# Applied Real Analysis

AMATH 331

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LaTeXed by Sibelius Peng

# Preface

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*Sibelius Peng*

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# Real Numbers

**Refs** 1 for review. 2.1-2.2, 2.9

## 1.1 Decimal expansions and the real number line

**finite decimal expansion**

A finite decimal expansion has the form

$$x = a_{0\circ}a_1a_2a_3\ldots a_N$$

where $a_0$ is an integer (positive, negative or zero) for $1 \leq n \leq N$ $a_n \in \{0, 1, \ldots, 9\}$

*Example:*

$$1_\circ 45$$

$$-38_\circ 298743$$

You can think of this as

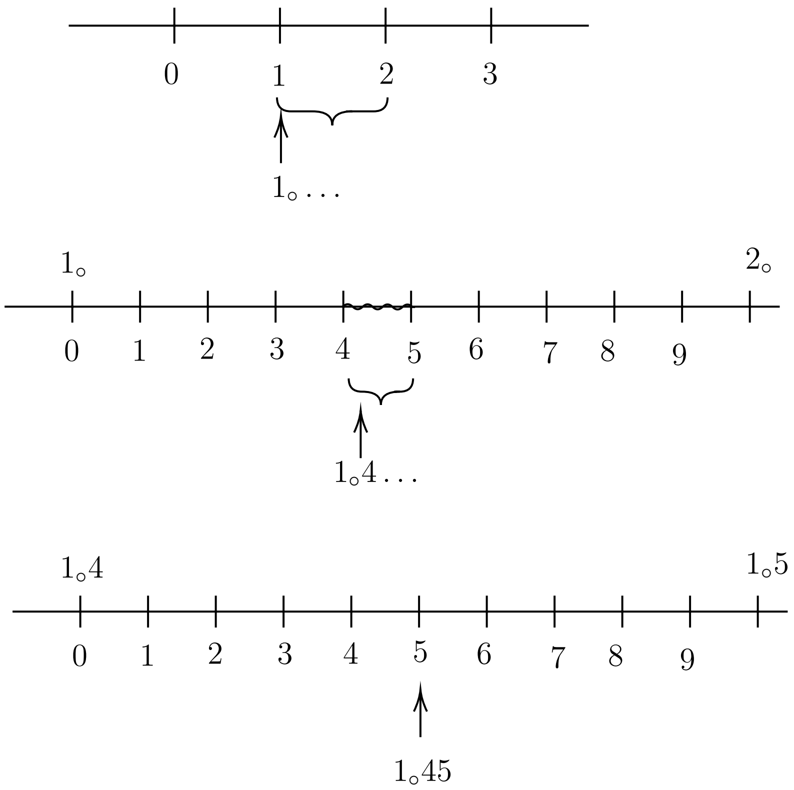
$$x = a_0 + a_1\left(\frac{1}{10}\right) + \ldots + a_N\left(\frac{1}{10^N}\right)$$

**Warning** This looks like the usual decimal representation but it is not the same for negative numbers.

Any finite decimal expansion can be replaced on the real number line.

*Example:*

Where is $1_\circ 45$?




We can similarly define infinite decimal expansions

**infinite decimal expansions**

$$x = a_{0\circ} a_1 a_2 \ldots$$

*Example:*

$$1_\circ 450000000\ldots$$

$$\pi = 3_\circ 1415926535\ldots$$

Assuming the real number line has no gaps, every infinite decimal expansion $x$ corresponds to a point on the line.

Given any positive integer $k$, let $y = a_{0\circ} a_1 a_2 \ldots a_k$ be the finite decimal expansion of $x$ to the $k$-th decimal space. Then, $x$ lies in the interval from $y$ to $(y + 10^{-k})$. So, $y$ approximates $x$ to an accuracy of $1/10^k$. As we increase $k$, we improve the accuracy; in fact, the error can be made arbitrarily small.

The converse direction: given a point on the real number line, can we find its decimal expansion?

Yes!

It is possible for two decimal expansions to represent the same point. This happens precisely when one ends in an infinite string of 0's.

*Example:*

$$1.000\ldots \qquad \text{and} \qquad 0.999\ldots$$
$$25.300\ldots \qquad \text{and} \qquad 25.2999\ldots$$

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