



# *Introduction to Optimization*

CO 255



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# Preface

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# Info

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Ricardo: MC 5036. OH: M 1:30 - 3pm  
TA: Adam Brown: MC 5462. OH: F 10-11am

## Books (not required)

- Intro to Linear Opt. Bertsimas
- Int Programming. Conforti

## Grading

- assns: 20% ( $\approx 5$ )
- mid: 30% (Feb 11 in class)
- final: 50%

# Introduction

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Given a set  $S$ , and a function  $f : S \rightarrow \mathbb{R}$ . An optimization problem is:

$$\begin{array}{ll} \max & f(x) \\ \underbrace{\text{s.t.}}_{\text{subject to}} & x \in S \end{array} \quad (\text{OPT})$$

- $S$  **feasible region**
- A point  $\bar{x} \in S$  is a **feasible solution**
- $f(x)$  is **objective function**

(OPT) means: “Find a feasible solution  $x^*$  such that  $f(x) \leq f(x^*), \forall x \in S$ ”

- Such  $x^*$  is an **optimal solution**
- $f(x^*)$  is **optimal value**

Other ways to write (OPT):

$$\max\{f(x), x \in S\}$$

$$\max_{x \in S} f(x)$$

Analogous problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}$$

## Note

$$\max_{\text{s.t.}} f(x) = -1 \left( \min_{\text{s.t.}} -f(x) \right)$$

**Problem**  $x^*$  may not exist

a) Problem is unbounded:

$$\forall M \in \mathbb{R}, \exists \bar{x} \in S, \text{ s.t. } f(\bar{x}) > M$$

b)  $S = \emptyset$ , i.e. (OPT) is **INFEASIBLE**

c) There may not exist  $x^*$  achieving supremum.

Example:

$$\begin{array}{ll} \max & x \\ \text{s.t.} & x < 1 \end{array}$$

#### supremum

$$\sup\{f(x) : x \in S\} = \begin{cases} +\infty & \text{if OPT unbounded} \\ -\infty & \text{if } S = \emptyset \\ \min\{x : x \geq f(x), \forall x \in S\} & \text{otherwise} \end{cases}$$

always exist and are well-defined

#### infimum

$$\inf\{f(x) : x \in S\} = -1 \cdot \sup\{-f(x) : x \in S\}$$

From this point on, we will abuse notation and say  $\max\{f(x) : x \in S\}$  is  $\sup\{f(x) : x \in S\}$ .

One way to specify that I want an opt. sol. (if exists) is

$$x^* \in \operatorname{argmax}\{f(x) : x \in S\}$$

# 2

## Linear Optimization (Programming) (LP)

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$$S = \{x \in \mathbb{R}^n : Ax \leq b\}$$

where  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$  and  $f(x) = c^T x$ ,  $c \in \mathbb{R}^n$ .

↓

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \quad (\text{LP})$$

### Note

$$A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix} \quad A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

### Clarifying

$$u, v \in \mathbb{R}^n, \quad u \leq v \iff u_j \leq v_j, \forall j \in 1, \dots, n$$

### Note

$u \not\leq v$  is not the same as  $u > v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \not\leq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

### Example:

$$\begin{array}{ll} \max & 2x_1 + 0.5x_2 \\ \text{s.t.} & x_1 \leq 2 \\ & x_1 + x_2 \leq 2 \\ & x \geq 0 \end{array}$$

- Strict ineq. not allowed

**halfspace, hyperplane, polyhedron**

Let  $h \in \mathbb{R}^n, h_0 \in \mathbb{R}$ .

$\{x \in \mathbb{R}^n : h^T x \leq h_0\}$  is a **halfspace**.

$\{x \in \mathbb{R}^n : h^T x = h_0\}$  is a **hyperplane**.

$Ax \leq b$  is a **polyhedron** (i.e. intersection of finitely many halfspaces).

**Example:**

$n$  products,  $m$  resources. Producing  $j \in \{1, \dots, n\}$  given  $c_j$  profit/unit and consumes  $a_{ij}$  units of resource  $i$ ,  $\forall i \in \{1, \dots, m\}$ . There are  $b_i$  units available  $\forall i \in \{1, \dots, m\}$ .

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i, \quad \forall i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

which is an LP.

## 2.1 Determining Feasibility

Given a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

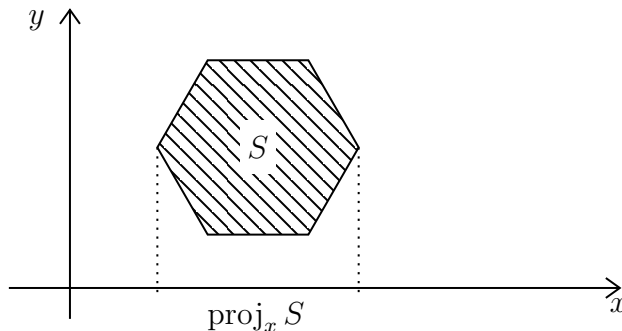
either find  $\bar{x} \in P$  or show  $P = \emptyset$ .

**Idea** In 1-d, easy.  $\rightarrow$  Reduce problem in dimension  $n$  to one in dimension  $n - 1$ .

**Notation** Let  $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \leq b\}$ , then

$$\text{proj}_x S := \{x \in \mathbb{R}^n : \exists y \text{ so that } (x, y) \in S\}$$

is the (orthogonal) *projection* of  $S$  onto  $x$ .





We will find if  $P = \emptyset$  by looking at  $\text{proj}_{x_1, \dots, x_{n-1}} \quad (P)$

## 2.2 Fourier-Motzkin Elimination

Call  $a_{ij}$  entries of  $A$ . Let

$$\begin{aligned} M &:= \{1, 2, \dots, m\} \\ M^+ &:= \{i \in M : a_{in} > 0\} \\ M^- &:= \{i \in M : a_{in} < 0\} \\ M^0 &:= \{i \in M : a_{in} = 0\} \end{aligned}$$

For  $i \in M^+$  (1):

$$a_i^T x \leq b_i \iff \sum_{j=1}^n a_{ij} x_j \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j + x_n \leq \frac{b_i}{a_{in}}, \quad \forall i \in M^+$$

For  $i \in M^-$  (2):

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j - x_n \leq \frac{b_i}{-a_{in}}, \quad \forall i \in M^-$$

For  $i \in M^0$  (3):

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} a_{ij} x_j \leq b_i, \quad \forall i \in M^0$$

$$P = \{x \in \mathbb{R}^n : (1)(2)(3)\}$$

Define (4):

$$\sum_{j=1}^{n-1} \left( \frac{a_{ij}}{a_{in}} - \frac{a_{kj}}{a_{kn}} \right) x_j \leq \frac{b_i}{a_{in}} - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, \forall k \in M^-$$

### Theorem 2.1

$$(\bar{x}_1, \dots, \bar{x}_{n-1}) \text{ satisfies (3), (4)} \iff \exists \bar{x}_n : (\bar{x}_1, \dots, \bar{x}_n) \in P$$

**Proof:**

$\Leftarrow$  If  $(\bar{x}_1, \dots, \bar{x}_n)$  satisfies (1), (2), (3) then  $(\bar{x}_1, \dots, \bar{x}_{n-1})$  satisfies (3) and adding (1), (2)  $\implies (\bar{x}_1, \dots, \bar{x}_{n-1})$  satisfies (4)

$\implies$  If  $(\bar{x}_1, \dots, \bar{x}_{n-1})$  satisfies (4)

$$\sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, k \in M^-$$

Let

$$\bar{x}_n := \max_{i \in M^+} \left\{ \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \right\}$$

$$\implies \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \bar{x}_j - \frac{b_i}{a_{in}} \leq -\bar{x}_n, \quad \forall i \in M^+$$

and

$$-\bar{x}_n \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \bar{x}_j - \frac{b_k}{a_{kn}}, \quad \forall k \in M^-$$

$$\implies (\bar{x}_1, \dots, \bar{x}_n) \in P$$

□

### Note

Proof assumes  $M^+, M^-$  are nonempty. But statement holds regardless.

(if  $M^+$  or  $M^- = \emptyset$  then (4) yields no constraints)

### Fourier-MotzKin

- $A^n = A, b^n = b$
- given  $A^i, b^i$  obtain  $A^{i-1}, b^{i-1}$  ( $A^{i-1}$  has one less column than  $A^i$  column than  $A^i$ ) by applying the steps described

$$P_i := \{x \in \mathbb{R}^i : A^i x \leq b^i\}$$

then

$$P_{i-1} = \text{proj}_{x_1, \dots, x_{i-1}} P_i$$

$$\text{and } P_{i-1} = \emptyset \iff P_i = \emptyset.$$

- Keep applying projection until  $i = 1$ .

$$P_0 = \emptyset \iff P_n = P = \emptyset$$

Let

$$P_i^n = P_i \times \mathbb{R}^{n-i} = \{x \in \mathbb{R}^n (A^i, 0)x \leq b^i\}$$

$$\text{not hard to see } P_i^n = \emptyset \iff P_i = \emptyset$$

Notice that

$$P_0 = \emptyset \iff P_0^n = \emptyset, P_0^n = \{0 \leq b^0\}$$

Example:

$$P_2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{rcl} x_1 & +x_2 & \leq 1 \\ -x_1 & & \leq 0 \\ & -x_2 & \leq -2 \\ -3x_1 & -3x_2 & \leq -6 \end{array} \right\}$$

draw the graph, clearly empty

$$M^+: \frac{1}{2}x_1 + x_2 \leq \frac{1}{2}$$

$$M^-: -x_2 \leq -2 \quad -x_1 - x_2 \leq -2$$

$$M^0: -x_1 \leq 0$$

$$P_1 = \left\{ x_1 \in \mathbb{R} : \begin{array}{rcl} & -x_1 & \leq 0 \\ \frac{1}{2}x_1 & & \leq -\frac{3}{2} \\ & -\frac{1}{2}x_1 & \leq -\frac{3}{2} \end{array} \right\}$$

$$M^+: x_1 \leq -3$$

$$M^-: -x_1 \leq 0 \text{ and } -x_1 \leq -3$$

$$P_0^2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{rcl} 0 & \leq & -3 \\ 0 & \leq & -6 \end{array} \right\} = \emptyset$$

Here  $b^0 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$

Remark:

Inequality in  $P_i^n$ :

- All inequalities are obtained by a nonnegative combination of inequality in  $P_{i+1}^n$   
 $\implies$  all nonnegative combination of inequalities in  $P$ .
- If all  $A, b$  are rational then so are all  $A^i, b^i$
- If  $b = 0, b_i = 0, \forall i$

### Theorem 2.2: Farkas' Lemma

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = \emptyset \iff \begin{array}{l} \exists u \in \mathbb{R}^m : u^T A = 0 \\ u^T b < 0 \\ u \geq 0 \end{array}$$

**Proof:** $(\Leftarrow)$  Suppose  $\bar{x}$  satisfies  $A\bar{x} \leq b$ .

$$0 = u^T A\bar{x} \leq u^T b < 0$$

which is impossible.

 $(\Rightarrow)$  If  $P = \emptyset$ . Apply Fourier-Motzkin until we get

$$P_0^n = \emptyset = \{x \in \mathbb{R}^n : 0x \leq b^0\}$$

i.e. there exists  $j$  for which  $b_j^0 < 0$ .If we look at corresponding constraint in  $P_0^n$  is

$$0^T x \leq b_j^0$$

which can be obtained by a vector  $u$  such that  $u^T A = 0, u^T b = b_j^0, u \geq 0$ .

□

**Farkas' Lemma (alternate statement)**

Exactly one of the following has a solution:

a)  $Ax \leq b$

$u^T A = 0$

b)  $u^T b < 0$

$u \geq 0$

**Farkas' Lemma (Different Form)**

Exactly one of the following has a solution:

a)  $Ax = b$

$x \geq 0$

b)  $u^T A \geq 0$

$u^T b < 0$

**Proof:**

(Sketch)

$$P = \left\{ x : \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\} = \left\{ x : \underbrace{\begin{pmatrix} A \\ -A \\ -I \end{pmatrix}}_{A'} x \leq \underbrace{\begin{pmatrix} b \\ -b \\ -0 \end{pmatrix}}_{b'} \right\}$$

Apply original Farkas' Lemma to get  $P = \emptyset \iff \exists u_1 \in \mathbb{R}^m, u_2 \in \mathbb{R}^m, v \in \mathbb{R}^n$ :

$$\begin{aligned} u_1^T A - u_2^T A - v &= 0 \\ u_1^T b - u_2^T b &< 0 \\ u_1, u_2, v &\geq 0 \end{aligned}$$

Let  $u = (u_1 - u_2)$

$$u^T A - v = 0 \implies u^T A \geq 0, \quad u^T b < 0$$

□

Consider a linear programming (LP):

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \quad (LP)$$

### Theorem 2.3: Fundamental Theorem of Linear Programming

(LP) has exactly one of 3 outcomes:

- a) Infeasible
- b) Unbounded
- c) There exists an optimal solution.

**Proof:**

Let's assume a), b) don't hold.

If  $n = 1$ , then (LP) has an optimal solution. (Why?)

Else, define

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c^T x \leq 0 \\ & Ax \leq b \end{aligned} \quad (LP')$$

(LP') is also not in case a) or b). (Why?)

Also if  $(x^*, z^*)$  is an optimal solution to (LP'), then  $x^*$  is an optimal solution to (LP). (Why?)

Apply Fourier-Motzkin to

$$\left\{ (x, z) : \begin{aligned} z - c^T x &\leq 0 \\ Ax &\leq b \end{aligned} \right\}$$

Until we are left with a polyhedron

$$\{z \in \mathbb{R} : A'z \leq b'\}$$

Now  $\max z \quad \text{s.t.} \quad A'z \leq b'$  is not cases a) or b). (Why?)

→ can get an optimal solution  $z^*$  to such problem. Apply Fourier-Motzkin back to get  $(x^*, z^*)$  optimal solution to (LP'). (Why?)  $\square$

## 2.3 Certifying Optimality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \quad (LP)$$

and let  $\bar{x} \in P = \{x : Ax \leq b\}$

**Question** Can we certify that  $\bar{x}$  is optimal?

**Example:**

$$\begin{array}{ll} \max & 2x_1 + x_2 \\ & x_1 + 2x_2 \leq 2 \\ \text{s.t.} & x_1 + x_2 \leq 2 \\ & x_1 - x_2 \leq 0.5 \end{array}$$

Consider  $\bar{x} = (0, 1)^T$  is clearly NOT optimal.

$x^* = (1, 0.5)^T$  and  $c^T x^* = 2.5$ . Any feasible solution satisfies

$$\begin{array}{rcl} x_1 + 2x_2 & \leq 2 & \times 1/3 \\ x_1 + x_2 & \leq 2 & \times 1 \\ + \quad x_1 - x_2 & \leq 0.5 & \times 2/3 \\ \hline 2x_1 + x_2 & \leq 3 & \end{array}$$

Instead do  $1 \times 1st$  constraint  $+ 1 \times 3rd$  constraint  $\implies 2x_1 + x_2 \leq 2.5$

In general:

$$\begin{array}{rcl} x_1 + 2x_2 & \leq 2 & \times y_1 \\ x_1 + x_2 & \leq 2 & \times y_2 \\ + \quad x_1 - x_2 & \leq 0.5 & \times y_3 \\ \hline (y_1 + y_2 + y_3)x_1 + (2y_1 + y_2 - y_3)x_2 & \leq 2y_1 + 2y_2 + 0.5y_3 \end{array}$$

As long as  $y_1, y_2, y_3 \geq 0$  and

$$\begin{array}{l} y_1 + y_2 + y_3 = 2 \\ 2y_1 + y_2 - y_3 = 1 \end{array}$$

This leads to the following linear program:

$$\begin{array}{ll} \min & 2y_1 + 2y_2 + 0.5y_3 \\ & y_1 + y_2 + y_3 = 2 \\ \text{s.t.} & 2y_1 + y_2 - y_3 = 1 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

This is called the dual LP.

In general:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \quad (P)$$

Dual of (P)

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & y^T A = c^T \\ & y \geq 0 \end{aligned} \quad (D)$$

**Remark:**

We call (P) primal LP.

### Theorem 2.4: Weak Duality

Let  $\bar{x}$  feasible for (P),  $\bar{y}$  feasible for (D). Then  $c^T \bar{x} \leq b^T \bar{y}$ .

**Proof:**

$$c^T \bar{x} = \bar{y}^T (A\bar{x}) \leq \bar{y}^T b$$

where we used  $A\bar{x} \leq b$  and  $\bar{y} \geq 0$ . □

### Corollary 2.5

Several results:

- If (P) is unbounded then (D) is infeasible.
- If (D) is unbounded then (P) is infeasible.

**Note**

(P) and (D) can both be infeasible.

- If  $\bar{x}$  is feasible for (P)  $\bar{y}$  feasible for (D)  $c^T \bar{x} = b^T \bar{y}$ , then  $\bar{x}$  optimal for (P),  $\bar{y}$  optimal for (D).

### Theorem 2.6: Strong Duality

$x^*$  is optimal for (P)  $\iff \exists y^*$  feasible for (D) such that  $c^T x^* = b^T y^*$ .

**Proof:**

( $\Leftarrow$ ) ✓

( $\Rightarrow$ ) Is (D) infeasible?

$$\text{Suppose } \left\{ y \in \mathbb{R}^n : \begin{array}{l} A^T y = c \\ y \geq 0 \end{array} \right\} = \emptyset$$

(Alternate version of Farkas' Lemma)  $\exists u : \begin{matrix} u^T A \geq 0 \\ u^T c < 0 \end{matrix} \iff \exists d : \begin{matrix} Ad \leq 0 \\ c^T d > 0 \end{matrix}$

Take look at  $x' = x^* + d$ , then

$$\begin{aligned} Ax' &= Ax^* + Ad \leq b \\ c^T x' &= c^T x^* + c^T d > c^T x^* \end{aligned}$$

Contradiction. Thus (D) has an optimal solution  $y^*$ .

Now let  $\gamma = b^T y^*$ , and let  $\theta := \left\{ x \in \mathbb{R}^n : \begin{matrix} Ax \leq b \\ -c^T x \leq -\gamma \end{matrix} \right\}$ .

If  $\theta = \emptyset$ , by Farkas'

$$\exists \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} : \begin{cases} \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix}^T \begin{pmatrix} A \\ -c^T \end{pmatrix} = 0 \\ \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix}^T \begin{pmatrix} b \\ -\gamma \end{pmatrix} < 0 \\ \begin{pmatrix} \bar{y} \\ \bar{\lambda} \end{pmatrix} \geq 0 \end{cases} \iff \begin{aligned} A^T \bar{y} &= c \bar{\lambda} \\ b^T \bar{y} &< \gamma \bar{\lambda} \\ \bar{y} &\geq 0 \\ \bar{\lambda} &\geq 0 \end{aligned}$$

Case 1:  $\bar{\lambda} > 0$ .

Let  $y' = \frac{\bar{y}}{\bar{\lambda}}$ . Then we have

$$A^T y' = A^T \frac{\bar{y}}{\bar{\lambda}} = c \quad \text{and} \quad b^T y' = b^T \frac{\bar{y}}{\bar{\lambda}} < \gamma \quad \text{and} \quad y' = \frac{\bar{y}}{\bar{\lambda}} \geq 0$$

Contradicts optimality of  $y^*$ .

$$A^T y = 0$$

Case 2:  $\bar{\lambda} = 0$ . Then  $b^T y < 0$

$$\bar{y} \geq 0$$

Now we can do the same thing previously. Let  $y' = y^* + \bar{y}$ , then

$$A^T y' = A^T y^* + A^T \bar{y} = c$$

and

$$\begin{aligned} y' &= y^* + \bar{y} \geq 0 \\ b^T y' &= b^T y^* + b^T \bar{y} < b^T y^* \end{aligned}$$

Contradicts optimality of  $y^*$ .

Thus  $\theta \neq \emptyset$ .



Let  $\bar{x} \in \theta$ ,

$$c^T x^* \underbrace{\leq}_{\text{weak duality}} b^T y^* = \gamma \underbrace{\leq}_{\bar{x} \in \theta} c^T \bar{x} \leq c^T x^*$$

where the last inequality is because  $\bar{x}$  feasible for (P),  $x^*$  optimal for (P).

□

## 2.4 Possible Outcomes

See [here](#).

## 2.5 Duals of generic LPs

$$\begin{array}{ll} \max & 2x_1 + 3x_2 - 4x_3 \\ & x_1 \quad \quad + 7x_3 \leq 5 \\ & \quad 2x_2 - x_3 \geq 3 \\ \text{s.t.} & x_1 \quad \quad + x_3 = 8 \\ & \quad x_2 \leq 6 \\ & x_1 \geq 0 \\ & \quad x_2 \leq 0 \end{array}$$

$$\begin{array}{ll} \max & (2, 3, -4)x \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 7 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 5 \\ -3 \\ 8 \\ -8 \\ 6 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

and dual

$$\begin{array}{ll} \min & (5, -3, 8, -8, 6, 0, 0)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0 \end{array} \quad (D_1)$$

$$\begin{array}{ll} \min & (5, -3, 8, -8, 6)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0 \end{array} \quad (D_2)$$

**Claim**  $(y_1^*, \dots, y_5^*)$  is optimal for  $(D_2) \iff (y_1^*, \dots, y_5^*, y_6^*, y_7^*)$  optimal for  $(D_1)$  with

$$\begin{aligned} y_6^* &= y_1^* + y_3^* - y_4^* - 2 \\ y_7^* &= 3 - (-2y_2^* + y_5^*) \end{aligned}$$

$$\begin{aligned} \min \quad & (5, 3, 8, 6)y \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 7 & -1 & 1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y_1 \geq 0, y_2 \leq 0 \quad y_4 \geq 0 \end{aligned} \quad (D_3)$$

**Claim** Opt value of  $(D_2)$  and  $(D_3)$  are same.

**In general**

$$\begin{array}{l|l} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \quad (P) \quad \left| \quad \begin{array}{l|l} \min & b^T y \\ \text{s.t.} & A^T y \leq c \\ & y \geq 0 \end{array} \quad (D)$$

## 2.5.1 Cheat Sheet

Here or

Primal (max)		Dual (min)	
Constraint	$\leq$	$\geq 0$	Variable
	$\geq$	$\leq 0$	
	$=$	free	
Variable	$\geq$	$\geq 0$	Constraint
	$\leq$	$\leq 0$	
	free	$=$	

**Remark:**

This is not symmetric... The way you can remember it is by thinking natural variables in real life, like you cannot have negative number of cars and so on...

**Q** What if you start with a minimization LP as primal?

**Example:**

$$\begin{aligned} \min \quad & x_1 - x_2 \\ & 2x_1 + 3x_2 \leq 5 \\ \text{s.t.} \quad & x_1 - x_2 \geq 3 \\ & x_1 + 5x_2 = 7 \\ & x_1 \geq 0, x_2 \leq 0 \end{aligned} \quad (P)$$

Rewrite as:

$$-1 \times \begin{pmatrix} \max & -x_1 + x_2 \\ \downarrow \\ \text{s.t.} & \dots \end{pmatrix}$$

Will lead to finding dual:

$$\begin{array}{ll} \max & 5y_1 + 3y_2 + 7y_3 \\ \downarrow & \\ & 2y_1 + y_2 \leq 1 \\ \text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\ & y_1 \leq 0, y_2 \geq 0, y_3 \text{ free} \end{array}$$

Also

- Weak duality holds.

If  $\bar{x}$  feasible for (P),  $\bar{y}$  feasible for (D), then  $c^T \bar{x} \geq b^T \bar{y}$ .

- Strong duality holds

### Note

The dual of the dual of (P) is (P).

### Example:

Given a simple undirected graph  $G = (V, E)$ .  $M \subseteq E$  is a *matching* if every vertex  $v \in V$  is incident to  $\leq 1$  edge in  $M$ .

See examples of matching in [CO 342](#) or [MATH 249](#).

## Max cardinality matching

Find matching  $M$  with largest  $|M|$ .

Define  $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{otherwise} \end{cases}$ .

$$\begin{array}{ll} \max & \sum_{e \in E} x_e \\ \downarrow & \\ & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\ \text{s.t.} & \\ & 0 \leq x_e, \quad \forall e \in E \end{array}$$

where  $\delta(v)$  = set of edges in  $E$  incident to  $v$ .

$$\begin{array}{ll} \min & \sum_{v \in V} y_v \\ \downarrow & \\ \text{s.t.} & y_u + y_v \geq 1, \quad \forall e = uv \in E \\ & y \geq 0 \end{array}$$

## 2.6 Other interpretations of dual

Example:

			Resources	
		Per unit Profit	Per unit consumption	
			A	B
Product	1	5	2	3
	2	3	4	1
Available Resources			15	10

$$\begin{aligned}
 & \max \quad 5x_1 + 3x_2 \\
 & \downarrow \\
 & \begin{aligned}
 & 2x_1 + 4x_2 \leq 15 \\
 \text{s.t.} \quad & 3x_1 + x_2 \leq 10 \\
 & x \geq 0
 \end{aligned}
 \end{aligned}$$

Suppose somebody wants to buy  $A, B$  from me. What is the lowest price I should ask?

Let  $y_A, y_B$  be prices:

$$\begin{aligned}
 & \min \quad 15y_A + 10y_B \\
 & \downarrow \\
 & \begin{aligned}
 & 2y_A + 3y_B \geq 5 \\
 \text{s.t.} \quad & 4y_A + y_B \geq 3 \\
 & y \geq 0
 \end{aligned}
 \end{aligned}$$

Example: Zero-Sum

Alice, Bob play game. A:  $m$  choices. B:  $n$  choices. Alice play  $i$ , Bob plays  $j$ , Bob pays Alice  $M_{ij}$  dollars.

		Alice		
		R	P	S
Bob	R	0	1	-1
	P	-1	0	1
	S	1	-1	0

Zero-sum: Amount won by Alice - Amount won by Bob = 0

Let  $y \in \mathbb{R}_+^m$ , Alice's probability distribution.

Let  $x \in \mathbb{R}_+^n$ , Bob's probability distribution.

Expected Amount Bob pays Alice:

$$\sum_{i=1}^m \sum_{j=1}^n y_i M_{ij} x_j = y^T M x$$

$$P = \left\{ x \in \mathbb{R}^n : \sum x_j = 1, x \geq 0 \right\}$$

$$Q = \left\{ y \in \mathbb{R}^m : \begin{array}{l} \sum y_i = 1 \\ y \geq 0 \end{array} \right\}$$

Alice wants  $\max_{y \in Q} \left\{ \min_{x \in P} y^T M_x \right\}$ . Bob wants  $\min_{x \in P} \left\{ \max_{y \in Q} y^T M_x \right\}$ .

Suppose  $\bar{y} \in Q$  is fixed. Bob's problem is

$$\begin{aligned} \min_{x \in P} \bar{y}^T M_x &= \min \sum_{j=1}^n \left( \sum_{i=1}^m M_{ij} \bar{y}_i \right) x_j \\ &\downarrow \\ \text{s.t.} \quad &\sum_{j=1}^n x_j = 1 \\ &x \geq 0 \end{aligned}$$

This is equivalent to picking smallest number in

$$\begin{aligned} &\left\{ \sum_{i=1}^m M_{ij} \bar{y}_i \right\}_{j=1}^n \\ \Rightarrow \max_{y \in Q} \min_{x \in P} y^T M_x &= \max_{y \in Q} \left\{ \begin{array}{l} \max u \\ \downarrow \\ \text{s.t.} \quad u \leq y^T M e_j, \quad \forall j = 1, \dots, n \end{array} \right\} \\ &= \begin{array}{l} \max u \\ \downarrow \\ \text{s.t.} \quad u \leq y^T M e_j, \quad \forall j = 1, \dots, n \\ y^T = 1 \\ y \geq 0 \end{array} \end{aligned}$$

Similarly Bob's problem:

$$\begin{aligned} \min v & \\ \downarrow & \\ \text{s.t.} \quad &v \geq e_i^T M x, \quad \forall i = 1, \dots, m \\ &x^T = 1 \\ &x \geq 0 \end{aligned}$$

There are  $x^*, y^*$  for which strategy values match  $\rightarrow$  Nash's Equilibrium.

Now get back to Farkas' Lemma Theorem 2.2. <sup>1</sup>

**Proof:**

$$\begin{aligned} \max \quad &0^T x \\ \downarrow & \\ \text{s.t.} \quad &Ax \leq b \end{aligned} \tag{P}$$

<sup>1</sup>Rephrase it a little bit: Exactly one of the two has a solution (i)  $Ax \leq b$  (ii)  $u^T \dots$

$$\begin{array}{ll}
\min & b^T u \\
\downarrow & \\
\text{s.t.} & u^T A = 0 \\
& u \geq 0
\end{array} \tag{D}$$

(D) is always feasible ( $u = 0$ ).

If  $\exists \bar{x} : A\bar{x} \leq b$ ,  $\bar{x}$  optimal for (P)  $\implies$  optimal for (D) has value 0.  
 $\implies \nexists u$  satisfying (i).

And the converse is also true. □

## 2.7 Complementary Slackness (C.S.)

Let  $x^*, y^*$  be feasible for primal and dual respectively.

### Complementary Slackness

Abbreviated as C.S.

- i) Either  $x_j^* = 0$  or corresponding dual constraint is tight at  $y^*$ ,  $\forall j = 1, \dots, n$ .
- ii) Either  $y_i^* = 0$  or corresponding primal constraint is tight at  $x^*$ ,  $\forall i = 1, \dots, m$ .

Example:

$$\begin{array}{ll}
\min & x_1 - x_2 \\
\downarrow & \\
& 2x_1 + 3x_2 \leq 5 \\
\text{s.t.} & x_1 - x_2 \geq 3 \\
& x_1 + 5x_2 = 7 \\
& x_1 \geq 0, x_2 \leq 0
\end{array} \tag{P}$$

$$\begin{array}{ll}
\max & 5y_1 + 3y_2 + 7y_3 \\
\downarrow & \\
& 2y_1 + y_2 + y_3 \leq 1 \\
\text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\
& y_1 \leq 0, y_2 \geq 0
\end{array} \tag{D}$$

- i)  $x_1^* = 0$  OR  $2y_1^* + y_2^* + y_3^* = 1$   
 $x_2^* = 0$  OR  $3y_1^* - y_2^* + 5y_3^* = -1$

- ii)  $y_1^* = 0$  OR  $2x_1^* + 3x_2^* = 5$   
 $y_2^* = 0$  OR  $x_1^* - x_2^* = 3$   
 $y_3^* = 0$  OR  $x_1^* + 5x_2^* = 7$

**Theorem 2.7**

Let  $x^*, y^*$  be feasible for primal/dual respectively. TFAE<sup>a</sup>

- a)  $x^*$  opt for primal AND  $y^*$  opt. for dual
- b) Obj. value of  $x^* =$  Obj. value of  $y^*$
- c)  $x^*, y^*$  satisfy C.S.

---

<sup>a</sup>the following are equivalent

**Proof:**

a)  $\iff$  b) done.

b)  $\iff$  c) Proof for

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \min & b^T y \\ \downarrow & \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

**Note**

$$A^T y \geq c \iff \sum_{i=1}^m a_{ij} y_i \geq c_j, \quad \forall j = 1, \dots, n$$

$$\begin{aligned} c^T x^* &= \sum_{j=1}^n c_j x_j^* \\ &\leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{ij} y_i^* \right) x_j^* \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{ij} x_j^* \right) y_i^* \\ &\leq \sum_{i=1}^m b_i y_i^* = b^T y^* \end{aligned}$$

where first and second inequalities come from  $x \geq 0, y \geq 0$  respectively.

(b)  $c^T x^* = b^T y^* \iff$  C.S. holds. (Just play with some strict inequality conditions)

□

**Example:**

$$\begin{array}{ll} \max & x_1 + x_2 \\ \downarrow & \\ \text{s.t.} & x_1 + x_2 \leq 1 \end{array} \qquad \begin{array}{ll} \min & y \\ \downarrow & \\ & y = 1 \\ \text{s.t.} & y = 1 \\ & y \geq 0 \end{array}$$

■ Consider a pair  $x^* = (0, 0), y^* = 1$  which violates CS.

### 2.7.1 Geometric Interpretation of C.S.

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \qquad \begin{array}{ll} \min & c^T y \\ \downarrow & \\ \text{s.t.} & A^T y = c \\ & y \geq 0 \end{array}$$

$$A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

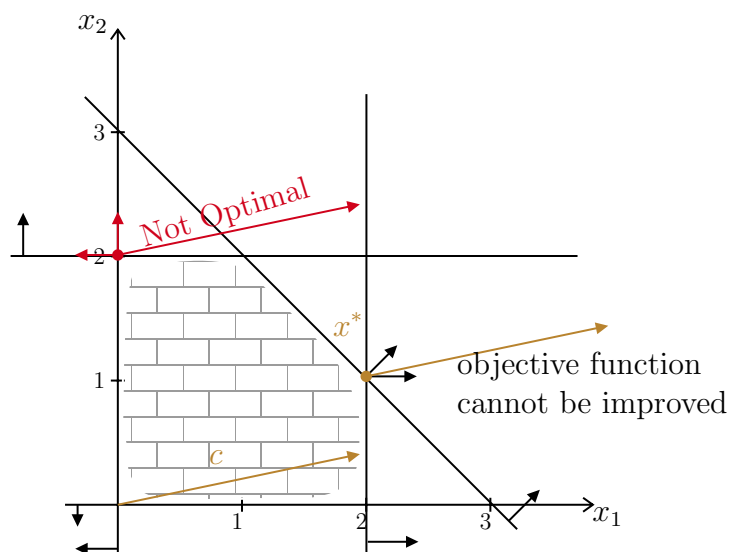
C.S. says  $a_i^T x^* = b_i$  or  $y_i^* = 0$ .

$$A^T y = c \implies \begin{pmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{pmatrix} y = c \implies \sum_{i=1}^m a_i y_i = c$$

C.S. says  $c$  is a nonnegative combination of tight constraint at  $x^*$ .

Example:

$$\begin{array}{ll} \max & 2x_1 + 0.5x_2 \\ \downarrow & \\ & x_1 \leq 2 \\ & x_2 \leq 2 \\ \text{s.t.} & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$





**Theorem 2.8**

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \quad (\text{P})$$

is unbounded iff (P) is feasible and  $\exists d \in \mathbb{R}^n : \begin{array}{l} c^T d > 0 \\ Ad \leq 0 \end{array}$ .

**Proof:**

$\Rightarrow$ ) Let  $\bar{x}$  feasible for (P),  $\bar{x} + \lambda d$  is also feasible for (P)  $\forall \lambda \geq 0$ .

$c^T(\bar{x} + \lambda d)$  can be made arbitrary large.

$\Leftarrow$ ) Hard exercise but doable.

□

## 2.8 Geometry of Polyhedra

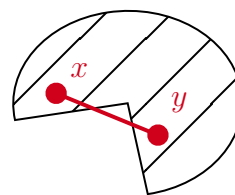
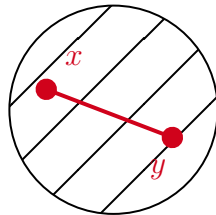
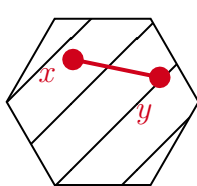
**line segment**

$\bar{x}, \bar{y} \in \mathbb{R}^n$  the line segment between  $\bar{x}, \bar{y}$  is

$$\left\{ x \in \mathbb{R}^n : \begin{array}{l} x = \lambda \bar{x} + (1 - \lambda) \bar{y} \\ \text{for some } \lambda \in [0, 1] \end{array} \right\}$$

**convex set**

$S$  is a convex set if  $\forall x, y \in S$ , line segment between  $x, y$  is contained in  $S$ .

**Example:**

NOT a convex set

Polyhedra are convex sets.  $P = \{x : Ax \leq b\}$ .  $\bar{x}, \bar{y} \in P$  then

$$A(\underbrace{\lambda}_{\geq 0} \bar{x} + \underbrace{(1 - \lambda)}_{\geq 0} \bar{y}) \leq \lambda b + (1 - \lambda)b = b$$

**convex combination**

Given  $x^1, \dots, x^k \in \mathbb{R}^n$ . We say  $\bar{x}$  is a convex combination of  $x^1, \dots, x^k$  if  $\exists \lambda$ :

$$\begin{aligned}\bar{x} &= \sum_{i=1}^k \lambda_i x^i \\ 1 &= \sum_{i=1}^k \lambda_i \\ \lambda &\geq 0\end{aligned}$$

Optimal solution seems to be happen at “corners”.

Let  $P$  be a polyhedron  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ .

**vertex**

$\bar{x}$  is a vertex of  $P$  if  $\exists c$ :  $\bar{x}$  is unique optimal solution to

$$\begin{aligned}\max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b\end{aligned}$$

**extreme point**

$\bar{x}$  is an extreme point of  $P$  if  $\nexists u, v \in P \setminus \{\bar{x}\}$  such that  $\bar{x}$  is in line segment between  $u, v$ .

**basic feasible solution**

$\bar{x} \in P$  is a basic feasible solution of  $P$  if there are  $n$  linearly independent tight constraints at  $\bar{x}$ .

**Note**

Constraints

$$a_i^T x \leq b_i, \quad \forall i = 1, \dots, m$$

are linearly independent if  $\{a_i\}_{i=1}^m$  are linearly independent.

**Theorem 2.9**

Let  $\bar{x} \in P$ . TFAE:

- a)  $\bar{x}$  is a vertex of  $P$ .
- b)  $\bar{x}$  is a basic feasible solution of  $P$ .
- c)  $\bar{x}$  is a extreme point of  $P$ .

**Proof:**

a)  $\implies$  c) Suppose  $\exists u, v \in P \setminus \{\bar{x}\}$  such that

$$\bar{x} = \lambda u + (1 - \lambda)v$$

for some  $\lambda \in (0, 1)$ . Consider  $c$  for which  $\bar{x}$  is an optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$$

$$\implies \begin{array}{l} c^T \bar{x} \geq c^T u \\ c^T \bar{x} \geq c^T v \end{array}$$

and

$$c^T \bar{x} = \underbrace{\lambda}_{\geq 0} c^T u + \underbrace{(1 - \lambda)}_{\geq 0} c^T v \leq \lambda c^T \bar{x} + (1 - \lambda) c^T \bar{x} = c^T \bar{x}$$

$$\implies c^T u = c^T v = c^T \bar{x}$$

$\implies \bar{x}$  NOT a vertex.

c)  $\implies$  b) Suppose  $\bar{x}$  is not a BFS. Let  $i \subseteq \{1, \dots, m\}$  be the index set of tight constraint at  $\bar{x}$ . Consider

$$a_i^T d = 0, \quad \forall i \in I \quad (*)$$

But since  $\bar{x}$  not BFS,  $\exists \bar{d} \neq 0$  satisfying  $(*)$ .<sup>a</sup>

$$x(\epsilon) = \bar{x} + \epsilon \bar{d}$$

$$a_i^T x(\epsilon) = a_i^T \bar{x} \leq b_i, \quad \forall i \in I$$

$$a_i^T x(\epsilon) = \underbrace{a_i^T \bar{x}}_{< b_i} + \epsilon a_i^T \bar{d} \leq b_i, \quad \forall i \notin I$$

which is satisfied if  $|\epsilon|$  is small enough.

$x(\epsilon) \in P$  if  $|\epsilon|$  is small enough.

But then

$$\bar{x} = \frac{1}{2}x(\epsilon) + \frac{1}{2}x(-\epsilon)$$

b)  $\implies$  a) Let  $I \subseteq \{1, \dots, m\}$  index set of tight constraint at  $\bar{x}$ .

Define

$$c := \sum_{i \in I} a_i$$

Then  $\forall x \in P$

$$c^T x = \sum_{i \in I} a_i^T x \leq \sum_{i \in I} b_i$$

And

$$c^T \bar{x} = \sum_{i \in I} a_i^T \bar{x} = \sum_{i \in I} b_i$$

$\implies \bar{x}$  is optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array} \quad (**)$$

If  $x' \in P$  is optimal solution to  $(**)$ , then

$$a_i^T x' = b_i, \quad \forall i \in I \quad (***)$$

But since there are  $n$  linear independent constraints in  $I$ ,  $\bar{x}$  is unique solution to  $(***)$ .  $\implies x' = \bar{x}$ .

□

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<sup>a</sup>by Rank-Nullity Theorem.

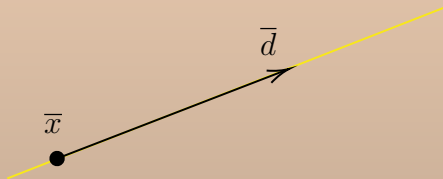
**Q** When does  $P$  have extreme points?

**line**

Let  $\bar{x}, \bar{d} \in \mathbb{R}^n$ ,  $\bar{d} \neq 0$ . The set

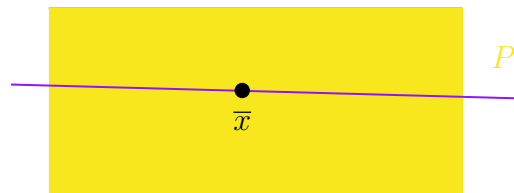
$$\{x \in \mathbb{R}^n : x = \bar{x} + \lambda \bar{d} \text{ for some } \lambda \in \mathbb{R}\}$$

is called a line.



We say a polyhedron  $P$  has a line if  $\exists \bar{x}, \bar{d}$  has a line if  $\exists \bar{x}, \bar{d}$  s.t.  $\bar{x} \in P, \bar{d} \neq 0$  and

$$\{x \in \mathbb{R}^n : x = \bar{x} + \lambda \bar{d} \text{ for some } \lambda \in \mathbb{R}\} \subseteq P$$



**Proposition 2.10**

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$  has a line iff  $P \neq \emptyset$  and  $\exists \bar{d} \neq 0$  such that  $A\bar{d} = 0$

$\iff P \neq \emptyset$  and  $\text{rank}(A) < n$

**Proof:**

Exercise.

□

**Theorem 2.11**

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$  has an extreme point

$\iff P \neq \emptyset$  and  $P$  has no lines.

**Proof:**

Exercise. □

**pointed polyhedron**

A non-empty polyhedron is called pointed if it has no lines.

**Note**

not pointed does not imply bounded. For example, in  $\mathbb{R}^2$ ,  $x \geq 0$  and  $y \geq 0$ .

**Theorem 2.12**

Let  $P \neq \emptyset$  pointed polyhedron. If  $\max_{x \in P} c^T x$  (LP) has an optimal solution, it has an optimal solution that is an extreme point.

**Proof:**

Let  $\bar{x}$  be an optimal solution to (LP) with largest number of linear independent tight constraints.

Suppose there are  $\leq n - 1$  linear independent tight constraints at  $\bar{x}$ .

Pick  $\bar{d} \neq 0$  such that  $a_i^T \bar{d} = 0, \forall i \in I$ , where  $I$  is the index set of tight constraints. By the exact same argument as before,  $\bar{x} \pm \epsilon \bar{d} \in P$  for  $\epsilon$  small enough. But

$$c^T(\bar{x} \pm \epsilon \bar{d}) = c^T \bar{x} \pm \epsilon c^T \bar{d}$$

$$\implies c^T \bar{d} = 0$$

$$\implies c^T d(\bar{x} \pm \epsilon d) = c^T \bar{x}$$



Since  $P$  is pointed,  $\exists \bar{\epsilon}$  for which

$$\bar{x} \pm \bar{\epsilon} \in P$$

and one of them not in  $P$  if  $|\epsilon| > \bar{\epsilon}$ . That can only happen if

$$a_k^T(\bar{x} + \bar{\epsilon} \bar{d}) = b_k \quad \text{or} \quad a_k^T(\bar{x} - \bar{\epsilon} \bar{d}) = b_k$$

for some  $k \notin I$ .

$\implies a_k^T \bar{d} \neq 0, \implies a_k$  is linear independent from  $\{a_i\}_{i \in I}$  since non-zero cannot be linear combination of zeros. Contradiction to choice of  $\bar{x}$ .  $\square$

## 2.9 Simplex Algorithm

### Standard Equality Form

A linear program is in Standard Equality Form (SEF) if it is of the form

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

### Proposition 2.13

Given any linear program, there exists an equivalent LP in SEF.

**Example:**

$$\begin{array}{ll} \max & x_1 + 2x_2 + x_3 \\ \downarrow & \\ & 3x_1 + x_2 \leq 5 \\ \text{s.t.} & -x_1 + x_3 \geq 6 \\ & x_1 \leq 0, x_3 \geq 0 \end{array} \quad (\text{P1})$$

$$\begin{aligned} x'_1 &= -x_1 \geq 0 \text{ and} \\ x_2 &= x_2^+ - x_2^- \text{ where } x_2^+ \geq 0, x_2^- \geq 0 \end{aligned}$$

We introduce

$$s_1 = 5 - 3x_1 - x_2 \geq 0, \quad s_2 = -x_1 + x_3 - 6 \geq 0$$

Then

$$\begin{array}{ll} \max & -x'_1 + 2x_2^+ - 2x_2^- + x_3 \\ \downarrow & \\ & -3x'_1 + 2x_2^+ - x_2^- + s_1 = 5 \\ \text{s.t.} & x'_1 + x_3 - s_2 = 6 \\ & x'_1, x_2^+, x_2^-, x_3, s_1, s_2 \geq 0 \end{array} \quad (\text{P2})$$

$x$  feasible for (P1)  $\iff (x'_1, x_2^+, x_2^-, x_3, s_1, s_2)$  feasible for (P2) and they have same cost.

**Assumption**  $A \in \mathbb{R}^{m \times n} \rightarrow \text{rank}(A) = m$ . This is WLOG. Since if

$$a_i = \sum_{k \neq i} \lambda_k a_k$$

Either

$$b_i \neq \sum_{k \neq i} \lambda_k b_k$$

in which case (SEF) is infeasible. Or  $a_i^T x = b_i$  is redundant. So it can be removed from (SEF).

**Note**

$\{x : Ax = b, x \geq 0\}$  is *pointed* polyhedron (if nonempty).

**Structure of BFS** Any feasible solution has  $m$  linear independent tight constraints ( $n - m$ ) extra tight constraint must come from  $x_j \geq 0$ .

Let  $B \subseteq \{1, \dots, n\}$  such that  $|B| = m$  and  $A_B$ <sup>2</sup> is invertible.

$N = \{1, \dots, n\} \setminus B$ .  $x_N = 0$ , i.e.  $x_j = 0, \forall j \in N$ .

Feasible solutions obtained this way are precisely BFS.

**Example:**

$$\begin{array}{ll} \max & (3 \quad 2 \quad 1 \quad 4) x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x \geq 0 \end{array}$$

If we pick

$$\begin{array}{ll} B = \{1, 2\} & A_B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ N = \{3, 4\} & A_N = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ C_B = (3 \quad 2)^T & C_N = (1 \quad 4)^T \end{array}$$

$$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

---


$$B = \{1, 3\}, B = \{2, 4\}, A_B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, A_N = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

$$C_B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, C_N = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$$

If we set  $x_N = 0$  (for  $B = \{1, 3\}$ ) we are left with

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

This has a unique solution  $x_1 = 3.5, x_3 = -1.5$ , but not feasible.

---

<sup>2</sup> $A_B$  is submatrix obtained by picking columns of  $A$  indexed by  $B$ . Such  $B$  is called a basis.

If we pick  $B = \{1, 2\}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$\underbrace{x_3 = x_4 = 0}_{x_N}, x_1 = 3, x_2 = 1$ , which is feasible.

In general,

$$Ax = b \iff A_B x_B + \cancel{A_N x_N}^0 = b$$

has unique solution  $x_b = A_B^{-1}b$ .

For any basis  $B$ , the corresponding *basic solution* is

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1}b \\ 0 \end{pmatrix}$$

If  $A_B^{-1}b \geq 0$ , then it is a *BFS*.

### 2.9.1 Canonical Form

Let  $B$  be a feasible basis (i.e. corresponding basis solution is feasible).

$$\begin{aligned} Ax = b &\iff A_B x_B + A_N x_N = b \\ &\iff x_B + A_B^{-1} A_N x_N = A_B^{-1}b \end{aligned}$$

Now let's take a look at objective.

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N - \cancel{c_B^T (x_B + A_B^{-1} A_N x_N - A_B^{-1}b)} \\ &= (c_N^T - c_B^T A_B^{-1} A_N) x_N + c_B^T A_B^{-1}b \end{aligned}$$

Thus (SEF) is said to be in canonical form for  $B$  if it is written as

$$\begin{aligned} \max \quad & \overbrace{(c_N^T - c_B^T A_B^{-1} A_N)}^{\text{Reduced costs}} x_N + c_B^T A_B^{-1}b \\ \downarrow \\ \text{s.t.} \quad & x_B + A_B^{-1} A_N x_N = A_B^{-1}b \\ & x_B, x_N \geq 0 \end{aligned}$$

**Example:**

Back to our previous example...

$B = \{1, 2\}$ . Rewriting in canonical form for  $B$ :

$$A_B^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

$$A_B A = \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix}$$



$$c_B^T A_B^{-1} A_N = (3 \quad 2) \begin{pmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{pmatrix} = (7/3 \quad -8/3)$$

$$c_N^T - c_B^T A_B^{-1} A_N = (-4/3 \quad 4/3)$$

Then

$$\begin{aligned} \max \quad & (0 \quad 0 \quad -4/3 \quad 4/3)x + 11 \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

is in canonical form for  $B = \{1, 2\}$ .

Example:

$$\begin{aligned} \max \quad & (1 \quad 3 \quad -2 \quad 0 \quad 0)x \quad \underbrace{+0}_{\text{obj. value}} \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned} \quad (\text{LP})$$

Canonical form for  $B = \{4, 5\}$ .

Corresponding BFS  $\begin{matrix} x_4 = 4 \\ x_5 = 1 \end{matrix}$ ,  $x_j = 0, \forall j \in N$

$$x = (0 \quad 0 \quad 0 \quad 4 \quad 1)^T$$

Objective value = 0

If increase  $x_1$  or  $x_2$ . Objective function increases.

Let's try to increase  $x_1$  from  $0 \rightarrow \theta$ . (Keep  $x_2 = x_3 = 0$ )

$$\begin{aligned} \theta + x_4 = 4 & \iff x_4 = 4 - \theta \\ \theta + x_5 = 1 & \iff x_5 = 1 - \theta \end{aligned}$$

New objective:  $0 + \theta$ . However, we have

$$\begin{aligned} x_4 \geq 0 & \implies \theta \leq 4 \\ x_5 \geq 0 & \implies \theta \leq 1 \end{aligned} \implies \text{Increase } x_1 \text{ by } 1$$

$x_5$  will be 0  $\rightarrow$   $\begin{matrix} x_1 \text{ enters basis} \\ x_5 \text{ leaves basis} \end{matrix}$ . Then new basis  $B = \{1, 4\}$ .

---

Rewriting (LP) in canonical form for  $B = \{1, 4\}$ .

$$\begin{array}{l}
 \max \quad (0 \quad 4 \quad -5 \quad 0 \quad -1) x + \underbrace{1}_{\text{obj. value}} \\
 \downarrow \\
 \text{s.t.} \quad \begin{pmatrix} 1 & -1 & 3 & 0 & 1 \\ 0 & 2 & -2 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
 x \geq 0
 \end{array}$$

Corresponding BFS:

$$x = (1 \quad 0 \quad 0 \quad 3 \quad 0)^T$$

Obj. value = 1

---

Pick  $j \in N$ :  $\bar{c}_j > 0$  ( $j = 2$ )

Increase  $x_2$  to  $\theta$ , keep  $x_3 = x_5 = 0$

$$\begin{aligned}
 x_1 - \theta &= 1 \iff x_1 = 1 + \theta \\
 x_4 + 2\theta &= 3 \iff x_4 = 3 - 2\theta
 \end{aligned}$$

and

$$\begin{aligned}
 x_1 \geq 0 &\implies \theta \geq -1 \\
 x_4 \geq 0 &\implies \theta \leq \frac{3}{2}
 \end{aligned}$$

Set  $\theta \leftarrow \frac{3}{2} \rightarrow$   $x_2$  enters basis  
 $x_4$  leaves basis

New basis  $B = \{1, 2\}$ .

---

(LP) in canonical form for  $B = \{1, 2\}$ .

$$\begin{array}{l}
 \max \quad (0 \quad 0 \quad -1 \quad -2 \quad 1) x + 7 \\
 \downarrow \\
 \text{s.t.} \quad \begin{pmatrix} 1 & 0 & 2 & 0.5 & 0.5 \\ 0 & 1 & -1 & 0.5 & -0.5 \end{pmatrix} x = \begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix} \\
 x \geq 0
 \end{array}$$

Corresponding BFS:

$$x = (2.5 \quad 1.5 \quad 0 \quad 0 \quad 0)^T$$

Obj. value = 7

---

Find  $j \in N$ ,  $\bar{c}_j > 0$  ( $j = 5$ )

$$\begin{aligned}
 x_1 = 2.5 - 0.5\theta \geq 0 &\implies \theta \leq 5 \rightarrow x_1 \text{ leaves basis} \\
 x_2 = 1.5 + 0.5\theta \geq 0 &\implies \theta \geq -3 \rightarrow x_5 \text{ enters basis}
 \end{aligned}$$

New basis  $B = \{2, 5\}$

---

(LP) in canonical form for  $B = \{2, 5\}$

$$\begin{aligned} & \max \quad (-2 \quad 0 \quad -5 \quad -3 \quad 0) x + 12 \\ & \downarrow \\ & \text{s.t.} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & \quad \quad \quad x \geq 0 \end{aligned}$$

BFS  $x = (0 \quad 4 \quad 0 \quad 0 \quad 5)^T$  } Optimal Solution  
 Obj. value = 12.

## 2.9.2 Iteration of simplex

- Start with feasible basis  $B$
- Rewrite LP in canonical form for  $B$
- Pick  $j \in N : \bar{c}_j > 0$  ( $x_j$  enters basis)
- Let  $\bar{b} = A_B^{-1}b$ ,  $\bar{A}_N = A_B^{-1}A_N$

Find largest  $\theta$  so that

$$\bar{b} - \theta \bar{A}_j \geq 0$$

Corresponding basic variable that becomes 0 (say  $x_k$ ) leaves basis.

- $B \leftarrow B \setminus \{k\} \cup \{j\}$ . Iterate.

If problem has optimal solution AND  $\theta$  is always  $> 0$ , simplex finishes.

### Note

If at current BFS we have a basic variable = 0, we may have  $\theta = 0$ .  $\rightarrow$  May lead to cycling. (i.e. return to current basis in future iteration)

### Bland's Rule

If there are multiple choices of entering or leaving variables, always pick lowest index variable.

Using Bland's Rule avoids cycling

**Observations** If  $\bar{c}_N \leq 0$ , then the (LP) obj. value in canonical form is

$$\underbrace{\bar{c}_N^T}_{\leq 0} \underbrace{x_N}_{\geq 0} + c_B^T A_B^{-1} b \leq c_B^T A_B^{-1} b$$

For any feasible solution  $\implies$  Current BFS is optimal

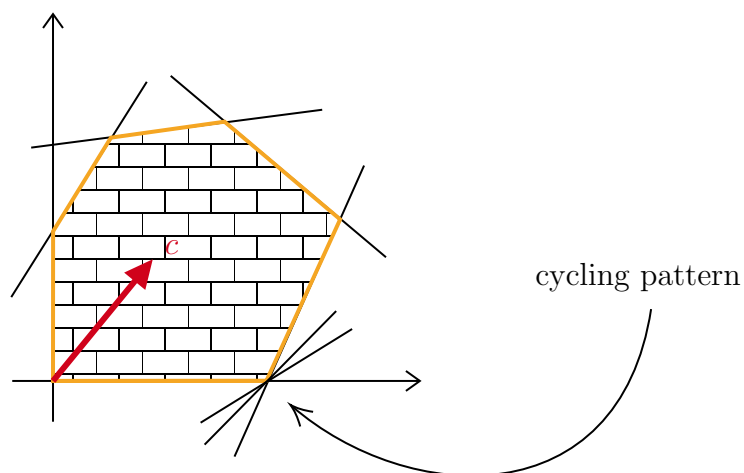


Figure 2.1: Simplex method

Original LP

$$\begin{aligned} \max \quad & c^T x \\ \downarrow \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \min \quad & b^T y \\ \downarrow \\ \text{s.t.} \quad & A^T y \geq c \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \min \quad & y^T b \\ \downarrow \\ \text{s.t.} \quad & y^T A \geq c^T \end{aligned}$$

$$\begin{aligned} \min \quad & y^T b \\ \downarrow \\ \text{s.t.} \quad & y^T A_B \geq c_B^T \\ & y^T A_N \geq c_N^T \end{aligned}$$

If satisfies C.S with BFS corresponding to  $B$

$$\begin{aligned} y^T A_B &= c_B^T \\ \implies y^T &= c_B^T A_B^{-1} \Longleftrightarrow c_B^T A_B^{-1} A_N \geq c_N^T \Longleftrightarrow \bar{c}_N \leq 0 \\ y^T A_N &\geq c_N^T \end{aligned}$$

### 2.9.3 Mechanics of Simplex

Example: 1

$$\begin{aligned} \max \quad & (1 \ 3 \ -2 \ 0 \ 0) x \\ \downarrow \\ \text{s.t.} \quad & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{aligned}$$

↑ enters basis ↑  $j$   
↑ pivot row  $\ell$

For  $\theta$

$$\theta \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and we have

$$\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 4 - \theta \\ 1 - \theta \end{pmatrix} \geq 0 \implies \boxed{\theta \leq 4}$$

We are actually picking  $\min \left\{ \frac{4}{1}, \frac{1}{1} \right\}$

Pick, out of all rows  $\min \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$  where  $j$  is entering variable.

Then now in row  $\ell$  (second row here). Make row operations so that pivot element become 1, all others in col  $j$  becomes 0.

→ Row  $2 \times 1$

→ Subtract row 2 from row 1

→ subtract row 2 from objective function (with RHS multiplied by  $-1$ )

$$\begin{array}{ll} \max & (0 \quad 4 \quad -5 \quad 0 \quad -1)x + 1 \\ \downarrow & \text{pivot} \leftarrow \begin{array}{c} j \\ \uparrow \end{array} \\ \text{s.t.} & \begin{pmatrix} 0 & 2 & -2 & 1 & -1 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{row } \ell \\ & x \geq 0 \end{array}$$

$$2\theta + x_4 = 3 \iff x_4 = 3 - 2\theta \geq 0 \implies \theta \leq \frac{3}{2}$$

$$-\theta + x_1 = 1 \iff x_1 = \theta + 1 \geq 0 \implies \theta \geq -1$$

where we are finding  $\min_{\bar{a}_{ij} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$ . Now follow the similar procedure, we have

$$\begin{array}{ll} \max & (0 \quad 0 \quad -1 \quad -2 \quad 1)x + 7 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 0 & 1 & -1 & 0.5 & -0.5 \\ 1 & 0 & 2 & 0.5 & 0.5 \end{pmatrix} x = \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} \end{array}$$

**In general** Pick  $j \in N : \bar{c}_j > 0$ .

Let  $\ell = \operatorname{argmin}_{\bar{a}_{ij} > 0} \left\{ \frac{\bar{b}_i}{\bar{a}_{ij}} \right\}$  (Ratio Test)

- Multiply row  $\ell$  by  $\frac{1}{\bar{a}_{\ell j}}$
- Add  $-\frac{\bar{a}_{ij}}{\bar{a}_{\ell j}}$  times row  $\ell$  to row  $i \neq \ell$ .

- Add  $-\frac{\bar{c}_j \cdot \bar{a}_{\ell k}}{\bar{a}_{\ell j}}$  to variable coeff in objective.  $\forall k \in 1, \dots, n$
- Add  $\frac{\bar{b}_\ell \cdot \bar{c}_j}{\bar{a}_{\ell j}}$  to objective value in objective function

Example: 2

$$\begin{array}{ll}
 \max & (2 \quad 1 \quad 1 \quad 0 \quad 0) x \\
 \downarrow & \\
 \text{s.t.} & \begin{array}{c} \text{pivot} \uparrow \\ \left( \begin{array}{ccccc} 1 & 2 & -1 & 1 & 0 \\ 2 & -2 & -1 & 0 & 1 \end{array} \right) x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ x \geq 0 \end{array} \quad \text{row } \ell
 \end{array}$$

**Ratio Test**  $\min \left\{ \frac{2}{1}, \frac{3}{2} \right\} = 1.5. \ell = 2. (x_2 \text{ enters, } x_5 \text{ leaves})$

$$\begin{array}{ll}
 \max & (0 \quad 3 \quad 2_j \quad 0 \quad -1) x + 3 \\
 \downarrow & \\
 \text{s.t.} & \begin{pmatrix} 0 & 3 & -0.5 & 1 & -0.5 \\ 1 & -1 & -0.5 & 0 & 0.5 \end{pmatrix} x = \begin{pmatrix} 0.5 \\ 1.5 \end{pmatrix} \\
 & x \geq 0
 \end{array}$$

If we increase  $x_3 \rightarrow \theta$  and keep  $x_2 = x_5 = 0$

$$\begin{array}{ll}
 -0.5\theta + x_4 = 0.5 & \Rightarrow x_1 = 1.5 + 0.5\theta \\
 -0.5\theta + x_1 = 1.5 & \Rightarrow x_4 = 0.5 + 0.5\theta \rightarrow \text{Problem is unbounded!}
 \end{array}$$

**In general** Let  $B$  be a basis

$$\begin{array}{ll}
 \max & \bar{c}_N^T x_N \\
 \downarrow & \\
 \text{s.t.} & \begin{array}{l} x_B + \bar{A}_N x_N = \bar{b} \\ x_B, x_N \geq 0 \end{array}
 \end{array}$$

Found  $j : \bar{c}_j > 0$  AND  $\bar{A}_j \leq 0$ .

Construct  $d \in \mathbb{R}^n$  to reflect what we are trying to do when we increase  $x_j \rightarrow \theta$ .

Right now, we are at BFS:

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}$$

We want:

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix} + \theta \begin{pmatrix} d_B \\ d_N \end{pmatrix}$$

where  $d_N = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \overset{j}{\uparrow} = e_j$  and  $d_B = -\bar{A}_j = -A_B^{-1}A_j$ .

Found  $d$ :  $d \geq 0$ , then

$$Ad = A_B d_B + A_N d_N = -A_B A_B^{-1} A_j + A_j = 0$$

and

$$c^T d = c_B^T d_B + c_N^T d_N = -c_B^T A_B^{-1} A_j + c_j = \bar{c}_j > 0$$

i.e.,

$$c^T d > 0$$

$$Ad = 0 \implies \text{Problem is unbounded}$$

$$d \geq 0$$

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