Ordinary Differential Equations 2

AMATH 351

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Preface

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Contents

1

Introduction and Review

1.1 Definitions and Terminology

A **differential equation** is any equation involving a function and derivatives of this function.

Ordinary differential equations contain only functions of a single variable, called the independent variable, and derivatives with respect to that variable.

Partial differential equations contain a function of two or more variables and some partial derivatives of this function.

The **order** of a differential equation in the order of the highest derivative in the equation.

A general n-th order ODE has the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.1}$$

where $y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}$ and so on. We assume further it can be written as

$$y^{(n)} = f(x, y', \ldots, y^{(n-1)}). \tag{1.2}$$

Eq. (1.2) is said to be **linear** when f is a linear function of $y, y', \ldots, y^{(n-1)}$. In this case, Eq. (1.2) can be written as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x). \tag{1.3}$$

A differential equation that is not linear is said to be **nonlinear**.

By a **solution** of Eq. (1.2) on an interval I we mean a function $y = \psi(x)$ such that $f(x, \psi(x), \psi'(x), \ldots, \psi^{(n-1)(x)})$ is defined for all x in I and is equal to $\psi^{(n)}(x)$ for all x in I.

A solution in which the dependent variable in expressed only in terms of the independent variable and constants is called an **explicit solution**.

A relation $G(x, y) = 0$ such that there exists at least one function $\psi(x)$ that satisfies the relation and Eq. (1.2) is called an **implicit solution**.

A solution which is free of arbitrary constants is called a **particular solution**.

A solutions that cannot be obtained by specializing any of the parameters in a family of solutions is called a **singular solution**.

Example:

Consider the DE $y' = xy^{1/2}$.

The explicit solution: $y = \left(\frac{x^2}{4} + c\right)^2$

A particular solution is $y = \frac{x^4}{16}$ obtained above for $c = 0$.

A singular solution is $y = 0$ which cannot be obtained from the explicit solution for any choice of constant c.

1.2 Initial-Value Problems

On some interval containing x_0, the problem

Solve $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$ subject to the initial conditions $y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_{n-1}$

where $y_0, \ldots, y_{n-1}$ are arbitrary specified real constants, is called an **initial-value problem** (IVP).

Consider the IVP $y' = f(x, y)$, $y(x_0) = y_0$.

Theorem 1.1: Picard

Let D be a rectangular region in the xy-plane defined by $D = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ and $(x_0, y_0) \in D$ the interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution $y(x)$ defined in an interval I centered at x_0.

1.3 First Order ODE

Separable variables

A first order DE of the form

$$\frac{dy}{dx} = g(x)h(y) \tag{1.4}$$

is said to be **separable** or to have **separable variables**. Solution method:

$$\frac{dy}{h(y)} = g(x)dx$$

Integrate both sides

$$\int \frac{1}{h(y)} dy = \int g(x)dx + C$$

Linear equations

A first order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1.5}$$

is called a **linear equation**.

Solution method:

- Write in its **standard form**

$$\frac{dy}{dx} + p(x) = f(x)$$

- Multiply both sides by the integrating factor $\mu(x) = \exp\left(\int p(x)dx\right)$, and rearrange into the exact form $\frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$
- Integrate both side with respect to x and get the general solution under the form

$$y(x) = \frac{1}{\mu(x)}\left(\int \mu(x)f(x)dx + C\right)$$

There are other type of ODEs that you learned how to solve in AMATH 251, such as homogeneous equations, exact equations, Bernouli equations.

2

Theory of Second-Order Linear DEs

2.1 2nd-Order Linear ODEs

The most general 2nd order linear DE is

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

In AMATH 251 we learned how to solve this equation where the coefficients a_2, a_1, a_0 are constants. This equation can be written in several different forms:

1. General form
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{2.1}$$

2. Standard form: If $a_2(x)$ is not identically zero then we obtain
$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.2}$$

3. 3) Associated homogeneous equation: This is the same as the standard form where RHS is zero,
$$y'' + P(x)y' + Q(x)y = 0 \tag{2.3}$$

   If RHS of Eq. (2.2) is non-zero the equation is said to be non-homogeneous or inhomogeneous.

2.2 Existence and Uniqueness

Existence and Uniqueness Before we try and find solutions to the DEs it is usually a good idea to know that a solution exists and it is unique. Otherwise we could be wasting out time. We state a theorem for existence and uniqueness. The ideas of the proof will be presented later when we discuss first-order systems.

Theorem 2.1: Existence and Uniqueness

Let $P(x), Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$ and if $y(x_0)$ and $y'(x_0)$ are any numbers, then Eq. (2.2) has one and only one solution $y(x)$ on the entire interval such that the initial conditions (ICs) are satisfied.

Remark:
If we are looking for a solution to the homogeneous equation with $y(0) = 0, y'(0) = 0$ observe that the trivial solution is an allowable solution. Therefore, by the existence and uniqueness theorem, it must be the only solution.

2.3 General Solutions to 2nd-order DEs

In AMATH 251 for the case of constant coefficients, we learned that the general solution to Eq. (2.2) is a superposition of any particular solution to the non-homogeneous problem and a general solution to the homogeneous one. This also holds true in the case of non-constant coefficients. Therefore, the method of attack is as follows:

1. Find the general solution to the homogeneous problem: In the case of constant coefficients we simply sub $y = e^{rx}$ find the characteristic equation, solve for the characteristic roots, r_1, r_2 form the two independent solutions $y_1(x), y_2(x)$ and get that the general solution is,

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

   where c_1, c_2 are arbitrary constants. In the case of non-constant coefficients we need to do more work to find y_1, y_2. In general we cannot find them explicitly.

2. Find a particular solution to the non-homogeneous problem. There are different methods that we can use.

The following Theorems will help us to find a unique solution of a general second-order scalar equation. First we look at the homogeneous problem and then at the more general DE.

Theorem 2.2: General solutions to 2nd-order homogeneous equations

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation (Eq. (2.3)) on the interval $[a, b]$, then the general solution to the same homogeneous problem is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

for arbitrary constants c_1, c_2.

Proof:
First we can sub y_1, y_2 and their linear superposition into the homogeneous equation to verify they are solutions.

Second we need to verify that this solution can satisfy any set of conditions, say $y(0)$

and $y'(0)$ [a]. We sub in our solution and find,

$$c_1 y_1(0) + c_2 y_2(0) = y(0),$$
$$c_1 y_1'(0) + c_2 y_2'(0) = y'(0).$$

This is a system of two equations and two unknowns c_1, c_2. To be able to solve this for any initial conditions we need that the matrix is non-singular,

$$\det \begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} = y_1(0)y_2'(0) - y_2(0)y_1'(0) \neq 0$$

This motivates the definition of **Wronskian**, $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.

To ensure that our expression is a general solution we need that the initial value of the Wronskian is nonzero, $W(y_1(0), y_2(0)) \neq 0$. □

[a] These should be replaced by $y(x_0) = y_0, y'(x_0) = y_1$ for some $x_0 \in [a, b]$

Also check the alternative proof on page 66 of `https://notes.sibeliusp.com/pdfs/1189/amath251.pdf`.

Therefore, the above tells us that if the initial value of the Wronskian of the two solutions is non-zero, we have a general solution. Next, we will show that if the Wronskian is non-zero at the initial time it is necessarily non-zero all time. The following theorem states and proves this result.

Index