



Introduction to Optimization

CO 255



Prof. Ricardo Fukasawa

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 255 during Winter 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

For any questions, send me an email via <https://notes.sibeliusp.com/contact/>.

You can find my notes for other courses on <https://notes.sibeliusp.com/>.

Sibeliusp Peng

Contents

Preface	1
0 Info	3
1 Introduction	4
1.1 Linear Optimization (Programming)	5
1.1.1 Determining Feasibility	7
1.1.2 Fourier-MotzKin Elimination	7

Info

Ricardo: MC 5036. OH: M 1:30 - 3pm
TA: Adam Brown: MC 5462. OH: F 10-11am

Books (not required)

- Intro to Linear Opt. Bertsimas
- Int Programming. Conforti

Grading

- assns: 20% (≈ 5)
- mid: 30% (Feb 11 in class)
- final: 50%

Introduction

Given a set S , and a function $f : S \rightarrow \mathbb{R}$. An optimization problem is:

$$\begin{array}{ll} \max f(x) \\ \underbrace{s.t.}_{\text{subject to}} x \in S & (\text{OPT}) \end{array}$$

- S **feasible region**
- A point $\bar{x} \in S$ is a **feasible solution**
- $f(x)$ is **objective function**

(OPT) means: “Find a feasible solution x^* such that $f(x) \leq f(x^*), \forall x \in S$ ”

- Such x^* is an **optimal solution**
- $f(x^*)$ is **optimal value**

Other ways to write (OPT):

$$\max\{f(x), x \in S\}$$

$$\max_{x \in S} f(x)$$

Analogous problem

$$\begin{array}{ll} \min f(x) \\ s.t. \quad x \in S \end{array}$$

Note

$$\max_{s.t. \quad x \in S} f(x) = -1 \left(\min_{s.t. \quad x \in S} -f(x) \right)$$

Problem x^* may not exist

a) Problem is unbounded:

$$\forall M \in \mathbb{R}, \exists \bar{x} \in S, \text{ s.t. } f(\bar{x}) > M$$

b) $S = \emptyset$, i.e. (OPT) is **INFEASIBLE**

c) There may not exist x^* achieving supremum.

Example:

$$\begin{array}{ll} \max & x \\ \text{s.t.} & x < 1 \end{array}$$

supremum

$$\sup\{f(x) : x \in S\} = \begin{cases} +\infty & \text{if OPT unbounded} \\ -\infty & \text{if } S = \emptyset \\ \min\{x : x \geq f(x), \forall x \in S\} & \text{otherwise} \end{cases}$$

always exist and are well-defined

infimum

$$\inf\{f(x) : x \in S\} = -1 \cdot \sup\{-f(x) : x \in S\}$$

From this point on, we will abuse notation and say $\max\{f(x) : x \in S\}$ is $\sup\{f(x) : x \in S\}$.

One way to specify that I want an opt. sol. (if exists) is

$$x^* \in \operatorname{argmax}\{f(x) : x \in S\}$$

1.1 Linear Optimization (Programming)

or (LP).

$$S = \{x \in \mathbb{R}^n : Ax \leq b\}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $f(x) = c^T x$, $c \in \mathbb{R}^n$.

$$\begin{array}{ll} \downarrow \\ \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \quad (LP)$$

Note

$$A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix} \quad A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

Clarifying

$$u, v \in \mathbb{R}^n, \quad u \leq v \iff u_j \leq v_j, \forall j \in 1, \dots, n$$

Note

$u \not\leq v$ is not the same as $u > v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \not\leq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example.

$$\begin{array}{lll} \max & 2x_1 + & 0.5x_2 \\ \text{s.t.} & x_1 & \leq 2 \\ & x_1 + & x_2 \leq 2 \\ & x & \geq 0 \end{array}$$

- Strict ineq. not allowed

halfspace, hyperplane, polyhedron

Let $h \in \mathbb{R}^n, h_0 \in \mathbb{R}$.

$\{x \in \mathbb{R}^n : h^T \leq h_0\}$ is a **halfspace**.

$\{x \in \mathbb{R}^n : h^T = h_0\}$ is a **hyperplane**.

$Ax \leq b$ is a **polyhedron** (i.e. intersection of finitely many halfspaces).

Example.

n products, m resources. Producing $j \in \{1, \dots, n\}$ given c_j profit/unit and consumes a_{ij} units of resource $i, \forall i \in \{1, \dots, m\}$. There are b_i units available $\forall i \in \{1, \dots, m\}$.

$$\begin{array}{ll} \max & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i = 1, \dots, m \\ & x \geq 0 \end{array}$$

which is an LP.

1.1.1 Determining Feasibility

Given a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

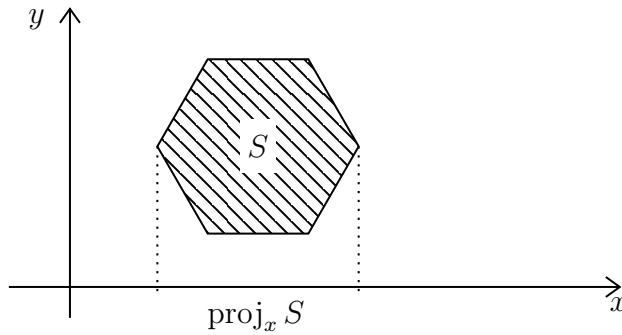
either find $\bar{x} \in P$ or show $P = \emptyset$.

Idea In 1-d, easy. \rightarrow Reduce problem in dimension n to one in dimension $n - 1$.

Notation Let $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \leq b\}$, then

$$\text{proj}_x S := \{x \in \mathbb{R}^n : \exists y \text{ so that } (x, y) \in S\}$$

is the (orthogonal) *projection* of S onto x .



We will find if $P = \emptyset$ by looking at $\text{proj}_{x_1, \dots, x_{n-1}}$ (P)

1.1.2 Fourier-MotzKin Elimination

Call a_{ij} entries of A . Let

$$\begin{aligned} M &:= \{1, 2, \dots, m\} \\ M^+ &:= \{i \in M : a_{in} > 0\} \\ M^- &:= \{i \in M : a_{in} < 0\} \\ M^0 &:= \{i \in M : a_{in} = 0\} \end{aligned}$$

For $i \in M^+$ (1):

$$a_i^T x \leq b_i \iff \sum_{j=1}^n a_{ij} x_j \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j + x_n \leq \frac{b_i}{a_{in}}, \quad \forall i \in M^+$$

For $i \in M^-$ (2):

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j - x_n \leq \frac{b_i}{-a_{in}}, \quad \forall i \in M^-$$

For $i \in M^0$ (3):

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} a_{ij} x_j \leq b_i, \quad \forall i \in M^0$$

$$P = \{x \in \mathbb{R}^n : (1)(2)(3)\}$$

Define (4):

$$\sum_{j=1}^{n-1} \left(\frac{a_{ij}}{a_{in}} - \frac{a_{kj}}{a_{kn}} \right) x_j \leq \frac{b_i}{a_{in}} - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, \forall k \in M^-$$

Theorem 1.1

$$(\bar{x}_1, \dots, \bar{x}_{n-1}) \text{ satisfies (3), (4)} \iff \exists \bar{x}_n : (\bar{x}_1, \dots, \bar{x}_n) \in P$$

Proof:

\Leftarrow If $(\bar{x}_1, \dots, \bar{x}_n)$ satisfies (1), (2), (3) then $(\bar{x}_1, \dots, \bar{x}_{n-1})$ satisfies (3) and adding (1), (2) \Rightarrow $(\bar{x}_1, \dots, \bar{x}_{n-1})$ satisfies (4)

\Rightarrow If $(\bar{x}_1, \dots, \bar{x}_{n-1})$ satisfies (4)

$$\sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, k \in M^-$$

Let

$$\bar{x}_n := \max_{i \in M^+} \left\{ \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \right\}$$

$$\Rightarrow \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq -\bar{x}_n, \quad \forall i \in M^+$$

and

$$-\bar{x}_n \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall k \in M^-$$

$$\Rightarrow (\bar{x}_1, \dots, \bar{x}_n) \in P$$

□

Note

Proof assumes M^+, M^- are nonempty. But statement holds regardless.

(if M^+ or $M^- = \emptyset$ then (4) yields no constraints)

Fourier-MotzKin

- $A^n = A, b^n = b$
- given A^i, b^i obtain A^{i-1}, b^{i-1} (A^{i-1} has one less column than A^i column than A^i) by applying the steps described

$$P_i := \{x \in \mathbb{R}^i : A^i x \leq b^i\}$$

then

$$P_{i-1} = \text{proj}_{x_1, \dots, x_{i-1}} P_i$$

$$\text{and } P_{i-1} = \emptyset \iff P_i = \emptyset.$$

- Keep applying projection until $i = 1$.

$$P_0 = \emptyset \iff P_n = P = \emptyset$$

Let

$$P_i^n = P_i \times \mathbb{R}^{n-i} = \{x \in \mathbb{R}^n (A^i, 0)x \leq b^i\}$$

$$\text{not hard to see } P_i^n = \emptyset \iff P_i = \emptyset$$

Notice that

$$P_0 = \emptyset \iff P_0^n = \emptyset, P_0^n = \{0 \leq b^0\}$$

Example.

$$P_2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{rrcl} x_1 & +x_2 & \leq & 1 \\ -x_1 & & \leq & 0 \\ & -x_2 & \leq & -2 \\ -3x_1 & -3x_2 & \leq & -6 \end{array} \right\}$$

draw the graph, clearly empty

$$M^+: \frac{1}{2}x_1 + x_2 \leq \frac{1}{2}$$

$$M^-: -x_2 \leq -2 \quad -x_1 - x_2 \leq -2$$

$$M^0: -x_1 \leq 0$$

$$P_1 = \left\{ x_1 \in \mathbb{R} : \begin{array}{rrcl} & -x_1 & \leq & 0 \\ \frac{1}{2}x_1 & & \leq & -\frac{3}{2} \\ & -\frac{1}{2}x_1 & \leq & -\frac{3}{2} \end{array} \right\}$$

$$M^+: x_1 \leq -3$$

$$M^-: -x_1 \leq 0 \text{ and } -x_1 \leq -3$$

$$P_0^2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{l} 0 \leq -3 \\ 0 \leq -6 \end{array} \right\} = \emptyset$$

Here $b^0 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

Remark:

Inequality in P_i^n :

- All inequalities are obtained by a nonnegative combination of inequality in P_{i+1}^n
 \implies all nonnegative combination of inequalities in P .
- If all A, b are rational then so are all A^i, b^i
- If $b = 0, b_i = 0, \forall i$

Theorem 1.2: Farkas' Lemma

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = \emptyset \iff \begin{array}{l} \exists u \in \mathbb{R}^m : u^T A = 0 \\ u^T b < 0 \\ u \geq 0 \end{array}$$

Index

H

halfspace, hyperplane, polyhedron.. 6

I

infimum 5

S

supremum 5