Graph Theory

CO 442

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LaTeXed by *Sibelius Peng*

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of CO 442 during Fall 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

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Sibelius Peng

Contents

First let's look at a proof example.

Theorem

Every two longest paths in a connected graph G intersect.

Proof:

Suppose note. That is, there exist two longest paths P_1 and P_2 of G such that $V(P_1) \cap V(P_2) = \varnothing$. For each $i \in \{1, 2\}$, let $v_{i,1}$ and $v_{i,2}$ be the ends of P_i. Since G is connected, there exists a shortest path P from $V(P_1)$ to $V(P_2)$. Since P is shortest, we have that $|V(P_i) \cap V(P)| = 1$ for each $i \in \{1, 2\}$.

For each $i \in \{1, 2\}$, let u_i be the end of P in $V(P_i)$. For each $i, j \in \{1, 2\}$, let $Q_{i,j}$ be the subpath of P_i from u_i to $v_{i,j}$. We assume without loss of generality that for each $i \in \{1, 2\}$, we have that $|E(Q_{i,1})| \geq |E(Q_{i,2})|$ and hence

$$|E(Q_{i,1})| \geq |E(P_i)|/2.$$

Let $P' = v_{1,1}Q_{1,1}u_1Pu_2Q_{2,1}v_{2,1}$. Note that P' is a path in G and

$$|E(P')| = |E(Q_{1,1})| + |E(P)| + |E(Q_{2,1})| \geq |E(P)| + |E(P_1)| > |E(P_1)|.$$

Hence P' is a longer path than P_1, contradicting that P_1 is a longest path. $\square$

Things to remember:

1. Correctness
2. Clarity/Precision
3. Ease of Reading

1

Colorings

1.1 Coloring and Brooks' Theorem

coloring

A **coloring** of a graph G is an assignment of colors to vertices of G such that no two adjacent vertices receive the same color.

k-coloring

Let G be a graph. We say $\phi : V(G) \to [k]$ is a k**-coloring** of G if $\phi(u) \neq \phi(v)$ for every $uv \in E(G)$.

Since every graph G has a $|V(G)|$-coloring, we are interested in the minimum numbers of colors needed to color G.

chromatic number

The **chromatic number** of a graph G, denoted $\chi(G)$, is the minimum number k such that G has a k-coloring.

Then why coloring?

- Coloring is a foundational problem in graph labeling, wherein we study functions on $V(G)$ according to constraints imposed on the graph (e.g. non-adjacent vertices are labeled differently)
- Coloring is a foundational problem in graph decomposition, wherein we seek to decompose $V(G)$ into certain kinds of subgraphs (e.g. independent sets)
- Applications to maps, scheduling, job processing, frequency assignment (e.g. cell networks)
- Applications to algorithms (e.g. distributed computing)

However, coloring is hard.

A graph being an **independent set** is by definition equivalent to being **1-colorable**.

A graph being **bipartite** is by definition equivalent to being **2-colorable**. (Indeed coloring is a generalization of partite)

Proposition 1.1

G is 2-colorable if and only if G does not contain an odd cycle.

Moreover, there exists a poly-time algorithm to decide if G is 2-colorable.

Theorem: Karp (1972)

For each $k \geq 3$, deciding if a graph G has a k-coloring is NP-complete.

Indeed, 3-coloring is NP-complete even for planar graphs. Any constant factor approximation is also NP-complete.

Then what about the bounds on chromatic number?

As mentioned $\chi(G) \leq |V(G)|$.

Greedy Upper bound: $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of vertices in G. Why? By a greedy algorithm:

- Order the vertices of G arbitrarily, $v_1, \ldots, v_{|V(G)|}$.
- Color the vertices in order avoiding the colors of previously colored neighbors.
- Since each vertex has at most $\Delta(G)$ neighbors, there is always at least one color for the current vertex.

Lower bound: $\chi(G) \geq \omega(G)$, where $\omega(G)$ denotes the clique number of H, that is the maximum size of a clique in G.

Can we do better than the greedy upper bound?
No! The bound is tight for complete graphs: $\omega(K_n) = \chi(K_n) = (n-1)+1 = \Delta(K_n)+1$.

Can we do better if the graph is not complete?
No! The graph could have a component that is complete.

Can we do better if the graph is connected and not complete?
No! The bound is tight for odd cycles: $\chi(C_{2k+1}) = 3 = 2+1 = \Delta(C_{2k+1}) + 1$.

Can we do better if the graph is connected and neither complete nor an odd cycle? **Yes!**

Theorem 1.2: Brooks 1941

If G is connected, then $\chi(G) \leq \Delta(G)$ if and only if G is neither complete nor an odd cycle.

1.2 An Informal Proof of Brooks' Theorem

How to prove Brook's Theorem?

Actually there are 8 to 10 distinct ways to prove Brooks' Theorem. See the nice survey *Brooks' Theorem and Beyond* by Cranston and Rabern from 2014 for more details. Here are some of those methods: Greedy Coloring, Kempe Chains, List Coloring, Alon-Tarsi Theorem, Kernel Perfection, Potential Method.

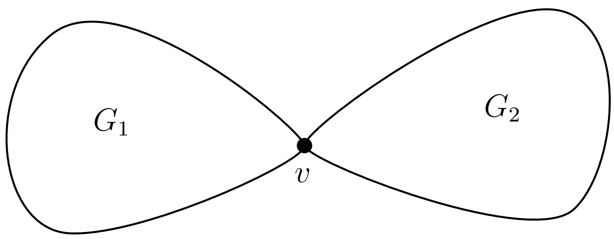
Today we give an informal proof sketch via the Greedy Coloring Method - arguably the most direct, brute-force of the approaches. (See Diestel for the Kempe Chain proof).

The idea is to try a method (greedy coloring) we know works for a similar problem ($\Delta+1$-coloring), and ask under what conditions can we use this to get the desired outcome (a Δ-coloring).

In the other cases we cannot apply greedy, we instead do **reductions**: that is, we show how to inductively color or to show that the graph is one of the exceptional outcomes (clique or odd cycle).

Alternatively, we could have built up a suite/library of reductions that work, and then tried to find a method to deal a finishing blow (i.e. to handle the cases we could not reduce).

First Reduction G has a cutvertex v. Then v separates G into two smaller graphs G_1 and G_2.



$\phi_1(v) = \phi_2(v)$

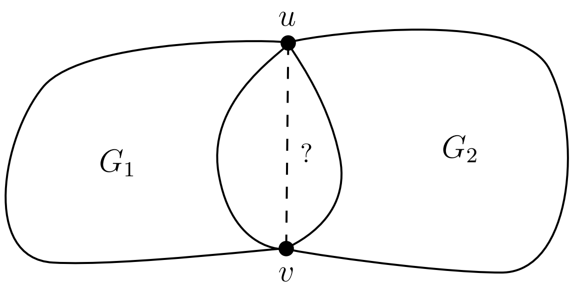
By minimality of G, G_i has a Δ-coloring ϕ_i, $i \in 1, 2$.

This only works if neither graph is $K_{\Delta+1}$ or odd cycle when $\Delta = 2$.

Now permute the colors in ϕ_2 so that $\phi_1(v) = \phi_2(v)$. Then $\phi_1 \cup \phi_2$ yields a Δ-coloring of G, a contradiction.

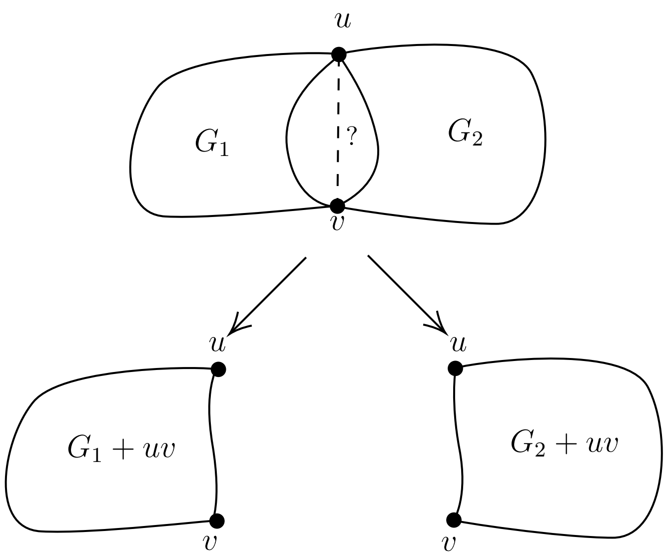
Second Reduction G has a cutset $\{u, v\}$.

Try the same trick. Say $\{u, v\}$ separates G into two smaller graphs G_1 and G_2. By induction or minimum counterexample, each of G_1, G_2 has a Δ-coloring ϕ_i, $i \in 1, 2$.



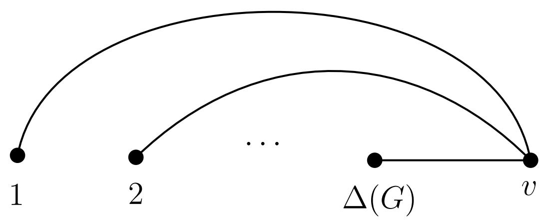
If $uv \in E(G)$, then we can permute the colorings so that $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$.

This fails if $uv \notin E(G)$. Because we may have u, v colored the same in one coloring and different in the order and no permuting will fix this! So we can add the edge uv to both G_1 and G_2!

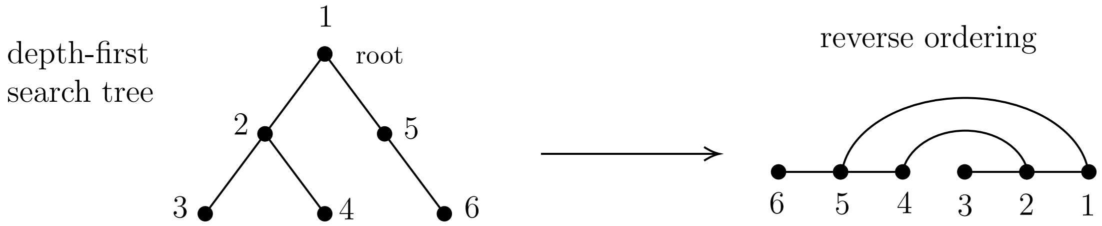


Have to show $\Delta(G_1 + uv), \Delta(G_2 + uv) \leq \Delta(G)$. We also have to ensure that neither G_1 nor G_2 is complete (or odd cycle in $\Delta(G) = 2$ case).

Then we assume G is 3-connected. We now turn to the finishing blow (greedy). The greedy *fails* when a vertex has $\Delta(G)$ earlier neighbors in the ordering, each with a different color from $\{1, \dots, \Delta(G)\}$.

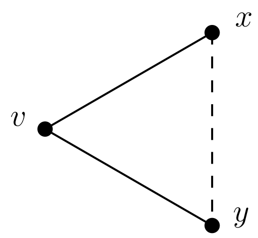


Can we find an ordering where most of the vertices have at most $\Delta(G) - 1$ earlier neighbors? Yes for all but the last vertex in the ordering! We can fix a root, then take a depth-first search tree ordering from the root. Reverse it!

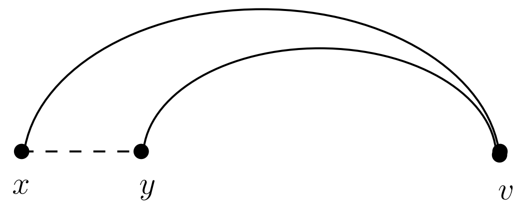


Now all vertices but the last will be fine in greedy.

If $\deg(v) \leq \Delta(G) - 1$, then we can ensure greedy does not fail at the last vertex v. Otherwise, we ensure that two of its neighbors x and y are colored the same (and hence there is a color left for v when it is v's turn). These two are two non-adjacent neighbors, which guaranteed to exist as G is not $K_{\Delta+1}$.



We can put x, y first in the ordering to guarantee x and y are colored the same. Then we can color them as we desire (since non-adjacent), say both with color 1.



$\phi(x) = 1 = \phi(y)$

Use the reverse of a depth-first search tree ordering of $G - \{x, y\}$ with root v, then we finish the ordering so every vertex in $V(G) \setminus \{x, y, v\}$ has at most $\Delta(G) - 1$ earlier neighbors. Since $G - \{x, y\}$ is connected as G is 3-connected, then this ordering exist.

1.3 A Formal Proof of Brooks' Theorem

Let us codify our ordering fact as a proposition.

Proposition 1.3: Ordering Proposition

If G is a connected graph on n vertices and $v \in V(G)$, then there exists an ordering $v_1, \ldots, v_n = v$ of $V(G)$ such that $|N(v_i) \cap \{v_{i+1}, \ldots, v_n\}| \geq 1$ for all $i \in [n-1]$.

Proof:
Reverse a depth-first search tree ordering from root v. Or more formally:

We proceed by induction on $|V(G)|$. If $|V(G)| = 1$, then the ordering v is as desired. So we assume that $|V(G)| \geq 2$. Let $G_1, \ldots, G_k$ be the components of $G - v$. As G is connected, there exists neighbors $u_1, \ldots, u_k$ of v such that $u_i \in V(G_i)$ for each $i \in [k]$.

For each $i \in [k]$, there exists by induction applied to G_i and u_i, an ordering σ_i of $V(G_i)$ as prescribed by the proposition. Let σ be the ordering of $V(G)$ obtained by concatenating the σ_i and finally v. Then σ is as desired. □

Now we are ready to prove Brooks' Theorem:

Suppose not. Let G a counterexample with $|V(G)|$ minimized. If $\Delta(G) \leq 2$, the result is standard. So we assume that $\Delta(G) \geq 3$.

Claim 1 There does not exist a cutvertex of G.

Proof:
Suppose not. That is, there exists a cutvertex v of G and two connected subgraphs G_1, G_2 of G such that $G_1 \cap G_2 = \{v\}, G_1 \cup G_2 = G$ and $|V(G_i)| < |V(G)|$ for each $i \in [2]$.

As G_1 and G_2 are subgraphs of G, we have that $\Delta(G_i) \leq \Delta(G)$ for each $i \in [2]$. Moreover, as G is connected, we have for each $i \in [2]$ that $\deg_{G_i}(v) \geq 1$ and hence $\deg_{G_i}(v) \leq \Delta(G) - 1$. Hence $G_i \neq K_{\Delta(G)+1}$ for each $i \in [2]$. Thus by the minimality of G, there exist $\Delta(G)$-colorings ϕ_i of G_i for each $i \in [2]$.

By permuting the colors of ϕ_2 as necessary, we assume without loss of generality that $\phi_1(v) = \phi_2(v)$. But then $\phi_1 \cup \phi_2$ is a $\Delta(G)$-coloring of G, a contradiction. □

Claim 2 There does not exist a 2-cut of G, or, there exists a vertex $v \in V(G)$ with $\deg_G(v) \leq \Delta(G) - 1$.

Proof:
Suppose not. Now let us suppose there exists a 2-cut $\{v_1, v_2\}$ of G and two connected subgraphs G_1, G_2 of G such that $G_1 \cap G_2 = \{v_1, v_2\}$, $G_1 \cup G_2 = G$ and $|V(G_i)| < |V(G)|$ for each $i \in [2]$. Choose v_1, v_2, G_1, G_2 such that neither $G_1 + v_1v_2$ nor $G_2 + v_1v_2$ is equal to $K_{\Delta(G)+1}$ if possible.

As G is connected and G does not have a cutvertex by Claim 1, we have for all $i, j \in [2]$ that $\deg_{G_i}(v_j) \geq 1$ and hence $\deg_{G_i}(v_j) \leq \Delta(G) - 1$. Thus $\Delta(G_i + v_1v_2) \leq \Delta(G)$ for all $i \in [2]$.

Next suppose that there exists $i \in [2]$ such that $G_i + v_1v_2 = K_{\Delta(G)+1}$. Without loss of generality, we assume that $i = 1$. Let v_1' be the neighbor of v_1 in $G_2 - v_2$. Let $G_1 = G_1 + v_1v_1'$ and $G_2' = G_2 \setminus \{v_1\}$. Now wither $\deg_G(v_1') \leq \Delta(G) - 1$, a contradiction, or we find that $G_i' + v_1'v_2 \neq K_{\Delta(G)+1}$ for each $i \in [2]$. But then v_1', v_2, G_1', G_2' contradict the choice of v_1, v_2, G_1, G_2.

So we assume that $G_1 + v_2v_2, G_2 + v_1v_2 \neq K_{\Delta(G)+1}$. Thus by the minimality of G, there exist $\Delta(G)$-colorings ϕ_i of G_i for each $i \in [2]$. By permuting the colors of ϕ_2 as necessary, we assume without loss of generality that $\phi_1(v_j) = \phi_2(v_j)$ for each $j \in [2]$. But then $\phi_2 \cup \phi_2$ is a $\Delta(G)$-coloring of G, a contradiction. □

Let $v \in V(G)$ with $\deg_G(v)$ minimized.

First suppose that $\deg_G(v) \leq \Delta(G) - 1$. By the Ordering Proposition, there exists an ordering $v_1, \ldots, v$ of $V(G)$ such that $|N(v_i) \cap \{v_{i+1}, \ldots, v\}| \geq 1$ for all $i \in [|V(G)| - 1]$. Now greedily color $V(G)$ in that order. This yields a $\Delta(G)$-coloring of G, a contradiction.

So we assume that $\deg_G(v) = \Delta(G)$. Since $G \neq K_{\Delta+1}$, there exist distinct $x, y \in N(v)$

such that $xy \notin E(G)$. By Claims 1 and 2, it follows that G is 3-connected and hence $G - \{x, y\}$ is connected. Hence by the Ordering Proposition, there exists an ordering $v_1, \ldots, v$ of $V(G) - \{x, y\}$ such that $|N(v_i) \cap \{v_{i+1}, \ldots, v\}| \geq 1$ for all $i \in [|V(G)| - 3]$. Now color x, y with color 1. Then greedily color $V(G) - \{x, y\}$ in that order. This yields a $\Delta(G)$-coloring of G, a contradiction.

1.4 Beyond Brooks' Theorem

Can we go further? Can we save more colors? Under what conditions?

Question (ω, Δ, χ paradigm)

What is the maximum chromatic number of graphs with $\omega(G) \leq \omega$ and $\Delta(G) \leq \Delta$?

Brooks' Reformulated

If G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.

Borodin-Kostochka Conjecture (1977)

If G is a graph with $\Delta(G) \geq 9$ and $\omega(G) \leq \Delta(G) - 1$, then $\chi(G) \leq (G) - 1$.

Why $\Delta \geq 9$?

Let $G = C_5 \boxtimes K_3$. (the blowup of every vertex in C_5 to a triangle K_3) Then $\Delta(G) = 8$, $\omega(G) = 6$, and yet $\chi(G) = 8$.

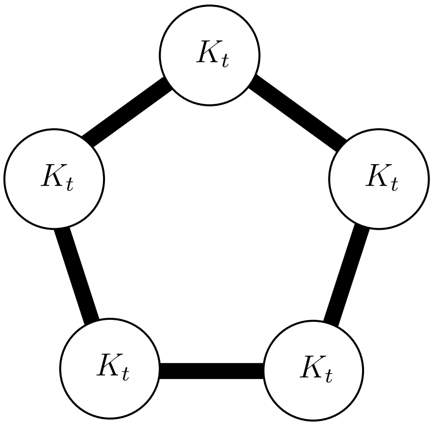
Theorem (Reed 1999)

True for $\Delta(G) \geq 10^{14}$.

Reed's conjecture

Reed's Conjecture (1998)

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil.$$



5-cycle blowup

$$\Delta = 3t - 1$$
$$\omega = 2t$$
$$\left\lceil \frac{1}{2}(\Delta + 1 + \omega) \right\rceil = \left\lceil \frac{5t}{2} \right\rceil$$
$$\alpha = 2$$

Theorem (Reed 1998)

The conjecture holds when $\Delta(G)$ is sufficiently large and

$$\omega(G) \geq (1 - 7 \cdot 10^{-7})\Delta(G).$$

Corollary (Reed)

There exists $\varepsilon > 0$ such that for every graph G,

$$\chi(G) \leq (1 - \varepsilon)(\Delta(G) + 1) + \varepsilon\omega(G).$$

Reed's value of ε was 10^{-8}.

Can we improve the ε for large enough Δ? Can we get closer to $\varepsilon = 1/2$?

For Large enough Δ, the following ε suffices:

- $\frac{1}{320e^6}$ (King and Reed 2012)
- $\frac{1}{26}$ (Bonamy, Perrett, Postle 2016+)
- $\frac{1}{13}$ (Delcourt and Postle 2017+)
- $\frac{1}{8.4}$ (Hurley, de Joannis de Verclos, Kang 2020+)

Large Girth

The **girth** of a graph G is the length of a shortest cycle in G.

Theorem (Erdös 1959)

$\forall g, k \geq 1$, there exists graphs of girth at least g and chromatic number at least k.

Theorem (Frieze and Luczak 1992)

Random d-regular graphs have chromatic number $(1-o(1))\frac{d}{2\ln d}$ with high probability.

Corollary

$\forall g, d \geq 1$, there exists a d-regular graph G of girth at least g with

$$\chi(G) \geq (1-o(1))\frac{d}{2\ln d}.$$

Girth-Five and Triangle-Free

Theorem (Kim 1995)

If G is a graph of girth five, then

$$\chi(G) \leq (1+o(1))\frac{\Delta(G)}{\ln \Delta(G)}.$$

Theorem (Johansson 1996)

If G is a triangle-free graph, then

$$\chi(G) \leq O\left(\frac{\Delta(G)}{\ln \Delta(G)}\right).$$

Theorem (Molloy 2017)

If G is a triangle-free graph, then

$$\chi(G) \leq (1+o(1))\frac{\Delta(G)}{\ln \Delta(G)}.$$

Small Clique Number

Theorem (Johansson 1999)

For every fixed r: if G is a graph with $\omega(G) \leq r$, then

$$\chi(G) \leq O\left(\frac{\Delta(G)}{\ln \Delta(G)} \cdot \ln\ln \Delta(G)\right).$$

Theorem (Molloy 2017)

$$\chi(G) \leq 200 \cdot \omega(G) \cdot \frac{\Delta(G)}{\ln \Delta(G)} \cdot \ln \ln \Delta(G).$$

Good for $\omega(G) \leq \frac{\ln \Delta(G)}{\ln \ln \Delta(G)}$. What if $\omega(G)$ is larger?

Question

For $k \geq 2$, what value of $\omega(G)$ guarantees $\chi(G) \leq \frac{\Delta(G)}{k}$?

Theorem (Bonamy, Kelly, Nelson, Postle 2018+)

$$\chi(G) \leq O\left(\Delta(G) \cdot \sqrt{\frac{\ln \omega(G)}{\ln \Delta(G)}}\right).$$

Corollary

$\forall k \geq 2$, if $\omega(G) \leq \Delta(G)^{\frac{1}{(192k)^2}}$, then

$$\chi(G) \leq \frac{\Delta(G)}{k}.$$

Ramsey theory constructions show that we cannot extend this beyond $\Delta(G)^{\frac{2}{k-1}}$.

Index