Game Theory

CO 456

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LaTeXed by *Sibelius Peng*

Preface

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Sibelius Peng

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Combinatorial games

1.1 Impartial games

Reference

- http://web.mit.edu/sp.268/www/nim.pdf
- https://ivv5hpp.uni-muenster.de/u/baysm/teaching/3u03/notes/14-games.pdf

Example: Game of Nim

We are given a collection of piles of chips. Two players play alternatively. On a player's turn, they remove at least 1 chip from a pile. First player who cannot move loses the game.

For example, we have three piles with 1, 1, 2 chips. Is there a winning strategy? In this case, there is one for the first player: Player I (p1) removes the pile of 2 chips. This forces p2 to move a pile of 1 chip. p1 removes the last chip. p2 has no move and loses the game. In this case, p1 has a winning strategy, so this is a **winning game** or **winning position**.

Now let's look at another example with two piles of 5 chips each. Regardless of what p1 does, p2 can make the same move on the other pile. p1 loses. If p1 loses regardless of their move (i.e., p2 has a winning strategy), then this is a **losing game** or **losing position**.

What if we have two piles have unequal sizes? say 5, 7. p1 moves to equalize the chip count (remove 2 from the pile of 7). p2 then loses, this is a winning game.

Lemma 1.1

In instances of Nim with two piles of n, m chips, it is a winning game if and only if $m \neq n$.

Solving Nim with only two piles is easy, but what about games with more than two piles?

This is more complicated.

Nim is an example of an **impartial game**. Conditions required for an impartial game:

1. There are 2 players, player I and player II.
2. There are several positions, with a starting position.
3. A player performs one of a set of allowable moves, which depends only on the current position, and not on the player whose turn it is. ("impartial") Each possible move generates an option.
4. The players move alternately.
5. There is complete information.
6. There are no chance moves.
7. The first player with no available move loses.
8. The rules guarantee that games end.

Example: Not an impartial game

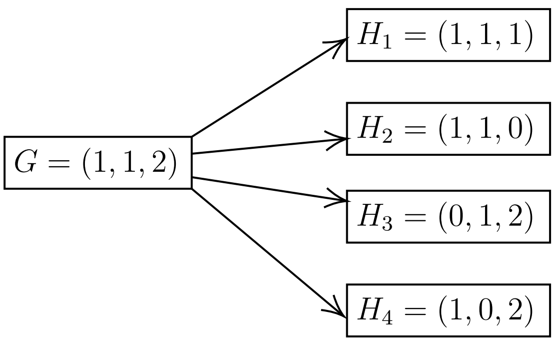
Tic-tac-toe: violates 7.

Chess: violates 3, since players can only move their own pieces.

Monopoly: violates 6. Poker: violates 5.

Example:

Let $G = (1, 1, 2)$ be a Nim game. There are 4 possible moves (hence 4 possible options):



Each option is by itself another game of Nim

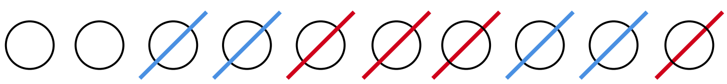
Note:
We can define an impartial game by its position and options recursively.

simpler

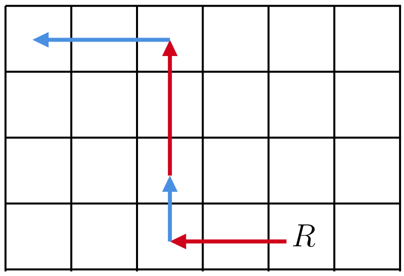
A game H that is reachable from game G by a sequence of allowable moves is **simpler** than G.

Other impartial games:

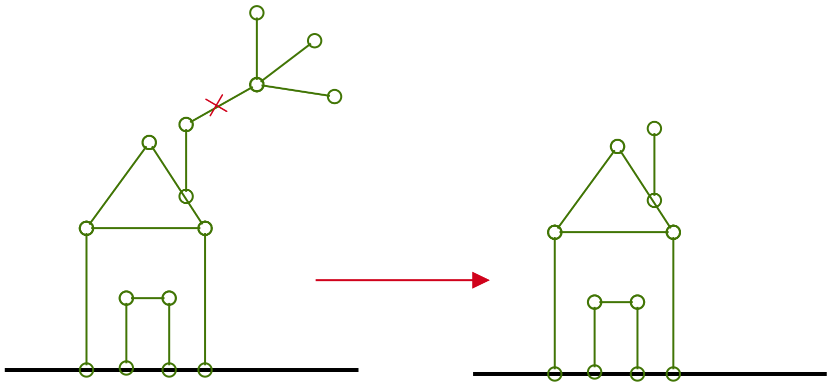
1. Subtraction game: We have one pile of n chips. A valid move is taking away 1, 2, or 3 chips. The first player who cannot move loses.



2. Rook game: We have an $m \times n$ chess board, and a rook in position (i, j). A valid move is moving the rook any number of spaces left or up. The first player who cannot move loses.



3. Green hackenbush game: We have a graph and the floor. The graph is attached to the floor at some vertices. A move consists of removing an edge of the graph, and any part of the graph not connected to the floor is removed. The first player who cannot move loses.



Spoiler A main result we will prove is that all impartial games are essentially like a Nim game.

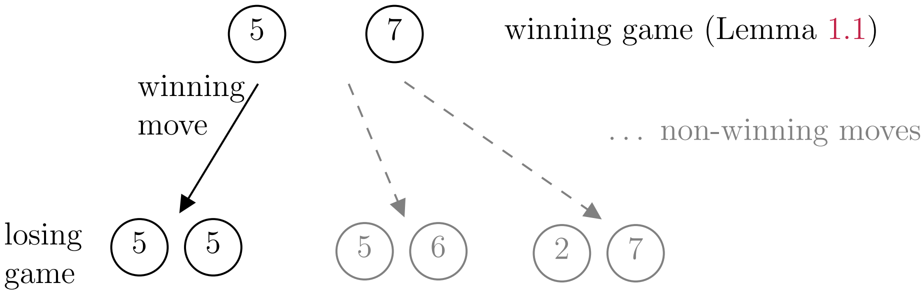
Lemma 1.2

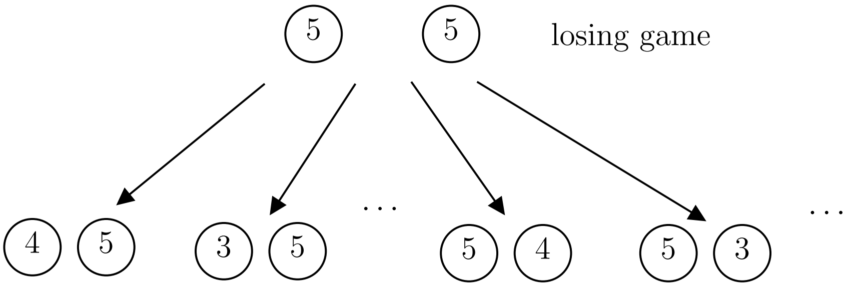
In any impartial game G, either player I or player II has a winning strategy.

Proof:
We prove by induction on the simplicity of G. If G has no allowable moves, then p1 loses, so p2 has a winning strategy. Assume G has allowable moves and the lemma holds for games simpler than G. Among all options of G, if p1 has a winning strategy in one of them, then p1 moves to that option and wins. Otherwise, p2 has a winning strategy for all options. So regardless of p1's move, p2 wins. $\square$

So every impartial game is either a winning game (p1 has a winning strategy) or a losing game (p2 has a winning strategy).

Example: Nim





all options are winning games $\implies$ p2 wins

Note:
We assume players play perfectly. If there is a winning move, then they will take it.

1.2 Equivalent games

game sums

Let G and and H be two games with options $G_1, \dots, G_m$ and $H_1, \dots, H_n$ respectively. We define $G + H$ as the games with options

$$G_1 + H, \dots, G_m + H, G + H_1, \dots, G + H_n.$$

Example:
We denote $*n$ to be a game of Nim with one pile of n chips. Then $*1 + *1 + *2$ is the game with 3 piles of 1, 1, 2 chips.

Example:
If we denote #2 to be the subtraction game with n chips, then $*5 + \#7$ is a game where a move consists of either removing at least 1 chip from the pile of 5 (Nim game), or removing 1, 2 or 3 chips from the pile of 7 (subtraction game).

Lemma 1.3

Let $\mathcal{G}$ be the set of all impartial games. Then for all $G, H, J \in \mathcal{G}$,

1. $G + H \in \mathcal{G}$ (closure)
2. $(G + H) + J = G + (H + J)$ (associative)
3. There exists an identity $0 \in \mathcal{G}$ (game with no options) where $G+0 = 0+G = G$
4. $G + H = H + G$ (symmetric)

Note:
This is an abelian group except the inverse element.

equivalent game

Two games G, H are **equivalent** if for any game J, $G+J$ and $H+J$ have the same outcome (i.e., either both are winning games, or both are losing games).

Notation: $G \equiv H$.

Example:
$*3 \equiv *3$ since $*3 + J$ is the same game as $*3 + J$ for any J, so they have the same outcome.

$*3 \not\equiv *4$ since $*3+*3$ is a losing game, but $*4+*3$ is a winning game from Lemma 1.1.

Lemma 1.4

$*n \equiv *m$ if and only if $n = m$.

Lemma 1.5

The relation $\equiv$ is an equivalence relation. That is, for all $G, H, K \in \mathcal{G}$,

1. $G \equiv G$ (reflexive)
2. $G \equiv G$ if and only if $H \equiv G$ (symmetric)
3. If $G \equiv H$ and $H \equiv K$, then $G \equiv K$ (transitive).

Exercise:
Prove that if $G \equiv H$, then $G + J \equiv H + J$ for any game J.

Note that the definition above only says they have the same outcome. To prove that they are equivalent, one needs to add another game on both sides to show they have the same outcome.

Nim with one pile $*n$ is a losing game if and only if $n = 0$.

Theorem 1.6

G is a losing game if and only if $G \equiv *0$.

Proof:

$\Leftarrow$ If $G \equiv *0$, then $G + *0$ has the same outcome as $*0 + *0$. But $*0$ is a losing game, so G is a losing game.

$\Rightarrow$ Suppose J is a losing game. (We want to show $G \equiv *0$, meaning $G + J$ and $*0 + J \equiv J$ have the same outcome.)

1. Suppose J is a losing game. (We want to show that $G+J$ is a losing game.)

   We will prove "If G and J are losing games, then $G + J$ is a losing game" by induction on the simplicity of $G + J$. When $G + J$ has no options, then G, J both have no options, so $G, J, G + J$ are all losing games.

   Suppose $G+J$ has some options. Then p1 makes a move on G or J. WLOG say p1 makes a move in G, and results in $G' + J$. Since G is a losing game, G' is a winning game. So p2 makes a winning move from G' to G'', and this results in $G'' + J$. Then G'' is a losing game, so by induction, $G'' + J$ is a losing game for p1. So p1 loses, and $G + J$ is a losing game.

2. Suppose J is a winning game. Then J has a winning move to J'. So p1 moves from $G + J$ to $G + J'$. Now both G, J' are losing games, so by case 1, $G + J'$ is a losing game. So p2 loses, meaning p1 wins, so $G + J$ is a winning game.

□

Corollary 1.7

If G is a losing game, then J and $J + G$ have the same outcome for any game J.

Proof:

Since G is a losing game, $G \equiv *0$ by Theorem 1.6. Then $J+G \equiv J+*0 \equiv J$ (previous exercise + Lemma 1.3). So J and $G + J$ have the same outcome. □

Example:

1. Recall $*5 + *5$ and $*7 + *7$ are losing games. Then Corollary 1.7 says $*5 + *5 + *7 + *7$ is also a losing game. (p1 moves in either $*5 + *5$ or $*7 + *7$. Then p2 makes a winning move from the same part, equalizing piles.)

2. $\underbrace{*1 + *1 + *2}_{\text{winning}} + \underbrace{*5 + *5}_{\text{losing}}$. Corollary 1.7 implies this is a winning game.

   (p1 makes a winning move in $*1 + *1 + *2$, therefore we have $\underbrace{*1 + *1}_{\text{losing}} + \underbrace{*5 + *5}_{\text{losing}}$. p2 loses.)

Lemma 1.8: Copycat principle

For any game G, $G + G \equiv *0$.

Proof:
Induction on the simplicity of G. When G has no options, $G+G$ has no options, so $G+G \equiv *0$ by Theorem 1.6. Suppose G has options, and WLOG suppose p1 moves from $G+G$ to $G'+G$. Then p2 can move to $G'+G'$. By induction, $G'+G' \equiv *0$, so it is a losing game for p1. Therefore, $G+G$ is a losing game, and $G+G \equiv *0$. □

Lemma 1.9

$G \equiv H$ if and only if $G+H \equiv *0$.

Proof:

$\Rightarrow$ From $G \equiv H$, we add H to both sides to get $G+H \equiv H+H \equiv *0$ by the copycat principle.

$\Leftarrow$ From $G+H \equiv *0$, we add H to both sides to get $G+H+H \equiv *0+H \equiv H$. But $G+G+G \equiv G+*0 \equiv G$ by the copycat principle. So $G \equiv H$. □

Example:
$*1+*2+*3$ is a losing game, so $*1+*2+*3 \equiv *0$. By Lemma 1.9, $*1+*2 \equiv *3$, or $*1+*3 \equiv *2$.

Another way to prove game equivalence is by showing that they have equivalent options.

Lemma 1.10

If the options of G are equivalent to options of H, then $G \equiv H$. (More precisely: There is a bijection between options of G and H where paired options are equivalent.)

Proof:
It suffices to show that $G+H \equiv *0$ by Lemma 1.9, i.e., $G+H$ is a losing game. This is true when G, H both have no options. Suppose G, H have options, and suppose WLOG p1 moves to $G'H$. By assumption, there exists an options of H, say H', such that $H' \equiv G'$. So p2 can move to $G'+H'$. Since $G' \equiv H'$, $G'+H' \equiv *0$ by Lemma 1.9. So $G'+H'$ is a losing game for p1. Hence $G+H$ is a losing game. □

Example:
We can show $*1+*2 \equiv *3$ using Lemma 1.10.

$$
\begin{tikzcd}
 & \ast 2 \arrow[r, dash] & \ast 2 & \\
\ast 1 + \ast 2 \arrow[ur] \arrow[r] \arrow[dr] & \ast 1 \arrow[r, dash] & \ast 1 & \ast 3 \arrow[ul] \arrow[l] \arrow[dl] \\
 & \ast 1 + \ast 1 \arrow[r, dash] & \ast 0 &
\end{tikzcd}
$$

Note:
The converse is false.

1.3 Nim and nimbers

Goal Show that every Nim game is equivalent to a Nim game with a single pile.

nimber

If G is a game such that $G \equiv *n$ for some n, then n is the **nimber** of G.

Example:
Any losing game has nimber 0 by Theorem 1.6.

Exercise:
Show that the notion of a nimber is well-defined. That is it is not possible for a game to have more than one nimber.

Theorem 1.11

Suppose $n = 2^{a_1} + 2^{a_2} + \ldots$ where $a_1 > a_2 > \ldots$, then $*n \equiv *2^{a_1} + *2^{a_2} + \ldots$.

Example:
$11 = 2^3 + 2^1 + 2^0$, $13 = 2^3 + 2^2 + 2^0$. Using this theorem, $*11 \equiv *2^3 + *2^1 + *2^0$ and $*13 \equiv *2^3 + *2^2 + *2^0$. Then

$$\begin{aligned} *11 + *13 &\equiv (*2^3 + *2^1 + *2^0) + (*2^3 + *2^2 + *2^0) \\ &\equiv (*2^3 + *2^3) + *2^2 + *2^1 + (*2^0 + *2^0) \quad \text{by assoc'y and commu'y} \\ &\equiv *0 + *2^2 + *2^1 + *0 \quad \text{by copycat principle} \\ &\equiv *2^2 + *2^1 \\ &\equiv *(2^2 + 2^1) \\ &\equiv *6 \end{aligned}$$

So the nimber of $*11 + *13$ is 6.

In general, how can we find the nimber for $*b_1 + *b_2 + \ldots + *b_n$? Look for binary expansions of each b_i. Copycat principle cancels any pair of identical powers of 2. So we look for powers of 2's that appear in odd number of expansions of the b_i's.

Use binary numbers: 11 in binary is 1011, 13 in binary is 1101. Take the XOR operation. We do normal addition except we do not carry over.

$$\begin{array}{rr} & 1011 \\ \oplus & 1101 \\ \hline & 0110 \end{array}$$

and 0110 is 6. So $11 \oplus 13 = 6$.

Example:
Consider $*25 + *21 + *11$. In binary they are $11001, 10101, 01011$.

$$\begin{array}{rr} & 11001 \\ & 10101 \\ \oplus & 01011 \\ \hline & 00111 \end{array}$$

and 00111 is 7. So $*25 + *21 + *11 \equiv *7$. (The nimber is 7)

Corollary 1.12

$*b_1 + *b_2 + \ldots + *b_n \equiv *(b_1 \oplus b_2 \oplus \ldots \oplus b_n)$.

This shows that every Nim game has a nimber.

Winning strategy for Nim

Example:

$*11 + *13 \equiv *6$. This is a winning game. How to find a winning move? Want to move a game equivalent to $*0$. Add $*6$ to both sides: $*11 + *13 + *6 \equiv *6 + *6 \equiv *0$ (copycat principle).

Consider $*11 + (*13 + *6)$. We see $13 \oplus 6 = 11$. So this is equivalent to $*11 + *11$, a losing game. Winning move: remove 2 chips from the pile of 13.

Example:

$*25 + *13 + *11 \equiv *7$. Add $*7$ to both sides. Consider $*25 + (*21 + *7) + *11$. We see $21 \oplus 7 = 18$, so this is equivalent to $*25 + *18 + *11$. Winning move: remove 3 chips from the pile of 21.

Why did we pair $*7$ with $*21$ instead of $*25$ or $*11$? $25 \oplus 7 = 31$, $11 \oplus 7 = 12$. This means that we are adding 6 chips to 25, or adding 1 chip to 11. Not allowed in Nim.

Lemma 1.13

If $*b_1 + \ldots + *b_n \equiv *s$ where $s > 0$, then there exists some b_i where $b_i \oplus s < b_i$.

Idea: Look for the largest power of 2 in s.

$*25 + *21 + *11 \equiv *7$

$$\begin{array}{rccccc l}
 & 1 & 1 & \boxed{0} & 0 & 1 & 25\\
 & 1 & 0 & \boxed{1} & 0 & 1 & 21\\
\oplus & 0 & 1 & \boxed{0} & 1 & 1 & 11\\ \hline
 & 0 & 0 & \boxed{1} & 1 & 1 & 7\\
 & & & \uparrow & \uparrow & \uparrow & \\
 & & & 4 & 2 & 1 &
\end{array}$$

$21 \oplus 7$: 4 is subtracted from 21

$25 \oplus 7$ or $11 \oplus 7$: 4 is added

$\leftarrow 4 > 2 + 1 \longrightarrow$

$\oplus$ reduces 21

$\oplus$ increases 25 or 11

Proof:

Suppose $s = 2^{a_1} + 2^{a_2} + \ldots$ where $a_1 > a_2 > \ldots$. Then 2^{a_1} appears in the binary expansions of $b_1, \ldots, b_n$ an odd number of times. Let b_i be one of them. Suppose $*b_i + *s \equiv *t$ for some t. Since 2^{a_1} is in the binary expansions of b_i and s, 2^{a_1} is not in the binary expansion of t. For $2^{a_2}, 2^{a_3}, \ldots$, at worse none of them are in the binary expansion of b_i, so all of them are in the binary expansion of t. So

$$t \leq b_i - 2^{a_1} + 2^{a_2} + 2^{a_3} + \ldots < b_i \quad \text{since } 2^{a_1} > 2^{a_2} + 2^{a_3} + \ldots$$

□

Finding winning moves in a winning Nim game: Say a game has nimber s. Look at the largest power of 2 in the binary expansion of s. Pair it up with any pile $*b_i$ containing this power of 2. Then $s \oplus b_i < b_i$. So a winning move is taking away $b_i - (s \oplus b_i)$ chips from the pile $*b_i$.

Now we wish to prove Theorem 1.11. The proof uses the following lemma:

Lemma 1.14

Let $0 \le p, q < 2^a$, and suppose Theorem 1.11 hold for all values less than 2^a. Then $p \oplus q < 2^a$.

Illustration for the proof of Theorem 1.11. Consider $*7$. $7 = 4 + 2 + 1$. Want to prove $*7 \equiv *4 + \underbrace{*2 + *1}_{\equiv *3 \text{ by induction}}$

Options of $*7$: $*0, *1, \ldots, *6$

Options of $*4 + *3$: (1) Move on $*4$ (2) Move on $*3$

(1)
$$\left.\begin{aligned} *0 + *3 &\equiv *3 \\ *1 + *3 &\equiv *2 \\ *2 + *3 &\equiv *1 \\ \underbrace{*3}_{<4} + \underbrace{*3}_{<4} &\equiv \underbrace{*0}_{\Rightarrow <4} \end{aligned}\right\} \text{distinct} \qquad \text{by Lemma 1.14}$$

(2)
$$\begin{aligned} *4 + *2 &\equiv *6 \\ *4 + *1 &\equiv *5 \\ \underbrace{*4 + \underbrace{*0}_{\text{binary expansion do not have 4}}}_{\text{each power of 2 appears at most once}} &\equiv *4 \end{aligned}$$

$\Rightarrow$ apply induction

Proof of Theorem 1.11:
We prove by induction on n.

When $n = 1$, $n = 2^0$ and $*1 \equiv *2^0$. Suppose $n = 2^{a_1} + 2^{a_2} + \ldots$ where $a_1 > a_2 > \ldots$. Let $q = n - 2^{a_1} = 2^{a_2} + 2^{a_3} + \ldots$.

If $q = 0$, then $n = 2^{a_1}$, so $*n \equiv 2^{a_1}$.

Assume $q \ge 1$. Since $q < n$, by induction, $*q \equiv *2^{a_2} + *2^{a_3} + \ldots$. It remains to show that $*n \equiv *2^{a_1} + *q$. The options of $*n$ are $*0, *1, \ldots, *(n-1)$. The options of $*2^{a_1} + *q$ can be partitioned into 2 types.

1. Consider options of the form $*i + *q$ where $0 \le i < 2^{a_1}$. Since $i, q < n$, by induction, the theorem holds for i, q. So $*i, *q$ are equivalent to sums of Nim piles by their binary expansions. Using arguments from Corollary 1.12, $*i + *q \equiv *r_i$ where $r_i = i \oplus q$. Since $i, q < 2^{a_1}, r_i < 2^{a_1}$ by Lemma 1.14. So $0 \le r_0, r_1, \ldots r_{2^{a_1}-1} < 2^{a_1}$.

   (We now show that these r_i's are distinct.) Suppose $r_i = r_j$ for some i, j. Then $*r_i \equiv *r_j$, so $*i + *q \equiv *j + *q$. Adding $*q$ on both sides, we get $*i \equiv *j$ (copycat principle), so $i = j$. So the r_i's are distinct.

   Also there are 2^{a_1} of these r_i's, and there are 2^{a_1} possible values (0 to $2^{a_1} - 1$). By Pigeonhole principle, for each $0 \le j \le 2^{a_1} - 1$, there is one r_i with $r_i = j$. So the options of this type are equivalent to $\{*0, *1, \ldots, *(2^{a_1} - 1)\}$.

2. Consider options of the form $*2^{a_1} + *i$ where $0 \le i < q$. Suppose $i = 2^{b_1} + 2^{b_2} + \ldots$ where $b_1 > b_2 > \ldots$. Then no b_i is equal to a_1 since $i < q = 2^{a_2} + \ldots$. So $2^{a_1} + 2^{b_1} + \ldots$ is a sum of distinct powers of 2. Then

$$\begin{aligned} *2^{a_1} + *i &\equiv *2^{a_1} + *2^{b_2} + \ldots && \text{by applying induction on } i \\ &\equiv *(2^{a_1} + 2^{b_1} + 2^{b_2} + \ldots) && \text{by applying induction on } 2^{a_1} + i \\ &\equiv *(2^{a_1} + i) \end{aligned}$$

Since $0 \le i < q$, the options of this type are equivalent to $\{*2^{a_1}, *(2^{a_1}+1), \ldots, \underbrace{(2^{a_1}+q-1)}_{n-1}\}$.

Combining the two types of options, we see that the options of $*2^{a_1} + *q$ are equivalent to the options of $*n$. So $*2^{a_1} + *q \equiv *n$. $\square$

1.4 Sprague-Grundy theorem

So far: All Nim games are equivalent to a Nim game of a single pile. Goal: Extend this to all impartial games.

Poker nim

Being equivalent does not mean that they play the same way.

Example:
$*11 + *13 \equiv *6$.

We move to $*11 + *11 \equiv *0$ by removing 2 chips from $*13$. RHS remove 6 chips.

There are other moves, say we move to $*11 + *8 \equiv *15$. We remove 5 chips from $*13$. RHS adding 9 chips.

Or, starting with $*11 + *11 \equiv *0$, any move on $*11 + *11$ will increase $*0$.

A variation on Nim: Poker nim consists of a regular Nim game plus a bag of B chips. We now allow regular Nim moves and adding $B' \le B$ chips to one pile. Example: $*3 + *4 \to *53 + *4$.

How does this change the game of Nim?

Nothing. Say we face a losing game, so any regular Nim move would lead to a loss. In poker nim, we now add some chips to one pile. The opposing player will simply remove the chips we placed, and nothing changed.

When we say that a game is equivalent to a Nim game with one pile, it is actually a game is equivalent to a Nim game with one pile, it is actually a game of poker nim with one pile.

Mex

Suppose a game G has options equivalent to $*0, *1, *2, *5, *10, *25$. We claim that G is equivalent to $*3$. The options of $*3$, which are $*0, *1, *2$, are all available. If we add chips to $*3$, then the opposing player can remove them to get back to $*3$. How do we get 3?

mex(S)

Given a set of non-negative integers S, $\text{mex}(S)$ is the smallest non-negative integer not in S. "**minimum excluded integer**"

Example:
$\text{mex}(\{0, 1, 2, 5, 15, 25\}) = 3$.

The mex function is the critical link between any impartial games and Nim games.

Theorem 1.15

Let G be an impartial game, and let S be the set of integers n such that there exists an option of G equivalent to $*n$. Then $G \equiv *(\text{mex}(S))$.

Example:

$$\begin{tikzcd}[column sep=large] & *1 + *2 \equiv *3 \\ *1 + *1 + *2 \arrow[ur] \arrow[r] \arrow[dr] & *1 + *1 \equiv *0 \\ & *1 + *1 + *1 \equiv *1 \end{tikzcd}$$

By theorem, $*1 + *1 + *2 \equiv *(\text{mex}(\{0, 1, 3\})) \equiv *2$.

Exercise:
A game cannot be equivalent to one of its options.

Proof of Theorem 1.15:
Let $m = \text{mex}(S)$. It suffices to show that $G + *m \equiv *0$.

1. Suppose we move to $G + *m'$ where $m' < m$. Since $m = \text{mex}(S)$, there exists an option G' of G such that $G' \equiv *m'$. p2 moves to $G' + *m'$, which is a losing game since $G' \equiv *m'$. So $G + *m$ is a losing game for p1, and $G + *m \equiv *0$.
2. Suppose we move to $G' + *m$, where G' is an option of G. Then $G' \equiv *k$ for some $k \in S$. So $G' + *m \equiv *k + *m \not\equiv *0$ since $k \neq \text{mex}(S)$. So $G' + *m$ is a winning game for p2. Then $G + *m$ is a losing game for p1, so $G + *m \equiv *0$.

$\square$

Theorem 1.16: Sprague-Grundy Theorem

Any impartial game G is equivalent to a poker nim game $*n$ for some n.

Proof (slightly sketchy):
If G has no options, then $G \equiv *0$. Suppose G has options $G_1, \ldots, G_k$. By induction, $G_i \equiv *n_i$ for some n_i. By Theorem 1.15, $G \equiv *(\text{mex}(\{n_1, \ldots, n_k\}))$. $\square$

So any impartial game has a nimber.

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