if

$$a_k^T(\bar{x} + \bar{\epsilon} \bar{d}) = b_k \quad \text{or} \quad a_k^T(\bar{x} - \bar{\epsilon} \bar{d}) = b_k$$

for some $k \notin I$.

$\implies a_k^T \bar{d} \neq 0, \implies a_k$ is linear independent from $\{a_i\}_{i \in I}$ since non-zero cannot be linear combination of zeros. Contradiction to choice of \bar{x} . \square

2.9 Simplex Algorithm

Standard Equality Form

A linear program is in Standard Equality Form (SEF) if it is of the form

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Proposition 2.13

Given any linear program, there exists an equivalent LP in SEF.

Example:

$$\begin{array}{ll} \max & x_1 + 2x_2 + x_3 \\ \downarrow & \\ & 3x_1 + x_2 \leq 5 \\ \text{s.t.} & -x_1 + x_3 \geq 6 \\ & x_1 \leq 0, x_3 \geq 0 \end{array} \quad (\text{P1})$$

$$\begin{aligned} x'_1 &= -x_1 \geq 0 \text{ and} \\ x_2 &= x_2^+ - x_2^- \text{ where } x_2^+ \geq 0, x_2^- \geq 0 \end{aligned}$$

We introduce

$$s_1 = 5 - 3x_1 - x_2 \geq 0, \quad s_2 = -x_1 + x_3 - 6 \geq 0$$

Then

$$\begin{array}{ll} \max & -x'_1 + 2x_2^+ - 2x_2^- + x_3 \\ \downarrow & \\ & -3x'_1 + 2x_2^+ - x_2^- + s_1 = 5 \\ \text{s.t.} & x'_1 + x_3 - s_2 = 6 \\ & x'_1, x_2^+, x_2^-, x_3, s_1, s_2 \geq 0 \end{array} \quad (\text{P2})$$

x feasible for (P1) $\iff (x'_1, x_2^+, x_2^-, x_3, s_1, s_2)$ feasible for (P2) and they have same cost.

Assumption $A \in \mathbb{R}^{m \times n} \rightarrow \text{rank}(A) = m$. This is WLOG. Since if

$$a_i = \sum_{k \neq i} \lambda_k a_k$$

Either

$$b_i \neq \sum_{k \neq i} \lambda_k b_k$$

in which case (SEF) is infeasible. Or $a_i^T x = b_i$ is redundant. So it can be removed from (SEF).

Note

$\{x : Ax = b, x \geq 0\}$ is *pointed* polyhedron (if nonempty).

Structure of BFS Any feasible solution has m linear independent tight constraints ($n - m$) extra tight constraint must come from $x_j \geq 0$.

Let $B \subseteq \{1, \dots, n\}$ such that $|B| = m$ and A_B ² is invertible.

$N = \{1, \dots, n\} \setminus B$. $x_N = 0$, i.e. $x_j = 0, \forall j \in N$.

Feasible solutions obtained this way are precisely BFS.

Example:

$$\begin{array}{ll} \max & (3 \quad 2 \quad 1 \quad 4) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x \geq 0 \end{array}$$

If we pick

$$\begin{array}{ll} B = \{1, 2\} & A_B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ N = \{3, 4\} & A_N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ C_B = (3 \quad 2)^T & C_N = (1 \quad 4)^T \end{array}$$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

$$B = \{1, 3\}, B = \{2, 4\}, A_B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, A_N = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$C_B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, C_N = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

If we set $x_N = 0$ (for $B = \{1, 3\}$) we are left with

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

This has a unique solution $x_1 = 3.5, x_3 = -1.5$, but not feasible.

² A_B is submatrix obtained by picking columns of A indexed by B . Such B is called a basis.

If we pick $B = \{1, 2\}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$\underbrace{x_3 = x_4 = 0}_{x_N}, x_1 = 3, x_2 = 1$, which is feasible.

In general,

$$Ax = b \iff A_B x_B + \cancel{A_N x_N}^0 = b$$

has unique solution $x_b = A_B^{-1}b$.

For any basis B , the corresponding *basic solution* is

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1}b \\ 0 \end{pmatrix}$$

If $A_B^{-1}b \geq 0$, then it is a *BFS*.

2.9.1 Canonical Form

Let B be a feasible basis (i.e. corresponding basis solution is feasible).

$$\begin{aligned} Ax = b &\iff A_B x_B + A_N x_N = b \\ &\iff x_B + A_B^{-1} A_N x_N = A_B^{-1}b \end{aligned}$$

Now let's take a look at objective.

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N - \textcolor{red}{c_B^T (x_B + A_B^{-1} A_N x_N - A_B^{-1}b)} \\ &= (c_N^T - c_B^T A_B^{-1} A_N) x_N + c_B^T A_B^{-1}b \end{aligned}$$

Thus (SEF) is said to be in canonical form for B if it is written as

$$\begin{aligned} \max \quad & (c_N^T - c_B^T A_B^{-1} A_N) x_N + c_B^T A_B^{-1}b \\ \downarrow \\ \text{s.t.} \quad & x_B + A_B^{-1} A_N x_N = A_B^{-1}b \\ & x_B, x_N \geq 0 \end{aligned}$$

Example:

Back to our previous example...

$B = \{1, 2\}$. Rewriting in canonical form for B :

$$A_B^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

$$A_B A = \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix}$$

$$c_B^T A_B^{-1} A_N = (3 \quad 2) \begin{pmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{pmatrix} = (7/3 \quad -8/3)$$

$$c_N^T - c_B^T A_B^{-1} A_N = (-4/3 \quad 4/3)$$

Then

$$\begin{aligned} & \max \quad (0 \quad 0 \quad -4/3 \quad 4/3)x + 11 \\ & \downarrow \\ & \text{s.t.} \quad \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ & \quad \quad \quad x \geq 0 \end{aligned}$$

is in canonical form for $B = \{1, 2\}$.

Index

B

basic feasible solution 25

C

C.S. 21
convex combination 25
convex set 24

E

extreme point 25

H

halfspace, hyperplane, polyhedron .. 7

I

infimum 5

L

line 27
line segment 24

P

pointed polyhedron 28

S

Standard Equality Form 29
supremum 5

V

vertex 25