



Introduction to General Relativity

AMATH 475



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Preface

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Some of the notes (especially special relativity part) are projected to the screen instead of using blackboards. They can be found on <https://sites.google.com/site/emmfis/teaching/gr>.

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Pre-Math

0.1 Index notation

$$A = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} \quad B = \begin{pmatrix} B^1_1 & B^1_2 \\ B^2_1 & B^2_2 \end{pmatrix}$$

$$(A \cdot B)^a_b = A^a_c B^c_b = B^c_b A^a_c \quad \text{sum over all possible } c$$

Identify followings:

$$B_\kappa^\nu A_\mu^\kappa = A_\mu^\kappa B_\kappa^\nu = C_\mu^\nu = (A \cdot B)_\mu^\nu$$

$$A^\kappa_\mu B_\kappa^\nu = D_\mu^\nu = (A^T)_\mu^\kappa B_\kappa^\nu = (A^T \cdot B)_\mu^\kappa$$

$$A_\kappa^\nu B_\mu^\kappa = E_\mu^\nu = (B \cdot A)_\mu^\nu$$

$$A^\kappa_\mu B^\nu_\kappa = (A^T)_\mu^\kappa (B^T)_\kappa^\nu = \left((B \cdot A)^T \right)_\mu^\nu$$

$$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 \quad \{\mathbf{e}_1, \mathbf{e}_2\} \text{ Basis 1.}$$

$$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a \quad \{\mathbf{e}'_1, \mathbf{e}'_2\} \text{ Basis 2.}$$

Change of basis matrix Λ

$$\mathbf{e}'_a = \Lambda_a^b \mathbf{e}_b$$

$$v'^a = \tilde{\Lambda}^a_b v^b$$

$$\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a_b v^b \Lambda_a^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a_b \Lambda_a^c v^b \mathbf{e}_c \\
&= \underbrace{\left(\tilde{\Lambda}^T \right)_b^a}_{\delta_b^c} \Lambda_a^c v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b \\
\\
\Rightarrow \left(\tilde{\Lambda}^T \right)_b^a \Lambda_a^c &= \delta_b^c \\
\tilde{\Lambda}^T \cdot \Lambda &= \mathbb{1}
\end{aligned}$$

$\tilde{\Lambda}^T$ is the inverse transpose of Λ

covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis. Λ . (sub-indices)

A contravariant object transforms like components of vectors. $(\tilde{\Lambda} = (\Lambda^T)^{-1})$. (super-indices)

0.2 Vectors and one-forms

one-form

Let V be a vector space. A one-form is a linear map $\omega : V \rightarrow \mathbb{R}$.

or we write: $(\omega, \cdot) : V \rightarrow \mathbb{R}$ and $(\omega, \mathbf{v}) \in \mathbb{R}$.

dual vector space

The set of all one-forms on V (call V^*) is a vector space as well called the dual vector space to V .

dual basis

Let $\{\Upsilon_1, \Upsilon_2, \dots\}$ (or $\{\Upsilon_i\}$) be a basis of V so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = v^i \Upsilon_i$.

We define the dual basis (of V^*) to $\{\Upsilon_i\}$ as $\{\omega^i\}$ such that $\omega^i(\Upsilon_j) = \delta_j^i$.

For a one form ω we denote its “components of the basis Υ ” as $(\omega, \Upsilon_m) = \omega_m$

Proposition 0.1

The dual basis of V^* is actually a basis of V^* .

The action of $\omega \in V^*$ on a vector $\mathbf{v} = v^\mu \Upsilon_\mu \in V$ is

$$(\omega, \mathbf{v}) = (\omega, v^\mu \Upsilon_\mu) = v^\mu \omega_\mu$$

Let’s prove $\{\Upsilon^a\}$ is linear independent.

Proof:

A linear comb. $c_a \Upsilon^a$ acts on a vector $\mathbf{v} = v^a \Upsilon_a$

$$\begin{aligned} (c_a \Upsilon^a, \mathbf{v}) &= c_a (\Upsilon^a, \mathbf{v}) \\ &= c_a (\Upsilon^a, v^b \Upsilon_b) \\ &= c_a v^b \underbrace{(\Upsilon^a, \Upsilon_b)}_{\delta_b^a} \\ &= c_a v^b \delta_b^a = c_a v^a \end{aligned}$$

For LI,

$$\begin{aligned} c_a \Upsilon^a = 0 &\iff c_a = 0 \quad \forall a \\ c_a v^a = 0 \quad \forall \mathbf{v} &\iff c_a = 0 \end{aligned}$$

□

vectors: take one-forms $\rightarrow \mathbb{R}$ one-forms: take vectors $\rightarrow \mathbb{R}$

0.3 Tensor

type (m, n) tensor

A type (m, n) tensor is a multilinear map that

$$\mathbf{T} : V^n \otimes (V^*)^m \rightarrow \mathbb{R}$$

Components of \mathbf{T} :

$$\mathbf{T}(\Upsilon_{a_1}, \dots, \Upsilon_{a_n}, \Upsilon^{b_1}, \dots, \Upsilon^{b_m}) = T_{a_1 \dots a_n}{}^{b_1 \dots b_m}$$

1. Tensor product takes $\binom{m}{n}$ and $\binom{m'}{n'} \rightarrow \binom{m+m'}{n+n'}$ tensor
2. Contraction takes $\binom{m}{n} \rightarrow \binom{m-1}{n-1}$

Example:

1. $T_a{}^b, S_c{}^d.$

$$(\mathbf{T} \otimes \mathbf{S})_a{}^b{}_c{}^d = T_a{}^d S_c{}^d = P_a{}^b{}_c{}^d$$

2. $T_a{}^{bc} \rightarrow c^b T_a{}^{ba}$

$$v^a, w_b \begin{cases} v^a \omega_b \\ v^a \omega_a \end{cases}$$

If you have a favorite type $(2, 0)$ symmetric tensor \mathbf{g}

$$v_\mu = g_{\mu\nu} v^\nu$$

$g^{\mu\nu} :=$ components of the inverse of $\mathbf{g}_{\mu\nu}$

$$v^\nu = g^{\mu\nu}$$

then

$$g^{\mu\nu} g_{\nu\sigma} = \delta_\sigma^\mu$$

$$g_{\mu\nu} v^\mu w^\nu = v_\mu w^\mu = \mathbf{v} \cdot \mathbf{w}$$

$$||\mathbf{v}||^2 = g_{\nu\mu} v^\mu v^\nu$$

Then we can define the angle

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}|| ||\mathbf{v}||} := \cos \theta$$

$$T_\mu{}^\nu = g^{\nu\sigma} T_{\mu\sigma}$$

$$T^{\mu\nu} = g^{\nu\sigma} g^{\mu\rho} T_{\sigma\rho}$$

$$g_\mu{}^\nu = g^{\nu\sigma} g_{\sigma\mu} = \sigma_\mu{}^\nu$$

0.4 Levi-Civita symbol

Levi-Civita symbol $\epsilon^{abc\dots}, \epsilon_{abc\dots}$

- is antisymmetric

- $\epsilon^{1234\dots} = 1, \epsilon_{1234} = 1$

$$\epsilon^{123} = 1, \quad \epsilon^{213} = -1, \quad \epsilon^{312} = 1, \quad \epsilon^{113} = 0$$

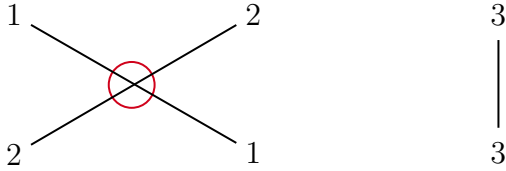
$$\epsilon^{123456} = 1, \quad \epsilon^{612453} = -1$$

Idea just see the permutations

Levi-Civita symbol

$$\varepsilon_{a_1 a_2 a_3 \dots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \dots, a_n) \text{ is an even permutation of } (1, 2, 3, \dots, n) \\ -1 & \text{if } (a_1, a_2, a_3, \dots, a_n) \text{ is an odd permutation of } (1, 2, 3, \dots, n) \\ 0 & \text{otherwise} \end{cases}$$

Here is a short-cut:



odd number crossings, so odd permutation.

Note that $\det(M) := \epsilon_{ijk\dots} M^i_1 M^j_2 M^k_3 \dots$

Exercise

prove $\epsilon^{i_1 i_2 \dots i_n} \epsilon_{j_1 j_2 \dots j_n} = n! \delta_{i_j}^{j_j} = 1, \dots, n$

$$\begin{aligned} \epsilon^{ijk} \epsilon_{ilm} &= \delta_l^j \delta_m^k - \delta_m^j \delta_l^k \\ \epsilon^{ijmn} \epsilon_{klmn} &= 2(\delta_k^i \delta_l^j - \delta_l^i \delta_k^j) \end{aligned}$$

Prove $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Proof:

Let $\vec{F} = \vec{A} \times (\vec{B} \times \vec{C})$ $\vec{D} = \vec{B} \times \vec{C}$

Then

$$\begin{aligned} D^k &= \epsilon^k_{ij} B^i C^j \\ F^l &= \epsilon^l_{mk} A^m D^k \implies F^l = \epsilon^l_{mk} \epsilon^k_{ij} A^m B^i C^j \end{aligned}$$

Then

$$\begin{aligned} F^l &= (\delta_i^l \delta_{mj} - \delta_j^l \delta_{mi}) A^m B^i C^j \\ &= \delta_i^l \delta_{mj} A^m B^i C^j - \delta_j^l \delta_{mi} A^m B^i C^j \\ &= B^l (A_j C^j) - C^l (A_i B^i) \end{aligned}$$

where we use

$$\vec{A} \cdot \vec{B} = A^i B_i$$

□

Special Relativity

1.1 Postulates of SR

Postulate 0

Newton's first law

Postulate 1: Principle of relativity

In the absence of gravity, all the laws of Physics are identical in all inertial reference frames.

Postulate 2

The speed of light in vacuum c is constant and the same from all inertial reference frames, regardless of their state of motion.

1.2 Lorentz Transformation

We define the spacetime interval Δs^2

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = -c^2 (t_2 - t_1)^2 + (\mathbf{x}_2 - \mathbf{x}_1)^2$$

Assuming the following:

1. The difference between the two frames is a constant speed $\lesssim c$
2. The transformation has to be linear.

$$t' = \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{x}}{c^2} \right), \quad \mathbf{x}' = \mathbf{x} + (\gamma - 1)(\mathbf{n} \cdot \mathbf{x})\mathbf{n} - \gamma \mathbf{v}t$$

and index notation

$$t' = \gamma \left(t - \frac{v_i x^i}{c^2} \right), \quad x^i = x^i + (\gamma - 1) \frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

1.3 Line element, proper time and spacelike, time-like and null separation

1.3.1 Classification of spacetime intervals

We can classify events according to the following criterion:

- Spacelike separated, $\Delta s^2 > 0$
- Timelike separated, $\Delta s^2 < 0$
- Lightlike (null) separated, $\Delta s^2 = 0$

Given the trajectory of a physical particle moving inertially, we will call co-moving frame (inertial) or proper frame (non-inertial) to the frame S_p where the particle is at rest.

1.3.2 Proper time and line element

$$ds^2 = -c^2 dt^2 + d\mathbf{x}^2$$

We will call ds^2 the spacetime line element.

$$\mathbf{v} := \frac{d\mathbf{x}}{dt}$$

P0, P1, P2 + linearity

$$\implies t' = t \left(t - \frac{v_i x^i}{c^2} \right) \tag{1}$$

$$x'^i = x^i + (\gamma - 1) \frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

Particle trajectory in a given inertial (Lab) frame $\mathbf{x}(t)$

Particle trajectory in its proper frame $\boldsymbol{\xi}(t) = 0$

Comoving frame's trajectories at each t (from lab frame) $\mathbf{x} = \mathbf{v}(t)t$.

$$d\tau = dt' = \gamma(t) \left(1 - \frac{\mathbf{v}(t)^2}{c^2} \right) dt \quad (2)$$

$$ds^2 = -c^2 dt^2 \left(1 - \frac{1}{c^2} \underbrace{\left(\frac{d\mathbf{x}}{dt} \right)^2}_{\mathbf{v}(t)} \right) = -c^2 \underbrace{\gamma^{-2} dt^2}_{d\tau^2} \implies ds^2 = -c^2 d\tau^2 \quad (3)$$

Example:

Find $\tau(t)$ for the three following trajectories.

1. $x(t) = v(t)$

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x}^2 \implies d\tau = \gamma^{-1} dt \implies \Delta\tau = \gamma^{-1} \Delta t$$

2. $x(t) = \frac{c^2}{a} \left[\sqrt{1 + \frac{a^2 t^2}{c^2}} - 1 \right]$

Then $\frac{dx}{dt} = \frac{at}{\sqrt{1 + \frac{a^2 t^2}{c^2}}}$

$$\left(\frac{d\tau}{dt} \right)^2 = 1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2$$

$$\implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2} \frac{a^2 t^2}{1 + \left(\frac{at}{c} \right)^2}}$$

$$\implies \tau(t) = \frac{c}{a} \operatorname{arcsinh} \left(\frac{at}{c} \right) \quad \text{and} \quad t(\tau) = \frac{c}{a} \sinh \left(\frac{a\tau}{c} \right)$$

3. $x(t) = L \sin(\omega t) \implies \frac{dx}{dt} = L\omega \cos(\omega t)$ with $L\omega < c$

$$\left(\frac{d\tau}{dt} \right)^2 = 1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2 \implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2} \implies d\tau = \sqrt{1 - \frac{L^2 \omega^2}{c^2}} dt$$

Then

$$\tau(t) = \frac{E \left(t\omega, \frac{1}{1 - \frac{L^2 \omega^2}{c^2}} \right)}{\omega \sqrt{\frac{1}{1 - \frac{L^2 \omega^2}{c^2}}}}$$

where

$$E(\phi|m) = \int_0^\phi (1 - m \sin^2 \theta)^{1/2} d\theta$$

1.4 Lorentzian Tensors

See notes for details.

A_μ transposes with Λ and it's covariant.

A^μ transposes with $\tilde{\Lambda} = (\Lambda^{-1})^T$ and it's contravariant.

1.5 Poincare group

The derivations are in notes.

1.6 Relativistic dynamics

1.6.1 Hamilton's principle and Euler-Lagrange equations

There exists at least one function (called action) of the trajectories that the degrees of freedom of a system may take in phase space. The physical trajectories are obtained demanding stationarity of this functional under variations that keep the initial and final positions constant.

Usually, the action S of a system of n particles can be written in terms of a Lagrangian $L(s, \mathbf{x}, \dot{\mathbf{x}})$ where $\dot{\mathbf{x}}$ represents $\frac{d\mathbf{x}}{ds}$ so that

$$S = \int_{s_1}^{s_2} ds L(s, \mathbf{x}, \dot{\mathbf{x}})$$

$$\delta S = \sum_n \int_{s_1}^{s_2} ds \left(\frac{\partial L}{\partial x_n^\mu} \delta x_n^\mu + \frac{\partial L}{\partial \dot{x}_n^\mu} \delta \dot{x}_n^\mu \right) = \sum_n \int_{s_1}^{s_2} ds \left(\frac{\partial L}{\partial x_n^\mu} - \frac{d}{ds} \frac{\partial L}{\partial \dot{x}_n^\mu} \right) \delta x_n^\mu + \sum_n \left[\frac{\partial L}{\partial \dot{x}_n^\mu} \delta x_n^\mu \right]_{s_1}^{s_2}$$

Impose Hamilton's Principle

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial x_n^\mu} - \frac{d}{ds} \frac{\partial L}{\partial \dot{x}_n^\mu} = 0$$

1.6.2 Conserved quantities and Noether's theorem

Noether's theorem

If the variation of the action around a physical trajectory under a continuous variation of the positions $\delta \mathbf{x}$ is zero, then the quantity

$$\delta Q = \sum_n \frac{\partial L}{\partial \dot{x}_n^\mu} \delta x_n^\mu$$

is conserved. That is

$$\frac{d(\partial Q)}{ds} = 0.$$

Proof:

See notes. □

1.6.3 Four-momentum

Let S be invariant under $\partial \mathbf{x} = \mathbf{n} \delta \alpha$.

$$\implies \delta Q = \frac{\partial L}{\partial \dot{x}^\mu} n^\mu \delta \alpha$$

is constant \implies the projection $\mathbf{n} \cdot \mathbf{p} = n^\mu p_\mu = \eta_{\mu\nu} n^\mu p^\nu$ (where $p_\mu := \frac{\partial L}{\partial \dot{x}^\mu}$) is conserved.

If the action is invariant under Lorentz transformation $\delta x^\mu = \delta \omega^\mu{}_\nu x^\nu$, then

$$J_{\mu\nu} := x_\mu p_\nu - x_\nu p_\mu$$

is conserved.

1.6.4 Angular momentum

The angular momentum \mathbf{J} associated to spatial rotations and the vector \mathbf{K} associated to boosts can be extracted directly from $J_{\mu\nu}$:

$$J^i = \frac{1}{2} \epsilon^{ijk} J_{jk}, \quad K_i = J_{i0}$$

1.6.5 Free particle dynamics

- S has to be a scalar (Invariant under Lorentz)
- Must coincide with the non-relativistic action in the limit $\frac{v}{c} \ll 1$.

$$\begin{aligned} S &= mc \int ds = -mc^2 \int d\tau = -mc^2 \int dt \frac{d\tau}{dt} = -mc^2 \int \frac{dt}{\gamma} = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} \\ &= -mc^2 \int dt \left[1 - \frac{1}{2} \frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right) \right] \end{aligned}$$

and

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} = -mc^2 + \frac{1}{2} m v^2 + \mathcal{O}\left(\frac{v^4}{c^4}\right)$$

Euler-Lagrange $\frac{d}{dt}(\gamma m \mathbf{v}) = 0$

$$p_i = \frac{\delta S}{\delta v^i} = \frac{\partial L}{\partial v^i} = m\gamma v_i, \quad \mathbf{p} = m\gamma \mathbf{v}$$

Hamiltonian

$$H = (\mathbf{p} \cdot \mathbf{v} - L)_{\mathbf{v} \rightarrow \mathbf{v}(\mathbf{p})} = \sqrt{m^2 c^4 + c^2 \mathbf{p}^2}$$

Let's introduce Four-velocity.

$$\frac{dx^\mu}{d\tau} =: \dot{x}^\mu \equiv u^\mu$$

solid dot means derivative w.r.t proper time.

$$\dot{x}^\mu := \frac{dx^\mu}{d\tau} = \frac{d}{d\tau} \begin{pmatrix} ct \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} c \frac{dt}{d\tau} \\ \frac{d\mathbf{x}}{d\tau} \end{pmatrix} = \begin{pmatrix} c \frac{dt}{d\tau} \\ \frac{d\mathbf{x}}{dt} \frac{dt}{d\tau} \end{pmatrix} = \gamma \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix}$$

If we choose action as (not four-velocity)

$$S = mv \int dt \sqrt{\eta_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}}$$

Lagrangian

$$L = mc \sqrt{-\eta_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}}$$

$$p_\mu = \frac{\delta S}{\delta \dot{x}^\mu} = m \dot{x}_\mu \implies p^\mu = m \dot{x}^\mu = m\gamma \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix}$$

p^0 in the proper frame: $p^0 = mc$, $\mathbf{p} = \mathbf{0}$. so cp^0 is energy.

Let's compute

$$p^\mu p_\mu = m^2 \dot{x}^\mu \dot{x}_\mu = -m^2 c^2$$

$$p^\mu p_\mu = -(p^0)^2 + \mathbf{p}^2$$

$$\implies -m^2 c^2 = -(p^0)^2 + \mathbf{p}^2 \implies p^0 = \frac{1}{c} \sqrt{m^2 c^4 + c^2 \mathbf{p}^2}$$

$$\implies E = \sqrt{m^2 c^4 + c^2 \mathbf{p}^2} = mc^2 \sqrt{1 + \gamma^2 \frac{v^2}{c^2}} = mc^2 + \frac{1}{2} m v^2 + \mathcal{O}\left(\frac{v^2}{c^2}\right)$$

- **Ultrarelativistic limit:** The kinetic term inside the square root is much larger than the rest energy of the particle: $pc \gg mc^2$, $E \approx cp$
- **Deep non-relativistic limit:** The rest energy is much larger than the kinetic energy of the particle: $mc^2 \gg pc$, $E \approx mc^2$

Two problems

You are designing a particle collider, you have two identical part of mass M and energy budget $E = 2\epsilon$. You have two strategies:

- a) spend $1/2E$ on each and accelerate them.
- b) spend E on one of them and accelerate it

Which one optimizes the center of mass energy?

Solution

a)



$$\text{Lab frame } p_1^\mu = \left(\frac{\epsilon}{c} + M_c, p, 0, 0\right) \quad p_2^\mu = \left(\frac{\epsilon}{c} + M_c, -p, 0, 0\right)$$

$$p_{lab}^\mu = p_1^\mu + p_2^\mu = \left(\frac{2\epsilon}{c} + 2M_c, 0, 0, 0\right) = p_{CM}^\mu$$

Then

$$E_{CM}^{(a)} = cp_{CM}^0 = 2\epsilon + 2Mc^2 = 2Mc^2 \left(1 + \frac{\epsilon}{\mu c^2}\right)$$

b) Here the p is different from the p above.

$$\text{Lab frame } p_1^\mu = \left(\frac{2\epsilon}{c} + Mc, p, 0, 0\right) \quad p_2^\mu = (Mc, 0, 0, c)$$

$$p_{lab}^\mu = p_1^\mu + p_2^\mu = \left(\frac{2\epsilon}{c} + 2Mc, p, 0, c\right)$$

Determine

$$p_1^\mu p_{1\mu} = -M^2 c^2 = -\left(\frac{4\epsilon^2}{c^2} + M^2 c^2 + M^2 c^2 + 4\epsilon M\right) + p^2$$

$$\implies p = \sqrt{\frac{4\epsilon^2}{c^2} + 4\epsilon M} = \frac{2\epsilon}{c} \sqrt{1 + \frac{Mc^2}{\epsilon}}$$

We want p_{cm}^0 , and we know $p_{CM}^\mu = (p_{CM}^0, \mathbf{0})$

Lorentz scalar: $p_{CM}^\mu p_{CM\mu} = -(p_{CM}^0)^2$ and lab frame $p_{lab}^\mu p_{lab\mu} = -(p_{CM}^0)^2$

$$\begin{aligned}
 p_{lab}^\mu p_{lab\mu} &= - \left(\frac{4\epsilon^2}{c^2} + 4M^2c^2 + 8\epsilon M \right) + \frac{4\epsilon^2}{c^2} \left(1 + \frac{Mc^2}{\epsilon} \right) \\
 &= -4M^2c^2 - 8\epsilon M + 4\epsilon M \\
 &= -4M(Mc^2 + \epsilon) \\
 &= -(p_{CM}^0)^2
 \end{aligned}$$

$$p_{CM}^0 = \sqrt{-p_{lab}^\mu p_{lab\mu}} = 2\sqrt{\mu(Mc^2 + \epsilon)} \implies E_{CM}^{(b)} = cp_{CM}^0 = 2Mc^2\sqrt{1 + \frac{\epsilon}{Mc^2}}$$

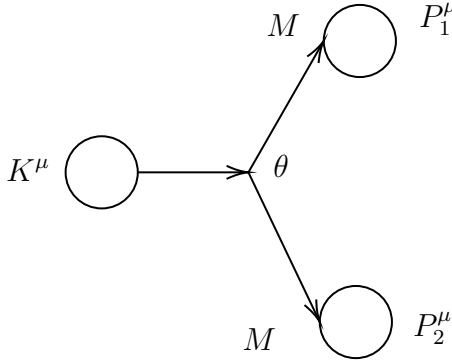
So

$$R = \frac{E_{CM}^{(a)}}{E_{CM}^{(b)}} = \sqrt{1 + \frac{\epsilon}{Mc^2}} > 1$$

In Deep non-real $\epsilon \ll Mc^2$, $\lim_{\frac{\epsilon}{Mc^2} \rightarrow 0} R = 1$.

In Ultra limit $\lim_{\epsilon \rightarrow pc} R = \sqrt{1 + \frac{pc}{Mc^2}} \rightarrow \infty$.

A massless particle cannot \rightarrow two identical mass particle. (converse is also true).



$$K^\mu = P_1^\mu + P_2^\mu$$

$$K^\mu K_\mu = P_1^\mu P_{1\mu} + P_2^\mu P_{2\mu} + 2P_1^\mu P_{2\mu} \quad (1)$$

where $K^\mu K_\mu = 0$, $P_1^\mu P_{1\mu} = -M^2c^2 = P_2^\mu P_{2\mu}$, and

$$P_1^\mu P_{2\mu} = \eta_{\mu\nu} P_1^\mu P_2^\nu = -P_1^0 P_2^0 + \mathbf{P}_1 \cdot \mathbf{P}_2 = -\frac{1}{c^2} + \mathbf{P}_1 \cdot \mathbf{P}_2$$

Sub them into (1), we get

$$\mathbf{P}_1 \cdot \mathbf{P}_2 = M^2c^2 + \frac{E_1 E_2}{c^2} \leq \|\mathbf{P}_1\| \|\mathbf{P}_2\| \implies \frac{E_1 E_2}{c^2} \leq \|\mathbf{P}_1\| \|\mathbf{P}_2\|$$

where we used

$$\|\mathbf{P}_1\| \|\mathbf{P}_2\| \cos \theta \leq \|\mathbf{P}_1\| \|\mathbf{P}_2\|$$

$$M^2c^2 + \frac{E_1 E_2}{c^2} \leq \|\mathbf{P}_1\| \|\mathbf{P}_2\| \geq \frac{E_1 E_2}{c^2} \quad (2)$$

$$E_1 = \sqrt{M^2 c^4 + \|\mathbf{P}_1\|^2 c^2} \implies \|\mathbf{P}_1\| = \sqrt{\frac{E_1^2}{c^2} - M^2 c^2} \implies \|\mathbf{P}_1\| < \frac{E_1}{c} \quad (3)$$

$$E_2 = \sqrt{M^2 c^4 + \|\mathbf{P}_2\|^2 c^2} \implies \|\mathbf{P}_2\| = \sqrt{\frac{E_2^2}{c^2} - M^2 c^2} \implies \|\mathbf{P}_2\| < \frac{E_2}{c} \quad (4)$$

$$M^2 c^2 + \frac{E_1}{c} \frac{E_2}{c} \leq \|\mathbf{P}_1\| \|\mathbf{P}_2\| \stackrel{(4)}{\leq} \frac{E_1}{c} \frac{E_2}{c} \implies M^2 c^2 < 0$$

which is impossible.

1.7 Accelerated observers and the Rindler metric

1.7.1 Four-acceleration

$$x^\mu \rightarrow \frac{dx^\mu}{d\tau} = \dot{x}^\mu \equiv e^\mu, \quad \frac{du^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} = \ddot{x}^\mu \equiv b^\mu$$

We know $(\dot{x}^\mu) = \gamma(c, \mathbf{v})$.

$$\frac{d}{d\tau} = \frac{dt}{d\tau} \frac{d}{dt}$$

then

$$\frac{d\gamma}{dt} = \gamma^2 \mathbf{v} \cdot \mathbf{a}$$

where $\mathbf{v} := \frac{d\mathbf{x}}{dt}$, $\mathbf{a} := \frac{d^2 \mathbf{x}}{dt^2}$. Then

$$(b^\mu) = \begin{pmatrix} b^0 \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \frac{\gamma^4}{c} \mathbf{v} \cdot \mathbf{a} \\ \frac{\gamma^4}{c^2} (\mathbf{v} \cdot \mathbf{a}) \mathbf{v} + \gamma^2 \mathbf{a} \end{pmatrix}$$

In the co-moving frame, we have $\mathbf{v} = \mathbf{0}$, $\gamma = 1$, then $(b^\mu) = \begin{pmatrix} 0 \\ \mathbf{a} \end{pmatrix}$.

In general,

$$b^\mu b_\mu = \gamma^4 \left[\frac{\gamma^2}{c^2} (\mathbf{v} \cdot \mathbf{a})^2 + \mathbf{a}^2 \right] \geq 0$$

In the co-moving frame, $b^\mu b_\mu = \mathbf{a}^2$, proper acceleration $|\mathbf{a}| = \sqrt{b^\mu b_\mu}$.

Now let's compute this

$$b_\mu \dot{x}^\mu = \frac{d\dot{x}^\mu}{d\tau} \dot{x}_\mu = \frac{1}{2} \frac{d}{d\tau} (\dot{x}^\mu \dot{x}_\mu) = \frac{1}{2} \frac{d}{d\tau} (-c^2) = 0$$

1.7.2 Constantly accelerated

From the co-moving frame (t, \mathbf{x}) , at time $t = 0$, $v(0) = 0$.

$$\begin{aligned} \frac{dp^i}{dt} &= mb^i \implies m \frac{d\gamma \mathbf{v}}{dt} = m \mathbf{a} \implies \mathbf{a} = \frac{d(\gamma \mathbf{v})}{dt} \\ \implies a dt &= \gamma \left(\gamma^2 \frac{v^2}{c^2} + 1 \right) dv \implies a dt = \frac{dv}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \\ \implies v &= \frac{dx}{dt} = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \implies x = \frac{c^2}{a} \left[\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1 \right] \end{aligned}$$

With initial condition $t = 0, \tau = t$, we get

$$\frac{d\tau}{dt} = \gamma^{-1} = \sqrt{1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2} = \sqrt{1 - \frac{1}{c^2} \frac{a^2 t^2}{1 + \left(\frac{at}{c}\right)^2}} \implies \tau = \frac{c}{a} \operatorname{asinh} \left(\frac{at}{c} \right) \implies t = \frac{c}{a} \sinh \left(\frac{a\tau}{c} \right)$$

which, using the properties of the hyperbolic functions

$$x = \frac{c^2}{a} \left[\cosh \left(\frac{a\tau}{c} \right) - 1 \right], \quad t = \frac{c}{a} \sinh \left(\frac{a\tau}{c} \right)$$

Let's find coordinates (τ, ξ) such that the particle going with trajectory $(t(\tau), \mathbf{x}(\tau))$ is always at $(0, 0)$.

$$t = \left(\frac{c}{a} + \frac{\xi}{c} \right) \sinh \left(\frac{a\tau}{c} \right), \quad x = \left(\frac{c^2}{a} + \xi \right) \cosh \left(\frac{a\tau}{c} \right) - \frac{c^2}{a}$$

(τ, ξ) are called Rindler coordinates.

1.7.3 Flat spacetime in Rindler coordinates

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \quad (\text{Minkowski})$$

Apply the transformation above $t = \dots \sinh(\dots), x = \dots$, we get

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \rightarrow ds^2 = - \left(1 + \frac{a\xi}{c^2} \right)^2 c^2 d\tau^2 + d\xi^2 + dy^2 + dz^2$$

Just use elementary calculus/arithmetic knowledge to derive it. This is “a” Rindler metric.

In Rindler, light

$$0 = - \left(1 + \frac{a\xi}{c^2} \right)^2 c^2 d\tau^2 + d\xi^2 \implies \frac{d\xi}{d\tau} = \left(1 + \frac{a\xi}{c^2} \right) c$$

So

$$0 < \frac{d\xi}{d\tau} < \infty$$

If we integrate it,

$$\xi(\tau) = \frac{c^2}{a} \left[\xi_c \exp\left(\frac{a\tau}{c}\right) - 1 \right]$$

which is exponential trajectory.

1.8 Bell's Paradox

The rope breaks, all right!

Solve the Paradox in Assignment 1.

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