ODE 2

AMATH 351

Xinzhi Liu

LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of AMATH 351 during Fall 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

I might not have time to complete the proof. For the complete proof, check lecture notes, course notes or https://notes.sibeliusp.com/pdfs/1195/amath351.pdf.

For any questions, send me an email via https://notes.sibeliusp.com/contact/.

You can find my notes for other courses on https://notes.sibeliusp.com/.

Sibelius Peng

Contents

1

Introduction and Review

1.1 Definitions and Terminology

A **differential equation** is any equation involving a function and derivatives of this function.

Ordinary differential equations contain only functions of a single variable, called the independent variable, and derivatives with respect to that variable.

Partial differential equations contain a function of two or more variables and some partial derivatives of this function.

The **order** of a differential equation in the order of the highest derivative in the equation.

A general n-th order ODE has the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.1}$$

where $y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}$ and so on. We assume further it can be written as

$$y^{(n)} = f(x, y', \ldots, y^{(n-1)}). \tag{1.2}$$

Eq. (1.2) is said to be **linear** when f is a linear function of $y, y', \ldots, y^{(n-1)}$. In this case, Eq. (1.2) can be written as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x). \tag{1.3}$$

A differential equation that is not linear is said to be **nonlinear**.

By a **solution** of Eq. (1.2) on an interval I we mean a function $y = \psi(x)$ such that $f(x, \psi(x), \psi'(x), \ldots, \psi^{(n-1)(x)})$ is defined for all x in I and is equal to $\psi^{(n)}(x)$ for all x in I.

A solution in which the dependent variable in expressed only in terms of the independent variable and constants is called an **explicit solution**.

A relation $G(x, y) = 0$ such that there exists at least one function $\psi(x)$ that satisfies the relation and Eq. (1.2) is called an **implicit solution**.

A solution which is free of arbitrary constants is called a **particular solution**.

A solutions that cannot be obtained by specializing any of the parameters in a family of solutions is called a **singular solution**.

Example:

Consider the DE $y' = xy^{1/2}$.

The explicit solution: $y = \left(\frac{x^2}{4} + c\right)^2$

A particular solution is $y = \frac{x^4}{16}$ obtained above for $c = 0$.

A singular solution is $y = 0$ which cannot be obtained from the explicit solution for any choice of constant c.

1.2 Initial-Value Problems

On some interval containing x_0, the problem

Solve $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ subject to the initial conditions $y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_{n-1}$

where $y_0, \ldots, y_{n-1}$ are arbitrary specified real constants, is called an **initial-value problem** (IVP).

Consider the IVP $y' = f(x, y)$, $y(x_0) = y_0$.

Theorem 1.1: Picard

Let D be a rectangular region in the xy-plane defined by $D = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ and $(x_0, y_0) \in D$ the interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution $y(x)$ defined in an interval I centered at x_0.

1.3 First Order ODE

Separable variables

A first order DE of the form

$$\frac{dy}{dx} = g(x)h(y) \tag{1.4}$$

is said to be **separable** or to have **separable variables**. Solution method:

$$\frac{dy}{h(y)} = g(x)dx$$

Integrate both sides

$$\int \frac{1}{h(y)}dy = \int g(x)dx + C$$

Linear equations

A first order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1.5}$$

is called a **linear equation**.

Solution method:

- Write in its **standard form**

$$\frac{dy}{dx} + p(x) = f(x)$$

- Multiply both sides by the integrating factor $\mu(x) = \exp\left(\int p(x)dx\right)$, and rearrange into the exact form $\frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$
- Integrate both side with respect to x and get the general solution under the form

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

There are other type of ODEs that you learned how to solve in AMATH 251, such as homogeneous equations, exact equations, Bernouli equations.

2

Theory of Second-Order Linear DEs

2.1 2nd-Order Linear ODEs

The most general 2nd order linear DE is

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

In AMATH 251 we learned how to solve this equation where the coefficients a_2, a_1, a_0 are constants. This equation can be written in several different forms:

1. General form
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{2.1}$$

2. Standard form: If $a_2(x)$ is not identically zero then we obtain
$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.2}$$

3. Associated homogeneous equation: This is the same as the standard form where RHS is zero,
$$y'' + P(x)y' + Q(x)y = 0 \tag{2.3}$$

   If RHS of Eq. (2.2) is non-zero the equation is said to be non-homogeneous or inhomogeneous.

2.2 Existence and Uniqueness

Existence and Uniqueness Before we try and find solutions to the DEs it is usually a good idea to know that a solution exists and it is unique. Otherwise we could be wasting out time. We state a theorem for existence and uniqueness. The ideas of the proof will be presented later when we discuss first-order systems.

Theorem 2.1: Existence and Uniqueness

Let $P(x), Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$ and if $y(x_0)$ and $y'(x_0)$ are any numbers, then Eq. (2.2) has one and only one solution $y(x)$ on the entire interval such that the initial conditions (ICs) are satisfied.

Remark:
If we are looking for a solution to the homogeneous equation with $y(0) = 0, y'(0) = 0$ observe that the trivial solution is an allowable solution. Therefore, by the existence and uniqueness theorem, it must be the only solution.

2.3 General Solutions to 2nd-order DEs

In AMATH 251 for the case of constant coefficients, we learned that the general solution to Eq. (2.2) is a superposition of any particular solution to the non-homogeneous problem and a general solution to the homogeneous one. This also holds true in the case of non-constant coefficients. Therefore, the method of attack is as follows:

1. Find the general solution to the homogeneous problem: In the case of constant coefficients we simply sub $y = e^{rx}$ find the characteristic equation, solve for the characteristic roots, r_1, r_2 form the two independent solutions $y_1(x), y_2(x)$ and get that the general solution is,

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

   where c_1, c_2 are arbitrary constants. In the case of non-constant coefficients we need to do more work to find y_1, y_2. In general we cannot find them explicitly.

2. Find a particular solution to the non-homogeneous problem. There are different methods that we can use.

The following Theorems will help us to find a unique solution of a general second-order scalar equation. First we look at the homogeneous problem and then at the more general DE.

Theorem 2.2: General solutions to 2nd-order homogeneous equations

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation (Eq. (2.3)) on the interval $[a, b]$, then the general solution to the same homogeneous problem is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

for arbitrary constants c_1, c_2.

Proof:
First we can sub y_1, y_2 and their linear superposition into the homogeneous equation to verify they are solutions.

Second we need to verify that this solution can satisfy any set of conditions, say $y(0)$ and $y'(0)$ [a]. We sub in our solution and find,

$$\begin{aligned} c_1 y_1(0) + c_2 y_2(0) &= y(0), \\ c_1 y_1'(0) + c_2 y_2'(0) &= y'(0). \end{aligned}$$

This is a system of two equations and two unknowns c_1, c_2. To be able to solve this for any initial conditions we need that the matrix is non-singular,

$$\det \begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} = y_1(0)y_2'(0) - y_2(0)y_1'(0) \neq 0$$

This motivates the definition of **Wronskian**, $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.

To ensure that our expression is a general solution we need that the initial value of the Wronskian is nonzero, $W(y_1(0), y_2(0)) \neq 0$. □

[a] These should be replaced by $y(x_0) = y_0, y'(x_0) = y_1$ for some $x_0 \in [a, b]$

Also check the alternative proof on page 66 of https://notes.sibeliusp.com/pdfs/1189/amath251.pdf.

Therefore, the above tells us that if the initial value of the Wronskian of the two solutions is non-zero, we have a general solution. Next, we will show that if the Wronskian is non-zero at the initial time it is necessarily non-zero all time. The following theorem states and proves this result.

Lemma 2.3: Uniformity of the Wronskian

If $y_1(x)$ and $y_2(x)$ are any two solutions to Eq. (2.3) on the interval $[a, b]$ then their Wronskian is either identically zero or never zero on $[a, b]$.

Proof:

$$\begin{aligned} W' &= y_1 y_2'' - y_2 y_1'' \\ &= y_1 \left[-P(x)y_2' - Q(x)y_2\right] - y_2 \left[-P(x)y_1' - Q(x)y_1\right] \\ &= -P(x) \left[y_1 y_2' - y_2 y_1'\right] \\ &= -P(x)W \end{aligned}$$

Then $W = W_0 \exp(-\int P(x)dx)$ is either zero everywhere or zero nowhere, depending on its initial values. $\square$

So far we know that the Wronskian is always zero or always non-zero for any two solutions. The next lemma shows the relation between the Wronskian and linear independence.

Lemma 2.4: Linear Dependence and the Wronskian

If $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous equation then they are linearly dependent on this interval if and only if their Wronskian is identically zero.

Proof:

$\Rightarrow$ Let $y_2(x) = cy_1(x)$ then calculate $W(y_1, y_2) = 0$

$\Leftarrow$ Assume Wronskian is zero, then

$$\det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = 0$$

If the determinant of the matrix is zero that means that the matrix is singular and that each column is a scalar multiple of the other. In particular, if we look at the first row we find that $y_2(x)$ is a linear multiple of $y_1(x)$. This is precisely the statement that the functions are linearly dependent.

$\square$

If we combine two lemmas we deduce that if two solutions are linearly independent at the initial time (or any other time for that matter) they are necessarily linear independent for all time. The next example is to give you some practice playing with the Wronskian.

Example:

Can show that $y = c_1 \sin x + c_2 \cos x$ is the general solution to $y'' + y = 0$ on any interval.

Now that we have a grasp on how to find general solutions to homogeneous equations, we can look at solving the non-homogeneous problem.

Theorem 2.5: General solutions to 2nd-order non-homogeneous equations

If $y_h(x)$ is the general solution to the homogeneous problem, Eq. (2.3), and $y_p(x)$ is any particular solution of the non-homogeneous problem, Eq. (2.2), then their superposition, $y(x) = y_h(x) + y_p(x)$, is a general solution to the non-homogeneous problem.

Proof:
To show that we have a general solution to the non homogeneous equation we must shows two things: 1) that it is in fact a solution and 2) we can reproduce any initial condition. With the second condition we can use our uniqueness theorem to guarantee that any two solutions to the DE with the same initial conditions must be equal.

1. skipped.
2. Now we must show that we can reproduce any IC with this solution. We do this as we did before by evaluating our solution and its derivatives at the initial time. First we do as before and define $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ and then we get,

$$\begin{aligned} c_1 y_1(0) + c_2 y_2(0) &= y(0) - y_p(x) \\ c_1 y_1'(0) + c_2 y_2'(0) &= y'(0) - y_p'(x) \end{aligned}$$

   Note that the effect of the particular solution is to offset how we pick our constants c_1 and c_2. By assumption, we have that our homogeneous solution consists of two linearly independent solutions, y_1 and y_2, and so their Wronskian is non-zero and so we can invert this 2×2 system to find a unique solution. This is enough to guarantee that our solution is a general solution to the non-homogeneous system.

□

2.4 BVPs versus IVPs

ODEs can be classified as either **Boundary Value Problems** (BVPs) or **Initial Value Problems** (IVPs). The equations themselves can be the same, what differs are the conditions that are imposed to determine the unknown constants. For IVPs the two conditions are imposed at the same time, for example $y(0) = \alpha, y'(0) = \beta$.

In contrast, in BVPs the two conditions are imposed at different times, or different locations, $y(0) = \alpha, y(1) = \beta$.

As the names suggest, IVPs usually have time as the independent variable and BVPs usually have space as the independent variable.

2.5 Examples of 2nd-Order DEs with non-constant coefficients

There is a long list of DEs with non-constant coefficients. Some of them are particularly famous and have special names. The solutions usually cannot be written in terms of simple functions and we define functions to be the solutions to such equations. They are usually referred to as **special functions**. Many of them arise in looking at solutions to Laplace's equations in different co-ordinate systems. The interested reader is directed to AM 353 for more details on how to obtain these equations.

1. Bessel's equation: p is a constant integer.
$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$
2. Legendre's equation: p integer.
$$(1 - x^2)y'' - 2xy' + p(p+1)y = 0$$
3. Laguerre's equation: constant a
$$xy'' + (1 - x)y' + ay = 0$$
4. Hermite's equation: constant a
$$y'' - 2xy' + 2ay = 0$$

2.6 Reduction of Order

There does not exist a general approach to find a solution for any general 2nd-order ODE. However, if we manage to find one solution to the homogeneous problem, say $y_1(x)$, there is a useful technique that allows us to find a second solution, $y_2(x)$. This is called Reduction of Order. The idea is very simple really. Look for a solution that is a product of some unknown function (that we have to determine) multiplied by the known solution, i.e.

$$y_2(x) = v(x)y_1(x)$$

some intermediate work... Sub it into DE and get

$$v' = \frac{1}{y_1^2}\exp\left(-\int P(x)dx\right)$$

Then

$$v = \int_0^x \frac{1}{y_1^2}\exp\left(-\int P(s)ds\right)dx$$

Therefore $y_2 = y_1(x)\int \frac{1}{y_1^2}\exp\left(-\int P(s)ds\right)dx$.

Then we can show y_2, y_1 are linearly independent.

> Example:
>
> $y_1 = x^2$ is an exact solution to the homogeneous DE $x^2y'' + xy' - 4y = 0$. Then by the procedure above, $v = -\frac{1}{4x^4}$. Hence $y_2 = \frac{1}{x^2}$. Thus the general solution is $y(x) = c_1x^2 + \frac{c_2}{x^2}$.

2.7 Method of Variation of Parameters

To find the particular solution to a non-homogeneous equation we can always use the method of variation of parameters.

Suppose that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions to the homogeneous problem. Next, we look for a trial solution to the non-homogeneous that is similar to the general solution to the homogeneous problem except that the constant coefficients are replaced with unknown functions, $v_1(x)$, $v_2(x)$, to be determined:

$$y = v_1y_1 + v_2y_2$$

A convenient choice to pick v_1, v_2 is

$$\boxed{v_1'y_1 + v_2'y_2 = 0}$$

Sub them into $y'' + Py' + Qy = R$ and we get in matrix form:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

y_1, y_2 are linearly independent, as we have assumed, that the system is invertible. The solution is,

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{R}{W}\begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

Then solution is thus

$$y = y_1\int^x \frac{-y_2R(x)}{W(y_1, y_2)}dx + y_2\int^x \frac{y_1R(x)}{W(y_1, y_2)}dx$$

In summary, given two linearly independent solutions to the homogeneous problem we can find a particular solution to the non-homogeneous equation where the inhomogeneity is $R(x)$.

Example:

Find a particular solution of $y'' + y = \csc x$.

We know homogeneous solution $y = c_1 \sin x + c_2 \cos x$. Using the formula above, we have

$$y = \sin x \log(\sin x) - x \cos x$$

3

Series Solutions and Special Functions

This Chapter discusses how to construct power series solutions of second-order ODEs with possibly non-constant coefficients. First we review some basic their of power series that is taught in MATH 138, then we show how to construction series solutions to various types of ODEs: ordinary points and singular points. When solving these equations we obtain special functions that arise naturally as solutions to famous equations.

3.1 Review of Power Series

Transcendental Function

These are elementary functions that consist of algebraic functions such as trigonometric, exponential, logarithmic and any of their combinations by addition and multiplication (and their inverses).

Special Functions

These are functions that cannot be expressed in terms of transcendental functions.

Examples of special functions are Bessel functions, which we will study in detail later in this chapter.

Review of Power Series:

1. A power series in x about x_0 is defined to be: $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. Usually we will set $x_0 = 0$ but that is not necessary.
2. A series is said to be **converge** at x if limit of $\lim_{m\to\infty} \sum_{n=0}^{m} a_n x^n$ exists and the sum of the series is the value of the limit.
3. Suppose a power series converges for $|x| < R$ for some $R > 0$ and let's denote its sum by $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

   The function $f(x)$ is smooth and we can differentiate term by term, and also integrate it term by term.

4. Given that $f(x)$ is a smooth function for $|x| < R$ with $R > 0$m then we can construct the power series for this function using Taylor's formula:

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

If it is not entirely smooth, then we can use Taylor's formula for the remainder to get a polynomial expansion of the function.

5. A function $f(x)$ has a power series expansion of

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

in a neighborhood of x_0 is said to be **analytic** at x_0. We can recover the coefficient using Taylor's formula,

$$a_n = \frac{f^{(n)}(x_0)}{n!}$$

to obtain Taylor series expansion of $f(x)$ about x_0.

3.2 Series Solutions of First-Order Equation

Consider $y' = y$. In AM 250 or AM 251 you learned how to solve this equation using various techniques, i.e. separable equations and integrating factors. Even though we already know how to solve it, let's try to solve it with a new technique, that of a power series solution. If we can solve a simple problem with this new method that should guide us along in solving more complicated problems.

Look for a solution of the following form,

$$y = \sum_{n=0}^{\infty} a_n x^n$$

where $a_0, \ldots, a_n, \ldots$ are to be determined.

Differentiate

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and sub into the equation,

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n = 0 \implies \sum_{n=0}^{\infty} [(n+1) a_{n+1} - a_n] x^n = 0$$

This equation is the power series representation of zero. But implies that each of the coefficients in the series is exactly zero and therefore,

$$a_{n+1} = \frac{a_n}{n+1} \quad \text{for } n = 0, 1, 2, \ldots$$

This is a recursion relation that defines a given coefficient in terms of the previous one:

$$a_n = \frac{a_0}{n!}$$

Therefore, after having done not so much work we have found our general solution to be,

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

which is nothing more than a constant times the power series representation of e^x.

3.3 2nd-Order Linear Equations: Ordinary Points

Now that we've seen the basic method in action let's try and tackle the more interesting homogeneous, second-order DE with non-constant coefficients,

$$y'' + P(x)y' + Q(x)y = 0$$

Not too surprisingly, the behaviour of the solution near x_0 is completely determined by the behaviour of $P(x)$ and $Q(x)$ near the same point.

A real number x_0 is called an **ordinary point** if $P(x)$ and $Q(x)$ are analytic at x_0, i.e.,

$$P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \quad |x - x_0| < r \qquad \text{and} \qquad Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n, \quad |x - x_0| < r$$

for some $r > 0$.

Any point that is not ordinary is referred to as a **singular point**.

Example: Simple harmonic oscillator

To get started finding power series solutions we begin with the simple harmonic oscillator,

$$y'' + \alpha^2 y = 0$$

We begin with a trial solution that is a power series and compute the second derivative:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

We can now substitute our solution into our governing DE and get,

$$\sum_{n=0}^{\infty} \alpha^2 a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n = 0$$

$$\sum_{n=0}^{\infty} [\alpha^2 a_n x^n + (n+2)(n+1)a_{n+2} x^n] = 0$$

For this to be zero we need that the coefficient of each term in the series is identically zero:

$$a_{n+2} = -\frac{\alpha^2 a_n}{(n+2)(n+1)}$$

One difference between this and the first-order example is that every coefficient is related with two coefficients before. Thus, to specify a unique solution we must specify a_0 and a_1. Using recursion we can determine all the coefficients in terms of the first two.

$$a_{2n} = (-1)^n \frac{\alpha^{2n} a_0}{(2n)!} \qquad a_{2n+1} = (-1)^n \frac{\alpha^{2n} a_1}{(2n+1)!}$$

The general pattern is readily observed. If we substitute this into the original power series and separate the solution into two linearly independent parts we get,

$$y = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n} a_0}{(2n)!} + a_1 \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n} a_1}{(2n+1)!}$$

Observe that the first term above is a scalar multiple of $\cos(\alpha x)$ and the second term is a scalar multiple of $\sin(\alpha x)$.

Therefore, even though we used a completely different method, we obtained the same solution we did in AM 250 or AM 251. That shouldn't be surprising it should be expected! Even though there wasn't a need to use this method on this relatively simple equation it was useful and sets up the machinery in how to deal with more general equations with non-constant coefficients.

To illustrate the real power behind power series solutions let's tackle Legendre's equation,

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$

Example: Legendre's equation

To get this into the standard form we divide through by $1-x^2$, we then get $P(x)$ and $Q(x)$. We conclude that both of these functions are analytic about $x = 0$ and thus it is an ordinary point. They are not analytic about $x = \pm 1$ and thus they are singular points.

Similarly, we have the recurrence relation:

$$a_{n+2} = -\frac{p(p+1) - n(n+1)}{(n+2)(n+1)} a_n = -\frac{(p-n)(p+n+1)}{(n+2)(n+1)} a_n$$

Then

$$\begin{aligned} a_{2n} &= \frac{(-1)^n [\prod_{i=0}^{n-1}(p-2i)][\prod_{i=1}^{n}(p-1+2i)]}{(2n)!} a_0 \\ a_{2n+1} &= \frac{(-1)^n [\prod_{i=0}^{n-1}(p-1-2i)][\prod_{i=1}^{n}(p+2i)]}{(2n+1)!} a_1 \end{aligned}$$

We can substitute this into our solution and obtain the following power series solution to Legendre's equation:

$$y = \ldots$$

The functions above in the last line following the constants a_0 and a_1 are defined to be Legendre functions of order p. We denote the even and odd Legendre polynomials as $L_p^0(x)$ and $L_p^1(x)$, respectively. For each p we have one such polynomial and another function which is a special function.

The following theorem is taken straight from the textbook by Simmons (Page 180) and is subsequently proven. Here we just state the theorem.

Theorem 3.1: Power Series Solutions at Ordinary Points

Let x_0 be an ordinary point of our standard homogeneous DE and let a_0 and a_1 be arbitrary constants. Then, there exists a unique function $y(x)$ that is analytic at x_0 that is a solution of the DE in a certain neighbourhood of this point and satisfies the initial conditions $y(x_0) = a_0$ and $y'(x_0) = a_1$. Furthermore, if the power series expansions of $P(x)$ and $Q(x)$ are valid on the interval $|x - x_0| < R$, with $R > 0$, then the power series expansion of this solution is also valid on the same interval.

3.4 2nd-Order Linear Equations: Singular Points

Let's consider the standard, homogeneous DE,

$$y'' + P(x)y' + Q(x)y = 0$$

We have previous said that x_0 is a singular point if either (or both) $P(x)$ and $Q(x)$ are not analytic at x_0, i.e., they are not continuous there. The following definition of a singular point that is not so bad and easier to work with.

regular/irregular singular

Suppose that x_0 is a singular point of above equation but that

$$(x - x_0)P(x) \quad \text{and} \quad (x - x_0)^2 Q(x)$$

are analytic at x_0, i.e.,

$$(x - x_0)P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \quad |x - x_0| < r \quad \text{and} \quad (x - x_0)Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n, \quad |x - x_0| < r$$

for some $r > 0$, then x_0 is said to be a **regular singular** point of the DE. Otherwise x_0 is an **irregular singular** point.

From this definition we gather that for x_0 to be a regular singular point we need the following:

1. $P(x)$ behaves no worse than $\frac{1}{x - x_0}$ at x_0 and
2. $Q(x)$ behaves no worse that $\frac{1}{(x - x_0)^2}$ at x_0

Example: Legendre's equation

Consider Legendre's equation in standard form,

$$y'' - \frac{2x}{1 - x^2} y' + \frac{p(p+1)}{1 - x^2} y = 0$$

We can show that $x = \pm 1$ are each regular singular points of the equation. For example consider $x = 1$ (the other point is analogous), both $(x - 1)P(x)$ and $(x - 1)^2 Q(x)$ are analytic and thus $x_0 = 1$ is a regular singular point.

Example: Bessel's equation

is usually written as $x^2 y'' + xy' + (x^2 - p^2)y = 0$. The standard form:

$$y'' + \frac{1}{x} y' + \frac{x^2 - p^2}{x^2} y = 0$$

We look at the limits of the following products of the coefficients as x_0 approaches zero to show that it is a regular singular point:

$$xP(x) = 1$$
$$x^2 Q(x) = x^2 - p^2$$

Clearly the RHSs are analytic around $x = 0$ and thus we are done.

To solve DEs about regular singular points consider the related Euler's equation,

$$x^2 y'' + p_0 x y' + q_0 y = 0$$

The general solution is,

$$y = x^r$$

We substitute and obtain the **indicial equation** in terms of r,

$$r(r-1) + p_0 r + q_0 = 0$$

This is a quadratic equation that has two solutions and there are three cases:

a) Distinct roots: r_1 and r_2,

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

b) Equal roots: $r_1 = r_2$

$$y(x) = c_1 x^{r_1} + c_2 x^{r_2} \ln x$$

c) Complex roots: $x^r = x^{\alpha \pm i\beta}$

$$x^r = e^{\ln x^r} = e^{r \ln x} = e^{(\alpha \pm i\beta)\ln x} = e^{\alpha \ln x}[\cos(\beta \ln x) + i \sin(\beta \ln x)]$$

The above is for $x > 0$. A similar solution can be found for $x < 0$.

In general the DE for a regular singular point can be thought of as looking like,

$$y'' + \left(\frac{p_0 + p_1 x + p_2 x^2 + \cdots}{x}\right) y' + \left(\frac{q_0 + q_1 x + q_2 x^2 + \cdots}{x^2}\right) y = 0$$

This suggests looking for a more general power series solution of the form,

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad \text{with } a_0 \neq 0$$

This approach is known as the **Method of Frobenius** and the solution is called a **Frobenius Series**.

Example:

$$2x^2 y'' + x(2x+1)y' - y = 0$$

In standard form,

$$y'' + \frac{\frac{1}{2} + x}{x} y' + \frac{-\frac{1}{2}}{x^2} y = 0$$

We first test that $x = 0$ is a regular singular point,

$$xP(x) = -\frac{1}{2}, \quad x^2 Q(x) = \frac{1+x}{2}$$

and

$$p_0 = \lim_{x\to 0} xP(x) = -\frac{1}{2}, \qquad q_0 = \lim_{x \to 0} x^2 Q(x) = \frac{1}{2}$$

The corresponding indicial notation for the associated Euler Equation is thus,

$$r(r-1) - \frac{1}{2} r + \frac{1}{2} = 0 \implies r = 1/2 \text{ or } 1$$

We try our Frobenius series solution:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

and $xy = \sum \ldots, xy' = \sum \ldots, 2x^2 y'' = \sum \ldots$

Sub them into equation and cancel the x^r,

$$a_0[2(r-1)r - r + 1] + \sum_{n=1}^{\infty} \{[2(n+r-1)(n+r) - (n+r-1)]a_n x^n + a_{n-1} x^n\} = 0$$

For $a_0 \neq 0$ we need either $r = 1/2$ or $r = 1$ as found above. For the other powers of x^n we require,

$$a_n = -\frac{1}{(n+r-1)(2(n+r)-1)} a_{n-1}$$

Two cases:

a) For $r = 1$, we have

$$a_n = (-1)^n \frac{1}{(3 \cdot 5 \cdot 7 \cdots (2n+1))n!} a_0$$

then $y_1 = x \sum_{n=0}^{\infty} a_n x^n$

b) For $r = \frac{1}{2}$

$$a_n = (-1)^n \frac{1}{n!(1 \cdot 3 \cdot 5 \cdots (2n-1))} a_0$$

then $y_2 = x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n$

Therefore, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

You can check using the ratio test that this is convergent for all x.

3.5 Extended Method of Frobenius

For the general homogeneous DE,

$$x^2 y'' + p(x) x y' + q(x) y = 0$$

where $p(x)$ and $q(x)$ are analytic functions at $x = 0$, the associated Euler equation is

$$x^2 y'' + p(0) x y' + q(0) y = 0$$

The indicial equation obtained after substituting in $y = x^r$ is,

$$r(r-1) + p(0) r + q(0) = 0$$

In general, let x_0 be a regular singular point of the homogeneous DE,

$$y'' + P(x) y' + Q(x) y = 0$$

i.e.,

$$(x - x_0) P(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n, \quad |x - x_0| < r$$

and

$$(x - x_0)^2 Q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n, \quad |x - x_0| < r$$

The indicial equation is

$$r(r-1) + p_0 r + q_0 = 0$$

where

$$p_0 = \lim_{x \to x_0} (x - x_0) P(x) \quad q_0 = \lim_{x \to x_0} (x - x_0)^2 Q(x)$$

Assume that the above indicial equation has two real roots, r_1, r_2 with $r_1 \geq r_2$. Then the following is true around the regular singular point x_0:

(a) There is always one Frobenius series solution of the form,

$$y_1 = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{with } a_0 \neq 0,\ 0 < x - x_0 < r$$

(b) If $r_1 - r_2$ is not an integer, there is a second linearly independent Frobenius series solution of the form,

$$y_2 = (x - x_0)^{r_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n, \quad \text{with } b_0 \neq 0,\ 0 < x - x_0 < r$$

(c) If $r_1 = r_2$, then there is a second linearly independent solution of the form,

$$y_2 = y_1 \log(x - x_0) + (x - x_0)^{r_1} \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad \text{with } c_0 \neq 0,\ 0 < x - x_0 < r$$

This can be obtained from the reduction of order.

(d) If $r_1 - r_2$ is a positive integer, then there is a second linearly independent solution of the form,

$$y_2 = \alpha y_1 \log(x - x_0) + (x - x_0)^{r_2} \sum_{n=0}^{\infty} d_n (x - x_0)^n, \quad \text{with } d_0 \neq 0,\ 0 < x - x_0 < r$$

where α is a constant that may be zero.

3.6 Bessel Functions

Bessel's Equation is

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

The point $x = 0$ is a regular singular point since

$$\begin{aligned} p_0 &= \lim_{x \to 0} xP(x) = \lim_{x \to 0} x \frac{1}{x} = 1 \\ q_0 &= \lim_{x \to 0} x^2 Q(x) = \lim_{x \to 0} x^2 \frac{x^2 - p^2}{x^2} = -p^2 \end{aligned}$$

we get that the indicial equation is

$$r(r-1) + r - p^2 = 0 \implies r = \pm p$$

To solve this problem we use the Frobenius method and try, $y = \sum_{n=0}^{\infty} a_n x^{n+r}, x^2 y = \sum \ldots, xy' = , x^2 y'' =$

We sub them into Bessel's equation and obtain,

$$\sum_{n=2}^{\infty} ([n(n+2r)]a_n + a_{n-2})x^n + (2r+1)a_1 x = 0$$

where we use the fact that $r^2 = p^2$ and also we divided through by x^r. For both indicial roots, this equation yields,

$$\begin{aligned} a_1 &= 0 \\ a_n &= -\frac{a_{n-2}}{n(n+2r)} \quad \text{for } n = 2, 3, \ldots \end{aligned}$$

This implies the odd polynomials all vanish. We can then use the recurrence relation to find the coefficients of the even polynomials.

The general solution for $k \geq 0$ is

$$a_{2k} = \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1+r)(2+r) \cdots (k+r)} a_0$$

Therefore one solution is

$$y = a_0 x^p \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1+p)(2+p) \cdots (k+p)} x^{2k} \right]$$

Now that we have one solution there are difference cases that must be considered to find the second.

(a) If $r_1 - r_2 = 2p$ is not an integer, then use the above analysis and write down the two linearly independent solutions:

$$y_1 = a_0 x^p \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1+p)(2+p)\cdots(k+p)} x^{2k}\right]$$
$$y_2 = a_0 x^{-p} \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} \cdot k! \cdot (1-p)(2-p)\cdots(k-p)} x^{2k}\right]$$

(b) If $r_1 = r_2$ then use reduction of order.

(c) If $r_1 - r_2 = 2p$ is a positive integer hen one solution is known, and the same as the above, and the second can be obtained from the previous formula. To see why this is a problem reconsider the recurrence relation before we solved it,

$$n(n+2r)a_n = -a_{n-2} \quad \text{for } n = 2, 3, \ldots$$

For concreteness, say $2p = N$. From the first root to the indicial equation we have,

$$n(n+2p)a_n = -a_{n-2}$$

where we notices that the coefficient on the LHS is nonzero. For the second indicial root we have something very similar,

$$n(n-2p)a_n = -a_{n-2}$$

The problem arises where $n = N$ since in this case the left hand side vanishes and we cannot determine a_n which puts us at a stand still. That is why in this case we go to the reduction of order to find a second linearly independent solution.

3.7 Bessel functions of the first kind of integer order

By convention, to define Bessel function of the first kind of order p, written $J_p(x)$, we use solution y_1 for the particular case where,

$$a_0 = \frac{1}{2^p p!}$$

If we substitute this into our solution we get

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot (k+p)!} \left(\frac{x}{2}\right)^{2k}$$

Looking at the first three terms of the first Bessel functions of the first kind of integer zero and one yields,

$$J_0(x) = \ldots, \quad J_1(x) = \ldots$$

Given this definition we can write down our general solution to Bessel's equation or order p as a linear superposition of Bessel functions of the first kind of order p if this parameter is not an integer,

$$y(x) = c_1 J_p(x) + c_2 J_{-p}(x)$$

This is not the convention however. The convention is instead to define the Bessel function or order p of the second kind to be,

$$Y_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}$$

If p is an integer m then we must take the limit,

$$Y_m(x) = \lim_{p \to m} Y_p(x)$$

and one can show that this limit exists and yields a linearly independent solution.

Therefore, the general solution to Bessel's equation or order p, for any p, is,

$$y(x) = c_1 J_p(x) + c_2 Y_p(x)$$

3.8 Bessel functions of the first kind of arbitrary order

To extend our solution to the case with p non-integer is pretty easy. The only complication with our previous solution is that it involves the factorial function. This function is perfectly well defined for integers but we need to extend this to non-integer values as well. This is done with the Gamma function. It is defined to be,

Gamma function

The **Gamma function** is the generalization of the factorial to the real line and is defined to be,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

It has the following properties:

$$\Gamma(x+1) = x\Gamma(x) \qquad \forall x > 0$$

$$\Gamma(n+1) = n! \quad \text{for } n = 0, 1, 2, \ldots$$

If we pick $a_0 = \dfrac{1}{2^p \Gamma(p+1)}$, then the Bessel function of the first kind of the p-th order (where p is non-integer) is,

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot \Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k}$$

This works for both p positive and negative. In the case where p is an integer we can still find an expression for $Y_p(x)$ but it is rather lengthy.

3.9 Asymptotic behaviour of the Bessel function of the first kind

From the above expression for $J_p(x)$ we can get an asymptotic expression for large and small values of x. In particular we find that,

$$J_p(x) \approx \frac{1}{\Gamma(p+1)} \left(\frac{x}{2}\right)^p, \quad \text{as } x \to 0^+$$

It can be shown, but it is not obvious at all, that we also have,

$$J_0(x) \approx \frac{1}{\sqrt{x}} (c_1 \sin x + c_2 \cos x), \quad \text{as } x \to \infty$$

3.10 Bessel functions where p is not an integer

The first solution is given by the above and the second solution is given by,

$$J_{-p}(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \cdot \Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k}$$

Other solutions exist but they are more complicated and will not be dealt with in lecture.

3.11 Gauss's hypergeometric equation

We study, in this section, the famous Gauss hypergemoetric equation

$$x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0$$

where a, b, c are constants.

In its standard form, we see

$$P(x) = \frac{c-(a+b+1)x}{x(1-x)} \quad \text{and} \quad Q(x) = -\frac{ab}{x(1-x)}$$

and $x = 0$ and $x = 1$ are only regular singular points.

Let's first consider its solution near $x = 0$. Since

$$xP(x) = \frac{c-(a+b+1)x}{(1-x)} = [c-(a+b+1)x](1+x+x^2+\ldots) = c+(c-a-b-1)x+\ldots, \quad |x|<1$$

and

$$x^2Q(x) = -\frac{abx}{1-x} = -abx(1+x+x^2+\ldots) = -abx-abx^2-\ldots, \quad |x|<1$$

we see that $p_0 = c$ and $q_0 = 0$, and the indicial equation is

$$m(m-1)+mc = 0$$

which has two roots: $m_1 = 0$ and $m_2 = 1-c$. If $1-c$ is not a positive integer, i.e., if c is not zero or a negative integer, then DE has a solution of the form

$$y = x^0\sum_{n=0}^{\infty} a_nx^n = \sum_{n=0}^{\infty} a_nx^n$$

Sub this into the DE,

$$x(1-x)\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + [c-(a+b+1)x]\sum_{n=1}^{\infty} na_nx^{n-1} - ab\sum_{n=0}^{\infty} a_nx^n = 0$$

which implies

$$\sum_{n=2}^{\infty} n(n-1)a_nx^{n-1} - \sum_{n=2}^{\infty} n(n-1)a_nx^n + c\sum_{n=1}^{\infty} na_nx^{n-1} - (a+b+1)\sum_{n=1}^{\infty} na_nx^n - ab\sum_{n=0}^{\infty} a_nx^n = 0$$

Equating the coefficient of x^n gives

$$[n(n+1)+(n+1)]a_{n+1} - [n(n-1)+(a+b+1)n+ab]a_n = 0$$

i.e.,

$$a_{n+1} = \frac{(n+a)(n+b)}{(n+1)(n+c)}a_n$$

Set $a_0 = 1$ and calculate the other a_n in succession, we found

$$a_n = \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}$$

Thus

$$y = 1+\sum_{n=1}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}x^n, \quad |x|<1$$

This is known as **hypergeometric series** and denoted by $F(a,b,c,x)$.

> Remark:
>
> $F(1,b,b,x) = 1+x+x^2+\ldots = \frac{1}{1-x}$, $|x|<1$, which is the familiar geometric series.
>
> If a or b is a nonpositive integer, $F(a,b,c,x)$ becomes a polynomial.

If $1-c \neq 0$ or a negative integer, then the DE has a second Frobenius series solution of the form

$$y = x^{1-c}\sum_{n=0}^{\infty} b_nx^n, \quad b_n \neq 0$$

One could compute b_n by sub it directly into DE. But an easier method is the following change of variable.

$$y = x^{1-c}z$$

DE then reduces to

$$x(1-x)z'' + [(2-c) - ([a-c+1] + [b-c+1] + 1)x]z' - (a-c+1)(b-c+1)z = 0$$

which is a hypergeometric equation with a, b, c replaced by $a-c+1, b-c+1$ and $2-c$. Thus

$$z = F(a-c+1, b-c+1, 2-c, x)$$

and hence by the change or variable, the second solution of DE is

$$y = x^{1-c}F(a-c+1, b-c+1, 2-c, x) \qquad 0 < x < 1$$

if $2-c$ is not zero or a negative integer. Thus if c is not an integer, the general solution of DE near $x = 0$ is

$$y = c_1F(a, b, c, x) + c_2x^{1-c}F(a-c+1, b-c+1, 2-c, x) \tag{$*$}$$

For the solution of DE near the singular point $x = 1$, we move $x = 1$ to the origin by letting $t = 1-x$. Then DE becomes

$$t(1-t)y'' + [(a+b-c+1) - (a+b+1)t]\, y' - aby = 0$$

which is again a hypergemoetric equation.

In a view of $(*)$, we see that if $a+b-c$ is not an integer, the general solution of DE near $x = 1$ is

$$y = c_1F(a, b, a+b-c+1, 1-x) + c_2(1-x)^{c-a+b}F(c-b, c-a, c-a-b+1, 1-x)$$

4

Oscillation theory and BVPs

4.1 Qualitative Analysis of ODEs

In this chapter, we will make some qualitative analysis of the second-order linear ODEs with variable coefficients. Let us start by considering the following simple homogeneous ODE with constant coefficients,

$$y'' + y = 0$$

We know that two linearly independent solutions to this equation are $\sin(x)$ and $\cos(x)$. Both of these solutions oscillate out of phase. Let's try and forget what we already know and instead see how we can derive this by simply looking at the DE itself.

1. Define $s(x)$ and $c(x)$ to be two linearly independent solutions to this equation $s(x)$ and $c(x)$ to be two linearly independent solutions with initial conditions $s(0) = 0, s'(0) = 1$ and $c(0) = 1, c'(0) = 0$.

2. The equation tells us the solutions are concave down or up depending on whether the solution is above or below the x-axis.

3. Initially: $s(x)$ starts off at the origin and is increasing but concave downwards. Soon afterwards it will start to decrease and eventually pass the x-axis at which point it will change its concavity and start to turn around. The $c(x)$ curve follows a very similar trajectory but starts at the maximal position. From this we see that both solutions will tend to oscillate about $x = 0$.

4. Observe that if $y(x)$ satisfies the DE than so does $y'(x)$. The initial conditions $s'(0) = 1$, $s''(0) = 0$ and $c'(0) = 0, c''(0) = -1$ implies that

$$s'(x) = c(x) \qquad \text{and} \qquad c'(x) = -s(x)$$

5. We saw a while ago that the Wronskian is equal to,

$$W = W_0 \exp\left(-\int P(x)dx\right)$$

In our special equation $P(x) = 0$ and thus in this case the Wronskian is constant everywhere. From the initial conditions we learn that $W_0(s,c) = -1$ and thus we deduce that

$$W(s,c) = s(x)c'(x) - s'(x)c(x) = -1$$

$$s^2(x) + c^2(x) = 1$$

If you recall from Calculus I one way to define the exponential function, e^x is as the unique solution to the DE $y' = y$ with $y(0) = 1$. Similarly, we could define the sine and cosine functions to be the unique solutions to the above DE with the prescribed initial conditions. This isn't typically done but if we do so we can derive many properties of these equations even without solving the DE.

The basic ideas that we used to analyze the simple harmonic oscillator can be used to study more general DEs. In particular, we can learn information about the zeros of the solutions of

$$y'' + P(x)y' + Q(x)y = 0 \tag{4.1}$$

Theorem 4.1: Sturm separation theorem

If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous equation Eq. (4.1), then the zeros of these solutions are distinct and occur alternatively. By that we mean that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$ and the converse is true.

Proof:
Assume y_1, y_2 are two linearly independent solutions. Lemma 2.3, Lemma 2.4 have shown that the Wronskian of these two functions is necessarily non-zero and thus has a constant sign,

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

1. The above equation shows that the two solutions cannot be zero at the same location for otherwise the Wronskian would be zero, which is a contradiction.
2. Say that x_1 and x_2 are successive zeros of y_2. We need to show that y_1 necessarily vanishes between these points. From above,

$$W(y_1(x_1), y_2(x_1)) = y_1(x_1)y_2'(x_1)$$
$$W(y_1(x_2), y_2(x_2)) = y_1(x_2)y_2'(x_2)$$

   which shows that y_1' and y_2' are both non-zero at x_1, x_2. Moreover, $y_2'(x_1)$ and $y_2'(x_2)$ are opposite in signs, i.e,m $y_2'(x_1)y_2'(x_2) < 0$, since if y_2 is increasing (decreasing) at x_1 it must be decreasing (increasing) at x_2. This implies $y_1(x_1)$ and $y_1(x_2)$ have opposite signs and therefore, by continuity, y_1 must have a zero somewhere in between x_1 and x_2.
3. From this argument that y_1 cannot vanish more than once between x_1 and x_2 but if it did we could deduce that y_2 has a zero between the two zeros, which is a contradiction to what we have supposed.

$\square$

One reason why the simple harmonic oscillator is so easy to understand is because y'' is a scalar multiple of y. This doesn't happen in general because of the y' term. It would be nice to make the y' term disappear so that we can use similar tricks.

Normal form (of the homogeneous equation): If we define $y(x) = u(x)v(x)$ we can easily compute its first two derivatives,

$$y' = vu' + v'u$$
$$y'' = vu'' + 2v'u' + uv''$$

If we then substitute this into the homogeneous equation, Eq. (4.1), we get

$$[vu'' + 2v'u' + v''u] + P(x)[vu' + v'u] + Q(x)vu = 0$$
$$vu'' + [2v' + P(x)v]u' + [v'' + P(x)v' + Q(x)v]u = 0$$

This equation is much more complicated but appreciate that there is a unique choice of $v(x)$ such that we eliminate the u' term in the above equation. This is done by choosing,

$$2v' + P(x)v = 0 \implies v' = -\frac{1}{2}P(x)v \implies v = \exp\left(-\frac{1}{2}\int P(x)dx\right)$$

If we make this choice than the resulting equation is said to be in **normal form** after we divide through by $v(x)$, as defined above,

$$u'' + q(x)u = 0 \tag{4.2}$$

where we have defined

$$q(x) = Q(x) - \frac{1}{4}P(x)^2 - \frac{1}{2}P'(x)$$

The original function $y(x)$ is recovered from the following equation,

$$y(x) = \exp\left(-\frac{1}{2}\int P(x)dx\right)u(x)$$

It's clear

$$y(x) = 0 \iff u(x) = 0$$

Thus the oscillatory properties of Eq. (4.1) remain unchanged after the transformation.

The following theorem uses the normal form to determine that some equations have at most one zero.

Theorem 4.2

If $q(x) < 0$ and if $u(x)$ is a nontrivial solution of the normal form Eq. (4.2), then $u(x)$ has at most one zero.

Proof:

Suppose that x_0 is a zero of $u(x)$, which is a non-trivial function. By the property above, we require that $u'(x_0) \neq 0$. Without loss of generality, assume that $u'(x_0) > 0$ By continuity, $u'(x)$ must be positive in a neighborhood of x_0.

Since we have assumed that $q(x) < 0$, we find using the equation that,

$$u''(x) = -q(x)u(x)$$

is positive on the interval to the right of x_0. Therefore, $u'(x)$ is an increasing function on this interval and consequently, there cannot be any zeros to the right of x_0. The same argument holds to the left of x_0 and also in the case where $u'(x_0) < 0$. □

Therefore, if we want to look at oscillating solutions we should consider $q(x) > 0$. This not surprising if we recall what we know about DEs with constant coefficients.

Theorem 4.3

Let $u(x)$ be any nontrivial solution of Eq. (4.2), where $q(x) > 0$ for all $x > 0$. If

$$\int_0^\infty q(x)dx = \infty,$$

then $u(x)$ has s infinitely many zeros on the positive x-axis.

Proof:

See page 30, theorem 1.11.3 in the course notes. □

Theorem 4.4

Let $u(x) \not\equiv 0$ be a solution of Eq. (4.2). Then $u(x)$ has at most a finite number of zeros on a closed interval $[a, b]$.

Proof:
If not, $\exists\{x_n\}_{n=0}^{\infty}$, $x_n \in [a,b]$, such that $u(x_n) = 0, \forall n$. Since $[a,b]$ is closed, $\exists x_0 \in [a,b]$ and $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x_0$ as $k \to \infty$. Then by continuity,

$$u(x_0) = \lim_{k\to\infty} u(x_{n_k}) = 0$$
$$u'(x_0) = \lim_{k\to\infty} \frac{u(x_{n_k}) - u(x_0)}{x_{n_k} - x_0}$$

By uniqueness, we have $u(x) \equiv 0$ which is a contradiction. Thus the proof is complete. □

Remark:
$u(x)$ can have an infinite number of zeros if $u(x)$ is not a solution of Eq. (4.2).

For example:

$$u(x) = \begin{cases} x\sin\frac{1}{x} & x \in (0,1] \\ 0 & x = 0 \end{cases}$$

$u(x)$ is continuous on $[0,1]$ and has an infinite number of zeros in $[0,1]$.

Example:
Consider the second-order linear ODE

$$y'' - y' + \frac{1}{4}\sin^2 xy = 0 \tag{4.3}$$

Let $u(x) = y(x)e^{\frac{1}{2}\int(-1)dx} = y(x)e^{-\frac{1}{2}x}$. Then DE reduces to

$$u'' + \frac{1}{4}(\sin^2 x - 1)u = 0 \tag{4.4}$$

Since $\frac{1}{4}(\sin^2 x - 1) \le 0$, it follows from Theorem 4.2 that all nonzero solutions of Eq. (4.4), thus of Eq. (4.3) are non-oscillating.

Example:
Consider the second-order linear ODE

$$y'' + \sin xy' + y = 0 \tag{4.5}$$

Let $u(x) = y(x) = e^{-\frac{1}{2}\cos x}$. Then Eq. (4.5) reduces to

$$u'' + \left(1 - \frac{1}{4}\sin^2 x - \frac{1}{2}\cos x\right)u = 0 \tag{4.6}$$

Since $1 - \frac{1}{4}\sin^2 x - \frac{1}{2}\cos x > 0$ and

$$\int_0^x \left(1 - \frac{1}{4}\sin^2 t - \frac{1}{2}\cos t\right)dt \ge \int_0^x \frac{1}{4}dt \longrightarrow \infty \text{ as } x \to \infty$$

It follows $\int_0^\infty \left(1 - \frac{1}{4}\sin^2 t - \frac{1}{2}\cos t\right)dt = \infty$. Thus by Theorem 4.3, all nonzero solutions of Eq. (4.6), thus of Eq. (4.5) are oscillatory.

4.2 The Sturm comparison theorem

The Sturm separation theorem tells us that all solutions of the second-order linear ODE

$$u'' + q(x)u = 0$$

oscillate with essentially the same rapidity. However, we see that solutions of this equation

$$y'' + 4y = 0$$

oscillate more rapidly than those of
$$y'' + y = 0$$
This is in fact a typical behaviour as shown in the following theorem.

Theorem 4.5: Sturm comparison theorem

Consider the second-order linear ODEs
$$y'' + q(x)y = 0 \tag{4.7}$$
$$z'' + r(x)z = 0 \tag{4.8}$$
where $q(x)$ and $r(x)$ are continuous on $(-\infty, \infty)$ and
$$q(x) > r(x) > 0, \quad \forall x \in (-\infty, \infty) \tag{4.9}$$
Then any solution $y(x) \not\equiv 0$ of Eq. (4.7) vanishes at least once between any two successive zeros of a nonzero solution $z(x)$ of Eq. (4.8), i.e., oscillation of solution of Eq. (4.8) implies that of Eq. (4.7).

Proof:
Let $x_1 < x_2$ be such that
$$z(x_1) = z(x_2) = 0, \quad z(x) \neq 0, \quad \forall x \in (x_1, x_2)$$
Suppose, for the sake of contradiction, that
$$y(x) \neq 0, \quad \forall x \in (x_1, x_2)$$
We may assume $z(x) > 0$ and $y(x) > 0, \forall x \in (x_1, x_2)$. Define $m(x) = y(x)z'(x) - z(x)y'(x)$. Then by Eq. (4.9)
$$m'(x) = y(x)z''(x) - z(x)y''(x) = (q(x) - r(x))y(x)z(x) > 0, \quad \forall x \in (x_1, x_2)$$
Thus $m(x_2) > m(x_1) = y(x_1)z'(x_1) > 0$ in view of the fact that $z'(x_1) > 0$ and $z'(x_2) < 0$.

But $m(x_2) = y(x_2)z'(x_2) < 0$, which is a contradiction. Thus we must have $x^* \in (x_1, x_2)$ such that $y(x^*) = 0$. □

Example:
Show that all solutions of the equation
$$u'' + (1 + \sin^2 x)u = 0$$
are oscillatory and the distance between any two successive zeros of a non zero solution of it is bounded by $\frac{\pi}{\sqrt{2}}$ and π.

Solution Let $\epsilon > 0$ be sufficiently small. Then
$$1 - \epsilon < 1 + \sin^2 x < 2 + \epsilon, \quad \forall x \in R$$
Thus by Sturm comparison theorem, the solutions of it oscillate more rapidly than those of
$$y'' + (1 - \epsilon)y = 0 \tag{4.10}$$
and less rapidly than those of
$$z'' + (2 + \epsilon)z = 0 \tag{4.11}$$
Since the distances of any two successive zeros of solutions of Eq. (4.10) and Eq. (4.11) are, respectively $\frac{\pi}{\sqrt{1-\epsilon}}$ and $\frac{\pi}{\sqrt{2+\epsilon}}$ and ϵ is arbitrary, it follows that e distance between any two successive zeros of a nonzero solution of the DE is bounded by $\frac{\pi}{\sqrt{2}}$ and π.

4.3 The vibrating string problem and BVPs

The vibrating string problem. A flexible string is pulled taut on x-axis and fastened at two points $x = 0, x = \pi$. Then it is drawn aside into a curve $y = f(x)$ in xy-plane and released. Let $y(x,t)$ be the motion of the string. Then it can be shown (the details are omitted here), for some positive constant α, that $y(x,t)$ satisfies the partial differential equation

$$\alpha \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{4.12}$$

subject to the initial condition

$$\frac{\partial y}{\partial t} = 0 \quad \text{and} \quad y(x,0) = f(x) \tag{4.13}$$

and the boundary condition

$$y(0,t) = y(\pi,t) = 0 \tag{4.14}$$

Eq. (4.12) - Eq. (4.14) is called the initial-boundary value problem of one-dimensional wave equation (IBVP). To solve the IBVP, we assume $y(x,t) = u(x)v(t)$. Then it can be shown (the details are omitted here), that $u(x)$ satisfies, for some constant λ, the equation

$$u'' + \lambda u = 0 \tag{4.15}$$

subject to the boundary condition

$$u(0) = u(\pi) = 0 \tag{4.16}$$

Eq. (4.15) - Eq. (4.16) are e called boundary value problems which are the prototype of a large class of important BVPs in applied mathematics.

4.4 Eigenvalues and eigenfunctions

In the vibrating string model discussed earlier, we need to solve the BVP

$$u'' + \lambda u = 0 \tag{4.17}$$
$$u(0) = 0, \quad u(\pi) = 0 \tag{4.18}$$

where λ is a parameter to be determined in such a way that Eq. (4.17) - Eq. (4.18) has a nontrivial solution.

If $\lambda \le 0$, then the BVP Eq. (4.17) - Eq. (4.18) has only trivial solutions since any nonzero solution of Eq. (4.17) has at most one zero. Since we are interested in the nonzero solutions of the BVP Eq. (4.17) - Eq. (4.18), we are restricted to the case $\lambda > 0$, and the general solution of Eq. (4.17) is

$$u(x) = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x$$

Condition $u(0) = 0$ yields $c_2 = 0$, thus the solution reduces to

$$u(x) = c_1 \sin \sqrt{\lambda} x$$

This together with the condition $u(\pi) = 0$, implies

$$\lambda = n^2, \quad n = 1, 2, \ldots$$

The values of λ are called **eigenvalues** of the BVP Eq. (4.17) - Eq. (4.18) and the corresponding solutions

$$\sin nx, \quad n = 1, 2, \ldots$$

are called **eigenfunctions**.

We observe the following:

1. Corresponding to each eigenvalue, there are infinite many solutions to BVP Eq. (4.17) - Eq. (4.18)the which forms a 1-dimensional vector space.
2. All eigenvalues form an increasing sequence of positive numbers that approaches ∞.
3. The eigenfunction corresponding to the n-th eigenvalue vanishes at the endpoints of $[0, \pi]$ and has exactly $n-1$ zeros inside this interval.
4. All eigenfunctions are orthogonal on $[0, \pi]$ in the sense

$$\int_0^\pi \sin nx \sin mx \, dx = 0, \quad \text{if } n \neq m \tag{4.19}$$

The formula Eq. (4.19) can be proved by elementary integration, but the following approach given by Euler in 1777 may apply to more general situation.

Proof (Euler 1777):

Let $u_n = \sin nx, u_m = \sin mx$. Then

$$u_n'' + n^2 u_n = 0 \qquad u_m'' + m^2 u_m = 0$$

Combining these two,

$$u_m u_n'' - u_n u_m'' = (m^2 - n^2) u_n u_m$$
$$(u_m u_n' - u_n u_m')' = (m^2 - n^2) u_n u_m$$

Integrating the above equation from 0 to π, we obtain

$$(m^2 - n^2) \int_0^\pi u_n(x) u_m(x) dx = 0$$

which gives Eq. (4.19) since $m \neq n$. □

We shall show later that the above properties can be extended to general Sturm-Liouville BVPs.

4.5 Regular Sturm-Liouville problems

Consider the Sturm-Lioville BVPs

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + \lambda q(x) y = 0, \quad y(a) = y(b) = 0$$

where $p(x)$ and $q(x)$ are positive continuous functions on $[a, b]$ and $p(x)$ is continuously differentiable in (a, b).

Theorem 4.6

Then there exists λ_n where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \quad \text{as } n \to \infty$$

such that BVP above has a nontrivial solution if and only if $\lambda = \lambda_n, n = 1, 2, \ldots$. The solution y_{λ_n} is unique except for an arbitrary constant factor, and y_{λ_n} has exactly $n-1$ zeros in the open interval (a, b).

Remark:

λ_n, y_{λ_n} for $n = 1, 2, 3, \ldots$, are the eigenvalues and eigenfunctions of BVP above, respectively.

4.6 General Sturm-Lioville problems

See the lecture notes below (cr. professor).

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \text{ as } n \to \infty$$

such that BVP (4.5.11) has a nontrivial solution if and only if $\lambda = \lambda_n, n = 1, 2, \cdots$. The solution y_{λ_n} is unique except for an arbitrary constant factor, and y_{λ_n} has exactly $n - 1$ zeros in the open interval (a, b).

Remark. $\lambda_n, y_{\lambda_n}, n = 1, 2, 3 \cdots$, are the eigenvalues and eigenfunctions of BVP (**??**), respectively.

4.6 General Sturm-Lioville problems

Let us return briefly to the general Sturm-Lioville BVPs of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[\lambda q(x) + r(x)\right]y = 0 \tag{4.6.1}$$

$$c_1 y(a) + c_2 y'(a) = 0, d_1 y(b) + d_2 y'(b) = 0 \tag{4.6.2}$$

where $p(x), q(x)$ and $r(x)$ are continuous on $[a, b]$, $p(x) > 0, q(x) > 0, \forall x \in [a, b], c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. Note that a special case of BVP (4.6.1) - (4.6.2) was discussed in Section 4.5. In the general case, the following result can be proved.

Theorem 4.6.1. There exist real numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \text{ as } n \to \infty \tag{4.6.3}$$

such that BVP (4.6.1) - (4.6.2) has a nonzero solution iff $\lambda = \lambda_n$, $n = 1, 2, \cdots$, and eigenfunctions

$$y_1(x), y_2(x), \cdots, y_n(x), \cdots \tag{4.6.4}$$

are orthogonal on $[a, b]$ with respect to the weight function $q(x)$, i.e.

$$\int_a^b q(x) y_m(x) y_n(x)\, dx = \begin{cases} 0 \text{ if } m \neq n, \\ \alpha_n \neq 0 \text{ if } m = n. \end{cases} \tag{4.6.5}$$

Proof. We shall only prove (4.6.5).
Let

$$m(x) = y_m(x)y_n'(x) - y_n(x)y_m'(x) = det \begin{bmatrix} y_m(x) & y_m'(x) \\ y_n(x) & y_n'(x) \end{bmatrix}.$$

Then from (4.6.2) we have

(4.6.6)
$$\left\{ \begin{array}{c} c_1 y_m(a) + c_2 y_m'(a) = 0, \\ c_2 y_n(a) + c_2 y_n'(a) = 0, \end{array} \right\}$$

and

(4.6.7)
$$\left\{ \begin{array}{c} d_1 y_m(b) + d_2 y_m'(b) = 0, \\ d_a y_n(b) + d_2 y_n'(b) = 0. \end{array} \right\}$$

Since $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$, it follows from (4.6.6) and (4.6.7) that

(4.6.8)
$$m(a) = 0 \text{ and } m(b) = 0.$$

Now consider

$$\begin{array}{rl} & y_n \left\{ \dfrac{d}{dx}\left[p\dfrac{dy_m}{dx}\right] + \left[\lambda_m q + r\right] y_m \right\} = 0 \\ - & y_m \left\{ \dfrac{d}{dx}\left[p\dfrac{dy_n}{dx}\right] + \left[\lambda_n q + r\right] y_n \right\} = 0 \\ \hline & y_n \dfrac{d}{dx}\left[p\dfrac{dy_m}{dx}\right] - y_m \dfrac{d}{dx}\left[p\dfrac{dy_n}{dx}\right] + (\lambda_m - \lambda_n) q y_m y_n = 0 \end{array}$$

which implies

$$d(\lambda_m - \lambda_n) q y_m y_n = y_m (p y_n')' - y_n (p y_m')'.$$

Integrating the above equation from a to b and using integration by parts, we obtain

$$\begin{aligned}(\lambda_m - \lambda_n)\int_a^b q y_m y_n\,dx &= \int_a^b y_m (p y_n')'\,dx - \int_a^b y_n (p y_m')'\,dx\\ &= \left[y_m(p y_n')\right]_a^b - \int_a^b y_m'(p y-_n)\,dx - \left[y_n(p y_m')\right]_a^b + \int y_n'(p y_m')\,dx\\ &= y_m(b)p(b)y_n'(b) - y_m(a)p(a)y_n'(a) - y_n(b)p(b)y_m'(b) + y_n(a)p(a)y_m'(a)\end{aligned}$$

$$\begin{aligned}&= p(b)[y_m(b) - y_n(b)y_m'(b)] - p(a)[y_m(b)y_n'(a) - y_m(a)y_m'(a)]\\ &= p(b)m(b) - p(a)m(a).\end{aligned}$$

Thus from (4.6.8)

$$(\lambda_m - \lambda_n \int_a^b q y_m y_n\,dx = 0$$

which implies (4.6.5).

The significance of property (4.6.5) of the eigenfunctions is that we can obtain expansions of the functions $f(x)$ in terms of the eigenfunctions given by (4.6.4). if we assume

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x), \tag{4.6.9}$$

then multiplying both sides of (4.6.9) by $q(x)y_m(x)$ and integrating term by term from a to b yields

$$\int_a^b f(x)q(x)y_m(x)\,dx = a_m \int_a^b q(x)y_m^2(x)\,dx$$

(4.6.1)

which implies

$$a_m = \frac{1}{\alpha_m}\int_a^b f(x)q(x)y_m(x)dx, \quad m = 1, 2, \cdots \tag{4.6.10}$$

Formula (4.6.9) with a_m given in (4.6.10) is called an eigenfunction expansion of $f(x)$.
Remarks:

Remark:

1. We didn't address the convergence of the series (4.6.9) whose study is beyond the scope of this course.

2. We call the BVP (4.6.1)-(4.6.2) a regular Sturm-Liouville problem because the interval $[a, b]$ is finite and the functions $p(x)$ and $q(x)$ are positive and continuous on $[a, b]$. Otherwise, it is called singular, which is considerably more difficult, and therefore not covered by our discussion here.

5

Systems of First-Order Differential Equations

5.1 Introduction

A general first order system of ODEs is of the form

$$\begin{aligned}
y_1' &= f_1\left(x, y_1, y_2, \cdots, y_n\right) \\
y_2' &= f_2\left(x, y_1, y_2, \cdots, y_n\right) \\
&\cdots \\
y_n' &= f_n\left(x, y_1, y_2, \cdots, y_n\right)
\end{aligned}$$

Vector notation is more compact:

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y})$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \qquad \mathbf{f}(x, \mathbf{y}) = \begin{bmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \\ \vdots \\ f_n(x, \mathbf{y}) \end{bmatrix}$$

A scalar ODE of order n can be written as an n-th order system by change of variables. Consider the following scalar equation,

$$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

let us define the following variables,

$$\begin{aligned}
y_1 &= y \\
y_2 &= y' \\
&\vdots \\
y_n &= y^{(n-1)}
\end{aligned}$$

then the equivalent system is

$$\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
&\vdots \\
y_n' &= f(x, y_1, \ldots, y_n)
\end{aligned}$$

Example: Lotka-Volterra predator-prey model

Consider the Lotka-Volterra predator-prey model. It describes the evolution of the concentration of prey x_1 and predators x_2:

$$\begin{aligned}\frac{dx_1}{dx} &= ax_1 - bx_1x_2 \\ \frac{dx_2}{dx} &= -cx_2 + dx_1x_2\end{aligned}$$

This is an example of two-first order nonlinear ODEs.

Example: Springs

If we have three springs in a horizontal line with walls at the two ends and two objects of mass m_1 and m_2 in the middle, we can describe the evolution of the position using Newton's second law. We define x_1 and x_2 to be the displacement of the center of mass from the rest position. We assume Hooke's law describes the force of the spring and that there could be additional forcing acting on the system. The two second order ODEs are,

$$\begin{aligned}m_1\frac{d^2x_1}{dt^2} &= k_2\left(x_2 - x_1\right) - k_1x_1 + F_1(t) \\ &= -\left(k_1 + k_2\right)x_1 + k_2x_2 + F_1(t) \\ m_2\frac{d^2x_2}{dt^2} &= -k_3x_2 - k_2\left(x_2 - x_1\right) + F_2(t) \\ &= k_2x_1 - \left(k_2 + k_3\right)x_2 + F_2(t)\end{aligned}$$

This is an example of two-second order ODEs that can be combined in terms of four first-order ODEs. To convert it to a system of first order equations we define,

$$y_1 = x_1, \quad y_2 = \frac{dx_1}{dt}, \quad y_3 = x_2, \quad y_4 = \frac{dx_2}{dt}$$

Following the procedure that we just mentioned we get,

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_1} & 0 & -\frac{k_2+k_3}{m_2} & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ F_1(t) \\ 0 \\ F_2(t) \end{bmatrix}$$

A general **linear system** of first-order DEs is of the form

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} + \mathbf{b}(x)$$

where $\mathbf{A}$ is an $n \times n$ matrix with coefficients $a_{ij}(x)$ and $\mathbf{b}(x)$ is a $1 \times n$ column vector with coefficients $b_i(x), i, j = 1, \ldots, n$

5.2 Existence-Uniqueness of IVPs

5.2.1 The Picard Method

COnsider the IVP

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y}, \quad \mathbf{y}(x_0) = \mathbf{y}_0$$

We can integrate the equation and combine it with the initial condition to get a single equation:

$$\begin{aligned}\int_{x_0}^{x}\frac{dy}{dx}(s)ds &= \int_{x_0}^{x} f(s, y(s))ds \\ y(x) - y\left(x_0\right) &= \int_{x_0}^{x} f(s, y(s))ds \\ y(x) &= y_0 + \int_{x_0}^{x} f(s, y(s))ds\end{aligned}$$

This equation is called the equivalent **integral equation**.

The main idea is to construct a sequence of functions using the above integral equation by iteration. Then prove that the sequence converge to a solution of the IVP. To begin, we use the roughest possible approximation of a solution

$$y_0(x) = y_0$$

This is a horizontal straight line that passes through the point (x_0, y_0). One advantage of this approximation is that it satisfies the initial condition. One disadvantage is that it usually doesn't equal the exact solution at any other points. If you recall from first year calculus that a Taylor series is a means of building up a complicated function using polynomials of all orders, this first guess can be see as the constant term in the Taylor series.

If we substitute this into our integral equation we obtain a new approximate solution,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))ds$$

Because $y_0(x)$ is constant this integral can be evaluated easily to compute $y_1(x)$ explicitly. This is then going to give us a linear function of x, which is like the second term in our Taylor series.

Then given $y_1(x)$ we can also plug it into the integral equation to obtain another approximation,

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))ds$$

This is going to be like the quadratic term in our Taylor series.

If we proceed inductively then at the n-th stage we have,

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))ds$$

This is called **Picard's method of successive approximations**. Because we are integrating it increases the order of the polynomial of the guess by one each time, therefore giving us more freedom to match the solution to the DE.

Example: toy problem

$$y' = y, \quad y(0) = 1$$

which we know has an exact solution $y(x) = e^x$. If we integrate the equation we find that the analogous integral equation is,

$$y(x) = 1 + \int_0^x y(s)ds$$

The n-th step of the Picard Method is of the form,

$$y_n(x) = 1 + \int_0^x y_{n-1}(s)ds$$

We can use this to construct our solution explicitly:

$$\begin{aligned}
y_0 &= 1 \\
y_1(x) &= 1 + \int_0^x y_0(s)ds = 1 + x \\
y_2(x) &= 1 + \int_0^x y_1(s)ds = 1 + x + \frac{x^2}{2} \\
y_3(x) &= 1 + \int_0^x y_2(s)ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \\
&\vdots \\
y_n(x) &= 1 + \int_0^x y_{n-1}(s)ds = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}
\end{aligned}$$

What we are doing is constructing the Taylor Series expansion of the solution. In this case the successive approximation converges to the exact solution e^x.

Example: slightly more complicated

$$y' = x + y, \quad y(0) = 1$$

The solution to this linear equation (using an integrating factor) is $y(x) = 2e^x - x - 1$.

The integral equation that is equivalent to the above equation is,

$$y(x) = y(0) + \int_0^x [s + y(s)]ds$$

and the n-th term in the approximation is

$$y_n(x) = y(0) + \int_0^x [s + y_{n=1}(s)]ds$$

We can use this to construct our solution as follows:

$$\begin{aligned}
&y_0(x) = 1 \\
&y_1(x) = 1 + \textstyle\int_0^x [s+1]ds = 1 + x + \frac{x^2}{2!} \\
&y_2(x) = 1 + \textstyle\int_0^x \left[1 + 2s + \frac{s^2}{2!},\right] ds = 1 + x + x^2 + \frac{x^3}{3!} \\
&y_3(x) = 1 + \textstyle\int_0^x \left[1 + 2s + s^2 + \frac{s^3}{3!},\right] ds = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4!} \\
&y_4(x) = 1 + \textstyle\int_0^x \left[1 + 2s + s^2 + \frac{s^3}{3!} + \frac{s^4}{4!},\right] ds = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3\cdot 4} + \frac{x^5}{5!} \\
&\dots \\
&y_n(x) = 1 + x + 2\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) + \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$

This converges to the following function,

$$1 + x + 2(e^x - x - 1) + 0 = 2e^x - x - 1$$

which is precisely our exact solution.

5.2.2 Picard's Theorem

Theorem 5.1: Picard's Theorem

Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a bounded and closed set D. If (x_0, y_0) is any interior point of D, then there exists a number $h > 0$ with the property that the IVP,

$$y' = f(x,y), \quad y(x_0) = y_0$$

has one and only one solution $y = y(x)$ on the interval $|x - x_0| \le h$.

Proof:
This proof is rather lengthy and technical and we will simply outline the main ideas or you could check page 37 of https://notes.sibeliusp.com/pdfs/1189/amath251.pdf.

Step 1: There exists a unique solution to the IVP if and only if there exists a unique solution to the associated integral equation. Construct a sequence of functions $\{y_n(x)\}$ by iteration.

Step 2: Our aim is to show that the sequence of functions, $y_n(x)$ converge to the exact solution. If we write down the following telescoping series we see that the convergence of $y_n(x)$ is equivalent to the convergence of the series,

$$y_0 + \sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)]$$

Step 3: The series above converges to the exact solution $y(x)$.

Step 4: $y(x)$ is a continuous solution of the integral equation.

Step 5: $y(x)$ is the only continuous solution of the integral equation. □

5.3 Linear Systems

5.3.1 Existence and Uniqueness

In this course, we will mainly study the following linear system

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} + \mathbf{b}$$

Example:

Consider the homogeneous second-order DE in general form,

$$y'' + P(x)y' + Q(x)y = 0$$

We can rewrite this as a system by defining,

$$y_1 = y(x), \qquad y_2 = y'(x)$$

then the governing equation is in the form of equation where the governing matrix is

$$\mathbf{A}(x) = \begin{bmatrix} 0 & 1 \\ -Q(x) & -P(x) \end{bmatrix}$$

Previously, we stated Picard's theorem for a general system of non-linear equations. Here we specialize it for the case of inhomogeneous linear systems, since that is what we will need throughout this chapter. Note that in the nonlinear version we needed that both $f(x, y)$ and $\frac{\partial f}{\partial y}$ were continuous. In the case of the linear system we simply need that the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ are continuous.

Theorem 5.2: Picard's Theorem for Linear Systems

Let $\mathbf{A}(x)$ and $\mathbf{text}b(x)$ be continuous functions on a closed interval $I = [\alpha, \beta]$, then for any point $x_0 \in I$ and any constant vector $\mathbf{y}_0$, there exists a unique solution $\mathbf{y}(x)$ to the IVP

$$\frac{d\mathbf{y}}{dx} = \mathbf{A}(x)\mathbf{y} + \mathbf{b}, \quad \mathbf{y}(x_0) = \mathbf{y}_0$$

defined throughout the interval I.

If the RHS of the DE depends on x explicitly, we say the system is **nonautonomous**. Otherwise, we say the system is **autonomous**.

Index