Introduction to General Relativity

AMATH 475

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Preface

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Some of the notes (especially special relativity part) are projected to the screen instead of using blackboards. They can be found on professor's course page.

For any questions, send me an email via `https://notes.sibeliusp.com/contact/`.

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Pre-Math

0.1 Index notation

$$A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \qquad B = \begin{pmatrix} B^1{}_1 & B^1{}_2 \\ B^2{}_1 & B^2{}_2 \end{pmatrix}$$

$$(A \cdot B)^a{}_b = A^a{}_c B^c{}_b = B^c{}_b A^a{}_c \qquad \text{sum over all possible } c$$

Identify followings:

$$B_\kappa{}^\nu A_\mu{}^\kappa = A_\mu{}^\kappa B_\kappa{}^\nu = C_\mu{}^\nu = (A \cdot B)_\mu{}^\nu$$

$$A^\kappa{}_\mu B_\kappa{}^\nu = D_\mu{}^\nu = (A^T)_\mu{}^\kappa B_\kappa{}^\nu = (A^T \cdot B)_\mu{}^\kappa$$

$$A_\kappa{}^\nu B_\mu{}^\kappa = E_\mu{}^\nu = (B \cdot A)_\mu{}^\nu$$

$$A^\kappa{}_\mu B^\nu{}_\kappa = (A^T)_\mu{}^\kappa (B^T)_\kappa{}^\nu = \Big((B \cdot A)^T\Big)_\mu{}^\nu$$

$\mathbf{v} = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 \qquad \{\mathbf{e}_1, \mathbf{e}_2\}$ Basis 1.

$\mathbf{v} = v^a\mathbf{e}_a = v'^a\mathbf{e}'_a \qquad \{\mathbf{e}'_1, \mathbf{e}'_2\}$ Basis 2.

Change of basis matrix Λ

$$\mathbf{e}'_a = \Lambda_a{}^b \mathbf{e}_b$$
$$v'^a = \tilde{\Lambda}^a{}_b v^b$$

$$
\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a{}_b v^b \Lambda_a{}^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a{}_b \Lambda_a{}^c v^b \mathbf{e}_c \\
&= \underbrace{\left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c}_{\delta_b^c} v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b
\end{aligned}
$$

$$
\begin{aligned}
\implies \left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c &= \delta_b^c \\
\tilde{\Lambda}^T \cdot \Lambda &= \mathbb{1}
\end{aligned}
$$

$\tilde{\Lambda}^T$ is the inverse transpose of Λ

covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis. Λ. (sub-indices)

A contravariant object transforms like components of vectors. ($\tilde{\Lambda} = (\Lambda^T)^{-1}$). (super-indices)

0.2 Vectors and one-forms

one-form

Let V be a vector space. A one-form is a linear map $\boldsymbol{\omega} : V \to \mathbb{R}$.

or we write: $(\boldsymbol{\omega}, \cdot) : V \to \mathbb{R}$ and $(\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}$.

dual vector space

The set of all one-forms on V (call V^*) is a vector space as well called the dual vector space to V.

dual basis

Let $\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2, \ldots\}$ (or $\{\boldsymbol{\Upsilon}_i\}$) be a basis of V so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = v^i \boldsymbol{\Upsilon}_i$.

We define the dual basis (of V^*) to $\{\boldsymbol{\Upsilon}_i\}$ as $\{\boldsymbol{\omega}^i\}$ such that $\boldsymbol{\omega}^i(\boldsymbol{\Upsilon}_j) = \delta^i_j$.

For a one form $\boldsymbol{\omega}$ we denote its "components of the basis $\boldsymbol{\Upsilon}$" as $(\boldsymbol{\omega}, \boldsymbol{\Upsilon}_m) = \omega_m$

Proposition 0.1

The dual basis of V^* is actually a basis of V^*.

The action of $\boldsymbol{\omega} \in V^*$ on a vector $\mathbf{v} = v^\mu \boldsymbol{\Upsilon} \in V$ is

$$(\boldsymbol{\omega}, \mathbf{v}) = (\boldsymbol{\omega}, v^\mu \boldsymbol{\Upsilon}_\mu) = v^\mu \omega_\mu$$

Let's prove $\{\boldsymbol{\Upsilon}^a\}$ is linear independent.

Proof:

A linear comb. $c_a \boldsymbol{\Upsilon}^a$ acts on a vector $\mathbf{v} = v^a \boldsymbol{\Upsilon}_a$

$$\begin{aligned}
(c_a \boldsymbol{\Upsilon}^a, \mathbf{v}) &= c_a (\boldsymbol{\Upsilon}^a, \mathbf{v}) \\
&= c_a (\boldsymbol{\Upsilon}^a, v^b \boldsymbol{\Upsilon}_b) \\
&= c_a v^b \underbrace{(\boldsymbol{\Upsilon}^a, \boldsymbol{\Upsilon}_b)}_{\delta^a_b} \\
&= c_a v^b \delta^a_b = c_a v^a
\end{aligned}$$

For LI,

$$\begin{aligned}
c_a \boldsymbol{\Upsilon}^a = 0 &\iff c_a = 0 \quad \forall a \\
c_a v^a = 0 \ \ \forall \mathbf{v} &\iff c_a = 0
\end{aligned}$$

□

vectors: take one-forms $\to \mathbb{R}$ one-forms: take vectors $\to \mathbb{R}$

0.3 Tensor

type (m, n) tensor

A type (m, n) tensor is a multilinear map that

$$\mathbf{T} : V^n \otimes (V^*)^m \to \mathbb{R}$$

Components of $\mathbf{T}$:

$$\mathbf{T}(\boldsymbol{\Upsilon}_{a1}, \ldots, \boldsymbol{\Upsilon}_{an}, \boldsymbol{\Upsilon}^{b1}, \ldots, \boldsymbol{\Upsilon}^{bm}) = T_{a_1 \ldots a_n}{}^{b_1 \ldots b_m}$$

1. Tensor product takes $\binom{m}{n}$ and $\binom{m'}{n'} \to \binom{m+m'}{n+n'}$ tensor
2. Contraction takes $\binom{m}{n} \to \binom{m-1}{n-1}$

Example:

1. $T_a{}^b, S_c{}^d$.

$$(\mathbf{T} \otimes \mathbf{S})_a{}^b{}_c{}^d = T_a{}^d S_c{}^d = P_a{}^b{}_c{}^d$$

2. $T_a{}^{bc} \to c^b T_a{}^{ba}$

$$v^a, w_b \begin{cases} v^a \omega_b \\ v^a \omega_a \end{cases}$$

If you have a favorite type $(2,0)$ symmetric tensor $\mathbf{g}$

$$v_\mu = g_{\mu\nu} v^\nu$$

$g^{\mu\nu} :=$ components of the inverse of $\mathbf{g}_{\mu\nu}$

$$v^\nu = g^{\mu\nu}$$

then

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$$

$$g_{\mu\nu} v^\mu w^\nu = v_\mu w^\nu = \mathbf{v}\mathbf{w}$$
$$||\mathbf{v}||^2 = g_{\nu\mu} v^\mu v^\nu$$

Then we can define the angle

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||\ ||\mathbf{v}||} := \cos\theta$$
$$T_\mu{}^\nu = g^{\nu\sigma} T_{\mu\sigma}$$
$$T^{\mu\nu} = g^{\nu\sigma} g^{\mu\rho} T_{\sigma\rho}$$
$$g^\nu_\mu = g^{\nu\sigma} g_{\sigma\mu} = \sigma^\nu_\mu$$

0.4 Levi-Civita symbol

Levi-Civita symbol $\epsilon^{abc\ldots}, \epsilon_{abc\ldots}$

- is antisymmetric
- $\epsilon^{1234\ldots} = 1$, $\epsilon_{1234} = 1$

$\epsilon^{123} = 1, \quad \epsilon^{213} = -1, \quad \epsilon^{312} = 1, \quad \epsilon^{113} = 0$

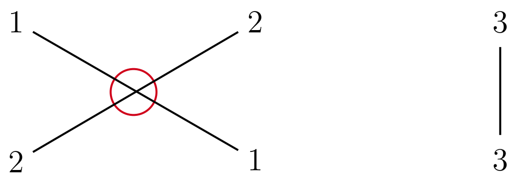
$\epsilon^{123456} = 1, \quad \epsilon^{612453} = -1$

Idea just see the permutations

Levi-Civita symbol

$$\varepsilon_{a_1a_2a_3\ldots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an even permutation of } (1, 2, 3, \ldots, n) \\ -1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an odd permutation of } (1, 2, 3, \ldots, n) \\ 0 & \text{otherwise} \end{cases}$$

Here is a short-cut:



odd number crossings, so odd permutation.

Note that $\det(M) := \epsilon_{ijk\ldots} M^i{}_1 M^j{}_2 M^j{}_3 \ldots$

Exercise:

prove $\epsilon^{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n} = n! i_j = 1, \ldots, n$

$$\epsilon^{ijk}\epsilon_{ilm} = \delta^j_l \delta^k_m - \delta^j_m \delta^k_l$$
$$\epsilon^{ijmn}\epsilon_{klmn} = 2(\delta^i_k \delta^j_l - \delta^j_k \delta^i_l)$$

Prove $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Proof:

Let $\vec{F} = \vec{A} \times (\vec{B} \times \vec{C})$ $\vec{D} = \vec{B} \times \vec{C}$

Then
$$D^k = \epsilon^k{}_{ij} B^i C_j$$
$$F^l = \epsilon^l{}_{mk} A^m D^k \implies F^l = \epsilon^l{}_{mk} \epsilon^k{}_{ij} A^m B^i C^j$$

Then
$$\begin{aligned} F^l &= (\delta^l_i \delta_{mj} - \delta^l_j \delta_{mi}) A^m B^i C^j \\ &= \delta^l_i \delta_{mj} A^m B^i C^j - \delta^l_j \delta_{mi} A^m B^i C^j \\ &= B^l (A_j C^j) - C^l (A_i B^i) \end{aligned}$$

where we use
$$\vec{A} \cdot \vec{B} = A^i B_i$$

$\square$

1

Special Relativity

1.1 Postulates of SR

Postulate 0

Newton's first law

Postulate 1: Principle of relativity

In the absence of gravity, all the laws of Physics are identical in all inertial reference frames.

Postulate 2

The speed of light in vacuum c is constant and the same from all inertial reference frames, regardless of their state of motion.

1.2 Lorentz Transformation

We define the spacetime interval Δs^2

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = -c^2 \left(t_2 - t_1\right)^2 + \left(\mathbf{x}_2 - \mathbf{x}_1\right)^2$$

Assuming the following:

1. The difference between the two frames is a constant speed $\succapprox$
2. The transformation has to be linear.

$$t' = \gamma \left(t - \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{c^2} \right), \quad \boldsymbol{x}' = \boldsymbol{x} + (\gamma - 1)(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - \gamma \boldsymbol{v} t$$

and index notation

$$t' = \gamma \left(t - \frac{v_i x^i}{c^2} \right), \quad x^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

1.3 Line element, proper time and spacelike, timelike and null separation

1.3.1 Classification of spacetime intervals

We can classify events according to the following criterion:

- Spacelike separated, $\Delta s^2 > 0$
- Timelike separated, $\Delta s^2 < 0$
- Lightlike (null) separated, $\Delta s^2 = 0$

Given the trajectory of a physical particle moving inertially, we will call co-moving frame (inertial) or proper frame (non-inertial) to the frame S_p where the particle is at rest.

1.3.2 Proper time and line element

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2$$

We will call ds^2 the spacetime line element.

$$\mathbf{v} := \frac{d\mathbf{x}}{dt}$$

P0, P1, P2 + linearity

$$\implies t' = t \left(t - \frac{v_i x^i}{c^2} \right) \tag{1}$$

$$x'^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

Particle trajectory in a given inertial (Lab) frame $\mathbf{x}(t)$

Particle trajectory in its proper frame $\boldsymbol{\xi}(t) = 0$

Comoving frame's trajectories at each t (from lab frame) $\mathbf{x} = \mathbf{v}(t)t$.

$$d\tau = dt' = \gamma(t)\left(1 - \frac{\mathbf{v}(t)^2}{c^2}\right) dt \tag{2}$$

$$ds^2 = -c^2 dt^2 \left(1 - \frac{1}{c^2}\underbrace{\left(\frac{d\mathbf{x}}{dt}\right)}_{\mathbf{v}(t)}^2\right) = -c^2 \underbrace{\gamma^{-2} dt^2}_{d\tau^2} \implies ds^2 = -c^2 d\tau^2 \tag{3}$$

Example:

Find $\tau(t)$ for the three following trajectories.

1. $x(t) = v(t)$

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x}^2 \implies d\tau = \gamma^{-1} dt \implies \Delta\tau = \gamma^{-1}\Delta t$$

2. $x(t) = \frac{c^2}{a}\left[\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right]$

Then $\frac{dx}{dt} = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2$$

$$\implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\frac{a^2t^2}{1 + \left(\frac{at}{c}\right)^2}}$$

$$\implies \tau(t) = \frac{c}{a}\operatorname{arcsinh}\left(\frac{at}{c}\right) \qquad \text{and} \qquad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

3. $x(t) = L\sin(\omega t) \implies \frac{dx}{dt} = Lw\cos(\omega t)$ with $L\omega < c$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 \implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2} \implies d\tau = \sqrt{1 - \frac{L^2\omega^2}{c^2}\omega t}\ dt$$

Then

$$\tau(t) = \frac{E\left(t\omega, \frac{1}{1-\frac{c^2}{L^2\omega^2}}\right)}{\omega\sqrt{\frac{1}{1-\frac{L^2\omega^2}{c^2}}}}$$

where

$$E(\phi|m) = \int_0^\phi (1 - m\sin^2\theta)^{1/2} d\theta$$

1.4 Lorentzian Tensors

See notes for details.

A_μ transposes with Λ and it's covariant.

A^μ transposes with $\tilde{\Lambda} = (\Lambda^{-1})^T$ and it's controvariant.

1.5 Poincare group

The derivations are in notes.

1.6 Relativistic dynamics

1.6.1 Hamilton's principle and Euler-Lagrange equations

There exists at least one function (called action) of the trajectories that the degrees of freedom of a system may take in phase space. The physical trajectories are obtained demanding stationarity of this functional under variations that keep the initial and final positions constant.

Usually, the action S of a system of n particles can be written in terms of a Lagrangian $L(s, \mathsf{x}, \overset{\circ}{\mathsf{x}})$ where $^\circ$ represents $\overset{\circ}{\mathsf{x}} = \frac{d\mathsf{x}}{ds}$ so that

$$S = \int_{s_1}^{s_2} ds\ L(s, \mathsf{x}, \overset{\circ}{\mathsf{x}})$$

$$\delta S = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} \delta x_n^\mu + \frac{\partial L}{\partial \overset{\circ}{x}{}_n^\mu} \delta \overset{\circ}{x}{}_n^\mu \right) = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \overset{\circ}{x}{}_n^\mu} \right) \delta x_n^\mu + \sum_n \left[\frac{\partial L}{\partial \overset{\circ}{x}{}_n^\mu} \delta x_n^\mu \right]_{s_1}^{s_2}$$

Impose Hamilton's Principle

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \overset{\circ}{x}{}_n^\mu} = 0$$

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