Applied Real Analysis

AMATH 331

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LATEXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of AMATH 331 during Winter 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

I gave up using definition blocks gradually since the professor uses a subsection to give all definition...

Also, I am not following the numbering convention in professor's lecture notes: Instead of setting the counter within the section (Theorem 13.1.1), I am using the counter within the chapter/lecture (Theorem 13.1).

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Sibelius Peng

Contents

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Real Numbers

Refs 1 for review. 2.1-2.2, 2.9

1.1 Decimal expansions and the real number line

finite decimal expansion

A finite decimal expansion has the form

$$x = a_{0\circ}a_1a_2a_3\dots a_N$$

where a_0 is an integer (positive, negative or zero) for $1 \leq n \leq N$ $a_n \in \{0, 1, \dots, 9\}$

Example:

$$1_\circ 45$$

$$-38_\circ 298743$$

You can think of this as

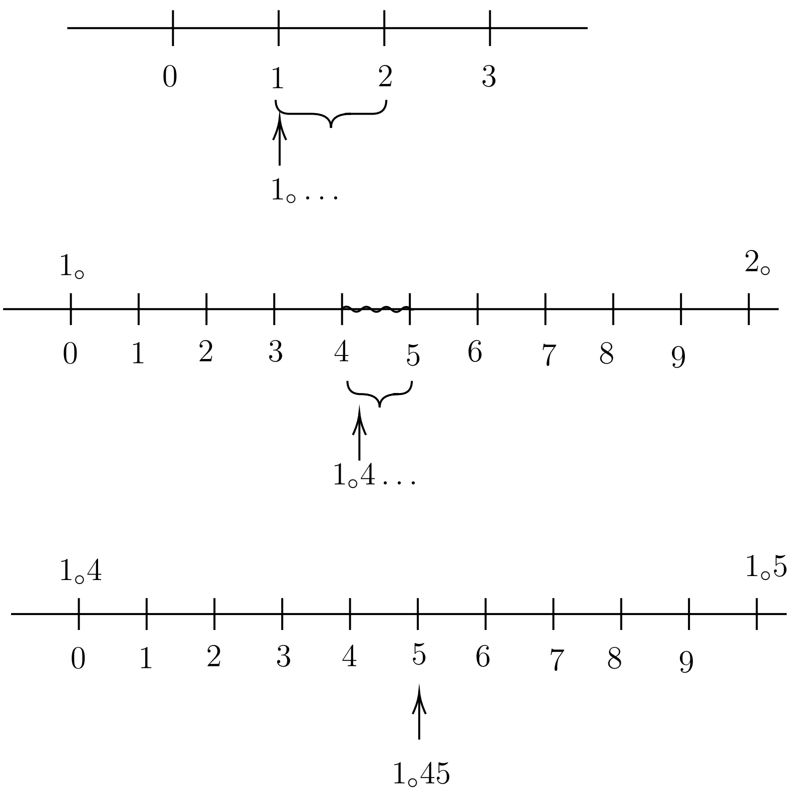
$$x = a_0 + a_1\left(\frac{1}{10}\right) + \dots + a_N\left(\frac{1}{10^N}\right)$$

Warning This looks like the usual decimal representation but it is not the same for negative numbers.

Any finite decimal expansion can be replaced on the real number line.

Example:

Where is $1_\circ 45$?



We can similarly define infinite decimal expansions

infinite decimal expansions

$$x = a_{0\circ} a_1 a_2 \ldots$$

Example:

$$1_\circ 450000000\ldots$$

$$\pi = 3_\circ 1415926535\ldots$$

Assuming the real number line has no gaps, every infinite decimal expansion x corresponds to a point on the line.

Given any positive integer k, let $y = a_{0\circ} a_1 a_2 \ldots a_k$ be the finite decimal expansion of x to the k-th decimal space. Then, x lies in the interval from y to $(y + 10^{-k})$. So, y approximates x to an accuracy of $1/10^k$. As we increase k, we improve the accuracy; in fact, the error can be made arbitrarily small.

The converse direction: given a point on the real number line, can we find its decimal expansion?

Yes!

It is possible for two decimal expansions to represent the same point. This happens precisely when one ends in an infinite string of 0's.

Example:

$$1.000\ldots \qquad \text{and} \qquad 0.999\ldots$$

$$25.300\ldots \qquad \text{and} \qquad 25.2999\ldots$$

We define the real numbers $\mathbb{R}$ as the set of all infinite decimal expansions.

1.2 Ordering of real numbers

Suppose

$$x = x_{0\circ}x_1x_2x_3\ldots, \qquad y = y_{0\circ}y_1y_2y_3\ldots$$

We say that x and y are equal and write $x = y$ if infinite decimal expansions are identical or equivalent, as discussed previously.

If x and y are not equal, then we say that x are not equal, then x is *less than* y and write $x < y$ if there exists integer $k \geq 0$ such that $x_k < y_k$ and $x_i = y_i$ for $i < k$. x is *greater than* y $(x > y)$ if ...

For any two real numbers x, y, exactly one of the following holds:

$$x = y \qquad x < y \qquad x > y$$

2

Bounds and Limits

2.1 Bounded sets of real numbers

upper bound

A set $S \subseteq \mathbb{R}$ is *bounded above* if there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. M is an *upper bound* of S.

lower bound

A set $S \subseteq \mathbb{R}$ is *bounded below* if there exists $m \in \mathbb{R}$ such that $s \geq m$ for all $s \in S$. m is an *lower bound* of S.

bounded

A set is *bounded* if it is both bounded above and bounded below.

supremum

The *supremum* or *least upper bound* of a nonempty set S that is bounded above is the upper bound L satisfies $L \leq M$ for all upper bounds M of S is written as $\sup S$.

infimum

The *infimum* or *greatest lower bound* of a nonempty set S is the lower bound ℓ satisfying $\ell \geq m$ for all lower bounds m of S. The infimum is denoted $\inf S$.

max

If there exists $M \in S$ such that $s \leq M$ for all $s \in S$, then M is called the *maximum* of S, $\max S$.

min

Analogous defn for $\min S$.

2.2 Examples

0. $S_0 = \varnothing$. Bounded above and below. No supremum or infimum.
1. $S_1 = \{n \in \mathbb{Z}^+\} = \{1, 2, 3, \ldots\}$ not bounded above, bounded below.

   1 is infimum and minimum
2. $S_2 = \{-3, -2, 0.5, 1.423\}$. Bounded above and below. Bounded. Has max, min.
3. $S_3 = \left\{1 - \frac{1}{n} : n \in \mathbb{Z}^+\right\} = \{0, \frac{1}{2}, \frac{2}{3}, \ldots\}$

   Bounded above by 1. Bounded below by 0.

   Supremum is 1, but there is no max.

2.3 Least Upper Bound Principle

Theorem 2.1: Least Upper Bound Principle

Every nonempty set S of $\mathbb{R}$ that is bounded above has a supremum. Every nonempty set that is bounded below has an infimum.

Sketch of proof for "infimum". There are only finitely many integers from m_0 to $s_0 + 2$. Choose the greatest integer lower bound $\to$ call it a_0.

$a_0 + 1$ is not a lower bound. Divide $[a_0, a_0 + 1]$ into 10, find a_1 such that $a_{0\circ}a_1$ is lower bound of S, but $a_{0\circ}a_1 + 1/10$ is not. Repeat infinitely many times to construct $L = a_{0\circ}a_1a_2a_3\ldots$.

Now, show that L is infimum.[1] $\square$

[1] See details in textbook.

3

Limits of Sequences

3.1 Sequences

An *infinite sequence of real* numbers is an infinite, enumerated list of real numbers, denoted by

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, \ldots)$$

Each $a_n \in \mathbb{R}$ is an *element* of the sequence.

We will just refer to them as sequences, and often write (a_n). Formally, a sequence is a function that maps positive integers to $\mathbb{R}$.

We say that a sequence is [bounded above/bounded below/bounded] if the set $A = \{a_n\}$ is respectively [bounded above/bounded below/bounded].

3.2 Examples

1. $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^n$ for $n \geq 1$.
2. $a_n = \frac{1}{n}$, for $n \geq 1$.
3. $(a_n) = (1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \ldots)$

3.3 Limits of Sequences

limit

Let $(a_n)_{n=1}^{\infty}$ be a sequence. We call $L \in \mathbb{R}$ the *limit* of the sequence if for all $\epsilon > 0$, there exists an integer N such that

$$|a_n - L| < \epsilon$$

for all $n \geq N$.

If such L exists, then we say that (a_n) is convergent, and converges to L and we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$.

If a sequence does not have such a limit, then we say it *diverges*, or is *divergent*.

A sequence (a_n) *diverges to* ∞ if for all $M > 0$, there exists N such that $a_n > M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = \infty$.

A sequence (a_n) *diverges to* $-\infty$ if for all $M < 0$, there exists N such that $a_n < M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = -\infty$.

Note
$\lim_{n\to\infty} a_n = \pm\infty$ does not mean limit exists.

3.4 Examples

1. $a_n = 1/n, \quad \lim_{n\to\infty} a_n = 0$

   For any $\epsilon > 0$, we need to show that there exists N such that $|a_n - 0| < \epsilon$ for all $n \geq N$.

   Choose N to be any integer greater than $1/\epsilon$. ($N > \frac{1}{\epsilon}$)

   For any $n \geq N$, $a_n = 1/n \leq \frac{1}{N} < \epsilon$. We also have $a_n \geq 0$

   $$\implies |a_n| < \epsilon$$

   for all $n \geq N$ as required.

3.5 Some basic properties of limits

Theorem 3.1: Squeeze Theorem

Let $(a_n), (b_n), (c_n)$ be sequences.

If $a_n \leq b_n \leq c_n$ for all $n \geq 1$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = 1$$

Proof:
We want to show that for all $\epsilon > 0$, there exists N such that $|b_n - L| < \epsilon$ for all $n \geq N$.

Let $\epsilon > 0$. Since $a_n \to L$, we can find N_1 such that $|a_n - L| < \epsilon$ for all $n \geq N_1$.

Similarly, there exists N_2 s.t. $|c_n - L| < \epsilon$ for all $n \geq N_2$.

Define $N := \max\{N_1, N_2\}$. Then, for $n \geq N$, $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$.

Equivalently,

$$L - \epsilon < a_n < L + \epsilon \qquad L - \epsilon < c_n < L + \epsilon$$

Since $a_n \leq b_n \leq c_n$. $L - \epsilon < b_n < L + \epsilon$, or

$$|b_n - L| < \epsilon$$

as required. □

Proposition 3.2

If a sequence converges to a limit L, then this limit is unique.

Proof:
See PDF. □

Proposition 3.3

If a sequence (a_n) converges, then the set $A := \{a_n : n \geq 1\}$ is bounded.

Proof:
Exercises. □

Theorem 3.4

Let (a_n) and (b_n) be two convergent sequences. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. $\lim_{n\to\infty}(a_n + b_n) = L + M$
2. for any $\alpha \in \mathbb{R}$, $\lim_{n\to\infty}(\alpha a_n) = \alpha L$
3. $\lim_{n\to\infty}(a_n b_n) = LM$, and
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$ and $b_n \neq 0$ for all n.

4

Monotone Sequence and Applications

4.1 Monotone Sequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. it is

1. monotone increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.
2. strictly monotone increasing if $a_{n+1} > a_n$ for all $n \geq 1$.
3. monotone decreasing if $a_{n+1} \leq a_n$
4. strictly monotone decreasing if $a_{n+1} < a_n$

monotone

A sequence is monotone is *monotone* if it is either (monotone) increasing or (monotone) decreasing.

Theorem 4.1: Monotone Convergence Theorem

Monotone Convergence Theorem:

(i) Every monotone increasing sequence that is bounded above converges

(ii) Every monotone decreasing sequence that is bounded below converges

Proof:
We will first show that (i) $\implies$ (ii).

Let (a_n) be a monotone decreasing sequence that is bounded below by m.

The sequence $(-a_n)_{n=1}^{\infty}$ is monotone increasing and is bounded above by $-m$. By part (i), $(-a_n)$ must converge. Call the limit $L = \lim_{n\to\infty}(-a_n)$.

By Theorem 3.4 Part 2,

$$\lim_{n\to\infty} = \lim_{n\to\infty}[(-1)(-a_n)] = (-1)\lim_{n\to\infty}(-a_n) = -L$$

To prove Part(i) of this theorem, suppose (a_n) is monotone increasing and bounded above.

The set $A = \{a_n | n \in \mathbb{Z}^+\}$ is bounded above, and nonempty.

By LUBP(Theorem 2.1), A has a supremum, which we call $L = \sup A$. We show that L is the limit of (a_n).

Given $\epsilon > 0$, we know that $L - \epsilon$ cannot be an upper bound of A.

So there exists N such that $a_n > L - \epsilon$.

Since (a_n) is increasing, $a_n > L - \epsilon$ for all $n \geq N$. Since L is an upper bound of A, $a_n \leq L$ for all $n \geq N$.

$$\implies L - \epsilon < a_n \leq L < L + \epsilon$$

That is $|a_n - L| \leq \epsilon$ for all $n \geq N$. □

4.2 Applications: Calculate Square Roots

The square root of a real number $a > 0$ can be obtained as the limit of the sequence defined recursively by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right), \qquad \text{for } n \geq 1$$

where the starting point x_0 is any positive number.

Moreover, for any $n \geq 1$, the error in approximating $\sqrt{a}$ by x_n satisfies the bound

$$0 \leq x_n - \sqrt{a} < x_n - \frac{a}{x_n}$$

Proof:
Strategy:

1. Prove that (x_n) is bounded below.
2. Prove that (x_n) is monotone decreasing.
3. Prove that (x_n) is monotone decreasing.
4. Use MCT to prove that (x_n) converges.

5. Use properties of limits to determine that $\sqrt{a}$ is the limit.

6. Look for upper and lower bounds for error.

See PDF for full proof. □

4.3 Warning about computing limits that don't exist

Example:

$a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$ for $n \geq 1$.

If we assume (a_n) has a limit L, then we can get nonsense.

$$a_{n+1} = \frac{1}{2}(a_n^2 + 1)$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n^2 + 1)$$
$$\implies L = \frac{1}{2}\left(\lim_{n\to\infty} a_n\right)^2 + \frac{1}{2} = \frac{1}{2}L^2 + \frac{1}{2}$$
$$L^2 - 2L + 1 = 0 \implies L = 1 \text{ is a solution}$$

However, it can be shown that (a_n) is monotone increasing. Since $a_1 = 2$, (a_n) cannot possibly converge to 1.

(In fact, it does not converge.)

5
Subsequences

5.1 Definitions of subsequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence. The sequence $(b_k)_{k=1}^{\infty}$ is a *subsequence* of (a_n) of there exist integers n_k with $1 \leq n_1 < n_2 < n_3 < \dots$ such that $b_k = a_{n_k}$ for each $k \geq 1$.

Example:

$(a_1, a_2, a_3, a_4, a_5, \dots)$

$(b_1, b_2, b_3, b_4, b_5, \dots)$

cannot do the following:

$(a_1, a_2, a_3, a_4, a_5, \dots)$

$(b_1, b_2, b_3, b_4, b_5, \dots)$

not allowed to change order

Example:

$$(a_n)_{n=1}^{\infty} = \left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \dots\right)$$

The sequence (b_k) with $b_k = a_k$ for all $k \geq 1$ is a subsequence of (a_n).

The sequence $\left(-1, -\frac{1}{3}, -\frac{1}{5}, \dots\right)$ is a subsequence.

The sequence $\left(\frac{1}{2}, \frac{1}{4}, \dots\right)$ is another subsequence.

5.2 Some properties of Subsequences

Lemma 5.1

Let n_k be integers satisfying $n_1 \geq 1$ and $n_k < n_{k+1}$ for all $k \geq 1$. Then $n_k \geq k$ for all $k \geq 1$.

Theorem 5.2

Suppose the sequence $(a_n)_{n=1}^{\infty}$ converges to the limit L. Then every subsequence of (a_n) also converges to L.

Proof:
By definition of limit, for every $\epsilon > 0$, there exists N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Let $(b_k)_{k=1}^{\infty}$ be any subsequence of (a_n), where $b_k = a_{n_k}$ for each $k \geq 1$.

From Lemma 5.1, we know that $n_k \geq k$ for each k. Given $\epsilon > 0$, chose N as in definition of $\lim_{n\to\infty} a_n = L$. For every $k \geq N$,

$$n_k \geq k \geq N \implies |b_k - L| = |a_{n_k} - L| < \epsilon$$

□

Example:

1. From 5.1, the theorem holds just as it is.
2. Converse is not true. If a subsequence converges, we cannot conclude that the original sequence converges.

5.3 Bolzano-Weierstrass

If for every integer $n \geq 1$, we have a nonempty, closed interval $I_n = [a_n, b_n]$ such that $I_{n+1} \subseteq I_n$, then we say that (I_n) is a *nested sequence of closed, bounded intervals.*

Lemma 5.3: Nested Intervals Lemma

If (I_n) is a nested sequence of closed bounded intervals, then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof:
Exercise. □

Theorem 5.4: Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Proof:
Outline.

1. Given a bounded sequence (a_n), construct a nested sequence of closed, bounded intervals I_n with lengths decreasing to zero, and such that each I_n contains infinitely many elements of the sequence (a_n).
2. Construct a subsequence (b_k) such that $b_k \in I_k$ for each $k \geq 1$.
3. Show that (b_k) converges.

□

Proof:

Step 1: Suppose $(a_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers. Let m_1 be a lower bound and M_1 be an upper-bound for $A = \{a_n : n \geq 1\}$.

Define an interval $I_1 = [m_1, M_1]$. Define the point $c_1 = \frac{1}{2}(m_1 + M_1)$. Choose one smaller interval either $[m_1, c_1]$ or $[c_1, M_1]$ that contains an infinite member of elements of $(a_n) \to$ call this interval $I_2 = [m_2, M_2]$.

We repeat this process for all $k \geq 2$. This gives a sequence of intervals $(I_k)_{k=1}^{\infty}$ such that $I_{n+1} \subseteq I_n$ for all $n \geq 1$, and lengths of I_n converges to zero. Also each I_k contains an infinite number of elements of (a_n).

Step 2: Let $n_1 = 2$ so $b_1 = a_1$. Suppose we have our subsequence (b_j) up to element k. Then we have $n_i \geq 1$ for all $i = 1, 2, \ldots, k$ and $n_i < n_{i+1}$ for all $i = 1, 2, \ldots, k-1$.

Since there are an infinite number of elements of (a_n) contained in I_{k+1}, we can choose n_{k+1} such that $n_{k+1} > n_k$ and $a_{n_{k+1}} \in I_{k+1}$, i.e. $b_{k+1} \in I_{k+1}$. In this way, we inductively define (b_j) as a subsequence of (a_n).

Step 3: By Nested Intervals Lemma (Lemma 5.3), $\displaystyle\bigcap_{k=1}^{\infty} I_k \neq \varnothing$, so there must exist a point $L \in \cap_{k=1}^{\infty} I_k$. The length of interval I_j is $\dfrac{(M_1 - m_1)}{2^{j-1}}$. For any $k \geq 1$, we have $L \in I_k$ and $b_k \in I_k$. Hence $|b_k - L| \leq \dfrac{(M_1 - m_1)}{2^{k-1}}$.

Consider sequence $(|b_k - L|)_{k=1}^{\infty}$. We can use Squeeze Theorem to show that $\lim_{n\to\infty} |b_k - L| = 0$ since

$$0 \leq |b_k - L| \leq \frac{(M_1 - m_1)}{2^{k-1}}.$$

Hence $\displaystyle\lim_{k\to\infty} b_k = L$.

□

6 Cauchy Sequences

6.1 Definition

A sequence (a_n) is *Cauchy* if for any $\epsilon > 0$, there exists an integer N such that

$$|a_n - a_m| < \epsilon$$

for all $n, m \geq N$.

Example:

$$(a_n)_{n=1}^{\infty} = (3, 3.1, 3.14, 3.141, \ldots)$$

More generally, if x is any real number with infinite decimal expression $x_{0\circ}x_1x_2x_3\ldots$, then the sequence of finite truncations, i.e., a_k is the truncation of x to k decimal places, is Cauchy.

$$a_k = x_{0\circ}x_1 \ldots x_k000\ldots$$

Given $\epsilon > 0$, we can find N such that $10^{-N} < \epsilon$.

For any $n \geq 1$, we have

$$a_n \leq x \leq a_n + 10^{-n}$$

In particular,

$$a_N \leq x \leq a_N + 10^{-N}$$

Note that (a_n) is monotone increasing, so $a_N \leq a_n, a_m \leq x \leq a_N + 10^{-N}$ for any $n, m \geq N$.

So

$$|a_n - a_m| \leq \text{length of interval} = 10^{-N} < \epsilon$$

$\implies (a_n)_{n=1}^{\infty}$ is Cauchy.

7

Cauchy and Completeness

7.1 Properties of Cauchy Sequences

Proposition 7.1

If a Cauchy sequence (a_n) has a convergent subsequence, then (a_n) converges. The limit is the same as the limit of the subsequence.

Proof:
Let $\epsilon > 0$. By definition of limit of $(b_k) = (a_{n_k})$ being L, i.e., $\lim_{k\to\infty} b_{n_k} = L$, there exists K such that

$$|b_k - L| = |a_{n_k} - L| < \frac{\epsilon}{2}$$

for all $k \geq K$.

By Cauchy property of (a_n), there exists N such that

$$|a_n - a_m| < \frac{\epsilon}{2}$$

for all $n, m \geq N$.

By Lemma 5.1, $n_k \geq k$ for all $k \geq 1$, so

$$|a_n - a_{n_k}| < \frac{\epsilon}{2}$$

for all $n, k \geq N$. Choose any $k \geq \max\{K, N\}$. Then, for all $n \geq N$,

$$|a_n - L| = |a_n - a_{n_k} + a_{n_k} - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Proposition 7.2

If a sequence (a_n) is Cauchy, then the set $\{a_n : n \geq 1\}$ is bounded.

Proof:
Exercise, or see PDF. □

7.2 Example of not quite Cauchy

Consider the sequence $(a_n)_{n=1}^{\infty}$, with $a_n = \log n$.

The difference between successive terms is

$$|a_{n+1} - a_n| = |\log(n+1) - \log(n)| = \left|\log\left(\frac{n+1}{n}\right)\right|$$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$, so $\lim |a_{n+1} - a_n| = 0$.

(a_n) is not bounded, since $\log(n) \to \infty$, hence by Proposition 7.2, (a_n) is not Cauchy.

7.3 Cauchy, Convergent and Complete

Proposition 7.3

Every convergent sequence is Cauchy.

Proof:
(Sketch)

N, K and use $\epsilon/2$. □

complete

We say that a subset X of $\mathbb{R}$ is *complete* if every Cauchy sequence in X has a limit in X.

Theorem 7.4: Completeness Theorem for Real Numbers

$\mathbb{R}$ is complete.

In other words, every Cauchy sequence of real numbers converges.

Proof:
Suppose (a_n) is any Cauchy sequence of real numbers. By Proposition 7.2, $\{a_n : n \geq 1\}$ is bounded. By Theorem 5.4, there must exist a convergent subsequence.

By Proposition 7.1, (a_n) must also converge. □

Remark:
The sequence of truncated decimal expansions of x (from Lecture 6) was shown to be Cauchy. Now we know, it must converge. It can be shown that the limit is x.

Note
$\mathbb{Q}$ is not a complete subset of $\mathbb{R}$. Using sequence of finite decimal expansions, we see that sequences of rational numbers can converge to an irrational limit.

7.4 Equivalent Statements of Completeness

We showed that construction of $\mathbb{R}$ as set of infinite decimal expansions leads to Least Upper Bound Principle.

$\implies$ Monotone Convergence Theorem

$\implies$ Nested Intervals Lemma

$\implies$ Bolzano-Weierstrass Theorem

$\implies$ Completeness Theorem

It is possible to show that Completeness $\implies$ LUBP. So all of these properties describe the same "behaviour" of $\mathbb{R}$.

7.5 Application: Proving convergence by Cauchy property

Sometimes it's easier to show that a sequence is Cauchy than convergent.

Example:
Consider a sequence $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n}$. We can show that $(a_n)_{n=1}^{\infty}$ is Cauchy. For $m > n$,

$$\begin{aligned} |a_m - a_n| &= \left| \frac{(-1)^{n+2}}{n+1} + \frac{-1^{n+3}}{n+2} + \ldots + \frac{(-1)^m}{m-1} + \frac{(-1)^{m+1}}{m} \right| \\ &= \ldots \end{aligned}$$

Suppose $m - n$ is even

$$|a_m - a_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \ldots + \frac{1}{m-1} - \frac{1}{m} \right|^{a}$$

[a] Sth wrong here... corrected in the lecture notes.

8 Series

8.1 Definitions for series

If $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers, we define its *sequence of partial sums* $(S_n)_{n=1}^{\infty}$ by $S_n = \sum_{k=1}^{n} a_k$.

The (infinite) series associated with (a_n) is $\sum_{n=1}^{\infty} a_n$. If the sequence of partial sums converges to a limit $L \in \mathbb{R}$, then we say the series $\sum_{n=1}^{\infty}$ converges. In this case, we say the sum or value of the series is L.

The series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

If a series does not converge, then it diverges.

A series that converges but is not absolutely convergent, then we say it is conditionally convergent.

Example:

1. $(a_n)_{n=1}^{\infty} = (1, 1, 1, 1, 1, \ldots)$. This sequence converges to 1.

   Sequence of partial sums is $(S_n) = (1, 2, 3, 4, 5, \ldots)$ does not converge (it diverges to ∞) so the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   Note
   $S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ forms a sequence such that

   $$S_{n+1} - S_n = \frac{1}{n+1} \to 0$$

   but (S_n) is not convergent, which means (S_n) is not Cauchy.

3. $a_n = \frac{1}{n(n+2)}$.

We will show that $\sum_{n=1}^{\infty} a_n$ converges.

Note
We can write
$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$
Then the sequence of partial sums is
$$S_n = \frac{1}{2}\sum_{k=1}^{n} \frac{2}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\left[\left(1+\frac{1}{2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)\right]$$
$$\lim_{n\to\infty} S_n = \frac{3}{4}$$
Hence,
$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$$

4. A geometric series $\sum_{n=0}^{\infty} a_n$ is one where the elements are of the form $a_n = a_0 r^n$ for some $a_0 \in \mathbb{R}, r \in \mathbb{R}$, for each $n \geq 0$.

   If $|r| < 1$, then the series converges
   $$\sum_{n=0}^{\infty} a_n = \frac{a_0}{1-r}$$
   If $|r| \geq 1$ and $a_0 \neq 0$, then the series diverges.

5. The alternating harmonic series
   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
   converges. It is not absolutely convergent. (See Example 2), so it is conditionally convergent.

Proposition 8.1

Every absolute convergent series is convergent.

Proof:
Trivial. □

8.2 Convergence Tests

Theorem 8.2: Cauchy criterion for series

Given a series $\sum_{n=1}^{\infty} a_n$, the following are equivalent:

1. The series converges.
2. Given $\epsilon > 0$, there exists an integer N such that
$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$$
for all $m > n \geq N$.

Note
If (S_n) is sequence of partial sums. Suppose $m > n$,
$$|S_m - S_n| = \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{m} a_k \right|$$

Theorem 8.3: Comparison Test for Series

Suppose $(a_n), (b_n)$ are two sequences and $|a_n| \leq b_n$ for all $n \geq 1$.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, and
$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} b_n$$
2. If $\sum_{n=1}^{n} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof:
Note that 2 follows from 1.

So, we just need to prove 1.

First, we show that
$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$
Let $\epsilon > 0$. By Cauchy criterion, there exists N such that
$$\left| \sum_{k=n+1}^{m} b_k \right| < \epsilon \text{ for all } m > n \geq N$$

Since $b_k \geq 0$ for all k, we can ignore absolute value sign.

$$\epsilon > \sum_{k=n+1}^{m} b_k \geq \sum_{k=n+1}^{m} |a_k| \geq \left| \sum_{k=n+1}^{m} a_k \right|$$

This is the Cauchy criterion for $\sum a_n$, so $\sum a_n$ converges.

The rest of proof is left as an exercise: Show remaining inequality. $\square$

9

Rearrangements of Series

9.1 Definition

A rearrangement is a series considering of the same terms as another series but in a different order. Suppose $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a permutation of the positive integers. Then, the series $\sum_{n=1}^{\infty} a_{\pi(n)}$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots$$

$$\sum_{n=1}^{\infty} a_{\pi(n)} = a_3 + a_4 + a_2 + a_1 + a_6 + \dots$$

9.2 Rearrangements of absolutely convergent series

Proposition 9.1

If an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ converges to L, then every rearrangement of $\sum_{n=1}^{\infty} a_n$ also converges to L.

Proof:
Let $\sum_{n=1} a_{\pi(n)}$ be a rearrangement. Fix $\epsilon > 0$. By absolute convergence of

$\sum_{n=1}^{\infty} a_n$, there exist N such that

$$\left|\sum_{n=1}^{N} |a_n| - \sum_{n=1}^{\infty} |a_n|\right| = \sum_{n=N+1}^{\infty} |a_n| < \frac{\epsilon}{2}$$

Since every term of the series $\sum_{n=1}^{\infty} a_n$ must appear in the rearrangement, there must exist $M \geq N$ such that $\sum_{n=1}^{M} a_{\pi(n)}$ includes all terms

$$a_1, a_2, a_3, \ldots, a_N$$

For any $m \geq M$,

$$\begin{aligned}
\left|\sum_{n=1}^{m} a_{\pi(n)} - L\right| &= \left|\sum_{n=1}^{m} a_{\pi(n)} - \sum_{n=1}^{N} a_{\pi(n)} + \sum_{n=1}^{N} a_{\pi(n)} - L\right| \\
&\leq \left|\sum_{n=1}^{m} a_{\pi(n)} - \sum_{n=1}^{N} a_{\pi(n)}\right| + \left|\sum_{n=1}^{N} a_{\pi(n)} - L\right| \\
&\leq \sum_{n=N+1}^{\infty} |a_n| + \sum_{n=N+1}^{\infty} |a_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

So $\sum_{n=1}^{\infty} a_{\pi(n)} = L$. □

9.3 Rearrangements of conditionally convergent series

Lemma 9.2

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series. Then there is an infinite number of non-negative terms and an infinite number of negative terms in the series.

Proof:
Use contrapositive.

Suppose there is a finite number of negative terms.

> Remark:
> Case with finite number of non-negative terms can be proved in the same way.

There must exist integer N such that N is the largest number for which $a_N < 0$. i.e. $a_n \geq 0$ for all $n > N$.

Case (i) $\sum_{n=1}^{\infty} a_n$ diverges. Trivially, not conditionally convergent.

Case (ii) $\sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_N$$

$$\implies \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

$$= \underbrace{\sum_{n=1}^{N} |a_n|}_{\substack{\text{finite sum} \\ \implies \text{ real number}}} + \underbrace{\sum_{n=1}^{\infty} a_n}_{\text{converges}} - \underbrace{\sum_{n=1}^{N} a_n}_{\substack{\text{finite sum} \\ \implies \text{ real number}}}$$

By properties of limits, $\sum_{n=1}^{\infty} |a_n|$ converges.

$\implies$ $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$\implies$ $\sum_{n=1}^{\infty} a_n$ not conditionally convergent.

□

Lemma 9.3

Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. For each $n \geq 1$, define b_n to be the n-th non-negative and c_n is the n-th negative term in the series. Then,

1. $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$, and
2. $\sum_{n=1}^{\infty} b_n = \infty$ and $\sum_{n=1}^{\infty} c_n = -\infty$.

Proof:
Exercise. □

Theorem 9.4

Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent series. Then, for any $L \in \mathbb{R}$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that convergent to L.

Proof:
Exercise. □

10

Euclidean Space

10.1 $\mathbb{R}^n$ Euclidean inner product and norm

We define the space $\mathbb{R}^n$ to be the set of all n-vectors $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$ where $x_i \in \mathbb{R}$ for each $i = 1, 2, 3, \ldots, n$.[1]

$\mathbb{R}^n$, equipped with vector addition and scalar multiplication, is a *vector* space.

We also define the Euclidean *inner product* of two vectors x and y is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

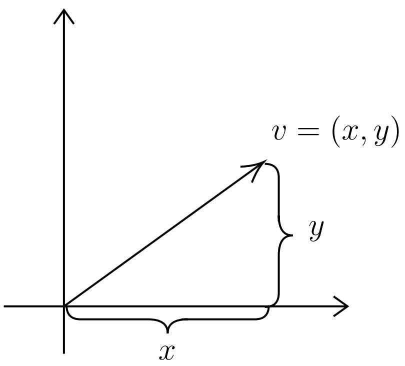
Also called dot product or scalar product.

The *Euclidean norm* of a vector $x \in \mathbb{R}^n$ is

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

Bt *Euclidean space*, we mean $\mathbb{R}^n$ with the structure imposed by the Euclidean inner product and norm.

[1] I will use $\mathbf{x}, \vec{x}$ or just x (from optimization courses) to represent vector. Readers should be clear when it is vector.



$$\|v\| = \sqrt{x^2 + y^2} = \left(\sum_{i=1}^{2} |v_i|^2 \right)^{1/2}$$

Figure 10.1: Example of $n = 2$

10.2 Properties of Euclidean inner product and norm

Proposition 10.1

Let $x, y, z\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. The Euclidean inner product satisfies

1. $\langle x, x \rangle \geq 0$ with equality iff $x = 0$. (positive definite)
2. $\langle x, y \rangle = \langle y, x \rangle$. (symmetry)
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$. (Bilinearty)

Proposition 10.2

Let $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. The Euclidean norm satisfies:

1. $\|x\| \geq 0$ with equality iff $x = 0$. (positive definite)
2. $\|\alpha x\| = |\alpha| \, \|x\|$. (homogeneous)
3. $\|x + y\| \leq \|x\| + \|y\|$. ($\triangle$ ineq)

Proof:
1,2 $\rightarrow$ Exercise.

3 $\rightarrow$ See Theorem 10.4. $\square$

10.3 Inequalities

Theorem 10.3: Cauchy-Shwarz Inequality

For any $x, y \in \mathbb{R}^n$,

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

Equality holds iff x and y are linearly dependent (i.e., there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 x + \alpha_2 y = 0$ and it is not true that $\alpha_1 = \alpha_2 = 0$).

Proof:

First, note that the result is trivial if $x = 0$ or $y = 0$.

Suppose $x \neq 0$ and $y \neq 0$. We define the unit vectors

$$\mathbf{u} = (u_1, \ldots, u_n) = \frac{x}{||x||}$$

and

$$\mathbf{v} = (v_1, \ldots, v_n) = \frac{y}{||y||}$$

For each $i = 1, 2, \ldots, n$,

$$0 \leq (u_i - v_i)^2 = u_i^2 - 2u_i v_i + v_i^2$$

$$u_i v_i \leq \frac{1}{2}(u_i^2 + v_i^2)$$

Adding together inequalities for all i:

$$\sum_{i=1}^{n} u_i v_i \leq \frac{1}{2} \sum_{i=1}^{n} (u_i^2 + v_i^2) \implies \langle u, v\rangle \leq \frac{1}{2}(||u||^2 + ||v||^2) = 1$$

We can do the same manipulation as above starting form

$$0 \leq (u_i + v_i)^2 = u_i + 2u_i v_i + v_i^2 \implies \langle u, v\rangle \geq -1$$

Hence $|\langle u, v\rangle| \leq 1$.

Exercise: Complete this proof. □

Theorem 10.4: Triangle Inequality

For any two vectors, $x, y \in \mathbb{R}^n$

$$\|x + y\| \leq \|x\| + \|y\|$$

Equality holds iff $x = 0$ or $y = \alpha x$ for some $\alpha \geq 0$.

Proof:

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle && \text{by bilinearity} \\
&\le \langle x,x\rangle + |\langle x,y\rangle| + |\langle x,y\rangle| + \langle y,y\rangle && \text{by properties of abs values} \\
&\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

Take square roots:

$$\|x+y\| \le \|x\| + \|y\|$$

Now prove "equality" statement.

$\Longrightarrow$) If equality holds, then

$$\langle x,y\rangle = |\langle x,y\rangle| = \|x\| \cdot \|y\|$$

The first "=": compare first introduction of inequality in proof.

The second "=": second inequality.

So we need C.S equality condition and we need $\langle x,y\rangle \ge 0$.

Case 1 $\alpha_2 \ne 0$, then $y = \alpha x$ where $\alpha = -\frac{\alpha_1}{\alpha_2}$.

Case 2 $\alpha_2 = 0$. Exercise.

$\square$

11

Convergence and Completeness in $\mathbb{R}^n$

11.1 Definitions of sequences and convergence in $\mathbb{R}^n$

An (infinite) sequence of vectors or points in $\mathbb{R}^n$ is an infinite enumerated list $(x_k)_{k=1}^{\infty} = (x_1, x_2, \ldots)$ where each $x_k \in \mathbb{R}^n$ for $k \geq 1$.

A sequence $(x_k)_{k=1}^{\infty}$ converges to a point $a \in \mathbb{R}^n$ if

Given $\epsilon > 0$, there exists N such that $\|x_k - a\| < \epsilon$ for all $k \geq N$.

If this holds, then a is called the limit of the sequence, and we write

$$\lim_{k \to \infty} x_k = a$$

Lemma 11.1

Let $(x_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. Then,

$$\lim_{k \to \infty} x_k = a \iff \lim_{k \to \infty} \|x_k - a\| = 0$$

Each $x_k = (\underbrace{x_{k,1}}_{\in \mathbb{R}}, \ldots, x_{k,n}) \in \mathbb{R}^n$.

Lemma 11.2

Let x_k be a sequence in $\mathbb{R}^n$. Then $\lim\limits_{k \to \infty} x_k = a = (a_1, a_2, \ldots, a_n)$ if and only if

$$\lim_{k \to \infty} x_{k,j} = a_j \quad \text{for } 1 \leq j \leq n$$

Proof:

$\Longrightarrow$) Suppose $\lim_{k\to\infty} x_k = a$. We must show that for each $j \in \{1, 2, \dots, n\}$ and for all $\epsilon > 0$, we can find N_j such that

$$|x_{k,j} - a_j| < \epsilon \quad \text{for all } k \geq N_j$$

Fix (arbitrary) $j \in \{1, \dots, n\}$ and let $\epsilon > 0$. By definition of $\lim_{k\to\infty} x = a$, there exists N such that $\|x_k - a\| < \epsilon$ for all $k \geq N$. By definition or norm,

$$\|x_k - a\|^2 = \sum_{i=1}^{n} |x_{k,i} - a_i|^2 \geq |x_{k,j} - a_j|^2$$

Hence, for $N_j := N$, we have

$$|x_{k,j} - a_j| < \epsilon \quad \text{for all } k \geq N$$

as required.

$\Longleftarrow$) Let $\epsilon > 0$. By convergence of components for each $j \in \{1, \dots, n\}$, there exists N_j such that

$$|x_{k,j} - a_j| < \bar{\epsilon} := \frac{\epsilon}{\sqrt{n}} \quad \text{for all } k \geq N_j$$

Define $N = \max\{N_j\}$. Then for all $k \geq N$

$$|x_{k,j} - a_j| < \bar{\epsilon} \quad \text{for all } j \in \{1, \dots, n\}$$

Then

$$\|x_k - a\|^2 = \sum_{j=1}^{n} |x_{k,j} - a_j|^2 < n\bar{\epsilon}^2 = \epsilon^2$$

So $\|x_k - a\| < \epsilon$ for all $k \geq N$, as required.

□

11.2 Cauchy sequences

A sequence $(x_k)_{k=1}^{\infty}$ is Cauchy if ...

Lemma 11.3

Let (x_k) be a sequence of points in $\mathbb{R}^n$. Then (x_k) is Cauchy if and only if $(x_{k,j})_{k=1}^{\infty}$ is Cauchy for all $j \in \{1, 2, \dots, n\}$.

11.3 Completeness

A subset S of $\mathbb{R}^n$ is complete if every Cauchy sequence in S converges to a limit in S.

Proposition 11.4

Every convergent sequence in $\mathbb{R}^n$ is Cauchy.

Theorem 11.5: Completeness Theorem for $\mathbb{R}^n$

$\mathbb{R}^n$ is complete.

Proof:

$$\begin{gathered}
(x_k)_{k=1}^{\infty} \text{ is Cauchy} \\
\Updownarrow \text{Lemma 11.3} \\
(x_{k,j})_{k=1}^{\infty} \text{ is Cauchy for all } j \in \{1, \ldots, n\} \\
\Updownarrow \text{Theorem 11.5} \\
(x_{k,j})_{k=1}^{\infty} \text{ is convergent for each } j \\
\Updownarrow \text{Lemma 11.2} \\
(x_k)_{k=1}^{\infty} \text{ is convergent}
\end{gathered}$$

Actually, we have shown iff. □

11.4 Closed subsets of $\mathbb{R}^n$

Let $X \subset \mathbb{R}^n$. We define a *limit point* of X as a point $a \in \mathbb{R}^n$ for which there exists a sequence in X converging to a.

Example:

$(0, 2] = X$

1, 2 is a limit point of X. 3 is not a limit point. 0 is a limit point, but not in X.

If X contains all of its limit points, we say it is *closed*.

So $(0, 2]$ is not closed.

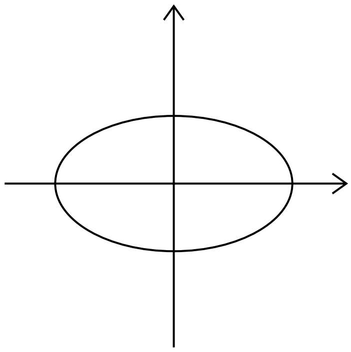
Given any subset $X \subseteq \mathbb{R}^n$, we define the closure of X, denoted $\overline{X}$, as the set of all limit points of X.

Example:

1. $\varnothing$ is closed since it contains all limit points.

2. $\mathbb{R}^n$ is closed.

3. The open interval $(0,1) \subseteq \mathbb{R}$. It is not closed.
4. The closure of $(0,1)$ is $[0,1]$ which is closed.
5.



$$X = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 < 1\right\}$$

X is not closed. Consider

$$x_k = \left(\left(0, 1 - \frac{1}{k}\right)\right)_{k=1}^{\infty} \to (0,1) \notin X$$

Proposition 11.6

A subset $S \in \mathbb{R}^n$ is closed if and only if it is complete.

Remark:
Not always equivalent - relies on completeness of $\mathbb{R}^n$.

12

Closed Subsets in $\mathbb{R}^n$

12.1 Properties of closed sets

Proposition 12.1

For any subset $X \subseteq \mathbb{R}^n$, the closure of X is closed, and is in fact, the smallest closed set that contains X.

Proof:
Exercise. $\square$

Proposition 12.2

If A and B are closed subsets of $\mathbb{R}^n$, then $A \cup B$ is closed.

Proof:
First note that $A \cup B$ is trivially closed if $A = B = \varnothing$. Now, suppose A and B are closed, and not both empty.

Let x be a limit point of $A \cup B$. We must show that $x \in A \cup B$.

By definition, there must be a sequence $(x_k)_{k=1}^{\infty}$ in $A \cup B$ converging to x. Either A or B (or both) must contain infinitely many terms of the sequence.

Suppose WLOG that A contains infinitely many terms in the sequence. Then, we can make a subsequence $(x_{k_j})_{j=1}^{\infty}$ in A.

Since this is a subsequence of a convergent sequence, it must converge, and its limit is x.

By closeness of A, $x \in A \implies x \in A \cup B \implies A \cup B$ is closed. $\square$

Remark:

By induction, $\bigcup_{i=1}^{N} X_i$ is closed for any integer N if X_i is closed for $1 \le i \le N$. But this does not extent to infinite unions.

Suppose $X_i = \left[0, 1 - \frac{1}{i+1}\right], \quad i = 1, 2, \ldots$

Proposition 12.3

If $A_i \subseteq \mathbb{R}^n$ is closed for each i in an arbitrary (possibly infinite) indexing set I, then $\bigcap_{i \in I} A_i$ is closed.

Proof:

Let $X = \bigcap_{i \in I} A_i$. If $X = \varnothing$, then it is closed. Now, suppose $X \neq \varnothing$. Let x be a limit point of X. Then, there is a sequence $(x_k)_{k=1}^{\infty}$ in X converges to x.

By definition of X, $x_k \in A_j$ for all $k \geq 1$, and for all $j \in I$. Hence $x \in A_j$ for all $j \in I$ (by closed property of A_j) $\implies x \in X = \bigcap_{j \in I} A_j$. $\square$

12.2 Closedness and Boundaries

We define the open ball of radius $r > 0$ about a point $a \in \mathbb{R}^n$ as the set

$$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

The complement of a set $X \subseteq \mathbb{R}^n$ is

$$X' = \mathbb{R}^n \setminus X = \{x \in \mathbb{R}^n : x \notin X\}$$

This can also be denoted by X^C.

A point $a \in \mathbb{R}^n$ is a boundary point of $S \subseteq \mathbb{R}^n$ if for every $r > 0$, the open ball $B_r(a)$ contains a point in S and a point in S'.

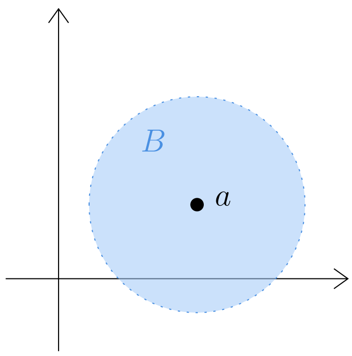


Figure 12.1: An example of open ball

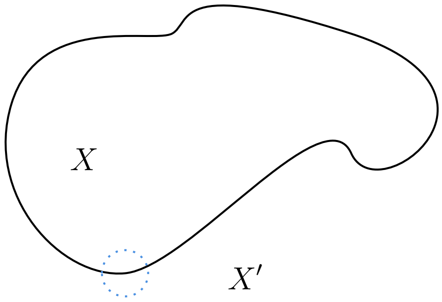


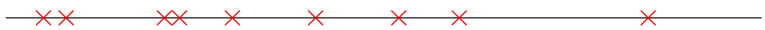
Figure 12.2: An example of boundary point

The boundary of a set $S \subseteq \mathbb{R}^n$ is the set of all boundary points of S, where we denote it by ∂S.

Example:

1. Let's take a look at $(0,1)$ and $[0,1)$. 0 and 1 are the (only) BPs for both cases. BPs may or may not in S.
2. $[0, 0.5) \cap (0.5, 1]$. 0.5 is also a boundary point.

   Note
   boundary point can be "in the middle" of a set.
3. X is a finite set.

   X is the boundary of itself, i.e., $\partial X = X$
4. $X = \{s \in \mathbb{Q} : |s| < 1\}$. 0,1, still BPs. Every number in $[0,1]$ is BP.

Proposition 12.4

A set $S \subseteq \mathbb{R}^n$ is closed if and only if it contains all of its boundary points.

Proof:
Exercise. $\square$

13

Open and Compact Subsets of $\mathbb{R}^n$

13.1 Open subsets in $\mathbb{R}^n$

As defined in the last lecture, the open ball of readius r about a point a in $\mathbb{R}^n$ is the set

$$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

A subset $U \subseteq \mathbb{R}^n$ is open if for all $a \in U$, there exists some $r > 0$ such that $B_r(a) \subseteq U$.

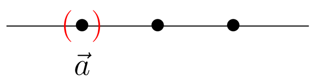
If U is an open set containing a point a, then we say that U is an open neighbourhood of a.

An interior point of a set $X \subseteq \mathbb{R}^n$ is a point $x \in X$ such that $B_r(x) \subseteq X$ for some $r > 0$.

The interior of a set $X \subseteq \mathbb{R}^n$ is the set of all interior points of X. It is denoted by $\text{int}(X)$. If $\text{int}(X)$ is empty, then we say X has empty interior. Otherwise, it has nonempty interior.

Example:

1. $X = \{1, 2, 3\}$



$B_r(a) = (a - r, a + r) \subseteq X$? No $\implies$ X not open.

2. $\varnothing$ is open

3. $\mathbb{R}^n$ is open

Note
The only subsets in $\mathbb{R}^n$ that are closed an open is $\mathbb{R}^n$ and $\varnothing$.

4. The open interval (a, b) is open
5. The close interval $[a, b]$ is not open.

   $\text{int}([a, b]) = (a, b)$
6. $(a, b]$, $[a, b)$ is not open and not closed.
7. $Br(a)$ is open for any $r > 0$ $a \in \mathbb{R}^n$
8. $X = \{s \in \mathbb{Q}, |s| < 1\}$. For any $r > 0$, $B_r(0) = (-r, r)$ contains irrational points. So it is not contained in X. In fact, it has empty interior.

   In fact, $\text{int}(X) = \varnothing$.

   Note
   "interior" doesn't always coincide with what you think of as the "inside" of a set.
9. $X = (-1, 1) \setminus \{0\}$. 0 is not an interior point.

13.2 Properties of open subsets

Proposition 13.1

If U and V are open subsets of $\mathbb{R}^n$, then $U \cap V$ is open.

Proof:
Exercise. □

Proposition 13.2

If U_i is an open subset of $\mathbb{R}^n$ for each i in an arbitrary (possible infinite) indexing set I, then $X = \bigcap_{i \in I}$ is open.

Proof:
Exercise. □

Note
Cannot take intersection of infinitely many open set and expect it to be open.

Example:

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

Theorem 13.3

A set $X \subseteq \mathbb{R}^n$ is open if and only if its complement is closed.

Proof:

$\Longrightarrow$) Let X be an open subset of $\mathbb{R}^n$ and suppose that a is a limit point of X'. Suppose for contradiction that $a \in X$. Since X is open, there exists an open ball $B_r(a) \subseteq X$. Then there is no point y in X' with $\|y - a\| < r$. No sequence in X' can converge to a, contradicting the assumption that a is a limit point of X'. Therefore, all limit points of X' must be in X', i.e., X' is closed.

$\Longleftarrow$) Suppose that X is not open. Then there must be a point $x \in X$ such that for every $r > 0$, the open ball $B_r(x)$ contains a point in X'. Construct a sequence $(a_k)_{k=1}^{\infty}$ in X' such that $a_k \in B_{r=1/k}(x)$ for each $k \geq 1$. Then, $\lim\limits_{k\to\infty} a_k = x \in X$, which means that there is a limit point of X' that is not in X'. This proves that X' is not closed.

□

Proposition 13.4

A set $X \subseteq \mathbb{R}^n$ is open if and only if it contains none of its boundary points.

Proof:
Exercise. □

13.3 Bounded sequences and subsets in $\mathbb{R}^n$

We say that a sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ is bounded if there exists a real number R such that $\|x_k\| < R$ for all k.

We say a set $X \subseteq \mathbb{R}^n$ is bounded if there exists a real number R such that $\|x\| < R$ for all $x \in X$.

Theorem 13.5: Bolzano-Weierstrass in $\mathbb{R}^n$

Every bounded sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ has a convergent subsequence $(x_{k_l})_{l=1}^{\infty}$.

Proof:
Immediate. □

Corrollary 13.6

If S is a closed and bounded subset of $\mathbb{R}^n$, then every sequence of points in S has a subsequence that converges in S.

Proof:
Trivial. □

13.4 Compact sets

Let K be a subset of $\mathbb{R}^n$. We say that K is compact if every sequence of points in K has a convergent subsequence with a limit in K.

Theorem 13.7: The Heine–Borel Theorem

A subset of $\mathbb{R}^n$ is compact if and only if it closed and bounded.

Proof:
The "if" part is Corollary 13.6. We now show that K is compact only if it is closed and bounded. Let K be a compact subset of $\mathbb{R}^n$.

To show that K is closed, suppose x is a limit point of K. Then there is a sequence of points in K such that $\lim\limits_{k\to\infty} x_k = x$. Since K is compact, there is a subsequence of this sequence that converges to a point in K. But every subsequence of a convergent sequence must converge to limit of the sequence. Hence $x \in K$.

We show that K is bounded by contradiction. Suppose that K is not bounded. Then, we can construct a sequence $(x_k)_{k=1}^{\infty}$ such that $\|x_k\| > k$ for each $k \geq 1$. If K is compact, then there must be a subsequence $(x_{k_j})_{j=1}^{\infty}$ that converges; denote the limit $x = \lim\limits_{j\to\infty} x_{k_j}$.

Choose $\epsilon = 1$ in the definition of convergence. There exists an integer N such that $\left\|x_{k_j} - x\right\| < \epsilon = 1$ for all $j \geq N$. By the Reverse $\triangle$ Ineq,

$$\left|\left\|x_{k_j}\right\| - \|x\|\right| \leq \left\|x_{k_j} - x\right\| < 1 \qquad \forall j \geq N$$

$$\implies \left\|x_{k_j}\right\| < \|x\| + 1 \qquad \forall j \geq N$$

But for any point $x \in \mathbb{R}^n$, there exists an integer $M > \|x\| + 1$. For sufficiently large j, we will have $k_j > M$ and by construction, $\left\|x_{k_j}\right\| > k + j > M > \|x\| + 1$. This contradicts the statement above, proving that K must be bounded. □

Proposition 13.8

If K is a compact subset of $\mathbb{R}^n$ and C is a closed subset of K, then C is compact.

Proof:
If $C \subseteq K$ and K is bounded, then C must be bounded. By assumption, C is closed. Hence, by Theorem 13.5, C is compact. □

13.4.1 Examples and nonexamples

1. $\varnothing$ is compact

2. $\mathbb{R}^n$ is not compact (not bounded)

3. $(0, 1]$ not compact

4. Intervals $[a, b]$ are compact for $a, b \in \mathbb{R}$

5. $[a, b]^n$ is compact in $\mathbb{R}^n$

   Proof:
   Sketch in $\mathbb{R}^2$.

   Suppose $X = [a, b] \times [a, b]$.

   Let $\boldsymbol{x}_k = (x_k, y_k)$ for $k \geq 1$ such that $(\boldsymbol{x}_k)_{k=1}^{\infty}$ is in X. Need to find subsequence that converges to $\boldsymbol{x} = (x, y) \in X$.

   Consider real sequence $(x_k)_{k=1}^{\infty}$ in $[a, b]$. Bounded, so we can apply Theorem 13.5. $\rightarrow (x_{k_j})_{j=1}^{\infty}$ that converges to x. Since $[a, b]$ is closed, $x \in [a, b]$.

   Now $(y_{k_j})_{j=1}^{\infty}$ is a sequence in $[a, b]$.

   By Theorem 13.5 and closed property of $[a, b]$, there is a subsequence $\left(y_{k_{j_l}}\right)_{l=1}^{\infty}$ that converges to $y \in [a, b]$.

   $\left(\boldsymbol{x}_{k_{j_l}}\right)_{l=1}^{\infty}$ must converge to (x, y)

   [convergence of each component to component of limit, + subsequence of (x_{k_j}) must converge to limit of (x_{k_j})]

   Since $x \in [a, b]$ and $y \in [a, b]$, $\boldsymbol{x} \in X$. Hence X is compact. $\square$

14

Limits and Continuity of Functions

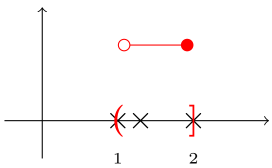
14.0 Preliminaries

Suppose $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$. $\lim_{x \to a} f(x) = L$?

For all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x such that $0 < |x - a| < \delta$.

Where does the notion of limit make sense?

Consider $f(x) = 1$ for all $x \in S$.



We want to be able to talk about limit at $x = 0$.

14.1 The limit of a function

Let $S \subseteq \mathbb{R}^n$. We say that $\boldsymbol{a} \in \mathbb{R}^n$ is an *accumulation point* of S if it is a limit point of $S \setminus \{\boldsymbol{a}\}$.

The set of all accumulation points of S is denoted by S^a.

A point $\boldsymbol{a} \in S \setminus S^a$ is called an *isolated point* of S.

Let $f : S \to \mathbb{R}^m$ be a function. Let $\boldsymbol{a} \in S^a$. The vector $\boldsymbol{v} \in \mathbb{R}^m$ is the limit of f at $\boldsymbol{a}$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\boldsymbol{x}) - \boldsymbol{v}\| < \epsilon$ for all $\boldsymbol{x} \in S$ satisfying $0 < \|\boldsymbol{x} - \boldsymbol{a}\| < \delta$.

If this holds, we write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = \boldsymbol{v}$.

Example:

1.
$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ |x-1|, & \text{otherwise} \end{cases}$$

   Then $\lim_{x\to 1} f(x) = 0$

2. $f(x, y) = \frac{x^3}{x^2+y^4}$ on $\mathbb{R}^2 \setminus \{0\}$

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} f(\boldsymbol{x}) = 0$$

   Note that this is equivalent to

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} |f(\boldsymbol{x})| = 0$$

   We can do as follows

$$|f(\boldsymbol{x})| = \frac{x^2}{x^2+y^4}|x| \leq |x|$$

   Show definition of limit holds (use δ, ϵ).

14.2 Infinite limits

Suppose $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$, $\boldsymbol{a} \in S^a$. The limit of f at $\boldsymbol{a}$ is $+\infty$ if for all $N \geq 1$, there exists $\delta > 0$ such that $f(\boldsymbol{x}) > N$ for all $\boldsymbol{x} \in S$ satisfying $0 < \|\boldsymbol{x} - \boldsymbol{a}\| < \delta$.

14.3 Continuity

Let $S \subseteq \mathbb{R}^n$. We say that a function $f : S \to \mathbb{R}^m$ is continuous at the point $\boldsymbol{a} \in S$ if:

> For all $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\boldsymbol{x}) - f(\boldsymbol{a})\| < \epsilon$ for all $\boldsymbol{x} \in S$ satisfying $\|\boldsymbol{x} - \boldsymbol{a}\| < \delta$.

If f is continuous at every $\boldsymbol{a} \in S$ then we say that it is continuous (on S). If f is not continuous at $\boldsymbol{a} \in S$, then we say it is discontinuous at $\boldsymbol{a}$.

Proposition 14.1

Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$. For every $\boldsymbol{a} \in S \cap S^a$, the function is continuous at $\boldsymbol{a}$ iff

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$$

Proof:
Exercise. $\square$

Example:
Show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous on its domain.

Proof:
First, choose arbitrary $a > 0$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all $x \in (a - \delta, a + \delta)$.

For any $x > 0$,

$$\begin{aligned} |f(x) - f(a)| &= \left|\frac{1}{x} - \frac{1}{a}\right| \\ &= \left|\frac{a - x}{ax}\right| = \frac{1}{ax}|x - a| \end{aligned}$$

Note
We *cannot* choose $\delta = \epsilon ax$ so that

$$|x - a| < \delta \implies |f(x) - f(a)| < \frac{1}{ax} \cdot \epsilon ax = \epsilon$$

δ is not allowed to depend on x since it must work for all x.

First, suppose $|x - a| < \frac{a}{2}$.

Then $x > \dfrac{a}{2} \implies ax > \dfrac{a^2}{2} \iff \dfrac{1}{ax} < \dfrac{2}{a^2}$.

So we have $|f(x) - f(a)| < \dfrac{2}{a^2}|x - a|$.

Pick $\delta = \min\{\dfrac{\epsilon a^2}{2}, \dfrac{a}{2}\}$ so that

$$|x - a| < \delta \implies |f(x) - f(a)| < \frac{2}{a^2}|x - a| < \frac{2}{a^2}\delta < \epsilon$$

$\square$

15 Discontinuous Functions

15.1 Examples

1. $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$

   This is discontinuous at $x = 0$, continuous everywhere else.

   This kind of discontinuity is called a *removable* discontinuity because you can remove it by changing the value of the function at $x = 0$ to $f(0) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Discontinuous at $x = 0$, not removable.

3. The Heaviside function

   $$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   Discontinuity at $x = 0$, not removable.

15.2 One-sided limits

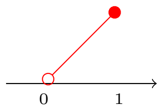
Let $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$. Consider a point ~~$a \in S$~~[a] a that is a limit point of $S \cap (a, \infty)$. We say that the *limit of f as x approaches a from the right* exists and is equal to L if given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in S$ satisfying $a < x < a + \delta$.

If this holds, then we write $\lim\limits_{x \to a^+} f(x) = L$.

The limit from the left is analogous: $\lim\limits_{x \to a^-} f(x) = L$.

Example:

1. Heaviside function: $\lim\limits_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^-} H(x) = 1$.
2. $f(x) = x$ on $(0, 1]$



15.3 Jump discontinuities and piecewise continuity

We say that a function f has a jump discontinuity at a point $a \in \mathbb{R}$ if the limits of f as x approaches a from the left and from the right both exist but are not equal. E.g. Heaviside function at $x = 0$.

A function on an interval is piecewise continuous if every finite subinterval contains a finite number of jump discontinuity and no other types of discontinuities.

Example: Thomae's function

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \notin Q \\ 1/q & \text{if } x = p/q \text{ in lowest terms, with } q > 0 \end{cases}$$

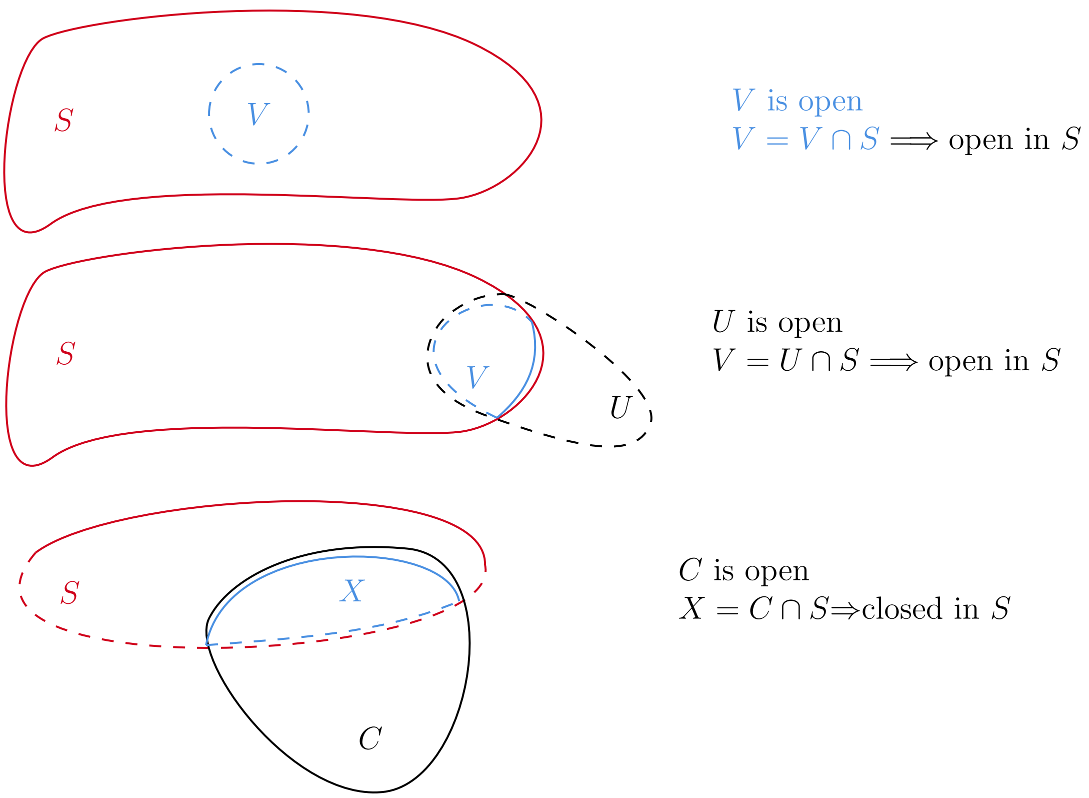
It can be shown that $\lim\limits_{x \to a} f(x) = 0$ at any point $a \in \mathbb{R}$. f is continuous at every irrational point and has a removable discontinuity at every rational point.

16

Properties of Continuous functions

16.1 Equivalent statements of continuity

We say that a subset V of a subset $S \subseteq \mathbb{R}^n$ is open in S, or relatively open with respect to S, if there exists an open set U such that $V = U \cap S$. Also, $X \subseteq S$ is closed in S if there exists a closed set C such that $X = C \cap S$.



Proposition 16.1

A subset $V \subseteq S$ is open in $S \subseteq \mathbb{R}^n$ if and only if the following holds:

For every $x \in V$, there exists $\delta > 0$ such that $B_\delta(x) \cap S \subseteq V$.

Theorem 16.2

For a function $f : S \to \mathbb{R}^m$ where $S \subseteq \mathbb{R}^n$, the following are equivalent:

1. f is continuous on S.
2. For any sequence $(x_k)_{k=1}^{\infty}$ in S that converges to a limit $a \in S$. then $\lim_{k\to\infty} f(x_k) = f(a)$
3. If U is an open subset of $\mathbb{R}^m$, then the preimage set $f^{-1}(U) = \{x \in S : f(x) \in U\}$ is open in S.

Remark:

- Statement 2 is called "sequential continuity".
- Statement 3 is "topological continuity".
- From the proof (below). Statements 1 and 2 can also be applied at each point a (pointwise) i.e., continuity at a is equivalent to sequential continuity at a.

Proof:

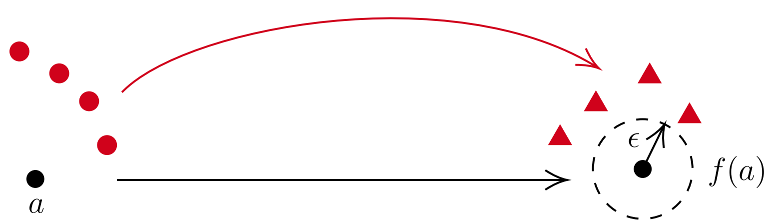
$1 \implies 2$ Continuity of f at every $a \in S$ means: fix $a \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$ such that $\|f(x) - f(a)\| < \epsilon$ for all $x \in S$ satisfying $\|x - a\| < \delta$.

Suppose $\lim_{k\to\infty} x_k = a$ for (x_k) is S.

There exists N such that $\|x_k - a\| < \delta$ for all $k \geq N$.

$$\implies \|f(x_k) - f(a)\| < \epsilon \text{ for all } k \geq N$$
$$\implies \lim_{k\to\infty} f(x_k) = f(a)$$

$\neg 1 \implies \neg 2$ If f is not continuous at $a \in S$, then there exists $\epsilon > 0$ such that for all $\delta > 0$, $\|x - a\| < \delta$ and $\|f(x) - f(a)\| \geq \epsilon$ for some $x \in S$.



For each integer $k \geq 1$, choose $\delta_k = \frac{1}{k}$ and choose $x_k \in S$ such that

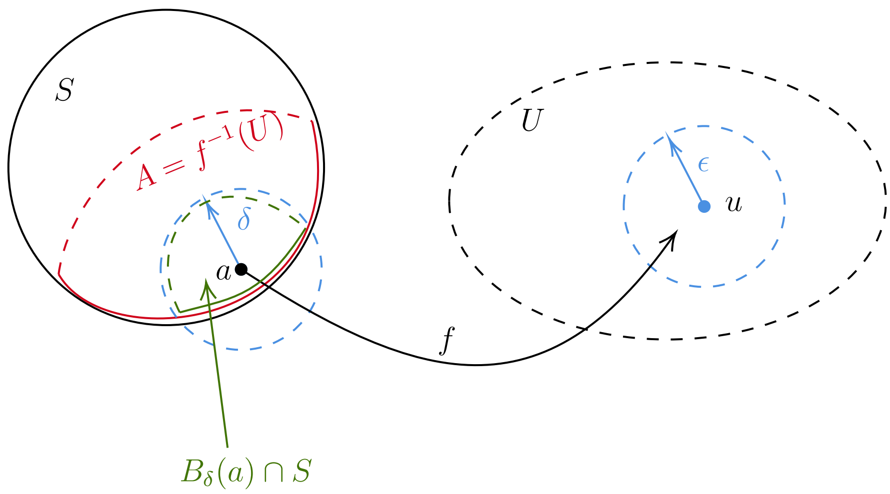
$\|x_k - a\| < \delta_k$ and $\|f(x_k) - f(a)\| \geq \epsilon$.

So, $\lim_{k\to\infty} f(x_k) \neq f(a)$ [It may not exist]

$1 \implies 3$ Suppose f is continuous on S. Let $U \subseteq \mathbb{R}^m$ be an arbitrary open set.

The pre-image $f^{-1}(U)$ is either empty or nonempty. If empty, then open, therefore open in S.

Otherwise, if $f^{-1}(U)$ is non-empty, then there exists a point $a \in A := f^{-1}(U)$.



Define $u = f(a) \in U$.

Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(u) \subseteq U$.

By continuity of f, there exists $\delta > 0$ such that $\|f(x) - f(a)\| < \epsilon$ for all $x \in S$ satisfying $\|x - a\| < \delta$.

Hence, $f(B_\delta(a) \cap S) \subseteq B_\epsilon(u) \subseteq U$. That is, $B_\delta(a) \cap S \subseteq f^{-1}(U) = A$.

From Proposition 16.1, A is open in S.

$3 \implies 1$ Let a be an arbitrary point in S and define $u = f(a)$. For arbitrary $\epsilon > 0$, the open ball $B_\epsilon(u)$ is open in $\mathbb{R}^m$ so Statement 3 $\implies$ $f^{-1}(B_\epsilon(u))$ is open in S.

Note that $a \in f^{-1}(B_\epsilon(u))$. By Proposition 16.1, there exists $\delta > 0$ such that $B_\delta(a) \cap S = \{x \in S : \|x - a\| < \delta\}$ is a subset of A.

Equivalently, $\|f(x) - f(a)\| < \epsilon$ for all $x \in S$ satisfying $\|x - a\| < \delta$.

$\square$

Example:

Find the limit of the sequence $a_n = \cos\left(\frac{1}{n}\right)$.

The function $f(x) = \cos(x)$ is continuous on $\mathbb{R}$. Since $\left(x_n = \frac{1}{n}\right)_{n=1}^{\infty}$ converges to

0, sequential continuity $\implies$

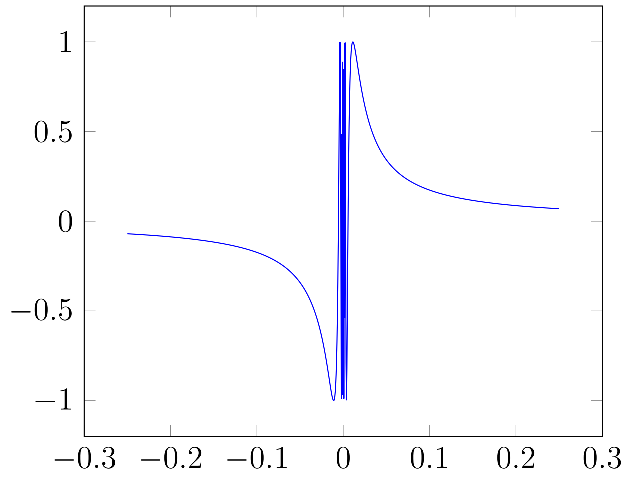
$$\lim_{n\to\infty} a_n = \cos\left(\lim_{n\to\infty} \frac{1}{n}\right) = \cos 0 = 1$$

Note
We only need continuity of f at $x = \lim_{n\to\infty} x_n$.

We can also use this theorem to disprove continuity.

Example:

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



16.2 Combining Limits

Theorem 16.3

Let f and g be two functions from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Let $a \in S$ and $u, v \in \mathbb{R}^m$ such that

$$\lim_{x\to a} f(x) = u \text{ and } \lim_{x\to a} g(x) = v$$

Then,

1. $\lim_{x\to a}(f+g)(x) = u + v$
2. $\lim_{x\to a} \alpha f(x) = \alpha u$ for any $\alpha \in \mathbb{R}$.

   In addition, if $m = 1$, we have
3. $\lim_{x\to a} f(x)g(x) = uv$, and
4. $\lim_{x\to a} f(x)/g(x) = u/v$, provided $v \neq 0$.

16.3 Combining continuous functions

Theorem 16.4

Let f and g be two functions from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. If there is a point $a \in S$ such that f and g continuous at a, then

1. $f + g$ is continuous at a,
2. αf is continuous at a,

   In addition, for $m = 1$

3. fg is continuous at a, and
4. f/g is continuous at a, provided $g(a) \neq 0$.

Theorem 16.5

Let $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$. Suppose we have functions $f : S \to T$ and $g : T \to \mathbb{R}^\ell$. If f is continuous at $a \in S$ and g is continuous at $f(a) \in T$, then the composition $g \circ f$ is continuous at a.

16.4 Examples

Every polynomial is continuous on $\mathbb{R}$.

17

Extreme and Intermediate Value Theorem

17.1 Extreme Values

Example:

1. Find the point at which the maximum is attained for the function $f : (0,2) \to \mathbb{R}$ defined by

$$f(x) = 2x - 3x^2 + x^3 = x(x-1)(x-2)$$

   We can find the value of $x_{\text{max}} = 1 - 1/\sqrt{3} \approx 0.42$.

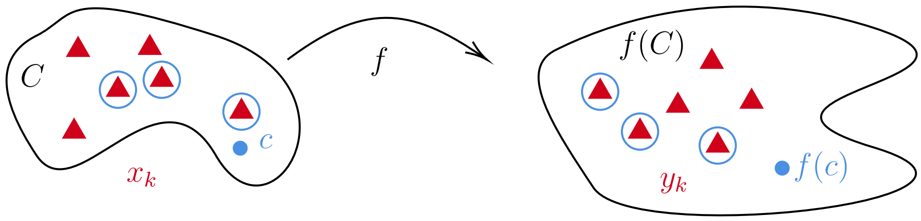
2. Consider the same function on domain $[1/2, 1]$. Now max is at $x = \frac{1}{2}$.
3. What if domain is $(1/2, 1]$? Now, there is no max.
4. What if domain is $(1/2, \infty)$? Unbounded so no max.

Note
If domain is closed an bounded, we can rule out behaviour like 3 and 4.

Theorem 17.1

Suppose C is a compact subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^m$ is continuous on C. Then, the image $f(C)$ is compact.

Proof:
We must show that any arbitrary sequence $(y_k)_{k=1}^{\infty}$ in $f(C)$ has a subsequence that converges to a point in $f(C)$.



For each $k \geq 1$, there exists x_k such that $f(x_k) = y_k$. So we have a sequence (x_k) in C. By compactness of C, there is a subsequence $(x_{k_j})_{j=1}^{\infty}$ that converges to $c \in C$.

By (sequential) continuity of f, $\left(f\left(x_{k_j}\right)\right)_{j=1}^{\infty}$ must converge to $f\left(\lim\limits_{j\to\infty} x_{k_j}\right) = f(c)$.

Since $c \in C$, $f(c) \in f(C)$. □

Theorem 17.2: Extreme Value Theorem

Let C be a nonempty compact subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ be a continuous function on C. Then, f attains its minimum and maximum values on C. That is there exists point $a, b \in C$ such that

$$f(a) \leq f(x) \leq f(b), \qquad \forall x \in C.$$

Proof:
If C is compact and f is continuous then $f(C)$ is compact (Theorem 17.1) and hence closed and bounded.

Bounded and nonempty (since C is nonempty) so there must be a supremum $M = \sup f(C)$.

By definition of supremum, we can define a sequence (y_k) in $f(C)$ that satisfies

$$M - \frac{1}{k} < y_k \leq M$$

for each k. Note $\lim\limits_{k\to\infty} y_k = M$.

But $f(C)$ is closed, so $M \in f(C)$. Hence, there exists $b \in C$ such that $f(b) = M$.

Similarly, we can show that infimum is attained at some $a \in C$. □

Remark:
You should be familiar with EVT expressed for continuous real-valued functions on the closed, bounded interval $[x_1, x_2]$.

We now generalize to functions of multiple variables, but still need function to be scalar-valued.

17.2 Intermediate Values

Theorem 17.3: Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $y \in \mathbb{R}$ satisfies $f(a) < y < f(b)$ or $f(b) < y < f(a)$, then there exists $c \in (a, b)$ such that $f(c) < y$.

Proof:
(Sketch)

Choose y satisfying hypothesis. Define $A = \{x \in [a, b] : f(x) < y\}$. $a \in A$ so A is non-empty. b is an upper bound of A.

So there is a supremum $c = \sup A$.

Strategy

1. Show that $c < b$
2. Show that $f(c) = y$ by contradiction.

   Suppose $f(c) > y$. Use continuity of f.

   Suppose $f(c) < y$.

□

Corrollary 17.4

Continuous functions map closed intervals to closed intervals. That is, if $f : [a, b] \to \mathbb{R}$ is continuous, then its image $f([a, b])$ is a closed interval.

Proof:
Theorem 17.2 can be applied to show that there exists $x_{\min}$ and $x_{\max}$ in the interval $[a, b]$ for which $m = f(x_{\min}) \leq f(x) \leq f(x_{\max}) = M$ for all $x \in [a, b]$. This implies that the image set $f([a, b]) \subseteq [m, M]$.

WLOG suppose $x_{\min} < x_{\max}$. Then from Theorem 17.3, we know that any point $y \in (m, M)$ has a pre-image in $(x_{\min}, x_{\max})$. Hence, $f([a, b]) \supseteq [m, M]$. In conclusion, $f([a, b]) = [m, M]$, as required. □

Corrollary 17.5

Let $S \subseteq \mathbb{R}^n$ and let f be a continuous function from S to $\mathbb{R}$. If a and b are two points in S that a re connected by a path in S, then for any $y \in \mathbb{R}$ between $f(a)$ and $f(b)$, there exists a point c on the path satisfying $f(c) = y$.

17.3 Applications of the Intermediate Value Theorem

1. Every polynomial of odd degree has at least one real root.

2. Finding roots of an equation $f(x) = 0$ (if f is continuous). E.g., the bisection method.

3. Mashed Potato Theorem: A plate of mashed potato can be evenly divided by a single straight vertical knife cut.

   In position K_1 less than half the potato is at the left of the knife, in position K_2 more than half is at the left. Hence (by the intermediate value theorem) there is an intermediate position where exactly half is at one side.

   You may care to think of how continuity should be involved in this argument.

4. The mashed potato and beans theorem: A plate of mashed potato and baked beans can be evenly divided by a single straight vertical knife cut.

   Choose one particular angle and the last result shows that you can divide the potatoes by a cut K_α at this angle. Then (say) there will be more than half the beans on the left of the cut. Now vary the angle continuously by π until the knife is in the same position as before, but pointing the other way. At each angle, make sure you bisect the potatoes. Now less than half the beans are on the left and so you passed through an intermediate position where both beans and potatoes were divided fairly.

   This result even holds true if you pile the beans on top of the potato (or vice versa).

5. The bowl of fruit theorem:

   An apple, a pear and a banana can be equally divided by a single knife-cut.

   > Remark:
   > This last result is usually called the Ham Sandwich Theorem (two pieces of bread and the ham).
   >
   > By putting the three pieces of fruit far apart, you should see that you do not have any freedom to deal with a fourth volume. If you move from three dimensions to four, though ...

   Src: http://www-groups.mcs.st-andrews.ac.uk/~john/analysis/Lectures/L20.html

18

More than Continuous

18.1 Uniformly continuous functions

We say that a function f from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous if

> for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ for every x and y in S satisfying $\|x - y\| < \delta$.

Remark:
For a function to be continuous at point a (pointwise continuity) we choose δ after fixing the point a and δ. For uniform continuity, the same δ must work for all points y. Hence, the "uniform". In particular, note that uniform continuity implies continuity.

18.2 Examples

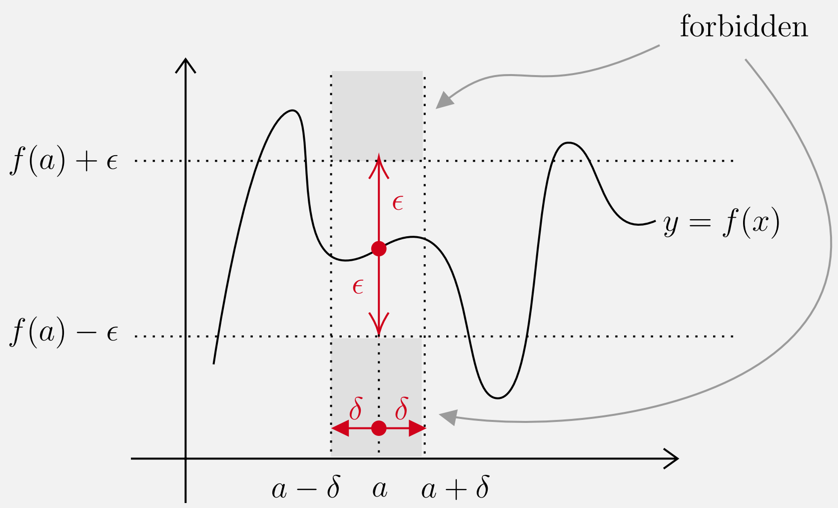
1. Prove $f(x) = x^2$ is uniformly continuous on $[a, b] \subset \mathbb{R}$ for some real numbers $a < b$.
2. $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. Check $f'(x) = x$.
3. $f(x) = 1/x$ on $(0, 1]$ is not uniformly continuous.
4. $f(x) = \sin(1/x)$ on $(0, 1]$ is not uniformly continuous.
5. $f(x) = x \sin(1/x)$ on $(0, 1]$ is uniformly continuous.

Difference between continuity and uniform continuity

For "regular" continuity, choose $y \in S$ first, then there must be $\delta > 0$.

For "uniform" continuity, there must be $\delta > 0$ that works for all $y \in S$ simultaneously/

Uniform continuity $\implies$ continuity.



For uniform continuity, for $\epsilon.\delta$. I can slide rect anywhere along graph, keeping it centered in graph. Graph cannot enter regions above or below rect.

18.3 Compactness and uniform continuity

Theorem 18.1

Let $K \subseteq \mathbb{R}^n$ be compact and $f : K \to \mathbb{R}^m$ be a continuous function. Then f is uniformly continuous on K.

Proof:
Suppose for contradiction: f not uni cont. This means that $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, there are points x and y in K satisfying $\|x - y\| < \delta$ and $\|f(x) - f(y)\| \geq \epsilon$.

Define a sequence of δ values, $\delta_k = \frac{1}{k}$ and choose points x_k and y_k satisfy the condition above. By compactness of K, there must be a subseq $(y_{k_j})_{j=1}^{\infty}$ that converges to a point $a \in K$.

The rest of the proof is left as an exercise. $\square$

18.4 Lipschitz functions

A function f from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$ is called a *Lipschitz function* if there exists a constant C such that

$$\|f(x) - f(y)\| \leq C\,\|x - y\| \text{ for all } x, y \in S.$$

Any constant C for which this condition is satisfied is called a Lipschitz constant for f. The smallest C for which this condition holds is called the (best) Lipschitz constant.

Loosely, this means that the function f cannot change too rapidly as you move a point x away from y.

If a Lipschitz function $f : [a, b] \to \mathbb{R}$ is differentiable, then its derivative must be bounded. In fact, it can also be shown that a differentiable function with bounded derivative must be Lipschitz.

Example:

The norm function $f(x) = \|x\|$ is Lipschitz on $\mathbb{R}^n$. By the Reverse triangle Ineq, $|\,\|x\| - \|y\|\,| \leq \|x - y\|$. So, 1 is a Lipschitz constant.

Equality holds when $y = \alpha x$ for some $\alpha > 0$. So the best Lipschitz constant is 1.

Proposition 18.2

Every Lipschitz function is uniformly continuous.

Proof:
Suppose f is a Lipschitz function with Lipschitz constant C. Given $\epsilon > 0$, let $\delta = \epsilon/C$. Then

$$\|\vec{x} - \vec{y}\| < \delta \Longrightarrow \|f(\vec{x}) - f(\vec{y})\| \leq C\|\vec{x} - \vec{y}\| < C\delta = \varepsilon$$

□

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear map (or linear transformation) if for any $\alpha, \beta \in \mathbb{R}$ and for any $x, y \in \mathbb{R}^n$, the function satisfies $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

Corrollary 18.3

Every linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly cont.

Proof:
Use Lipschitz. □

19

Normed Vector Space

19.1 Normed vector spaces

We now consider vectors that are not necessarily in $\mathbb{R}^n$.

We will write x, y, etc for vectors instead of $\vec{x}, \vec{y}$, etc. For the zero vector, we still write $\vec{0}$ to distinguish it from the number 0.

Let V be a vector space over $\mathbb{R}$ (i.e., a real vector space).

A *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies the following properties for all x, y and for all $\alpha \in \mathbb{R}$:

1. $\|x\| \geq 0$ with equality iff $x = \vec{0}$
2. $\|\alpha x\| = |\alpha| \, \|x\|$
3. $\|x + y\| \leq \|x\| + \|y\|$

These are the same properties we discussed for Euclidean norm (Proposition 10.2).

Norms are not unique!

19.2 Examples of norms on $\mathbb{R}^n$

- Euclidean
- p-norm
- $p = 1$, "Manhattan" or "taxicab" norm or "chess"
- infinity norm.

Exercise
Think about the shape of balls of unit radius for different norms.

19.3 Vector spaces of continuous and differentiable functions

Let $C[a,b]$ denote the vector space of continuous, real-valued functions on $[a,b]$.

Exercise
Verify that is a vector space.

Recall a function $f:[a,b]\to\mathbb{R}$ is n-times continuously differentiable if its derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a,b]$.

Let $C^n[a,b]$ denote the vector space of n-times continuously differentiable functions on $[a,b]$.

We can also generalize notation to $C(K)$, the space of continuous functions on a compact set $K\subseteq\mathbb{R}^n$.

19.4 Norms on vector spaces of functions

For a function $f\in C[a,b]$ and any real number $p\geq 1$, we define the L^p norm by $\|f\|_p = \left(\int_a^b |f(x)|^p\, dx\right)^{1/p}$

For functions $f\in C(K)$ with $K\subseteq\mathbb{R}^n$ compact, we define the uniform norm

$$\|f\|_\infty = \sup_{x\in K}|f(x)|$$

Exercise
Check that this is a norm.

For functions $f\in C^n[a,b]$, we define

$$\|f\|_{C^n} = \max_{0\leq j\leq n}\|f^{(j)}\|_\infty = \max\{\|f\|_\infty, \|f'\|_\infty, \ldots, \left\|f^{(n)}\right\|_\infty\}$$

Example:
Find the uniform norm for the function $f(x)=x(2-x)$ on $[0,2]$.

It's 1!.

$g(x)=\sin(\pi x)+\frac{1}{2}$ on $[0,2]$.

It's $3/2$.

20

Normed and Inner Product Spaces

20.1 Topology in normed vector spaces

Many defns and results are exactly as they were defined for vectors in $\mathbb{R}^n$. Let $(V, \|\cdot\|)$ be a normed vector spaces.

Remark:
We will commonly just refer to V instead of $(V, \|\cdot\|)$ for short.

Convergence, Cauchy seq, open balls, open/closed subsets, bounded subsets and compact subsets are as before.

Every convergent sequence is a Cauchy sequence.

As in $\mathbb{R}^n$, a subset of V is open iff its complement is closed.

The Heine-Borel Theorem (a subset is compact if and only if closed and bounded) is not valid in infinite-dimensional vector spaces.

20.2 Counterexample to Heine-Borel Theorem

Consider the normed space $(C[-1,1], \|\cdot\|_\infty)$

We'll find a closed and bounded subset of $C[-1,1]$ is not compact (w.r.t. uniform norm).

Let $(f_n)_{n=1}^\infty$ be the seq of funcs in $C[-1,1]$ defined by

$$f_n(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq 0 \\ nx, & \text{if } 0 < x < 1/n \\ 1, & \text{if } 1/n \leq x \leq 1 \end{cases}$$

Note that $\|f_n\|_\infty = 1$ for all $n \geq 1$ so $(f_n)_{n=1}^\infty$ is a seq in X. If we can show that

(f_n) has no convergent subseq, then this will imply that it is not compact. We will prove this by showing that no subsequence can be Cauchy.

Suppose $(f_{n_j})_{j=1}^{\infty}$ is an arbitrary subseq. Given any pos int j, choose any k s.t. $n_k \geq 2n_j$. Then $\frac{1}{2n_j} \geq \frac{1}{n_k}$, then

$$\begin{aligned}
\left\|f_{n_j} - f_{n_k}\right\|_\infty &= \sup_{x\in[-1,1]} \left|f_{n_j}(x) - f_{n_k}(x)\right| \\
&\geq \left|f_{n_j}\left(\frac{1}{2n_j}\right) - f_{n_k}\left(\frac{1}{2n_j}\right)\right| \\
&= |1/2 - 1| = 1/2
\end{aligned}$$

So, choosing any postive $\epsilon \leq \frac{1}{2}$, there does not exists any int N s.t. $\left\|f_{n_j} - f_{n_k}\right\|_\infty < \epsilon$ whenever $j, k \geq N$. Hence, the subseq is not Cauchy.

20.3 Inner products on vector spaces

V: vector space. An inner product on V is a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ that satisfies the following three properties for all $x, y, z \in V$ and for all $\alpha, \beta \in \mathbb{R}$:

IP1: $\langle x, x\rangle \geq 0$ with equality iff $x = 0$ (pos definite)

IP2: $\langle x, y\rangle = \langle y, x\rangle$ (symmetry)

IP3: $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$ (bilinearity)

We call $(V, \langle\cdot,\cdot\rangle)$ an inner product space. We also refer to V as the inner product space (with an implied inner product).

A norm can be induced by any inner product according to the definition: $\|x\| = \langle x, x\rangle^{1/2}$.

Theorem 20.1: Cauchy-Schwarz Ineq

$\forall x, y \in V$

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|$$

with equality holding iff x and y are linearly dependent (collinear).

Proof:
Exercise. □

Corrollary 20.2

Let $(V, \langle\cdot,\cdot\rangle)$ be an IPS and let $\|\cdot\|$ be defined as $\|x\| = \langle x, x\rangle^{1/2}$ for any $x \in V$. Then $\forall x, y \in V$,

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof:
Exercise. □

As we have just shown, any inner product space has a natural norm so you can think of V as being both an inner product space and a normed space.

20.4 An inner product for $C[a, b]$

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$

The norm is L^2 norm

$$\|f\|_2 = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$$

21

Convergence of Functions

21.1 Example of a limit in $C[a, b]$

omitted

21.2 Pointwise convergence of functions

Let S be a subset of $\mathbb{R}^n$ and let $(f_k)_{k=1}^{\infty}$ be a seq of functions $f_k : S \to \mathbb{R}^m$ for all k. We say that (f_k) converges pointwisely to a function f if

$$\lim_{k\to\infty} f_k(x) = f(x) \qquad \forall x \in S$$

Properties:

1. Pointwise limit of continuous functions can be discontinuous.
2. Limit of integral may not be integral of limit.
3. Pointwise limit of discontinuous functions can be continuous. $f_k(x) = f\left(\frac{[xk]}{k}\right)$

21.3 Uniform convergence of functions

Given $\epsilon > 0$, $\exists$ integer N s.t.

$$\|f_k(x) - f(x)\| < \epsilon \quad \forall x \in S \text{ and } \forall k \geq N$$

Note that uniform convergence implies pointwise convergence.

Remark:
We see that if $\|\cdot\|_\infty$ is a valid norm for the functions f_k and f, then uniform convergence of $(f_k)_{k=1}^{\infty}$ to f is equivalent to convergence under the uniform norm.

We defined uniform convergence of a seq of functions to a function f. We did not refer to the norms, i.e., $\|f_k - f\|$. Recall $C(K)$ is a normed space (usually use $\|\cdot\|_\infty$).

If $f_k \in C(K)$ for all $k \geq 1$, and $f \in C(K)$, then uniform convergence of (f_k) to f is the same (by defn) as convergence under the uniform norm.

$\|\cdot\|_\infty$ is not well defined norm for $C(\mathbb{R})$. E.g. $f(x) = x$. f not bounded, so $|f(x)|$ does not have supremum.

Can you think of $f_k : \mathbb{R} \to \mathbb{R}$ s.t. (f_k) converges to $f(x) = x$?

$$f_k(x) = x + \frac{1}{k}$$

So uniform convergence can still hold for sequences that are not from a normed space of functions.

21.4 Uniform convergence and continuity

Theorem 21.1

$S \subseteq \mathbb{R}^n$, (f_k) a seq of cont functions from S to $\mathbb{R}^m$. If (f_k) converges uniformly to f, then f is cont.

Proof:

$$\begin{aligned}\|f(\vec{x}) - f(\vec{a})\| &\leq \|f(\vec{x}) - f_N(\vec{x})\| + \|f_N(\vec{x}) - f_N(\vec{a})\| + \|f_N(\vec{a}) - f(\vec{a})\| \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon\end{aligned}$$

□

Remark:
If $K \subseteq \mathbb{R}^n$ is compact, then vector space can be given the uniform norm $\|\cdot\|_\infty$. This can be extended to the space $C(K, \mathbb{R}^m)$ of vector valued, cont. functions from K to $\mathbb{R}^m$. The uniform is defined by

$$\|f\|_\infty = \sup_{x \in K} \|f(x)\|_2$$

where $\|\cdot\|_2$ is the Euclidean norm for vectors in $\mathbb{R}^m$.

Theorem 21.2: Completeness Theorem for $C(K, \mathbb{R}^m)$

Let $K \subseteq \mathbb{R}^n$ be compact. Then $C(K, \mathbb{R}^m)$ is complete w.r.t the uniform norm.

Proof:
We need to show that every Cauchy sequence converges uniformly to a function in $C(K, \mathbb{R}^m)$.

The rest of proof is left as an exercise to the readers. $\square$

Remark:
We only needed K compact to guarantee uniform is well defined. We can relax this restriction slightly, e.g., to the set of all bounded functions.

22 Discrete Dynamical Systems

Now we are more in the "application" side of this course...

22.1 Introduction to dynamical systems

We consider a system whose state at any given time is described by a vector $x \in X$, where X is any normed vector space.

The initial state is x_0.

In this course, we consider x to evolve in discrete time steps, so the state at time n is denoted x_n.

The dynamical system is governed by a map $T : X \to X$ such that

$$x_{n+1} = T(x_n) \quad \text{for } n \geq 0$$

> **Note**
> We also write Tx for $T(x)$.

Using this map repeatedly gives

$$\begin{aligned} x_1 &= Tx_0 \\ x_2 &= Tx_1 = T^2x_0 \\ &\vdots \\ x_n &= T(x_{n-1}) = \ldots = T^n(x_0) \end{aligned}$$

> **Note**
> $T^n(x) = T(T(T \ldots T(x)))$
>
> not $[T(x)]^n$.

22.2 Basic definitions for discrete dynamical systems

Let X ve a subset of a normed vector space and let $T : X \to X$ be a continuous map. We call (X, T) a discrete dynamical system.

For any point $x \in X$, we define the forward orbit of x as the sequence of points $O(x) = (T^n x)_{n=0}^{\infty}$.

The forward orbit tells you that how the system behaves (evolves) over time starting from the initial point x.

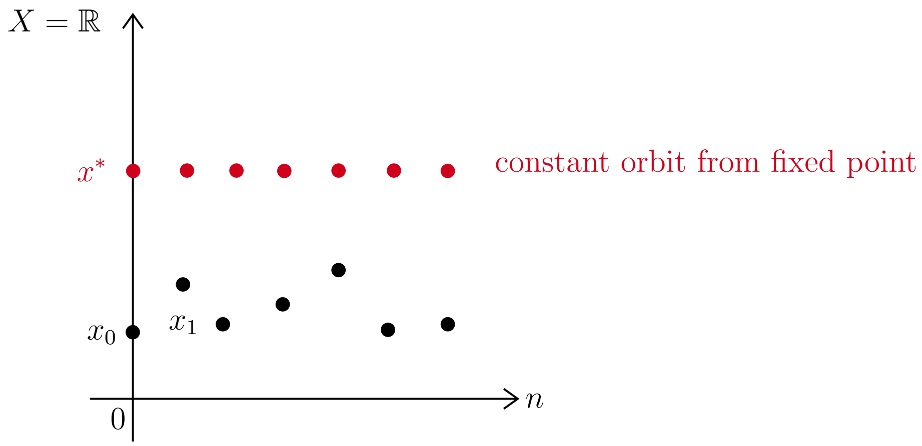
A fixed point of a dynamical system is $x^* \in X$ that satisfies $Tx^* = x^*$.

Any fixed point x^* has forward orbit $O(x^*) = (x^*, x^*, \ldots)$

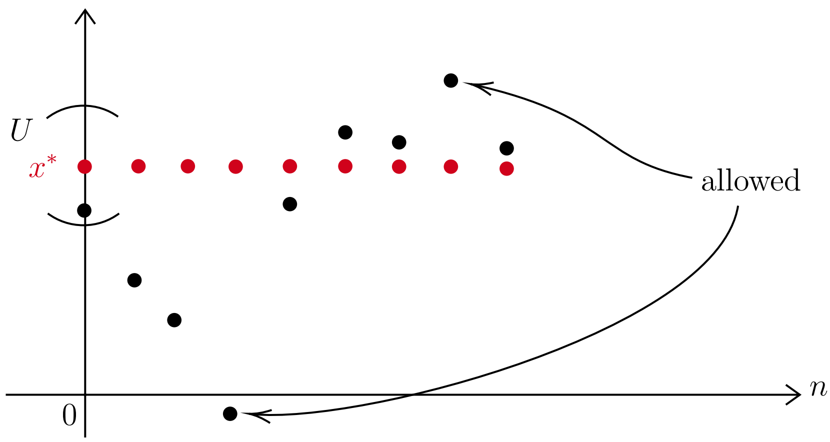
A fixed point x^* is called attractive (or a sink) if there exists an open neighbourhood U of x^* such that $O(x)$ converges to x^* for any $x \in U \cap X$.

A fixed point x^* is repelling (or a source) if there is an open neighbourhood U such that $O(x)$ is not contained in U for any $x \in U \cap X \setminus \{x^*\}$.

We say that T is a C^1 dynamical system on $X \subseteq \mathbb{R}$ if T is continuously differentiable on X.

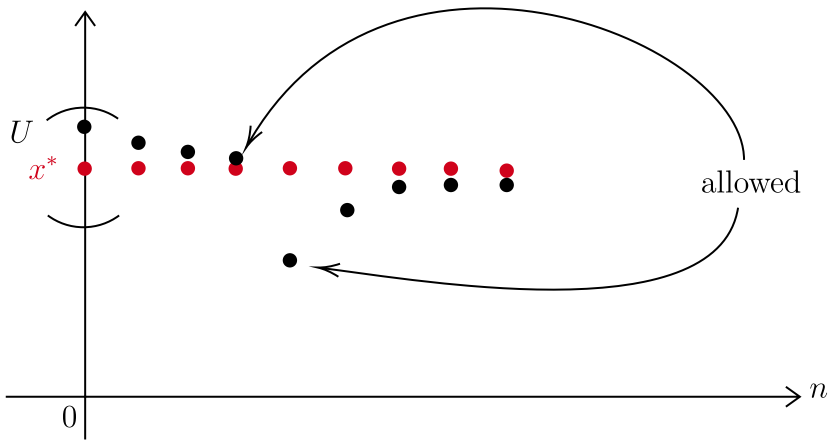


Suppose x^* attractive fixed point



"Attractive" does not mean $\|x^* - T^n x\|$ is strictly decreasing.

Suppose x^* repelling



"repelling" does not mean distance from x^* is monotone increasing.

22.3 Examples

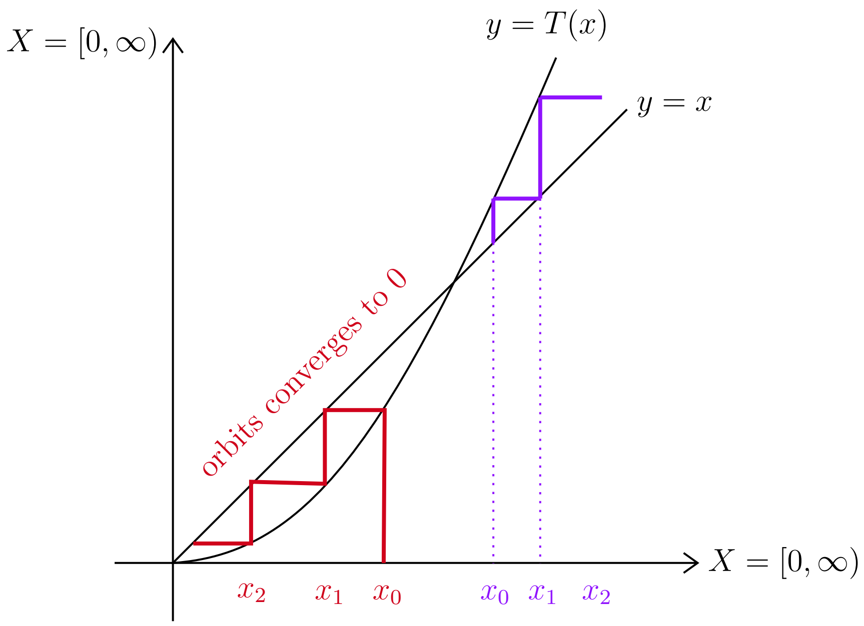
1. $T(x) = x$ and consider $X = \mathbb{R}$. $x_n = T(x_{n-1}) = x_{n-1}$

   Every point is a fixed point.

   Every point is (not repelling), (not attractive).

   > Remark:
   > fixed point does not to be either attractive or repelling.

2. $T(x) = x^2, X = [0, \infty)$



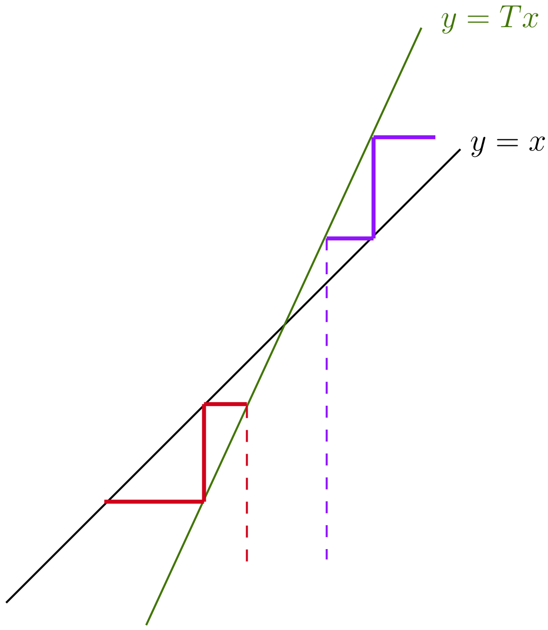
Start between 0 and 1 $\implies$ converges to 0

Start $> 1 \implies$ diverges to ∞

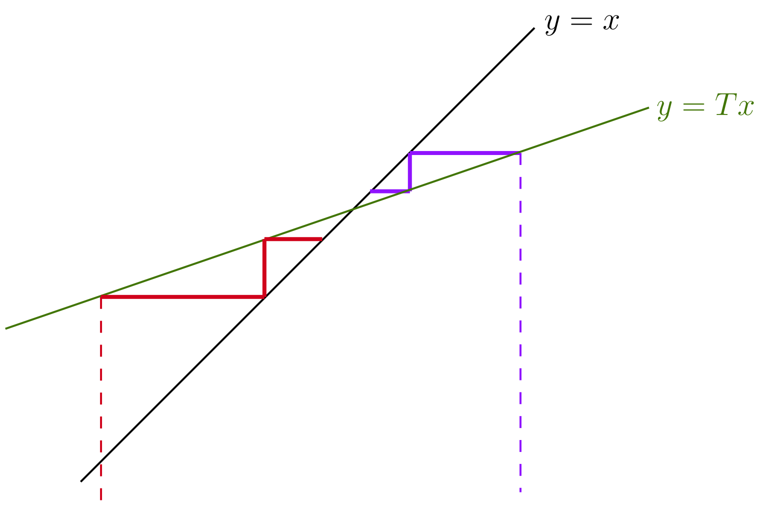
$x_0 \implies$ fixed point

23

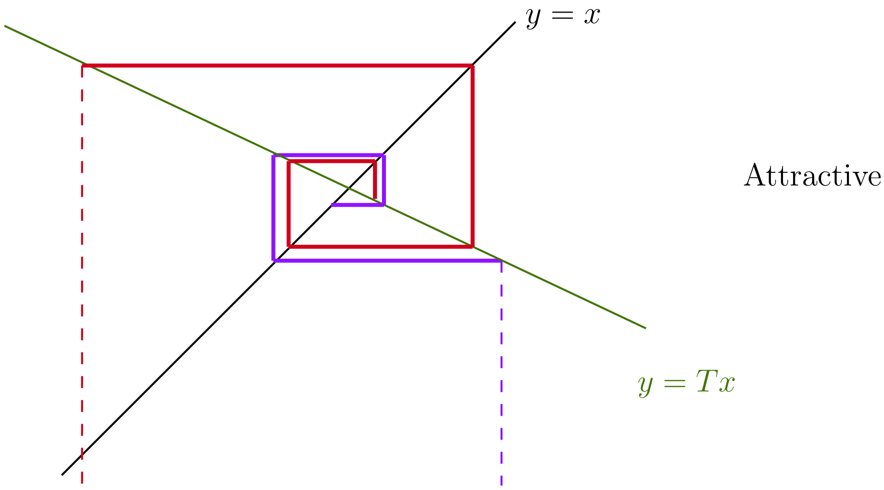
Fixed points and Contractions

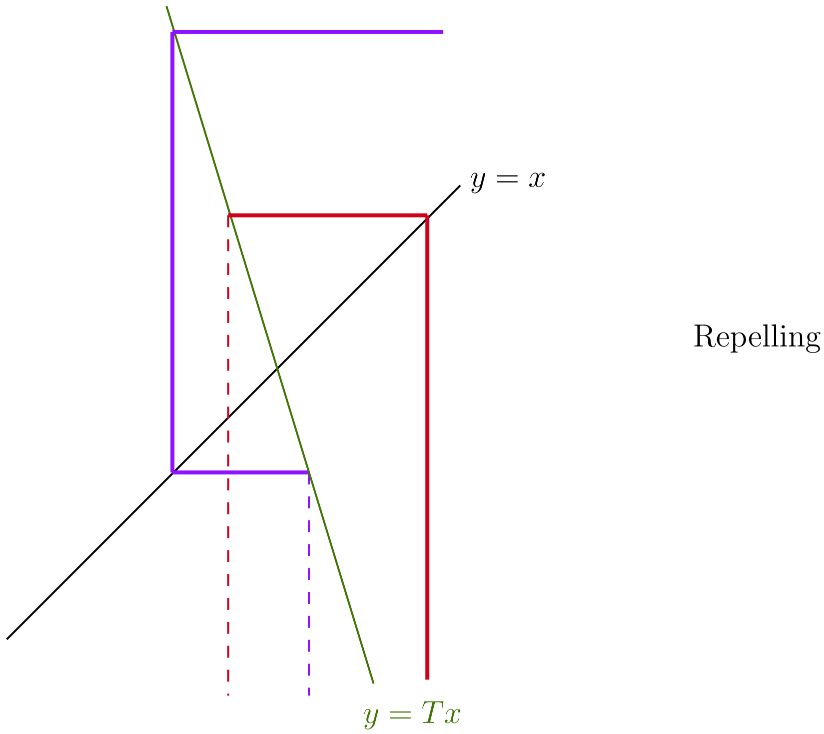


Repelling



Attractive





23.1 Stability Theorem for fixed points

Theorem 23.1

Let T be C^1 dynamical system on $X \subseteq \mathbb{R}$ with a fixed point x^*.

1. If $|T'(x)| > 1$, then x^* is a repelling fixed point.
2. If $|T'(x)| < 1$, then x^* is an attractive fixed point.

Further more, for any c satisfying $|T'(x)| < c < 1$, then there exists an interval $U = (x^* - \delta, x^* + \delta)$ with $\delta > 0$ such that for any $x_0 \in U \cap X$, the orbit of x_0

satisfies

$$|x_n - x^*| \le c^n |x_0 - x^*| \le \frac{c^n}{1-c}|x_1 - x_0|$$

Exercise

Check the examples from sec 22.3 to see if they are consistent with this theorem.

23.2 Contractions

Let X be a subset of a normed vector space. A map $T : X \to X$ is a **contraction** on X if there exists a number $c \in [0, 1)$ such that $\|T_x - T_y\| \le c\,\|x - y\|$ for all $x, y \in X$. The number c is called the contraction constant.

Remark:

Contractions are Lipschitz functions with Lipschitz constant $c < 1$.

Example:

$T(x) = x^2$ on $[0, 1/4]$ is a contraction.

Note that for differentiable functions, Lipschitz constant is the supremum of $|T'(x)|$.

$T'(x) = 2x$

$$\sup_{x\in[0,1/4]} |T'(x)| = \frac{1}{2} = c < 1 \implies \text{Contraction}$$

Note that $T(x) = x^2$ is not a contraction on $[0, 1]$ even though $|Tx - 0| \le |x - 0|$ for all $x \in [0, 1]$.

Also be careful, $\|Tx - Ty\| < \|x - y\|$ for all x, y is not sufficient for contraction.

24

Banach Contraction Principle

Theorem 24.1: Banach Contraction Principle

Let X be a closed subset of a complete normed space.

If $T : X \to X$ is a contraction with contraction constant c, then:

1. T has a unique fixed point x^*,
2. the forward orbit of any point $x \in X$ converges to x^*, i.e., $\lim_{n\to\infty} T^n x = x^*$, and
3. $\|T^n x - x^*\| \leq c^n \|x - x^*\| \leq \dfrac{c^n}{1-c} \|x - Tx\|$

Proof:
We want to show that for any $x \in X$, the sequence (x_n) is Cauchy.

Applying $\triangle$ Ineq for a number of times:

$$\begin{aligned}\|x_{n+m} - x_n\| &= \|x_{n+m} - x_{n+m-1} + x_{n+m-1} - x_{n+m-2} + \ldots - x_n\| \\ &\leq \sum_{i=0}^{m-1} \|x_{n+i+1} - x_{n+i}\|\end{aligned}$$

Each term of this sum can be bounded by noting that

$$\|x_{k+1} - x_k\| = \|Tx_k - Tx_{k-1}\| \leq c\|x_k - x_{k-1}\| \leq \ldots \leq c^k \underbrace{\|x_1 - x_0\|}_{D} = c^k D$$

So,

$$\|x_{n+m} - x_n\| \leq \sum_{i=0}^{m-1} c^{n+1} D \leq \sum_{i=0}^{\infty} c^{n+i} D = \frac{c^n D}{1-c}$$

Hence, (x_n) is Cauchy (show this step). $\square$