



# *Introduction to General Relativity*

AMATH 475



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# Preface

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# Contents

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Preface	1
0 Pre	3
0.1 Index notation . . . . .	3
0.2 Vectors and one-forms . . . . .	4

# Pre

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## 0.1 Index notation

$$A = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} \quad B = \begin{pmatrix} B^1_1 & B^1_2 \\ B^2_1 & B^2_2 \end{pmatrix}$$

$$(A \cdot B)^a_b = A^a_c B^c_b = B^c_b A^a_c \quad \text{sum over all possible } c$$

Identify followings:

$$\begin{aligned} B_\kappa^\nu A_\mu^\kappa &= A_\mu^\kappa B_\kappa^\nu = C_\mu^\nu = (A \cdot B)_\mu^\nu \\ A^\kappa_\mu B_\kappa^\nu &= D_\mu^\nu = (A^T)_\mu^\kappa B_\kappa^\nu = (A^T \cdot B)_\mu^\kappa \\ A_\kappa^\nu B_\mu^\kappa &= E_\mu^\nu = (B \cdot A)_\mu^\nu \\ A^\kappa_\mu B^\nu_\kappa &= (A^T)_\mu^\kappa (B^T)_\kappa^\nu = \left( (B \cdot A)^T \right)_\mu^\nu \end{aligned}$$

$$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 \quad \{\mathbf{e}_1, \mathbf{e}_2\} \text{ Basis 1.}$$

$$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a \quad \{\mathbf{e}'_1, \mathbf{e}'_2\} \text{ Basis 2.}$$

Change of basis matrix  $\Lambda$

$$\begin{aligned} \mathbf{e}'_a &= \Lambda_a^b \mathbf{e}_b \\ v'^a &= \tilde{\Lambda}^a_b v^b \end{aligned}$$

$$\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a_b v^b \Lambda_a^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a_b \Lambda_a^c v^b \mathbf{e}_c \\
&= \underbrace{\left( \tilde{\Lambda}^T \right)_b^a}_{\delta_b^c} \Lambda_a^c v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b \\
\\
\Rightarrow \left( \tilde{\Lambda}^T \right)_b^a \Lambda_a^c &= \delta_b^c \\
\tilde{\Lambda}^T \cdot \Lambda &= \mathbb{1}
\end{aligned}$$

$\tilde{\Lambda}^T$  is the inverse transpose of  $\Lambda$

#### covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis.  $\Lambda$ . (sub-indices)

A contravariant object transforms like components of vectors. ( $\tilde{\Lambda} = (\Lambda^T)^{-1}$ ). (super-indices)

## 0.2 Vectors and one-forms

#### dual vector space

Let  $V$  be a vector space. A one-form is a linear map  $\omega : V \rightarrow \mathbb{R}$ .

The set of all one-forms on  $V$  (call  $V^*$ ) is a vector space as well called the dual vector space to  $V$ .

#### dual basis

Let  $\{\Upsilon_1, \Upsilon_2, \dots\}$  (or  $\{\Upsilon_i\}$ ) be a basis of  $V$  so that any  $\mathbf{v} \in V$  can be written as  $\mathbf{v} = v^i \Upsilon_i$ .

We define the dual basis (of  $V^*$ ) to  $\{\Upsilon_i\}$  as  $\{\omega^i\}$  such that  $\omega^i(\Upsilon_j) = \delta_j^i$ .

#### Proposition 0.1

The dual basis of  $V^*$  is actually a basis of  $V^*$ .

# Index

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C

covariant and contravariant object . 4

D

dual basis ..... 4  
dual vector space ..... 4