CO 342

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Spring 2019

Contents

Intro

Tips

- Review Math 239
- start assignments **WELL IN ADVANCE**
- Know definitions
- Attend Class

Emphasis on theory and proofs
Official material for the course is what is presented in class (Course Notes just for reference)
Occasional (frequent) quizzes will test definitions from te course

Course Preview

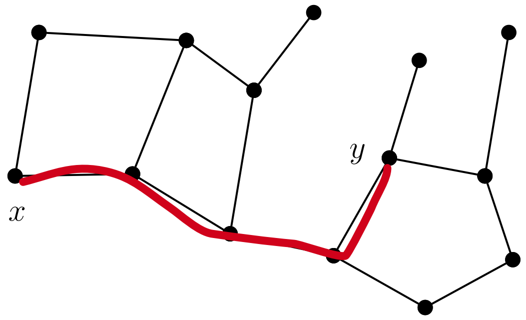
(a) Connectivity

(b) planar graphs

(c) matchings

(d) Vector Spaces

(a) A graph G is said to be connected if for every pair x, y of vertices of G, there exists a path from x to y in G.



Higher connectivity even after removing a few vertices, the graph remains connected

(b) A graph G is said to be planar if it can be drawn in the plane such that

- no two vertices coincide
- no two edges intersect (except at their common end point, if it exists)

Kuratowski's Theorem A graph G is planar iff it does not contain a subdivision of K_5 or $K_{3,3}$. (Proof later)

(c) A matching M in a graph G is set of edges, no two of which share common endpoint.

In MATH 239, we saw Hall's Theorem: a necessary and sufficient condition for the existence of a perfect matching in a bipartite graph.

Hungarian algorithm An algorithm that finds a maximum matching in a bipartite graph.

We will see Tutte's Theorem and Edwards Algorithm that do the same things for general graphs (i.e. not just bipartite)

(d) One example of a vector space (associated with graphs.)
Let g be a graph. A subgraph H of G is called even if every vertices of H has even degree. The set of all even subgraphs G forms a vector space over $\mathbb{Z}_2$, called flow space (link to coding theory).

Some Specifics from MATH 239

Definition A graph G consists of a (finite) set $V(G)$ of vertices, and a set of $E(G)$ of edges, where each edge is a subset of $V(G)$ of size 2, We usually write uv as shorthand for $\{u, v\}$. (So $uv = vu$)

Note that we do not allow loops (an edge that is a multiset of size 2) or multiple edges (where $E(G)$ is allowed to be a multiset).

If we want to consider loops and multiple edges, we use the term <u>multigraph</u>.

Recall Terminology

- adjacent
- incident
- neighbour
- degree $d(v)$
- complete graph
- bipartite graph
- path
- cycle
- subgraph
- connected
- component (maximal connected subgraph)
- tree
- subdivision

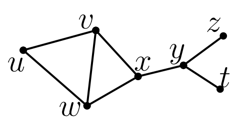
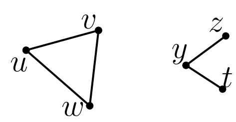
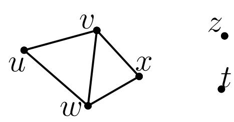
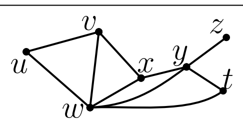
CHAPTER 1

Connectivity & Planar Graphs

cut-vertex

Let G be connected graph. A cut-vertex in G is a vertex such that the graph $G - x$ (obtained by removing x from G) is disconnected.

Example

G_1		cut vertex x, 2 components 	cut vertex y, 3 components 
G_2		no cut vertices	

vertex cut

A vertex cut in a connected graph G is a set W of vertices such that $G - W$ is disconnected. (where $G - W$ denotes the graph obtained by removing all vertices in W from G)

Some vertex cuts for G_2:

$$\{y, w\} \ \{y, t\} \ \{v, w, x, y, t\} \ \{v, w, x\}$$

k-connected

Let $k \geq 1$, a connected graph G is said to be k-connected if

- G has at least $k+1$ vertices, AND
- G has no vertex cut of size less than k

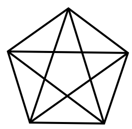
So 1-connected is (almost) the same as connected (except for • and (empty))
2-connected: means (connected and) no cut-vertex.

G_2 is 2-connected, G_1 is not 2-connected, G_2 is not 3-connected

connectivity $\kappa(G)$

The connectivity $\kappa(G)$ of a connected graph G is the largest k for which G is k-connected.
(If G is not connected, we say $\kappa(G) = 0$.)

Q Why not define $\kappa(G)$ as "the size of a smallest vertex cut in G"?



$\kappa(K_5) = 4$

A Because it would not be defined for any complete graph.

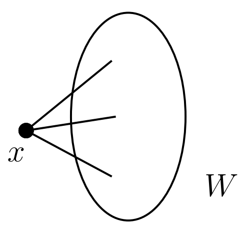
minimum degree

The minimum degree $\delta(G)$ of G is

$$\delta(G) = \min\{d(v) : v \in V(G)\}$$

Lemma For any graph G we have $\kappa(G) \leq \delta(G)$

Proof Let $x \in V(G)$ be such that $d(x) = \delta(G)$. Let W denote the neighbourhood in G.



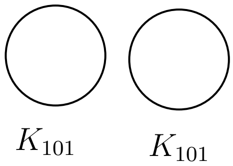
- If $V(G) = \{x\} \cup W$, then $\kappa(G) \leq |W|$, because $|V(G)| \leq |W| + 1$
- If $|V(G)| > |W|+1$, then $V(G) \setminus (\{x\} \cup W)$ is non-empty, and so W is a vertex cut in which x is a component of size 1 in $G - W$. Hence $\kappa(G) \leq |W| = \delta(G)$

■

Note Minimum degree k does NOT IMPLY k-connected!!!

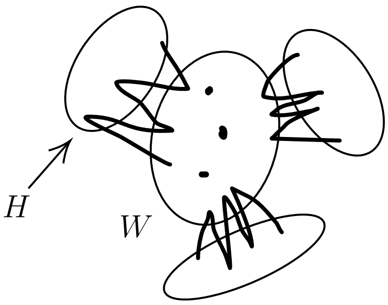
$\delta = 100$

$\kappa = 0$



Lemma Let G be a graph, with n vertices, and let $1 \leq k \leq n-1$. If $\delta(G) \geq \frac{n+k-2}{2}$ then G is k-connected.

Proof We have $n \geq k+1$ by assumption. Suppose on the contrary that G has a vertex cut W of size $< k$.



Then $G - W$ has at least 2 components

Let H be the <u>smallest</u> component of $G - W$, then H has at most $\frac{n-|W|}{2}$ vertices. So any vertex x of H has all its neighbours in $(H \setminus \{x\}) \cup W$. So

$$\begin{aligned} d(x) \leq |H| - 1 + |W| &\leq \frac{n-|W|}{2} + |W| - 1 \\ &= \frac{n+|W|-2}{2} \\ &\leq \frac{n+k-2}{2} \end{aligned}$$

This contradicts the assumption of the lemma. Hence G is k-connected. ■

Note If G is k-connected then it is l-connected for every $l \leq k$.

1.1 2-connected graphs

Recall that a relation $\sim$ on a set X is an equivalence relation if

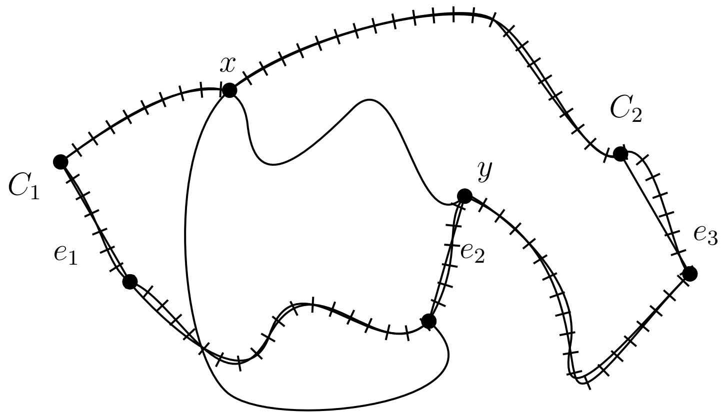
1. $x \sim x$ for every $x \in X$ (reflexive)
2. If $x \sim y$, then $y \sim x$ $\quad \forall x, y \in X$ (symmetric)
3. If $x \sim y$ and $y \sim z$, then $x \sim z$, $\forall x, y, z \in X$ (transitive)

Lemma Let G be a graph. Let the relation $\sim$ on $E(G)$ defined by $e \sim f$ means "$e = f$ or e and f lie in a common cycle of G", then $\sim$ is an equivalence relation on $E(G)$.

Proof It is clear that $\sim$ is reflexive and symmetric.

To check transitivity, let e_1, e_2, e_3 be edges such that $e_1 \sim e_2$ and $e_2 \sim e_3$ (We may assume all are distinct).

Let C_1 be a cycle of G contains e_1 and e_2 and C_2 be a cycle of G contains e_2 and e_3.



Let x and y be the first two vertices of C_1 encountered when starting at the two end points of e_3 and walk along C_2 in either direction. Then x and y are distinct because e_2 is an edge of both C_1 and C_2.

Then the x, y segment p of C_2 contains e_3 is disjoint from C_1 except for the vertices x and y.

Then P together with the x, y segment of C_1 containing e_1 is a cycle containing e_1 and e_3. Hence $e_1 \sim e_3$. So $\sim$ is transitive. ■

Theorem The following are equivalent for any graph G with $|V(G)| \geq 3$:

(a) G is 2-connected

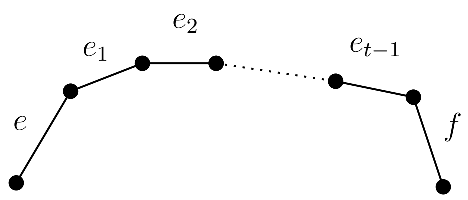
(b) G has no isolated vertices [1] and any two edges lie in a common cycle

(c) Any two vertices lie in a common cycle

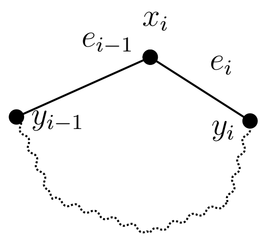
Proof

(a) $\Longrightarrow$ (b) Assume G is 2-connected, then G has no isolated vertices. Let e and f be edges of G. Since G is connected, there is a path in G from an endpoint of e to an endpoint of f.

[1] of degree 0

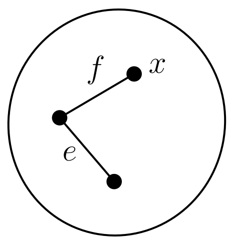


Then there exists a sequence of distinct edges, $e_0, e_1, \dots, e_t$: where $e_0 = e$ and $e_t = f$ and e_i and e_{i-1} share a vertex for each i. Since G is 2-connected, it has no cut-vertex. For each i, the vertex x_i common to e_{i-1} and e_i is such that $G - x_i$ is connected. So the other endpoints y_{i-1} and y_i of e_{i-1} and e_i are joined by a path P_i in $G - x_i$. So P_i together with e_{i-1} and e_i form a cycle contains e_{i-1} and e_i. Hence $e_{i-1} \sim e_i$ (for the relation from previous Lemma)



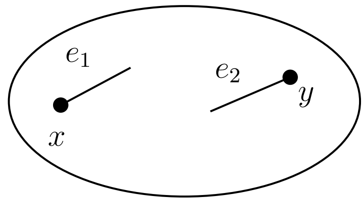
So $e \sim e_1, e_1 \sim e_2, \dots, e_{t-1} \sim f$, hence by transitivity $e \sim f$.

If $e = f$, then e is contained in a cycle: we just to check there exists some edge f different from e (then use previous proof).



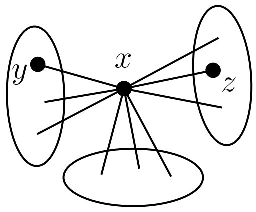
Since $|V(G)| \geq 3$, there exists $x \in V(G)$ not incident to e, and it is not isolated so it is incident to some $f \neq e$.

(b) $\implies$ (c) Assume (b). Let x and y be vertices of G.

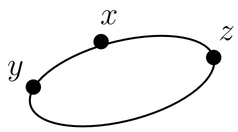


Since x and y are not isolated, there exists an edge e_1 incident to x, an edge e_2 incident to y, then a common cycle C contains x and y [If $e_1 = e_2$, we can choose a different edge e_3 to some $z \neq x, y$ and say the same argument.]

(c) $\implies$ (a) Assume (c). We know by assumption that $|V(G)| \geq 3$. Any two vertices are joined by a path (in the cycle containing them both) so G is connected.



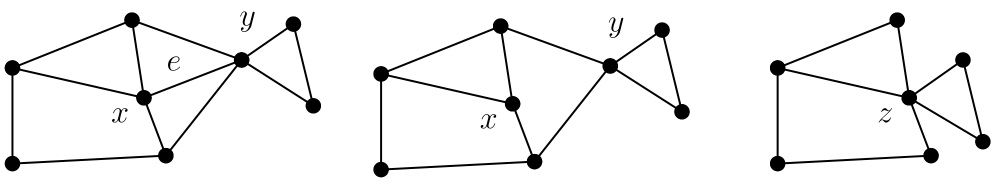
Suppose on the contrary that x is a cut-vertex of G. Let y and z be vertices in two different components of $G - x$, then there is a cycle C in G containing y and z. But then there is a path in C between y and z that does not containing x.



This contradicts the fact that y and z are in different components of $G - x$. Hence G is 2-connected. ■

1.2 Minors, contraction and deletion

Let G be a graph, and let $e = xy$ be an edge of G.



G $\qquad$ $G \setminus e$ $\qquad$ G/e

- The graph $G \setminus e$ has vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. We say $G \setminus e$ is obtained from G by deleting e.
- The graph G/e has vertex set $(V(G) \setminus \{x, y\}) \cup \{z\}$, and edge set

  $$\{uv \in E(G) : \{u, v\} \cap \{x, y\} = \varnothing\} \cup \{uz : ux \in E(G) \setminus \{e\} \text{ or } uy \in E(G) \setminus \{e\}\}$$

  We say G/e is the graph obtained from G by contracting e

Notes

- Both $G \setminus e$ and G/e have fewer edges than G.
- In G/e, the degree of z is at least $\max\{d_G(x) - 1, d_G(y) - 1\}$

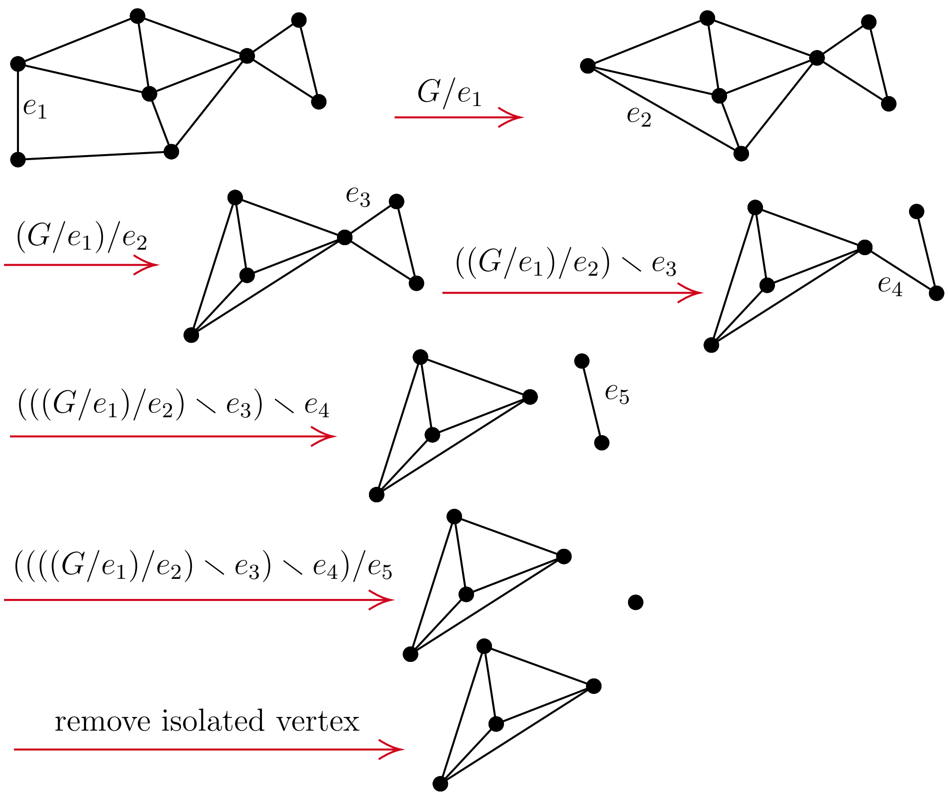
minor

We say a graph H is a minor of graph G if we can obtain H from G by a sequence of

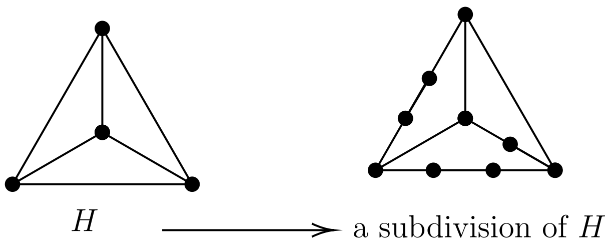
- edge deletions
- edge contractions

- (removal of isolated vertices)

Eg Is K_4 a minor of this graph?



Recall the notion of <u>subdivision</u> of a graph H



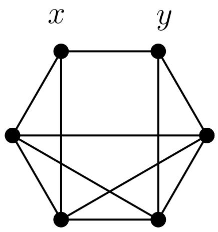
Any graph obtained by replacing each edge of H by a path of length at least 1, such that all paths are disjoint except for their endpoints.

Note that if g contains a subdivision of H then G contains H as a minor:

- contract (one by one) all paths in the subdivision of H down to edges
- delete all other edges (not in this copy of H)
- remove any isolated vertices

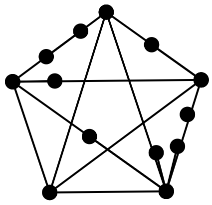
Last class If G contains a subdivision of H, then G has H as a minor. The converse is NOT true in general.

For example,



G

This graph does not contain a subdivision of K_5

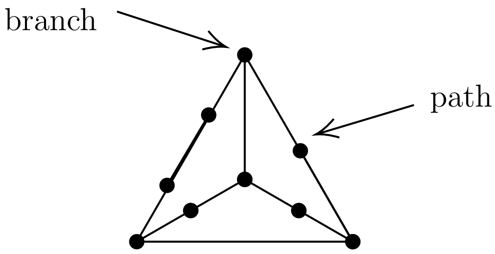


K_5: it does not contain 5 vertices of degree ≥ 4

But G has K_5 as a minor (just contract xy to get K_5)

cubic

We call a graph H cubic if it is 3-regular (i.e. every vertex has degree 3)



a subdivision of K_4

branch/path vertices

For a subdivision H_1 of a cubic H, we call the vertices of degree 3 in H_1 branch vertices, the vertices of degree 2 in H_1 path vertices.

Lemma Let H be a cubic graph. If G has H as a minor, then G contains a subdivision of H.

Proof We use induction on $|E(G)|$.

When $|E(G)| = |E(H)|$, then $G = H$. So G has H as a minor (trivially).

IH [2] Assume $|E(G)| > |E(H)|$ and every graph with fewer than $|E(G)|$ edges satisfies the statement.

Let $e = xy$ be the first edge in a sequence of edge deletions/contractions that takes G to H.

[2] Inductive Hypothesis

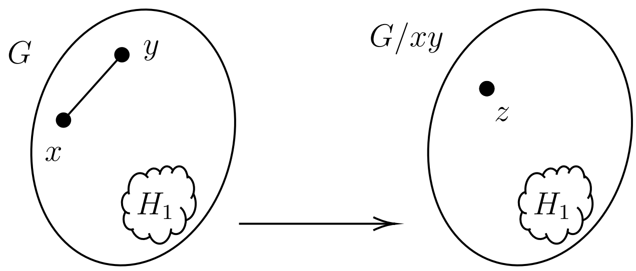
- If the sequence σ starts by deleting xy, then $G \setminus xy$ has H as a minor (via the rest of G) and has fewer edges than G.

  So by IH, $G \setminus xy$ contains a subdivision H_1 of H. Then H_1 is a subgraph of G, so G contains a subdivision of H as required.

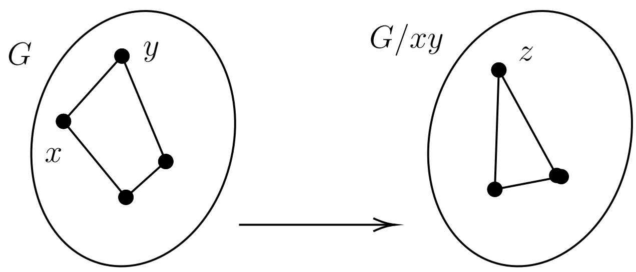
- If starts by contracting xy, then G/xy has H as a minor (as before) and G/xy has fewer edges than G.

  So by IH, G/xy contains a subdivision H_1 of H.

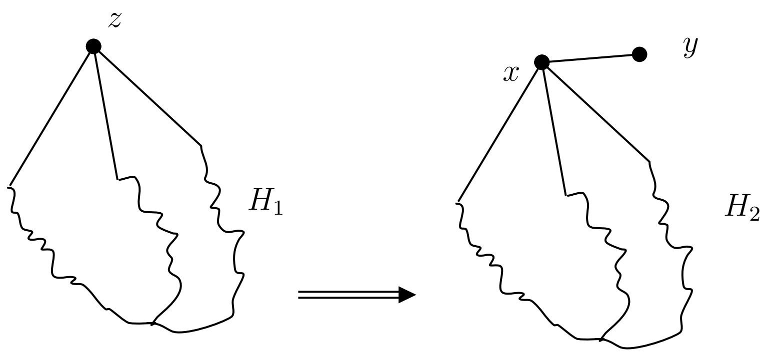
If the new vertex z in G/xy is not in H_1, then H_1 is a subgraph of G and were done.



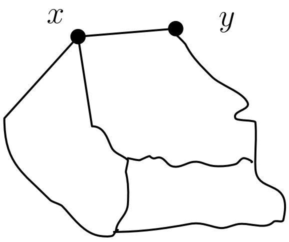
Let H_2 be the subgraph of G contains x and y such that $H_2/xy = H_1$.



If z is a branch vertex of H_1: then each of the 3-edges incident to z in H_1 was incident to x or y in G.



- If all 3 edges were incident to x in H_2, then H_2 contains H_1 (with z replaced by x) as a subgraph. Done
- If 2 edges were incident to x and are to y.



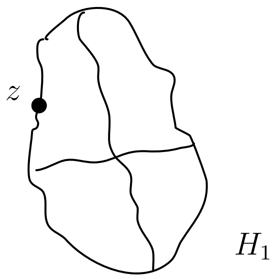
Then H_2 is a subdivision of H_1 (in which x replaces z and y a new path vertex).

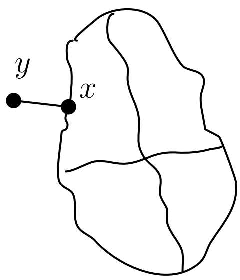
But a subdivision (H_2) of a subdivision (of H_1) is a subdivision (of H).

Hence G contains a subdivision (H_2) of H.

If z is a path vertex of H_1:

- If Both edges of H_1 incident to z were incident to x in G, then H_2 contains H_1 with z replaced by x.





- If one edge is incident to x and the other to y, then H_2 is a subdivision of H_1 (in which the path containing z becomes a path containing $e = xy$)

Hence by induction, the statement holds. ■

1.3 Blocks

block

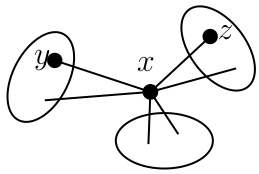
A block is a connected graph with no cut-vertex.
(So a block is
- 2-connected, or
- $\bullet$——$\bullet$ or
- $\bullet$ (trial block)
)

block of G

Let G be a graph: A block of G is a subgraph of G that is maximal with respect to being a block. (In particular, if G is 2-connected, then it has exactly one block, namely itself)

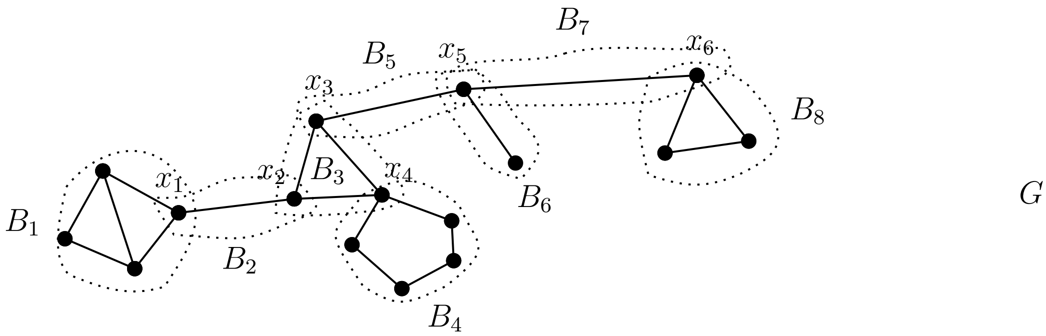
Notes

- Two distinct blocks B_1, B_2 of G share at most one vertex: if B_1 is a single edge then it can't be contained in B_2 (by def). If B_1 and B_2 are both 2-connected then $|V(B_1) \cap V(B_2)| \geq 2$, then $B_1 \cup B_2$ is 2-connected, contradicts the definition of block.
- any cut-vertex of G is in at least two blocks: if x is a cut-vertex of G. let y and z be in distinct components of $G - x$ then xy is contained in a block B_1 of G, xz is contained in a block B_2 of G.



If $B_1 = B_2$, then $|V(B_1)| \geq 3$ so B_1 is 2-connected and contains y and z. So (by our lemma) there is a cycle C in B_1 containing y and z. But then $C - x$ and hence $G - x$ contains a (y, z)-path, contradiction

Example

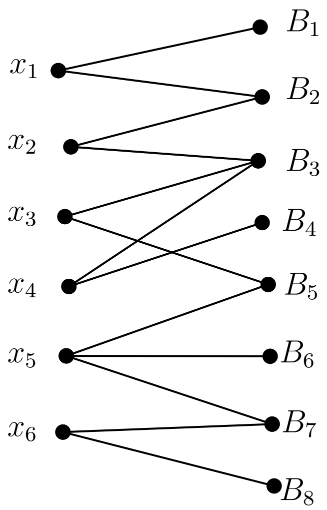


block-cut vertex forest

The block-cut vertex forest $BCV(G)$ of a graph G is a graph with vertex set $C \cup B$ where C is the set of all cut vertices of G, and B is the set of all blocks of G. The edges of $BCV(G)$ are

$$\{xb : x \in C, b \in B, \text{ and } x \in V(B)\}$$

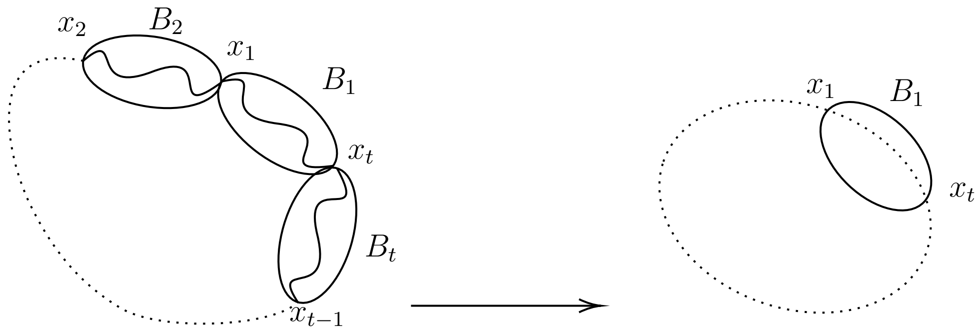
Example



Note that any vertex of degree 1 in $BCV(G)$ is a block. These are called <u>end-blocks</u>.

Theorem Let G be a connected graph. Then $BCV(G)$ is a tree.

Proof We show $BCV(G)$ has no cycles. Suppose on the contrary that $x_1 B_1 x_2 B_2 \ldots x_t B_t$ is a cycle in $BCV(G)$, where $t \geq 2$, in G.



In B_i from x_i to x_{i+1}, so their union forms a cycle C containing all of $x_1 \ldots x_t$.

But then $C \cup B_1$ is a 2-connected subgraph of G that strictly contains B_1:

- If B_1 is 2-connected then B_1 and C share at least 2 vertices so $B_1 \cup C$ is 2-connected.
- If B_1 is a single edge then $B_1 \cup C = C$, which is 2-connected.

This contradicts the definition of B_1 being a block. So $BVC(G)$ has no cycles.

Exercise: show $BCV(G)$ is connected. ■

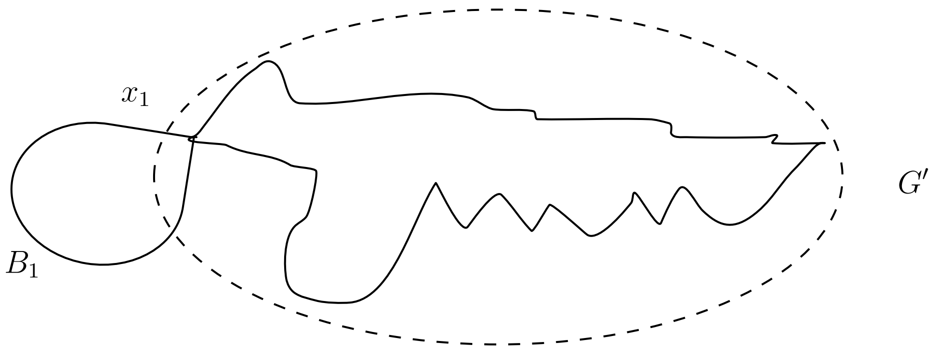
Theorem $(K1)$ Let G be a connected graph. Suppose every block of G is planar, then G is planar.

Proof By induction on the number of blocks b of G.

If $b = 1$, then G is its own single block, hence planar.

IH Assume $b \geq 2$ and any graph with $b - 1$ blocks is planar if all its blocks are planar.

Consider an end block B_1 of G (which exists since $BVC(G)$ is a tree). Then B_1 contains exactly one cut vertex x_1.



Then $G' = G-(B-\{x_1\})$ has $b-1$ blocks, all of which are planar by assumption. Hence by IH, G' is planar.

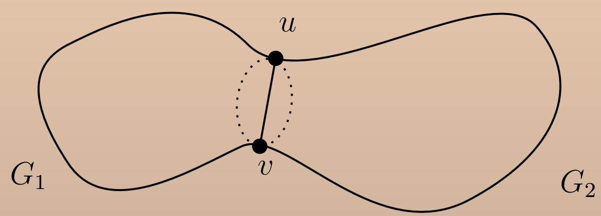
Then we get a planar drawing of G by gluing a planar drawing of B_1 with x_1 an outer face and a planar of G' with x_1 an outer face. Hence G is planar. ■

Recall from 239 any planar graph can be drawn with any face chosen as the outer face.

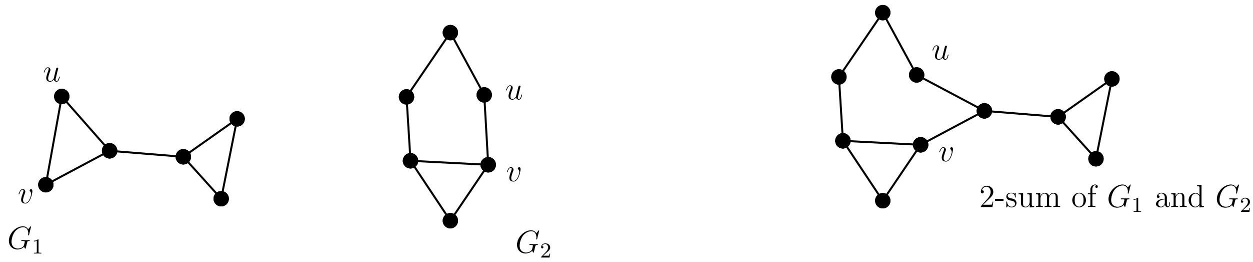
1.4 2-sums

2-sum

Suppose G_1 and G_2 are graphs with at least 3 vertices each, and suppose $V(G_1) \cap V(G_2) = \{u, v\}$. Suppose also that $uv \in E(G_1) \cap E(G_2)$, then the 2-sum of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \setminus \{u, v\}$.



Example

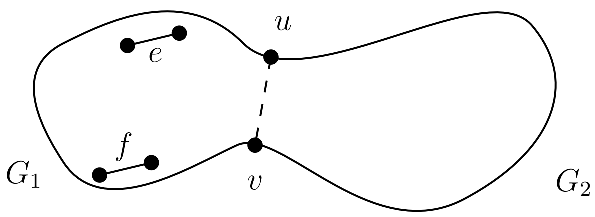


Lemma Suppose the graph G_2 is 2-connected, then the 2-sum G of G_1 and G_2 contains a subdivision of G_1.

Proof Since G_2 is 2-connected, we know that any two edges are in a common cycle in G_2. Let C be in a cycle in G_2 containing uv. Then $C \setminus uv$ together with $G_1 \setminus uv$ forms a subdivision of G_1 in G. ■

Note The 2-sum of two cycles is a cycle.

Lemma If G_1 and G_2 are 2-connected then the 2-sum G of G_1 and G_2 is 2-connected.



Proof

We know G has no isolated vertices because $\delta(G_1) \geq 2$ and $\delta(G_2) \geq 2$ and we only remove one edge when forming G. We show that every pair e, f of edges of G lie in a common cycle of G.

If $e, f \in E(G_1 \setminus uv)$, then there exists a cycle C in G_1 containing e and f.

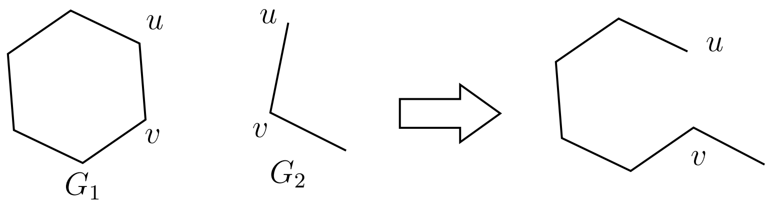
If $uv \notin E(C)$, then C is a cycle in G containing e and f.

If $uv \in E(C)$ then let C_2 be a cycle in G_2 containing uv (which exists since G_2 is 2-connected), then the 2-sum C' of C and C_2 is a cycle in G containing e and f.

Similarly if $e, f \in E(G_2 \setminus uv)$. If $e \in E(G_1 \setminus uv)$ and $f \in E(G_2 \setminus uv)$, let C_1 be a cycle in G_1 containing e and uv, C_2 be a cycle in G_2 containing f and uv.

Then the 2-sum C' of C_1 and C_2 is a cycle in G containing e and f. Hence G is 2-connected. ■

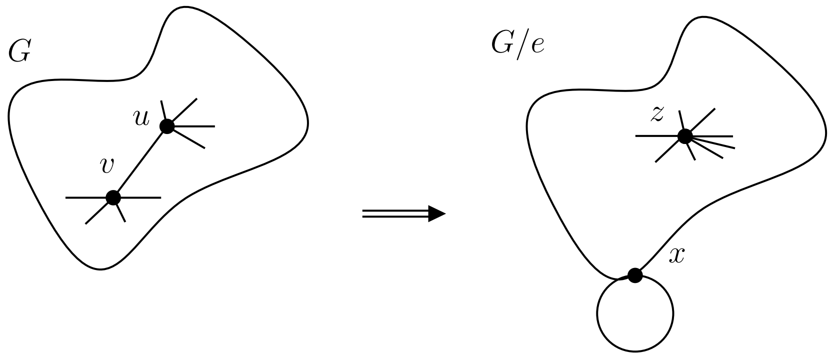
Eg Suppose G_2 is not 2-connected, then 2-sum of G_1 and G_2 might not contain a subdivision of G_1:



Theorem Let G be a 2-connected graph with at least 4 vertices. For every edge e of G_1, either $G \setminus e$ is 2-connected, or G/e is 2-connected.

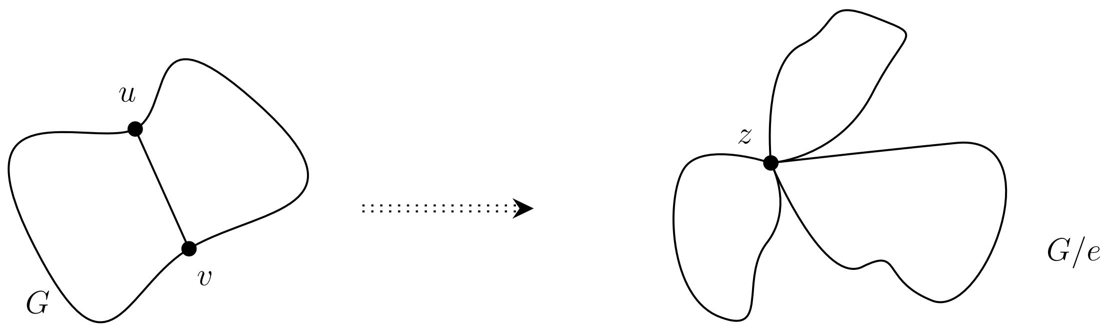
Proof Let $e = uv$ be an edge of G. Assume G/e is not 2-connected.

Since G/e has ≥ 3 vertices, and is connected, it has a cut-vertex. Let z be the image of uv in G/e.

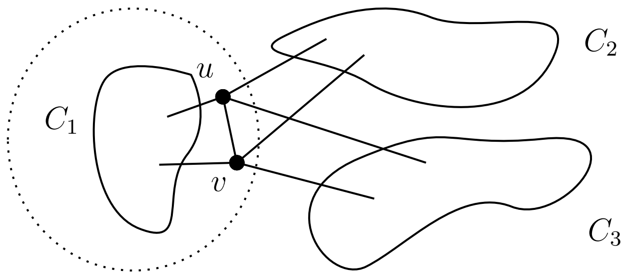


If $x \neq z$ is a cut-vertex of G/e then x is a cut-vertex of G which is a contradiction.

Hence the only cut-vertex in G/e is z.



So $\{u, v\}$ is a vertex cut of G of size 2. For each component C_i, $G - \{u, v\}$.



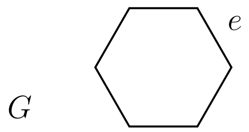
There's an edge of G joining u to C_i and an edge of G joining v to C_i. Otherwise v (respectively u) would be a cut-vertex of G.

Let $\overline{C_1}$ be the subgraph of G induced by $C_1 \cup \{u, v\}$ then $\overline{C_1}$ does not contain a cut-vertex (otherwise it would be a cut-vertex of G; or it would be u or v, neither can be cut-vertices.)

So $\overline{C_1}$ is 2-connected.

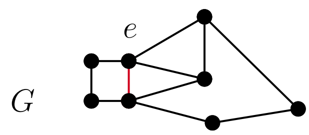
Similarly $\overline{C_0}$ the subgraph of G induced by $\{u, v\} \cup C_2 \cup \ldots \cup C_r$ is 2-connected, then $G \setminus e$ the 2-sum of $\overline{C_1}$ and $\overline{C_0}$, hence is 2-connected. ■

Eg



$G \setminus e$ is not 2-connected

G/e is 2-connected



$G \setminus e$ is 2-connected

G/e is not 2-connected

In fact, any 2-connected graph with ≥ 4 vertices has an edge e such that G/e is 2-connected. (See assignment 2).

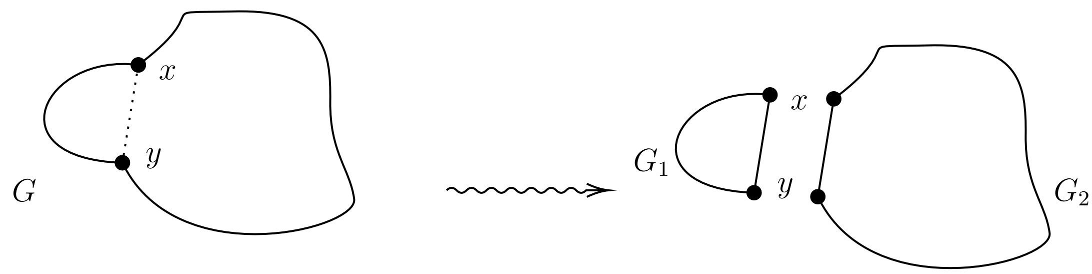
Theorem Let G be a 2-connected graph and suppose G has a vertex cut $\{x, y\}$ of size 2, then there exist graphs G_1 and G_2 such that

(1) G_1 and G_2 are both 2-connected

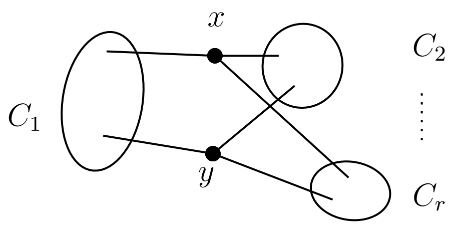
(2) $V(G_1) \cap V(G_2) = \{x, y\}$

(3) $xy \in E(G_1) \cap E(G_2)$

(4)
- If $xy \notin E(G)$, then G is the 2-sum of G_1 and G_2.
- If $xy \in E(G)$, then $G \setminus e$ is the 2-sum of G_1 and G_2.



Proof First suppose $xy \notin E(G)$.



Let $C_1, \ldots, C_r$ be the components of $G - \{x, y\}$. As in our previous proof from Wednesday, $G_1 = C_2 \cup \{x, y\} + xy$ [3] is 2-connected. We let G_2 be the subgraph of G induced by $V(G) \smallsetminus V(C_1)$, together with the edge xy. Then as before, G_2 is 2-connected, and G is the 2-sum of G_1 and G_2.

If $xy \in E(G)$, then $G \smallsetminus e$ is 2-connected, because (by previous lemma) G/e is not 2-connected, and $|V(G)| \geq 4$ (since it has a vertex cut). So we can apply the first case to find G_1 and G_2 as required. ■

We'll prove later:

Theorem ($K3$) Let G be a 3-connected graph, that does not contain a subdivision of K_5 or $K_{3,3}$, then G is planar.

Theorem ($K2$) Let G be a 2-connected graph, that does not contain a subdivision of K_5 or $K_{3,3}$, then G is planar.

Proof By induction on $V(G)$.

If $|V(G)| \leq 5$, then since K_5 is the only non-planar graph with ≤ 5 vertices, the theorem is true.

IH Assume $|V(G)| \geq 6$ and that the theorem holds for any graph with fewer than $|V(G)$ vertices.

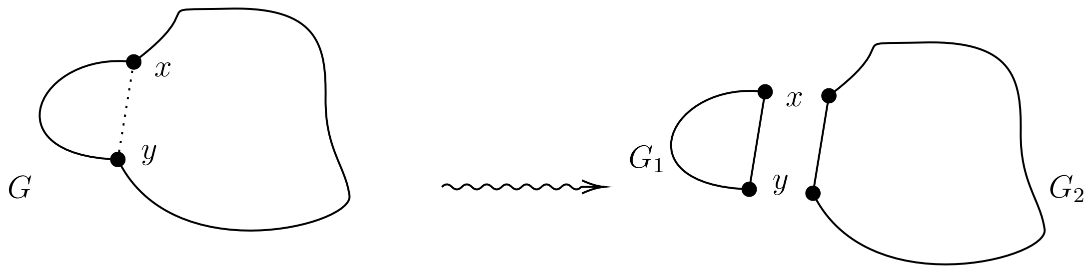
Let G be 2-connected without subdivision of K_5 or $K_{3,3}$.

- If G is 3-connected, then by theorem $K3$, G is planar.

[3] $C_1 \cup \{x, y\}$ together with the edge xy

- If G is not 3-connected, then it has a vertex cut $\{x, y\}$ of size 2.

  By our previous theorem, there exist 2-connected graphs G_1 and G_2, with $xy \in E(G_1) \cap E(G_2)$, such that either G is 2-sum of G_1 and G_2, for $G \setminus xy$ is 2-sum of



  G_1 and G_2.

  Then $|V(G_1)| < |V(G)|$ since $|V(G_2)| \geq 3$ (it is 2-connected) and $|V(G)| = |V(G_1)| + |V(G_2)| - 2$ (same for G_2).

  Also G_1 does not contain a subdivision of K_5 or $K_{3,3}$, by our lemma which said that if G_2 is 2-connected, then the 2-sum of G_1 and G_2 contains a subdivision of G_1. So if G_1 contained a subdivision of K_5 or $K_{3,3}$, then so does G (a subdivision of a subdivision of H is a subdivision of H). Therefore by IH, G_1 is planar. Similarly, G_2 is planar.

  Then we can obtain a planar drawing of G (or of $G + xy$, if $xy \notin E(G)$) by giving

  - a planar drawing of G_1 with xy on the outer face.
  - a planar drawing of G_2 with xy on the outer face. ■

Theorem Let G be a 2-connected graph. Then one of the following holds:

(a) G is a cycle.

(b) there exists an edge $e \in E(G)$ such that $G \setminus e$ is 2-connected.

(c) G is 2-sum of a 2-connected graph and a cycle.

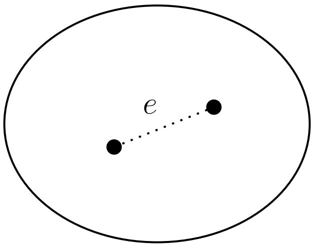
Proof Call a graph <u>good</u> if it satisfies one of a, b, c. Let G be given.

Since G is 2-connected, in particular it contains a cycle, so it has some good subgraphs. Let H be a subgraph of G that is good, and has the largest possible number of edges.

If $|E(H)| = |E(G)|$, then $H = G$, and were done.

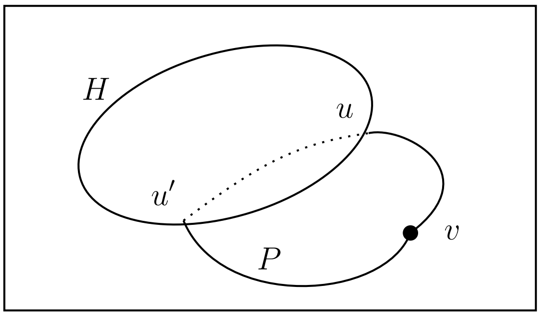
If $|E(H)| < |E(G)|$:

- If $|V(G)| = |V(H)|$. Noting that any good graph is 2-connected. (for (c) we apply our earlier lemma), we know H is 2-connected. But then $H + e$ where $e \in E(G) \setminus E(H)$ satisfies (b) and hence is good. Since $H + e$ is a subgraph of G, this contradicts the definition of H.



$V(H) = V(G)$

- We may assume $|V(H)| < |V(G)|$, then there exists $u \in V(G)$ and $v \in V(G) \setminus V(H)$ such that $uv \in E(G)$ (since G is connected).

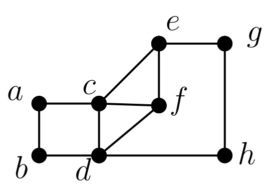


Since u is not a cut-vertex of G, there is a path from v to $V(H) \setminus \{u\}$ in $G - u$. Let P be a shortest such path, so that the end vertex u' is the only vertex of H on P.

Let $H' = \begin{cases} H & \text{if } uu' \in E(H) \\ H + uu' & \text{otherwise} \end{cases}$

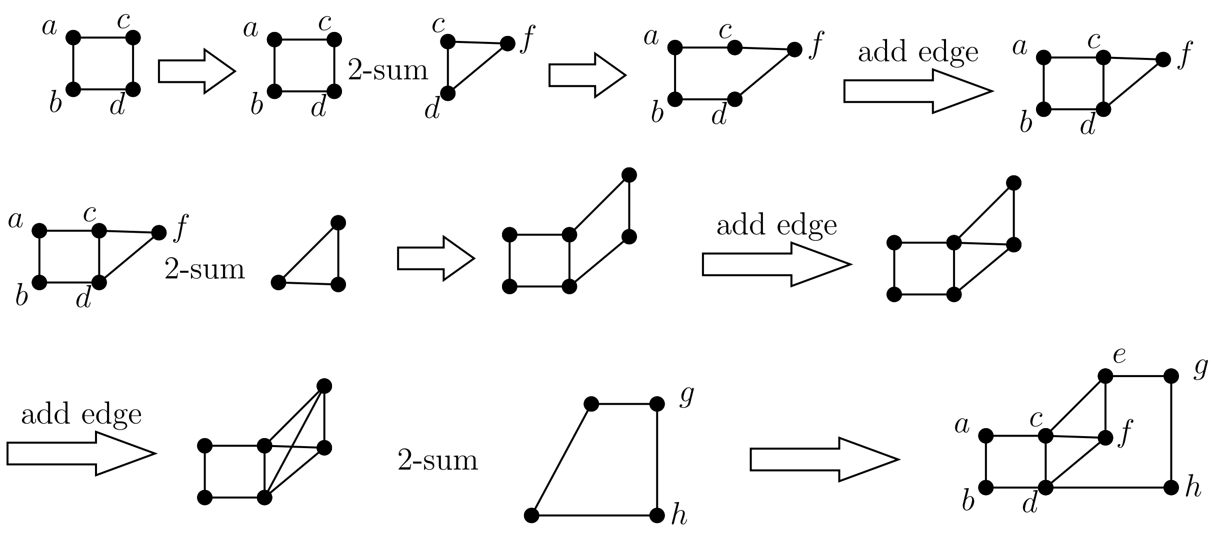
Then the 2-sum J of H' and the cycle $P + \{u\} \cup \{uu', uv\}$ is a subgraph of G, and it is good.

Since $|E(J)| \geq |E(H)| - 1 + 2 > |E(H)|$, this contradicts our choice of H. Hence $H = G$ as required. ■

Example 

Any 2-connected graph can built from any one of its cycles by

- adding edges, and
- taking 2-sum with a cycle.



Lemma Let G be a 2-connected planar graph. In every planar drawing of G, every face is bounded by a cycle and every edge is in exactly two faces.

Proof We use induction on $|V(G)|$ and $|E(G)|$.

- If $|V(G)| = 3$, then G is a triangle and the statement is true. $\triangle$
- Since $\delta(G) \geq 2$ because G is 2-connected.

$$E(G) = \frac{1}{2} \sum_{v \in V(G)} d(v) \geq |V(G)|$$

If $|E(G)| = |V(G)|$ then G is a cycle, so again the theorem holds.

IH Assume $|V(G)| \geq 4$ and the theorem holds for every graph with $< |V(G)|$ vertices, and for every graph with $= |V(G)|$ vertices but fewer edges than G.

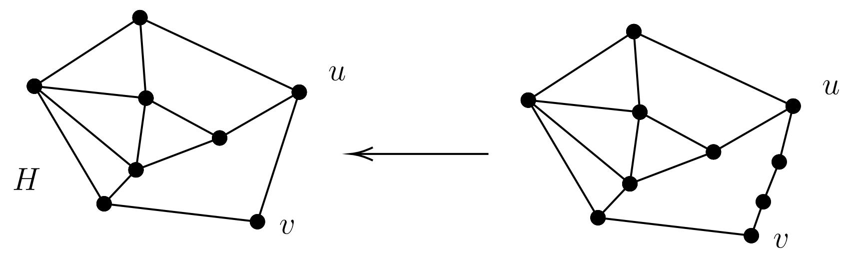
Let G as in the statement be given. Fix a planar drawing of G. Then G is not a cycle (i.e. not(a)).

If there exists $e = xy \in E(G)$ such that $G \setminus e$ is 2-connected, then the planar drawing of G gives a planar drawing of $G \setminus e$ with x and y on the same face $\mathcal{F}$ of $G \setminus e$.

Note If G is 2-sum of a graph H and a cycle, then G is a subdivision of H, with more vertices than H.

Recall we were proving (by induction on # of vertices and # of edges) that in any planar drawing of a 2-connected graph g, every face is bounded by a cycle, and every edge is in exactly two faces.

In the proof, we reached the case in which G is the 2-sum of a 2-connected graph H and a cycle. Then G is a subdivision of H. The planar drawing of G gives a planar drawing of H.



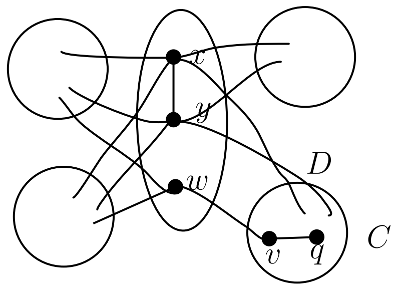
Since $|V(H)| < |V(G)|$, by IH, every face of H is a cycle and every edge is in exactly two faces, then the same is true for G, since the edge uv of H is replaced by a u, v path in G. ■

Theorem Let G be a 3-connected graph with at least 5 vertices, then G has an edge e such that G/e is 3-connected. (We call e a contractible edge)

Proof Suppose on the contrary that no such edge e exists. Then for every $e \in E(G)$, since G/e has ≥ 4 vertices, it must contain a vertex cut of size at least 2.

Let $e = xy$ be an edge, and let z denote the image of xy in G/e. Then the vertex cit S of G/e of size ≤ 2 must contain z (otherwise it was a vertex cut of G), and $|S| = 2$ (otherwise $\{x, y\}$ was a vertex cut of G).

Say $S = \{z, w\}$, then $\{x, y, w\}$ is a vertex cut of G. Then for every component C of $G - \{x, y, w\}$, there is an edge from each of x, y, w into C.



Choose $e = xy$, w and C such that over all such choices, C has the smallest number of vertices.

Let v be a neighbour of w in C. We will find a choice contradicts to previous paragraph.

Since G/wv is not 3-connected, there exists a vertex u such that $\{w, v, u\}$ is a vertex cut of G, and for each component D of $G - \{w, v, u\}$, there is an edge from each of $\{w, v, u\}$ to D.

There exists a component D of $G - \{u, v, w\}$ that is disjoint from $\{x, y\}$. This is because $G - \{w, v, u\}$ has ≥ 2 components, and xy is an edge. We know v has a neighbour q in D and $q \notin \{w, x, y\}$. Thus $q \in C$. Then $D \cap C \neq \varnothing$, and since $D \cap \{x, y, w\} \neq \varnothing$, we find $D \subseteq C$.

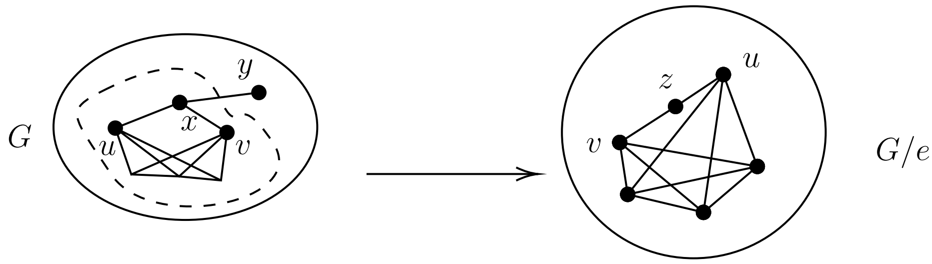
But $v \in C$, and $v \notin D$. Thus $D \subsetneq C$. Thus the choice (wv, u, D) contradicts our choice of (xy, w, C). ■

This theorem will help us prove Kuratowski's Theorem: the graph G is planar $\iff$ it does not contain a subdivision of K_5 or $K_{3,3}$.

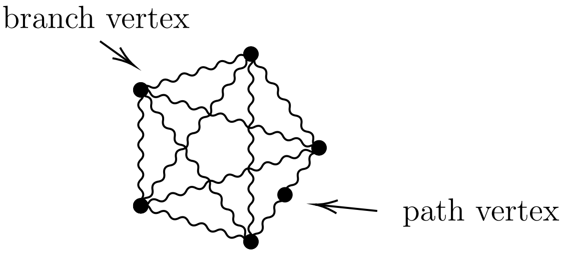
Lemma Let e be an edge of G, and suppose G/e contains a subdivision of K_5, then G contains a subdivision of K_5 or $K_{3,3}$.

Proof Let $e = xy$ and let z denote the image of e under contraction.

Let K be a subdivision of K_5 in G/e.

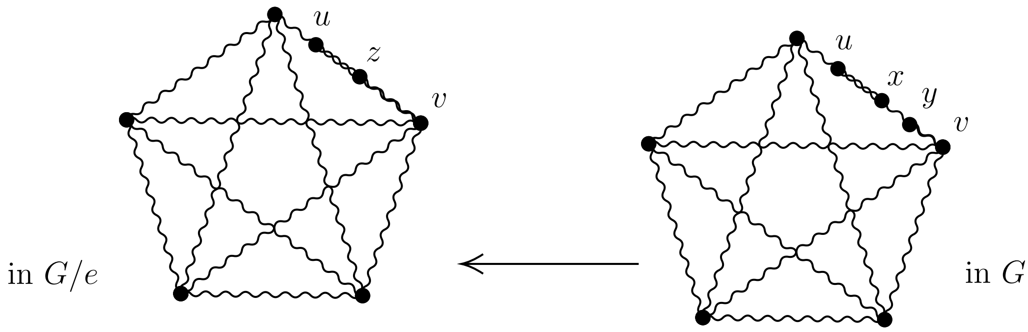


If $z \notin V(K)$, then K is a subgraph of G as needed.



If $z \in V(K)$: if z is a path vertex of K. Let $u, v \in V(K)$ be its neighbours in K.

- If u, v were both neighbours of x in G, then K (with x in place of z) is a subgraph of G.
- If u is a neighbour of x and v is a neighbour of y in G then G contains a subdivision of k, hence of K_5.

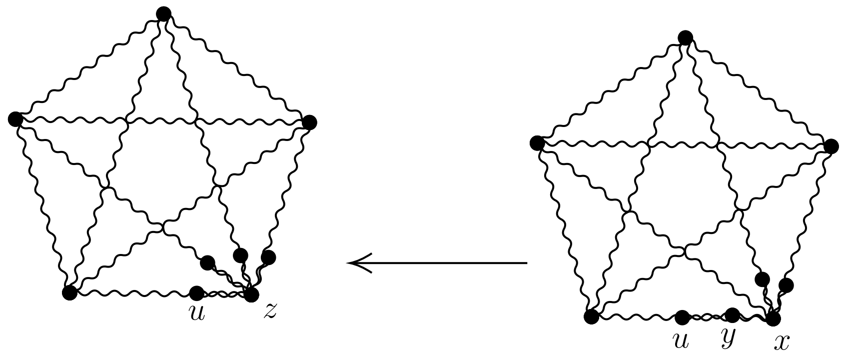


If z is branch vertex of K:

- If all 4 neighbours of z in K are all 4 neighbours of x in G: then as before, G contains K (with x in place of z)
- If 3 neighbours of z in K are x in G, then it is a neighbour of y, then G

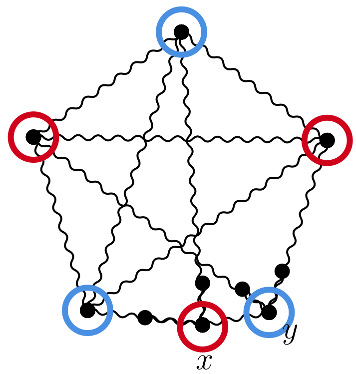
contains a subdivision of K, hence K_5.

in G/e



in G

- If 2 neighbours of z in k are neighbours of x in G and neighbours of y in G, then we get



which contains a subdivision of $K_{3,3}$. ■

Theorem ($K3$) Let G be a 3-connected graph that does not contain a subdivision of K_5 and $K_{3,3}$. Then G is planar.

Proof By induction on $n = |V(G)|$.

Base Case When $n \leq 5$: the only non-planar graph with ≤ 5 vertices is K_5, which is a subdivision of K_5.

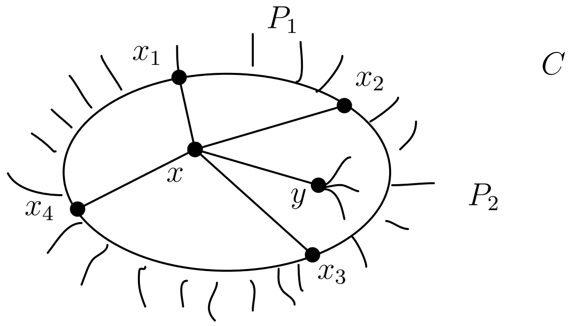
IH $n \geq 6$ and statement holds for any $< n$ vertices. Let G be given, then (by theorem) G has a contractible edge $e = xy$, i.e. G/e is 3-connected.

If G/e contains a subdivision of K_5, then G contains a subdivision of K_5 or $K_{3,3}$ by lemma.

Hence G/e does not contain a subdivision of K_5. If G/e contains a subdivision of $K_{3,3}$, then G/e has $K_{3,3}$ as a minor. Hence so does G. So by earlier lemma, since $K_{3,3}$ is 3-regular (cubic), G contains a subdivision of $K_{3,3}$. Hence G/e does not contain a subdivision of $K_{3,3}$.

Therefore, $G' = G/e$ satisfies the conditions of OH, hence by induction it is planar. Fix a planar drawing of G'.

This gives a planar drawing of $G' - z$. Then the face of $G' - z$ that contains (the location of) z is bounded by a cycle C (using lemma), since $G' = G/e$ is 3-connected, so $G' - z$ is 2-connected.



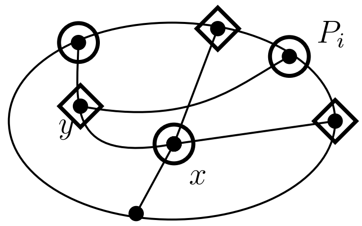
Then all neighbours of z (and hence all neighbours of x and y except x and y themselves) are vertices of C. Let $x_1, x_2, \dots, x_k$ be the neighbours of x, in order on C.

Let P_i denote the path on C from x_i to $x_{i+1} (\mod k)$. Draw x in the drawing, joined to all x_i. We want to add y to complete the drawing.

case 1: All neighbours of y are in same P_i, then add y and edges to its neighbours in the face containing x and P_i, done.

case 2: y has a neighbour in the interior of some P_i (i.e. not x_i or x_{i+1}) AND a neighbour outside P_i.

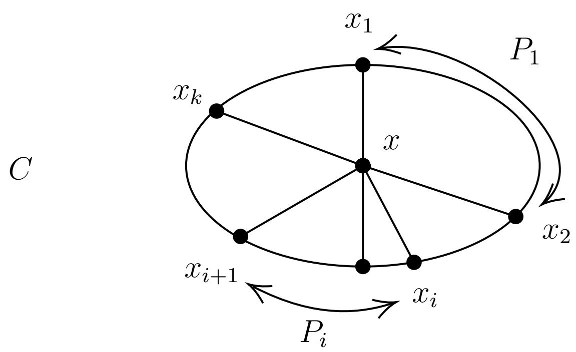
Then G contains a subdivision of $K_{3,3}$. $\star$



So if y has a neighbour in the interior of any P_i, it is covered by case 1 or case 2.

———————————Continued from last class ———————————

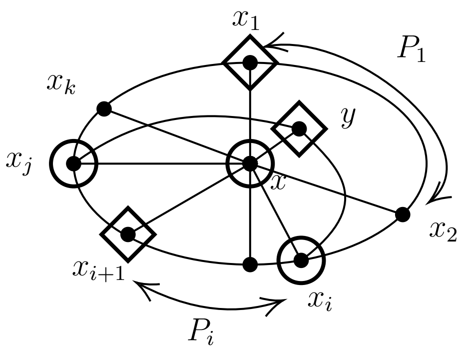
$G, G' = G/e$ 3-connected, $e = xy$, z image of xy in G/e, planar drawing of $G' - z$, cycle C.



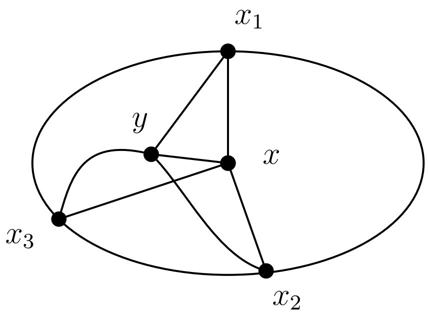
Placed x inside C, its neighbour $x_1, \ldots, x_k$. Path P_i from x_i to x_{i+1} on C.

If y has a neighbour in the interior of any P_i we have dealt with this case.

case 3: Suppose y has neighbours x_i and x_j that are not endpoints of the same path P_t. Then G contains a subdivision of $K_{3,3}$. $\star$



case 4: y has 3 neighbours among x_i's, every pair of which is on the same P_i.



Then $k = 3$, and yx_1, yx_2 and yx_3 ar edges.

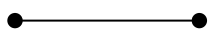
Then G contains a subdivision of K_5. $\star$

Therefore, G is planar. ∎

1.5 Kuratowski's Theorem

A graph G is planar if and only if it does not contains a subdivision of K_5 or $K_{3,3}$.

Proof

- If G is 2-connected (in particular if G is 3-connected) then it is planar by theorem $K2$.
- If G is connected but not 2-connected: If G contains no subdivision of K_5 or $K_{3,3}$, then every block of G contains no subdivision of K_5 or $K_{3,3}$. Each block of G is 2-connected (hence planar by theorem $K2$) or is 

  Hence G is planar by theorem $K1$.

If G is not connected, then we consider each component separately.

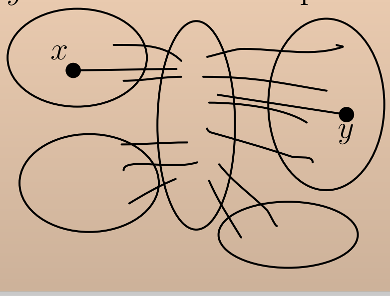
Conversely if G is planar then it does not contain a subdivision of K_5 or $K_{3,3}$

since neither of these is planar. ■

1.6 k-connected graphs

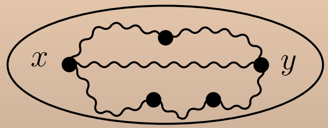
separate

A vertex cut W in a graph G is said to separate vertices x and y in G if x and y are in different components of $G - W$.



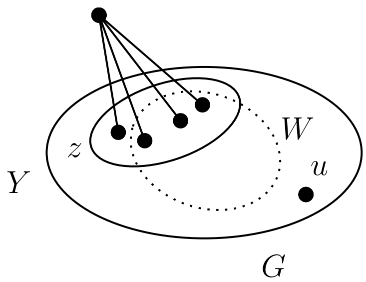
internal disjoint

Let x and y be vertices of a graph G. A set S of paths $(P_1, \ldots, P_k)$, each with endpoints x and y, is called internally disjoint if $V(P_i) \cap V(P_j) = \{x, y\}$.



Theorem (Menger)
Let x and y be disjoint non-adjacent vertices of a graph G. Then the maximum size of a set of internally-disjoint paths between x and y in G is equal to the minimum size of a vertex cut separating x and y in G. (we will prove this later.)

Lemma Let G be a k-connected graph, and let $Y \subset V(G)$ be of size $|Y| = k$. The graph H obtained by adding a new vertex y and all edges $\{yw : w \in Y\}$ is k-connected.



Proof Since $|V(H)| \geq |V(G)| + 1 \geq k + 2$, if H is not k-connected, then it has a vertex cut W with $|W| \leq k - 1$.

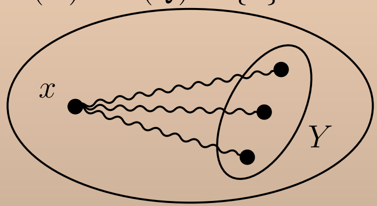
Then $y \notin W$, otherwise $W \setminus \{x\}$ is a vertex cut of G of size $< k$. $\star$.

Let $z \in Y \setminus W$. Let C be the component of $H - W$ containing y, and let u be a vertex of G separated from y by W. But then W separates u from z in G, contradicting the k-connectivity of G (since $z \in G$). Hence H is k-connected. ■

Theorem (Menger-Whitney Theorem)
Let G be a graph with $|V(G)| \geq 2$, then G is k-connected iff every pair of distinct vertices is joined by a set of k internally-disjoint paths. (Proof: postponed, but easy from Menger's theorem)

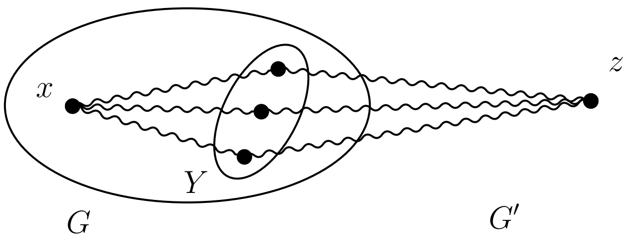
fan

Let G be a graph, let x be a vertex of G and let Y be a subset of $V(G) \setminus \{x\}$. And (x, Y)-fan in G is a set of paths from x to Y, where $|S| = |Y|$, and $V(P) \cap V(Q) = \{x\}$ for all $P \neq Q$ in S.



Lemma (Fan Lemma)
Let $k \geq 2$, and let G be a k-connected graph. Then for any $x \in V(G)$ and $Y \subseteq V(G) \setminus \{x\}$, there exists an (x, Y)-fan in G.

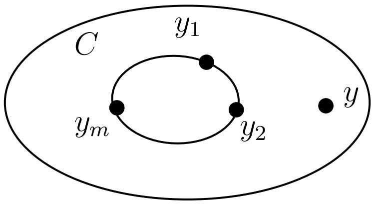
Proof Let G' be the graph formed from G by adding a new vertex z and edges $\{zy, y \in Y\}$.



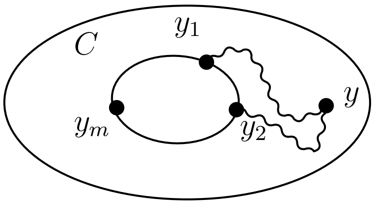
Then by our previous lemma we know G' is k-connected. Then by the Menger-Whitney Theorem, there exists a set $\{P_1, \ldots, P_k\}$ of internally-disjoint paths from x to z in G'. Then $\{P_1 - z, \ldots, P_k - z\}$ is an (x, Y)-fan in G. ■

Lemma (cycle lemma)
Let $k \geq 2$ and let G be a k-connected graph. Let $Y \subseteq V(G)$ be a set of k vertices in G. Then there exists a cycle in G containing all vertices in Y.

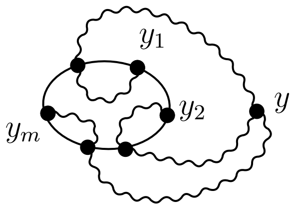
Proof Let C be a cycle containing as many of the vertices of Y as possible: let this set be $\{y_1, \ldots, y_m\}$. If $m = k$, were done. So assume $m < k$. Since $k \geq 2$, we know that $m \geq 2$. Let $y \in Y \setminus \{y_1, \ldots, y_m\}$.

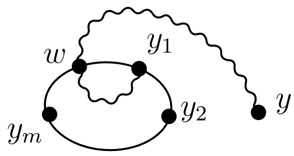


case 1: $V(C) = \{y_1, \ldots, y_m\}$. Then by fan lemma applied to y and $Y' = \{y_1, \ldots, y_m\}$. Since G is m-connected, there exists an (y, Y')-fan in G, with paths P_i from y to y_i, for $1 \leq i \leq m$. Then (assuming y_1 and y_2 are adjacent on C) the cycle $y_1 P_1 y P_2 y_2 y_3 \ldots y_m$ is a cycle containing $\{y, y_1, \ldots, y_m\}$ contradicting the choice of C.



case 2: C contains other vertices besides $Y' = \{y_1, \ldots, y_m\}$. Let $x \in V(C) \setminus \{y_1, \ldots, y_m\}$. Let $\{P_0, P_1, \ldots, P_m\}$ be a $(y, Y' \cup \{x\})$-fan in G.





Let $w_i (0 \leq i \leq m)$ be (resp) the first vertex on C encountered when moving along P_i from y to C. Then since $P_0 \ldots P_m$ are disjoint except for y, we know: $w_0, w_1, \ldots, w_m$ are all distinct.

Then for some w_j and w_ℓ there is no vertex of Y' strictly inside (w_j, w_ℓ)-segment of C, once there are $k+1$ w's and only k y's. Then the cycle obtained from C by replacing the (w_j, w_ℓ)-segment by two paths from y to w_j and w_ℓ from the fan contains more vertices of Y. This contradicts our choice of C. ■

1.7 Menger's Theorem

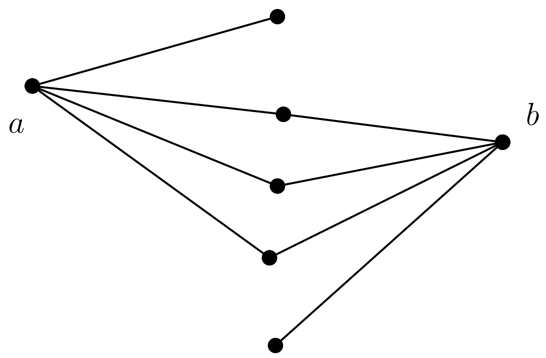
Theorem Let a and b be distinct non-adjacent vertices in a graph G. Let s be the minimum size of a vertex cut separating a and b in G. Then G contains a set of s internally disjoint (a, b)-paths.

Proof The statement is clearly true when $s \leq 1$, so we may assume $s \geq 2$. We use induction on $|E(G)|$.

Base case $|E(G)| = 0$ true.

IH Assume $|E(G)| \geq 1$ and the theorem is true for all graphs with fewer than $|E(G)|$ edges.

case 1: Suppose all edges of G are incident to a or b.

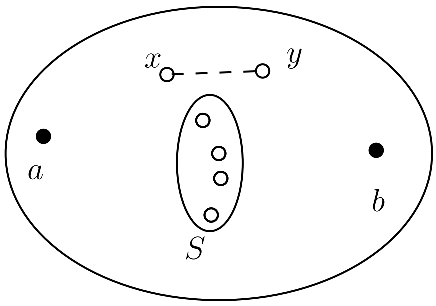


Then $s = |\{u \in V(G) : ua \in E(G) \text{ and } ub \in E(G)\}|$

Here is a set of exactly internally disjoint (a, b) paths of length 2 one for each u.

case 2: There exists an edge $e = xy \in E(G)$ such that $\{x, y\} \cap \{a, b\} = \varnothing$.

Let $H = G \setminus e$. Let S be a vertex cut of H of minimum size that separates a and b in H.

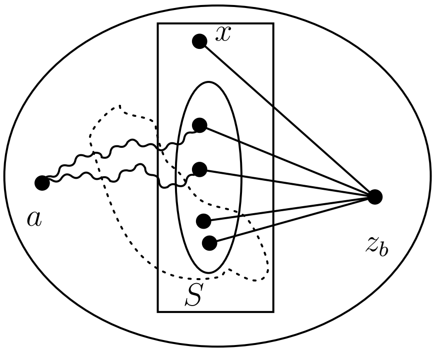


If $|S| \geq s$, by IH applied to H, there exists s internally disjoint (a, b)-paths in H, hence also in G.

So we may assume that $|S| < s$. Note that S is not a vertex cut separating a and b in G, but $S \cup \{x\}$ (and also $S \cup \{y\}$) <u>are</u> vertex cuts separating a and b in G, since removing x (or y) removes the edge e. Therefore $|S| = s - 1$.

Also, any paths (we will fix one arbitrary) from a to b in $G - S$ must use the edge xy. Assume WLOG that x is closer to a than y is, on this path.

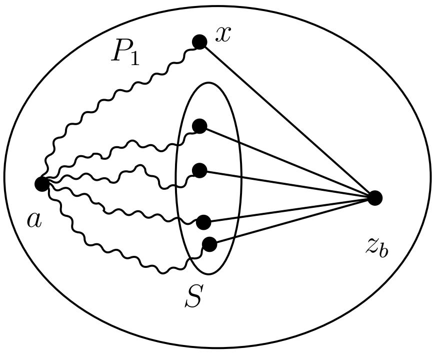
Let G_b be the graph obtained from G by contracting all edges in the component C_b of $G - (S \cup \{x\})$. Call the new vertex z_b.



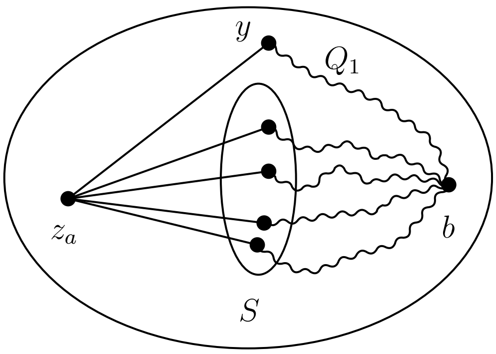
G_b

If T is a vertex cut of G_b separating a and z_b, then T is a vertex cut of G separating a and b. Therefore the minimum size of a vertex of G_b separating a from z_b is $\geq s$.

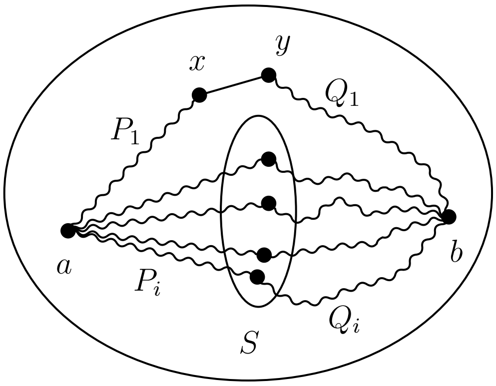
Note that $|E(G_b)| < |E(G)|$ because both y and b disappear in the contraction, and there is a path from y to b of length ≥ 1. Hence by IH, there are $\geq s$ internally disjoint (a, z_b)-paths $P_1, P_2, \ldots, P_s$ in G. So these paths have as second-last vertex all vertices of $S \cup \{x\}$. Let $P_i' = P_i - z_b$.



Similarly there exists $\geq s$ internally disjoint (z_a, b)-paths $Q_1 \ldots Q_s$ in the graph G_a obtained by the same procedure using vertex cut $S \cup \{y\}$. Let $Q_i' = Q_i - \{z_a\}$.



Then concatenating each P_i' and Q_i' (for $i \geq 2$) and $P_1 \cup Q_1 \cup \{xy\}$ give s (a, b)-paths, which are internally disjoint provided $V(P_i') \cap V(Q_j') \subseteq S \cup \{x, y\}$. But this follows because $V(P_i') \setminus (S \cup \{x, y\})$ for each i is contained in C_a ? (which is contained in the component of $H - S$ containing a), and similarly for $V(Q_i')$. So they are disjoint.



■

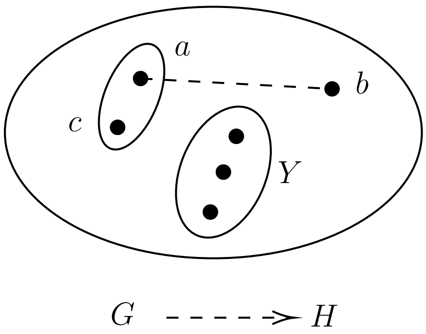
1.8 Menger-Whitney Theorem

Theorem Let G be a graph with $|V(G)| \geq 2$, then G is k-connected if and only if for every pair of distinct vertices a and b, there exists a set of k internally disjoint (a, b)-paths in G.

Proof The statement is true by definition for $k = 1$. So assume $k \geq 2$.

First suppose G is k-connected. Let a and b be distinct vertices.

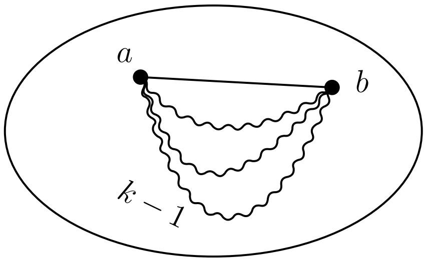
If $ab \notin E(G)$ then since G is k-connected, the min size of a vertex cut separating a and b is $\geq k$. Thus by Menger's Theorem there exists a set of k internally disjoint (a, b)-paths as required.



If $ab \in E(G)$, consider $H = G \setminus ab$. We claim that H is $(k-1)$-connected.

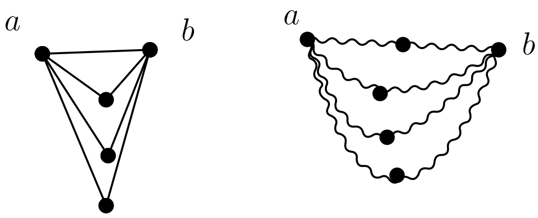
Suppose not. then $|V(H)| \geq |V(G)| \geq k+1$ so there would exist a vertex cut Y in H with $|Y| \leq k-2$, then Y separates a and b in H, otherwise Y would be a vertex cut of G. So WLOG (since $|V(H)| \geq k+1$) some component of $H - Y$ not containing b contains a vertex c different from a. But then $Y \cup \{a\}$ is a vertex cut of size $\leq k-1$ separating b from c in G. $\star$

Hence we can apply Menger's Theorem to H to find $k-1$ internally disjoint (a, b) paths in H. These together with edge ab completes the proof.



Conversely, suppose every (a, b) is joined by k internally disjoint (a, b)-paths. Then there is no vertex cut in G of size $\leq k-1$.

Also, $|V(G)| \geq k+1$ because for any a, b distinct, at least $k-1$ of the internally disjoint paths must contain a vertex not in any other. So G is k-connected.



■

CHAPTER 2

Matchings

matching

A matching in a graph G is a set of edges, no two of which share a vertex.

saturate

A matching M saturates a vertex v if v is incident to an edge of M.
A matching M in a graph G is perfect if M saturates all vertices in $V(G)$.
(So if G has a perfect matching then $|V(G)|$ is even.)

We will see a characterizations of graphs with perfect matchings (Tutte's Theorem) and an algorithm for finding a maximum matching (a matching of maximum size) in general graphs (Edmond's Algorithm).

2.1 Greedy Algorithm for Matching

Algorithm 1: Greedy Algorithm

Input: Graph G
Output: A matching of G

1 $M := \varnothing$ $H := G$
2   If H has no edges, stop. Output M
3   Pick an edge xy in H. Set $M := M \cup \{xy\}$
4   Set $H = H - \{x, y\}$, go to 2.

We denote by $\nu(G)$ the size of maximum matching in G. Then the Greedy Algorithm finds a matching in G of size at least $\frac{\nu(G)}{2}$.

Let M be a matching in G obtained by the Greedy Algorithm, then each edge of G shares a vertex with an edge of M. Let M' be a maximum matching in G. Then each edge of M can share a vertex with ≤ 2 edges of M'. Thus $|M'| \leq 2|M|$. Hence $|M| \geq \frac{\nu(G)}{2}$.

neighbourhood

For a set S of vertices of a graph G, the neighbourhood of S is

$$\Gamma(S) = \{y \in V(G) : xy \in E(G) \text{ for some } x \in S\}$$

(sometimes written as $N(S)$)

2.2 Hall's Theorem

Theorem (Hall's Theorem)
Let G be a bipartite graph with vertex classes X and Y then G has a matching saturating X if and only if

$$(*) \quad |\Gamma(S)| \geq |S| \quad \text{for each } S \subseteq X$$

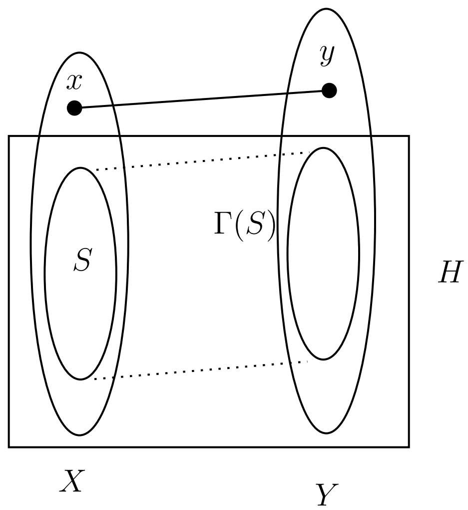
$(*)$ is called "Hall's Condition"

Proof ($\Longrightarrow$) If G has a matching M saturating X, then for every S, just the set of neighbours of S via edges of M has size $|S|$, so $(*)$ holds.

($\Longleftarrow$) Assume $(*)$ holds. We use induction on $|X|$. If $|X| = 1$, then the statement is clearly true.

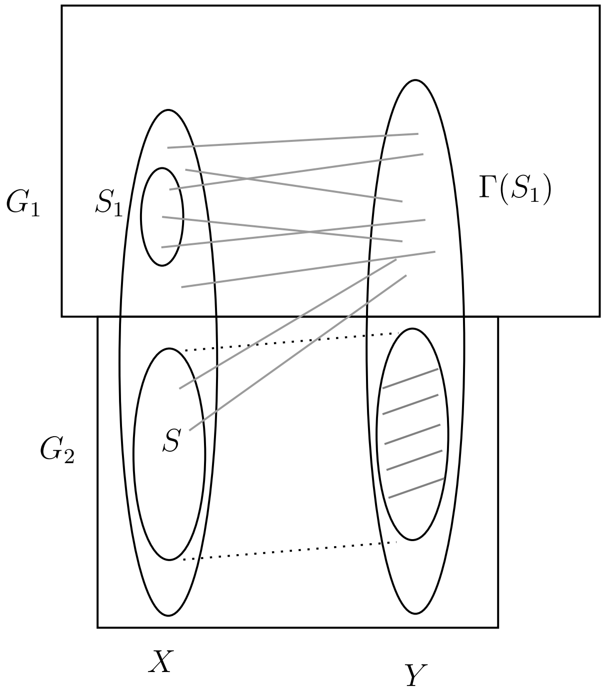
IH Assume $|X| \geq 2$ and statement holds for any graph satisfying $(*)$ with smaller X. Let G be given.

case 1: Suppose $|\Gamma(S)| > |S|$ for every $\varnothing \neq S \neq X$. Let $x \in X$ be arbitrary. Choose a neighbour y of x in Y (exists since $(*)$ holds.)



Let $H = G - \{x, y\}$. Then $(*)$ holds for H, since $|\Gamma_H(S)| \geq |\Gamma_G(S)| - 1 \geq |S|$. Thus by IH, H has a matching M_H saturating $X \setminus \{x\}$. Thus $M_H \cup \{xy\}$ is a matching in G saturating X.

case 2: There exists a subset $S_1 \subset X$ with $\varnothing \neq S_1 \neq X$ for which $|\Gamma(S_1)| = |S_1|$. Let G_1 be the subgraph of G induced by $S_1 \cup \gamma(S_1)$, and G_2 induced by $(X \setminus S_1) \cup (Y \setminus \Gamma(S_1))$.



Then G_1 satisfies $(*)$ since $\Gamma(S) \leq \Gamma(S_1)$ for each $S \subseteq S_1$.

We claim that G_2 also satisfies $(*)$: Let $S \subseteq X \setminus S_1$. Then

$$\Gamma_{G_2}(S) = \Gamma_G(S \cup S_1) \setminus \Gamma_G(S_1)$$

So

$$\begin{aligned} |\Gamma_{G_2}(S)| &= |\Gamma_G(S \cup S_1)| - |\Gamma_G(S_1)| \\ &\geq \underbrace{|S \cup S_1|}_{by\ (*)} - \underbrace{|S_1|}_{by\ case\ 2} \\ &= |S| \end{aligned}$$

Hence both G_1 and G_2 have matchings M_1, M_2 saturating S_1 and $X \setminus S_1$ respectively by IH. Since M_1 and M_2 are in disjoint graphs, $M_1 \cup M_2$ is the required matching in G. ∎

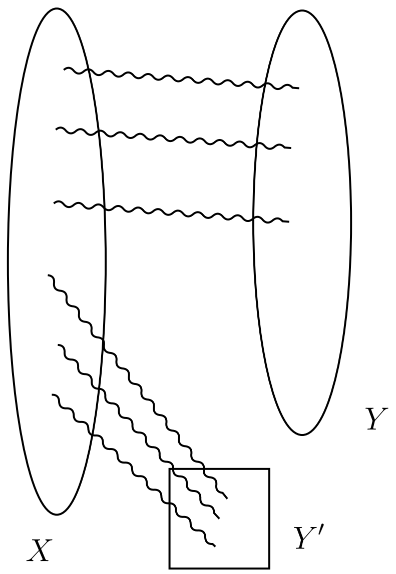
2.2.1 defect version

Theorem (defect version of Hall's Theorem)
Let d be a non-negative integer, then a bipartite graph G with vertex classes X and Y has a matching of size $\geq |X| - d$ if and only if

$$(**) \quad |\Gamma(S)| \geq |S| - d \ \text{ for all } S \subseteq X$$

Proof ($\Longleftarrow$) as before

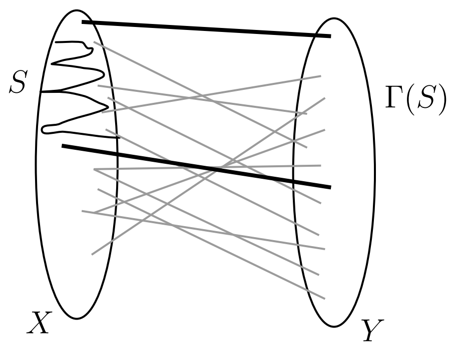
($\Longrightarrow$): Construct a graph H by adding a set Y' of d vertices to Y and joining them all to all vertices in X.



Then $(*)$ holds in H, so by Hall's Theorem, H has a matching H of size $|X|$. Then the subset of M of edges not incident to Y' is the required matching in G. ■

Theorem Any regular bipartite graph with degree $k \geq 1$ has a perfect matching.

Proof Let X and Y be the vertex class of G. Let $S \subseteq X$ be an arbitrary subset of X. Let $E(S, \Gamma(S))$ denote the set of edges of G from S to $\Gamma(S)$. Then $|E(S, \Gamma(S))| = k|S|$.



But also $|E(S, \Gamma(S))| \leq k|\Gamma(S)|$ since G is k-regular. Hence

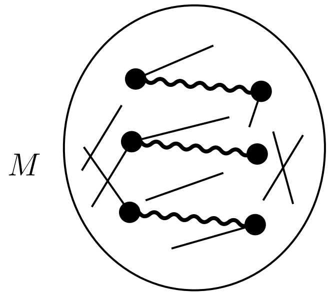
$$k|S| \leq k|\Gamma(S)| \implies |\Gamma(S)| \geq |S|$$

So Hall's Theorem implies G has a matching of size $|X|$. Similarly, it has a matching of size $|Y|$. Hence $|X| = |Y|$ and G has a perfect matching. ■

Recall $\nu(G)$ is the maximum size of matching in G.

vertex cover

A vertex cover of a graph G is a set $C \subseteq V(G)$ such that every edge of G is incident to a vertex of C. We denote by $\tau(G)$ the minimum size of vertex cover of G.



Note that $\nu(G) \leq \tau(G) \overset{*}{\leq} 2\nu(G)$ for every graph.

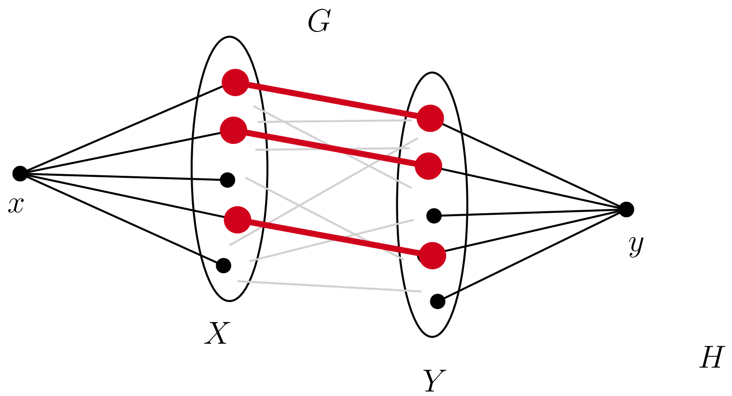
$(*)$: since all vertices saturated by a maximal matching form a vertex cover.

Note that $\tau(G) = 2\nu(G)$ holds for K_3.

2.3 König's Theorem

Theorem (König's Theorem)
If G is bipartite, then $\tau(G) = \nu(G)$.

Proof Let X and Y denote the vertex classes of G. Construct H by adding new vertices x and y, and edges $\{xz : z \in X\} \cup \{yz : z \in Y\}$.



Note that if C is a vertex cut of H separating x and y, then C must contain a vertex cover of G. (otherwise a path from x to y would remain in $H - C$)

If S is a set of internally disjoint (x, y)-paths in H, then the set consisting of the second edges of all paths in S forms a matching in G.

Therefore

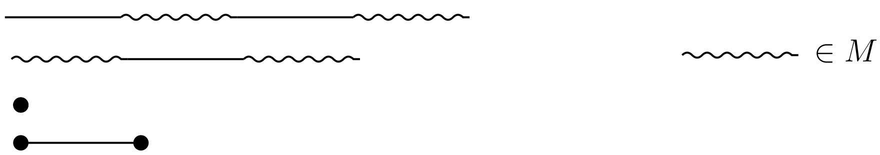
$$\tau(G) \le \begin{array}{l}\text{min size of}\\ \text{vertex cut}\\ \text{separating}\\ x, y \text{ in } H\end{array} \overset{\text{Menger}}{\overbrace{=}} \begin{array}{l}\text{max size of a set of}\\ \text{internally disjoint}\\ (x, y) \text{ paths in } H\end{array} \le \nu(G)$$

■

M-alternating path

Let M be a matching in graph G. An M-alternating path is a path in G with every second edges in M.

Ex

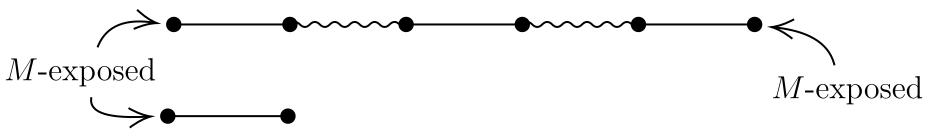


M-exposed

We say a v is M-exposed if it is not M-saturated.

M-augmenting

An M-augmenting path is an M-alternating path of length ≥ 1 that starts and ends with an M-exposed vertex.



If P is an M-augmenting path in G, then

$$M' = M \Delta E(P) = M \smallsetminus (M \cap E(P)) \cup (E(P) \smallsetminus M)$$

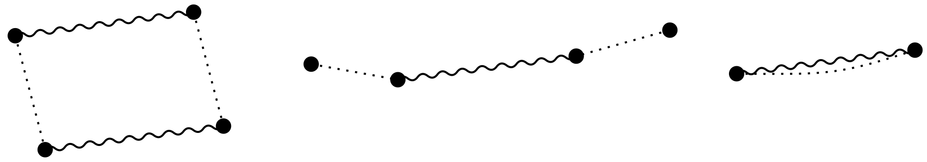
is a matching in G with $|M'| = |M| + 1$. (We sometimes say M' is "M switched on P")

Theorem (Berge's Theorem)
A matching M in graph G is a maximum matching if and only if there is no M-augmenting path in G.

Proof If M is maximum, then there is no M-augmenting path P (otherwise $M \Delta E(P)$ contradicts the maximality of M).

Conversely, suppose there is no M-augmenting path in G. Let M^* be a maximum matching in G. Consider the subgraph H of G with edge set $M \cup M^*$. The components of H can be

- single edges in $M \cap M^*$
- even cycles that are (M, M^*) alternating
- paths that are (M, M^*)-alternating



No path component can have more M^*-edges than M-edges, since it would be M-augmenting path. So the number of M^*-edges is at most the number M-edges in every component of H.

Hence $|M^*| \leq |M| \implies M$ is maximum. ∎

independent

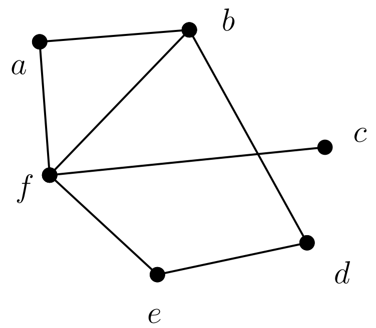
Recall that an independent set[a] of vertices in a graph G is a set $W \subseteq V(G)$ such that the subgraph $G[W]$ induced by W has no edges. We denote by $\alpha(G)$ the maximum size of an independent set of vertices in G.

[a] In graph theory, an independent set, stable set, coclique or anticlique is a set of vertices in a graph, no two of which are adjacent. That is, it is a set S of vertices such that for every two vertices in S, there is no edge connecting the two. Equivalently, each edge in the graph has at most one endpoint in S. (from wiki)

edge cover

An edge cover of a graph G is a set $S \subseteq E(G)$ such that every vertex of G is incident to an edge of S. We denote by $\rho(G)$ the minimum size of an edge cover of G. (If G has isolated vertices we can say it is $\begin{cases} \infty \\ \text{undefined} \end{cases}$)

$\{a, c, e\}$ is indepedent
$\{fc, de, ab\}$ is an edge cover
$\{b, f, d\}$ is a vertex cover



Lemma For every graph G,

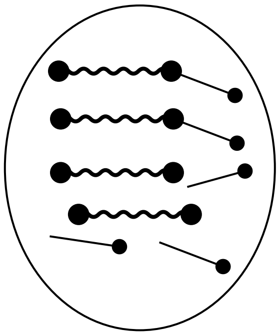
$$\alpha(G) + \tau(G) = |V(G)|$$

Proof Any vertex cover C is such that $G - C$ has no edges, i.e. the subgraph induced by $V(G) \setminus C$ is an independent set. So C is minimum vertex cover if and only if $V(G) \setminus C$ is a maximum independent set. ■

Lemma (Gallai)
Let G be a graph with no isolated vertices, then

$$\nu(G) + \rho(G) = |V(G)|$$

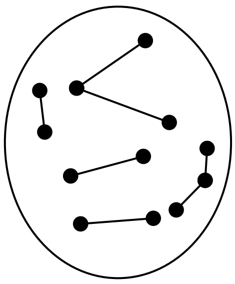
Proof Let $|V(G)| = n$ and let M be a maximum matching (so $|M| = \nu(G)$). Let $|V(M)|$ be the set of vertices saturated by M, then $V(G) \setminus V(M)$ is independent.



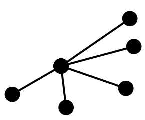
Form an edge cover S of G by taking one edge incident to each $x \in V(G) \setminus V(M)$, plus the matching edge of M. This gives $n - 2|M| + |M| = n - |M| = n - \nu(G)$ edges in S. So

$$\rho(G) \leq |S| = n - \nu(G) \implies \rho(G) + \nu(G) \leq n$$

Conversely, suppose F is an edge cover of G of size $\rho(G)$. Then each edge of F is incident to a vertex that is not incident to any other edge of F.



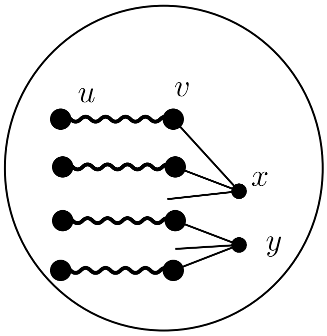
Therefore the spanning subgraph H of G with $E(H) = F$ has the property that every component is a star (so H has no cycles) Then there is a matching in G formed by taking one edge from each component of H.



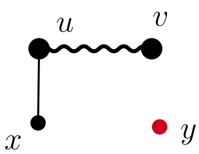
Since H has n vertices and $\rho(G)$ edges, and no cycles, it has exactly $n - \rho(G)$ components (MATH 239 fact...). Thus $\nu(G) \geq n - \rho(G)$. So $\nu(G) + \rho(G) \geq n$. ■

Theorem (Erdös-Posa)
For every graph G, $\nu(G) \geq \min\{\delta(G), \left\lfloor \frac{|V(G)|}{2} \right\rfloor\}$.

Proof Let M be a maximum matching and let $V(M)$ denote the set of vertices saturated by M. If $|M| \geq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$, then we are done. So assume $|M| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor - 1$, so $|V(M)| \leq |V(G)| - 2$.



Let $x, y \in V(G) \smallsetminus V(M)$, then all edges incident to x or y are incident to some vertex of $V(M)$. For any $uv \in M$, note that



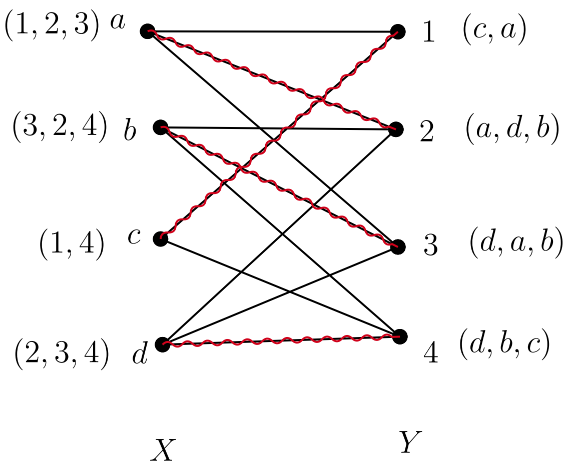
If $xu \in E(G)$ then $yv \notin E(G)$, so the number of edges from $\{x, y\}$ to $\{u, v\}$ is at most 2. So the total number N of edges from $\{x, y\}$ to $V(M)$ is at most $2|M|$. Thus $d(x) + d(y) \leq 2|M|$.

$$\implies 2\delta(G) \leq 2|M| \implies |M| \geq \delta(G)$$

■

2.4 Stable Matching

Let G be a bipartite graph with vertex classes X and Y. Suppose for each vertex z of G, there is a linear order $L(z)$ of the vertices in $\Gamma(z)$. Hence $L(z)$ is called the preference list for z.



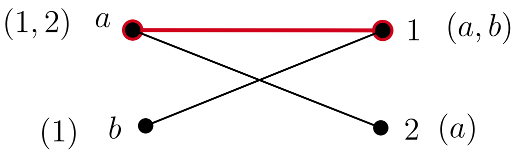
stable matching

A stable matching in (G, L) is a matching M such that for every edge $xy \notin M$, either

- $xy' \in M$ for some $y' > y$ in $L(x)$ (i.e. x is matched to a neighbour it prefers to y)
  or
- $x'y \in M$ for some $x' > x$ in $L(y)$ (i.e. y is matched to a neighbour it prefers to x)

We will see that every (G, L) has a stable matching.

Note A stable matching is not necessarily a maximum matching. For example,



$\{1a\}$ is stable but not maximum

2.4.1 Gale-Shapley Algorithm

Algorithm 2: Gale-Shapley Algorithm

Input: A bipartite graph G with vertex classes X and Y and preference lists L

Output: A stable matching M^* in G

1 Set $K(x) := L(x)$ for each $x \in X$. Set $M := \varnothing$

2 If for each $x \in X$, either $K(x) = \varnothing$ or x is M-saturated then STOP. Set $M^* := M$ and OUTPUT M^*

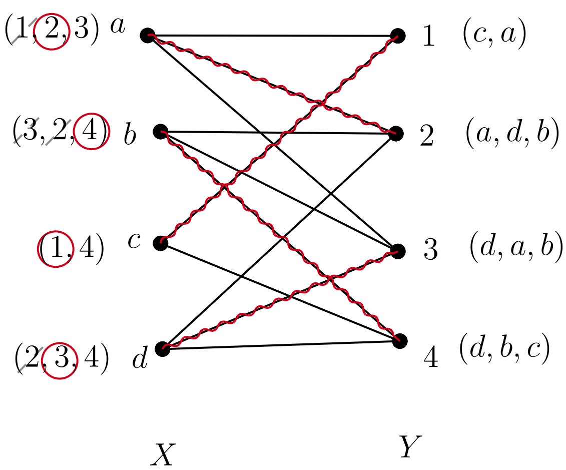
3 Otherwise choose $x \in X$ where $K(x) \neq \varnothing$ and x is M-exposed. Let y be the largest element of $K(x)$. So $xy \in E(G)$.

4 If $x'y \in M$ and $x > x'$ in $L(y)$ (i.e. y prefers x to x'), then set $M := M \setminus \{x'y\} \cup \{xy\}$

5 If y is M-exposed then set $M := M \cup \{xy\}$

6 Set $K(x) = K(x) \setminus \{y\}$

7 Go to 2.



Theorem Gale-Shapley Algorithm finds a stable matching in (G, L).

Proof First note that the quantity $\sum_{x\in X} |K(x)|$ decreases by 1 at each iteration, so the algorithm terminates in at most $\sum_{x\in X} |L(x)| \sum_{x\in X} |L(x)| = |E(G)|$.

To see M^* is a matching, note that in steps 4 and 5, if an edge xy is added to M then x is M-exposed, and edge is incident to y is removed.

To show M^* is stable, consider an edge $x_0y_0 \notin M^*$.

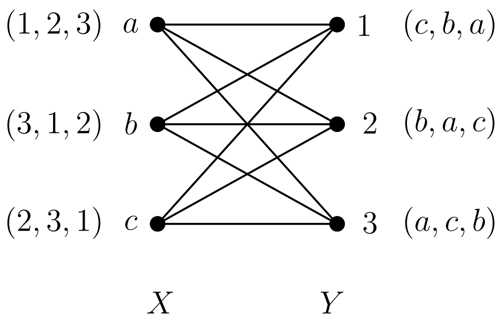
case 1: $y_0 = y$ at line 3 in some iteration I with $x_0 = x$.
If x_0y_0 is put into M in step 4, then at some later iteration I' it was removed. Thus in I' there was $x_1 > x_0$ in $L(y)$ and x_1y_0 was put into M. Hence in M^*, $x_fy_0 \in M^*$ for some $x_f > x_0$ in $L(y_0)$.

NOTE: At all times in the algorithm, the situation for each $y \in Y$ only improves (or stays the same) and for each $x \in X$ it only deteriorates (or stays the same).

If x_0y_0 is not put into M in I, then x_1y_0 already for some $x_1 > x_0$ in $L(y_0)$.

case 2: y_0 is never considered at line 3 when $x = x_0$. Since $y_0 \in K(x_0)$ at termination, then x_0 is matched to some $y_1 > y_0$ in $L(x_0)$ at termination.

Hence, M is stable. ■



$M_1 = \{a1, b3, c2\}$ stable, found by G.S. alg with X

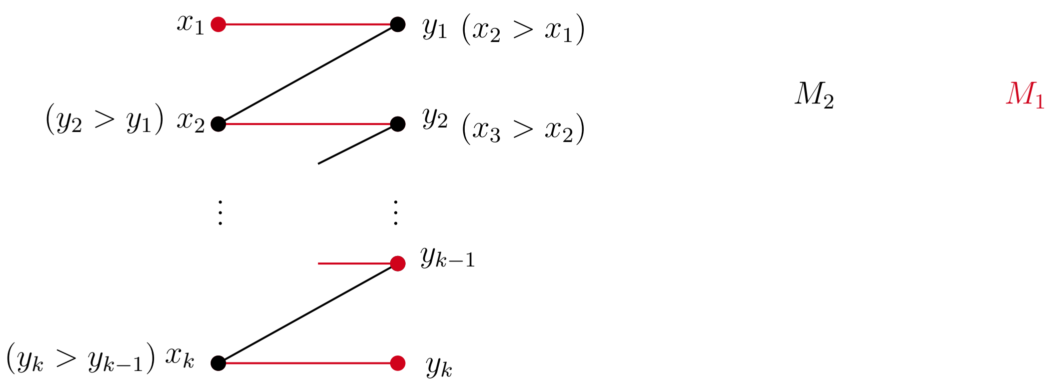
$M_2 = \{a2, b1, c3\}$ stable

$M_1 = \{a3, b2, c1\}$ stable, found by G.S. alg with Y

Theorem All stable matchings in (G, L) have the same size.

Proof Let M_1 and M_2 be two stable matchings in G with preference lists L. Let H be the subgraph of G with edge set $M_1 \cup M_2$, then every component of H is a path or an even cycle.

If $|M_1| > |M_2|$, then some path component $P = x_1y_1x_2y_2 \dots x_ky_k$ must have more M_1-edges than M_2-edges. Thus $x_iy_i \in M_1$, for each i, $y_ix_{i+1} \in M_2$ for each i, and x_1 and y_k are M_2-exposed.



Since $x_1y_1 \notin M_2$, and x_1 is M_2-exposed in $L(y_1)$ we have $x_2 > x_1$. Since $x_2y_1 \notin M_1$, and $x_2 > x_1$ in $L(y_1)$ we find $y_2 > y_1$ in $L(x_2)$. Suppose for $2 \le i \le k-1$ we know that $y_i > y_{i-1}$ in $L(x_i)$, then since $x_iy_i \notin M_2$ we conclude $x_{i+1} > x_i$ in $L(y_i)$. Then since $x_{i+1}y_i \notin M_1$, we find $y_{i+1} > y_i$ in $L(x_{i+1})$. But then $y_k > y_{k-1}$ in $L(x_k)$ and $x_ky_k \notin M_2$. But y_k is M_2-exposed, this contradicts the fact M_2 is stable.

Hence $|M_1| = |M_2|$. ■

X-optimal

Let G be a bipartite graph with vertex classes X and Y and preference lists L. A stable matching M is X-optimal if each $x \in X$ is matched by M to the best possible neighbour it could set in ANY stable matching.

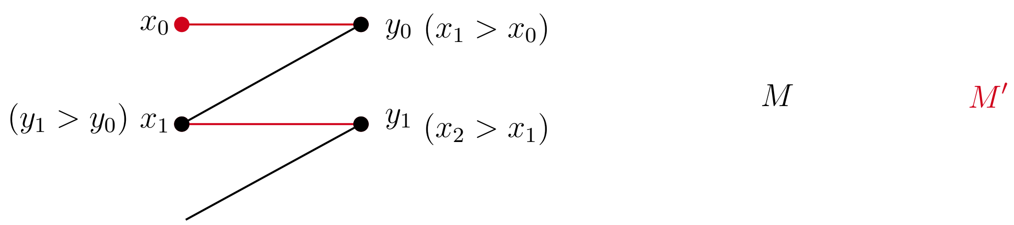
Theorem The matching found by GS Alg is X-optimal.

Proof Let M' be an arbitrary stable matching in G. Let M^* be the GS matching. Suppose on the contrary that $x_0y_0 \in M'$ for some $x_0 \in X$, where x_0 is either M^*-exposed or $x_0y^* \in M^*$ for some $y^* < y_0$ in $L(x_0)$

[we say x_0 is "worse off" in M^* than in M']

Let I be the first iteration of GS Alg such that

- there exists $x_0y_0 \in M'$ where x_0 is worse off in M^*
- an edge y_0x_1 is put into M in iteration I where $x_1 > x_0$ in $L(y_0)$



Since $x_1y_0 \notin M'$ and M' is stable, for some y_1 we have $x_1y_1 \in M'$ for some $y_1 > y_0$ in $L(x_1)$. Then in some earlier iteration I' of GSA, x_1 proposed to y_1 and was rejected, because x_2y_1 was already in M for some $x_2 > x_1$ by y_1.

So iteration I' we had

- there exists $x_1y_1 \in M'$ where x_1 is worse off in M^* (recall situation only deteriorates for x in GSA)
- y_1x_2 is put into M in iteration I' where $x_2 > x_1$ in $L(y_1)$

This contradicts our choice of I.

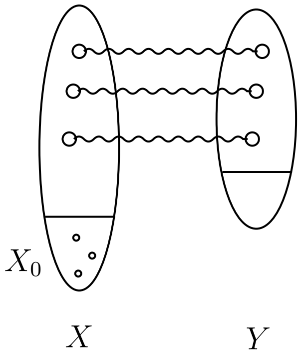
Hence M^* is X-optimal. ■

Y-pessimal

Similar to the last theorem, the GS alg finds the Y-pessimal stable matching in (G, L): for each $y \in Y$, either y is unmatched or is matched to the worst possible neighbour it could set in any stable matching.

Corollary Every stable matching in (G, L) saturates the same set of vertices.

Proof Suppose $x \in X$ is not saturated by the GS matching (call the subset of X of unmatched vertices in the GS matching X_0).



Since GS is X-optimal, each $x \in X_0$ is unmatched in every stable matching. Since every stable matching M has size $|X| - |X_0|$, we find that M saturates exactly the set $X \setminus X_0$. Similarly for $y \in Y$, if y is saturated by the GS matching, then since GS is y-pessimal, y is saturated by every stable matching. ■

Corollary The matching found by GS alg is independent of the order in which the "proposal" steps are executed.

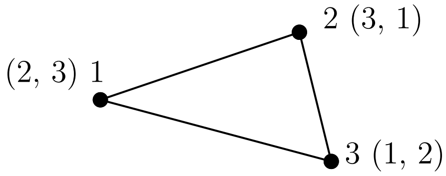
Proof The X-optimal matching is unique. ■

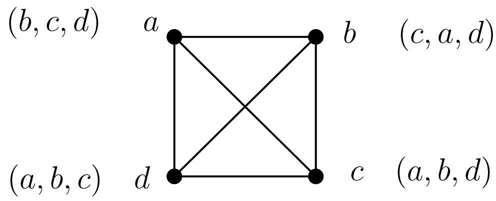
2.5 Stable Roommates Problem

Stable matching in general (i.e. not bipartite) graphs: each vertex v has a linear order $L(v)$ on its neighbour, and a matching M is stable if for each edge $xy \notin M$, either $xy_1 \in M$ for some $y_1 > y$ in $L(x)$ or $x_1y \in M$ for some $x_1 > x$ in $L(y)$.

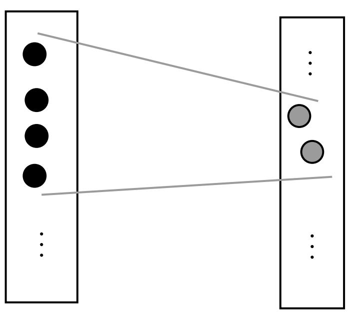
A stable matching in this setting might not exist.

Eg:



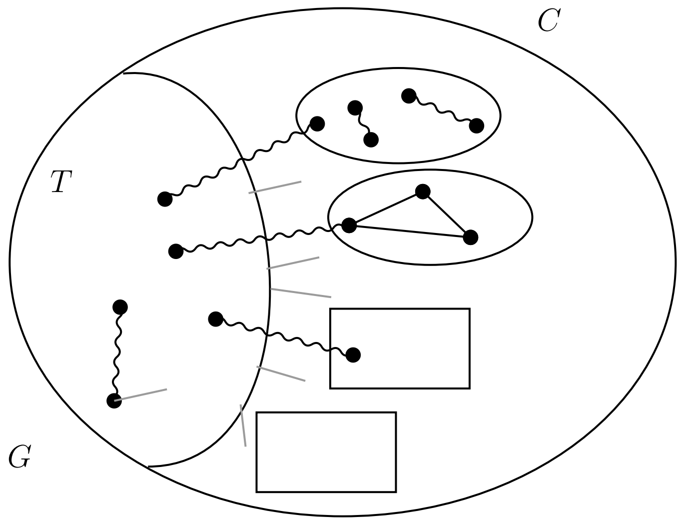


2.6 Tutte's Theorem



When does a graph have a perfect matching?

Let G be a graph with a perfect matching M, then in particular, $|V(G)|$ is even.



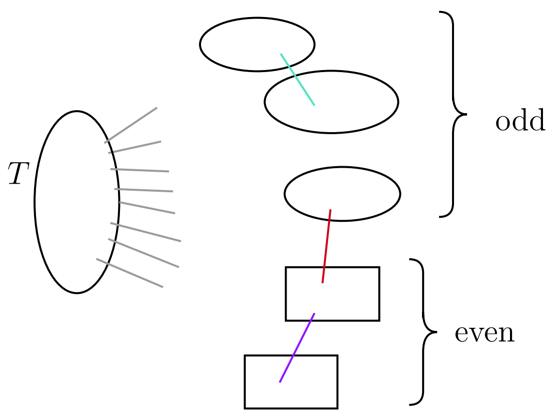
Suppose $T \subseteq V(G)$. Consider the components of $G - T$ (the subgraph of G induced by $V(G) \smallsetminus T$). If C is an odd component of $G - T$ (i.e. $|V(C)|$ is odd), at least one edge of M must join C to T.

For each odd component there is such an edge of M, so the number of odd components in $G - T$ is at most $|T|$. We denote by odd(H) the number of odd components in the graph H. Thus if G has a perfect matching then for every $T \subseteq V(G)$, odd$(G - T) \subseteq |T|$.

Theorem (Tutte's Theorem)
A graph G has a perfect matching if and only if $(**)$ for every $T \subseteq V(G)$, odd$(G - T) \leq |T|$.

Notes $(**)$ is called Tutte's Condition. If G has an odd number of vertices then it fails $(**)$ when $T = \varnothing$. (Since G must have a component, e.g. itself if it is connected that has an odd number of vertices).

Lemma A Let G be a graph satisfying $(**)$. If H is a graph with $V(H) = V(G)$ and $E(G) \subseteq E(H)$, then H satisfies $(**)$.



Proof Let $T \subseteq V(G)$, adding a new edge cannot increase the number of odd components (consider the cases). ■

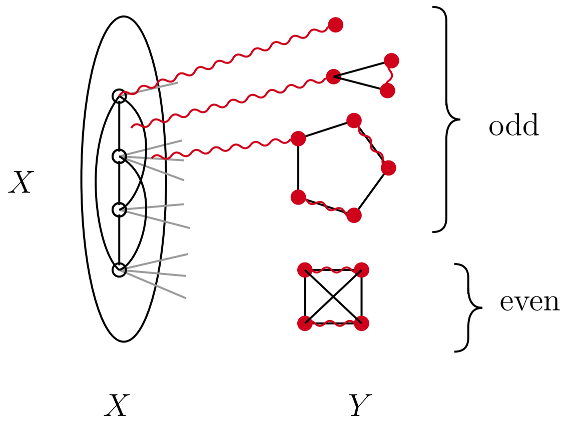
Type-0

Let G be a graph. Let X be the set of vertices $x \in V(G)$ such that $\Gamma(x) = V(G) \setminus \{x\}$. Let $Y = V(G) \setminus X$. We say G is Type-0 if every component of the subgraph $G[Y]$ is a complete graph.

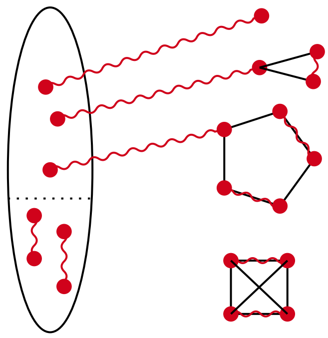
(Note X or Y could be empty)

Lemma B Let G be a graph satisfying (∗∗) $\mathrm{odd}(G-T) \leq |T|$ for every $T \subseteq V(G)$. If G is Type-0 then G has a perfect matching.

Proof Taking $T = X$ in (∗∗) tells us that $\mathrm{odd}(G-X) \leq |X|$. But $G - X = G[Y]$, so $G[Y]$ has at most $|X|$ odd components.



Begin constructing a matching by matching one vertex of each odd component of $G[Y]$ to a vertex of X. Match the rest of vertices in each C inside C.



Then the even components C can all be matched inside C and the rest of X inside X since $|V(G)|$ is even (consequence of (∗∗) with $T = \varnothing$). ■

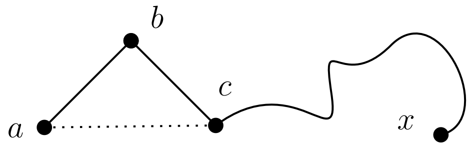
2.6.1 Proof of Tutte's Theorem (Lovász)

Suppose a graph G satisfies $(**)$, but on the contrary does not have a perfect matching. By (possibly) adding edges one by one, construct a graph H with $V(H) = V(G)$ and such that H has NO perfect matching but $H + e$ has a perfect matching for every $e \notin E(H)$. Then by Lemma A, H satisfies $(**)$. We will show H is Type-0, contradicting Lemma B.

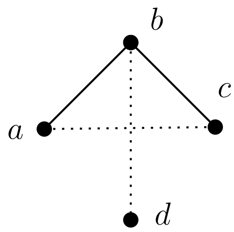
Let $X \subseteq V(H)$ be the set of $x \in V(H)$ where $\Gamma(x) = V(H) \setminus \{x\}$ ($X = \varnothing$ is possible). Let $Y = V(H) \setminus X$. If $Y = \varnothing$ then H is Type-0. Let C be a component of $G[Y]$. If $|V(C)| = 1$ or 2, then done. Suppose $|V(C)| \geq 3$.

Claim If C is not complete, then there exist $a, b, c \in V(C)$ with $ab, bc \in E(C)$ and $ac \notin E(C)$.

Proof of the claim Since C is not complete, there exists a and x with $ax \notin E(C)$. Take a shortest path P from a to x in C, and take b, c to be the second and third vertices in P.

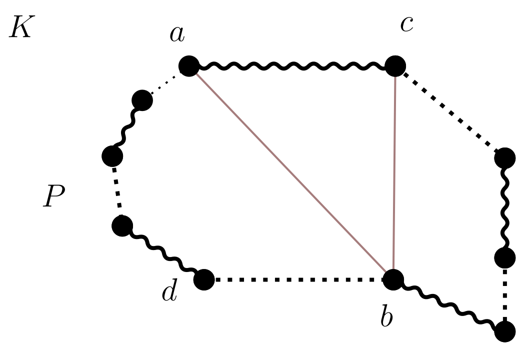


Let $d \notin \{a, b, c\}$ where $bd \notin E(H)$ such a d exists since $b \notin X$.



By the property of H, $H + ac$ has a perfect matching M_1 and $H + bd$ has a perfect matching M_2. Then graph J with edge set $M_1 \cup M_2$ is a disjoint union of single edges (in $M_1 \cap M_2$) and (M_1, M_2)-alternating cycles.

Since $ac \in M_1 \setminus M_2$, the component K of J containing ac is a cycle component. If $bd \notin K$, then the matching $M_1 \Delta E(K)$ is a perfect matching of H. So $bd \in K$ also.



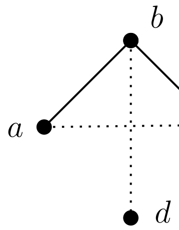
Then in K, there is a path P from b to (WLOG) a that does not contain d or c. Then $P \cup \{bd\} \cup \{da\}$ is an M_2-alternating cycle C'.

But then $M_2 \Delta E(C')$ is perfect matching of H. This contradiction show H is Type-0, contradicting Lemma B. ■

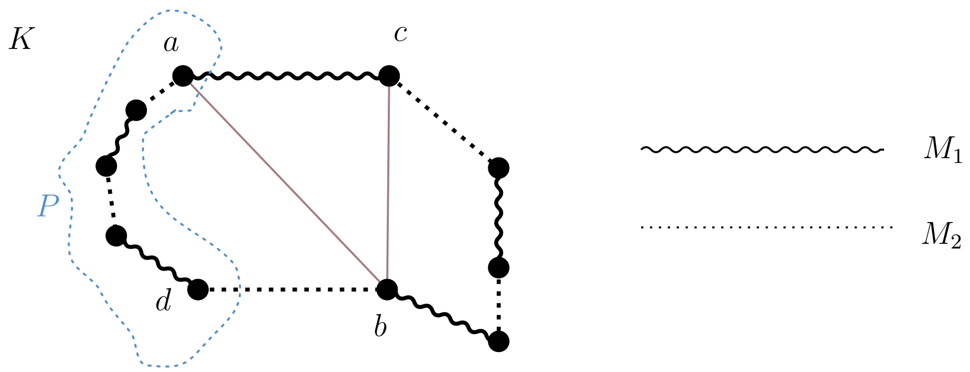
More explanation...

We had

- a graph with no perfect matching
- edges ab, bc



- non-edges bd, ac
- $H + ac$ has a perfect matching M_1, $H + bd$ has a perfect matching M_2.
- alternating (M_1, M_2)-cycle containing both ac and bd.
- path P in K from d to a that does not contain b or c.



Consider the cycle $K' = P \cup \{db, ba\}$ in $H + bd$. This is an M_2-alternating cycle. Then $M_2 \Delta E(K')$ is a perfect matching of H, contradicting our assumption on H. Thus H is Type-0, and hence has a perfect matching by Lemma B. Again a contradiction. ■

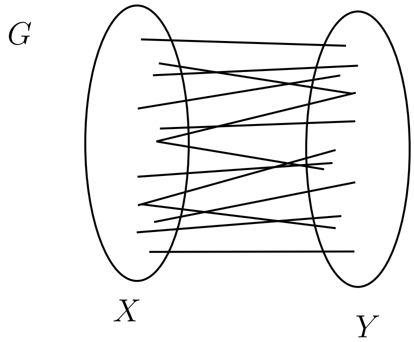
2.6.2 Applications

Hall's Theorem implies Tutte's Theorem when our graph G is bipartite:

Assume G is bipartite graph with vertex classes X and Y. Suppose that for every $T \subseteq V(G)$.

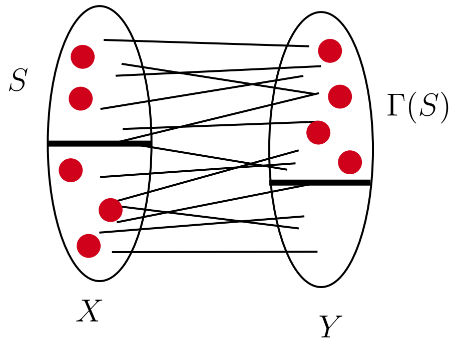
$$(**) \quad \text{odd}(G - T) \leq T$$

We prove that G has a perfect matching (using Hall's Theorem).



Taking $T = X$: then $G - T$ consists of $|Y|$ isolated vertices. Each vertex of Y is then an odd component of $G - X$. So $\mathrm{odd}(G - X) \geq |Y|$. Hence by $(**)$, $|Y| \leq \mathrm{odd}(G - X) \leq |X|$. Similarly, taking $T = Y$ tells us $|X| \leq |Y|$. Hence $|X| = |Y|$.

To verify Hall's condition: take a subset $S \subseteq X$, then the graph $G - \Gamma(S)$ has at least $|S|$ isolated vertices (the vertices in S).



So $|\Gamma(S)| \overset{(**)}{\geq} \mathrm{odd}(G - \Gamma(S)) \geq |S|$. Hence Hall's condition holds, so G has a perfect matching.

Lemma If G is a graph with $|V(G)|$ even, and $T \subseteq V(G)$, then $\mathrm{odd}(G - T) \equiv |T| \mod 2$.

Proof Since $|V(G)|$ is even, $|T|$ and $|V(G - T)|$ have the same parity. But

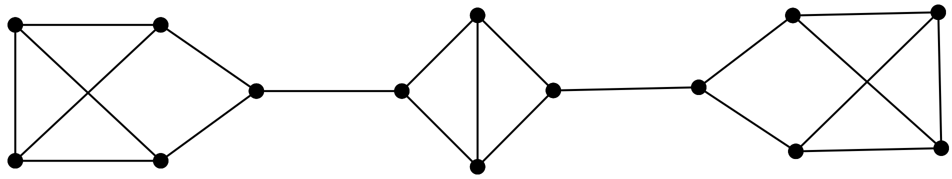
$$\mathrm{odd}(G - T) \equiv |V(G - T)| \mod 2 \equiv |T| \mod 2$$

■

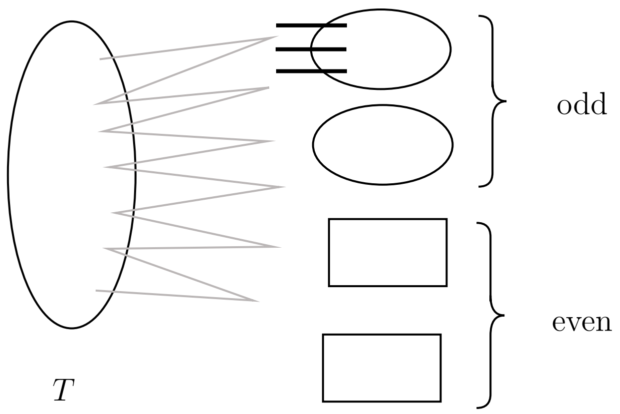
cut-edge

A cut-edge (or bridge) of a connected graph is an edge e such that $G \setminus e$ is disconnected.

Theorem (Peterson's Theorem)
Let G be a connected 3-regular graph with at most 2 cut-edges. Then G has a perfect matching.



Proof Suppose G does not have a perfect matching. Then by Tutte's theorem, there exists $T \subseteq V(G)$ such that $\text{odd}(G-T) > T$. Let C be an odd component of $G-T$.



Claim The number $|E(C,T)|$ of edges joining C to T is odd.

Proof of Claim

$$\sum_{v \in C} d(v) = 2|E(G[C])| + |E(C,T)|$$

But $\sum_{v \in C} = 3|V(C)|$ which is odd. $\star$

If $|E(C,T)| = 1$, then the edge in $E(C,T)$ is a cut-edge. Hence at least $\text{odd}(C-T)-2$ odd components C have $|E(C,T)| \geq 3$. So the number of edges from $V(G-T)$ to T is at least

$$3(\text{odd}(G-T)-2)+2$$

But by our lemma, $\text{odd}(G-T) \geq |T|+2$. (Since also $|V(G)|$ is even because G is 3-regular).

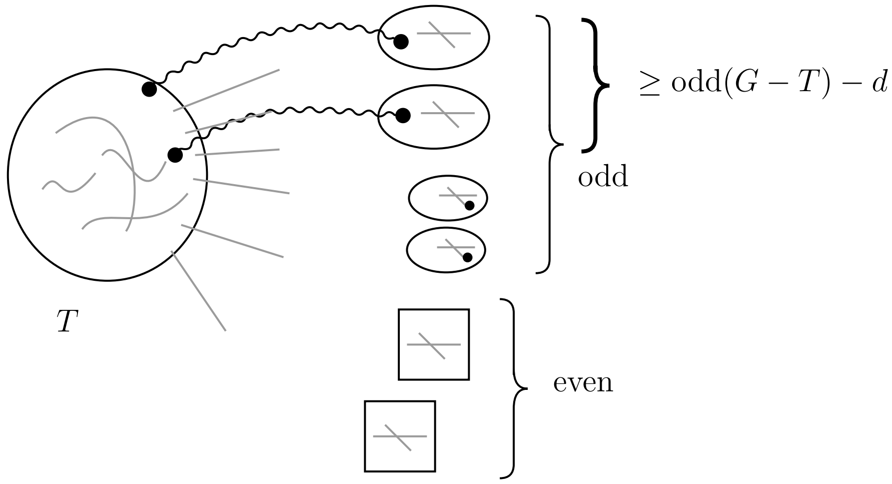
So we get $\geq 3(|T|)+2$ edges from $G-T$ to T. This contradicts the fact that every vertex in T has degree 3. Hence G has a perfect matching. ∎

Theorem (Defect version of Tutte's theorem)
Let G be a graph and let d be a non-negative integer such that $d \equiv |V(G)| \mod 2$. Then G has a matching saturating at most $|V(G)|-d$ vertices if and only if

$$\text{odd}(G-T) \leq |T|+d \qquad \text{for all } T \subseteq V(G)$$

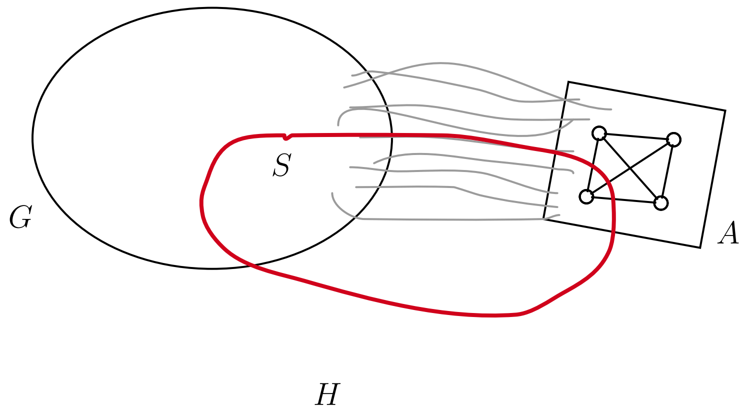
Proof Note that if $d=0$, then this is Tutte's Theorem, so we may assume $d \geq 1$.

($\implies$) If G has a matching M saturating $\geq |V(G)| - d$ vertices, and $T \subseteq V(G)$, then at most d of the odd components of $G - T$ contain an M-exposed vertex. Hence at least $\mathrm{odd}(G - T) - d$ odd components have an edge of M joining it to T. So $|T| \geq \mathrm{odd}(G - T) - d$.



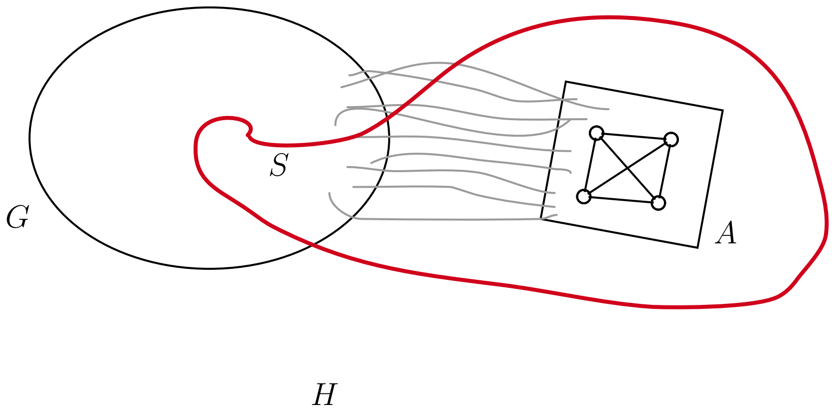
($\impliedby$) Assume $\mathrm{odd}(G - T) \leq |T| + d$ for each $|T| \subseteq V(G)$. Construct a graph H by adding a set A of d new vertices to G, and joining each $a \in A$ to all $V(H) \setminus \{a\}$. Note $|V(H)|$ is even. We show H has a perfect matching by verifying Tutte's condition: Let $S \subseteq V(H)$.

- If $\varnothing \neq S \subseteq V(H)$ and $A \nsubseteq S$, then $\mathrm{odd}(H - S) \leq 1$ (since $H - S$ is connected via A). Hence $\mathrm{odd}(H - S) \leq 1 \leq |S|$.



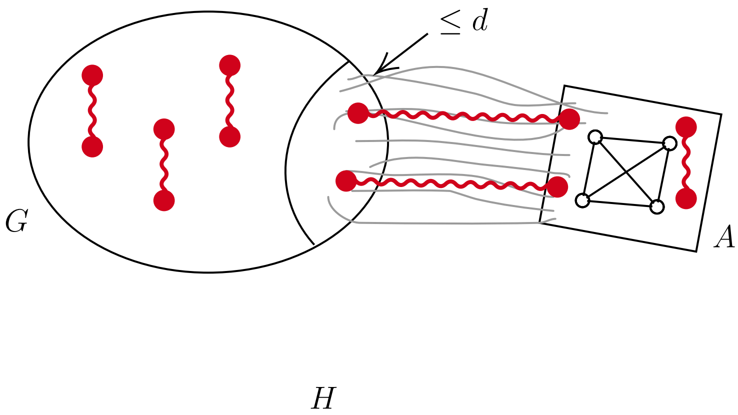
- If $A \subseteq S$ then $\mathrm{odd}(H - S) = \mathrm{odd}(G - (S \setminus A))$. So

$$\mathrm{odd}(H - S) = \mathrm{odd}(G - (S \setminus A)) \leq |S \setminus A| + d = |S| - d + d = |S|$$



- If $S = \varnothing$ then $\mathrm{odd}(H - S) = 0$ since H is connected (via A) and $|V(H)|$ is even.

Hence by Tutte's theorem, H has a perfect matching M.

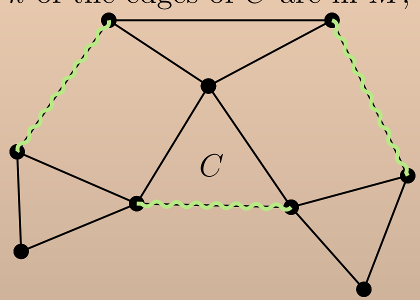


Then M saturates at least $|V(G)| - d$ vertices of G with edges of $M \cap E(G)$ as required. ■

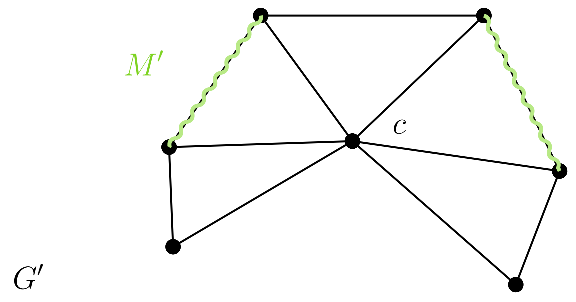
2.7 Edmonds Algorithm

shrinkable odd cycle

Let G be a graph and let M be a matching in G. Let C be an odd cycle in G of length $2k+1$. We say C is shrinkable odd cycle with respect to M if exactly k of the edges of C are in M, and C has one M-exposed vertex.

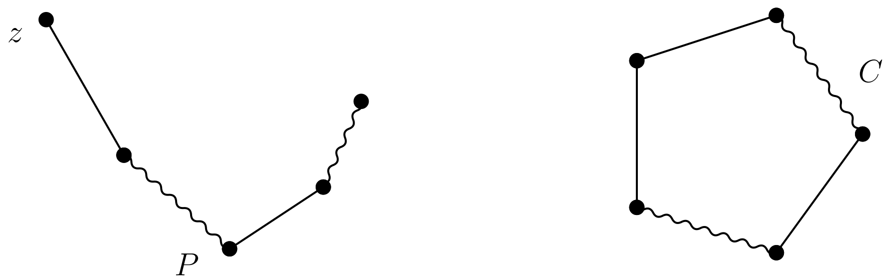


Note that in the graph G' obtained by contracting all edges of C into a single vertex c, the vertex c is M'-exposed where $M' = M \setminus E(C)$.



Lemma (cycle shrinking)
Let M be a matching in G and let C be a shrinkable odd cycle with respect to M. Let G' be the graph obtained from G by contracting $E(C)$, and set $M' = M \setminus E(C)$, then M is maximum in G if and only if M' is maximum in G'.

Proof ($\Longleftarrow$) Assume M' is maximum in G'. Suppose on the contrary that M is not maximum in G. Then by Berge's Theorem, G contains an M-augmenting path P. Then P intersects C, otherwise it is an M'-augmenting path in G'. Since C is shrinkable, and P has 2 M-exposed vertices, one endpoint z of P is not on C. But then the (z, c)-segment of P in G' is an M'-augmenting path, contradicting the maximality of M' in G'.



Intermission

[a] G graph, M matching in G, C odd cycle is shrinkable w.r.t. M means

- max possible number of edges of C are in M
- last vertex of C is M-exposed

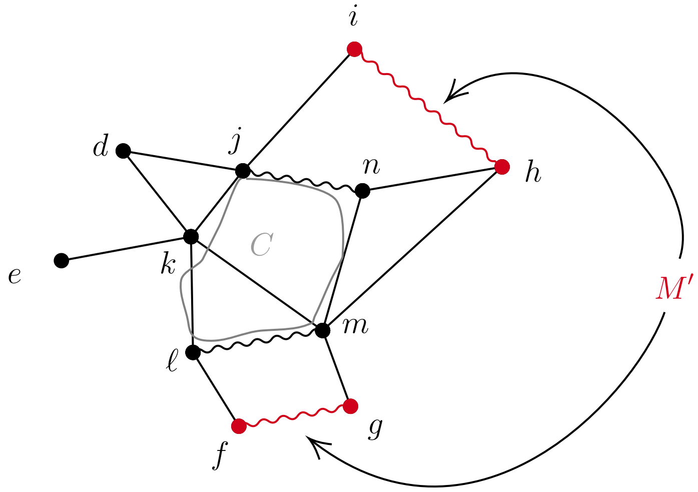
And we have

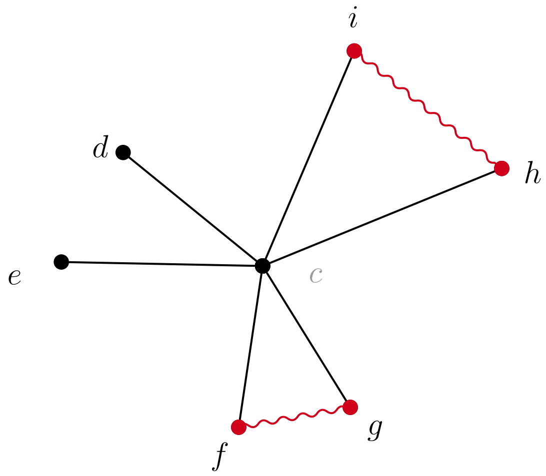
- $G' = G/E(C)$
- $M' = M \setminus E(C)$

Cycle shrinking lemma M is maximum in G iff M' is maximum in G'.

Proof ($\Longleftarrow$) did

[a] Due to the annoying quiz, the second half of proof was postponed until the next lecture





($\Longrightarrow$) Assume M is maximum in G, C has length $2k+1$. Suppose on the contrary that M' is not maximum in G. Let N' be a matching in G' with $|N'| > |M'|$. Then each edge of N' corresponds to an edge in G (i.e. is either itself an edge of G, or it corresponds to some edge of G with one endpoint in C).

Then at most one edge of N' is incident to a vertex of C, we can add k edges of C to N' to set a matching of size

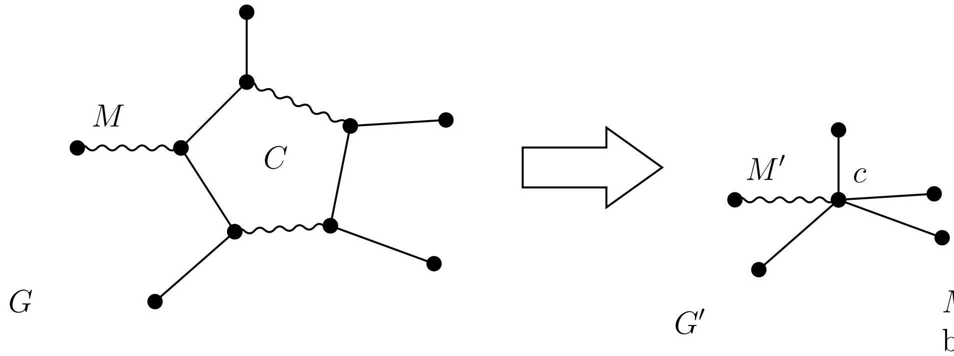
$$|N'| + k > |M'| + k = |M|.$$

This contradicts the maximality of M. Hence M' is maximum in G'. ■

Notes

- The condition that C contains an M-exposed vertex is necessary:

eg

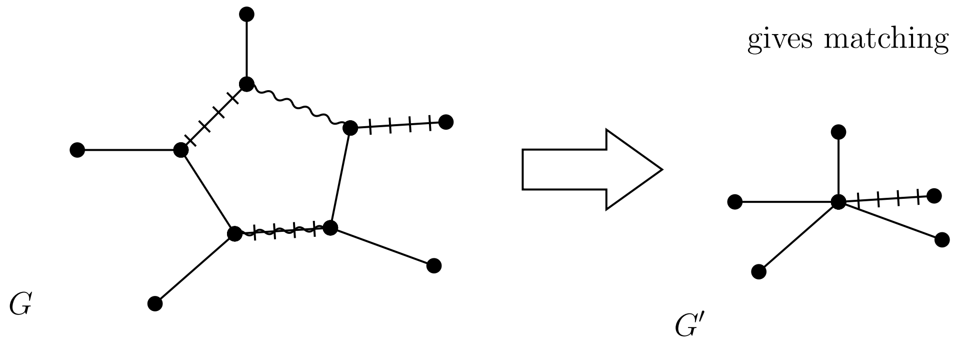


M' is maximum in G', but M is not in G

- The proof does NOT say that if N' is a maximum matching in G' then we can get a <u>maximum</u> matching of G by adding k edges of C to N'. It only says that if $|N'| > |M'|$ then this gives a <u>bigger</u> matching (than $|M|$) in G.

eg

gives matching G of size $3 < \nu(G)$



Main ingredients of Edmond's Algorithm

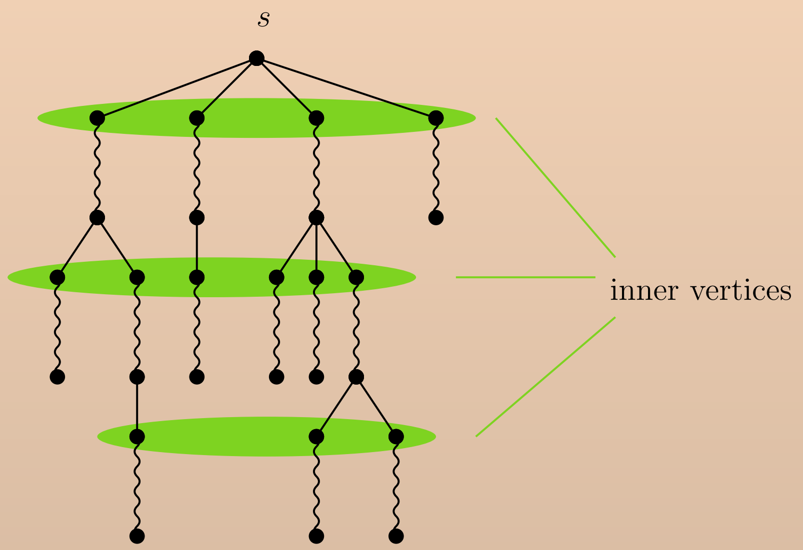
- cycle shrinking
- alternating forests

The algorithm constructs a special subgraph in G with given matching M, called an M-alternating forest F.

M-alternating forest

Each component T of F is a tree of the following type:

- T contains exactly one M-exposed vertex
- every edge of T at an odd distance from s in T has degree 2 in T.

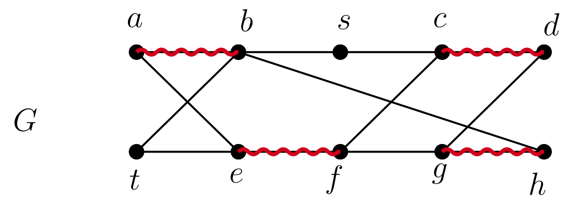


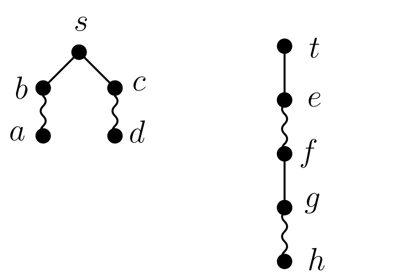
inner/outer vertices

The vertices of T at an odd distance from s in T are called the inner vertices of T.
The rest (including s) are called outer vertices.

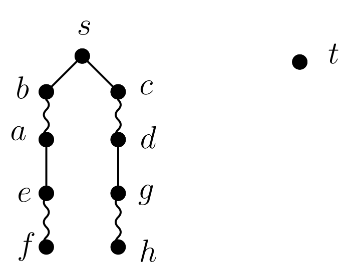
An M-alternating forest F is any subgraph whose components are all of this form. We say F is a <u>maximal</u> M-alternating forest if it is not contained in any strictly larger (i.e. more vertices) M-alternating forest.

eg





a maximal M-alternating forest



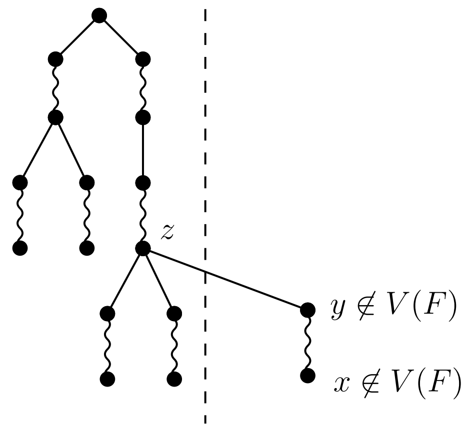
another one

Lemma Let G be a graph and let M be a matching in G. Let F be a maximal M-alternating forest in G. Suppose there is no edge of G joining two outer vertices of F. Then M is a maximum matching in G.

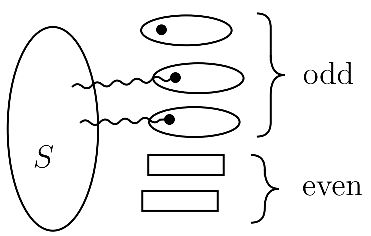
Proof Let F be a maximal M-alternating forest as in the statement. Then every M-exposed vertex of G is in F, since F is maximal. Also, if $xy \in M$ and x is in same component T of F, then y is also in T.

Claim If z is an outer vertex of F, then $\Gamma_G(z)$ is contained in the set of inner vertices of F.

Pf of claim To see this, let $y \in \Gamma_G(z)$, then y is not an outer vertex of F. Suppose $y \notin V(F)$. Then y is not M-exposed, so $xy \in M$ for some x. Hence $x \notin V(F)$. But then F is not maximal, since y and x could be added to the same component of F that contains z. Hence y is an inner vertex of F.

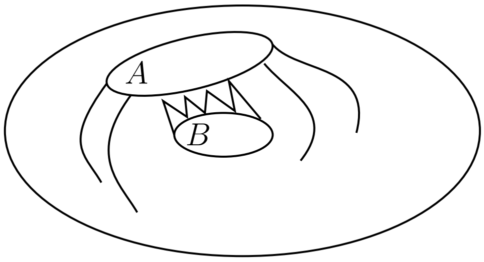


Let A_T and B_T denote the sets of inner and other vertices of component T of F respectively. Let $A = \bigcup_{\substack{T \text{ comp} \\ \text{of } F}} A_T$ and $B = \bigcup_{\substack{T \text{ comp} \\ \text{of } F}} B_T$ be the sets of inner and outer vertices of F. So by the claim, $\Gamma_G(B) \subseteq A$. Then for each component T of F we have $|A_T| + 1| = |B_T|$. So $|B| = |A| + c$ where c is the number of components of F. Note also that c is the number of M-exposed vertices in G.

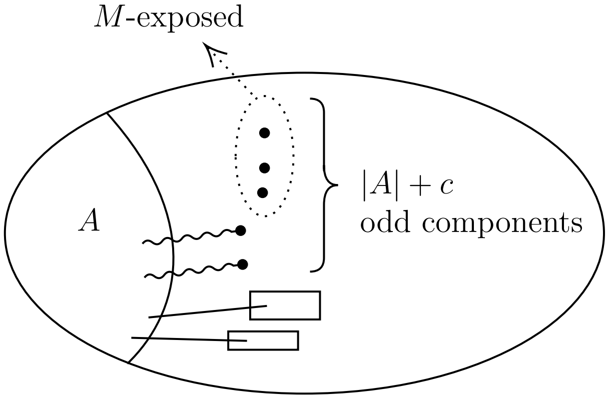


Recall (from the defect version of Tutte's Theorem) that for any subset $S \subseteq V(G)$ in each odd component of $G - S$ there must be a vertex that is either M-exposed or matched to a vertex of S.

Consider $A \subseteq V(G)$. The number of odd components of $G - A$ is at least $|B| = |A| + c$ since each vertex of B is a component of size 1 in $G - A$ (by the claim).



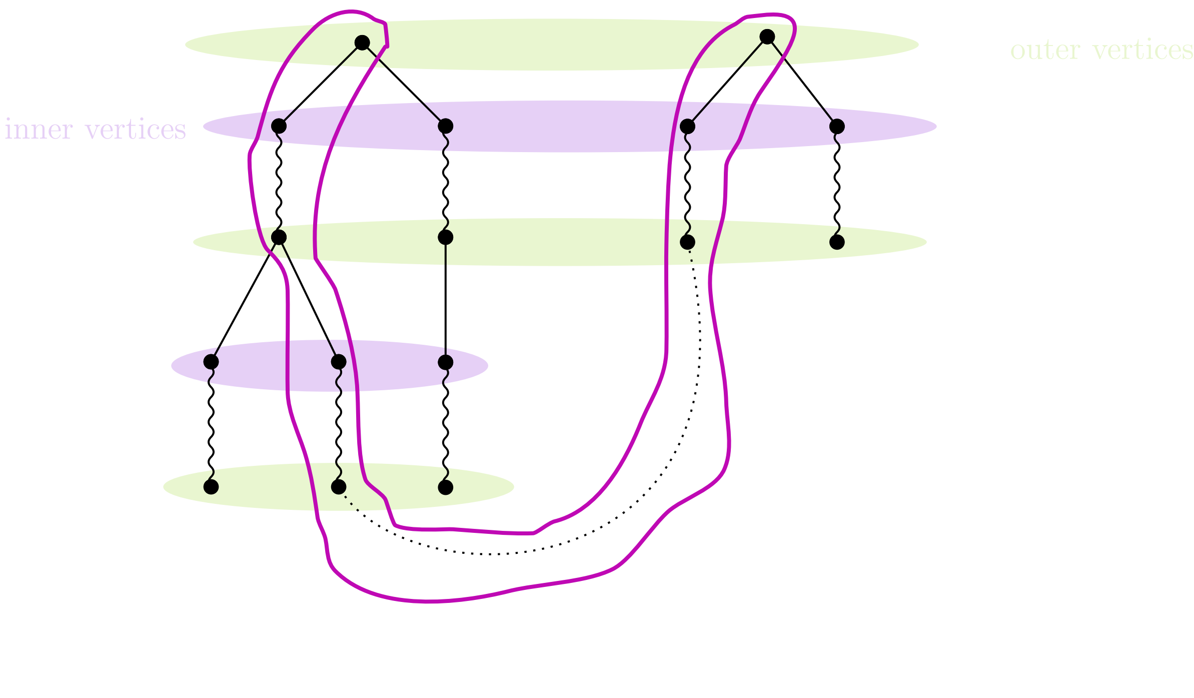
So $\mathrm{odd}(G - A) \geq |A| + c$. But the number of M-exposed vertices is exactly c, so M is a maximum matching in G.



$\blacksquare$

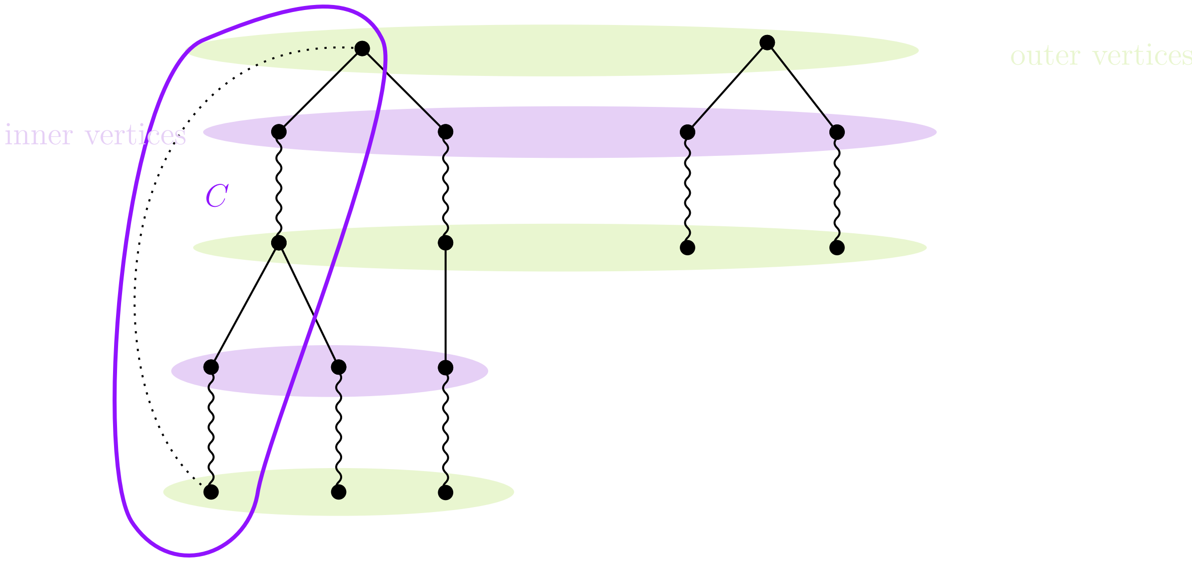
Lemma (from last time) if there is no edge of G joining two outer vertices in a maximal M-alternating forest then M is maximum. (gives the stopping rule for Edmond's Algorithm)

Lemma Let G be a graph, let M be a matching in G, and let F be an M-alternating forest in G. Suppose there exists an edge of G joining two outer vertices of F, that are in distinct components of F, then G contains an M-augmenting path.



Proof (see pic.) or: let xy be the edge, then the path from M-exposed vertex in the component T_x of F containing x to x, together with xy and and corresponding path in T_y to its exposed vertex, is M-augmenting. ■

Lemma With the same assumptions on G, M and F. Suppose there exists an edge e of G joining two outer vertices of F that lie in the same component of F, then there exists a matching $\overline{M}$ in G with $|M| = |\overline{M}|$ and an odd cycle C that is shrinkable with respect to $\overline{M}$.



Proof Let C be the odd cycle formed by adding the edge e to the component T of F containing both its endpoints. Let P denote path in T from C to M-exposed from C to the M-exposed root of T. Set $\overline{M} = M \Delta E(P)$, then $|\overline{M}| = |M|$, and C is shrinkable w.r.t. $\overline{M}$.

2.7.1 Idea of Edmond's Algorithm

- start with a graph G and a matching M in G.

  Aim : to either

  - certify M is maximum, or
  - find a matching of size $> |M|$

- start constructing an M-alternating forest F in G
- If you find an edge of G joining two outer vertices of F that are in the same component of F:
  - switch M to $\overline{M}$.
  - shrink the shrinkable odd cycle C to get $G' = G/E(C)$.
  - run algorithm recursively on G' and $M' = \overline{M} \setminus E(C)$.

This either certifies M' is maximum in G' (hence $\overline{M}$ is maximum in G by cycle shrinking lemma) or finds a larger matching in G', leading to a larger matching in G (cycle shrinking lemma)

- If you find an edge of G joining two outer vertices in different components of F, then find an M-augmenting path. Switch on it to get a bigger matching.
- If you never find an edge joining two outer vertices after completing the construction of a maximal M-alternating forest, then (by our lemma), M is maximum. Stop, output M.

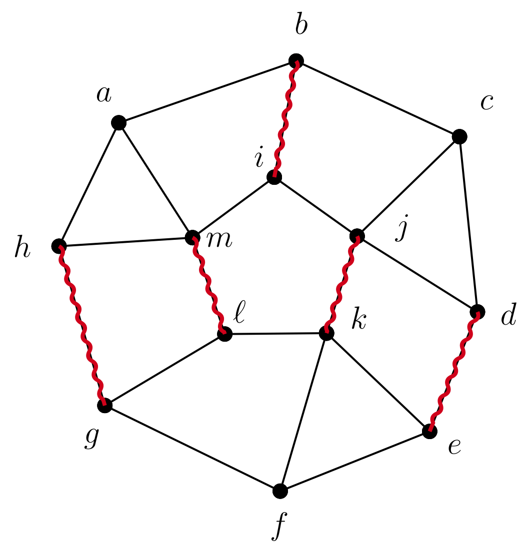
2.7.2 Algorithm

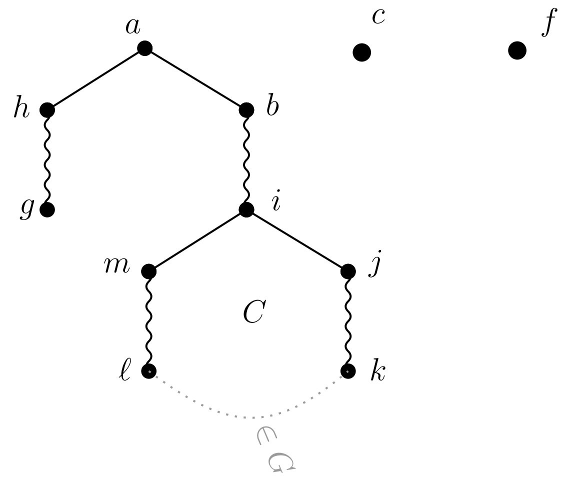
Algorithm 3: Edmond's Matching Algorithm

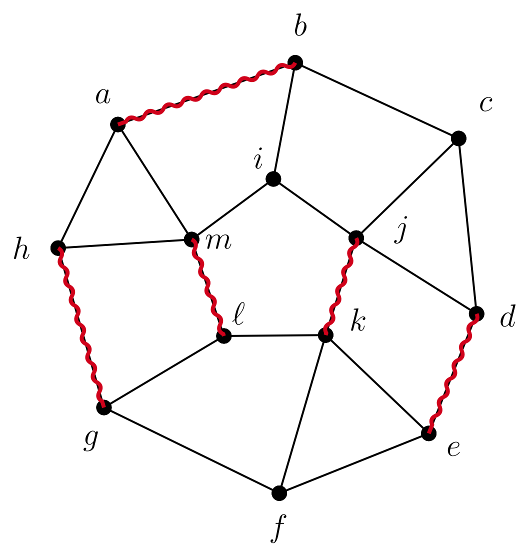
Input: A graph G and a matching M
Output: A maximum matching M^* in G

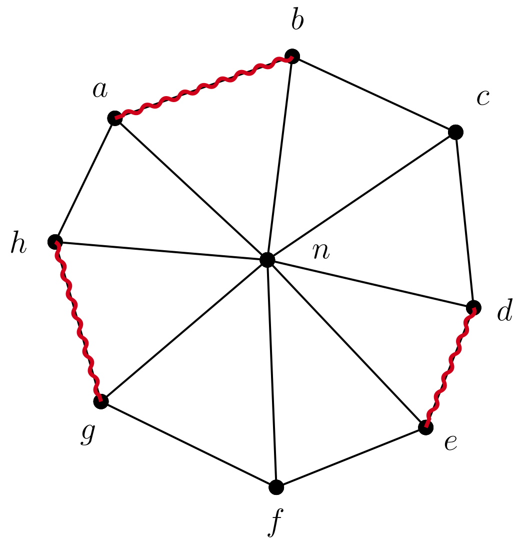
0 Set $G_0 := G$ and $M_0 := M$
1 Let S be the set $\{s_1, \ldots, s_T\}$ of M-exposed vertices.
2 Set *Outer* $:= S$, *Inner* $:= \varnothing$, *Checked* $:= \varnothing$, $F := \varnothing$. Label each $s_i \in$ *Outer* with $\ell(s_i) := i$.
3 Choose $v \in$ *Outer* and edge $vw \notin$ *Checked* if they exist. If none exists, then M_0 is maximum in G_0. Set $M^* := M_0$. Stop
4 Otherwise, if $w \in$ *Inner*, add $wv \in$ *Checked*. Go to 3.
5 If $w \in$ *Outer* and $\ell(w) = \ell(v)$ then the cycle C in $F \cup \{vw\}$ is shrinkable w.r.t. $\overline{M}$ (as in cycle shrinking lemma). Record the subgraph induced by $V(C)$, and the edges joining $V(C)$ to the rest of G. Shrink C: replace G by $G/E(C)$ and M by $\overline{M} \setminus E(C)$. Go to 1.
6 If $w \in$ *Outer* and $\ell(w) \neq \ell(v)$ then there is an M-augmenting path P. Switch on P to set a bigger matching, expand all shrunken cycles and add k edges in each $(2k+1)$-cycle to the matching, replace M by this new bigger matching and G by the original graph. Go to 0.
7 If $w \in$ *Inner* $\cup$ *Outer*, then there exists $wx \in M_0$. Add vw and wx to F and to *Checked*. Add w to *Inner*, x to *Outer*. Set $\ell(w) = \ell(x) = \ell(v)$. Go to 3.

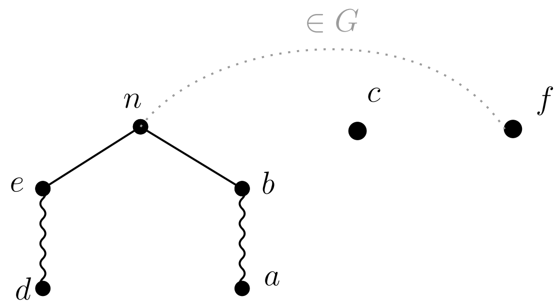
Example



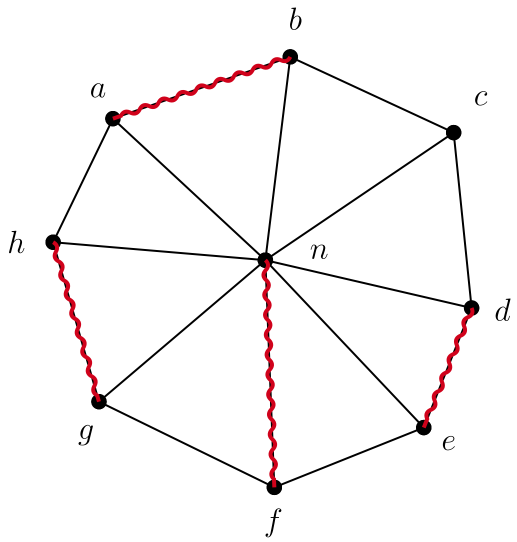




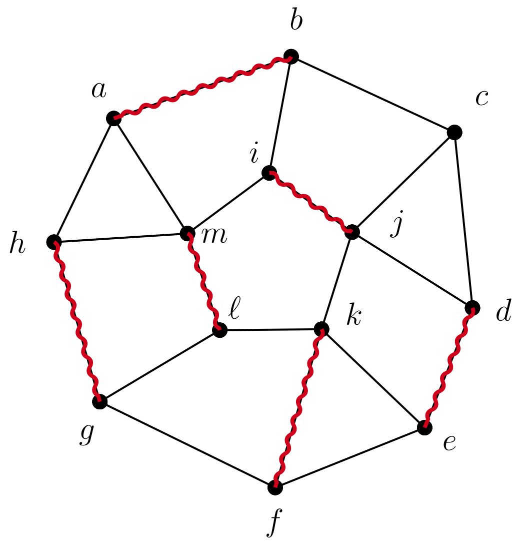


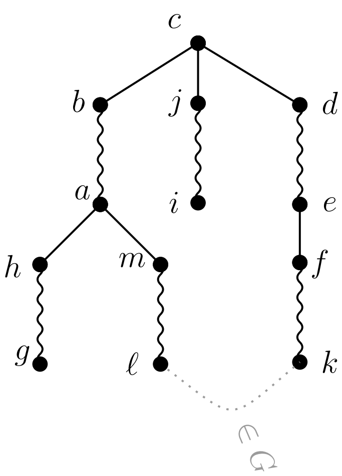


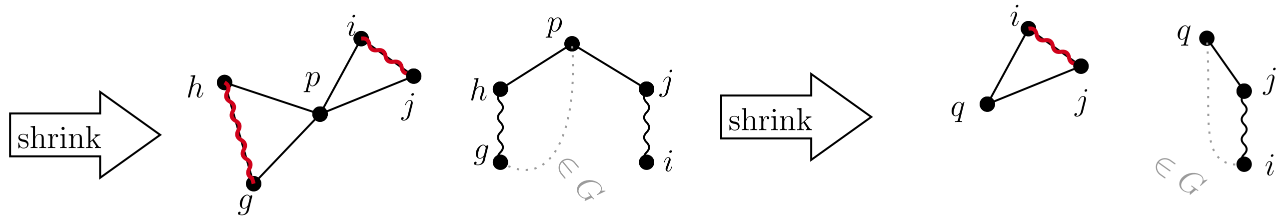
after step 5.

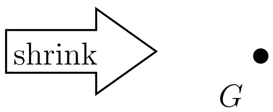


START OVER









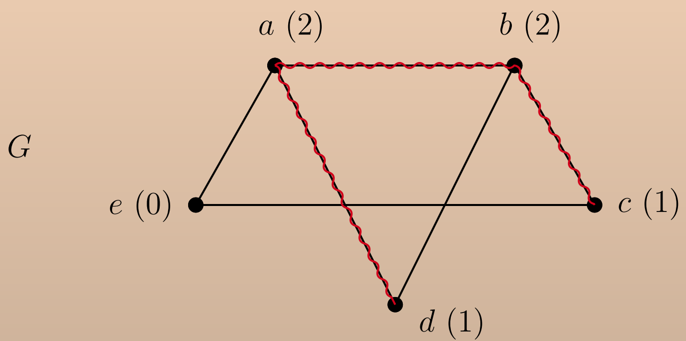
no edges joining outer vertices

Conclude matching M_0 is maximum in Input Graph

2.8 f-Factors

f-factor

Let G be a graph and let $f\colon V(G) \to \{0, 1, 2, \ldots\}$ be a function. An f-factor in G is a subgraph H of G such that $d_H(v) = f(v)$.

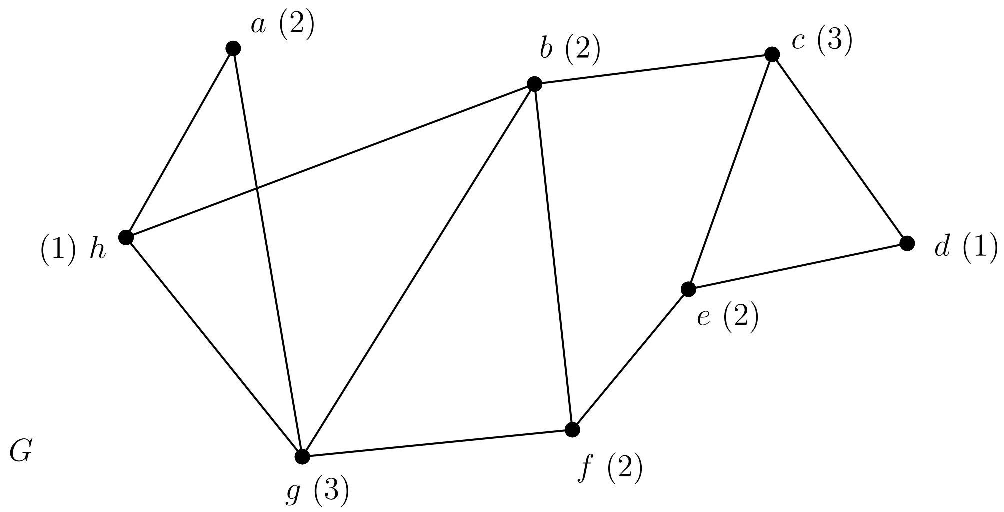


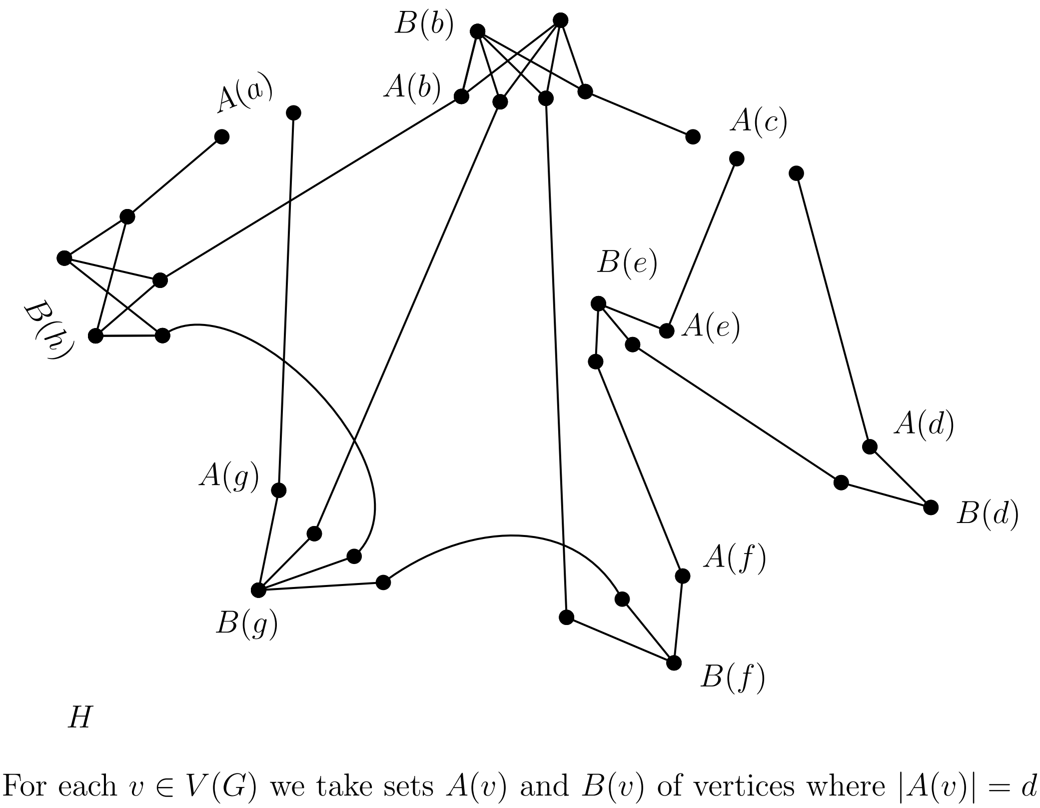
For example, a perfect matching in G is a $\mathbb{1}$-factor, where $\mathbb{1}$ denotes the function that assigns 1 to each $v \in V(G)$. When f is a constant function, we often write k-factor, where $f(v) = k, \quad \forall v$.

Let G be a graph and f a function on $V(G)$. If $f(v) > d(v)$ for some v then clearly G has no f-factor. So we may assume $f(v) \le d(v)$ for each v.

We will reduce the problem of finding an f-factor in G (and in particular, determining if one exists) to the same problem in the following auxiliary graph.

Define $H(G, f)$:



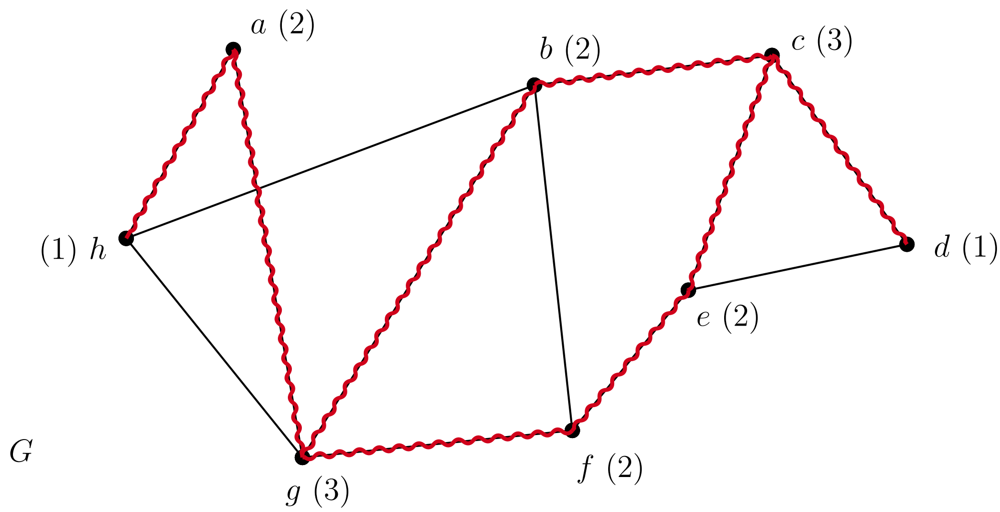


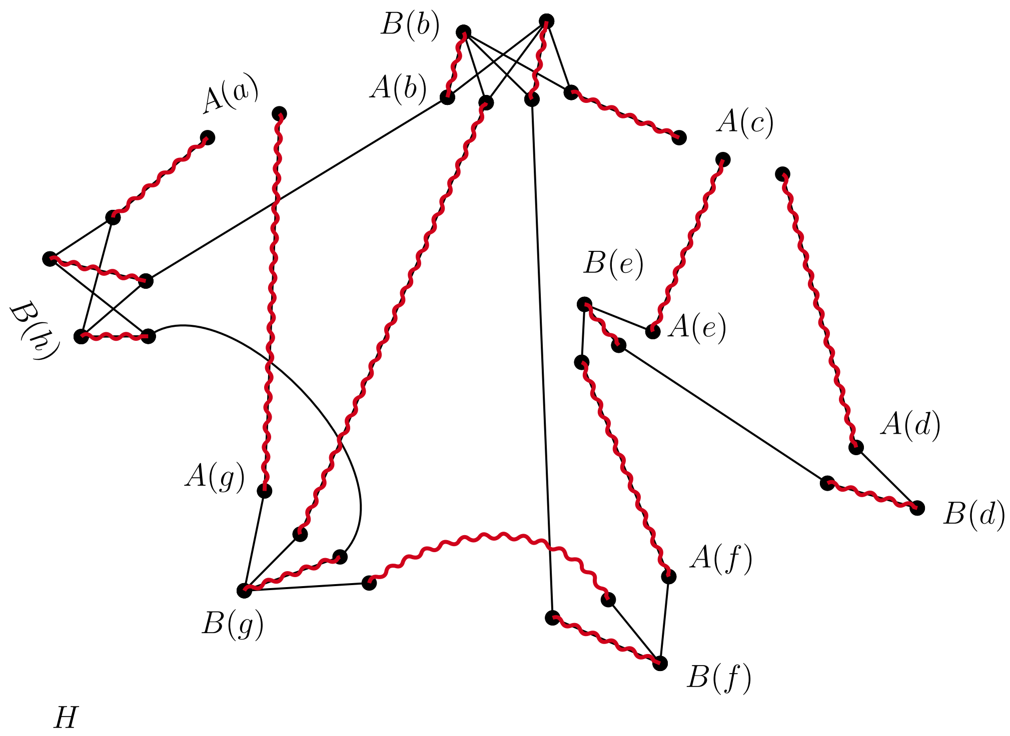
For each $v \in V(G)$ we take sets $A(v)$ and $B(v)$ of vertices where $|A(v)| = d(v)$ and $|B(v)| = d(v) - f(v)$. All of $A(v)$ is joined to all of $B(v)$ for each v. For each edge xy of G, we put on edge from $A(x)$ to $A(y)$ in $H(G, f)$ so that all of these edges are vertex-disjoint.

Theorem (Tutte)
A graph G has an f-factor if and only if $H(G, f)$ has a perfect matching.

Proof First suppose G has an f-factor J, then the edges of J correspond to a matching M in $H(G, f)$, such that exactly $f(v)$ vertices of $A(v)$ are saturated by M. Then M completes to a perfect matching of $H(G, f)$ by matching the remaining $d(v) - f(v)$ vertices in $A(v)$ to $B(v)$.

Conversely, suppose H has a perfect matching M, then the vertices in $B(v)$ must be matched to exactly $|B(v)| = d(v) - f(v)$ vertices in $A(v)$ for each v. Then M saturates exactly $f(v)$ vertices of $A(v)$ via edges that correspond to edges of G. Thus these edges form an f-factor of G. ■





2.9 Eulerian Circuits

Eulerian Circuits

A sequence $v_0 e_1 v_2 \ldots e_n v_n$ of vertices and edges in a graph G is an Eulerian circuit if

- $v_0 = v_n$
- $e_i = v_i v_{i-1}$ for each i
- each $e \in E(G)$ appears exactly once.

even graph

A graph is even if every vertex has even degree.

Since each visit of Eulerian circuit to a vertex v uses 2 edges incident to v, if G has an Eulerian circuit then it is even. Also G must be connected, to contain an Eulerian circuit.

Theorem A connected graph G has an Eulerian circuit if and only if it is even.

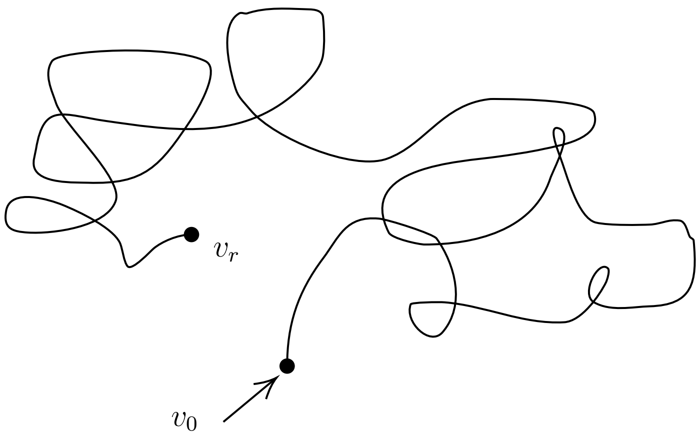
Proof ($\Longrightarrow$) done

($\Longleftarrow$) Assume G is even, we'll prove by induction on $m = E(G)$ that it has an Eulerian circuit.

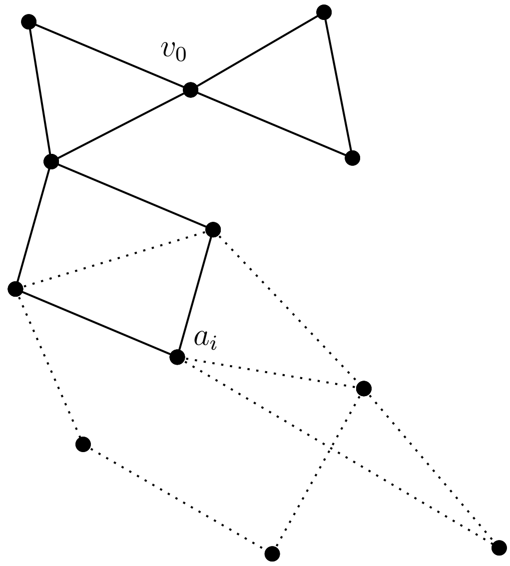
Base case $m = 0$ then $V(G) = \{x\}$, which is an Eulerian Circuit with 0 edges.

IH Assume $m \geq 1$ and every connected even graph with fewer than m edges has an Eulerian circuit.

Induction Step $m \geq 1$ and G is an even graph with m vertices. Let v_0 be an arbitrary vertex and let $W = v_0 e_0 v_1 e_2 v_2 \ldots e_r v_r$ be a maximal sequence such that $e_i = v_i v_{i-1}$ for each i, and all the e_i are distinct. Then since G is even, $v_r = v_0$ (Note W contributes 2 to every intermediate vertex, and hence also to $v_r = v_0$).

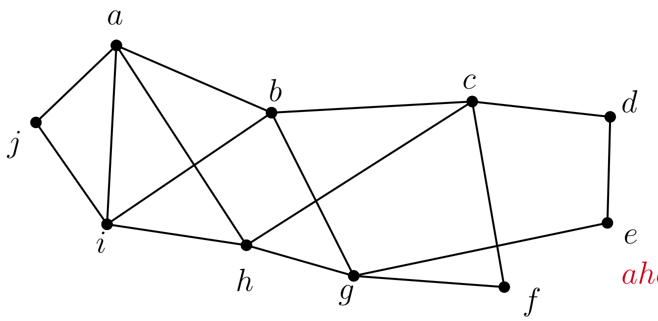


If $\{e_1, \ldots, e_r\} = E(G)$ then W is an Eulerian circuit. Otherwise consider G' where $V(G') = V(G)$ and $E(G') = E(G) \setminus \{e_1, \ldots, e_r\}$. Then since G is even, G' is also even. Thus by IH, every component G_i of G' has an Eulerian Circuit T_i. Since G is connected, every G_i that is not isolated vertex shares a vertex a_i with W. So we may insert T_i at a_i into W for each i to get an Eulerian circuit of G.



$\blacksquare$

Example:



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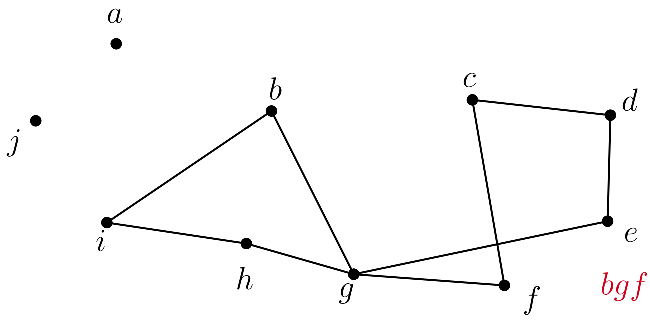


bgfcdeghib

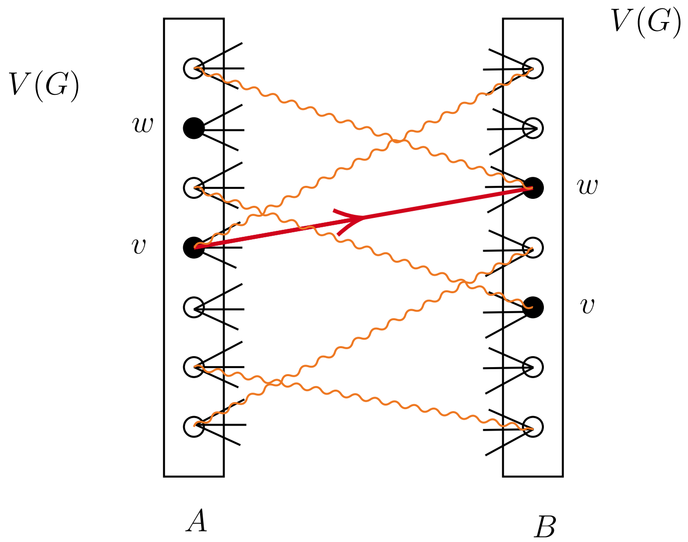
$\boxed{ahcbgfcdeghibajia}$

k-factor

For a positive integer k, a k-factor in a graph G is a (spanning) subgraph H such that $d_H(v) = k$ for each $v \in V(G)$. (i.e. an f-factor for the function $f(v) = k \quad \forall v$)

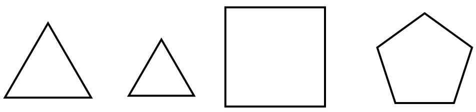
Theorem (Peterson)
Every r-regular graph where $r \geq 2$ is even, has a 2-factor.

Proof We may assume that G is connected r-regular graph by considering each component separately. Then G has an Eulerian Circuit C. Construct a bipartite graph J with vertex classes A and B, both of which are copies of $V(G)$.



We join $v \in A$ to $w \in B$ for each vw such that C uses vw in the direction $v \to w$. So every edge of G is represented exactly once in J. Then each vertex of J has degree $\frac{r}{2}$. Hence by our corollary of Hall's Theorem, J has a perfect matching M. Each edge of M is an edge of G, so M represents $|V(G)|$ edges in G. So M represents $|V(G)|$ edges in G. The copy of v in A is incident to 1 edge of M, and the copy of v in B is incident to 1 edge of M and these are different edges by construction. Hence M is a 2-factor in G. ■

eg



CHAPTER 3

Vector Spaces

Recall that what is a vector space? Field $\mathbb{F}$ $(+, \cdot, 0, 1, \text{inverses})$. We use the field $\mathbb{Z}_2$.

a	b	$a+b$
0	0	0
0	1	1
1	0	1
1	1	0

Table 3.1: addition

a	b	$a \cdot b$
0	0	0
0	1	0
1	0	0
1	1	1

Table 3.2: multiplication

vector space

A vector space W over a field $\mathbb{F}$ is a set of vectors satisfying the following
- W contains 0 vector
- if $u \in W$ and $c \in \mathbb{F}$, then $cu \in W$.
- if $u, v \in W$ then $u + v \in W$.

subspace

If $U \subseteq W$ then U is a vector space over the same field, we say U is a subspace of W.

Eg Set W to be the set of all n-dimensional vectors over $\mathbb{Z}_2$ that have an even number of 1's.

$$\text{Let } n = 3,\ W = \left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}. \text{ Then } \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} \in W.$$

Idea of proving third bullet Let $u, v \in W$.

$$\begin{array}{r|c|c|}
 & u & v \\
\text{1 is in both } u \text{ and } v & 1 & 1 \\ \hdashline
\text{1 is only in } u & 1 & 0 \\ \hdashline
\text{1 is only in } v & 0 & 1 \\ \hdashline
\text{0 is in both } u \text{ and } v & 0 & 0
\end{array}$$

By considering the parity of the number of rows with a 1in both u and v, we find $u + v \in W$.

3.1 Flow Space

flow space

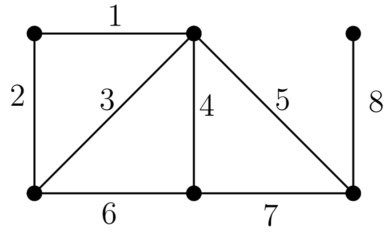
Let G be a connected graph. The flow space of G is the vector space over $\mathbb{Z}_2$ whose elements are the $\{0, 1\}$-vectors indexed by $E(G)$ that are characteristic vectors of the spanning subgraphs of G, where every vertex has even degree.

even spanning subgraph

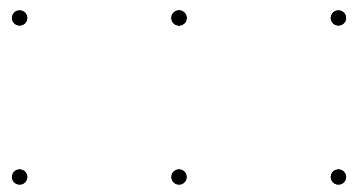
A spanning subgraph H of G ($V(H) = V(G)$) is called even if every vertex has even degree in H.

Eg Take G:

G



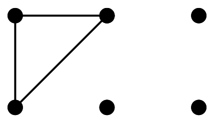
H



and its corresponding vector is $\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$

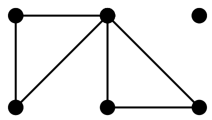
We could also take

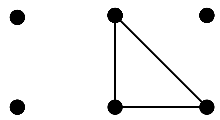
H_1



or

H_2





$$\begin{bmatrix}1\\1\\1\\0\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\\0\\1\\0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\1\\0\\1\\0\end{bmatrix}$$

symmetric difference

If S and T are even spanning subgraphs, then the symmetric difference $S \oplus T$ is the spanning subgraph with edge set

$$(E(S) \cup E(T)) \smallsetminus (E(S) \cap E(T))$$

characteristic vector

The characteristic vector of a subgraph H of G is the vector χ_H which has an entry for each edge of G, which is 1 if the edge is in H and 0 otherwise.

Observation If S and T are spanning even subgraphs, then $\chi_S + \chi_T = \chi_{S \oplus T}$.

Let W be a flow space of a connected graph G, and let $C_1, C_2, \ldots, C_k$ be cycles of G. Then

(i) $H = C_1 \oplus C_2 \oplus \ldots \oplus C_k$ is an even subgraph of G.

(ii) $\chi_H = \sum_{i=1}^{k} \chi_{C_i}$

We will prove converse of (i) also holds.

Lemma Let W be a flow space of a connected graph G. Let H be an even subgraph of G. Then there exists a collection $\{C_1, \dots, C_k\}$ of pairwise edge-disjoint cycles of G, such that $H = C_1 \oplus C_2 \oplus \dots \oplus C_k$.

Proof Sketch

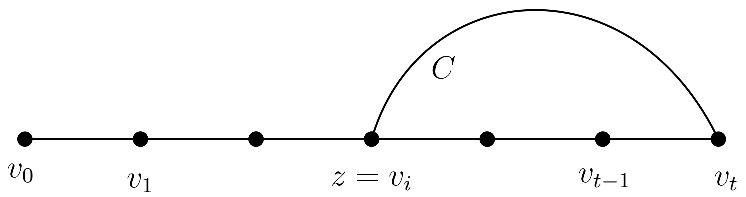
1. use indunction on $|E(H)|$
2. Find a cycle C in H
3. Apply induction for $H \setminus E(C)$
4. Put back C

Proof

Base Case If $|E(H)| = 0$, our collection is the empty set.

IH IT holds for $|E(H)| = 0, 1, \dots, m$.

Induction Step Assume that $|E(H)| = m + 1$. Let $P = (v_0, v_1, \dots, v_t)$ be a longest path in H.



$|E(H)| = m + 1 \geq 1$, so $t \geq 1$. Since H is even, v_t has a neighbour z other than v_{t-1}, and $z \in \{v_0, \dots, v_{t-2}\}$; otherwise, $P \cup \{v_t, z\}$ is a longer path, giving a contradiction.

Let $z = v_i (0 \leq i \leq t - 2)$. Let $C = (v_i, v_{i+1}, \dots, v_t, v_i)$. Now consider $H' = H \setminus E(C)$. Then H' is even and $|E(H')| < |E(H)|$. By IH, there exists a collection $\{C_1, \dots, C_k\}$ of pairwise edge-disjoint cycles of G. Then

$$\begin{aligned} H &= H' \oplus C & E(H') \cap E(C) = \varnothing \text{ since } E(H') = E(H) - E(C) \\ &= (C_1 \oplus C_2 \oplus \dots \oplus C_k) \oplus C \end{aligned}$$

■

Alternatively, we can prove (2) by using the fact that H is not a forest.

Corollary A flow space W of a connected graph G is spanned by characteristic vectors of cycles of G.

Proof Let H be an even subgraph of G. Then by previous Lemma, there are cycles $C_1, \ldots, C_k$ of G such that $H = C_1 \oplus C_2 \oplus \ldots \oplus C_k$. $\chi_H = \sum_{i=1}^{k} \chi_{C_i}$ ■

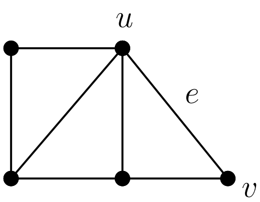
Corollary There exists a basis [1] of W that contains only cycles of G.

We will identify a basis of W.

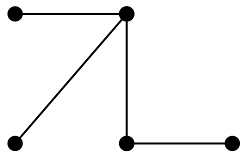
3.2 Bases

Ex

G



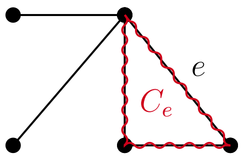
T



spanning tree of G

Let $e = uv \in E(G) - E(T)$

$T \cup \{e\}$



In T, there is the unique uv-path P. Then $P \cup \{e\}$ is the unique cycle C_e of $T \cup \{e\}$.

fundamental cycle

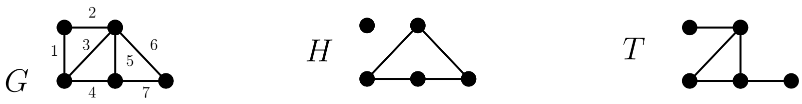
Let G be a connected graph, and let T be a spanning tree of G. For $e \in E(G) - E(T)$, the unique cycle C_e in $T \cup \{e\}$ is called the fundamental cycle for e w.r.t T.

Denote by $F(T)$ the set of all fundamental cycles w.r.t T

$$F(T) = \{C_e : e \in E(G) - E(T)\}$$

Example

[1] 1. linear indep 2. spanning



$E(G) - E(T) = \{1, 4, 6\}$
$F(T) = \{C_1, C_4, C_6\}$

$$\begin{array}{ll} C_1 = \{1,2,3\} & (1,1,1,0,0,0,0) \\ C_4 = \{3,4,5\} & (0,0,1,1,1,0,0) \\ C_6 = \{5,6,7\} & (0,0,0,0,1,1,1) \\ H = \{3,4,6,7\} & (0,0,1,1,0,1,1) \end{array}$$

Question $F(T)$ spans H? Find $e \subseteq F(T)$ whose symmetric difference is H.

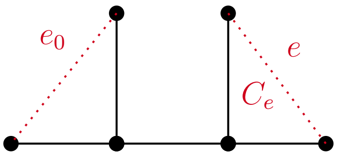
$C_1 \in e$? No, only C_1 contains the edge 1. $C_4 \in e, C_6 \in e$. In fact, $H = C_4 \oplus C_6$.

Lemma Let W be a flow space of a connected graph G, and let T be a spanning tree of G. Then characteristic vectors of $F(T)$ is a basis for W.

Proof Let X be the set of characteristic vectors of $F(T)$.

- X is linearly independent.

  Proof Assume $\displaystyle\sum_{e \in E(G)-E(T)} \lambda_e \cdot \chi_{C_e} = \mathbf{0}$

  T

  Fix $e_0 \in E(G) - E(T)$, then $e_0 \in C_e \iff e_0 = e$. Thus if $e \neq e_0$, e_0th-entry of χ_{C_e} is 0. $\implies \lambda_{e_0} = 0$. Since we picked e_0 arbitrarily, $\lambda_e = 0$ for every $e \in E(H) - E(T)$. ⋆

- Let H be an even subgraph of G. We aim to show that $H = H'$ where $H' = \displaystyle\bigoplus_{e \in E(H)-E(T)} C_e$.

  For $e \in E(H) - E(T)$, only H and C_e contains e among $\{H\} \cup \{C_f | f \in E(H) - E(T)\}$. Thus $e \notin H + H'$ for all $e \in E(H) - E(T)$. For $e \in E(G) - (E(H) \cup E(T))$, none of $\{H\} \cup \{C_f | f \in E(H) - E(T)\}$. Thus $e \notin H \oplus H'$ for all $e \in E(G) - (E(G) \cup E(T))$. Therefore, $H \oplus H'$ is a subgraph of T.

  Claim $H \oplus H' = \varnothing$.

Proof of Claim $H \oplus H'$ is even. There is a collection C of edge-disjoint cycles whose symmetric difference is $H \oplus H'$, but $H \oplus H'$ is a subgraph of T. Hence $C = \varnothing \implies H \oplus H' = \varnothing$.

Alternative Proof Suppose $H \oplus H' \neq \varnothing$. Let X be a component with max number of edges. X is not an isolated vertex, so $\delta(X) \geq 1$. X is even since $H \oplus H'$ is even $\implies \delta(X) \geq 2$. Thus X contains a cycle, contradicting $X \subseteq T$. ■

binary code

A binary code of length m us a subspace U of $\mathbb{Z}_2{}^m$.

minimum distance

The minimum distance of U is the smallest integer t such that for some $u \in U$, $u \neq 0$ has exactly t coordinates equal to 1.

Example $U = \{(0,0,0),\ (1,0,0),\ (1,0,1),\ (0,0,1)\}$ is a binary code of length 3. minimum distance of U is 1.

Hamming Distance

The Hamming Distance between $u, v \in U$ is the number of coordinates of $u+v$ equal to 1.

Ex Hamming distance of $(1,0,0),(1,0,1)$ is 1. Since $(1,0,0)+(1,0,1) = (0,0,1)$.

Lemma Let U be a binary code with minimum distance t. Then any two vectors $u \neq v$ in U are at Hamming distance $\geq t$.

Proof Let us count the number of 1's in $u+v$. $u+v \in U$ and $u+v \neq 0$ (since $u \neq v$). Thus the number of 1's in $u+v$ is at least t. ■

girth

The girth of a graph G is the length of a shortest cycle. (If G has no cycle, then the girth is ∞)

Lemma Let G be a connected graph of girth g. Then the flow space W of G is a binary code of length $m = |E(G)|$ of dimension $m - |V(G)| + 1$ that has minimum distance g.

Proof $\dim(W) = |F(T)| = |E(G)| - E(T)| = m - |V(G)| + 1$.

Since every nonempty even subgraph $H \subseteq G$ contains a cycle, $|E(H)| \geq g$. Take H to be a shortest cycle in G, then $|E(H)| = g$, so the minimum distance of W is g. ■

changing track...

3.3 Vector Spaces of Incidence Matrix

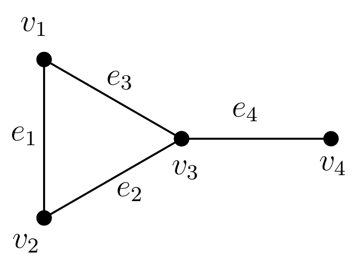
incidence matrix

Let G be a graph. The incidence matrix of G is the $n \times m$ matrix $B(G)$ where $n = |V(G)|$, $m = |E(G)|$, and

$$[B(G)]_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident to } e_j \\ 0 & \text{otherwise} \end{cases}$$

ex

G



$$B(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Trivial Observations

- every column has exactly two non-zero entries
- row i has exactly $\deg(v_i)$ 1's

Recall from Linear Algebra

There are two very natural vector spaces over $\mathbb{Z}_2$ associated with B.

- $\text{Ker}(B)$: the set of all vectors $v \in \mathbb{Z}_2^m$ for which $Bv = 0$.
- $\text{row}(B)$: the subspace of $\mathbb{Z}_2^m$ spanned by the rows of B.

1. WHAT DOES KER(B) LOOK LIKE?

   Let $w \in \text{Ker}(B)$. Since $w \in \mathbb{Z}_2^m$, we can of course think of w as being the characteristic vector of some spanning subgraph H of G. Since $w \in \text{Ker}(B)$, $Bw = 0$ and so every entry of Bw is equal to zero. Since "1" entries in row i of B correspond to edges incident with v_i, it follows that i^{th} entry of Bw is the sum of an even number of "1"s, and so that v_i has even degree in H. Thus H is even.

(Conversely, if w is the characteristic vector of any even subgraph $H \subseteq G$, then $w \in \mathrm{Ker}(B)$.)

Thus $Ker(B)$ is the flow space of G. If G is connected, it follows that $\mathrm{Ker}(B)$ has dimension $m - n + 1$.

What about row(B)? Well, the dimension of the rowspace is the rank of B. (i.e. the size of largest set of linearly independent rows of B)

From linear algebra: $\mathrm{rank}(B) + \dim \mathrm{Ker}(B) = m$. So if G is connected, $\mathrm{rank}(B) = m - (m - n + 1) = n - 1$.

Easy induction $\mathrm{rank}(B) = n - c$ if G has c components.

2. WHAT DOES ROW(B) LOOK LIKE?

If B has rows $r_1, \ldots, r_n$, then the elements of row(B) are linear combinations of $r_1, \ldots, r_n$. Say $w \in \mathrm{row}(B)$, with $w = \lambda_1 r_1 + \ldots + \lambda_n r_n$, then

$$w = \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} \underbrace{\begin{bmatrix} r_{11} & \ldots & r_{1m} \\ \vdots & \ddots & \vdots \\ r_{n1} & \ldots & r_{nm} \end{bmatrix}}_{B}$$

and so every vector $w \in \mathrm{row}(B)$ is of the form $w = y^T B$ for some $y \in \mathbb{Z}_2^n$. Note y corresponds to some $S \subseteq V(G)$ – that is, it is the charateristic vector of some $S \subseteq V(G)$.

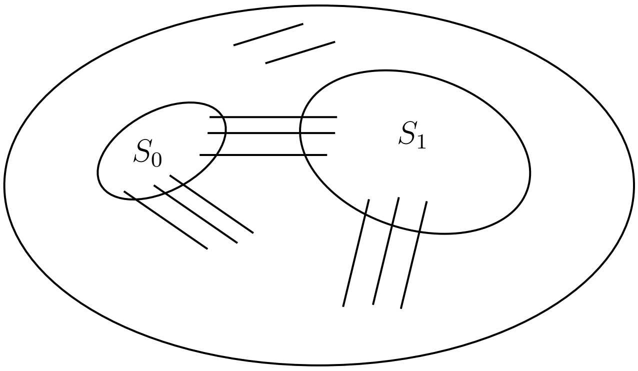
Now which edges e_i are such that the i^{th} coordinate of w is 1? Turns out, it's precisely the edges that have exactly one endpoint in S.

Thus each $w \in \mathrm{row}(B)$ is the characteristic vector of ∂S for some $S \subseteq V(G)$ (∂S denotes the edge boundary (edge cut) of SA - the set of edges with precisely one endpoint in S).

Also each $S \subseteq V(G)$ gives rise to a vector $w \in y_S^T B$ that is the characteristic vector of ∂S. So row(B) is precisely the vector space of characteristic vectors of edge boundaries in G. (The 0-vector is the characteristic vector of the edge cut of $S = \varnothing$)

Lemma Suppose ∂S_0 and ∂S_1 are two edge boundaries in a graph G. Then the symmetric difference $(\partial S_0)\Delta(\partial S_1)$ is also an edge boundary in G.

Proof The characteristic vectors w_0 & w_1 of ∂S_0 & ∂S_1 are vectors in row(B), where B is the incidence matrix of G. Then $w_0 + w_1 \in \mathrm{row}(B)$, since row$(B)$ is a vector space. Note that $w_0 + w_1$ is the characteristic vector of $(\partial S_0)\Delta(\partial S_1)$. Thus $(\partial S_0)\Delta(\partial S_1)$ is also an edge boundary. ■



row(B) is called the cut-space of G

Index

M

N

P

S

T

V

X

Y