Coding Theory

CO 331

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LaTeXed by Sibelius Peng

Preface

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Sibelius Peng

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Pre

Example: Replication code

source msgs		codewords
0	$\to$	0
1	$\to$	1

\# of errors/codeword that be detected: 0
\# errors/codeword that can be corrected: 0
Rate: 1

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source msgs		codewords
0	$\to$	00
1	$\to$	11

\# of errors/codeword that be detected: 1
\# errors/codeword that can be corrected: 0
Rate: 1/2

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source msgs		codewords
0	$\to$	000
1	$\to$	111

\# of errors/codeword that be detected: 2
\# errors/codeword that can be corrected: 1 (nearest neighbour decoding)
Rate: 1/3

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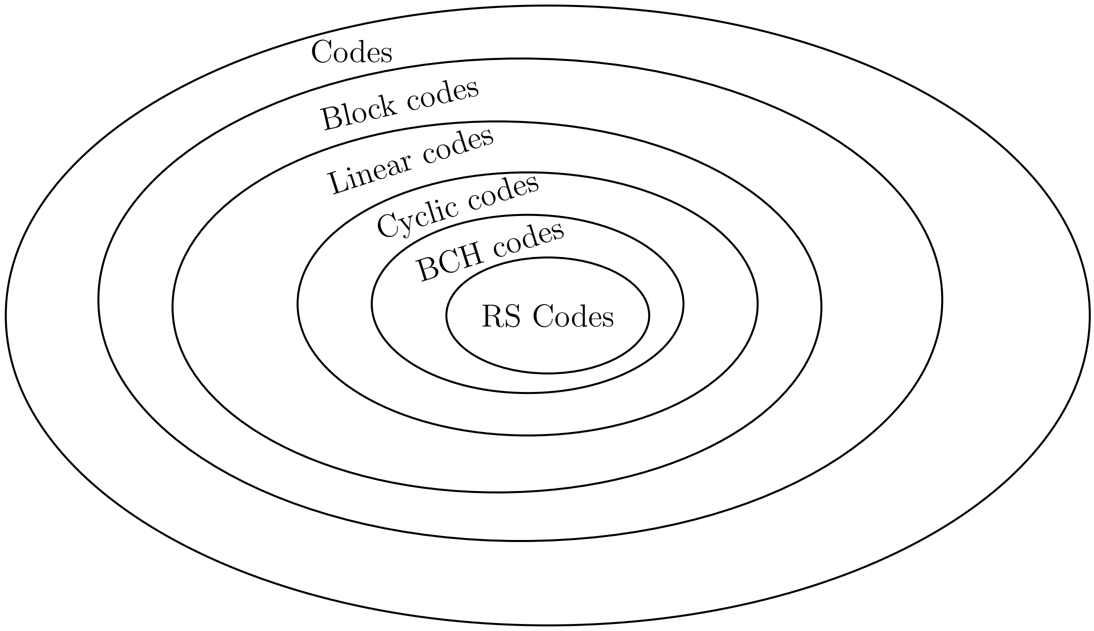
source msgs		codewords
0	$\to$	00000
1	$\to$	11111

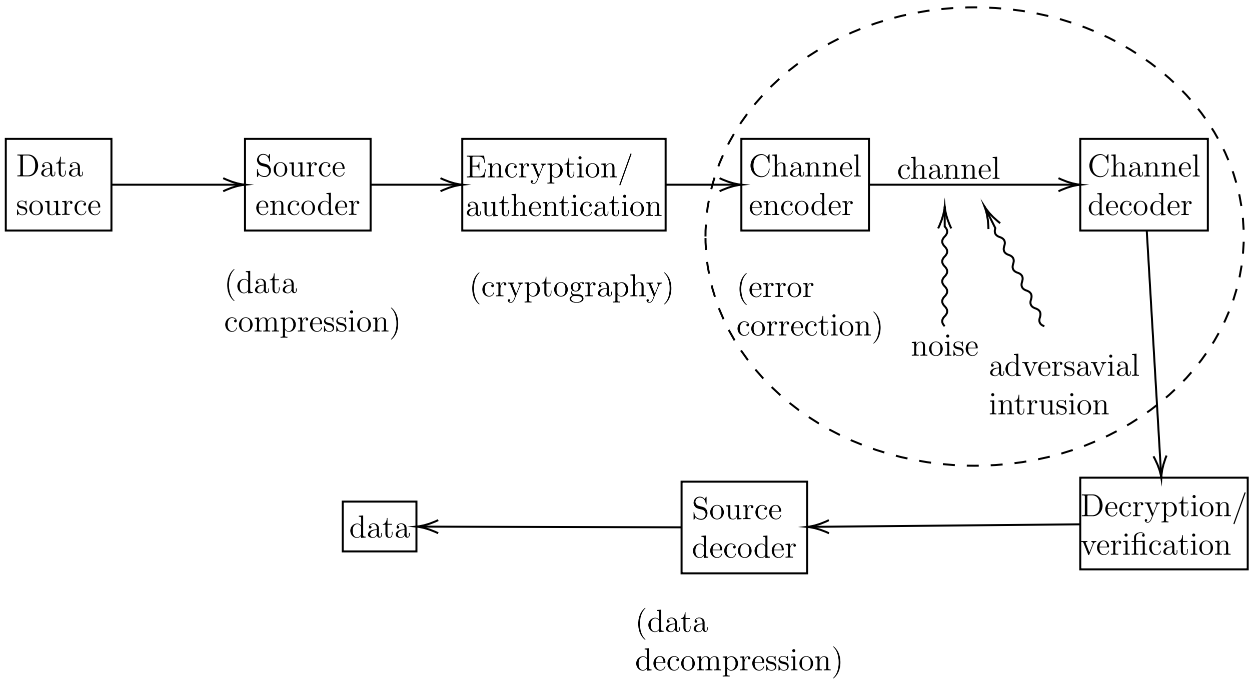
\# of errors/codeword that be detected: 4
\# errors/codeword that can be corrected: 2 (nearest neighbour decoding)
Rate: 1/5

Goal of Coding Theory Design codes so that:

1. High information rate
2. High error-correcting capability
3. Efficient encoding & decoding algorithms



The big picture In its broadest sense, coding deals with the reliable, efficient, secure transmission of data over channels that are subject to inadvertent noise and malicious intrusion.



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Introduction & Fundamentals

alphabet, word, length...

An *alphabet* A is a finite set of $q \geq 2$ symbols. E.g. $A = \{0, 1\}$.

A *word* is a finite sequence of symbols from A. (tuples or vectors)

The *length* of a word is the number of symbols in it.

A *code* C over A is a finite set of words over A (of size ≥ 2).

A *codeword* is a word in C.

A *block code* is a code where all codewords have the same length.

A block code C of length n containing M codewords over A is a subset $C \subseteq A^n$, with $|C| = M$. This is denoted by $[n, M]$.

Example:

$A = \{0, 1\}$. $C = \{00000, 11100, 00111, 10101\}$ is a $[5, 4]$-code over $\{0, 1\}$.

Messages		Codewords
00	$\to$	00000
10	$\to$	11100
01	$\to$	00111
11	$\to$	10101

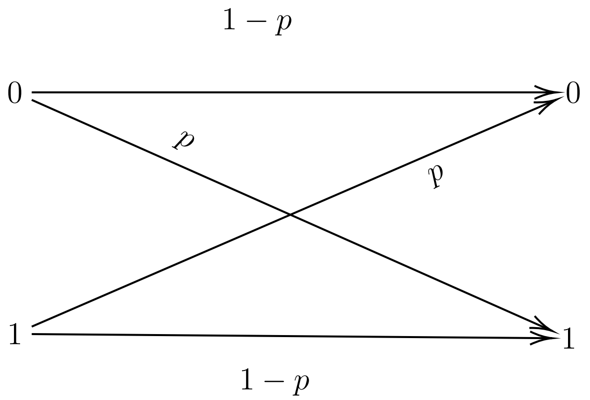
Encoding 1-1 map

The channel encoder transmits only codewords. But, what's received by the channel decoder might not be codeword.

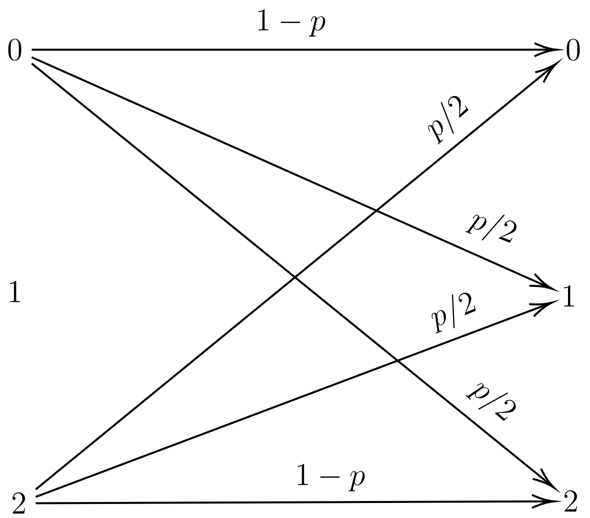
Example:

Suppose the channel decoder receives $r = 11001$. What should it do?

Example: $q = 2$ (Binary symmetric channel, BSC)



Example: $q = 3$



Assumptions about the communications channel

1) The channel only transmits symbols from A.

2) No symbols are deleted, added, or transposed.

3) (Errors are "random") Suppose the symbol transmitted are $X_1, X_2, X_3, \ldots$. Suppose the symbols received and $Y_1, Y_2, Y_3, \ldots$. Then for all $i \geq 1$, and all $i \leq j, k \leq q$,

$$Pr(Y_i = a_j | X_i = a_k) = \begin{cases} 1 - p, & \text{if } j = k \\ \frac{p}{q-1}, & \text{if } j \neq k \end{cases}$$

where $p =$ symbol error prob.

Notes about BSC

(i) If $p = 0$, the channel is perfect.

(ii) If $p = \frac{1}{2}$, the channel is useless.

(iii) If $1 \geq p > \frac{1}{2}$, then simply flip all bits that are received.

(iv) WLOG, we will assume that $0 < p < \frac{1}{2}$.

(v) Analogously, for a q-ary channel, we can assume that $0 < p < \frac{q-1}{q}$. (Optional exercise)

Hamming distance

If $x, y \in A^n$, the *Hamming distance* $d(x, y)$ is the # of coordinate positions in which x&y differ.

The *distance of a code C* is

$$d(C) = \min\{d(x, y) \in C, x \neq y\}$$

Example:

$$d(10111, 01010) = 4$$

Theorem 1.1

d is a metric. For all $x, y, z \in A^n$

(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$

(iii) $\triangle$ inequality $d(x, z) \leq d(x, y) + d(y, z)$

rate

The *rate* of an $[n, M]$-code C over A with $|A| = q$ is

$$R = \frac{\log_q M}{n}.$$

If the source messages are all k-tuples over A,

$$R = \frac{\log_q(q^k)}{n} = \frac{k}{n}.$$

Example:

$$C = \{00000, 11100, 00111, 10101\} \qquad A = \{0, 1\}$$

Here $R = \frac{2}{5}$ and $d(C) = 2$.

1.1 Decoding Strategy

Let C be an $[n, M]$-code over A of distance d. Suppose some codeword is transmitted, and $r \in A^n$ is received. The channel decoder has to decide the following:

(i) no errors have occurred, accept r.

(ii) errors have occurred, and (decode) correct r to some codeword.

(iii) errors has occurred, correction is not possible.

1.1.1 Nearest Neighbour Decoding

Incomplete Maximum Likelihood Decoding (IMLD). Correct r to the unique codeword c for which $d(r, c)$ is smallest. If c is not unique, reject r. Complete MLD (CMLD). Same as IMLD, accept ties are broken arbitrarily.

Question Is IMLD a reasonable strategy?

Theorem 1.2

IMLD selects the codeword c that maximizes $P(r|c)$ prob. that r is received given that c was sent.

Proof:
Suppose $c_1, c_2 \in C$ with $d(c_1, r) = d_1$ and $d(c_2, r) = d_2$. Suppose $d_1 > d_2$.

Now

$$P(r|c_1) = (1-p)^{n-d_1}\left(\frac{p}{q-1}\right)^{d_1}$$

and

$$P(r|c_2) = (1-p)^{n-d_2}\left(\frac{p}{q-1}\right)^{d_2}$$

So,

$$\frac{P(r|c_1)}{P(r|c_2)} = (1-p)^{d_2-d_1}\left(\frac{p}{q-1}\right)^{d_1-d_2} = \left(\frac{p}{(1-p)(q-1)}\right)^{d_1-d_2}$$

Recall

$$p < \frac{q-1}{q} \implies pq < q-1 \implies 0 < q - pq - 1$$

$$\implies p < p + q - pq - 1 \implies p < (1-p)(q-1) \implies \frac{p}{(1-p)(q-1)} < 1$$

Hence

$$\frac{P(r|c_1)}{P(r|c_2)} < 1$$

and so

$$P(r|c_1) < P(r|c_2)$$

□

The ideal strategy is to correct r to $c \in C$ that minimizes $P(c|r)$. This is Minimum error decoding (MED).

Example: (IMD is not the same as MED)

Let $C = \{\underbrace{000}_{c_1}, \underbrace{111}_{c_2}\}$. (corresponding to 0, 1).

Suppose $P(c_1) = 0.1, P(c_2) = 0.9$. Suppose $p = 1/4$ and $r = 100$.

IMLD $\quad r \to 000$

MED

$$\begin{aligned}
P(c_1|r) &= \frac{P(r|c_1) \cdot P(c_1)}{P(r)} \\
&= p(1-p)^2 \times 0.1/P(r) \\
&= \frac{9}{640 \cdot P(r)}
\end{aligned}$$

Similarly

$$\begin{aligned}
P(c_2|r) &= \frac{P(r|c_2) \cdot P(c_2)}{P(r)} \\
&= p(1-p)^2 \times 0.9/P(r) \\
&= \frac{27}{640 \cdot P(r)}
\end{aligned}$$

So MED: $r \to 111$

Note

1. IMLD: Select c. s.t. $P(r|c)$ is maximum
   MED: Select c. s.t. $P(c|r)$ is maximum

2. MED has the drawback that it requires knowledge of $P(c_i)$, $1 \leq i \leq M$

3. Suppose source messages are equally likely, so $P(c_i) = \frac{1}{M}$, for each $1 \leq i \leq M$. Then

   $$P(r|c_i) = P(c_i|r) \cdot P(c_i)/P(r) = P(c_i|r) \cdot \underbrace{\left[\frac{1}{M \cdot P(r)}\right]}_{\text{does not depend on } i}$$

   So IMLD is the same as MED.

4. In the remainder of the course, we will use IMLD/CMLD.

1.2 Error Correcting & Detecting Capabilities of a Code

- If C is used for error correction, the strategy is IMLD/CMLD.
- If C is used for error detection (only), the strategy is:

If $r \notin C$, then reject r; otherwise accept r.

e-error correcting code

A code C is called an *e-error correcting code* if the decoding always makes the correct decision if at most e errors per codeword are introduced. (Similarly: e-error detecting code)

Example:

$C = \{0000, 1111\}$ is 1-error correcting code, but not a 2-error correcting code.

$C = \{\underbrace{0\ldots0}_{m}, \underbrace{1\ldots1}_{m}\}$ is a $\lfloor \frac{m-1}{2} \rfloor$-error correcting code.

$C = \{0000, 1111\}$ is a 3-error detecting code.

Theorem 1.3

Suppose $d(C) = d$. Then C is a $(d-1)$-error detecting code.

Proof:
Suppose $c \in C$ is transmitted and r is received.

- If no error occur, then $r = c \in C$ and the decoder accepts r.
- If ≥ 1 and $\leq (d-1)$ errors occur, then $1 \leq d(r,c) \leq d-1$. So, $r \notin C$, and hence the decoder rejects r.

$\square$

Theorem 1.4

If $d(C) = d$, then C is not a d-error detecting code.

Proof:
Since $d(C) = d$, there exist $c_1, c_2 \in C$ with $d(c_1, c_2) = d$. If c_1 is sent, it is possible that d errors occur and c_2 is received. In this case, the decoder accepts c_2. $\square$

Theorem 1.5

If $d(C) = d$, then C is a $\lfloor \frac{d-1}{2} \rfloor$-error correcting code.

Proof:

Suppose $c \in C$ is transmitted, at most $\frac{d-1}{2}$ errors are introduced, and r is received. Let $c_1 \in C, c_1 \neq c$.

By $\triangle$ ineq, $d(c, c_1) \leq d(c, r) + d(r, c_1)$. So

$$d(r, c_1) \geq d(c, c_1) - d(c, r) \geq d - \frac{d-1}{2} = \frac{d+1}{2} \geq \frac{d-1}{2}$$

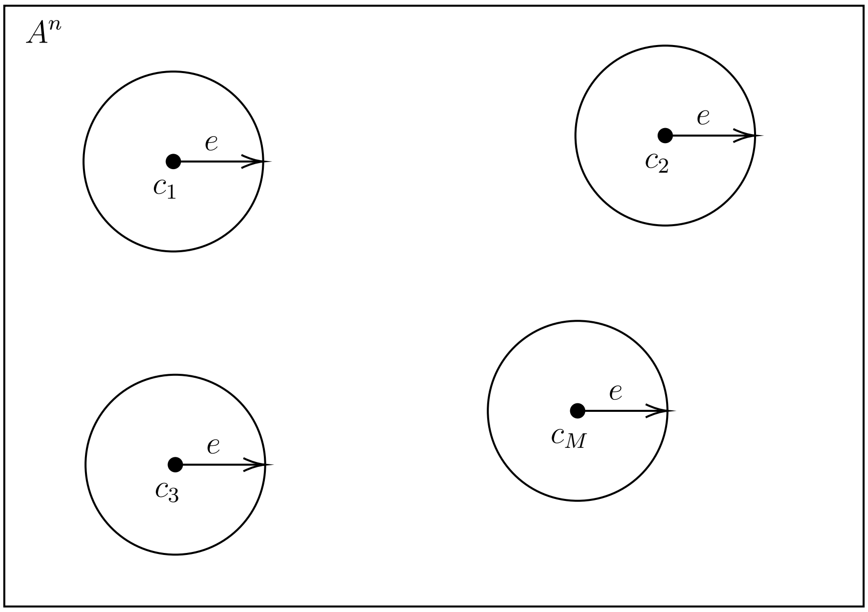
So c is the unique codeword closest to r.

So IMLD/CMLD will decode r to c. $\square$

Theorem 1.6

If $d(C) = d$, then C is not a $\left(\left\lfloor \frac{d-1}{2} \right\rfloor + 1\right)$-error correcting code.

Question Given q, n, M, d, does there exist an $[n, M]$-code C over A (with $|A| = q$), with $d(C) = d$?



$C = \{c_1, c_2, \ldots, c_M\}$. Let $e = \left\lfloor \frac{d-1}{2} \right\rfloor$. For $c \in C$, let S_c =sphere of radius e centered at $c = \{r \in A^n : d(r, c) \leq e\}$. We proved: If $c_1, c_2 \in C, c_1 \neq c_2$, then $S_{c_1} \cap S_{c_2} \neq \varnothing$. The question can be viewed as a *sphere packing problem*: Can we place M spheres of radius e in A^n (such that no 2 spheres overlap)? This is purely combinatorial problem.

Example:

Take $q = 2$, $n = 128$, $M = 2^{64}$, $d \geq 22$. Does a code with these parameters exist?

Answer YES.

Question What are the codewords?

Question How do we encode and decode efficiently?

Preview We'll view $\{0,1\}^{128}$ as a vector space of dimension 128 over $\mathbb{Z}_2$. We'll choose C to be a 64-dimensional subspace of this vector space.

2

Introduction to Finite Fields

field

A *field* $(F, +, \cdot)$ consists of a set F and two operations

$$+ : F \times F \to F$$

and

$$\cdot : F \times F \to F,$$

such that

(i) $a + (b + c) = (a + b) + c \quad \forall a, b, c \in F.$

(ii) $a + b = b + a, \quad \forall a, b \in F.$

(iii) $\exists 0 \in F$ such that $a + 0 = a$, $\forall a \in F$.

(iv) $\forall a \in F$, $\exists - a \in F$ such that $a + (-a) = 0$.

(v) $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall a, b, c \in F.$

(vi) $a \cdot b = b \cdot a, \quad \forall a, b \in F.$

(vii) $\exists 1 \in F, 1 \neq 0$, such that $a \cdot 1 = a \quad \forall a \in F.$

(viii) $\forall a \in F, a \neq 0, \exists a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.

(ix) $a \cdot (b + c) = a \cdot b + b \cdot c, \quad \forall a, b, c \in F.$

infinite, finite, order

A field F is *infinite* if $|F|$ is infinite. F is *finite* if $|F|$ is finite, in which case $|F|$ is the *order* of F.

Example:

$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are infinite fields. $\mathbb{Z}$ is *not* a field.

Q For what integers $n \geq 2$ do there exist finite fields of order n? if a field of order n exists, how do we "construct"?

Recall Let $n \geq 2$, the integers modulo n, $\mathbb{Z}_n$, is the set of all equivalent classes mod n,

$$\mathbb{Z}_n = \{[0], [1], [2], \ldots, [n-1]\}$$

where $[a] + [b] = [a+b]$, $\qquad [a] \cdot [b] = [a \cdot b]$.

More simply $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with addition & multiplication performed mod n.

Example:

$\mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

In $\mathbb{Z}_9, 5 + 7 = 3,\ 5 \cdot 7 = 8$.

Fact $\mathbb{Z}_n$ is a *commutative ring*. (i.e. field axioms (i)-(ix) are satisfied, except possibly (viii)).

Theorem 2.1

$\mathbb{Z}_n$ is a field if and only if n is prime.

Proof:

$(\Longleftarrow)$ Suppose n is prime. Let $a \in \mathbb{Z}_n, a \neq 0$ (so $1 \leq a \leq n-1$). Since n is prime, $\gcd(a, n) = 1$, so $\exists s, t \in \mathbb{Z}$ such that $as + nt = 1$. Reducing both sizes $\pmod{n}$, gives

$$as \equiv 1 \pmod{n}$$

So $a^{-1} = s$. So (viii) is satisfied, so $\mathbb{Z}_n$ is a field (of order n).

$(\Longrightarrow)$ Suppose n is composite, say $n = a \cdot b$. where $2 \leq a, b \leq n-1$. Suppose a^{-1} exists, $a^{-1} = s$. Then $as \equiv 1 \pmod{n}$. So

$$abs \equiv b \pmod{n},$$

so

$$ns \equiv b \pmod{n},$$

so $0 \equiv b \pmod{n}$, so $n | b$ which is impossible.

$\therefore a^{-1}$ does not exist, so $\mathbb{Z}_n$ is not a field.

$\square$

Q Do there exist finite fields of orders 4 and 6?

characteristic

The *characteristic* of a field denoted $\mathrm{char}(F)$, is the smallest positive integer m such that

$$\underbrace{1+1+1+\ldots+1}_{m}=0.$$

If no such m exists, then $\mathrm{char}(F)=0$.

Example:

$\mathrm{char}(\mathbb{Q})=0,\ \ \mathrm{char}(\mathbb{R})=0,\ \ \mathrm{char}(\mathbb{C})=0.$

$\mathrm{char}(\mathbb{Z}_p)=p$ (p is prime)

Theorem 2.2

If $\mathrm{char}(F)=0$, then F is infinite.

Proof:

Consider 1, 1+1, 1+1+1, 1+1+1+1,...

Then no 2 elements in this list are equal, because if

$$\underbrace{1+1+1+\ldots+1}_{a}=\underbrace{1+1+1+\ldots+1}_{b} \qquad \text{where } a<b$$

then $0=\underbrace{1+1+1+\ldots+1}_{b-a}$ which contradicts $\mathrm{char}(F)=0$.

So F is infinite. $\square$

Theorem 2.3

If F is a finite field, then $\mathrm{char}(F)$ is prime.

Proof:

Suppose $\mathrm{char}(F)=m$, which is composite. Say, $m=a\cdot b$, where $2\le a,b\le m-1$. Now $(\underbrace{1+1+1+\ldots+1}_{a})\cdot(\underbrace{1+1+1+\ldots+1}_{b})=\underbrace{1+1+1+\ldots+1}_{m}=0$ since $\mathrm{char}(F)=m$.

Let $\underbrace{1+\ldots+1}_{a}=s$ and $\underbrace{1+\ldots+1}_{b}=t$, so $s\cdot t=0$.

But $s\neq 0$, and so s^{-1} exists, thus $s^{-1}\cdot s\cdot t=0$, therefore $t=0$, which contradicts $\mathrm{char}(F)=m$. $\square$

Next class Let F be a finite field of order n. Then $\mathrm{char}(F)=p$ (prime). Then $\mathbb{Z}_p$ is a "subfield" of F. And F is a vector space over $\mathbb{Z}_p$ say of dimension k. Then order of F is p^k.

2.1 Non-existence of finite fields

Let F be a finite field of characteristic p. Consider

$$E = \{0, 1, 1+1, 1+1+1, \ldots, \underbrace{1+1+1+\ldots+1}_{p-1}\} \subseteq F$$

Check: E is a field w.r.t the field operations of F. Also, E has order p. If we label the elements of E in a natural way

$$1+1 \leftrightarrow 2, 1+1+1 \leftrightarrow, \ldots, \underbrace{1+1+1+\ldots+1}_{p-1} \leftrightarrow p-1,$$

then E is really just $\mathbb{Z}_p$. (E is *isomorphic* to $\mathbb{Z}_p$).

Theorem 2.4

If F be a finite field of order n, then $\mathrm{char}(F) = p$ (prime). Then $\mathbb{Z}_p$ is a "subfield" of F.

So let's identify:

$$\begin{aligned} \text{elements of } F &\leftrightarrow \text{vectors} \\ \text{elements of } \mathbb{Z}_p &\leftrightarrow \text{scalars} \\ \text{addition in } F &\leftrightarrow \text{vector addition} \\ \text{multiplication in } F &\leftrightarrow \text{scalar multiplication} \end{aligned}$$

Theorem 2.5

If F is a finite char P, then F is a vector space over $\mathbb{Z}_p$.

Proof:
Read Appendix A (of the textbook). □

Theorem 2.6

If F is a finite field of char P, then order of F is p^n for some $n \geq 1$.

Proof:
Let n be the dimension of (the vector space) F over $\mathbb{Z}_p$. Let $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis. Then every element in F can be written uniquely as

$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n, \tag{*}$$

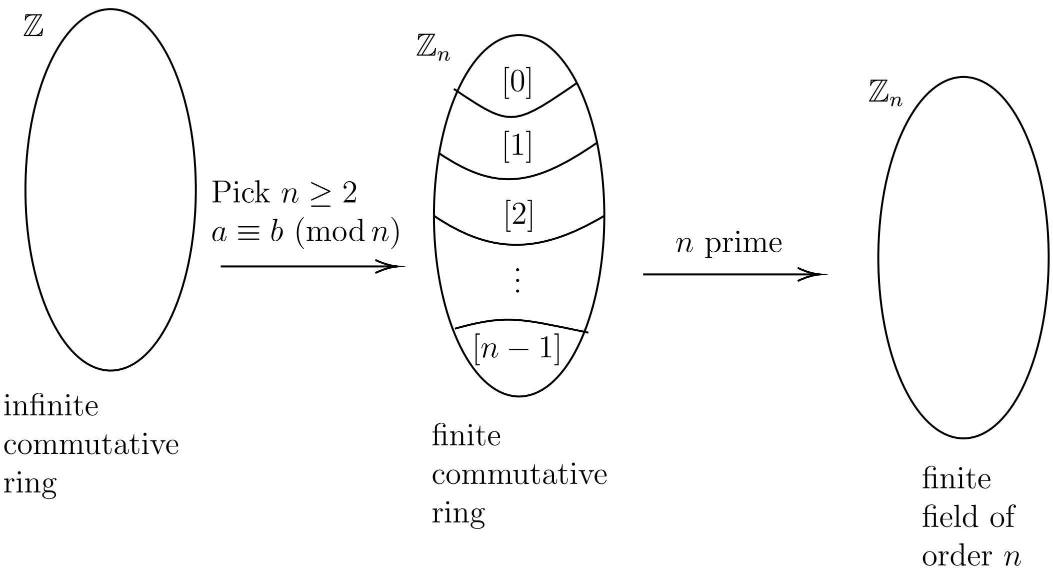
where $c_i \in \mathbb{Z}_p$.

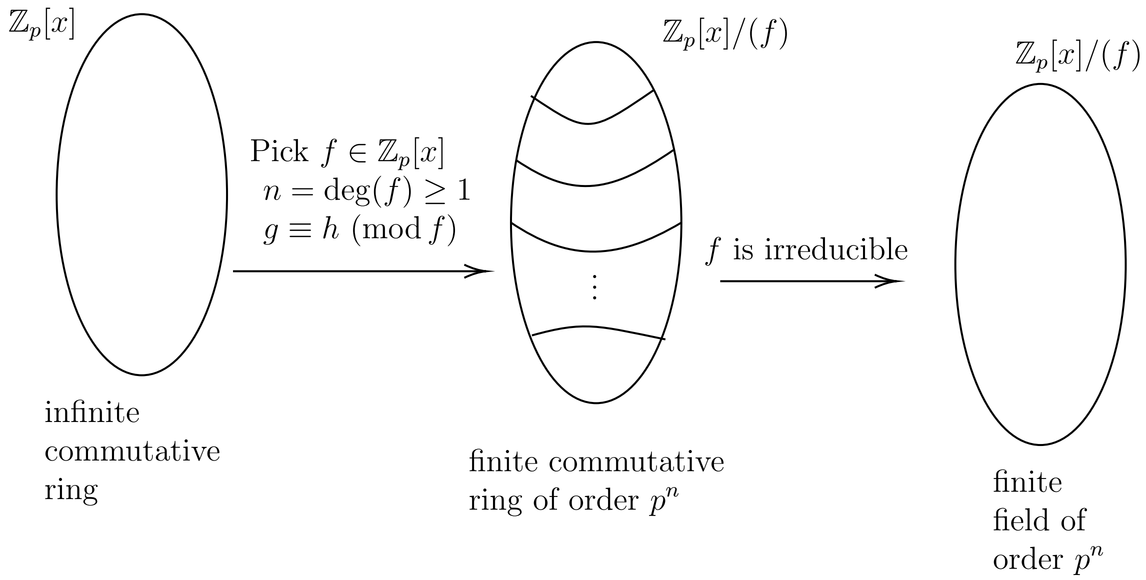
Also every element (*) is in F. Hence $\mathrm{ord}(F) = p^n$. □

Example:

There is no field of order 6.

Q Is there a finite field of order 4? 8? 9? Yes.





F[x]

If F is a field, then $F[x]$ is the set of all polynomials in x with coefficients from F.

Addition and multiplication is done in the usual way, with coefficient arithmetic in F.

Example:

In $\mathbb{Z}_{11}[x]$, $(2 + 5x + 6x^2) + (3 + 9x + 5x^2) = 5 + 3x$.

Theorem 2.7

$F[x]$ is an infinite commutative ring.

Some notations

Let $f \in F[x]$, $\deg(f) \geq 1$.

If $g, h \in F[x]$, we write $g \equiv h \pmod f$.

If $g - h = \ell f$ for some $\ell \in F[x]$, we write $(f | g - h)$.

Facts

1. $\equiv$ is an equivalence relation.
2. The equivalence class containing $g \in F[x]$ is
$$[g] = \{h \equiv g \pmod f : h \in F[x]\}$$
3. We define $[g_1] + [g_2] = [g_1 + g_2] \qquad [g_1] \cdot [g_2] = [g_1 \cdot g_2]$
4. The set of all equivalence classes, denoted $F[x]/(f)$ (where $f \in F[x], \deg(f) \geq 1$) is a commutative ring.
5. The polynomials in $F[x]$ of degree $< \deg(f)$ are a system of distinct representatives of the equivalence classes in $F[x]/(f)$.

Justification Let $g \in F[x]$. By division algorithm for polynomials, we can write $g = \ell f + r$ where $\deg(r) < \deg(f)$. [Convention: $\deg(0) = -\infty$]

Then $g - r = \ell f$. So $g \equiv r \pmod f$. So $[g] = [r]$.

Also if $r_1, r_2 \in F[x], r_1 \neq r_2$ and $\deg(r_1), \deg(r_2) < \deg(f)$, then $f \nmid r_1 - r_2$, so $r_1 \not\equiv r_2 \pmod f$. Hence $[r_1] \neq [r_2]$.

2.2 Constructing finite fields

We proved A system of distinct representatives for $\mathbb{Z}_p[x]/(f)$ is $[r(x)] : r \in \mathbb{Z}_p[x], \deg(r) < \deg(f)$. Therefore, $|\mathbb{Z}_p[x]/(f)| = p^n$.

irreducible

Let F be a field and $f(x) \in F[x]$ of degree $n \geq 1$. Then f is *irreducible (over F)* if f cannot be written as $f = gh$, where $g, h \in F[x]$ and $\deg(g), \deg(h) \geq 1$.

Example:

$x^2 + 1$ is irreducible over $\mathbb{R}$.

$x^2 + 1$ is reducible over $\mathbb{C}$, since $(x^2 + 1) = (x + i)(x - i)$.

$x^2 + 1$ is reducible over $\mathbb{Z}_2$, since $x^2 + 1 = (x + 1)^2$.

$x^2 + 1$ is irreducible over $\mathbb{Z}_3$.

Theorem 2.8

Let F be a field, and $f \in F[x]$ of degree $n \geq 1$. Then $F[x]/(f)$ is a field if and only if f irreducible over F.

Proof:

$F[x]/(f)$ is a commutative ring.

($\Longleftarrow$) Suppose $g \in F[x]/(f)$, $g \neq 0$, (and $\deg(g) < \deg(f)$). Then $\gcd(g, f) = 1$, and by the EEA for polynomials, there exist $s, t \in F[x]$ such that $gs + ft = 1$. Reducing both sides mod f gives $gs \equiv \pmod f$. So $g^{-1} = s$. Hence $F[x]/(f)$ is a field.

($\Longrightarrow$) Exercise.

$\square$

So, to construct a finite field of order $p^n (n \geq 2)$, we need an irreducible polynomial $f \in \mathbb{Z}_p[x]$ of degree n. Then $\mathbb{Z}_p[x]/(f)$ is a finite field of order p^n.

Fact For any prime p, integer $n \geq 2$, there exists an irreducible polynomial degree n in $\mathbb{Z}_p[x]$.

Theorem 2.9

There exists a finite field of order q iff q is a prime power.

Example: Construct a finite field of order 4.

Take $f(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$, which is irreducible over $\mathbb{Z}_2$. So, the field is $\mathbb{Z}_2[x]/(x^2 + x + 1) = \{0, 1, x, x + 1\}$.

- $x + (x + 1) = 1$.
- $x \cdot (x + 1) = x^2 + x = 1$.
- So, $x^{-1} = x + 1$.
- $1^{-1} = 1$
- $x^{-1} = x + 1$
- $(x + 1)^{-1} = x$

Example: Field of order $8 = 2^3$

We need an irreducible polynomial of degree 3 over $\mathbb{Z}_2$. Take $f(x) = x^3 + x + 1$ which is irreducible over $\mathbb{Z}_2$. Then a field of order 8 is

$$F_1 = \mathbb{Z}_2[x]/(x^3 + x + 1) = \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$$

- $x^2 + (x^2 + x + 1) = x + 1$
- $x^2 \cdot (x^2 + x + 1) = x^4 + x^3 + x^2 = 1$.

$$\begin{array}{r|l}
 & \qquad\qquad\quad x + 1 \\ \hline
x^3 + x + 1 & x^4 + x^3 + x^2 \\
 & -x^4 \qquad - x^2 \; - x \\ \cline{2-2}
 & \qquad x^3 \qquad\; - x \\
 & \quad\; - x^3 \qquad - x - 1 \\ \cline{2-2}
 & \qquad\qquad\; - 2x - 1
\end{array}$$

- $(x^2)^{-1} = x^2 + x + 1$
- $x^{-1} = x^2 + 1$

Example: Finite field of order 8

Take $f_2(x) = f(x) = x^3 + x^2 + 1$. Then $F_2 = \mathbb{Z}_2[x]/(x^3 + x^2 + 1)$ is a finite field of order 8. Its elements are $F_2 = \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$.

- $x^{-1} = x^2 + x$.

Note

F_1 and F_2 are two different field of order 8. In fact, they are "essentially the same", i.e., they are *isomorphic*, i.e., there is a bijection $\alpha : F_1 \to F_2$ such that $\alpha(a + b) = \alpha(a) + \alpha(b)$ and $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b)$, $\forall a, b \in F$.

Fact Any two fields of order q are isomorphic.

GF(q)

We will denote *the* finite field of order q by GF(q).

We saw two different representations of GF(2^3).

Recall A finite field of order q exists iff $q = p^n$ for some prime p and $n \geq 1$. (p =characteristic)

- Also GF(q) $= \mathbb{Z}_p[x]/(f)$, where $f \in \mathbb{Z}_p[x]$ is irreducible and has degree n.

Example: Construct GF(16)

Take $f(x) = x^4 + x + 1 \in \mathbb{Z}_2[x]$.

f has no roots in $\mathbb{Z}_2$, and hence no linear factors.

Long division shows that $x^2+x+1 \nmid x^4+x+1$, so f has no irreducible quadratic factor.

$\therefore$ f is irreducible over $\mathbb{Z}_2$. So $\mathrm{GF}(16) = \mathbb{Z}_2[x]/(x^4+x+1)$.

2.3 Properties of finite fields

Theorem 2.10: Frosh's Dream

Let $\alpha, \beta \in \mathrm{GF}(q)$, where $\mathrm{char}(\mathrm{GF}(q)) = p$. Then $(\alpha+\beta)^p = \alpha^p + \beta^p$.

Proof:

$$(\alpha+\beta)^p = \alpha^p + \sum_{i=1}^{p-1} \binom{p}{i} \alpha^i \beta^{p-i} + \beta^p$$

Now, $\displaystyle\binom{p}{i} = \frac{p(p-1)\ldots(p-i+1)}{1\cdot 2\cdot \ldots \cdot i} \in \mathbb{N}$.

If $1 \le i \le p-1$, then $p|$ numerator; but $p \nmid$ denominator. $\therefore p|\binom{p}{i}$. So,

$$\begin{aligned}
\binom{p}{i}\alpha^i\beta^{p-i} &= \underbrace{\alpha^i\beta^{p-i} + \ldots + \alpha^i\beta^{p-i}}_{\binom{p}{i}} \\
&= \alpha^i\beta^{p-i}(\underbrace{1+1+1+\ldots+1}_{\binom{p}{i}}) \\
&= \alpha^i\beta^{p-i}\cdot 0 \qquad \text{since char} = p \text{ and } p \left| \binom{p}{i} \right. \\
&= 0
\end{aligned}$$

$\square$

Note

More generally,

$$(\alpha+\beta)^{p^m} = \alpha^{p^m} + \beta^{p^m}$$

for all $m \ge 1$.

Theorem 2.11

Let $\alpha \in \mathrm{GF}(q)$. Then $\alpha^q = \alpha$.

Proof:

- If $\alpha = 0$, then of course $\alpha^q = \alpha$.
- Suppose $\alpha \neq 0$. Let $\alpha_1, \ldots, \alpha_{q-1}$ be the nonzero elements in $\mathrm{GF}(q)$. Consider $\alpha\alpha_1, \ldots, \alpha\alpha_{q-1}$. The elements in this list are pairwise distinct because

if $\alpha\alpha_i = \alpha\alpha_j \quad (i \neq j)$, then $\alpha^{-1}\alpha\alpha_i = \alpha^{-1}\alpha\alpha_j$, so $\alpha_i = \alpha_j$. Also

$$\alpha\alpha_i \neq 0, \quad \forall 1 \leq i \leq q-1.$$

Hence

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{q-1}\} = \{\alpha\alpha_1, \ldots, \alpha\alpha_{q-1}\}$$

$\therefore \alpha_1 \ldots \alpha_{q-1} = (\alpha\alpha_1) \ldots (\alpha\alpha_{q-1})$

$\therefore \alpha^{q-1} = 1$

$\therefore \alpha^q = \alpha$

□

GF$(q)^*$

Let $\mathrm{GF}(q)^* = \mathrm{GF}(q) \setminus \{0\}$.

ord(alpha)

Let $\alpha \in \mathrm{GF}(q)^*$. The order of α, denoted $\mathrm{ord}(\alpha)$, is the smallest, positive integer t such that $\alpha^t = 1$.

Example:
How many elements of order 1 are there in GF(q)?

$\alpha = 1$

Example:
Find $\mathrm{ord}(x)$ in $\mathrm{GF}(16) = \mathbb{Z}_2[x]/(x^4 + x + 1)$.

$x^1 = 1, x^2 = x^2, x^3 = x^3, x^4 = x + 1, x^5 = x^2 + x, \ldots, x^{15} = 1.$

Since $\mathrm{ord}(x) \neq 1, 3, 5, \mathrm{ord}(x) | 15$, we have $\mathrm{ord}(x) = 15$.

Lemma 2.12

Let $\alpha \in \mathrm{GF}(q)^*, \mathrm{ord}(\alpha) = t, s \in \mathbb{Z}$. $\alpha^s = 1 \iff t|s$.

Proof:
Let $s \in \mathbb{Z}$. Long division g gives $s = \ell t + r$, where $0 \leq r \leq t-1$.

Then $\alpha^s = \alpha^{\ell t + r} = (\alpha^t)^\ell \alpha^r = \alpha^r$.

So

$$\begin{aligned} \alpha^s = 1 &\iff \alpha^r = 1 \\ &\iff r = 0 \qquad \text{since } 0 \leq r \leq t-1 \\ &\iff t|s \end{aligned}$$

□

Corrollary 2.13

If $\alpha \in \mathrm{GF}(q)^*$, then $\mathrm{ord}(\alpha)|q-1$.

Proof:
We know that $\alpha^{q-1} = 1$. So $\mathrm{ord}(\alpha)|q-1$ by previous lemma. □

generator

An element $\alpha \in \mathrm{GF}(q)$ is a *generator of* $\mathrm{GF}(q)^*$ (primitive element in $\mathrm{GF}(q)$). If $\mathrm{ord}(\alpha) = q-1$.

Lemma 2.14

If α is a generator of $\mathrm{GF}(q)^*$ then $\{\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{q-2}\} = \mathrm{GF}(q)^*$.

Lemma 2.15

If $\alpha \in \mathrm{GF}(q)^*$ has order t, then $\alpha^0, \alpha^1, \ldots, \alpha^{t-1}$ are pairwise distinct.

Proof:
Suppose $\alpha^i = \alpha^j$, where $0 \leq i < j \leq t-1$. Then $\alpha^{j-1} = 1$ which contradicts $\mathrm{ord}(\alpha) = t$ since $1 \leq j - i \leq t-1$. □

So, if α is a generator of $\mathrm{GF}(q)^*$ then $\{\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{q-2}\} = \mathrm{GF}(q)^*$.

Theorem 2.16

$\mathrm{GF}(q)^*$ has at least one generator.

Proof:
See LEARN (optional). □

Example:
Find a generator of $\mathrm{GF}(8) = \mathbb{Z}_2[x]/(x^3+x+1)$.

x is a generator.

3

Linear Codes

Let $F = \mathrm{GF}(q)$.

Let $V_n(F) = F \times F \times \ldots \times F = F^n$

Then $V_n(F)$ is an n-dimensional vector space over F.

We have $|V_n(F)| = q^n$.

linear (n, k)-code over F

A *linear (n, k)-code over F* is a k-dimensional subspace of $V_n(F)$.

subspace

A subspace of of a vector space V over F is a subset $S \subseteq V$ such that

(i) $S \neq \varnothing$.

(ii) $v_1 + v_2 \in S \qquad \forall v_1, v_2 \in S$.

(iii) $\lambda v \in S, \qquad \forall v \in S, \lambda \in F$.

Note
S is also a vector space over F.

$0 \in S$.

3.1 Properties of Linear Codes

Let C be an (n, k)-code over F. Let $v_1, v_2, \ldots, v_k$ be an ordered basis for C.

1) The codewords in C are precisely:

$$mv_1 + m_2v_2 + \ldots + m_kv_k,$$

where $m_i \in F$.

So $|C| = M = q^k$.

2) The rate of C is $R = \frac{\log_q M}{n} = \frac{k}{n}$,

3) Distance

weight

The (Hamming)*weight* of $v \in V_n(F)$, $\omega(v)$, is the number of nonzero coordinate positions in vv.

The weight of C is $\omega(C) = \min\{\omega(c) : c \in C, c \neq 0\}$.

Theorem 3.1

If C is a linear code, then $d(C) = \omega(C)$.

Proof:

$$\begin{aligned} d(C) &= \min\{d(x,y) : x, y \in C, x \neq y\} \\ &= \min\{\omega(x-y) : x, y \in C, x \neq y\} \\ &= \min\{\omega(c) : c \in C, c \neq 0\} \\ &= \omega(C) \end{aligned}$$

□

4) Encoding.

Since $M = q^k$, there are q^k source messages. We'll assume that the source messages are elements of $V_k(F)$. A natural encoding rule is: Given $(m_1, m_2, \ldots, m_k) \in V_n(F)$. We will encode it as $c = m_1v_1 + m_2v_2 + \ldots + m_kv_k$.

Note
The encoding rule depends on the basis chosen for C.

5) Note if $m = (m_1, \ldots, m_k)$, then the encoding rule can be written as follows.

$$c = (m_1, m_2, \ldots, m_k) \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_k & - \end{bmatrix}_{k \times n}$$

$$c = mG$$

generator matrix

Let C be an (n,k) code. A *generator matrix* G for C is a $k \times n$ matrix whose rows form a basic for C.

Note
An encoding rule for C w.r.t. G is $c = mG$.

Note
Performing elementary row operations on G gives a different matrix for the same code C.

Example: Consider a binary $(5,3)$-code C

where binary means "over $F = \mathrm{GF}(2) = \mathbb{Z}_2$. 5 is n, length of code. 3 is k, dimension.

Then $M = q^k = 2^3$ and $R = \frac{k}{n} = \frac{3}{5}$. and

$$C = \langle \underbrace{10010}_{v_1}, \underbrace{01011}_{v_2}, \underbrace{00101}_{v_3} \rangle$$

$$G = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]_{3\times 5}$$

indeed has rank 3 so G is *a* GM for C.

Encoding rule is $c = mG$.

m source msgs	$\to$	c codewords
000	$\to$	00000
001	$\to$	00101
010	$\to$	01011
011	$\to$	01110
100	$\to$	10010
101	$\to$	10111
110	$\to$	11001
111	$\to$	11100

$d(C) = 2, e = 0$

Note
Any matrix row equivalent to G is also a GM for C, but yields a different encoding rule.

systematic, standard form

Let matrix $[I_k|A]_{k\times n}$ is a GM for an (n,k)-code C. If an (n,k)-code has a GM of this form, then C is *systematic*, and the GM is in *standard form.*

Example:
$C = \langle 100011, 101010, 100110 \rangle$ is a non-systematic $(6,3)$-code. A GM for C is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Another GM for C is

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Another GM for C:

$$G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

C is not systematic.

However, if every codeword is permuted by moving the second bit to a new fourth bit, then we get a new code C' that is linear, and has the same n, k, d as C.

equivalent

Let C be an (n,k)-code. If π is a permutation on $\{1, 2, \ldots, n\}$, Then $\pi(C)$[a] is an (n,k)-code and is said to be *equivalent* to C.

[a] i.e. apply π to each codeword

Fact

1. If C, C' are equivalent codes, then $d(C) = d(C')$.

2. Every linear code is equivalent to a systematic code.

   Proof:
   Let C be an (n,k)-code. Let G be a GM for C in row reduced form. Then one can permute to columns of G to get a matrix $G' = [I_k | A]$ in standard form.

   Then G' is a GM for a code C' that is equivalent to C. $\square$

3.2 Dual Codes

inner product

Let $x, y \in V_n(F)$. The *inner product* of x and y is

$$x\dot{y} = \sum_{i=1}^{n} x_i y_i \in F$$

Properties For all $x, y, z \in V_n(F)$ and all $\lambda \in F$

1. $x \cdot y = y \cdot x$
2. $x \cdot (y + z) = x \cdot y + x \cdot z$
3. $(\lambda x) \cdot y = \lambda(x \cdot y)$
4. $x \cdot x = 0$ does **not** imply that $x = 0$.

Example:

Consider $V_2(\mathbb{Z}_2)$

Then $(1, 1) \cdot (1, 1) = 0$.

dual code

Let C be an (n, k)-code over F. The *dual code* of C is

$$C^\perp = \{x \in V_n(F) : x \cdot c = 0, \quad \forall c \in C\}$$

orthogonal

If $x, y \in V_n(F)$ and $x \cdot y = 0$, then x, y are *orthogonal*.

Theorem 3.2

If C is an (n, k)-code over F, then $C^\perp$ is an $(n, n - k)$-code over F.

Proof:
Let $v_1, v_2, \ldots, v_k$ be a basis for C.

Claim Let $x \in V_n(F)$. Then $x \in C^\perp$ iff $v_1 \cdot x = v_2 \cdot x = \ldots = v_k \cdot x = 0$.

$(\implies)$ If $x \in C^\perp$, then $x \cdot c \ \ \forall c \in C$. In particular, $x \cdot v_1 = 0, \ldots, x \cdot v_k = 0$.

$(\impliedby)$ Suppose $x \cdot v_1 = x \cdot v_2 = \ldots = x \cdot v_k = 0$. Let $c \in C$. We can write

$$c = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k, \qquad v_i \in F$$

Then $x \cdot c = \lambda_1(x \cdot v_1) + \ldots + \lambda_k(x \cdot v_k) = 0$. Hence $x \in C^\perp$.

Consider

$$G = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}_{k \times n}$$

Then $x \in C^\perp$ iff $Gx^T = 0$. So $C^\perp$ is the nullspace of G. Hence $C^\perp$ is an $(n-k)$-dimensional subspace of $V_n(F)$. □

Theorem 3.3

If C is a linear code, then $(C^\perp)^\perp = C$.

Proof:
Let C be an (n, k)-code, then $C^\perp$ is an $(n, n-k)$-code. So $(C^\perp)^\perp$ is an (n, k)-code. But $C \subseteq (C^\perp)^\perp$ by definition of $C^\perp$.

Suppose C is a code over $F = \mathrm{GF}(q)$.Then $|C| = q^k$ and $|(C^\perp)^\perp| = q^k$.

$\therefore C = (C^\perp)^\perp$. □

Theorem 3.4: Constructing a GM for $C^\perp$

Let C be an (n, k)-code with GM $G = \left[I_k | A_{k \times (n-k)}\right]_{k \times n}$. Then *a* GM for $C^\perp$ is

$$H = \left[-A^T | I_{n-k}\right]_{(n-k) \times n}$$

Proof:
$\mathrm{rank}(H) = n - k$, so H is indeed a GM for some $(n, n-k)$-code $\overline{C}$.

Now,

$$GH^T = [I_k | A] \left[\frac{-A}{I_{n-k}}\right] = -A + A = 0$$

Since $GH^T = 0$, every row of H is orthogonal to every row of G. So, every vector in the row space of H is orthogonal to every vector in the row space of G. Hence $\overline{C} \subseteq C^\perp$. Since $\dim(\overline{C}) = \dim(C^\perp)$, we have $\overline{C} = C^\perp$. □

parity-check matrix

A GM for $C^\perp$ is called a *parity-check matrix* (PCM) for C.

Example:

Consider a $(5,2)$-code C over $\mathbb{Z}_3$ with GM

$$G = \begin{bmatrix} 2 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}_{2\times 5}$$

For C: $q = 3, n = 5, k = 2, M = 3^2 = 9$.

$$C = \{00000, \underset{c_1}{20210}, \underset{2c_1}{10120}, \underset{c_2}{11001}, \underset{2c_2}{22002}, \underset{c_1+c_2}{01211}, \underset{c_1+2c_2}{12212}, \underset{2c_1+c_2}{21121}, \underset{2c_1+2c_2}{02122}\}$$

Now find a GM for $C^\perp$

$$\begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row reductions}} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

So,

$$H = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

is a GM for $C^\perp$ which is an $(5,3)$-code over $\mathbb{Z}_3$.

Note

Let C be an (n,k)-code over F with GM G:

1. $C^\perp$ is the nullspace of G.
2. $C^\perp$ is an $(n, n-k)$-code over F.
3. $(C^\perp)^\perp = C$
4. Let H be a GM for $C^\perp$, then H is a PCM for C (by definition).
5. G is a PCM for $C^\perp$.
6. $GH^T = 0$.
7. $\boxed{\text{For } x \in V_n(F), x \in C \text{ iff } Hx^T = 0.}$

   [C is the nullspace of H.]

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