Game Theory

CO 456

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LaTeXed by Sibelius Peng

Preface

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Sibelius Peng

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Combinatorial games

1.1 Impartial games

- http://web.mit.edu/sp.268/www/nim.pdf
- https://ivv5hpp.uni-muenster.de/u/baysm/teaching/3u03/notes/14-games.pdf

Example: Game of Nim

We are given a collection of piles of chips. Two players play alternatively. On a player's turn, they remove at least 1 chip from a pile. First player who cannot move loses the game.

For example, we have three piles with 1, 1, 2 chips. Is there a winning strategy? In this case, there is one for the first player: Player I (p1) removes the pile of 2 chips. This forces p2 to move a pile of 1 chip. p1 removes the last chip. p2 has no move and loses the game. In this case, p1 has a winning strategy, so this is a **winning game** or **winning position**.

Now let's look at another example with two piles of 5 chips each. Regardless of what p1 does, p2 can make the same move on the other pile. p1 loses. If p1 loses regardless of their move (i.e., p2 has a winning strategy), then this is a **losing game** or **losing position**.

What if we have two piles have unequal sizes? say 5, 7. p1 moves to equalize the chip count (remove 2 from the pile of 7). p2 then loses, this is a winning game.

Lemma 1.1

In instances of Nim with two piles of n, m chips, it is a winning game if and only if $m \neq n$.

Solving Nim with only two piles is easy, but what about games with more than two piles? This is more complicated.

Nim is an example of an **impartial game**. Conditions required for an impartial game:

1. There are 2 players, player I and player II.
2. There are several positions, with a starting position.
3. A player performs one of a set of allowable moves, which depends only on the current position, and not on the player whose turn it is. ("impartial") Each possible move generates an option.
4. The players move alternately.
5. There is complete information.

6. There are no chance moves.
7. The first player with no available move loses.
8. The rules guarantee that games end.

Example: Not an impartial game

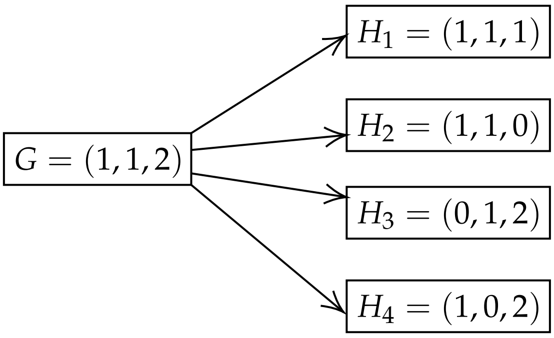
Tic-tac-toe: violates 7.

Chess: violates 3, since players can only move their own pieces.

Monopoly: violates 6. Poker: violates 5.

Example:

Let $G = (1,1,2)$ be a Nim game. There are 4 possible moves (hence 4 possible options):



Each option is by itself another game of Nim

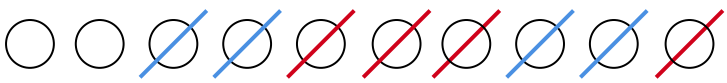
Note:
We can define an impartial game by its position and options recursively.

simpler

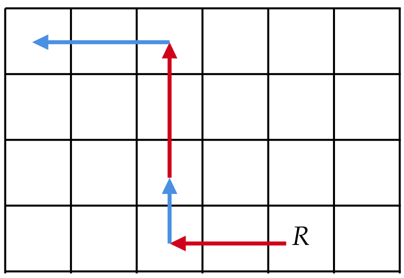
A game H that is reachable from game G by a sequence of allowable moves is **simpler** than G.

Other impartial games:

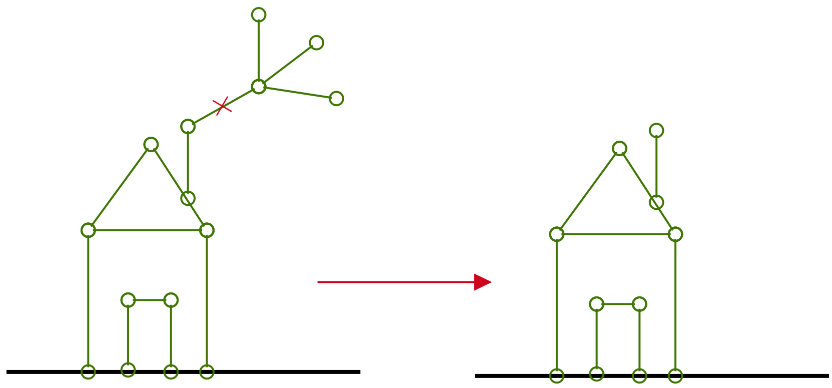
1. Subtraction game: We have one pile of n chips. A valid move is taking away 1, 2, or 3 chips. The first player who cannot move loses.



2. Rook game: We have an $m \times n$ chess board, and a rook in position (i, j). A valid move is moving the rook any number of spaces left or up. The first player who cannot move loses.



3. Green hackenbush game: We have a graph and the floor. The graph is attached to the floor at some vertices. A move consists of removing an edge of the graph, and any part of the graph not connected to the floor is removed. The first player who cannot move loses.



Spoiler A main result we will prove is that all impartial games are essentially like a Nim game.

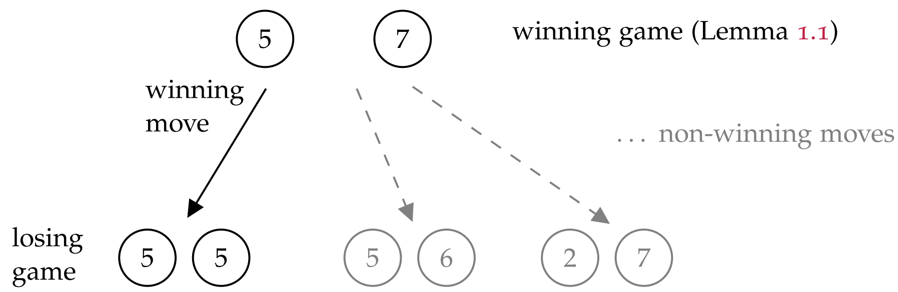
Lemma 1.2

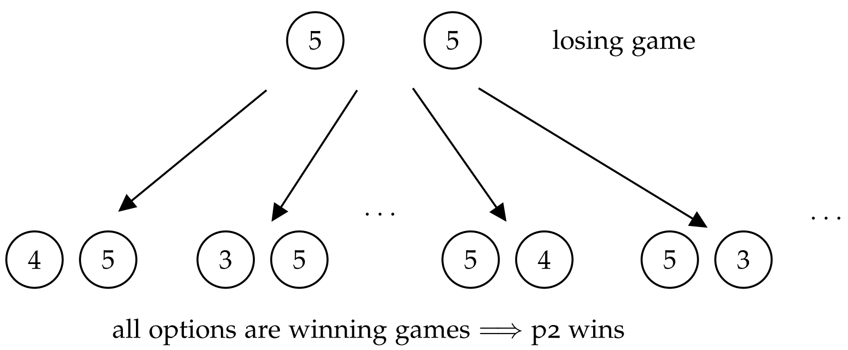
In any impartial game G, either player I or player II has a winning strategy.

Proof:
We prove by induction on the simplicity of G. If G has no allowable moves, then p1 loses, so p2 has a winning strategy. Assume G has allowable moves and the lemma holds for games simpler than G. Among all options of G, if p1 has a winning strategy in one of them, then p1 moves to that option and wins. Otherwise, p2 has a winning strategy for all options. So regardless of p1's move, p2 wins. □

So every impartial game is either a winning game (p1 has a winning strategy) or a losing game (p2 has a winning strategy).

Example: Nim





Note:
We assume players play perfectly. If there is a winning move, then they will take it.

1.2 Equivalent games

game sums

Let G and and H be two games with options $G_1, \ldots, G_m$ and $H_1, \ldots, H_n$ respectively. We define $G + H$ as the games with options

$$G_1 + H, \ldots, G_m + H, G + H_1, \ldots, G + H_n.$$

Example:
We denote $*n$ to be a game of Nim with one pile of n chips. Then $*1 + *1 + *2$ is the game with 3 piles of 1, 1, 2 chips.

Example:
If we denote $\#2$ to be the subtraction game with n chips, then $*5 + \#7$ is a game where a move consists of either removing at least 1 chip from the pile of 5 (Nim game), or removing 1, 2 or 3 chips from the pile of 7 (subtraction game).

Lemma 1.3

Let $\mathcal{G}$ be the set of all impartial games. Then for all $G, H, J \in \mathcal{G}$,

1. $G + H \in \mathcal{G}$ (closure)
2. $(G + H) + J = G + (H + J)$ (associative)
3. There exists an identity $0 \in \mathcal{G}$ (game with no options) where $G + 0 = 0 + G = G$
4. $G + H = H + G$ (symmetric)

Note:
This is an abelian group except the inverse element.

equivalent game

Two games G, H are **equivalent** if for any game J, $G + J$ and $H + J$ have the same outcome (i.e., either both are winning games, or both are losing games).

Notation: $G \equiv H$.

Example:
$*3 \equiv *3$ since $*3 + J$ is the same game as $*3 + J$ for any J, so they have the same outcome.

$*3 \not\equiv *4$ since $*3 + *3$ is a losing game, but $*4 + *3$ is a winning game from Lemma 1.1.

Lemma 1.4

$*n \equiv *m$ if and only if $n = m$.

Lemma 1.5

The relation $\equiv$ is an equivalence relation. That is, for all $G, H, K \in \mathcal{G}$,

1. $G \equiv G$ (reflexive)
2. $G \equiv G$ if and only if $H \equiv G$ (symmetric)
3. If $G \equiv H$ and $H \equiv K$, then $G \equiv K$ (transitive).

Exercise:
Prove that if $G \equiv H$, then $G + J \equiv H + J$ for any game J.

Note that the definition above only says they have the same outcome. To prove that they are equivalent, one needs to add another game on both sides to show they have the same outcome.

Nim with one pile $*n$ is a losing game if and only if $n = 0$.

Theorem 1.6

G is a losing game if and only if $G \equiv *0$.

Proof:

- $\Leftarrow$ If $G \equiv *0$, then $G + *0$ has the same outcome as $*0 + *0$. But $*0$ is a losing game, so G is a losing game.
- $\Rightarrow$ Suppose J is a losing game. (We want to show $G \equiv *0$, meaning $G + J$ and $*0 + J \equiv J$ have the same outcome.)
  1. Suppose J is a losing game. (We want to show that $G + J$ is a losing game.)

     We will prove "If G and J are losing games, then $G + J$ is a losing game" by induction on the simplicity of $G + J$. When $G + J$ has no options, then G, J both have no options, so $G, J, G + J$ are all losing games.

     Suppose $G + J$ has some options. Then p1 makes a move on G or J. WLOG say p1 makes a move in G, and results in $G' + J$. Since G is a losing game, G' is a winning game. So p2 makes a winning move from G' to G'', and this results in $G'' + J$. Then G'' is a losing game, so by induction, $G'' + J$ is a losing game for p1. So p1 loses, and $G + J$ is a losing game.
  2. Suppose J is a winning game. Then J has a winning move to J'. So p1 moves from $G + J$ to $G + J'$. Now both G, J' are losing games, so by case 1, $G + J'$ is a losing game. So p2 loses, meaning p1 wins, so $G + J$ is a winning game.

$\square$

Corollary 1.7

If G is a losing game, then J and $J + G$ have the same outcome for any game J.

Proof:
Since G is a losing game, $G \equiv *0$ by Theorem 1.6. Then $J + G \equiv J + *0 \equiv J$ (previous exercise + Lemma 1.3). So J and $G + J$ have the same outcome. $\square$

Example:

1. Recall $*5 + *5$ and $*7 + *7$ are losing games. Then Corollary 1.7 says $*5 + *5 + *7 + *7$ is also

a losing game. (p1 moves in either $*5 + *5$ or $*7 + *7$. Then p2 makes a winning move from the same part, equalizing piles.)

2. $\underbrace{*1 + *1 + *2}_{\text{winning}} + \underbrace{*5 + *5}_{\text{losing}}$. Corollary 1.7 implies this is a winning game.

   (p1 makes a winning move in $*1 + *1 + *2$, therefore we have $\underbrace{*1 + *1}_{\text{losing}} + \underbrace{*5 + *5}_{\text{losing}}$. p2 loses.)

Lemma 1.8: Copycat principle

For any game G, $G + G \equiv *0$.

Proof:
Induction on the simplicity of G. When G has no options, $G + G$ has no options, so $G + G \equiv *0$ by Theorem 1.6. Suppose G has options, and WLOG suppose p1 moves from $G + G$ to $G' + G$. Then p2 can move to $G' + G'$. By induction, $G' + G' \equiv *0$, so it is a losing game for p1. Therefore, $G + G$ is a losing game, and $G + G \equiv *0$. □

Lemma 1.9

$G \equiv H$ if and only if $G + H \equiv *0$.

Proof:

$\Rightarrow$ From $G \equiv H$, we add H to both sides to get $G + H \equiv H + H \equiv *0$ by the copycat principle.

$\Leftarrow$ From $G + H \equiv *0$, we add H to both sides to get $G + H + H \equiv *0 + H \equiv H$. But $G + G + G \equiv G + *0 \equiv G$ by the copycat principle. So $G \equiv H$. □

Example:
$*1 + *2 + *3$ is a losing game, so $*1 + *2 + *3 \equiv *0$. By Lemma 1.9, $*1 + *2 \equiv *3$, or $*1 + *3 \equiv *2$.

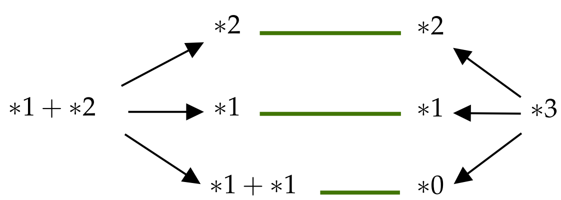
Another way to prove game equivalence is by showing that they have equivalent options.

Lemma 1.10

If the options of G are equivalent to options of H, then $G \equiv H$. (More precisely: There is a bijection between options of G and H where paired options are equivalent.)

Proof:
It suffices to show that $G + H \equiv *0$ by Lemma 1.9, i.e., $G + H$ is a losing game. This is true when G, H both have no options. Suppose G, H have options, and suppose WLOG p1 moves to $G'H$. By assumption, there exists an options of H, say H', such that $H' \equiv G'$. So p2 can move to $G' + H'$. Since $G' \equiv H'$, $G' + H' \equiv *0$ by Lemma 1.9. So $G' + H'$ is a losing game for p1. Hence $G + H$ is a losing game. □

Example:
We can show $*1 + *2 \equiv *3$ using Lemma 1.10.



Note:
The converse is false.

1.3 Nim and nimbers

Goal Show that every Nim game is equivalent to a Nim game with a single pile.

nimber

If G is a game such that $G \equiv *n$ for some n, then n is the **nimber** of G.

Example:
Any losing game has nimber 0 by Theorem 1.6.

Exercise:
Show that the notion of a nimber is well-defined. That is it is not possible for a game to have more than one nimber.

Theorem 1.11

Suppose $n = 2^{a_1} + 2^{a_2} + \ldots$ where $a_1 > a_2 > \ldots$, then $*n \equiv *2^{a_1} + *2^{a_2} + \ldots$.

Example:
$11 = 2^3 + 2^1 + 2^0$, $13 = 2^3 + 2^2 + 2^0$. Using this theorem, $*11 \equiv *2^3 + *2^1 + *2^0$ and $*13 \equiv *2^3 + *2^2 + *2^0$. Then

$$\begin{aligned}
*11 + *13 &\equiv (*2^3 + *2^1 + *2^0) + (*2^3 + *2^2 + *2^0) \\
&\equiv (*2^3 + *2^3) + *2^2 + *2^1 + (*2^0 + *2^0) \quad \text{by assoc'y and commu'y} \\
&\equiv *0 + *2^2 + *2^1 + *0 \quad \text{by copycat principle} \\
&\equiv *2^2 + *2^1 \\
&\equiv *(2^2 + 2^1) \\
&\equiv *6
\end{aligned}$$

So the nimber of $*11 + *13$ is 6.

In general, how can we find the nimber for $*b_1 + *b_2 + \ldots + *b_n$? Look for binary expansions of each b_i. Copycat principle cancels any pair of identical powers of 2. So we look for powers of 2's that appear in odd number of expansions of the b_i's.

Use binary numbers: 11 in binary is 1011, 13 in binary is 1101. Take the XOR operation. We do normal addition except we do not carry over.

$$\begin{array}{rr}
 & 1011 \\
\oplus & 1101 \\ \hline
 & 0110
\end{array}$$

and 0110 is 6. So $11 \oplus 13 = 6$.

Example:
Consider $*25 + *21 + *11$. In binary they are $11001, 10101, 01011$.

$$\begin{array}{rr}
 & 11001 \\
 & 10101 \\
\oplus & 01011 \\ \hline
 & 00111
\end{array}$$

and 00111 is 7. So $*25 + *21 + *11 \equiv *7$. (The nimber is 7)

Corollary 1.12

$$*b_1 + *b_2 + \ldots + *b_n \equiv *(b_1 \oplus b_2 \oplus \ldots \oplus b_n).$$

This shows that every Nim game has a nimber.

Winning strategy for Nim

Example:

$*11 + *13 \equiv *6$. This is a winning game. How to find a winning move? Want to move a game equivalent to $*0$. Add $*6$ to both sides: $*11 + *13 + *6 \equiv *6 + *6 \equiv *0$ (copycat principle).

Consider $*11 + (*13 + *6)$. We see $13 \oplus 6 = 11$. So this is equivalent to $*11 + *11$, a losing game. Winning move: remove 2 chips from the pile of 13.

Example:

$*25 + *13 + *11 \equiv *7$. Add $*7$ to both sides. Consider $*25 + (*21 + *7) + *11$. We see $21 \oplus 7 = 18$, so this is equivalent to $*25 + *18 + *11$. Winning move: remove 3 chips from the pile of 21.

Why did we pair $*7$ with $*21$ instead of $*25$ or $*11$? $25 \oplus 7 = 31$, $11 \oplus 7 = 12$. This means that we are adding 6 chips to 25, or adding 1 chip to 11. Not allowed in Nim.

Lemma 1.13

If $*b_1 + \ldots + *b_n \equiv *s$ where $s > 0$, then there exists some b_i where $b_i \oplus s < b_i$.

Idea: Look for the largest power of 2 in s.

$$*25 + *21 + *11 \equiv *7$$

$$\begin{array}{rrl} & 1\;1\;0\;0\;1 & 25 \\ & 1\;0\;1\;0\;1 & 21 \longleftarrow 21 \oplus 7\text{: 4 is subtracted from 21} \\ \oplus & 0\;1\;0\;1\;1 & 11 \longleftarrow 25 \oplus 7 \text{ or } 11 \oplus 7\text{: 4 is added} \\ \hline & 0\;0\;\boxed{1}\;1\;1 & 7 \\ & \uparrow\;\uparrow\;\uparrow & \\ & 4\;2\;1 & \leftarrow 4 > 2 + 1 \longrightarrow \begin{array}{l} \oplus \text{ reduces 21} \\ \oplus \text{ increases 25 or 11} \end{array} \end{array}$$

Proof:

Suppose $s = 2^{a_1} + 2^{a_2} + \ldots$ where $a_1 > a_2 > \ldots$. Then 2^{a_1} appears in the binary expansions of $b_1, \ldots, b_n$ an odd number of times. Let b_i be one of them. Suppose $*b_i + *s \equiv *t$ for some t. Since 2^{a_1} is in the binary expansions of b_i and s, 2^{a_1} is not in the binary expansion of t. For $2^{a_2}, 2^{a_3}, \ldots$, at worse none of them are in the binary expansion of b_i, so all of them are in the binary expansion of t. So

$$t \leq b_i - 2^{a_1} + 2^{a_2} + 2^{a_3} + \ldots < b_i \quad \text{since } 2^{a_1} > 2^{a_2} + 2^{a_3} + \ldots$$

□

Finding winning moves in a winning Nim game: Say a game has nimber s. Look at the largest power of 2 in the binary expansion of s. Pair it up with any pile $*b_i$ containing this power of 2. Then $s \oplus b_i < b_i$. So a winning move is taking away $b_i - (s \oplus b_i)$ chips from the pile $*b_i$.

Now we wish to prove Theorem 1.11. The proof uses the following lemma:

Lemma 1.14

Let $0 \leq p, q < 2^a$, and suppose Theorem 1.11 hold for all values less than 2^a. Then $p \oplus q < 2^a$.

Illustration for the proof of Theorem 1.11. Consider $*7$. $7 = 4+2+1$. Want to prove $*7 \equiv *4 + \underbrace{*2 + *1}_{\equiv *3 \text{ by induction}}$

Options of $*7$: $*0, *1, \ldots, *6$

Options of $*4 + *3$: (1) Move on $*4$ (2) Move on $*3$

(1)

$$\left.\begin{aligned} *0 + *3 &\equiv *3 \\ *1 + *3 &\equiv *2 \\ *2 + *3 &\equiv *1 \end{aligned}\right\} \text{distinct}$$

$$\underbrace{*3}_{<4} + \underbrace{*3}_{<4 \Rightarrow} \equiv \underbrace{*0}_{<4} \quad \text{by Lemma 1.14}$$

(2)

$$\begin{aligned} *4 + *2 &\equiv *6 \\ *4 + *1 &\equiv *5 \\ \underbrace{*4 + \underbrace{*0}_{\text{binary expansion do not have 4}}}_{\substack{\text{each power of 2 appears at most once} \\ \Rightarrow \text{ apply induction}}} &\equiv *4 \end{aligned}$$

Proof of Theorem 1.11:
We prove by induction on n.

When $n = 1$, $n = 2^0$ and $*1 \equiv *2^0$. Suppose $n = 2^{a_1} + 2^{a_2} + \ldots$ where $a_1 > a_2 > \ldots$. Let $q = n - 2^{a_1} = 2^{a_2} + 2^{a_3} + \ldots$.

If $q = 0$, then $n = 2^{a_1}$, so $*n \equiv 2^{a_1}$.

Assume $q \geq 1$. Since $q < n$, by induction, $*q \equiv *2^{a_2} + *2^{a_3} + \ldots$. It remains to show that $*n \equiv *2^{a_1} + *q$. The options of $*n$ are $*0, *1, \ldots, *(n-1)$. The options of $*2^{a_1} + *q$ can be partitioned into 2 types.

1. Consider options of the form $*i + *q$ where $0 \leq i < 2^{a_1}$. Since $i, q < n$, by induction, the theorem holds for i, q. So $*i, *q$ are equivalent to sums of Nim piles by their binary expansions. Using arguments from Corollary 1.12, $*i + *q \equiv *r_i$ where $r_i = i \oplus q$. Since $i, q < 2^{a_1}, r_i < 2^{a_1}$ by Lemma 1.14. So $0 \leq r_0, r_1, \ldots r_{2^{a_1}-1} < 2^{a_1}$.

   (We now show that these r_i's are distinct.) Suppose $r_i = r_j$ for some i, j. Then $*r_i \equiv *r_j$, so $*i + *q \equiv *j + *q$. Adding $*q$ on both sides, we get $*i \equiv *j$ (copycat principle), so $i = j$. So the r_i's are distinct.

   Also there are 2^{a_1} of these r_i's, and there are 2^{a_1} possible values (0 to $2^{a_1} - 1$). By Pigeonhole principle, for each $0 \leq j \leq 2^{a_1} - 1$, there is one r_i with $r_i = j$. So the options of this type are equivalent to $\{*0, *1, \ldots, *(2^{a_1} - 1)\}$.

2. Consider options of the form $*2^{a_1} + *i$ where $0 \leq i < q$. Suppose $i = 2^{b_1} + 2^{b_2} + \ldots$ where $b_1 > b_2 > \ldots$. Then no b_i is equal to a_1 since $i < q = 2^{a_2} + \ldots$. So $2^{a_1} + 2^{b_1} + \ldots$ is a sum of distinct powers of 2. Then

$$\begin{aligned} *2^{a_1} + *i &\equiv *2^{a_1} + *2^{b_2} + \ldots && \text{by applying induction on } i \\ &\equiv *(2^{a_1} + 2^{b_1} + 2^{b_2} + \ldots) && \text{by applying induction on } 2^{a_1} + i \\ &\equiv *(2^{a_1} + i) \end{aligned}$$

   Since $0 \leq i < q$, the options of this type are equivalent to
   $\{*2^{a_1}, *(2^{a_1} + 1), \ldots, \underbrace{(2^{a_1} + q - 1)}_{n-1}\}$.

Combining the two types of options, we see that the options of $*2^{a_1} + *q$ are equivalent to the options of $*n$. So $*2^{a_1} + *q \equiv *n$. $\square$

1.4 Sprague-Grundy theorem

So far: All Nim games are equivalent to a Nim game of a single pile. Goal: Extend this to all impartial games.

Poker nim

Being equivalent does not mean that they play the same way.

> Example:
>
> $*11 + *13 \equiv *6$.
>
> We move to $*11 + *11 \equiv *0$ by removing 2 chips from $*13$. RHS remove 6 chips.
>
> There are other moves, say we move to $*11 + *8 \equiv *15$. We remove 5 chips from $*13$. RHS adding 9 chips.
>
> Or, starting with $*11 + *11 \equiv *0$, any move on $*11 + *11$ will increase $*0$.

A variation on Nim: Poker nim consists of a regular Nim game plus a bag of B chips. We now allow regular Nim moves and adding $B' \leq B$ chips to one pile. Example: $*3 + *4 \to *53 + *4$.

How does this change the game of Nim?

Nothing. Say we face a losing game, so any regular Nim move would lead to a loss. In poker nim, we now add some chips to one pile. The opposing player will simply remove the chips we placed, and nothing changed.

When we say that a game is equivalent to a Nim game with one pile, it is actually a game is equivalent to a Nim game with one pile, it is actually a game of poker nim with one pile.

Mex

Suppose a game G has options equivalent to $*0, *1, *2, *5, *10, *25$. We claim that G is equivalent to $*3$. The options of $*3$, which are $*0, *1, *2$, are all available. If we add chips to $*3$, then the opposing player can remove them to get back to $*3$. How do we get 3?

$\mathbf{mex}(S)$

Given a set of non-negative integers S, $\text{mex}(S)$ is the smallest non-negative integer not in S. "**minimum excluded integer**"

> Example:
>
> $\text{mex}(\{0, 1, 2, 5, 15, 25\}) = 3$.

The mex function is the critical link between any impartial games and Nim games.

Theorem 1.15

Let G be an impartial game, and let S be the set of integers n such that there exists an option of G equivalent to $*n$. Then $G \equiv *(\text{mex}(S))$.

> Example:
>
> $$\begin{tikzcd} & *1 + *2 \equiv *3 \\ *1 + *1 + *2 \arrow[ur] \arrow[r] \arrow[dr] & *1 + *1 \equiv *0 \\ & *1 + *1 + *1 \equiv *1 \end{tikzcd}$$
>
> By theorem, $*1 + *1 + *2 \equiv *(\text{mex}(\{0, 1, 3\})) \equiv *2$.

Exercise:
A game cannot be equivalent to one of its options.

Proof of Theorem 1.15:
Let $m = \text{mex}(S)$. It suffices to show that $G + *m \equiv *0$.

1. Suppose we move to $G + *m'$ where $m' < m$. Since $m = \text{mex}(S)$, there exists an option G' of G such that $G' \equiv *m'$. p2 moves to $G' + *m'$, which is a losing game since $G' \equiv *m'$. So $G + *m$ is a losing game for p1, and $G + *m \equiv *0$.

2. Suppose we move to $G' + *m$, where G' is an option of G. Then $G' \equiv *k$ for some $k \in S$. So $G' + *m \equiv *k + *m \not\equiv *0$ since $k \neq \text{mex}(S)$. So $G' + *m$ is a winning game for p2. Then $G + *m$ is a losing game for p1, so $G + *m \equiv *0$.

□

Theorem 1.16: Sprague-Grundy Theorem

Any impartial game G is equivalent to a poker nim game $*n$ for some n.

Proof (slightly sketchy):
If G has no options, then $G \equiv *0$. Suppose G has options $G_1, \ldots, G_k$. By induction, $G_i \equiv *n_i$ for some n_i. By Theorem 1.15, $G \equiv *(\text{mex}(\{n_1, \ldots, n_k\}))$. □

So any impartial game has a nimber.

Finding nimbers is recursive: Games with no options have nimber 0. Move backwords and use mex to determine other nimbers.

Example: Rock game

	1	2	3	4	5	
1	$*0$	$*1$	$*2$	$*3$	$*4$	
2	$*1$	$*0$	$*3$	$*2$	$*5$	
3	$*2$	$*3$	$*0$	$*1$	$*6$	
4	$*3$	$*2$	$*1$	$*0$	R	$\leftarrow *7$
					$(4,5)$	

Winning move: move to $(4,4)$, an options with nimber 0.

This is like a 2-pile Nim game.

Example: Subtraction game (remove 1,2, or 3 chips)
Let s_n be the nimber of a subtraction game with n chips. Then $s_n = \text{mex}(\{s_{n-1}, s_{n-2}, s_{n-3}\})$ (if they exist)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	...
s_n	0	1	2	3	0	1	2	3	0	1	2	3	0	...

Losing game if and only if $n \equiv 0 \mod 4$. When $n \not\equiv 0 \mod 4$, the winning move is remove just enough chips to the next multiple of 4.

Example:
Subtraction game with removing 2, 5, or 6 chips Then $s_n = \text{mex}(\{s_{n-2}, s_{n-5}, s_{n-6}\})$ (if they exist)

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
s_n	0	0	1	1	0	2	1	3	0	2	1	0	0	1	1	...

repeats (not proved here)

Losing game if and only if $n \equiv 0, 1, 4, 8 \mod 11$. Winning move from 9: move to 4.

Example: Combining games

Let G be the rook game at $(4, 2)$. Let H be the second subtraction with $n = 7$.

Then $G \equiv *2, H \equiv *3$, so $G + H \equiv *2 + *3 \equiv *1$. Winning game.

Winning move:

- From H, $3 \oplus 1 = 2$. Move to $*2$. Remove 2 chips in the subtraction game.
- From G, $2 \oplus 1 = 3$. Move to $*3$. Move to $(4, 1)$ or $(3, 2)$.

2

Strategic games

Example: Prisoner's dilemma

Game show version: 2 players won $10,000. They each need to make a final decision: "share" or "steal".

- If both pick "share", then they each win $5,000.
- If one picks "steal" and the other picks "share", then the one who picks "steal" gets $10,000, the other gets nothing.
- If both pick "steal", then they both get a consolation price with $10.

How would players behave? The benefit a player receives is dependent on their own decision and the decisions of other players.

strategic game

A **strategic game** is defined by specifying a set $N = \{1, \ldots, n\}$ of players, and for each player $i \in N$, then there is a set of possible strategies s_i to play, and a utility function: $u_i : s_1 \times \cdots \times s_n \to \mathbb{R}$.

Example:

With prisoner's dilemma above, $s_1 = s_2 = \{\text{share,steal}\}$. Samples of the utility functions: $u_1(\text{share,share}) = 5000$, $u_2(\text{steal,share}) = 0$. We can summarize the utility functions in a payoff table.

		PII	
		share	steal
PI	share	5k, 5k	0, 10k
	steal	10k, 0	10, 10

Each cell records the utilities of PI, PII in this order given the strategies played in that row (PI) and column (PII).

Assumptions about strategic games;

1. All players are rational and selfish (want to maximize their own utility).
2. All players have knowledge of all game parameters.
3. All players move simultaneously.
4. Player i plays a strategy $s_i \in S_i$, this forms a strategy profile $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$. Player i earns $u_i(s)$.

Given a strategic game, what are we looking for? One answer is we want to know how are the players expected to behave?

Resolving prisoner's dilemma

Recall the payoff table from a previous example. What would a rational and selfish player choose to play?

1. If you know that the other player chooses to "share", then choosing "share" gives 5k, choosing "steal" gives 10k. Steal is better.
2. If you know that the other player chooses "steal", then choosing "share" gives 0, choosing "steal" gives 10. Steal is better.

In both cases, it is better to steal than to share. So we expect both players to choose "steal".

This is an example of a **strictly dominating strategy**: regardless of how other players behave, this strategy gives the best utility over all other possible strategies. If a strictly dominating strategy exists, then we expect the players to play it.

In this case, playing a strictly dominating strategy "steal" yields very little benefit. They could get more if there is some cooperation (both share). So even though we expect strictly dominating strategy is played, it might not have the best "social welfare" (the overall utility of the players).

2.1 Nash equilibrium

There are many games with "no" strictly dominating strategies.

> Example: Bach or Stravinsky?
>
> Two players want to go to a concert. Player I likes Bach, player II likes Stravinsky, but they both prefer to be with each other. Payoff table:
>
> | | | PII Bach | PII Stravinsky |
> |---|---|---|---|
> | PI | Bach | 2, 1 | 0, 0 |
> | PI | Stravinsky | 0, 0 | 1, 2 |
>
> No strict dominating strategy exists.
>
> What do we expect to happen? If both choose "Bach", then there is no reason for one player to switch their strategy (which gives utility 0). Similar if both choose "Stravinsky".

These are steady states, which we call **Nash equilibria**: a strategy profile where no player is incentivized to change strategy.

Mixed strategies

There are many games with no Nash equilibria.

> Example: Rock paper scissors
>
> R beats S, S beats P, P beats R. Utility 1 if they win, -1 if they lose, 0 if they tie.
>
> | | | PII R | PII P | PII S |
> |---|---|---|---|---|
> | PI | R | 0, 0 | -1, 1 | 1, -1 |
> | PI | P | 1, -1 | 0, 0 | -1, 1 |
> | PI | S | -1, 1 | 1, -1 | 0, 0 |
>
> "No" NE exist: regardless what they play, someone is incentivized to switch strategy so that they

win.

How would we expect players to play this? Randomly, probability $\frac{1}{3}$ each. This is a **mixed strategy**. IT is also a NE, there is no incentive to change to a different probability distribution.

Nash's Theorem

Every strategic game with finite number of strategies has a Nash equilibrium (could be mixed strategies).

Notation

Recall: Strategic game is defined by

- Players $N = \{1, \dots, n\}$.
- Strategy set S_i for player i.
- Utility for player i: $u_i : s_1 \times \cdots s_n \to \mathbb{R}$. A strategy profile is a vector $s = (s_1, \dots, s_n) \in S_1 \times \cdots S_n$ which records what the players played.

Let $S = S_1 \times S_n$ be the set of all strategy profiles. We will often compare the utilities of a player's strategies when we fix the strategies of the remaining players. Let S_{-i} be the set of all strategy profiles of all players except player i (we drop S_i from the cartesian product $S_1 \times \cdots S_n$). If $s \in S$, then the profile obtained from s by dropping s_i is denoted $s_{-i} \in S_{-i}$. If player i switches their strategy from s_i to s_i', then the new strategy profile is denoted $(s_i', s_{-i}) \in S$.

Nash equilibrium

A strategy profile $s^* \in S$ is a **Nash equilibrium** if $u_i(s^*) \geq u_i(s_i', s_{-i}^*)$ for all $s_i' \in S_i$ and for all $i \in N$.

Example: Prisoner's dilemma

		PII share	PII steal
PI	share	5k, 5k	0, 10k
	steal	10k, 0	10, 10

Let $s^* = (\text{steal}, \text{steal})$.

From PI: $u_1(s^*) = 10$, $u_1(\underbrace{\text{share}}_{s_1'}, \underbrace{\text{steal}}_{s_{-1}'}) = 0 < u_1(s^*)$.

Similar for PII. So s^* is a NE.

Example: Guess 2/3 average game

3 players, a positive integer k. Each player simultaneously pick an integer from $\{1, \dots, k\}$, producing the strategy profile $s = (s_1, s_2, s_3)$. There is \$1 which is split among all players whose choices are closest to $\frac{2}{3}$ of the 3 numbers. Other players get \$0.

If $s = (5, 2, 4)$, then the average is $\frac{11}{3}$, and $\frac{2}{3}$ average is $\frac{22}{9} = 2 + \frac{4}{9}$. p2 is the closest, so $u_2(s) = 1, u_1(s) = u_3(s) = 0$. Is s a NE? No. If p1 switches to 2, the $u_1(2, s_{-1}) = u_1(2, 2, 4) = \frac{1}{2}$. ($\frac{2}{3}$ average is $\frac{16}{9}$, closer to 2 than 4).

Is there a NE? Idea: Lowering the guess generally pulls the $\frac{2}{3}$ average closer. Try $(1, 1, 1)$. If a player switches to $t \geq 2$, then the $\frac{2}{3}$ average is $\frac{4+2t}{9} = \frac{4}{9} + \frac{2}{9}t$, which is closer to 1 than t.

Prove that $(1,1,1)$ is the only NE of this game.

2.1.1 Best response function

For a NE, a player does not want to switch. If you fix the strategies of the remaining players, then you play a strategy that maximizes utility for yourself, i.e., it is a "best response" to the fixed strategies.

best response function

Player i's **best response function** for $s_{-i} \in S_{-i}$ is given by

$$B_i(s_{-i}) = \{s_i' \in S_i : \underbrace{u_i(s_i', s_{-i})}_{\text{utility of a best response}} \geq \underbrace{u_i(s_i, s_{-i})}_{\text{utility of all possible responses to } s_{-i}} \quad \forall s_i \in S_i\}.$$

Example: Prisoner's dilemma

$B_1(\text{share}) = \{\text{steal}\}, \qquad B_1(\text{steal}) = \{\text{steal}\}.$

Example: 2/3 average game

$$B_1(5,5) = \{1,2,3,4\} \qquad u_1(x,5,5) = \begin{cases} 1 & x < 5 \quad \text{best response} \\ 1/3 & x = 5 \\ 0 & x > 5 \end{cases}$$

If s^* is a NE, then each player i must have played a best response to s^*_{-i}. Changing s_i^* cannot increase utility for i. Converse is also true.

Lemma 2.1

$s^* \in S$ is a Nash equilibrium if and only if $s_i^* \in B_i(s^*_{-i})$ for all $i \in N$.

This lemma helps us find NE by looking for strategies in the BRF.

Example:

		PII share	PII steal
PI	share	5k, 5k	0, 10k$^\circ$
PI	steal	10k*, 0	$\boxed{10^*, 10^\circ}$

These are best responses to each other. So this is a NE

$$\begin{array}{lll} B_1(\text{share}) = \{\text{steal}\} & B_1(\text{steal}) = \{\text{steal}\} & * \\ B_2(\text{share}) = \{\text{steal}\} & B_2(\text{steal}) = \{\text{steal}\} & \circ \end{array}$$

Example: Arbitrary game

		PII X	PII Y	PII Z
PI	A	1, 2°	2*, 1	1*, 0
PI	B	$\boxed{2^*, 1^\circ}$	0, 1°	0, 0
PI	C	0, 1	0, 0	$\boxed{1^*, 2^\circ}$

$$\begin{array}{llll} B_1(X) = \{B\} & B_1(Y) = \{A\} & B_1(Z) = \{A, C\} & * \\ B_2(A) = \{X\} & B_2(B) = \{X, Y\} & B_2(C) = \{Z\} & \circ \end{array}$$

NE are (B, X) and (C, Z), as they are best responses to each other. The rest are not NE as one is not a best response to the other.

2.2 Cournot's oligopoly model

We have a set $N = \{1, \dots, n\}$ of n firms producing a single type of goods sold on the common market. Each firm i needs to decide the number of units of goods q_i to produce. (variables)

Production cost is $C_i(q_i)$ where C_i is a given increasing function.

Given a strategy profile $q = (q_1, \dots, q_n)$, a unit of the goods sell for the price of $P(q)$, where P is a given non-increasing function on $\sum_i q_i$ (more goods in the market = low price)

The utility of firm i in the strategy profile q is $u_i(q) = \underbrace{q_i P(q)}_{\substack{\text{revenue for} \\ \text{selling } q_i \text{ units}}} - \underbrace{c_i(q_i)}_{\substack{\text{production} \\ \text{production cost}}}$

Szidarovszky and Yakowitz proved that a Nash equilibrium always exists under some continuity and differentiability assumptions on P, C.

Special case: linear costs and prices

Suppose we assume $C_i(q_i) = cq_i, \forall i \in N$ (the cost is linear, same unit cost c for all firms). $P(q) = \max\{0, \alpha - \sum_j q_j\}$ (prices starts at α, decreases 1 for each unit produced, min price 0) where $0 < c < \alpha$.

Utility is

$$u_i(q) = q_i P(q) - C_i(q_i) = \begin{cases} q_i(\alpha - c - \sum_j q_j) & \alpha - \sum_j q_j \geq 0 \\ -cq_i & \alpha - \sum_j q_j < 0 \end{cases}$$

When is it possible to make a profit? When $\alpha - c - \sum_j q_j > 0$. Separate q_i from the sum: $\alpha - c - q_i - \sum_{i \neq j} q_j > 0$. So $q_i < \alpha - c - \sum_{j \neq i} q_j$. Does not make sense for q_i if RHS ≤ 0, so assume RHS > 0.

The utility is $q_i(\alpha - c - q_i - \sum_{j \neq i} q_j)$. Treating q_i as the variable, this utility is maximized when $q_i = (\alpha - c - \sum_{j \neq i} q_j)/2$. So the best response function for firm i given the production of other firms q_{-i} is

$$B_i(q_{-i}) = \begin{cases} \left\{(\alpha - c - \sum_{j \neq i} q_j)/2\right\} & \alpha - c - \sum_j q_j > 0 \\ \{0\} & \text{otherwise} \end{cases}$$

Two-firm case

Suppose we simplify to 2 firms. Suppose $q^* = (q_1^*, q_2^*)$ is a Nash equilibrium. By Lemma 2.1, a player's choice must be the best response to the other player's choice. So $q_1^* \in B_1(q_2^*)$ and $q_2^* \in B_2(q_1^*)$.

Verify that we may assume $q_1^*, q_2^* > 0$. Then $q_1^* = (\alpha - c - q_2^*)/2$ and $q_2^* = (\alpha - c - q_1^*)/2$.

Solving this gives $q_1^* = q_2^* = (\alpha - c)/3$. This is the amount we expect each firm to produce at equilibrium.

Price at equilibrium: $P(q^*) = \alpha - q_1^* - q_2^* = \alpha - \frac{2}{3}(\alpha - c) = \frac{\alpha}{3} + \frac{2c}{3}$.

Profit at equilibrium: $u_i(q^*) = q_i^*(\alpha - c - q_1^* - q_2^*) = (\alpha - c)^2/9$.

Note:

1. Suppose the two firms can collude, and together they produce Q units total. Total profit is $Q(\alpha - c - Q)$, which is maximized at $Q = (\alpha - c)/2$. The profit is $\left(\frac{\alpha - c}{2}\right)\left(\alpha - c - \frac{\alpha - c}{2}\right) = (\alpha - c)^2/4$. Each firm gets $\frac{(\alpha-c)^2}{8} > \frac{(\alpha-c)^2}{9}$.
2. In the general case with n firms, if q^* is a NE, then $q_i^* = (\alpha - c - \sum_{j \neq i} q_j^*)/2$. Solving this system gives $q_j^* = \frac{\alpha - c}{n+1}$. Price is

$$P(q^*) = \alpha - \sum_j q_j^* = \alpha - \frac{n}{n+1}(\alpha - c) = \frac{1}{n+1}\alpha + \frac{n}{n+1}c$$

As $n \to \infty, P(q^*) \to c$. As more firms are involved, the expected market price gets closer to the production cost.

2.3 Dominance

2.3.1 Strict dominance

strict dominance

For two strategies $s_i^{(1)}, s_i^{(2)} \in S_i$ for player i, we say that $s_i^{(1)}$ **strictly dominates** $s_i^{(2)}$ if for all $s_{-i} \in S_{-i}$, $u_i(s_i^{(1)}, s_{-i}) > u_i(s_i^{(2)}, s_{-i})$.

If there exists a strategy that strictly dominates s_i, then s_i is **strictly dominated**.

If s_i strictly dominates all strategies $s_i' \in S_i \setminus \{s_i\}$, then s_i is a **strictly dominating strategy**.

In prisoner's dilemma, "steal" is a strictly dominating strategy for both players.

Lemma 2.2

If $s_i \in S_i$ is a strictly dominating strategy for player i and $s^* \in S$ is a NE, then $s_i^* = s_i$.

In any NE, the strictly dominating strategy is played whenever it exists. A game is easy to play if such a strategy exists.

Now we look at strictly dominated strategies.

Example:

	X	Y	Z
A	4, 2	1, 3	2, 1
B	2, 3	0, 1	3, 1

Z is strictly dominated by X since $u_2(A, X) > u_2(A, Z)$ and $u_2(B, X) > u_2(B, Z)$

Z is a strictly dominated strategy: There is no reason to play it.

Lemma 2.3

IF $s^* \in S$ is a NE, then s_i^* is not strictly dominated for any $i \in N$.

Iterated elimination of strictly dominated strategies (IESDS)

Example:

	X	Y	Z
A	4, 2	1, 3	2, 1
B	2, 3	0, 1	3, 1

Z is strictly dominated, so it will not appear in any NE.

$\xrightarrow{\text{eliminate Z}}$

	X	Y
A	4, 2	1, 3
B	2, 3	0, 1

B is strictly dominated. (Notice this was not the case before)

$\xrightarrow{\text{eliminate B}}$

	X	Y
A	4, 2	1, 3

X is strictly dominated.

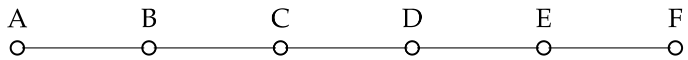
$\xrightarrow{\text{eliminate X}}$

	Y
A	1, 3

NE?

IESDS: Repeatedly eliminate strictly dominated strategies until we have only one strategy profile. We claim that if this works, then the surviving profile is the unique NE of the game.

Example: Facility location game



Two firms are each given a permit to open one store in one of 6 towns along a high way. Firm I can open in A, C or E, firm II can open in B, D or F. Assume towns are equally spaced and equally populated. Customers in a town will go to the closest store. Where to open stores?

		Firm II		
		B	D	F
Firm I	A	1, 5	2, 4	3, 3
	C	4, 2	3, 3	4, 2
	E	3, 3	2, 4	5, 1

Firm I, A is strictly dominated by C.
Firm II, F is strictly dominated by D.
Eliminate these two strategies.

		Firm II	
		B	D
Firm I	C	4, 2	3, 3
	E	3, 3	2, 4

Firm I, E is strictly dominated by C.
Firm II, B is strictly dominated by D.
Eliminate these two strategies.

		Firm II
		D
Firm I	C	3, 3

(C, D) is a NE.

Note: Extend this to 1000 towns with alternating options. The two ends are strictly dominated by the centre towns. Eliminate them to get 998 towns. Repeat. End with the two towns in the centre as NE.

Results in IESDS

Theorem 2.4

Suppose G is a strategic game. If IESDS ends with only one strategy profile s^*, then s^* is the unique Nash equilibrium of G.

This is a consequence of the following result.

Theorem 2.5

Let H be a strategic game where s_i is a strictly dominated strategy for player i. Let G' be obtained from G by removing s_i from S_i. Then s^* is a Nash equilibrium of G if and only if s^* is a Nash equilibrium of G'.

Proof Sketch:
Suppose s^* is a NE of G. Since s_i is strictly dominated, it cannot appear in s^* (Lemma 2.3). So s^* is a valid strategy profile in G'. If s^* is not a NE of G', then a player can deviate to get a higher utility. However, all strategies in G' are available in G, so such a player can do it in G as well. This contradicts s^* is a NE of G.

Suppose s^* is a NE of G'. Suppose s^* is not a NE of G. Then a player can deviate to get a higher utility. This can be replicated in G' (which results in a contradiction) unless it is player i switching to strategy s_i (the only strategy in G not in G'). Then player i could switch to the strategy that strictly dominates s_i (available in G') to get a higher utility in G'. This contradicts s^* is a NE in G'. $\square$

2.3.2 Weak dominance

weak dominance

For two strategies $s_i^{(1)}, s_i^{(2)} \in S_i$ for player i, we say that $s_i^{(1)}$ **weakly dominates** $s_i^{(2)}$ if for all $s_{-i} \in S_{-i}$, $u_i(s_i^{(1)}, s_{-i}) \geq u_i(s_i^{(2)}, s_{-i})$, and this inequality is strict for at least one $s_{-i} \in S_{-i}$.

If some strategy weakly dominates s_i, then s_i is **weakly dominated**.

If s_i weakly dominates all strategies $s_i' \in S_i \setminus \{s_i\}$, then s_i is a **weakly dominating strategy**.

Example:

PII

PI		X	Y	Z
	A	3, 3	1, 1	4, 1
	B	2, 1	0, 1	3, 1

Z is weakly dominated by X, $u_2(A, X) > u_2(A, Z)$ and $u_2(B, X) \geq u_2(B, Z)$. Z is not weakly dominated by Y, no strict inequality.

Iterated elimination of weakly dominated strategies (IEWDS)

Remove weakly dominated strategies until there is only one strategy profile.

Example:

Z and Y are weakly dominated by X above. Eliminating them gives

	X
A	3, 3
B	2, 1

A weakly dominates B.

	X
A	3, 3

(A, X) is a NE.

Theorem 2.6

Suppose G is a strategy game. If IEWDS ends with only one strategy profile s^*, then s^* is a Nash equilibrium of G.

Note:
Compared with Theorem 2.4, here we can no longer claim that the NE is unique. A different sequence of eliminations can result in a different NE.

Exercise:

	X	Y	Z
A	1, 1	1, 0	2, 1
B	1, 1	0, 0	0, 0
C	0, 0	0, 0	1, 1

Show that two different applications of IEWDS here could end with two different profiles.

Key difference Unlike strictly dominated strategies, weakly dominated strategies can appear in a NE.

Some NE cannot found through IEWDS, e.g., *Bach or Stravinsky* has no weakly dominated strategies.

Just like strictly dominating strategies, weakly dominating strategies are good to play.

Lemma 2.7

If for all players i, s_i^* is a weakly dominating strategy, then s^* is a Nash equilibrium.

2.4 Auctions

Set up of an auction: A seller puts one item up for an auction. Potential buyers put in bids to buy the item. Seller decides who wins (usually highest bidder) and the prices they pay.

Typical auction: Open bid auction. Buyers bid repeatedly until no one else bids. Highest bid wins and pays their bid price. Another type: Closed bid auctions. Each buyer submits one secret bid to the seller. (Easier to analyze).

First price auction: Highest bid wins, winner pays their bid. For example, 3 bidders: 150, 100, 200, pays 200. Does this simulate an open auction? No, in the open auction setting, the winner will bid slightly over 150 and win, so they pay ∼150.

Second price auction: Highest bid wins, winner pays 2nd highest bid. For example, 3 bidders: 150, 100, 200, pays 150. We will analyze second price closed bid auction.

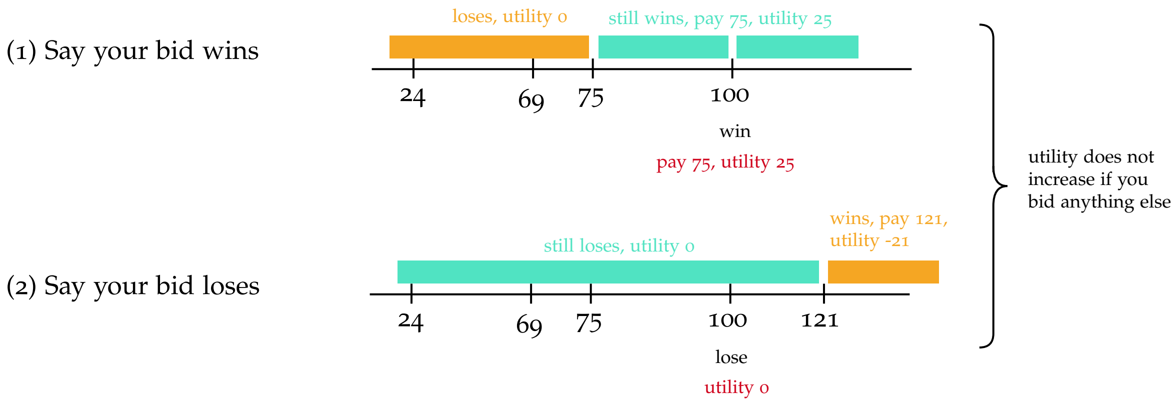
Set up

We have buyers $N = \{1, \dots, n\}$. Buyer i thinks the item has value v_i "valuation". Suppose buyer i submits the bid b_i, giving strategy profile $b = (b_1, \dots, b_n)$. The winner is the buyer who submits the highest bid, pays price equal to the second highest bid. If there is a tie, then the winner is the buyer with the lowest index i among all tied buyers.

Given a strategy profile b, the utility for buyer i is

$$u_i(b) = \begin{cases} v_i - \max_{j \neq i} b_j & i \text{ wins in } b \\ 0 & \text{otherwise} \end{cases}$$

Suppose your valuation of the item is 100. Would you bid anything other than 100?



Theorem 2.8

In the second price auction, v_i is a weakly dominating strategy for player $i \in N$.

Proof:
We first show that $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all $b_i \in S_i$ and $b_{-i} \in S_{-i}$. 2 cases.

1. v_i is a winning bid in (v_i, b_{-i}). Let b_j be the second highest bid (could equal v_i). The utility for player i is $u_i(v_i, b_{-i}) = v_i - b_j \geq 0$. Suppose player i changes their bid to b_i.

   If $b_i > b_j$ or ($b_i = b_j$ and $i < j$), then b_i is still the winning bid in (b_j, b_{-i}). Payment is b_j, so utility remains the same. Otherwise, b_i is a losing bid, so the utility is 0, which is at most $u_i(b_i, b_{-i})$.

   So $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for any b_i.

2. v_i is a losing bid in (v_i, b_{-i}). Let b_j be the winning bid (so $b_j \geq b_i$). The utility for player i is $u_i(v_i, b_{-i}) = 0$. Suppose player i changes their bid to b_i.

   If $b_i < b_j$ or ($b_i = b_j$ and $i > j$), then b_i is still a losing bid in (b_i, b_{-i}). Utility is still 0. Otherwise, b_i is a winning bid, with payment b_j. The utility is $u_i(b_i, b_{-i}) = v_i - b_j \leq 0$ (since $b_j \geq v_i$). So $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for any b_i.

In both cases, bidding v_i gives the highest utility among all possible bids of player i.

We still need to show that for all $b_i \neq v_i$, there exists $s_{-i} \in S_{-i}$ such that $u_i(v_i, b_{-i}) > u_i(v_i, b_{-i})$. Two cases:

1. Suppose $b_i < v_i$. Let k be in $b_i < k < v_i$. Set $b_j = k$ for all $j \neq i$.

   When v_i is played against b_{-i}, player i wins ($v_i > k$) and pays k. Utility $u_i(v_i, b_{-i}) = v_i - k > 0$. When b_i is played against b_{-i}, player i loses ($b_i < k$) and utility $u_i(b_i, b_{-i}) = 0$. So $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$.

2. Suppose $b_i > v_i$. Let k be in $v_i < k < b_i$. Set $b_j = k$ for all $j \neq i$.

   When v_i is played against b_{-i}, player i loses ($v_i < k$) and utility $u_i(v_i, b_{-i}) = 0$. When b_i is played against b_{-i}, player i wins ($b_i > k$) and pays k. Utility $u_i(b_i, b_{-i}) = v_i - k < 0$. So $u_i(v_i, b_{-i}) > u_i(b_i, b_{-i})$.

Therefore, playing v_i is a weakly dominating strategy. $\square$

Note:
The way we play this game does not depend on knowing how other players value the item. So it is easy to play: simply bid your valuation.

Exercise:
Suppose buyer 1 has highest valuation v_1, and buyer 2 has second highest valuation v_2, then $(v_2, v_1, 0, 0, \ldots, 0)$ is a NE.

2.5 Mixed strategies

Example: Matching pennies
Two players each has a penny. They simultaneously show heads or tails. If they match, then player I gains the penny from player II. If they don't match, then player II gets the penny from player I.

		PII H	PII T
PI	H	1, -1	-1, 1
PI	T	-1, 1	1, -1

There's no Nash equilibrium here (in the way NE has been described so far). Allow players to play this probabilistically. For example, PI might play H $\frac{1}{3}$ of the time, and play T $\frac{2}{3}$ of the time. PII might play $\frac{3}{4}$ on H, $\frac{1}{4}$ on T.

Is there an equilibrium here? If p1 plays $\frac{1}{3}$H, $\frac{2}{3}$T, then p2 wants to play H more often than T. Then p1 wants to play H more often than T. Then p2 wants to play T more often than H, ... etc. Seems that it is stable only if both players play $\frac{1}{2}$H, $\frac{1}{2}$T.

mixed strategy

A **mixed strategy** for player i is a vector $x_i \in \mathbb{R}_+^{S_i}$ such that $\sum_{s\in S_i} x_s^i = 1$. The set of all mixed strategies for player i is denoted Δ^i.

mixed strategy profile

A **mixed strategy profile** is a vector $x = (x^1, \dots, x^n)$ where $x^i \in \Delta^i$ is a mixed strategy for player i. The set of all mixed strategy profiles is denoted $\Delta = \Delta^1 \times \cdots \times \Delta^n$. The mixed strategy profile with player i removed is $x^{-i} \in \Delta^{-i}$.

Note:

- If we play a strategy with probability 1, then it is a **pure strategy** (this is the way we play previously).
- As convention for this course, we use s's to represent pure strategies, x's to represent mixed strategies.

Example:
In matching pennies, if we order the pure strategies in the order H, T, then we had

$$x^1 = (x_H^1, x_T^1) = \left(\frac{1}{3}, \frac{2}{3}\right), x^2 = (x_H^2, x_T^2) = \left(\frac{3}{4}, \frac{1}{4}\right)$$

as mixed strategies. The strategy profile is $x = (x^1, x^2) = \left((\frac{1}{3}, \frac{2}{3}), (\frac{3}{4}, \frac{1}{4})\right)$.

Why mixed strategies?

1. Introduce unpredictability in games that are played repeatedly. Examples: In penalty kicks, you do not always kick to the same side; in politics, you do not always want to make major announcements on Tuesdays. Then the oppositions and preempt you on their announcements on Mondays.
2. Think of a player as representing a population, with probability of a strategy being proportional to the portion of the population who prefer it. Example: Say 55% like donkeys and 45% like elephants, perhaps there will be more donkeys in zoos.

Utility

We will use expected value as utility.

Example:

		PII: H	PII: T
PI	H	1, -1	-1, 1
	T	-1, 1	1, -1

$$x^1 = \left(\frac{1}{3}, \frac{2}{3}\right), \quad x^2 = \left(\frac{3}{4}, \frac{1}{4}\right)$$

Two cases for p1:

1. If p1 plays H as pure strategy, then $\frac{3}{4}$ chance we get 1, $\frac{1}{4}$ chance we get -1. We expect to get $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot (-1) = \frac{1}{2}$.
2. If p1 plays T as pure strategy, then $\frac{3}{4}$ chance we get -1, $\frac{1}{4}$ chance we get 1. We expect to get $\frac{3}{4} \cdot (-1) + \frac{1}{4} \cdot 1 = -\frac{1}{2}$.

Overall, p1 plays H $\frac{1}{3}$ of the time and T $\frac{2}{3}$ of the time. So the expected utility is $\frac{1}{3}\cdot(\frac{1}{2})+\frac{2}{3}\cdot(-\frac{1}{2}) = -\frac{1}{6}$.

expected utility of a pure strategy

We are given a strategy profile $x = (x^1, \ldots, x^n) \in \Delta$. The **expected utility of a pure strategy** $s_i \in S_i$ for player i is

$$u_i(s_i, x^{-i}) = \sum_{s_{-i}\in S_{-i}} \underbrace{u_i(s_i, s_{-i})}_{\substack{\text{utility of}\\ \text{playing } s_i}} \underbrace{\prod_{j\neq i} x^j_{s_j}}_{\substack{\text{probability that}\\ \text{the remaining}\\ \text{players play } s_{-i}}}$$

where $u_i(s_i, x^{-i})$ is the utility from the pure strategy game.

expected utility

The **expected utility** of player i in x is

$$u_i(x) = \sum_{s_i\in S_i} \underbrace{x^i_{s_i}}_{\substack{\text{prob. that}\\ p_i \text{ plays } s_i}} \underbrace{u_i(s_i, x^{-i})}_{\substack{\text{utility } p_i \text{ gets}\\ \text{for playing } s_i}}$$

Example:

For matching pennies above, $u_1(H, x^2) = \frac{1}{2}$, $u_1(T, x^2) = -\frac{1}{2}$, $u_1(x) = -\frac{1}{6}$

Example:

Suppose 3 players each make a choice between A and B. A \$1 prize is split among players who pick the majority choice. Suppose $x^1 = (p, 1-p), x^2 = (\frac{1}{2}, \frac{1}{2}), x^3 = (\frac{2}{5}, \frac{3}{5})$. What is the expected utility for p1?

When p1 plays A, there are 4 cases:

1. $u_1(A, A, A) = \frac{1}{3}$. The probability that this happens is $x^2_A \cdot x^3_A = (\frac{1}{2})(\frac{2}{5}) = \frac{1}{5}$.
2. $u_1(A, A, B) = \frac{1}{2}$. The probability that this happens is $x^2_A \cdot x^3_B = (\frac{1}{2})(\frac{3}{5}) = \frac{3}{10}$.
3. $u_1(A, B, A) = \frac{1}{2}$. The probability that this happens is $x^2_B \cdot x^3_A = (\frac{1}{2})(\frac{2}{5}) = \frac{1}{5}$.
4. $u_1(A, B, B) = 0$. Does not matter.

Utility for playing A is $u_1(A, x^{-1}) = (\frac{1}{5})(\frac{1}{3}) + (\frac{3}{10})(\frac{1}{2}) + (\frac{1}{5})(\frac{1}{2}) + 0 = \frac{19}{60}$

And $u_1(B, x^{-1}) = \frac{7}{20}$. Then expected utility for p1 is $u_1(x) = p\cdot\frac{19}{60} + (1-p)\frac{7}{20} = \frac{7}{20} - \frac{1}{15}p$.

It would make sense to pick $p = 0$, so p1 always plays B. (p3 is more likely to pick B, letting us form a majority more often.)

2.5.1 Mixed equilibria

mixed Nash equilibrium

A mixed strategy profile $\bar{x} \in \Delta$ is a **mixed Nash equilibrium** if for each player $i \in N$, $u_i(\bar{x}) \geq u_i(x^i, \bar{x}^{-i})$ for all $x^i \in \Delta^i$.

We often omit the word "mixed", so it is also a Nash equilibrium.

best response function

Given a profile $\bar{x}^{-i} \in \Delta^{-i}$, the **best response function** for player i, $B_i(\bar{x}^{-i})$, is the set of all mixed strategies of player i that have maximum utility against $\bar{x}^{-i}$, i.e.,

$$B_i(\bar{x}^{-i}) = \left\{ \bar{x}^i \in \Delta^i : u_i(\bar{x}^i, \bar{x}^{-i}) \geq u_i(x_i, \bar{x}^{-i}) \quad \forall x^i \in \Delta^i \right\}$$

Proposition 2.9

$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^n) \in \Delta$ is a Nash equilibrium if and only if $\bar{x}^i \in B_i(\bar{x}^{-i})$ for all $i \in N$.

Example: Matching pennies

		PII H	PII T
PI	H	1, -1	-1, 1
	T	-1, 1	1, -1

Suppose $x^1 = (p, 1-p)$ and $x^2 = (q, 1-q)$.

For p1, the expected utility for playing H is $q \cdot 1 + (1-q) \cdot (-1) = 2q - 1$. The expected utility for playing T is $q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q$. Utility for p1 is $p(2q-1) + (1-p)(1-2q) = p(-2+4q) + (1-2q)$.

Given q, which p maximizes this utility? $1 - 2q$ is constant, so we maximize $p(-2+4q)$. 3 cases:

1. If $q < \frac{1}{2}$, then $-2 + 4q < 0$. So we maximize with $p = 0$.
2. If $q = \frac{1}{2}$, then $-2 + 4q = 0$. Then any p maximizes it, so $p \in [0, 1]$.
3. If $q > \frac{1}{2}$, then $-2 + 4q > 0$. Maximize with $p = 1$.

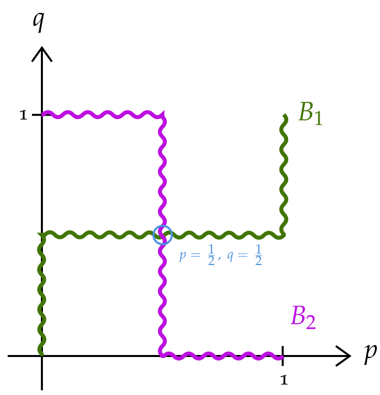
BRF for p1:

$$B_1(x^2) = \begin{cases} \{(0,1)\} & q < \frac{1}{2} \\ \{(p, 1-p) : p \in [0,1]\} & q = \frac{1}{2} \\ \{(1,0)\} & q > \frac{1}{2} \end{cases}$$

Similarly, for p2, the utility is $q(2-4p) + (2p-1)$. Divide cases with $p = \frac{1}{2}$. Then

$$B_2(x^1) = \begin{cases} \{(1,0)\} & p < \frac{1}{2} \\ \{(q, 1-q) : q \in [0,1]\} & p = \frac{1}{2} \\ \{(0,1)\} & p > \frac{1}{2} \end{cases}$$

We look for p, q such that x^1, x^2 are best responses to each other. Draw B_1, B_2 on a "graph".



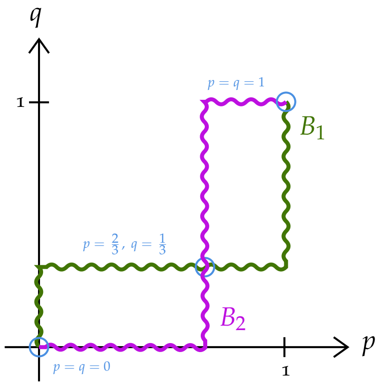
The intersection is where they are best responses simultaneously, hence a Nash equilibrium. $x^1 = (1/2, 1/2), x^2 = (1/2, 1/2)$ and (x^1, x^2) is a NE.

Example: Bach or Stravinsky

		PII	
		B	S
PI	B	2, 1	0, 0
	S	0, 0	1, 2

Suppose $x^1 = (p, 1-p), x^2 = (q, 1-q)$. We have

$$B_1(x^2) = \begin{cases} \{(0,1)\} & q < \frac{1}{3} \\ \{(p, 1-p) : p \in [0,1]\} & q = \frac{1}{3} \\ \{(1,0)\} & q > \frac{1}{3} \end{cases} \qquad B_2(x^1) = \begin{cases} \{(0,1)\} & p < \frac{2}{3} \\ \{(q, 1-q) : q \in [0,1]\} & p = \frac{2}{3} \\ \{(1,0)\} & p > \frac{2}{3} \end{cases}$$



3 NE: 2 pure strategies $((0,1),(0,1))$ and $((1,0),(1,0))$. 1 mixed strategy $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$

2.5.2 Support characterization

Suppose $\bar{x}^{-i}$ is fixed. Which $x^i \in \Delta^i$ maximizes $u_i(x^i, \bar{x}^{-i})$? Write a LP:

$$\begin{aligned} \max \quad & \sum_{x \in S_i} x^i_s u_i(s, \bar{x}^{-i}) \\ \text{s.t.} \quad & \sum_{s \in S_i} x^i_s = 1 \\ & x^i \geq \mathbf{0} \end{aligned} \tag{P}$$

Variables: x^i_s for each $s \in S_i$. What is the dual? One dual variable y.

$$\begin{aligned} \min \quad & y \\ \text{s.t.} \quad & y \geq u_i(s, \bar{x}^{-i}) \quad \text{for all } s \in S_i \end{aligned} \tag{D}$$

(P) is feasible (set x^i to be any probability distribution). (D) is feasible (set y to be max value of $u_i(s, \bar{x}^{-i})$). Therefore, (P) and (D) both have optimal solutions, and their optimal values are equal.

(D) is easy to solve: $y = \max_{s \in S_i} u_i(s, \bar{x}^{-i})$, maximum utility when pure strategies are played against $\bar{x}^{-i}$. (P) also has optimal value y. So the maximum utility of all mixed strategies is equal to the max utility of pure strategies.

Complementary slackness conditions: $x^i_s = 0$ or $y = u_i(s, \bar{x}^{-i})$ for all $s \in S_i$. Equivalently, $x^i_s > 0$ implies $y = u_i(s, \bar{x}^{-i})$. Translation: only pure strategies with maximum utility could have positive probabilities in a best response.

Theorem 2.10: Support characterization

Given $\bar{x}^{-i} \in \Delta^{-i}$, a mixed strategy $x^i \in B_i(\bar{x}^{-i})$ if and only if $x^i_s > 0$ implies $s \in S_i$ is a pure strategy of maximum utility against $\bar{x}^{-i}$.

support

For a mixed strategy $x^i \in \Delta^i$, the **support** is the set of strategies with positive probability in x^i.

Rephrasing of Theorem 2.10: x^i is in the BRF if and only if the support of x^i are strategies with maximum utility.

Example: Bach or Stravinsky

		PII: B	PII: S
PI	B	2, 1	0, 0
	S	0, 0	1, 2

Suppose p2 plays $x^2 = (q, 1-q)$. The utilities of p1 using pure strategies are: $u_1(B, x^2) = 2q$, $u_1(S, x^2) = 1 - q$. Depending on q, the strategies with maximum utility are different.

1. If $2q < 1 - q$, then $q < \frac{1}{3}$, and B is not in the support and gets probability 0. BRF $\{(0, 1)\}$.
2. If $2q = 1 - q$, then $q = \frac{1}{3}$, and both B, S could be in the support. Any combination works, so BRF $\{(p, 1-p) : p \in [0, 1]\}$.
3. If $2q > 1 - q$, then $q > \frac{1}{3}$, and S is not in the support. BRF $\{(1, 0)\}$.

This matches the BRF we calculated previously.

Example:

Consider a 2-player game with this payoff table. Suppose p2 plays $x^2 = (0, \frac{1}{3}, \frac{2}{3})$. What is $B_1(x^2)$?

	D	E	F
A	2, 2	3, 3	1, 1
B	3, 1	0, 4	2, 1
C	3, 4	5, 1	0, 7

$$u_1(A, x^2) = 0 + \frac{1}{3} \cdot 3 + \frac{2}{3} \cdot 1 = \frac{5}{3}$$
$$u_1(B, x^2) = 0 + 0 + \frac{2}{3} \cdot 2 = \frac{4}{3}$$
$$u_1(C, x^2) = 0 + \frac{1}{3} \cdot 5 + 0 = \frac{5}{3}$$

By support characterization, $x^1_B = 0$. Any distribution over x^1_A and x^1_C works.

So $B_1(x^2) = \{(p, 0, 1-p) : p \in [0, 1]\}$.

The maximum utility for p1 is $p \cdot \frac{5}{3} + (1-p) \cdot \frac{5}{3} = \frac{5}{3}$, which is equal to the max utility for a pure strategy.

Any strategy in $B_1(x^2)$ maximizes utility for p1. Which of these maximizes utility for p2? This will give a NE.

Suppose $x^1 = (p, 0, 1-p)$. Calculate the utilities for p2: $u_2(D, x^1) = 4 - 2p$, $u_2(E, x^1) = 1 + 2p$, $u_2(F, x^1) = 7 - 6p$. If $x^2 = (0, \frac{1}{3}, \frac{2}{3})$ is in the best response, then E, F must have maximum utility. $1 + 2p = 7 - 6p$, so $p = \frac{3}{4}$. Utility for E, F is $\frac{5}{2}$. Utility for D is also $\frac{5}{2}$, so indeed E, F have max utility. (So does D, but this is fine.)

So $x^1 = (\frac{3}{4}, 0, \frac{1}{4})$ and $x^2 = (0, \frac{1}{3}, \frac{2}{3})$ are in the best responses for each other, and (x^1, x^2) is a NE.

Note:
One "algorithm" for finding NE is by looking at possible combinations of the supports for each player. In example above, if we ask "suppose support for p1 is $\{A, C\}$ and support for p2 is $\{E, F\}$" then we can use support characterization to find a NE or prove that none exist for these supports.

Problem: There are exponentially many support sets each player ($\sim 2^k$ if there are k pure strategies). Not practical.

Exercise:
Show that in the game of rock paper scissors, both players playing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is the only Nash equilibrium.

2.6 Voting game

Downs paradox Voting has costs. The probability that one vote is a decisive vote is very small. Costs outweigh benefits.

Expectation: People don't vote. Reality: people do vote.

Model for voter participation

Suppose there are two candidates A, B, and the number of supporters are a, b, respectively.

WLOG, assume $a \geq b$. Each person can choose to "vote" or "abstain". If they vote, then they incur a cost of c where $0 < c < 1$. Regardless voting or abstaining, each person gets a payoff of 2 if their supporting candidate wins, 1 for a tie, 0 for a loss.

Pure NE

Suppose $a = b = 1$.

		PII (B) A	PII (B) V
PI (A)	A	1, 1	0, 2-c
	V	2-c, 0	1-c, 1-c

It's like prisoner's dilemma: both players vote, get lower utility than both players abstain.

Now suppose $a = b \geq 2$. 4 cases:

1. Everyone votes. There is a tie, everyone has utility $1 - c$, switching gives 0. NE
2. Not everyone votes, and there is a tie. One who abstains can vote, $1 \to 2 - c > 1$. Not NE
3. One candidate wins by 1 vote. One who abstains for the losing candidate can vote, $0 \to 1 - c > 0$. Not NE
4. One candidate wins by at least 2 votes. One who votes for the winning candidate can abstain, $2 - c \to 2$. Not NE

In a close election, we expect more people to vote.

Exercise:
Show that when $a > b$, there is no pure Nash equilibrium.

Mixed NE

Then we consider mixed Nash equilibrium: one possible scenario for a mixed NE.

Suppose $a > b$. Among all A supporters, b of them will vote and $a - b$ of them will abstain. Suppose every B supporter will vote with the same probability p. So the best that B can do is a tie. It is easy to check that $p = 0$ or $p = 1$ is not a NE. Assume $p \in (0, 1)$.

Consider a B supporter. If they abstain, then B cannot win. So utility of "abstain" as pure strategy is 0. If they vote, then B ties only if all other B supporters vote (utility $1 - c$), otherwise B loses (utility $-c$). Expected utility of "vote" as pure strategy is

$$\underbrace{p^{b-1}}_{b-1\text{ vote}} \underbrace{(1-c)}_{\substack{\text{utility}\\\text{of a tie}}} + \underbrace{(1-p^{b-1})}_{\substack{\text{not all}\\ b-1\text{ vote}}} \underbrace{(-c)}_{\substack{\text{utility of}\\\text{a loss}}} = p^{b-1} - c$$

When is it possible that this is in a NE? $p \in (0, 1)$, so both strategies have positive probabilities. To be in the best response, support characterization implies the two utilities are equal. So $0 = p^{b-1} - c$, or $p = c^{\frac{1}{b-1}}$.

Given this p, are A supporters incentivized to change their mixed strategies? Currently, all of them are playing pure strategies. In order to switch, the utility of switching to the other pure strategy must be greater.

1. Consider an A who abstained. Expected utility is $\underbrace{p^b}_{b\text{ vote}} \cdot \underbrace{1}_{\substack{\text{utility}\\\text{of a tie}}} + \underbrace{(1-p^b)}_{<b\text{ vote}} \cdot \underbrace{2}_{\substack{\text{utility of}\\\text{a win}}} = 2 - p^b$

   Expected utility of voting is $2 - c$ (A guaranteed to win). $2 - c = 2 - p^{b-1} \leq 2 - p^b \quad (0 < p < 1)$
   .

   Switching to a pure strategy does not increase utility. So switching to any mixed strategy does not increase utility. No reason to switch.

2. Consider an A supporter who voted. Expected utility is $\underbrace{p^b(1-c)}_{\text{tie}} + \underbrace{(1-p^b)(2-c)}_{\text{win}} = 2 - p^b - c$

   If they abstain...

   - A loses if all B supporters vote;
   - A ties if $b - 1$ B supporters vote, 1 abstain;
   - A wins otherwise.

   Utility of abstaining is

   $$p^b \cdot 0 + \underset{\substack{\text{choices of}\\\text{who abstains}}}{b} \cdot \underset{b-1\text{ votes}}{p^{b-1}} \cdot \underset{1\text{ abstain}}{(1-p)} \cdot 1 + \underbrace{(1 - p^b - b \cdot p^{b-1} \cdot (1-p))}_{\text{remaining probability}} \cdot 2 = 2 - 2p^b - bp^{b-1}(1-p)$$

   and we know: $2 - p^b - c \geq 2 - 2p^b - bp^{b-1}(1-p)$. No reason to switch.

When $p = c^{\frac{1}{b-1}}$, this is a mixed NE.

Q What happens to voter participation as cost increase?

If c increase, then p increase, so more voters will vote.

2.7 Two-player zero-sum game

zero-sum

A strategic game is a **zero-sum** game if for all strategy profiles $s \in S$, $\sum_{i \in N} u_i(s) = 0$.

Examples: Matching pennies and rock paper scissors.

For a two-player zero-sum game, let $s_1 = \{1, \ldots, m\}$ and $s_2 = \{1, \ldots, n\}$. Define such a game with a payoff matrix $A \in \mathbb{R}^{m \times n}$ where $u_1(i,j) = A_{ij}$ and $u_2(i,j) = -A_{ij}$.

Example:

PII

PI		1	2	3
	1	3	5	-2
	2	-5	7	1

$= A$

payoff for PI

PII

PI		1	2	3
	1	-3	-5	2
	2	5	-7	-1

$= -A$

payoff for PII

Note:
For a mixed strategy profile $x = (x^1, x^2)$, $u_1(x^1, x^2) = -u_2(x^1, x^2)$.

We use min-max argument for finding a NE: Given a strategy that we play, the opposing player will maximize their utility, which maximizes our utility. Knowing how they would play, what can we do to maximize our own utility?

Player I's perspective: Suppose player I plays x^1. They expect player II to play from their best response.

PII's expected utility for playing pure strategy j is $-(x^1)^T A_{\cdot j}$ ($A_{\cdot j}$ is the j-th column of A)

Utility of PII's best response is equal to the maximum of these values,

$$\max_{j \in \{1,\ldots,n\}} -(x^1)^T A_{\cdot j} = -\min_{j \in \{1,\ldots,n\}} (x^1)^T A_{\cdot j}$$

So utility for PI is $\min_{j \in \{1,\ldots,n\}} (x^1)^T A_{\cdot j}$

PI wants to maximize this:

$$\begin{aligned} \max \quad & \min_{j \in \{1,\ldots,n\}} (x^1)^T A_{\cdot j} \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_i^1 = 1 \\ & x^1 \geq \mathbf{0} \end{aligned}$$

which is not an LP. So we turn it into

$$\begin{aligned} \max \quad & u_1 \\ \text{s.t.} \quad & u_1 \leq (x^1)^T A_{\cdot j} \quad \forall j \in \{1, \ldots, n\} \\ & \sum_{i=1}^{m} x_i^1 = 1 \\ & x^1 \geq \mathbf{0} \end{aligned}$$

Example:
Expected utilities for PII's 3 strategies are

$$\begin{aligned} u_2(1, x^1) &= -3x_1^1 + 5x_2^1, \\ u_2(2, x^1) &= -5x_1^1 - 7x_2^1, \\ u_2(3, x^1) &= 2x_1^1 - x_2^1 \end{aligned}$$

Look for

$$\begin{aligned} &\max\{-3x_1^1 + 5x_2^1, -5x_1^1 - 7x_2^1, 2x_1^1 - x_2^1\} \\ = &\min\{3x_1^1 - 5x_2^1, 5x_1^1 + 7x_2^1, -2x_1^1 + x_2^1\} \end{aligned}$$

$$\begin{aligned} \max \quad & u_1 \\ \text{s.t.} \quad & u_1 \leq 3x_1^1 - 5x_2^1 \\ & u_1 \leq 5x_1^1 + 7x_2^1 \\ & u_1 \leq -2x_1^1 + x_2^1 \\ & x_1^1 + x_2^1 = 1 \\ & x_1 \geq \mathbf{0} \end{aligned}$$

Player II's perspective: Suppose PII plays x^2. Then PI will play from their best response.

Utility of PI's best response is $\max_{i\in\{1,\ldots,m\}} -(x^2)^T A_{i\cdot}$ where $A_{i\cdot}$ is the i-th row of A.

PII's utility is $-\max_{i\in\{1,\ldots,m\}} (x^2)^T A_{i\cdot}$

Maximizing this is equivalent to minimizing $\max_{i\in\{1,\ldots,m\}} (x^2)^T A_{i\cdot}$

PI wants to maximize this:

$$\begin{aligned} \min \quad & \max_{i\in\{1,\ldots,m\}} (x^2)^T A_{i\cdot} \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_j^2 = 1 \\ & x^2 \geq \mathbf{0} \end{aligned}$$

which is not an LP. So we turn it into

$$\begin{aligned} \min \quad & u_2 \\ \text{s.t.} \quad & (x^2)^T A_{i\cdot} \leq u_2 \quad \forall i \in \{1,\ldots,m\} \\ & \sum_{i=1}^{n} x_j^2 = 1 \\ & x^2 \geq \mathbf{0} \end{aligned}$$

Example:
PI's best response has utility

$$\max\{3x_1^2 + 5x_2^2 - 2x_3^2, -5x_1^2 + 7x_2^2 + x_3^2\}$$

Thus

$$\begin{aligned} \min \quad & u_2 \\ \text{s.t.} \quad & 3x_1^2 + 5x_2^2 - 2x_3^2 \leq u_2 \\ & -5x_1^2 + 7x_2^2 + x_3^2 \leq u_2 \\ & x_1^2 + x_2^2 + x_3^2 = 1 \\ & x_2 \geq \mathbf{0} \end{aligned}$$

Exercise:
The LPs for player I and player II are duals of each other.

Both LPs are feasible (take x^1, x^2 to be any probability distribution, u_1, u_2 as max/min values).

So both have optimal solutions with the same objective value. (Note: obj value of PI's LP is the utility of PI, so the obj value of PII's LP is the negative of the utility of PII.) The optimal solutions are best responses to each other, so they form a NE. Solve this using simplex (a modified version of simplex is provably polynomial time).

Theorem 2.11

Assume finite pure strategies, any two-player zero-sum game has a mixed Nash equilibrium, and this can be efficiently computed.

Example:
For our 2 LPs above, an optimal solution is

$$\begin{aligned} &\text{PI:} && x_1^1 = \tfrac{6}{11}, x_2^1 = \tfrac{5}{11}, u_1 = -\tfrac{7}{11} && (u_1 \text{ is the utility of PI}) \\ &\text{PII:} && x_1^2 = \tfrac{3}{11}, x_2^2 = 0, x_3^2 = \tfrac{8}{11}, u_2 = -\tfrac{7}{11} && (-u_2 \text{ is the utility of PII}) \end{aligned}$$

Note:
Computing NE in general is difficult. Even in the 3-player zero-sum game or 2-player general-sum game, no polynomial time algorithm is known.

2.8 Nash's theorem

Theorem 2.12: Nash

Every strategies game with finitely many players and pure strategies has a Nash equilibrium.

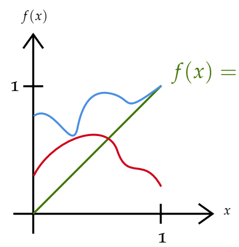
2.8.1 Brouwer's fixed point theorem

Brouwer

Let X be a convex and compact set in a finite-dimensional Euclidean space, and let $f : X \to X$ be a continuous function. Then there exists $x_0 \in S$ such that $f(x_0) = x_0$ ("fixed point")

Example:

Let $X = [0, 1]$. Consider any continuous function $f : [0, 1] \to [0, 1]$.



The graph of f will always intersect $f(x) = x$, producing a fixed point. This is a consequence of the intermediate value theorem (apply to $f(x) - x$)

Terminology from the theorem:

- We will think of an Euclidean space as $\mathbb{R}^n$ with the standard dot product, which defines how we measure distance and angle.
- A set is convex if for any two points in the set, the line segment joining them is also in the set.

  Precise definition: S is convex if for all $u, v \in S$, $\lambda u + (1 - \lambda)v \in S$ for all $\lambda \in [0, 1]$.

  Note: The convex combination of any set of points is convex.

  $$S = \{\lambda_1 v_1 + \ldots + \lambda_n v_n : \lambda_1, \ldots, \lambda_n \geq 0, \lambda_1 + \ldots + \lambda_n = 1\}$$

- A set is compact if it is closed and bounded[1].

Note:
This is a deep theorem from analysis. We will not prove it here, though there are many fascinating proofs of it (suggestion: look into the combinatorial proof using Sperner's Lemma). None of the proofs are constructive: we know that a fixed point exists, but the proofs do not tell us how to find one.

Illustrations

1. Print a world map and place it on your desk. This is a continuous mapping from the surface of Earth to the part of the surface occupied by the map on your desk. The theorem implies there is a fixed point: some point on the map is directly on top of the point it represents on your desk.
2. Take a cup of tea and stir it. Let it settle. Then some part of the liquid is in the same spot before the stir.

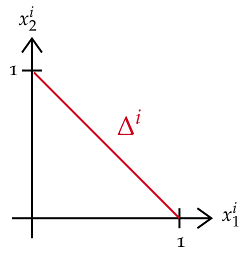
[1] This is not true in general, but it works for a subset of $\mathbb{R}^n$ by The Heine–Borel Theorem. See more in lec 13 and 20 in https://notes.sibeliusp.com/pdfs/1201/amath331.pdf

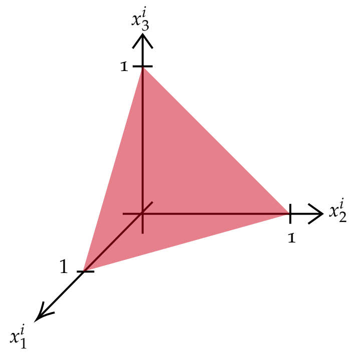
Relation to strategic games

We want to use Brouwer's fixed point theorem when X is the set of all mixed strategy profiles of a finite strategic game. Need to verify that Δ is convex and compact.

Start with just one player i and their set of mixed strategies Δ^i. If the set of pure strategies is $\{1,\ldots,k\}$, then $\Delta^i = \{(x_1^i,\ldots,x_k^i) : x_j^i \geq 0, x_1^i + \ldots + x_k^i = 1\}$

$k = 2 : \Delta^i = \{(p, 1-p) : p \in [0,1]\}$ $\qquad$ $k = 3 : \Delta^i = \{(p,q,r) : p+q+r = 1, 0 \leq p,q,r \leq 1\}$



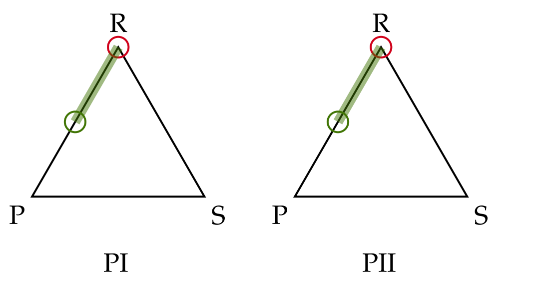


In the case of $k = 3$, it is a triangle, that's why we call it Δ. We can see (without proof) that Δ^i is compact: it is closed and any 2 points have distance at most 1. Δ^i is convex: it is the convex combination of the standard basis vectors $e_1,\ldots,e_k$. (An element of Δ^i has the form $x_1^i e_1 + \ldots + x_k^i e_k$ where $x_1^i + \ldots + x_k^i = 1, x_j^i \geq 0$.) These $e_1,\ldots,e_k$ are the pure strategies of player i.

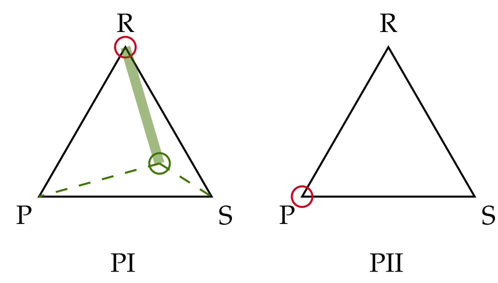
The set of all strategy profiles is $\Delta = \Delta^1 \times \cdots \Delta^n$. We can "pretend" that this is a set in $\mathbb{R}^{|S_1|+\cdots+|S_n|}$. It is still compact (a result, Tychonoff's Theorem, from analysis is that the cartesian product of compact sets is compact). It is also convex. So we can use Δ as the set in Brouwer's fixed point theorem. Now we need to find a continuous function $f : \Delta \to \Delta$ that relates fixed points to mixed Nash equilibria.

Given a strategy profile $x = (x^1,\ldots,x^n)$, a player i will look at possibly switching to a pure strategy to gain utility against x^{-i}. If pure strategy s improves utility, then player i wants to shift the probability distribution so that s receives higher probability. The function will take x, and map it to another strategy profile where each player improves their utility.

Example: Rock paper scissors



Suppose both play rock as a pure strategy.
They can increase utility by moving toward paper.



Suppose PI plays rock, PII plays paper. PII cannot improve utility by moving to paper or scissors. PI will move more towards scissors than paper.

What is the meaning of a fixed point? No player can improve their utility. So it must be a Nash equilibrium.

2.8.2 Defining the function

First define Φ which records the improvement of a player in switching to a pure strategy. Given strategy profile $x \in \Delta$, a player i, and a pure strategy $s \in S_i$, define $\Phi_s^i(x) = \max\{0, u_i(s, x^{-i}) - u_i(x)\}$. If playing s increases utility for player i, then $\Phi_s^i(x)$ represents this increase. Otherwise $\Phi_s^i(x) = 0$.

For player i and strategies s where $\Phi_s^i(x) > 0$, we want to increase probability on s. We want to replace x_s^i by $x_s^i + \Phi_s^i(x)$. But the sum of probabilities is greater than 1. We can normalize this by dividing by $\sum_{s' \in S_i}(x_{s'}^i + \Phi_{s'}^i(x)) = 1 + \sum_{s' \in S_i} \Phi_{s'}^i(x)$

We define $f : \Delta \to \Delta$ by $f(x) = \bar{x}$ where for each player i and strategy $s \in S_i$, $\bar{x}_s^i = \dfrac{x_s^i + \Phi_s^i}{1 + \sum_{s' \in S_i} \Phi_{s'}^i(x)}$

We can verify that $f(x) \in \Delta$.

Example:

In rock paper scissors where PI plays rock and PII plays paper, the strategy profile is $x = ((1, 0, 0), (0, 1, 0))$. For PII, $\Phi_s^2(x) = 0$ for each $s \in \{R, P, S\}$. For PI, $\Phi_R^1(x) = 0, \Phi_P^1(x) = 1, \Phi_S^1(x) = 2$. So the new strategy for PI is

$$\bar{x}_R^1 = \frac{1+0}{1+3} = \frac{1}{4}, \quad \bar{x}_P^1 = \frac{0+1}{1+3} = \frac{1}{4}, \quad \bar{x}_S^1 = \frac{0+2}{1+3} = \frac{1}{2}.$$

Thus, $f(x) = \left((\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (0, 1, 0)\right)$.

2.8.3 Completing the proof of Nash's theorem

Given $x \in \Delta$, consider Φ and $f : \Delta \to \Delta$ defined above. We see that f is continuous since Φ is continuous. By Brouwer's fixed point theorem, there exists $\hat{x} \in \Delta$ such that $f(\hat{x}) = \hat{x}$. We prove that $\hat{x}$ is a NE by showing $\hat{x}^i \in B_i(\hat{x}^{-i})$.

For player i, let $s \in S_i$ be a pure strategy such that $\hat{x}_s^i > 0$ and $u_i(s, \hat{x}^{-i}) \leq u_i(\hat{x})$. (Exercise: show such s exists.) Then $\Phi_s^i(\hat{x}) = 0$. Since $\hat{x}$ is a fixed point, $\hat{x}_s^i = (f(\hat{x}))_s^i = \hat{x}_s^i / (1 + \sum_{s' \in S_i} \Phi_{s'}^i(\hat{x}))$. Since $\hat{x}_s^i > 0$, the denominator must be 1. So $\sum_{s' \in S_i} \Phi_{s'}^i(\hat{x}) = 0$. But Φ is non-negative, so $\Phi_{s'}^i(\hat{x}) = 0$ for all $s' \in S_i$. This means that $u_i(s', \hat{x}^{-i}) \leq u_i(\hat{x})$ for all $s' \in S_i$. So playing $\hat{x}^i$ gives the highest utility against $\hat{x}^{-i}$, so $\hat{x}^i \in B_i(\hat{x}^{-i})$. Since this holds for all players, $\hat{x}$ is a Nash equilibrium. □

Note:
This proves that a NE always exists, but the proof does not show us how to find such a NE, as it depends on Brouwer's fixed point theorem.

3

Cooperative games

3.1 Introduction

There are games where group of players can work together to obtain higher utility.

Example: Ice cream

Alice, Bob, Carol want to buy ice cream. Three sizes: 1L, 1.5L, 2L with costs \$6, \$9, \$11 respectively. A has \$3, B has \$4, C has \$5. On their own, they cannot buy any. But if they pool money together, they can get some ice cream. (e.g. B + C can buy 1.5L)

cooperative game

A **cooperative game** is given by a set of players N and a characteristic function $v : 2^N \to \mathbb{R}$ that assigns a value $v(S)$ to each coalition $S \subseteq N$ of players. We use (N, V) to represent this game. The set N is the **grand coalition**.

Example:

In the ice cream game, $N = \{A, B, C\}$, and v is defined by

S	$\varnothing, \{A\}, \{B\}, \{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	$\{A, B, C\}$
$v(S)$	0	1	1	1.5	2

General assumptions: $v(\varnothing) = 0$, $v(S) \geq 0$ for all $S \subseteq N$.

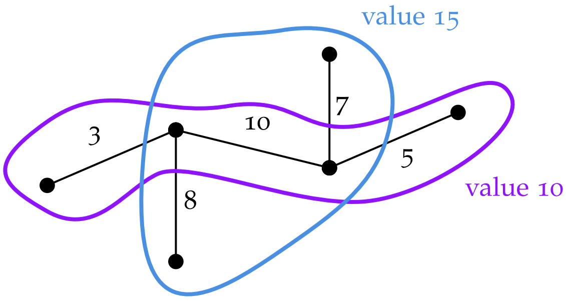
Example: 101-member parliament

A country has a 101-member parliament. There are 4 parties A, B, C, D with 40, 22, 30, 9 members, respectively. They need to decide how to spend a \$1 billion windfall, they need to form a majority to spend it. Thus $N = \{A, B, C, D\}$ and

$$v(S) = \begin{cases} 10^9 & \text{parties in } S \text{ have} \geq 51 \text{ members} \\ 0 & \text{otherwise} \end{cases}$$

Example: matching game

In a matching game, we are given a graph $G = (V, E)$ and edge weights $\omega : E \to \mathbb{R}$. The players are the vertices, $V = N$. The weight of an edge represents the benefits if two vertex players work together. For any subset $S \subseteq N$, the value is the maximum weight of a matching using vertices in S.



Outcomes of cooperative games

Outcomes of Strategic games: Strategy profiles (pure or mixed). Which strategy is played by each player?

Outcomes of Cooperative games:

1. Divide the players into groups, we call them coalitions. "coalition structure"

   Each coalition will generate their assigned value.

2. Distribute the value that each coalition generates among its members. "payoff vector"

coalition structure

Given a cooperative (N, v), a **coalition structure** is a partition π of N, i.e., $\pi = (C^1, \ldots, C^k)$ where each $C^i \subseteq N$, $C^i \cap C^j = \varnothing$ whenever $i \neq j$, and $C^1 \cup \cdots \cup C^k = N$.

payoff vector

A **payoff vector** is a vector $x \in \mathbb{R}^n$ such that $x \geq \mathbf{0}$ and

$$\sum_{i \in C^j} x_i \leq v(C^j)$$

for all $j = 1, \ldots, k$.

Notation: For any $T \subseteq N$, $x(T) = \sum_{i \in T} x_i$. So the inequality here is $x(C^j) \leq v(C^j)$.

efficient outcome

An **outcome** consists of (π, x). Such an outcome is **efficient** if $x(C^j) = v(C^j)$ for all j.

Example:
An outcome of the ice cream game is (π, x) where $\pi = (\{A, B\}, \{C\})$, and $x_A = x_B = 0.5$, $x_C = 0$. This outcome is efficient: $v(\{A, B\}) = 1 = x_A + x_B$, $v(\{C\}) = 0 = x_C$.

Some classes of games

1. Monotone games: $S \subseteq T \implies v(S) \leq v(T)$. "more games produce more value"
2. Superadditive games: for disjoint S, T, $v(S) + v(T) \leq v(S \cup T)$.

   "forming coalitions is always better"

   Superadditive $\implies$ monotome, converse is not true.

We usually only consider the grand coalition: $\pi = (N)$.

3. Convex games: for any S, T, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. (supermodularity inequality)

   Convexity $\implies$ superadditivity, converse is not true.

Proposition 3.1

A game (N, v) is convex if and only if for every S, T where $S \subseteq T \subseteq N$ and for every player $i \in N \setminus T$,

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$$

"A player is more useful in larger coalitions"

3.2 Shapley values

Two desirable properties of an outcome in cooperative games:

1. Fairness. The payoff vector should reflect the contribution of the players to their coalitions.
2. Stability. We want to incentivize the players to stay in their assigned coalition in the coalition structure.

Shapley values deal with the fairness of the payoff vector. Assume players form the grand coalition. (If not, look at individual coalitions separately.)

How to quantify a player's contribution?

Idea 1: Compare the value of the coalition before and after joins it.

Example: Ice cream game. The contribution of A is $v(\{A, B, C\}) - v(\{B, C\}) = 0.5$

Problem: The sum of the payoffs may exceed the value of coalition $x(N) > v(N)$.

Idea 2: Fix a sequence of players, and see their contribution sequentially.

Example: Use sequence A, B, C. $v(\{A\}) = 0$, so A contributes 0. $v(\{A, B\}) = 1$, so B contributes 1. $v(\{A, B, C\}) = 2$, so C contributes 1. This is efficient, $x(N) = v(N)$.

Problem: Different orderings produce different results.

Shapley's idea: Look at all possible orderings of players, average a player's contributions.

S_N

A permutation of N has the form $\sigma = (\sigma_1, \ldots, \sigma_n)$ where each σ_i is a distinct element of N. The element σ_i is at the i-th position of σ. The set of all permutations of N is denoted S_N.

marginal contribution

Given a permutation $\sigma \in S_N$, the **marginal contribution** of player σ_i is

$$\Delta_\sigma(\sigma_i) = v(\{\sigma_1, \ldots, \sigma_i\}) - v(\{\sigma_1, \ldots, \sigma_{i-1}\})$$

Shapley value

The **Shapley value** of player i is

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(i)$$

Example:

In the ice cream game, $N = \{A, B, C\}$, and there are 6 permutations:

$$\sigma^1 = (A, B, C), \quad \sigma^2 = (A, C, B), \quad \sigma^3 = (B, A, C),$$

$$\sigma^4 = (B, C, A), \quad \sigma^5 = (C, A, B), \quad \sigma^6 = (C, B, A)$$

We calculate the marginal contribution of A in each of the permutations:

$$\begin{aligned}
\Delta_{\sigma^1}(A) &= v(\{A\}) - v(\varnothing) = 0 \\
\Delta_{\sigma^2}(A) &= v(\{A\}) - v(\varnothing) = 0 \\
\Delta_{\sigma^3}(A) &= v(\{B, A\}) - v(\{B\}) = 1 \\
\Delta_{\sigma^4}(A) &= v(\{B, C, A\}) - v(\{B, C\}) = 0.5 \\
\Delta_{\sigma^5}(A) &= v(\{C, A\}) - v(\{C\}) = 1 \\
\Delta_{\sigma^6}(A) &= v(\{C, B, A\}) - v(\{C, B\}) = 0.5
\end{aligned}$$

So the Shapley value for A is $\varphi_A = \frac{1}{6}(0 + 0 + 1 + 0.5 + 1 + 0.5) = \frac{1}{2}$

Other Shapley values; $\varphi_B = \varphi_C = \frac{3}{4}$

4 good properties of Shapley values

1. **Efficiency**: it distributes $v(N)$ to all players.

Proposition 3.2

$$\sum_{i \in N} \varphi_i = v(N)$$

Proof:

For any $\sigma \in S_N$, the sum of all marginal contributions is

$$\begin{aligned}
\sum_{i \in N} \Delta_\sigma(i) &= \sum_{i=1}^{n} \Delta_\sigma(\sigma_i) \qquad \text{since permutation is a bijection} \\
&= [v(\{\sigma_1\}) - v(\varnothing)] + [v(\{\sigma_1, \sigma_2\}) - v(\{\sigma_1\})] + \cdots + [v(\{\sigma_1, \ldots, \sigma_n\}) - v(\sigma_1, \ldots, \sigma_{n-1})] \\
&= v(\{\sigma_1, \ldots, \sigma_n\}) - v(\varnothing) \\
&= v(N)
\end{aligned}$$

So the sum of Shapley values is

$$\sum_{i \in N} \varphi_i = \sum_{i \in N} \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(i) = \frac{1}{n!} \sum_{\sigma \in S_N} \sum_{i \in N} \Delta_\sigma(i) = \frac{1}{n!} \sum_{\sigma \in S_N} v(N) \underset{|S_N| = n!}{=} \frac{1}{n!}(n!)v(N) = v(N)$$

$\square$

2. **Symmetric**.

symmetric

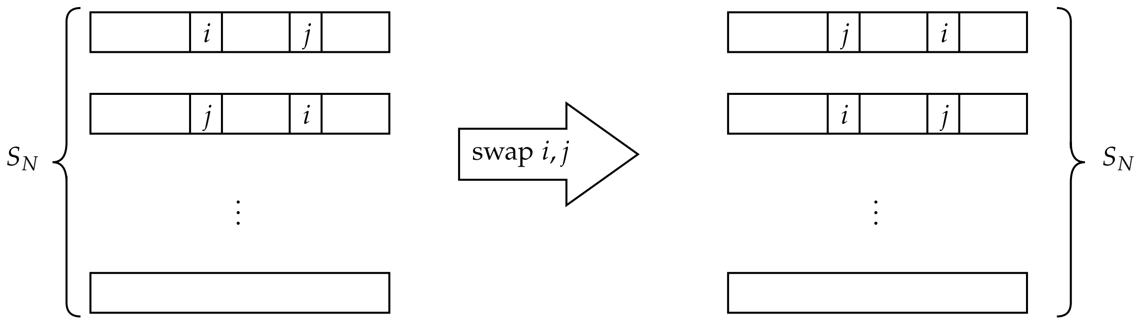
Two players i, j are **symmetric** if $v(C \cup \{i\}) = v(C \cup \{j\})\ \forall N \setminus \{i, j\}$. (they contribute to coalitions equally)

Example:

In the ice cream game, B, C are symmetric. $v(\varnothing \cup \{B\}) = v(\varnothing \cup \{C\}) = 0$, $v(\{A\} \cup \{B\}) = v(\{A\} \cup \{C\}) = 1$.

Proposition 3.3

If i, j are symmetric players, then $\varphi_i = \varphi_j$.



calculate marg cont of i here = calculate marg cont of j here

Proof:

Define $f : S_N \to S_N$ where $f(\sigma)$ is obtained from σ by swapping i and j. This is a bijection $f^{-1} = f$. We claim $\Delta_\sigma(i) = \Delta_{f(\sigma)}(j)$. Two cases:

- Suppose i precedes j in σ. Let C be all elements preceding i. In $f(\sigma)$, C is also the elements preceding j. So

$$\Delta_\sigma(i) = v(C \cup \{i\}) - v(C) \text{ and } \Delta_{f(\sigma)}(j) = v(C \cup \{j\}) - v(C)$$

  Since $C \subseteq N \setminus \{i, j\}$ and i, j are symmetric, $v(C \cup \{i\}) = v(C \cup \{j\})$. So $\Delta_\sigma(i) = \Delta_{f(\sigma)}(j)$.

- Suppose j precedes i in σ. Let C be all elements preceding i except j. In $f(\sigma)$, C is also the elements that precedes j except i. So

$$\Delta_\sigma(i) = v(C \cup \{j\} \cup \{i\}) - v(C \cup \{j\}) \text{ and } \Delta_{f(\sigma)}(j) = v(C \cup \{i\} \cup \{j\}) - v(C \cup \{i\})$$

  Since $C \subseteq N \setminus \{i, j\}$ and i, j are symmetric, $v(C \cup \{j\}) = v(C \cup \{i\})$, so $\Delta_\sigma(i) = \Delta_(f(\sigma))(j)$.

Therefore,

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(i) \underset{\text{from above}}{=} \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_{f(\sigma)} j \underset{\text{since } f \text{ is a bijection}}{=} \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(j) = \varphi(j)$$

□

Example: Unanimity Game

Suppose $|N| = n$ and $v(S) = \begin{cases} 1 & S = N \\ 0 & \text{otherwise} \end{cases}$

Any pair of players is symmetric, so $\varphi_i = \varphi_j$ for any i, j. Since φ is efficient, the sum is $v(N) = 1$. So $\varphi_i = \frac{1}{n}$ for each $i \in N$.

3. **Dummy player.**

dummy player

i is a **dummy player** if $v(S \cup \{i\}) = v(S), \forall S \subseteq N \setminus \{i\}$. The player does not contribute to any coalition.

Example: 101-seat parliament

A, B, C, D with 40, 22, 30, 9 seats. Party D is a dummy player: no combination of A, B, C exists where it is not a majority, but adding 9 gives a majority.

Proposition 3.4

If i is a dummy player, then $\varphi_i = 0$.

Proof:

For any $\sigma \in S_N$, say i is at the j-th position ($\sigma_j = i$), the marginal contribution of i is $\Delta_\sigma(i) = v(\{\sigma_1, \ldots, \sigma_{j-1}, i\}) - v(\{\sigma_1, \ldots, \sigma_{j-1}\}) = 0$ by definition of a dummy player. So

$$\varphi_i = \frac{1}{n!} \sum_{\sigma \in S_N} \Delta_\sigma(i) = 0$$

$\square$

Note:

The converse is not true. If a game is monotone, then the converse is true.

4. **Additivity**: Suppose there are multiple games with the same set of players. We add the values together to get a new game. Then the Shapley values are also added together.

Proposition 3.5

Let $(N, v^1), (N, v^2)$ be two cooperative games. Define $v^3(S) = v^1(S) + v^2(S), \forall S \subseteq N$. Let φ_i^j be the Shapley values of player i in (N, v^j), $j = 1, 2, 3$. Then $\varphi_i^3 = \varphi_i^1 + \varphi_i^2$ for all i.

Summary The Shapley values satisfy 4 good properties: efficiency, symmetric, dummy player, additivity. *Deep result*: The Shapley value function is the only one that satisfies all 4 properties. (If f is a function that maps (N, v) to a real vector $\mathbb{R}^n$ and all properties hold, then f gives the Shapley values.)

3.3 The core

Stability: Given an outcome, what would be a reason that players want to deviate from it? A group of players could generate more value than what they are receiving. $x(C) < v(C)$

core

The **core** of a cooperative game (N, v) consists of all outcomes (π, x) such that $x(C) \geq v(C)$ for all $C \subseteq N$.

Example: Ice cream game

Consider (π, x) with $\pi = (\{A, B\}, \{C\})$ and $x_A = 0.5, x_B = 0.5, x_C = 0$. If C joins with $\{A, B\}$, then they produce value 2, while currently their combined payoffs is 1. Better if they form $\{A, B, C\}$, not in the core.

Same reasoning gives $\pi = (N)$ if (π, x) is in the core. If $x_A = 2, x_B = x_C = 0$, then $\{B, C\}$ can get more value. If $x_A = 0, x_B = x_C = 1$, then (π, x) is in the core. This satisfies these inequalities:

$$x_A + x_B + x_C \geq 2, \quad x_A + x_B \geq 1, \quad x_A + x_C \geq 1, \quad x_B + x_C \geq 1.5, \quad x_A \geq 0, \quad x_B \geq 0, \quad x_C \geq 0$$

3.3.1 Properties of the outcomes in the core

1. *They are efficient within the coalition structure.*

Proposition 3.6

If (π, x) is in the core, then $x(C) = v(C)$ for each $C \in \pi$.

Proof:
Let $C \in \pi$. By the definition of the core, $x(C) \geq v(C)$. Since (π, x) is a valid outcome, $x(C) \leq v(C)$. Therefore, $x(C) = v(C)$ for all $C \in \pi$. □

2. *The coalition structure generates the maximum amount of total value among all outcomes.* "social welfare"

$v(\pi)$

$$v(\pi) = \sum_{C \in \pi} v(C)$$

Proposition 3.7

If (π, x) is in the core, then $v(\pi) \geq v(\pi')$ for all partitions π' of N.

Proof:

$$v(\pi) = \sum_{C\in\pi} v(C) \underset{\substack{\uparrow \\ \text{By Proposition 3.6}}}{=} \sum_{C\in\pi} x(C) \underset{\substack{\uparrow \\ \text{Since } \pi \text{ is a} \\ \text{partition of } N}}{=} \sum_{i\in N} x_i \underset{\substack{\uparrow \\ \text{Since } \pi' \text{ is a} \\ \text{partition of } N}}{=} \sum_{C'\in\pi'} x(C') \underset{\substack{\uparrow \\ \text{Since } (\pi,x) \\ \text{is in the core}}}{\geq} \sum_{C'\in\pi'} v(C') = v(\pi')$$ □

Note:
This proposition only says that coalition structures that maximize total value are eligible to be in the core. It does not mean that there exists an outcome in the core with this structure.

3.3.2 Games with empty cores

Example: 3-player majority game

$$N = \{1, 2, 3\},\ v(S) = \begin{cases} 1 & |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

We claim that no outcome is in the core. Suppose (π, x) is in the core. Then $x_1 + x_2 + x_3 \geq 1$, $x_1, x_2, x_3 \geq 0$. So $x_i \geq \frac{1}{3}$ for some i. The value of any coalition structure is at most 1, so $x_1 + x_2 + x_3 \leq 1$. This means $x(N \setminus \{i\}) \leq \frac{2}{3}$. However, $v(N \setminus \{i\}) = 1 > x(N \setminus \{i\})$. This contradicts the assumption that (π, x) is in the core, which implies core is empty.

Main question: which games have non-empty cores?

3.3.3 Cores of superadditive games

Goal: Determine when a superadditive game has a non-empty core. We can narrow the search: we only need to consider outcomes that form the grand coalition.

Proposition 3.8

Let (N, v) be a superaddtive game. If (π, x) is in the core, then $((N), x)$ is in the core.

Proof:
We need to prove: the core conditions holds, and $((N), x)$ is a valid outcome.

Since (π, x) is in the core, $x(C) \geq v(C)$ for all $C \subseteq N$. This still holds for $((N), x)$.

To show that $((N), x)$ is a valid outcome, we need to show that $x(N) \leq v(N)$.

$$x(N) \underset{\substack{\uparrow \\ \text{Since } \pi \text{ is a} \\ \text{partition of } N}}{=} \sum_{C\in\pi} x(C) \underset{\substack{\uparrow \\ \text{Since } (\pi, x) \text{ is} \\ \text{a valid outcome}}}{\leq} \sum_{C\in\pi} v(C) \underset{\substack{\uparrow \\ \text{Superadditivity}}}{\leq} v(N)$$

□

Example: Unanimity game

$$v(S) = \begin{cases} 1 & S = N \\ 0 & \text{otherwise} \end{cases}$$

To determine if the core is non-empty, we only need to consider $((N), x)$. x satisfies $\sum_{i\in N} x_i = 1$ (Proposition 3.6) and $\sum_{i\in S} x_i \geq 0$ for all $S \subseteq N$. e.g., $x_i = \frac{1}{n}, \forall i$, or $x_1 = 1, x_i = 0$ if $i \neq 1$.

Characterizing superaddtive games with non-empty cores

Given an outcome $((N), x)$, what must x satisfy to be in the core? We claim that x must be in the set

$$\mathcal{C} = \{x \in \mathbb{R}^N : \underset{\substack{\uparrow \\ \text{Proposition 3.6}}}{x(N) = v(N)}, \quad \underset{\substack{\uparrow \\ \text{Definition of the core}}}{x(C) \geq v(C)} \quad \forall C \subseteq N\}$$

Example: Ice cream game

$$\mathcal{C} = \left\{ x \in \mathbb{R}^N : \begin{array}{l} x_A + x_B + x_C = 2, \\ x_A \geq 0, \\ x_B \geq 0, \\ x_C \geq 0, \\ x_A + x_B \geq 1, \\ x_A + x_C \geq 1, \\ x_B + x_C \geq 1.5, \\ x_A + x_B + x_C \geq 2 \end{array} \right\}$$

Now $\mathcal{C}$ is the intersection of halfspaces, so it is a polyhedron. *Mini-result*: (N, v) has a non-empty core if and only if $\mathcal{C}$ is non-empty.

We can solve the problem of "is $\mathcal{C}$ non-empty" using a linear program.

Let (P) be the following LP:

$$\begin{aligned} \min \quad & x(N) \\ \text{s.t.} \quad & x(C) \geq v(C) \quad \forall C \subseteq N \end{aligned}$$

Take the dual (D):

$$\begin{aligned} \max \quad & \sum_{C\subseteq N} y_C v(C) \\ \text{s.t.} \quad & \sum_{C\subseteq N, i\in C} y_C = 1 \quad \forall i \in N \\ & y \geq \mathbf{0} \end{aligned}$$

Example:

$$\begin{array}{llllll}
\min & x_A & +x_B & +x_C & \\
\text{s.t.} & x_A & & & \geq 0 \\
& & x_B & & \geq 0 \\
& & & x_C & \geq 0 \\
& x_A & +x_B & & \geq 1 \\
& x_A & & +x_C & \geq 1 \\
& & x_B & +x_C & \geq 1.5 \\
& x_A & +x_B & +x_C & \geq 2
\end{array}$$

$$\begin{array}{lllllllll}
\max & & & & y_{AB} & +y_{AC} & +1.5y_{BC} & +2y_{ABC} & \\
\text{s.t.} & y_A & & & +y_{AB} & +y_{AC} & & +y_{ABC} & = 1 \\
& & y_B & & +y_{AB} & +y_{AC} & & +y_{ABC} & = 1 \\
& & & y_C & & +y_{AC} & +y_{BC} & +y_{ABC} & = 1 \\
& & & & & & & y & \geq \mathbf{0}
\end{array}$$

Rationale: (P) is feasible (take large x), and $x(N) \geq v(N)$ is a constraint, so (P) is bounded. $\implies$ (P) has an optimal solution. If optimal value is $v(N)$, then we have optimal solution x with $x(N) = v(N)$ and $x(C) \geq v(C)$, $\forall C \subseteq B$, so $x \in \mathcal{C}$.

(P) has an optimal value $v(N)$ $\Leftrightarrow$ (D) has optimal value $v(N)$ $\Leftrightarrow$ $\sum_{C\subseteq N} y_C v(C) \leq v(N)$ for all feasible y.

(Subtle pt: Is it possible that $\sum\limits_{C\subseteq N} y_C v(C) < v(N)$ for all y?)

What is the meaning of the dual?

1. Feasible solution: $\sum\limits_{C\subseteq N, i \in C} y_C = 1$.

   Example:

   $$\underset{(\{A,B\},\{C\})}{\overset{\uparrow}{y_{AB} = y_C = 1}} \text{ or } \underset{(\{A,B,C\})}{\overset{\uparrow}{y_{ABC} = 1}} \text{ or } y_{AB} = y_{BC} = y_{AC} = \tfrac{1}{2}$$

   "fractional" partition
   A works for
   $\{A,B\}$ 1/2 the time,
   $\{A,C\}$ 1/2 the time,
   $\implies$ Total 1 for A

   A feasible solution y is a generalized coalition structure. y_C represents the fraction of time each member will contribute to C, with a total time of 1 from each player. Any such feasible y is called a **balancing weight**.

2. Objective. $\sum\limits_{C\subseteq N} y_C v(C) \leq v(N)$.

   Example:

   $$y_{AB} v(\{A,B\}) + y_C v(\{C\}) \leq 2$$

   $$v(\{A,B\}) + v(\{C\}) \leq 2$$

   value of the coalition structure $(\{A,B\},\{C\})$ — value of the grand coalition (N)

   By Proposition 3.7, maximize social welfare.

   Then $\sum_{C\subseteq N} y_C v(C)$ is the total value of the fractional partition represented by y. Then inequality $\sum_{C\subseteq N} y_C v(C) \leq v(N)$ means the value of the grand coalition is maximum over the values of any fractional partition. This generates Proposition 3.7.

   A game that satisfies this inequality for all balancing weight y is called a **balanced game**.

Theorem 3.9: Bondareva-Shapley

A superaddtive game has a non-empty core if and only if it is balanced.

3.3.4 Game with non-empty cores

In superadditive games with non-empty cores, there is always an outcome in the core with the grand coalition. THis is not necessarily the case for cooperative games in general.

Example:

Let $N = \{1,2,3,4\}$, $v(S) = \begin{cases} 2 & |S| \geq 2 \\ 0 & \text{otherwise} \end{cases}$

By Proposition 3.7, coalition structure in the core has highest value. $v(N) = 2$. But $v(\{1,2\},\{3,4\}) = 4$. So the grand coalition cannot be in any outcome of the core. The core is non-empty: $\pi = (\{1,2\},\{3,4\})$ with $x = (1,1,1,1)$ is in the core.

Checking if the core is non-empty cannot be reduced to checking only the grand coalition. But we can relate this to superadditive games.

superadditive cover

For any cooperative game (N, v), its **superadditive cover** is (N, v^*) where, for each $S \subseteq N$,

$$v^*(S) = \max\{v(\pi) : \pi \text{ is a partition of } S\}$$

Example:

The superaddtive cover for example above is (N, v^*) where $v^*(S) = \begin{cases} 4 & |S| = 4 \\ 2 & |S| = 2,3 \\ 0 & |S| = 0,1 \end{cases}$

For example,

$$v^*(\{1,2,3\}) = \max\{v(\{1,2,3\}), v(\{1\},\{2,3\}), v(\{2\},\{1,3\}), v(\{3\},\{1,2\}), v(\{1\},\{2\},\{3\})\} = 2$$

Then we can prove that superadditive cover is superaddtive.

Proposition 3.10

A cooperative game (N, v) has a non-empty core if and only if its superaddtive cover (N, v^*) has a non-empty core.

Example:

Check that $((N),(1,1,1,1))$ is in the core of (N, v^*) above.

Proof:

($\Rightarrow$) Let (π, x) be in the core of (N, v).

Note that by Proposition 3.7, $v(\pi)$ has the maximum value among all partitions of N. So by the definition of superadditive cover, $v^*(N) = v(\pi)$.

1. $$v^*(N) = v(\pi) = \sum_{C\in\pi} v(C) \underset{\substack{\uparrow \\ \text{Proposition 3.6} \\ (\pi,x)\text{ is in the core}}}{=} \sum_{C\in\pi} x(C) \underset{\substack{\uparrow \\ \text{Since } \pi \text{ is a} \\ \text{partition of } N}}{=} \sum_{i\in N} x_i = x(N)$$

2. Let $C \subseteq N$. Suppose $v^*(C) = v(\pi')$ for some partition π' of C. Then

$$v^*(C) = v(\pi') = \sum_{C'\in\pi'} v(C') \underset{\substack{\uparrow \\ (\pi,x)\text{ is} \\ \text{in the core}}}{\leq} \sum_{C'\in\pi'} x(C') \underset{\substack{\uparrow \\ \text{Since } \pi' \text{ is a} \\ \text{partition of } C}}{=} x(C)$$

So $((N), x)$ is in the core of (N, v^*).

($\Leftarrow$) Since (N, v^*) is superadditive, there exists a payoff vector x such that $((N), x)$ is in the core Proposition 3.8. By the definition of the superadditive cover, $v^*(N) = v(\pi)$ for some partition π of N.

1. It is a valid outcome, $v(C) \geq x(C)$ for all $C \in \pi$; and
2. core condition is satisfied, $v(C) \leq x(C)$ for all $C \subseteq N$. We leave the proof of them as exercises.

□

Corollary 3.11

A cooperative game has a non-empty core if and only if its superadditive cover is balanced.

Proof:
Combine the Bondareva-Shapley theorem with Proposition 3.10. □

3.3.5 Cores of convex games

Example: Bankruptcy game
Bob the Banker is bankrupt. He owes three people money, \$100, \$200, \$300 each. Bob only has \$200. How should Bob's money be divided? \$50, \$50, \$100? \$0, \$0, \$200? which ones are stable?

General model: Bob has \$$M$, he owes money to n people N, amounts owed are $d \in \mathbb{R}^N$. Assume $0 \leq M \leq \sum_{i \in N} d_i$.

Cooperative game: (N, v) where $v(S) = \max\{0, M - \sum_{i \in N \setminus S} d_i\}$ for each $S \subseteq N$.

Meaning: Players are taking a pessimistic view, $v(S)$ is the amount left if the remaining players take what they owed.

With numbers above, $M = 200, d = (100, 200, 300)$. Examples of values: $v(\{2,3\}) = 200 - d_1 = 100, v(\{1,2\}) = 0$.

Exercise:
Show that is a convex game.

Proposition 3.12

Convex games have non-empty cores.

Idea: The marginal contributions of the players in any permutation form the payoff vector in the core.

Proof:
Since convex games are superadditive, it suffices to find x such that $((N), x)$ is in the core (Proposition 3.8). Let $\sigma \in S_N$. WLOG, assume $\sigma = (1, 2, \ldots, n)$. Define x by $x_i = \Delta_\sigma(i)$.

1. In the proof of Proposition 3.2, $x(N) = \sum_{i=1}^n \Delta_\sigma(i) = v(N)$.
2. Let $C \subseteq N$. Suppose $C = \{i_1, \ldots, i_k\}$ where $i_1 < \cdots < i_k$.

   For any i_j, the "equivalent convex condition" implies

$$v(\{1, \ldots, i_j\}) - v(\{1, \ldots, i_j - 1\}) \geq v(\{i_1, \ldots, i_j\}) - v(\{i_1, \ldots, i_{j-1}\})$$

So

$$\begin{aligned}
x(C) &= \sum_{j=1}^{k} \Delta_\sigma\left(i_j\right) \\
&= \sum_{j=1}^{k} \left(v\left(\{1,\dots,i_j\}\right) - v\left(\{1,\dots,i_j-1\}\right)\right) \\
&\geq \sum_{j=1}^{k} \left(v\left(\{i_1,\dots,i_j\}\right) - v\left(\{i_1,\dots,i_j-1\}\right)\right) \\
&= v\left(\{i_1,\dots,i_k\}\right) - v(\varnothing) \\
&= v(C)
\end{aligned}$$

□

Note:
Any vector of marginal contributions is in the core with the grand coalition. The set of all vectors in the core is $\mathcal{C}$, which is a polyhedron, which is convex. The vector of Shapley values φ is a convex combinations of the marginal contributions, so it is also in the core.

Stronger result: $\mathcal{C}$ is precisely equal to the convex combinations of all marginal contribution vectors. (We will not prove this)

Example: Bankruptcy game

Shapley values: $(33+\frac{1}{3}, 83+\frac{1}{3}, 83+\frac{1}{3})$. This is in the core with the grand coalition. Order the players $\sigma = (3,2,1)$, the marginal contributions $(100,100,0)$ is also in the core.

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