Introduction to General Relativity

AMATH 475

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LaTeXed by Sibelius Peng

Preface

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Some of the notes (especially special relativity part) are projected to the screen instead of using blackboards. They can be found on https://sites.google.com/site/emmfis/teaching/gr.

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0 Pre-Math

0.1 Index notation

$$A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \qquad B = \begin{pmatrix} B^1{}_1 & B^1{}_2 \\ B^2{}_1 & B^2{}_2 \end{pmatrix}$$

$$(A \cdot B)^a{}_b = A^a{}_c B^c{}_b = B^c{}_b A^a{}_c \qquad \text{sum over all possible } c$$

Identify followings:

$$B_\kappa{}^\nu A_\mu{}^\kappa = A_\mu{}^\kappa B_\kappa{}^\nu = C_\mu{}^\nu = (A \cdot B)_\mu{}^\nu$$

$$A^\kappa{}_\mu B_\kappa{}^\nu = D_\mu{}^\nu = (A^T)_\mu{}^\kappa B_\kappa{}^\nu = (A^T \cdot B)_\mu{}^\kappa$$

$$A_\kappa{}^\nu B_\mu{}^\kappa = E_\mu{}^\nu = (B \cdot A)_\mu{}^\nu$$

$$A^\kappa{}_\mu B^\nu{}_\kappa = (A^T)_\mu{}^\kappa (B^T)_\kappa{}^\nu = \left((B \cdot A)^T\right)_\mu{}^\nu$$

$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 \qquad \{\mathbf{e}_1, \mathbf{e}_2\}$ Basis 1.

$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a \qquad \{\mathbf{e}'_1, \mathbf{e}'_2\}$ Basis 2.

Change of basis matrix Λ

$$\mathbf{e}'_a = \Lambda_a{}^b \mathbf{e}_b$$
$$v'^a = \tilde{\Lambda}^a{}_b v^b$$

$$
\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a{}_b v^b \Lambda_a{}^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a{}_b \Lambda_a{}^c v^b \mathbf{e}_c \\
&= \underbrace{\left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c}_{\delta_b^c} v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \quad \left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c &= \delta_b^c \\
\tilde{\Lambda}^T \cdot \Lambda &= \mathbb{1}
\end{aligned}
$$

$\tilde{\Lambda}^T$ is the inverse transpose of Λ

covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis. Λ. (sub-indices)

A contravariant object transforms like components of vectors. ($\tilde{\Lambda} = (\Lambda^T)^{-1}$). (super-indices)

0.2 Vectors and one-forms

one-form

Let V be a vector space. A one-form is a linear map $\boldsymbol{\omega} : V \to \mathbb{R}$.

or we write: $(\boldsymbol{\omega}, \cdot) : V \to \mathbb{R}$ and $(\boldsymbol{\omega}, \mathbf{v}) \in \mathbb{R}$.

dual vector space

The set of all one-forms on V (call V^*) is a vector space as well called the dual vector space to V.

dual basis

Let $\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2, \ldots\}$ (or $\{\boldsymbol{\Upsilon}_i\}$) be a basis of V so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = v^i \boldsymbol{\Upsilon}_i$.

We define the dual basis (of V^*) to $\{\boldsymbol{\Upsilon}_i\}$ as $\{\boldsymbol{\omega}^i\}$ such that $\boldsymbol{\omega}^i(\boldsymbol{\Upsilon}_j) = \delta^i_j$.

For a one form $\boldsymbol{\omega}$ we denote its "components of the basis $\boldsymbol{\Upsilon}$" as $(\boldsymbol{\omega}, \boldsymbol{\Upsilon}_m) = \omega_m$

Proposition 0.1

The dual basis of V^* is actually a basis of V^*.

The action of $\boldsymbol{\omega} \in V^*$ on a vector $\mathbf{v} = v^\mu \boldsymbol{\Upsilon} \in V$ is

$$(\boldsymbol{\omega}, \mathbf{v}) = (\boldsymbol{\omega}, v^\mu \boldsymbol{\Upsilon}_\mu) = v^\mu \omega_\mu$$

Let's prove $\{\boldsymbol{\Upsilon}^a\}$ is linear independent.

Proof:
A linear comb. $c_a \boldsymbol{\Upsilon}^a$ acts on a vector $\mathbf{v} = v^a \boldsymbol{\Upsilon}_a$

$$\begin{aligned}
(c_a \boldsymbol{\Upsilon}^a, \mathbf{v}) &= c_a (\boldsymbol{\Upsilon}^a, \mathbf{v}) \\
&= c_a (\boldsymbol{\Upsilon}^a, v^b \boldsymbol{\Upsilon}_b) \\
&= c_a v^b \underbrace{(\boldsymbol{\Upsilon}^a, \boldsymbol{\Upsilon}_b)}_{\delta^a_b} \\
&= c_a v^b \delta^a_b = c_a v^a
\end{aligned}$$

For LI,

$$c_a \boldsymbol{\Upsilon}^a = 0 \iff c_a = 0 \quad \forall a$$

$$c_a v^a = 0 \quad \forall \mathbf{v} \iff c_a = 0$$

□

vectors: take one-forms $\to \mathbb{R}$ one-forms: take vectors $\to \mathbb{R}$

0.3 Tensor

type (m, n) tensor

A type (m, n) tensor is a multilinear map that

$$\mathbf{T} : V^n \otimes (V^*)^m \to \mathbb{R}$$

Components of $\mathbf{T}$:

$$\mathbf{T}(\boldsymbol{\Upsilon}_{a1}, \ldots, \boldsymbol{\Upsilon}_{an}, \boldsymbol{\Upsilon}^{b1}, \ldots, \boldsymbol{\Upsilon}^{bm}) = T_{a_1 \ldots a_n}{}^{b_1 \ldots b_m}$$

1. Tensor product takes $\binom{m}{n}$ and $\binom{m'}{n'} \to \binom{m+m'}{n+n'}$ tensor
2. Contraction takes $\binom{m}{n} \to \binom{m-1}{n-1}$

Example:

1. $T_a{}^b, S_c{}^d$.

$$(\mathbf{T} \otimes \mathbf{S})_a{}^b{}_c{}^d = T_a{}^d S_c{}^d = P_a{}^b{}_c{}^d$$

2. $T_a{}^{bc} \to c^b T_a{}^{ba}$

$$v^a, w_b \begin{cases} v^a \omega_b \\ v^a \omega_a \end{cases}$$

If you have a favorite type $(2,0)$ symmetric tensor $\mathbf{g}$

$$v_\mu = g_{\mu\nu} v^\nu$$

$g^{\mu\nu} :=$ components of the inverse of $\mathbf{g}_{\mu\nu}$

$$v^\nu = g^{\mu\nu}$$

then

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma$$

$$g_{\mu\nu} v^\mu w^\nu = v_\mu w^\nu = \mathbf{v}\mathbf{w}$$
$$||\mathbf{v}||^2 = g_{\nu\mu} v^\mu v^\nu$$

Then we can define the angle

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}|| \; ||\mathbf{v}||} := \cos\theta$$
$$T_\mu{}^\nu = g^{\nu\sigma} T_{\mu\sigma}$$
$$T^{\mu\nu} = g^{\nu\sigma} g^{\mu\rho} T_{\sigma\rho}$$
$$g^\nu_\mu = g^{\nu\sigma} g_{\sigma\mu} = \sigma^\nu_\mu$$

0.4 Levi-Civita symbol

Levi-Civita symbol $\epsilon^{abc\dots}, \epsilon_{abc\dots}$

- is antisymmetric
- $\epsilon^{1234\dots} = 1$, $\epsilon_{1234} = 1$

$\epsilon^{123} = 1, \quad \epsilon^{213} = -1, \quad \epsilon^{312} = 1, \quad \epsilon^{113} = 0$

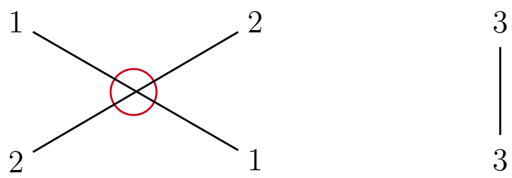
$\epsilon^{123456} = 1, \quad \epsilon^{612453} = -1$

Idea just see the permutations

Levi-Civita symbol

$$\varepsilon_{a_1 a_2 a_3 \dots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \dots, a_n) \text{ is an even permutation of } (1, 2, 3, \dots, n) \\ -1 & \text{if } (a_1, a_2, a_3, \dots, a_n) \text{ is an odd permutation of } (1, 2, 3, \dots, n) \\ 0 & \text{otherwise} \end{cases}$$

Here is a short-cut:



odd number crossings, so odd permutation.

Note that $\det(M) := \epsilon_{ijk\dots} M^i{}_1 M^j{}_2 M^j{}_3 \dots$

Exercise

prove $\epsilon^{i_1 i_2 \dots i_n} \epsilon_{j_1 j_2 \dots j_n} = n! i_j = 1, \dots, n$

$$\epsilon^{ijk} \epsilon_{ilm} = \delta^j_l \delta^k_m - \delta^j_m \delta^k_l$$
$$\epsilon^{ijmn} \epsilon_{klmn} = 2(\delta^i_k \delta^j_l - \delta^j_k \delta^i_l)$$

Prove $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Proof:

Let $\vec{F} = \vec{A} \times (\vec{B} \times \vec{C})$ $\vec{D} = \vec{B} \times \vec{C}$

Then

$$D^k = \epsilon^k{}_{ij} B^i C_j$$
$$F^l = \epsilon^l{}_{mk} A^m D^k \implies F^l = \epsilon^l{}_{mk} \epsilon^k{}_{ij} A^m B^i C^j$$

Then

$$\begin{aligned} F^l &= (\delta^l_i \delta_{mj} - \delta^l_j \delta_{mi}) A^m B^i C^j \\ &= \delta^l_i \delta_{mj} A^m B^i C^j - \delta^l_j \delta_{mi} A^m B^i C^j \\ &= B^l (A_j C^j) - C^l (A_i B^i) \end{aligned}$$

where we use

$$\vec{A} \cdot \vec{B} = A^i B_i$$

$\square$

1 Special Relativity

1.1 Postulates of SR

Postulate 0

Newton's first law

Postulate 1: Principle of relativity

In the absence of gravity, all the laws of Physics are identical in all inertial reference frames.

Postulate 2

The speed of light in vacuum c is constant and the same from all inertial reference frames, regardless of their state of motion.

1.2 Lorentz Transformation

We define the spacetime interval Δs^2

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = -c^2 \left(t_2 - t_1\right)^2 + \left(\mathbf{x}_2 - \mathbf{x}_1\right)^2$$

Assuming the following:

1. The difference between the two frames is a constant speed $\succapprox$
2. The transformation has to be linear.

$$t' = \gamma \left(t - \frac{\boldsymbol{v} \cdot \boldsymbol{x}}{c^2} \right), \quad \boldsymbol{x}' = \boldsymbol{x} + (\gamma - 1)(\boldsymbol{n} \cdot \boldsymbol{x})\boldsymbol{n} - \gamma \boldsymbol{v} t$$

and index notation

$$t' = \gamma \left(t - \frac{v_i x^i}{c^2} \right), \quad x^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

1.3 Line element, proper time and spacelike, timelike and null separation

1.3.1 Classification of spacetime intervals

We can classify events according to the following criterion:

- Spacelike separated, $\Delta s^2 > 0$
- Timelike separated, $\Delta s^2 < 0$
- Lightlike (null) separated, $\Delta s^2 = 0$

Given the trajectory of a physical particle moving inertially, we will call co-moving frame (inertial) or proper frame (non-inertial) to the frame S_p where the particle is at rest.

1.3.2 Proper time and line element

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2$$

We will call ds^2 the spacetime line element.

$$\mathbf{v} := \frac{d\mathbf{x}}{dt}$$

P0, P1, P2 + linearity

$$\implies t' = t \left(t - \frac{v_i x^i}{c^2} \right) \tag{1}$$

$$x'^i = x^i + (\gamma - 1)\frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

Particle trajectory in a given inertial (Lab) frame $\mathbf{x}(t)$

Particle trajectory in its proper frame $\boldsymbol{\xi}(t) = 0$

Comoving frame's trajectories at each t (from lab frame) $\mathbf{x} = \mathbf{v}(t)t$.

$$d\tau = dt' = \gamma(t)\left(1 - \frac{\mathbf{v}(t)^2}{c^2}\right) dt \tag{2}$$

$$ds^2 = -c^2 dt^2 \left(1 - \frac{1}{c^2}\underbrace{\left(\frac{d\mathbf{x}}{dt}\right)^2}_{\mathbf{v}(t)}\right) = -c^2 \underbrace{\gamma^{-2} dt^2}_{d\tau^2} \implies ds^2 = -c^2 d\tau^2 \tag{3}$$

Example:

Find $\tau(t)$ for the three following trajectories.

1. $x(t) = v(t)$

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x}^2 \implies d\tau = \gamma^{-1} dt \implies \Delta\tau = \gamma^{-1}\Delta t$$

2. $x(t) = \frac{c^2}{a}\left[\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right]$

   Then $\frac{dx}{dt} = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2$$

$$\implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\frac{a^2t^2}{1 + \left(\frac{at}{c}\right)^2}}$$

$$\implies \tau(t) = \frac{c}{a}\operatorname{arcsinh}\left(\frac{at}{c}\right) \qquad \text{and} \qquad t(\tau) = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

3. $x(t) = L\sin(\omega t) \implies \frac{dx}{dt} = L w\cos(\omega t)$ with $L\omega < c$

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2 \implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\left(\frac{dx}{dt}\right)^2} \implies d\tau = \sqrt{1 - \frac{L^2\omega^2}{c^2}\omega t}\ dt$$

   Then

$$\tau(t) = \frac{E\left(t\omega, \frac{1}{1-\frac{c^2}{L^2\omega^2}}\right)}{\omega\sqrt{\frac{1}{1-\frac{L^2\omega^2}{c^2}}}}$$

   where

$$E(\varphi|m) = \int_0^\varphi (1 - m\sin^2\theta)^{1/2} d\theta$$

1.4 Lorentzian Tensors

See notes for details.

A_μ transposes with Λ and it's covariant.

A^μ transposes with $\tilde{\Lambda} = (\Lambda^{-1})^T$ and it's controvariant.

1.5 Poincare group

The derivations are in notes.

1.6 Relativistic dynamics

1.6.1 Hamilton's principle and Euler-Lagrange equations

There exists at least one function (called action) of the trajectories that the degrees of freedom of a system may take in phase space. The physical trajectories are obtained demanding stationarity of this functional under variations that keep the initial and final positions constant.

Usually, the action S of a system of n particles can be written in terms of a Lagrangian $L(s, \mathsf{x}, \mathring{\mathsf{x}})$ where $^\circ$ represents $\mathring{\mathsf{x}} = \frac{d\mathsf{x}}{ds}$ so that

$$S = \int_{s_1}^{s_2} ds\ L(s, \mathsf{x}, \mathring{\mathsf{x}})$$

$$\delta S = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} \delta x_n^\mu + \frac{\partial L}{\partial \mathring{x}_n^\mu} \delta \mathring{x}_n^\mu \right) = \sum_n \int_{s_1}^{s_2} \mathrm{d}s \left(\frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \mathring{x}_n^\mu} \right) \delta x_n^\mu + \sum_n \left[\frac{\partial L}{\partial \mathring{x}_n^\mu} \delta x_n^\mu \right]_{s_1}^{s_2}$$

Impose Hamilton's Principle

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial x_n^\mu} - \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial \mathring{x}_n^\mu} = 0$$

1.6.2 Conserved quantities and Noether's theorem

Noether's theorem

If the variation of the action around a physical trajectory under a continuous variation of the positions $\partial \mathsf{x}$ is zero, then the quantity

$$\delta Q = \sum_n \frac{\partial L}{\partial \mathring{x}_n^\mu} \delta x^\mu$$

is conserved. That is

$$\frac{d(\partial Q)}{ds} = 0.$$

Proof:
See notes. □

1.6.3 Four-momentum

Let S be invariant under $\partial \mathsf{x} = \mathsf{n}\delta\alpha$.

$$\implies \delta Q = \frac{\partial L}{\partial \overset{\circ}{x}^{\mu}} n^{\mu} \delta\alpha$$

is constant $\implies$ the projection $\mathsf{n} \cdot \mathsf{p} = n^{\mu} p_{\mu} = \eta_{\mu\nu} n^{\mu} p^{\nu}$ (where $p_{\mu} := \frac{\partial L}{\partial \overset{\circ}{x}^{\mu}}$) is conserved.

If the action is invariant under Lorentz transformation $\delta x^{\mu} = \delta\omega^{\mu}{}_{\nu} x^{\nu}$, then

$$J_{\mu\nu} := x_{\mu} p_{\nu} - x_{\nu} p_{\mu}$$

is conserved.

1.6.4 Angular momentum

The angular momentum $\boldsymbol{J}$ associated to spatial rotations and the vector $\boldsymbol{K}$ associated to boosts can be extracted directly from $J_{\mu\nu}$:

$$J^{i} = \frac{1}{2}\epsilon^{ijk} J_{jk}, \quad K_{i} = J_{i0}$$

1.6.5 Free particle dynamics

- S has to be a scalar (Invariant under Lorentz)
- Must coincide with the non-relativistic action in the limit $\frac{v}{c} \ll 1$.

$$S = mc \int ds = -mc^2 \int d\tau = -mc^2 \int dt \frac{d\tau}{dt} = -mc^2 \int \frac{dt}{\gamma} = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}}$$
$$= -mc^2 \int dt \left[1 - \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right)\right]$$

and

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} = -mc^2 + \frac{1}{2} m v^2 + \mathcal{O}\left(\frac{v^4}{c^4}\right)$$

Euler-Lagrange $\frac{d}{dt}(\gamma m \boldsymbol{v}) = 0$

$$p_i = \frac{\delta S}{\delta v^i} = \frac{\partial L}{\partial v^i} = m\gamma v_i, \qquad \boldsymbol{p} = m\gamma \boldsymbol{v}$$

Hamiltonian

$$H = (\boldsymbol{p} \cdot \boldsymbol{v} - L)_{\boldsymbol{v} \to \boldsymbol{v}(\boldsymbol{p})} = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$$

Let's introduce Four-velocity.

$$\frac{dx^\mu}{d\tau} =: \dot{x}^\mu \equiv u^\mu$$

solid dot means derivative w.r.t proper time.

$$\dot{x}^\mu := \frac{dx^\mu}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d\boldsymbol{x}}{d\tau} \end{pmatrix} = \begin{pmatrix} c\frac{dt}{d\tau} \\ \frac{d\boldsymbol{x}}{dt}\frac{dt}{d\tau} \end{pmatrix} = \gamma \begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$$

If we choose action as (not four-velocity)

$$S = mv \int dt \sqrt{\eta_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}$$

Lagrangian

$$L = mc\sqrt{-\eta_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}}$$

$$p_\mu = \frac{\delta S}{\delta \dot{x}^\mu} = m\dot{x}_\mu \implies p^\mu = m\dot{x}^\mu = m\gamma\begin{pmatrix} c \\ \boldsymbol{v} \end{pmatrix}$$

p^0 in the proper frame: $p^0 = mc, \quad \boldsymbol{p} = \boldsymbol{0}$. so cp^0 is energy.

Let's compute

$$p^\mu p_\mu = m^2\dot{x}^\mu\dot{x}_\mu = -m^2c^2$$

$$p^\mu p_\mu = -(p^0)^2 + \boldsymbol{p}^2$$

$$\implies -m^2c^2 = -(p^0)^2 + \boldsymbol{p}^2 \implies p^0 = \frac{1}{c}\sqrt{m^2c^4 + c^2\boldsymbol{p}^2}$$

$$\implies E = \sqrt{m^2c^4 + c^2\boldsymbol{p}^2} = mc^2\sqrt{1 + \gamma^2\frac{v^2}{c^2}} = mc^2 + \frac{1}{2}mv^2 + \mathcal{O}\left(\frac{v^2}{c^2}\right)$$

- **Ultrarrelativistic limit**: The kinetic term inside the square root is much larger than the rest energy of the particle: $pc \gg mc^2, \quad E \approx cp$
- **Deep non-relativistic limit**: The rest energy is much larger than the kinetic energy of the particle: $mc^2 \gg pc, \quad E \approx mc^2$

Two problems

You are designing a particle collider, you have two identical part of mass M and energy budget $E = 2\varepsilon$. You have two strategies:

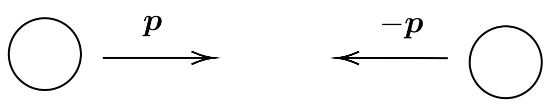
a) spend $1/2E$ on each and accelerate them.

b) spend E on one of them and accelerate it

Which one optimizes the center of mass energy?

Solution

a)



Lab frame $p_1^\mu = \left(\frac{\epsilon}{c} + M_c, p, 0, 0\right)$ $\quad$ $p_2^\mu = \left(\frac{\epsilon}{c} + M_c, -p, 0, c\right)$

$$p^\mu_{lab} = p_1^\mu + p_2^\mu = \left(\frac{2\epsilon}{c} + 2M_c, 0, 0, 0\right) = p^\mu_{CM}$$

Then

$$E^{(a)}_{CM} = cp^0_{CM} = 2\epsilon + 2Mc^2 = 2Mc^2\left(1 + \frac{\epsilon}{\mu c^2}\right)$$

b) Here the p is different from the p above.

Lab frame $p_1^\mu = \left(\frac{2\epsilon}{c} + Mc, p, 0, 0\right)$ $\quad$ $p_2^\mu = (Mc, 0, 0, c)$

$$p^\mu_{lab} = p_1^\mu + p_2^\mu = \left(\frac{2\epsilon}{c} + 2M_c, p, 0, 0\right)$$

Determine

$$p_1^\mu p_{1\mu} = -M^2c^2 = -\left(\frac{4\epsilon^2}{c^2} + M^2c^2 + M^2c^2 + 4\epsilon M\right) + p^2$$

$$\implies p = \sqrt{\frac{4\epsilon^2}{c^2} + 4\epsilon M} = \frac{2\epsilon}{c}\sqrt{1 + \frac{Mc^2}{\epsilon}}$$

We want p^0_{cm}, and we know $p^\mu_{CM} = (p^0_{CM}, \mathbf{0})$

Lorentz scalar: $p_{CM}{}^\mu p_{CM\mu} = -(p^0_{CM})^2$ and lab frame $p_{lab}{}^\mu p_{lab\mu} = -(p^0_{CM})^2$

$$
\begin{aligned}
{p_{lab}}^{\mu} p_{lab\mu} &= -\left(\frac{4\epsilon^2}{c^2} + 4M^2c^2 + 8\epsilon M\right) + \frac{4\epsilon^2}{c^2}\left(1 + \frac{Mc^2}{\epsilon}\right) \\
&= -4M^2c^2 - 8\epsilon M + 4\epsilon M \\
&= -4M(Mc^2 + \epsilon) \\
&= -(p^0_{CM})^2
\end{aligned}
$$

$$p^0_{CM} = \sqrt{-p^{\mu}_{lab} p_{lab\mu}} = 2\sqrt{\mu(Mc^2 + \epsilon)} \implies E^{(b)}_{CM} = cp^0_{CM} = 2Mc^2\sqrt{1 + \frac{\epsilon}{Mc^2}}$$

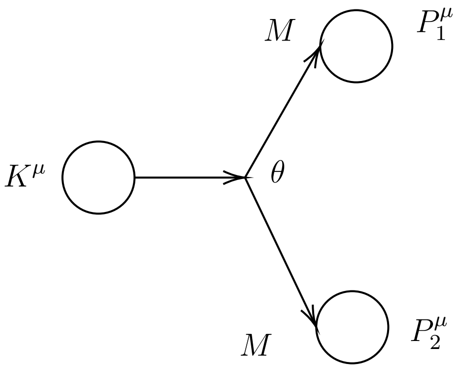
So

$$R = \frac{E^{(a)}_{CM}}{E^{(b)}_{CM}} = \sqrt{1 + \frac{\epsilon}{Mc^2}} > 1$$

In Deep non-real $\epsilon \ll Mc^2$, $\lim_{\frac{\epsilon}{Mc^2} \to 0} R = 1$.

In Ultra limit $\lim_{\epsilon \to pc} R = \sqrt{1 + \frac{pc}{Mc^2}} \to \infty$.

A massless particle cannot $\to$ two identical mass particle. (converse is also true).



$$K^\mu = P^\mu_1 + P^\mu_2$$

$$K^\mu K_\mu = P^\mu_1 P_{1\mu} + P^\mu_2 P_{2\mu} + 2P^\mu_1 P_{2\mu} \tag{1}$$

where $K^\mu K_\mu = 0$, $P^\mu_1 P_{1\mu} = -M^2c^2 = P^\mu_2 P_{2\mu}$, and

$$P^\mu_1 P_{2\mu} = \eta_{\mu\nu} P^\mu_1 P^\nu_2 = -P^0_1 P^0_2 + \boldsymbol{P}_1 \cdot \boldsymbol{P}_2 = -\frac{1}{c^2} + \boldsymbol{P}_1 \cdot \boldsymbol{P}_2$$

Sub them into (1), we get

$$\boldsymbol{P}_1 \cdot \boldsymbol{P}_2 = M^2c^2 + \frac{E_1E_2}{c^2} \leq \|\boldsymbol{P}_1\|\|\boldsymbol{P}_2\| \implies \frac{E_1E_2}{c^2} \leq \|\boldsymbol{P}_1\|\|\boldsymbol{P}_2\|$$

where we used

$$\|\boldsymbol{P}_1\|\|\boldsymbol{P}_2\| \cos\theta \leq \|\boldsymbol{P}_1\|\|\boldsymbol{P}_2\|$$

$$M^2c^2 + \frac{E_1E_2}{c^2} \leq \|\boldsymbol{P}_1\|\|\boldsymbol{P}_2\| \geq \frac{E_1E_2}{c^2} \tag{2}$$

$$E_1 = \sqrt{M^2c^4 + \|\boldsymbol{P}_1\|^2 c^2} \implies \|\boldsymbol{P}_1\| = \sqrt{\frac{E_1^2}{c^2} - M^2c^2} \implies \|\boldsymbol{P}_1\| < \frac{E_1}{c} \tag{3}$$

$$E_2 = \sqrt{M^2c^4 + \|\boldsymbol{P}_2\|^2 c^2} \implies \|\boldsymbol{P}_2\| = \sqrt{\frac{E_2^2}{c^2} - M^2c^2} \implies \|\boldsymbol{P}_2\| < \frac{E_2}{c} \tag{4}$$

$$M^2c^2 + \frac{E_1}{c}\frac{E_2}{c} \leq \|\boldsymbol{P}_1\|\|\boldsymbol{P}_2\| \overset{(4)}{\leq} \frac{E_1}{c}\frac{E_2}{c} \implies M^2c^2 < 0$$

which is impossible.

1.7 Accelerated observers and the Rindler metric

1.7.1 Four-acceleration

$$x^\mu \to \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \dot{x}^\mu \equiv e^\mu, \qquad \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = \ddot{x}^\mu \equiv b^\mu$$

We know $(\dot{x}^\mu) = \gamma(c, \boldsymbol{v})$.

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}$$

then

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \gamma^2 \boldsymbol{v} \cdot \boldsymbol{a}$$

where $\boldsymbol{v} := \dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}, \quad \boldsymbol{a} := \dfrac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}$. Then

$$(b^\mu) = \begin{pmatrix} b^0 \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} \frac{\gamma^4}{c}\boldsymbol{v}\cdot\boldsymbol{a} \\ \frac{\gamma^4}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})\boldsymbol{v} + \gamma^2\boldsymbol{a} \end{pmatrix}$$

In the co-moving frame, we have $\boldsymbol{v} = \boldsymbol{0}, \gamma = 1$, then $(b^\mu) = \binom{0}{\boldsymbol{a}}$.

In general,

$$b^\mu b_\mu = \gamma^4 \left[\frac{\gamma^2}{c^2}(\boldsymbol{v}\cdot\boldsymbol{a})^2 + \boldsymbol{a}^2\right] \geq 0$$

In the co-moving frame, $b^\mu b_\mu = \boldsymbol{a}^2$, proper acceleration $|\boldsymbol{a}| = \sqrt{b^\mu b_\mu}$.

Now let's compute this

$$b_\mu \dot{x}^\mu = \frac{\mathrm{d}\dot{x}_\mu}{\mathrm{d}\tau}\dot{x}^\mu = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{x}^\mu \dot{x}_\mu) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(-c^2) = 0$$

1.7.2 Constantly accelerated

From the co-moving frame $(t, \boldsymbol{x})$, at time $t = 0$, $v(0) = 0$.

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = mb^i \implies m\frac{\mathrm{d}\gamma\boldsymbol{v}}{\mathrm{d}t} = m\boldsymbol{a} \implies \boldsymbol{a} = \frac{\mathrm{d}(\gamma\boldsymbol{v})}{\mathrm{d}t}$$
$$\implies a\mathrm{d}t = \gamma\left(\gamma^2\frac{v^2}{c^2} + 1\right)\mathrm{d}v \Rightarrow a\mathrm{d}t = \frac{\mathrm{d}v}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}$$
$$\implies v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{at}{\sqrt{1 + \left(\frac{at}{c}\right)^2}} \Rightarrow x = \frac{c^2}{a}\left[\sqrt{1 + \left(\frac{at}{c}\right)^2} - 1\right]$$

With initial condition $t = 0, \tau = t$, we get

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \gamma^{-1} = \sqrt{1 - \frac{1}{c^2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2} = \sqrt{1 - \frac{1}{c^2}\frac{a^2t^2}{1 + \left(\frac{at}{c}\right)^2}} \Rightarrow \tau = \frac{c}{a}\,\mathrm{asinh}\left(\frac{at}{c}\right) \Rightarrow t = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

which, using the properties of the hyperbolic functions

$$x = \frac{c^2}{a}\left[\cosh\left(\frac{a\tau}{c}\right) - 1\right], \quad t = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right)$$

Let's find coordinates $(\tau, \boldsymbol{\xi})$ such that the particle going with trajectory $(t(\tau), \boldsymbol{x}(\tau))$ is always at $(0, 0)$.

$$t = \left(\frac{c}{a} + \frac{\xi}{c}\right)\sinh\left(\frac{a\tau}{c}\right), \quad x = \left(\frac{c^2}{a} + \xi\right)\cosh\left(\frac{a\tau}{c}\right) - \frac{c^2}{a}$$

$(\tau, \boldsymbol{\xi})$ are called Rindler coordinates.

1.7.3 Flat spacetime in Rindler coordinates

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \qquad \text{(Minkowski)}$$

Apply the transformation above $t = \ldots\sinh(\ldots), x = \ldots$, we get

$$\mathrm{d}s^2 = -c\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \to \mathrm{d}s^2 = -\left(1 + \frac{a\xi}{c^2}\right)^2 c^2\mathrm{d}\tau^2 + \mathrm{d}\xi^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

Just use elementary calculus/arithmetic knowledge to derive it. This is "a" Rindler metric.

In Rindler, light

$$0 = -\left(1 + \frac{a\xi}{c^2}\right)^2 c^2\mathrm{d}\tau^2 + \mathrm{d}\xi^2 \Rightarrow \frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \left(1 + \frac{a\xi}{c^2}\right)c$$

So

$$0 < \frac{d\xi}{d\tau} < \infty$$

If we integrate it,

$$\xi(\tau) = \frac{c^2}{a}\left[\xi_c \exp\left(\frac{a\tau}{c}\right) - 1\right]$$

which is exponential trajectory.

1.8 Bell's Paradox

The rope breaks, all right!

Solve the Paradox in Assignment 1.

Problem A ship takes off from Earth on May 15th 2075 and travels per five years (as docks onboard measure) with constant acceleration $a = 1g = 9.8m \cdot s^2$. (also measured by instruments on board), then it slows down for another five years at the same rate, and returns in the same way (for a total trip time of 20 years). What is the distance the ship travelled? How long did the whole trip take as measure on Earth? (take $c = 3 \times 10^8 m/s$)

Bonus: If you were the ship's captain, would you believe it if, on arrival to Earth, somebody told you that while you were absent Earth was scheduled to be demolished to make way for a hyperspace bypass? Why?

Solution On Earth coordinate (t, x).

$$a(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t} = \frac{a'}{\gamma^3 u(t)} \implies dt = du\frac{\gamma^3(u)}{g} \implies t = \frac{1}{g}\int \frac{du}{\sqrt{1 - \left(\frac{u}{c}\right)^2}}$$

substitute in $\frac{u}{c} = \sin\theta, u(0) = 0$

$$\implies u(t) = \frac{gt}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}}$$

and

$$x(t) = \int dtu(t) = \int \frac{gt}{\sqrt{1 + \left(\frac{gt}{c}\right)^2}} dt = \frac{c^2}{g}\left(\sqrt{1 + \left(\frac{gt}{c}\right)^2} - 1\right)$$

$$\gamma(t) = \gamma(u(t)) = \sqrt{1 + g^2t^2}$$

These two implies

$$d\tau = \frac{dt}{\gamma(t)}, \qquad t = \frac{c}{g}\sinh\left(\frac{g^2}{c}\right)$$

Then

$$x(\tau) = x(t(\tau)) = \frac{c^2}{g}\left(\cosh\frac{g^2}{c} - 1\right)$$

Now plug all numbers in, we get

$$t(\tau = 5 \text{ years}) = 83.76\text{years}$$

$$x(t = 83.76 \text{ years}) = x(\tau = 5 \text{ years}) = 82.79 \text{ lightyears}$$

... to Bonus April 1st 2406

2

Differential Geometry

2.1 Differentiable manifolds

A chart is a pair $(\mathcal{U}, \varphi)$ where φ is a *homeomorphism* from an open subset $\mathcal{U} \subseteq \mathcal{M}$ and an open subset $\varphi(\mathcal{U}) \subseteq \mathbb{R}^n$.

An infinitely differentiable atlas (smooth atlas) is a set of continuous chart $\{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ and transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ such that their union covers the whole manifold and the transition functions are C^∞.

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

A differentiable manifold (C^∞) is a topological manifold $\mathcal{M}$ with a C^∞ atlas.

We will call coordinates $x^\mu = (x^1, \dots, x^n)$ of the point $p \in \mathcal{M}$ in a chart $(\mathcal{U}, \varphi)$, the coordinates of its image $p \in \mathcal{U}$, x^μ are the $\mathbb{R}^n$ coordinates of $\varphi(p) \in \varphi(\mathcal{U}) \subseteq \mathbb{R}^n$.

A function $f : \mathcal{M} \to \mathbb{R}$ is smooth iff the function $\overline{f} := f \circ \varphi^{-1} : \mathbb{R}^n \to \mathbb{R}$ is C^∞.

2.2 Vectors

Given a parametric curve $\gamma(s) \in \mathcal{M}, s \in \mathbb{R}$. We define the tangent vector to the curve γ at the point $p = \gamma(_0)$ as the operator $\mathsf{v}_{\gamma(s_0)}$ that assigns to each smooth function $f : \mathcal{M} \to \mathbb{R}$ the number

$$\mathsf{v}|_{\gamma(s_0)}(f) := \partial_s (f \circ \gamma)|_{s_0} \equiv \frac{\partial}{\partial s}(f \circ \gamma)\bigg|_{s_0}$$

$$\mathsf{v}|_{\gamma(s_0)}(f) = \partial_s(\overline{f}(y^\mu(s)))|_{s_0} = \partial_s y^\mu|_{s_0} \partial_\mu \overline{f}\big|_{y(s_0)} = \mathsf{v}|_{\gamma(s_0)} f = \partial_s y^\mu_{s_0} \partial_\mu \overline{f}|_{y^\mu_{s_0}}$$

where $\partial_\mu := \frac{\partial}{\partial y^\mu}$.

$$\mathsf{v}|_p = \underbrace{\partial_s y^\mu}_{v^\mu} \partial_\mu = v^\mu \Upsilon_\mu$$

The set $T_p\mathcal{M}$ of all the vectors v_p at the point $p \in \mathcal{M}$ is a vector space of the same dimension as the manifold. We call $T_p\mathcal{M}$ the tangent vector space to $\mathcal{M}$ at p.

Change of basis

$$\Upsilon'_\mu = \Lambda^\nu_\mu \Upsilon_\nu \quad \Leftrightarrow \quad \partial'_\mu = \Lambda^\nu_\mu \partial_\nu = \Lambda_\mu{}^\nu \frac{\partial}{\partial y^\nu}$$

where

$$\Lambda_\mu{}^\nu = \frac{\partial y^\nu}{\partial y'^\nu}$$

is Jacobian.

Vectors (elements of $T_p\mathcal{M}$)

- act on functions (of $\mathcal{M}$) return $\mathbb{R}$.
- are defined as a set of directional derivatives

A vector field over $\mathcal{M}$ is a set of vectors of the tangent space $T_p\mathcal{M}$ per each point $p \in \mathcal{M}$ such that their components in any coordinate basis are smooth.

Let v, w be vector fields on $\mathcal{M}$.

Linear map

$$(\mathsf{v} \circ \mathsf{w})(f) := \mathsf{v}[\mathsf{w}(f)]$$

$$\mathsf{v} = v^\mu_{(p)} \Upsilon_\mu$$

Then

$$(\mathsf{v} \circ \mathsf{w})(fg) = \ldots = f \cdot (\mathsf{v} \circ \mathsf{w})(g) + g \cdot (\mathsf{v} \circ \mathsf{w})(f) + \mathsf{w}(f) \cdot \mathsf{v}(g) + \mathsf{v}(f) \cdot \mathsf{w}(g)$$

By eliminating terms, we get

$$[\mathsf{v}, \mathsf{w}] := \mathsf{v} \circ \mathsf{w} - \mathsf{w} \circ \mathsf{v}$$

Lie bracket, commutator.

- Antisymmetric: $[\mathsf{v}, \mathsf{w}] = -[\mathsf{w}, \mathsf{v}]$
- Jacobi Identity: $[\mathsf{u}, [\mathsf{v}, \mathsf{w}]] + [\mathsf{w}, [\mathsf{u}, \mathsf{v}]] + [\mathsf{v}, [\mathsf{w}, \mathsf{u}]] = 0$

> Exercise
> Show that in a coordinate basis the composite of the commutator of two vector fields v, w are
>
> $$([\mathsf{v}, \mathsf{w}])^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu$$
>
> $$[\mathsf{v}, \mathsf{w}] = (v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu) \partial_\mu$$

Proof:

First see that these two are equivalent: $\mathsf{v} = \mathsf{v}^\mu \Upsilon_\mu = \mathsf{v}^\mu \partial_\mu$.

$$\begin{aligned}
[\mathsf{v}, \mathsf{w}]\, f &= \mathsf{v}[\mathsf{w}(\mathsf{f})] - \mathsf{w}[\mathsf{v}(\mathsf{f})] \\
&= v^\mu \partial_\mu (w^\nu \partial_\nu f) - w^\mu \partial_\mu (v^\nu \partial_\nu f) \\
&= v^\mu (\partial_\mu w^\nu) \partial_\nu f + v^\mu w^\nu \partial_\mu \partial_\nu f - w^\nu (\partial_\mu v^\nu) \partial_\nu f - w^\mu v^\nu \partial_\mu \partial_\nu f \\
&= v^\nu (\partial_\mu w^\nu) \partial_\nu f - w^\mu (\partial_\mu v^\nu) \partial_\nu f \\
&= [\mathsf{v}, \mathsf{w}] f
\end{aligned}$$

$$[\mathsf{v}, \mathsf{w}] = (v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu) \partial_\mu$$

$\square$

Coordinate Υ_μ are tangent vectors to coordinate curves $\{\partial_x, \partial_y, \partial_z\}$

2.2.1 Coordinate vs Non-coordinate basis

Non-coordinate basis Υ_μ[1] are not tangent vectors to coordinate curves $\{\partial_x, x\partial_y\}$.

a set of vector fields that span $T_p\mathcal{M}$ $\forall p \in \mathcal{M}$ and a coordinate basis iff they are mutually commute in $\mathcal{M}$.

Notation

- Υ^a: a, b, c for arbitrary basis
- Υ^μ: $\mu, \nu, \eta, \ldots$ for coordinate basis
- Υ^i: i, j, k spatial components

Exercise

Consider Cartesian Vs "Polar" coordinates in $\mathbb{R}^2 \setminus \{0\}$.

Note that $x = r\cos\theta, y = r\sin\theta$.

2 basis of $\mathbb{R}^2$ are $\begin{cases} \{\partial_x, \partial_y\} \\ \{\partial_r, \partial_\theta\} \end{cases}$

$$\begin{aligned}
\partial_r &= \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y = \cos\theta\partial_x + \sin\theta\partial_y \\
\partial_\theta &= \frac{\partial x}{\partial \theta}\partial_x + \frac{\partial y}{\partial \theta}\partial_y = -r\sin\theta + r\cos\theta\partial_y
\end{aligned}$$

Is $\{\partial_x, \partial_y\}$ a coordinate basis? Is $\{\partial_x, \partial_y\}$ a coordinate basis?

$$[\partial_x, \partial_y] = \partial_x\partial_y - \partial_y\partial_x = 0$$

[1]For simplicity, I write Υ instead of $\boldsymbol{\Upsilon}$. Readers should be clear on this.

which commutes. Let's check the other one.

$$\begin{aligned}[\partial_r, \partial_\theta] &= \partial_r\partial_\theta - \partial_\theta\partial_r \\ &= \underbrace{\partial_r(-r\sin\theta\partial_x + r\cos\theta\partial_y)}_{\partial_r\partial_\theta} - \underbrace{\partial_\theta(\cos\theta\partial_x + \sin\theta\partial_y)}_{\partial_\theta\partial_r} \\ &= -\sin\theta\partial_x + \cos\theta\partial_y - (-\sin\theta\partial_x + \cos\theta\partial_y) \\ &= 0\end{aligned}$$

Now let's define a new polar "coordinates"[a]

$$\begin{aligned}\Upsilon'_r &= \Upsilon_r = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \\ \Upsilon'_\theta &= \frac{1}{r}\Upsilon_\theta = \frac{1}{r}\begin{pmatrix}-r\sin\theta \\ r\cos\theta\end{pmatrix}\end{aligned}$$

Is $\{\Upsilon'_r, \Upsilon'_\theta\}$ a coordinate basis?

$$\begin{aligned}[\Upsilon'_r, \Upsilon'_\theta] &= \Upsilon'_r\Upsilon'_\theta - \Upsilon'_\theta\Upsilon'_r \\ &= \partial_r\frac{\partial_\theta}{r} - \frac{\partial_\theta\partial_r}{r} \\ &= \frac{\partial_r\partial_\theta}{r} - \frac{\partial_\theta}{r^2} - \frac{\partial_\theta\partial_r}{r} \\ &= -\frac{\partial_\theta}{r^2} \\ &\neq 0\end{aligned}$$

thus not coordinate basis after we normalize.

[a] Note the difference...

2.3 One-forms

linear functional over $T_p\mathcal{M}$.

$$\boldsymbol{\omega} : T_p\mathcal{M} \to \mathbb{R}, \qquad \boldsymbol{\omega} : \mathsf{v} \to \langle \boldsymbol{\omega}, \mathsf{v}\rangle \in \mathbb{R}$$

Given an arbitrary basis of $T_p\mathcal{M}$, $\{\Upsilon_a\}$ there is a unique set of one-forms $\{\Upsilon^a\}$ such that $\langle \Upsilon^a, \Upsilon_b\rangle = \delta^a_b$. The set $\{\Upsilon^a\}$ is linear independent and forms a basis (called the dual basis) of the vector space $T^*_p\mathcal{M}$ (cotangent vector space to $\mathcal{M}$ at p).

Given an arbitrary vector $\mathsf{v} = v^a\Upsilon_a$ and an arbitrary one-form $\boldsymbol{\omega} = \omega_a\Upsilon^a$.

$$\langle \boldsymbol{w}, \mathsf{v}\rangle = \left\langle \omega_a\Upsilon^a, v^b\Upsilon_b\right\rangle = \omega_a v^b \left\langle \Upsilon^a, \Upsilon_b\right\rangle = \omega_a v^b \delta^b_a = \omega_a v^a$$

Each function f over $\mathcal{M}$ defines a one-form $\mathrm{d}f|_p$ that we call the differential if f:

$$\langle df, \mathsf{v}\rangle := \mathsf{v}(f)$$

Given now the coordinate basis

$$\langle df, \Upsilon_\mu \rangle = \langle df, \partial_\mu \rangle = \partial_\mu f$$

Given coordinate function $x^\mu(p) = x^\mu$

$$\langle dx^\mu, \Upsilon_\mu \rangle = \langle dx^\mu, \partial_\nu \rangle = \partial_\nu x^\mu = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu$$

$$df = \langle df, \Upsilon_\mu \rangle \Upsilon^\nu = \partial_\mu f dx^\mu = \frac{\partial f}{\partial x^\mu} dx^\mu = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \dots$$

2.4 Tensors

A tensor T of type(r, s) is multilinear map that acts on $(T_p\mathcal{M})^{\times r}$ and $(T_p\mathcal{M})^{\times s}$ and return real number. We call T a s times covariant and r times contravariant tensor.

$$\mathsf{T}(\Upsilon^{a_1}, \dots, \Upsilon^{a_r}, \Upsilon_{b_1}, \dots, \Upsilon_{b_s}) := T^{a_1 \dots a_r}{}_{b_1 \dots b_s}$$

Example:

$$T'^{ab}_c = \Lambda_c{}^d \tilde{\Lambda}^a{}_e \tilde{\Lambda}^b{}_f T_d{}^{ef}$$

$$T'^{a'b'}_c = \Lambda'^{\,c}_c \tilde{\Lambda}^{a'}{}_a \Lambda^{b'}{}_b T_a{}^{ab}$$

$$\mathsf{T} = T^{\mu\nu} \Upsilon_\nu \otimes \Upsilon_\nu$$

$$\mathsf{T}(\boldsymbol{\omega}, \boldsymbol{\sigma}) = T(\omega_a \Upsilon^a, \sigma_b \Upsilon^b) = T^{ab} \omega_a \omega_b$$

T type (r, s), "eats r one-forms, and s vectors and spits out real numbers"

a tenser of type $(0, 1)$ eats one vector and returns a number is a one-form.

a tensor of type $(1, 0)$ eats one one-form and returns a number is a vector.

2.4.1 Some tensor operations

- – Symmetrization:

  $$T_{(a,b)} = \frac{1}{2}(T_{ab} + T_{ba})$$

  – Antisymmetrization:

  $$T_{[a,b]} = \frac{1}{2}(T_{ab} - T_{ba})$$

  $$T_{a[bCc]} = \frac{1}{2}[T_{abCc} - T_{acCb}]$$

  See the full formula in the notes.

- Contraction: $f = v^a w_a$

- Tensor product: $(\mathsf{v} \otimes \mathsf{w})^a{}_b = v^a w_b$

  $\mathsf{v} = v^a \Upsilon_a$, $\mathsf{w} = w_b \Upsilon^b$

  $$(v^a \Upsilon_a) \otimes (w_b \Upsilon^b) = v^a w_b \Upsilon_a \otimes \Upsilon^b$$

2.5 Smooth maps and Diffeomorphisms

2.5.1 Smooth maps

A map $\varphi : \mathcal{M} \to \mathcal{M}'$ is a smooth map iff given two atlases $\{\mathcal{U}_\alpha, \varphi_\alpha\}$, $\{\mathcal{U}'_\beta, \varphi'_\beta\}$ of $\mathcal{M}$ and $\mathcal{M}'$ respectively, the functions $\varphi'_\beta \circ \varphi \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ are smooth. In other words, given two sets of local coordinates for p, p' the coordinates of $p' = \varphi(p) \in \mathcal{M}'$, the image of $p \in \mathcal{M}$ are smooth functions of the coordinates of p.

The map φ indices a linear map φ^*, which we call pull-back between the space of functions $\mathcal{F}\{\mathcal{M}'\}$ (maps $p' \in \mathcal{M}'$ to $\mathbb{R}$) and $\mathcal{F}\{\mathcal{M}\}$ (maps $p \in \mathcal{M}$ to $\mathbb{R}$) according to the following rule:

> Given a function $f' : \mathcal{M}' \to \mathbb{R}$ we define its pull-back $\varphi^* f' : \mathcal{M} \to \mathbb{R}$ such that
> $$\varphi^* f'(p) := f' \circ \varphi(p) = f'(p') \in \mathbb{R}$$

Can we $\varphi_* f(p') := f \circ \varphi^{-1}(p') = f(p)$? No, since φ^{-1} may not exist.

The map φ does induce a push-forward between $T_p\mathcal{M}$ and $T_{p'}\mathcal{M}'$ according to the following rule:

> Given $\mathsf{v} \in T_p\mathcal{M}$ then $\varphi_*\mathsf{v} \in T_{p'}\mathcal{M}', p' = \varphi(p) \in M'$ such that its action on a $f' = \mathcal{F}\{\mathcal{M}\}$ as given by
> $$(\varphi_*\mathsf{v})\big|_{\varphi(p)}(f') := \mathsf{v}\big|_p(\varphi^* f')$$

The map φ induces a pull-back between $T^*_{\varphi(p)}\mathcal{M}'$ and $T^*_p\mathcal{M}$ according to this rule:

> Given a one-form $\boldsymbol{\omega}' \in T^*_{\varphi(p)}\mathcal{M}'$, then we define $\varphi^*\omega' \in T^*_P\mathcal{M}$ as the one-form whose action on vectors of $T_p\mathcal{M}$ is given by
> $$\langle \varphi^*\boldsymbol{\omega}', \mathsf{v}\rangle\big|_p := \langle \boldsymbol{\omega}', \varphi_*\mathsf{v}\rangle\big|_{\varphi(p)}$$

> Exercise
> $$\varphi^*(\mathrm{d}f')\big|_p = \mathrm{d}(\varphi^* f')\big|_{\varphi(p)}$$

2.5.2 Diffeomorphisms

A map $\varphi : \mathcal{M} \to \mathcal{M}'$ is a diffeomorphism if both φ and φ^{-1} are smooth bijection.

Let us focus on the diffeomorphisms mapping from $\mathcal{M}$ to itself. Let $\gamma_p(s)$ be a curve in $\mathcal{M}$ such that $\gamma_p(0) = p$, and whose tangent vector at every point $\gamma_p(s)$ is the vector field k, k is the "generator of (a set of) local diffeomorphisms $\varphi_s : p \in \mathcal{M} \to \varphi_s(p) \in \mathcal{M}'$ iff $\varphi_s(p) = \gamma_p(s)$.