Introduction to General Relativity

AMATH 475

Eduardo Martin-martinez

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of AMATH 475 during Winter 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

Some of the notes (especially special relativity part) are projected to the screen instead of using blackboards. They can be found on https://sites.google.com/site/emmfis/teaching/gr.

For any questions, send me an email via https://notes.sibeliusp.com/contact/.

You can find my notes for other courses on https://notes.sibeliusp.com/.

Sibelius Peng

Contents

0

Pre-Math

0.1 Index notation

$$A = \begin{pmatrix} A^1_{\ 1} & A^1_{\ 2} \\ A^2_{\ 1} & A^2_{\ 2} \end{pmatrix} \qquad B = \begin{pmatrix} B^1_{\ 1} & B^1_{\ 2} \\ B^2_{\ 1} & B^2_{\ 2} \end{pmatrix}$$

$$(A \cdot B)^a_{\ b} = A^a_{\ c} B^c_{\ b} = B^c_{\ b} A^a_{\ c} \qquad \text{sum over all possible } c$$

Identify followings:

$$B_\kappa^{\ \nu} A_\mu^{\ \kappa} = A_\mu^{\ \kappa} B_\kappa^{\ \nu} = C_\mu^{\ \nu} = (A \cdot B)_\mu^{\ \nu}$$

$$A^\kappa_{\ \mu} B_\kappa^{\ \nu} = D_\mu^{\ \nu} = (A^T)_\mu^{\ \kappa} B_\kappa^{\ \nu} = (A^T \cdot B)_\mu^{\ \kappa}$$

$$A_\kappa^{\ \nu} B_\mu^{\ \kappa} = E_\mu^{\ \nu} = (B \cdot A)_\mu^{\ \nu}$$

$$A^\kappa_{\ \mu} B^\nu_{\ \kappa} = (A^T)_\mu^{\ \kappa} (B^T)_\kappa^{\ \nu} = \left((B \cdot A)^T \right)_\mu^{\ \nu}$$

$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 \qquad \{\mathbf{e}_1, \mathbf{e}_2\}$ Basis 1.

$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a \qquad \{\mathbf{e}'_1, \mathbf{e}'_2\}$ Basis 2.

Change of basis matrix Λ

$$\mathbf{e}'_a = \Lambda_a^{\ b} \mathbf{e}_b$$

$$v'^a = \tilde{\Lambda}^a_{\ b} v^b$$

$$
\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a{}_b v^b \Lambda_a{}^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a{}_b \Lambda_a{}^c v^b \mathbf{e}_c \\
&= \underbrace{\left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c}_{\delta_b^c} v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b
\end{aligned}
$$

$$
\implies \left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c = \delta_b^c
$$

$$
\tilde{\Lambda}^T \cdot \Lambda = \mathbb{1}
$$

$\tilde{\Lambda}^T$ is the inverse transpose of Λ

---

covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis. Λ. (sub-indices)

A contravariant object transforms like components of vectors. $(\tilde{\Lambda} = (\Lambda^T)^{-1})$. (super-indices)

0.2 Vectors and one-forms

---

one-form

Let V be a vector space. A one-form is a linear map $\boldsymbol{\omega} : V \to \mathbb{R}$.

---

or we write: $(\boldsymbol{\omega}, \cdot) : V \to \mathbb{R}$ and $(\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}$.

---

dual vector space

The set of all one-forms on V (call V^*) is a vector space as well called the dual vector space to V.

---

dual basis

Let $\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2, \ldots\}$ (or $\{\boldsymbol{\Upsilon}_i\}$) be a basis of V so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = v^i \boldsymbol{\Upsilon}_i$.

We define the dual basis (of V^*) to $\{\boldsymbol{\Upsilon}_i\}$ as $\{\boldsymbol{\omega}^i\}$ such that $\boldsymbol{\omega}^i(\boldsymbol{\Upsilon}_j) = \delta_j^i$.

For a one form $\boldsymbol{\omega}$ we denote its "components of the basis $\boldsymbol{\Upsilon}$" as $(\boldsymbol{\omega}, \boldsymbol{\Upsilon}_m) = \omega_m$

> **Proposition 0.1**
>
> The dual basis of V^* is actually a basis of V^*.

The action of $\boldsymbol{\omega} \in V^*$ on a vector $\mathbf{v} = v^\mu \boldsymbol{\Upsilon} \in V$ is

$$(\boldsymbol{\omega}, \mathbf{v}) = (\boldsymbol{\omega}, v^\mu \boldsymbol{\Upsilon}_\mu) = v^\mu \omega_\mu$$

Let's prove $\{\boldsymbol{\Upsilon}^a\}$ is linear independent.

Proof:
A linear comb. $c_a \boldsymbol{\Upsilon}^a$ acts on a vector $\mathbf{v} = v^a \boldsymbol{\Upsilon}_a$

$$\begin{aligned}
(c_a \boldsymbol{\Upsilon}^a, \mathbf{v}) &= c_a (\boldsymbol{\Upsilon}^a, \mathbf{v}) \\
&= c_a (\boldsymbol{\Upsilon}^a, v^b \boldsymbol{\Upsilon}_b) \\
&= c_a v^b \underbrace{(\boldsymbol{\Upsilon}^a, \boldsymbol{\Upsilon}_b)}_{\delta^a_b} \\
&= c_a v^b \delta^a_b = c_a v^a
\end{aligned}$$

For LI,

$$c_a \boldsymbol{\Upsilon}^a = 0 \iff c_a = 0 \quad \forall a$$
$$c_a v^a = 0 \ \forall \mathbf{v} \iff c_a = 0$$

$\square$

vectors: take one-forms $\to \mathbb{R}$ one-forms: take vectors $\to \mathbb{R}$

0.3 Tensor

> **type (m, n) tensor**
>
> A type (m, n) tensor is a multilinear map that
>
> $$\mathbf{T} : V^n \otimes (V^*)^m \to \mathbb{R}$$
>
> Components of $\mathbf{T}$:
>
> $$\mathbf{T}(\boldsymbol{\Upsilon}_{a1}, \dots, \boldsymbol{\Upsilon}_{an}, \boldsymbol{\Upsilon}^{b1}, \dots, \boldsymbol{\Upsilon}^{bm}) = T_{a_1 \dots a_n}{}^{b_1 \dots b_m}$$

1. Tensor product takes $\binom{m}{n}$ and $\binom{m'}{n'} \to \binom{m+m'}{n+n'}$ tensor

2. Contraction takes $\binom{m}{n} \to \binom{m-1}{n-1}$

> Example:
>
> 1. $T_a{}^b, S_c{}^d$.
> $$(\mathbf{T} \otimes \mathbf{S})_a{}^b{}_c{}^d = T_a{}^d S_c{}^d = P_a{}^b{}_c{}^d$$
>
> 2. $T_a{}^{bc} \to c^b T_a{}^{ba}$
> $$v^a, w_b \begin{cases} v^a \omega_b \\ v^a \omega_a \end{cases}$$

If you have a favorite type $(2,0)$ symmetric tensor $\mathbf{g}$

$$v_\mu = g_{\mu\nu} v^\nu$$

$g^{\mu\nu} :=$ components of the inverse of $\mathbf{g}_{\mu\nu}$

$$v^\nu = g^{\mu\nu}$$

then

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$$

$$g_{\mu\nu} v^\mu w^\nu = v_\mu w^\nu = \mathbf{v}\mathbf{w}$$
$$||\mathbf{v}||^2 = g_{\nu\mu} v^\mu v^\nu$$

Then we can define the angle

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}|| \, ||\mathbf{v}||} := \cos\theta$$

$$T_\mu{}^\nu = g^{\nu\sigma} T_{\mu\sigma}$$

$$T^{\mu\nu} = g^{\nu\sigma} g^{\mu\rho} T_{\sigma\rho}$$

$$g^\nu_\mu = g^{\nu\sigma} g_{\sigma\mu} = \sigma^\nu_\mu$$

0.4 Levi-Civita symbol

Levi-Civita symbol $\epsilon^{abc\cdots}, \epsilon_{abc\ldots}$

- is antisymmetric

- $\epsilon^{1234\cdots} = 1$, $\epsilon_{1234} = 1$

$\epsilon^{123} = 1, \quad \epsilon^{213} = -1, \quad \epsilon^{312} = 1, \quad \epsilon^{113} = 0$
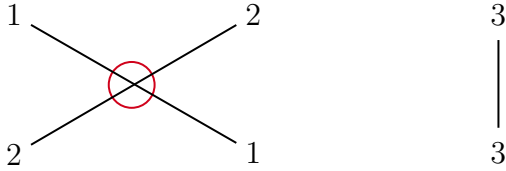
$\epsilon^{123456} = 1, \quad \epsilon^{612453} = -1$

Idea   just see the permutations

> **Levi-Civita symbol**
>
> $$\varepsilon_{a_1 a_2 a_3 \ldots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an even permutation of } (1, 2, 3, \ldots, n) \\ -1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an odd permutation of } (1, 2, 3, \ldots, n) \\ 0 & \text{otherwise} \end{cases}$$

Here is a short-cut:

odd number crossings, so odd permutation.

Note that $\det(M) := \epsilon_{ijk\ldots} M^i{}_1 M^j{}_2 M^j{}_3 \ldots$

Exercise

prove $\epsilon^{i_1 i_2 \ldots i_n} \epsilon_{j_1 j_2 \ldots j_n} = n! i_j = 1, \ldots, n$

$$\epsilon^{ijk} \epsilon_{ilm} = \delta^j_l \delta^k_m - \delta^j_m \delta^k_l$$

$$\epsilon^{ijmn} \epsilon_{klmn} = 2(\delta^i_k \delta^j_l - \delta^j_k \delta^i_l)$$

Prove $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Proof:

Let $\vec{F} = \vec{A} \times (\vec{B} \times \vec{C})$ $\vec{D} = \vec{B} \times \vec{C}$

Then

$$D^k = \epsilon^k{}_{ij} B^i C_j$$

$$F^l = \epsilon^l{}_{mk} A^m D^k \implies F^l = \epsilon^l{}_{mk} \epsilon^k{}_{ij} A^m B^i C^j$$

Then

$$F^l = (\delta^l_i \delta_{mj} - \delta^l_j \delta_{mi}) A^m B^i C^j$$
$$= \delta^l_i \delta_{mj} A^m B^i C^j - \delta^l_j \delta_{mi} A^m B^i C^j$$
$$= B^l (A_j C^j) - C^l (A_i B^i)$$

where we use

$$\vec{A} \cdot \vec{B} = A^i B_i$$

□

Special Relativity

1.1   Postulates of SR

> **Postulate 0**
>
> Newton's first law

> **Postulate 1: Principle of relativity**
>
> In the absence of gravity, all the laws of Physics are identical in all inertial reference frames.

> **Postulate 2**
>
> The speed of light in vacuum c is constant and the same from all inertial reference frames, regardless of their state of motion.

1.2   Lorentz Transformation

We define the spacetime interval Δs^2

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = -c^2 \left(t_2 - t_1\right)^2 + \left(\mathbf{x}_2 - \mathbf{x}_1\right)^2$$

Assuming the following:

1. The difference between the two frames is a constant speed $\gtrsim\!\!\!\lessgtr$

2. The transformation has to be linear.

$$t' = \gamma \left(t - \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{c^2} \right), \quad \boldsymbol{x}' = \boldsymbol{x} + (\gamma - 1)(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - \gamma \boldsymbol{v} t$$

and index notation

$$t' = \gamma \left(t - \frac{v_i x^i}{c^2} \right), \quad x^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

1.3 Line element, proper time and spacelike, time-like and null separation

1.3.1 Classification of spacetime intervals

We can classify events according to the following criterion:

- Spacelike separated, $\Delta s^2 > 0$

- Timelike separated, $\Delta s^2 < 0$

- Lightlike (null) separated, $\Delta s^2 = 0$

Given the trajectory of a physical particle moving inertially, we will call co-moving frame (inertial) or proper frame (non-inertial) to the frame S_p where the particle is at rest.

1.3.2 Proper time and line element

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2$$

We will call ds^2 the spacetime line element.

$$\mathbf{v} := \frac{d\mathbf{x}}{dt}$$

P0, P1, P2 + linearity

$$\implies t' = t\left(t - \frac{v_i x^i}{c^2} \right) \tag{1}$$

$$x'^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

Particle trajectory in a given inertial (Lab) frame $\mathbf{x}(t)$

Particle trajectory in its proper frame $\boldsymbol{\xi}(t) = 0$

Comoving frame's trajectories at each t (from lab frame) $\mathbf{x} = \mathbf{v}(t)t$.

$$d\tau = dt' = \gamma(t)\left(1 - \frac{\mathbf{v}(t)^2}{c^2}\right)dt \tag{2}$$

$$ds^2 = -c^2 dt^2 \left(1 - \frac{1}{c^2}\underbrace{\left(\frac{d\mathbf{x}}{dt}\right)^2}_{\mathbf{v}(t)}\right) = -c^2 \underbrace{\gamma^{-2} dt^2}_{d\tau^2} \implies ds^2 = -c^2 d\tau^2 \tag{3}$$

Example:

Find $\tau(t)$ for the three following trajectories.

1. $x(t) = v(t)$

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x}^2 \implies d\tau = \gamma^{-1} dt \implies \Delta\tau = \gamma^{-1}\Delta t$$

2. $x(t) = \frac{c^2}{a}\left[\sqrt{1 + \frac{a^2 t^2}{c^2}} - 1\right]$

Then $\frac{dx}{dt} = \frac{at}{\sqrt{1 + \frac{a^2 t^2}{c^2}}}$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2$$

$$\implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\frac{a^2 t^2}{1 + \left(\frac{at}{c}\right)^2}}$$

$$\implies \tau(t) = \frac{c}{a}\operatorname{arcsinh}\left(\frac{at}{c}\right) \quad \text{and} \quad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

3. $x(t) = L\sin(\omega t) \implies \frac{dx}{dt} = L\omega\cos(\omega t)$ with $L\omega < c$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 \implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2} \implies d\tau = \sqrt{1 - \frac{L^2\omega^2}{c^2}\omega t}\ dt$$

Then

$$\tau(t) = \frac{E\left(t\omega, \frac{1}{1 - \frac{c^2}{L^2\omega^2}}\right)}{\omega\sqrt{\frac{1}{1 - \frac{L^2\omega^2}{c^2}}}}$$

where

$$E(\phi|m) = \int_0^\phi (1 - m\sin^2\theta)^{1/2}d\theta$$

1.4 Lorentzian Tensors

See notes for details.

A_μ transposes with Λ and it's covariant.

A^μ transposes with $\tilde{\Lambda} = (\Lambda^{-1})^T$ and it's controvariant.

1.5 Poincare group

The derivations are in notes.

1.6 Relativistic dynamics

1.6.1 Hamilton's principle and Euler-Lagrange equations

There exists at least one function (called action) of the trajectories that the degrees of freedom of a system may take in phase space. The physical trajectories are obtained demanding stationarity of this functional under variations that keep the initial and final positions constant.

Usually, the action S of a system of n particles can be written in terms of a Lagrangian $L(s, \mathsf{x}, \overset{\circ}{\mathsf{x}})$ where $\circ$ represents $\overset{\circ}{\mathsf{x}} = \frac{d\mathsf{x}}{ds}$ so that

$$S = \int_{s_1}^{s_2} ds \; L(s, \mathsf{x}, \overset{\circ}{\mathsf{x}})$$

$$\delta S = \sum_n \int_{s_1}^{s_2} ds \left(\frac{\partial L}{\partial x_n^\mu} \delta x_n^\mu + \frac{\partial L}{\partial \overset{\circ}{x}_n^\mu} \delta \overset{\circ}{x}_n^\mu \right) = \sum_n \int_{s_1}^{s_2} ds \left(\frac{\partial L}{\partial x_n^\mu} - \frac{d}{ds} \frac{\partial L}{\partial \overset{\circ}{x}_n^\mu} \right) \delta x_n^\mu + \sum_n \left[\frac{\partial L}{\partial \overset{\circ}{x}_n^\mu} \delta x_n^\mu \right]_{s_1}^{s_2}$$

Impose Hamilton's Principle

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial x_n^\mu} - \frac{d}{ds} \frac{\partial L}{\partial \overset{\circ}{x}_n^\mu} = 0$$

1.6.2 Conserved quantities and Noether's theorem

Noether's theorem

If the variation of the action around a physical trajectory under a continuous variation of the positions $\partial \mathsf{x}$ is zero, then the quantity

$$\delta Q = \sum_n \frac{\partial L}{\partial \overset{\circ}{x}_n^\mu} \delta x^\mu$$

is conserved. That is

$$\frac{d(\partial Q)}{ds} = 0.$$

Proof:
See notes. □

1.6.3 Four-momentum

Let S be invariant under $\partial \mathsf{x} = \mathsf{n}\delta\alpha$.

$$\implies \delta Q = \frac{\partial L}{\partial \overset{\circ}{x}{}^{\mu}} n^{\mu} \delta\alpha$$

is constant $\implies$ the projection $\mathsf{n} \cdot \mathsf{p} = n^{\mu} p_{\mu} = \eta_{\mu\nu} n^{\mu} p^{\nu}$ (where $p_{\mu} := \frac{\partial L}{\partial \overset{\circ}{x}{}^{\mu}}$) is conserved.

If the action is invariant under Lorentz transformation $\delta x^{\mu} = \delta\omega^{\mu}{}_{\nu} x^{\nu}$, then

$$J_{\mu\nu} := x_{\mu} p_{\nu} - x_{\nu} p_{\mu}$$

is conserved.

1.6.4 Angular momentum

The angular momentum $\boldsymbol{J}$ associated to spatial rotations and the vector $\boldsymbol{K}$ associated to boosts can be extracted directly from $J_{\mu\nu}$:

$$J^{i} = \frac{1}{2}\epsilon^{ijk} J_{jk}, \quad K_{i} = J_{i0}$$

1.6.5 Free particle dynamics

- S has to be a scalar (Invariant under Lorentz)

- Must coincide with the non-relativistic action in the limit $\frac{v}{c} \ll 1$.

$$S = mc \int ds = -mc^2 \int d\tau = -mc^2 \int dt \frac{d\tau}{dt} = -mc^2 \int \frac{dt}{\gamma} = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}}$$

$$= -mc^2 \int dt \left[1 - \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right) \right]$$

and

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} = -mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{v^4}{c^4}\right)$$

Euler-Lagrange $\frac{d}{dt}(\gamma m\boldsymbol{v}) = 0$

$$p_i = \frac{\delta S}{\delta v^i} = \frac{\partial L}{\partial v^i} = m\gamma v_i, \qquad \boldsymbol{p} = m\gamma \boldsymbol{v}$$

Hamiltonian

$$H = (\boldsymbol{p} \cdot \boldsymbol{v} - L)_{v \to v(p)} = \sqrt{m^2 c^4 + c^2 \boldsymbol{p}^2}$$

Let's introduce Four-velocity.

$$\frac{dx^\mu}{d\tau} =: \dot{x}^\mu \equiv u^\mu$$

solid dot means derivative w.r.t proper time.

$$\dot{x}^\mu := \frac{dx^\mu}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d\boldsymbol{x}}{d\tau} \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d\boldsymbol{x}}{dt}\frac{dt}{d\tau} \end{pmatrix} = \gamma \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$$

If we choose action as (not four-velocity)

$$S = mv \int dt \sqrt{\eta_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}$$

Lagrangian

$$L = mc\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}$$

$$p_\mu = \frac{\delta S}{\delta \dot{x}^\mu} = m\dot{x}_\mu \implies p^\mu = m\dot{x}^\mu = m\gamma\begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$$

p^0 in the proper frame: $p^0 = mc, \quad \boldsymbol{p} = \boldsymbol{0}$. so cp^0 is energy.

Let's compute

$$p^\mu p_\mu = m^2 \dot{x}^\mu \dot{x}_\mu = -m^2 c^2$$
$$p^\mu p_\mu = -(p^0)^2 + \boldsymbol{p}^2$$
$$\implies -m^2 c^2 = -(p^0)^2 + \boldsymbol{p}^2 \implies p^0 = \frac{1}{c}\sqrt{m^2 c^4 + c^2 \boldsymbol{p}^2}$$
$$\implies E = \sqrt{m^2 c^4 + c^2 \boldsymbol{p}^2} = mc^2\sqrt{1 + \gamma^2 \frac{v^2}{c^2}} = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{v^2}{c^2}\right)$$

- **Ultrarrelativistic limit**: The kinetic term inside the square root is much larger than the rest energy of the particle: $pc \gg mc^2, \quad E \approx cp$

- **Deep non-relativistic limit**: The rest energy is much larger than the kinetic energy of the particle: $mc^2 \gg pc, \quad E \approx mc^2$

Two problems

You are designing a particle collider, you have two identical part of mass M and energy budget $E = 2\varepsilon$. You have two strategies:

a) spend $1/2E$ on each and accelerate them.

b) spend E on one of them and accelerate it

Which one optimizes the center of mass energy?

Solution

a)

Lab frame $p_1^\mu = \left(\frac{\epsilon}{c} + M_c, p, 0, 0\right)$ $p_2^\mu = \left(\frac{\epsilon}{c} + M_c, -p, 0, c\right)$

$$p_{lab}^\mu = p_1^\mu + p_2^\mu = \left(\frac{2\epsilon}{c} + 2M_c, 0, 0, 0\right) = p_{CM}^\mu$$

Then

$$E_{CM}^{(a)} = cp_{CM}^0 = 2\epsilon + 2Mc^2 = 2Mc^2\left(1 + \frac{\epsilon}{\mu c^2}\right)$$

b) Here the p is different from the p above.

Lab frame $p_1^\mu = \left(\frac{2\epsilon}{c} + Mc, p, 0, 0\right)$ $p_2^\mu = (Mc, 0, 0, c)$

$$p_{lab}^\mu = p_1^\mu + p_2^\mu = \left(\frac{2\epsilon}{c} + 2M_c, p, 0, 0\right)$$

Determine

$$p_1^\mu p_{1\mu} = -M^2c^2 = -\left(\frac{4\epsilon^2}{c^2} + M^2c^2 + M^2c^2 + 4\epsilon M\right) + p^2$$

$$\implies p = \sqrt{\frac{4\epsilon^2}{c^2} + 4\epsilon M} = \frac{2\epsilon}{c}\sqrt{1 + \frac{Mc^2}{\epsilon}}$$

We want p_{cm}^0, and we know $p_{CM}^\mu = (p_{CM}^0, \mathbf{0})$

Lorentz scalar: $p_{CM}{}^\mu p_{CM\mu} = -(p_{CM}^0)^2$ and lab frame $p_{lab}{}^\mu p_{lab\mu} = -(p_{CM}^0)^2$

$$p_{lab}{}^{\mu}p_{lab\,\mu} = -\left(\frac{4\epsilon^2}{c^2} + 4M^2c^2 + 8\epsilon M\right) + \frac{4\epsilon^2}{c^2}\left(1 + \frac{Mc^2}{\epsilon}\right)$$
$$= -4M^2c^2 - 8\epsilon M + 4\epsilon M$$
$$= -4M(Mc^2 + \epsilon)$$
$$= -(p_{CM}^0)^2$$

$$p_{CM}^0 = \sqrt{-p_{lab}^{\mu}p_{lab\,\mu}} = 2\sqrt{\mu(Mc^2 + \epsilon)} \implies E_{CM}^{(b)} = cp_{CM}^0 = 2Mc^2\sqrt{1 + \frac{\epsilon}{Mc^2}}$$

So

$$R = \frac{E_{CM}^{(a)}}{E_{CM}^{(b)}} = \sqrt{1 + \frac{\epsilon}{Mc^2}} > 1$$

In Deep non-real $\epsilon \ll Mc^2$, $\lim\limits_{\frac{\epsilon}{Mc^2}\to 0} R = 1$.

In Ultra limit $\lim\limits_{\epsilon\to pc} R = \sqrt{1 + \frac{pc}{Mc^2}} \to \infty$.

A massless particle cannot $\to$ two identical mass particle. (converse is also true).

Index