Introduction to Optimization

CO 255

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LaTeXed by Sibelius Peng

Preface

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Info

Ricardo: MC 5036. OH: M 1:30 - 3pm
TA: Adam Brown: MC 5462. OH: F 10-11am

Books (not required)

- Intro to Linear Opt. Bertsimas
- Int Programming. Conforti

Grading

- assns: 20% ($\approx$ 5)
- mid: 30% (Feb 11 in class)
- final: 50%

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Introduction

Given a set S, and a function $f : S \to \mathbb{R}$. An optimization problem is:

$$\begin{aligned} &\max f(x) \\ &\underbrace{s.t.}_{\text{subject to}} \quad x \in S \end{aligned} \qquad \text{(OPT)}$$

- S **feasible region**
- A point $\overline{x} \in S$ is a **feasible solution**
- $f(x)$ is **objective function**

(OPT) means: "Find a feasible solution x^* such that $f(x) \leq f(x^*), \forall x \in S$"

- Such x^* is an **optimal solution**
- $f(x^*)$ is **optimal value**

Other ways to write (OPT):

$$\max\{f(x), x \in S\}$$

$$\max_{x \in S} f(x)$$

Analogous problem

$$\begin{aligned} &\min f(x) \\ &s.t. \quad x \in S \end{aligned}$$

Note

$$\begin{aligned} &\max f(x) \\ &s.t. \quad x \in S \end{aligned} = -1 \begin{pmatrix} \min -f(x) \\ s.t. \quad x \in S \end{pmatrix}$$

Problem x^* may not exist

a) Problem is unbounded:

$$\forall M \in \mathbb{R}, \exists \overline{x} \in S, \ s.t. \ f(\overline{x}) > M$$

b) $S = \phi$, i.e. (OPT) is **INFEASIBLE**

c) There may not exist x^* achieving supremum.

Example:

$$\begin{aligned} &\max x \\ &s.t. \ x < 1 \end{aligned}$$

supremum

$$\sup\{f(x) : x \in S\} = \begin{cases} +\infty & \text{if OPT unbounded} \\ -\infty & \text{if } S = \varnothing \\ \min\{x : x \geq f(x), \forall x \in S\} & \text{otherwise} \end{cases}$$

always exist and are well-defined

infimum

$$\inf\{f(x) : x \in S\} = -1 \cdot \sup\{-f(x) : x \in S\}$$

From this point on, we will abuse notation and say $\max\{f(x) : x \in S\}$ is $\sup\{f(x) : x \in S\}$.

One way to specify that I want an opt. sol. (if exists) is

$$x^* \in \text{argmax}\{f(x) : x \in S\}$$

1.1 Linear Optimization (Programming)

or (LP).

$$S = \{x \in \mathbb{R}^n : Ax \leq b\}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $f(x) = c^T x, c \in \mathbb{R}^n$.

$$\downarrow$$

$$\begin{aligned} &\max c^T x \\ &s.t. \ Ax \leq b \end{aligned} \qquad (LP)$$

Note

$$A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix} \qquad A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

Clarifying

$$u, v \in \mathbb{R}^n, \qquad u \leq v \iff u_j \leq v_j, \forall j \in 1, \dots, n$$

Note $u \not\leq v$ is not the same as $u > v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \not\leq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example:

$$\begin{aligned} \max \quad & 2x_1 + \; 0.5x_2 \\ s.t. \quad & x_1 \qquad\quad \leq 2 \\ & x_1 + \quad x_2 \leq 2 \\ & x \qquad\quad\; \geq 0 \end{aligned}$$

- Strict ineq. not allowed

halfspace, hyperplane, polyhedron

Let $h \in \mathbb{R}^n, h_0 \in \mathbb{R}$.

$\{x \in \mathbb{R}^n : h^T \leq h_0\}$ is a **halfspace**.

$\{x \in \mathbb{R}^n : h^T = h_0\}$ is a **hyperplane**.

$Ax \leq b$ is a **polyhedron** (i.e. intersection of finitely many halfspaces).

Example:

n products, m resources. Producing $j \in \{1, \dots, n\}$ given c_j profit/unit and consumes a_{ij} units of resource i, $\forall i \in \{1, \dots, m\}$. There are b_i units available $\forall i \in \{1, \dots, m\}$.

$$\begin{aligned} \max & \sum_{j=1}^{n} c_j x_j \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad \forall i = 1, \dots, m \\ & x \geq 0 \end{aligned}$$

which is an LP.

1.1.1 Determining Feasibility

Given a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

either find $\overline{x} \in P$ or show $P = \varnothing$.

Idea In 1-d, easy. $\rightarrow$ Reduce problem in dimension n to one in dimension $n-1$.

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