Introduction to Optimization

CO 255

Ricardo Fukasawa

LaTeXed by Sibelius Peng

Preface

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Sibelius Peng

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Info

Ricardo: MC 5036. OH: M 1:30 - 3pm
TA: Adam Brown: MC 5462. OH: F 10-11am

Books (not required)

- Intro to Linear Opt. Bertsimas
- Int Programming. Conforti

Grading

- assns: 20% ($\approx$ 5)
- mid: 30% (Feb 11 in class)
- final: 50%

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Introduction

Given a set S, and a function $f : S \to \mathbb{R}$. An optimization problem is:

$$\begin{array}{ll} \max & f(x) \\ \underbrace{\text{s.t.}}_{\text{subject to}} & x \in S \end{array} \tag{OPT}$$

- S **feasible region**
- A point $\overline{x} \in S$ is a **feasible solution**
- $f(x)$ is **objective function**

(OPT) means: "Find a feasible solution x^* such that $f(x) \leq f(x^*), \forall x \in S$"

- Such x^* is an **optimal solution**
- $f(x^*)$ is **optimal value**

Other ways to write (OPT):

$$\max\{f(x), x \in S\}$$

$$\max_{x \in S} f(x)$$

Analogous problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}$$

Note:

$$\begin{array}{ll} \max & f(x) \\ \text{s.t.} & x \in S \end{array} = -1 \begin{pmatrix} \min & -f(x) \\ \text{s.t.} & x \in S \end{pmatrix}$$

Problem x^* may not exist

a) Problem is unbounded:

$$\forall M \in \mathbb{R}, \exists \overline{x} \in S, \text{ s.t. } f(\overline{x}) > M$$

b) $S = \varnothing$, i.e. (OPT) is **INFEASIBLE**

c) There may not exist x^* achieving supremum.

Example:

$$\begin{array}{ll} \max & x \\ \text{s.t.} & x < 1 \end{array}$$

supremum

$$\sup\{f(x) : x \in S\} = \begin{cases} +\infty & \text{if OPT unbounded} \\ -\infty & \text{if } S = \varnothing \\ \min\{x : x \geq f(x), \forall x \in S\} & \text{otherwise} \end{cases}$$

always exist and are well-defined

infimum

$\inf\{f(x) : x \in S\} = -1 \cdot \sup\{-f(x) : x \in S\}$

From this point on, we will abuse notation and say $\max\{f(x) : x \in S\}$ is $\sup\{f(x) : x \in S\}$.

One way to specify that I want an opt. sol. (if exists) is

$$x^* \in \text{argmax}\{f(x) : x \in S\}$$

2 Linear Optimization (Programming) (LP)

$$S = \{x \in \mathbb{R}^n : Ax \leq b\}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and $f(x) = c^T x, c \in \mathbb{R}^n$.

$$\downarrow$$

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array} \tag{LP}$$

Note:

$$A = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix} \qquad A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

Clarifying

$$u, v \in \mathbb{R}^n, \qquad u \leq v \iff u_j \leq v_j, \forall j \in 1, \dots, n$$

Note:
$u \not\leq v$ is not the same as $u > v$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \not\leq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Example:

$$\begin{array}{llll} \max & 2x_1 + & 0.5x_2 & \\ \text{s.t.} & x_1 & & \leq 2 \\ & x_1 + & x_2 & \leq 2 \\ & x & & \geq 0 \end{array}$$

- Strict ineq. not allowed

halfspace, hyperplane, polyhedron

Let $h \in \mathbb{R}^n, h_0 \in \mathbb{R}$.

$\{x \in \mathbb{R}^n : h^T x \leq h_0\}$ is a **halfspace**.

$\{x \in \mathbb{R}^n : h^T x = h_0\}$ is a **hyperplane**.

$Ax \leq b$ is a **polyhedron** (i.e. intersection of finitely many halfspaces).

Example:

n products, m resources. Producing $j \in \{1, \dots, n\}$ given c_j profit/unit and consumes a_{ij} units of resource i, $\forall i \in \{1, \dots, m\}$. There are b_i units available $\forall i \in \{1, \dots, m\}$.

$$\begin{aligned} \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i, & \qquad \forall i = 1, \dots, m \\ x \ge 0 & \end{aligned}$$

which is an LP.

2.1 Determining Feasibility

Given a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \le b\}$$

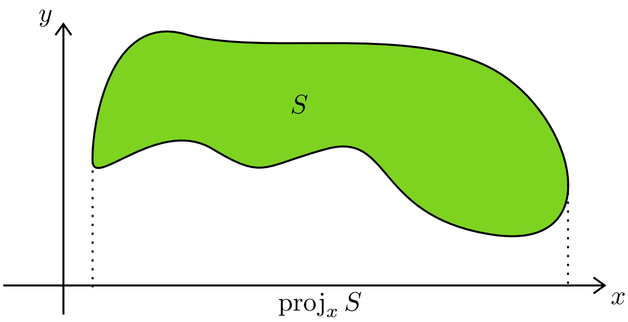
either find $\overline{x} \in P$ or show $P = \varnothing$.

Idea In 1-d, easy. $\to$ Reduce problem in dimension n to one in dimension $n-1$.

Notation Let $S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p : Ax + Gy \le b\}$, then

$$\operatorname{proj}_x S := \{x \in \mathbb{R}^n : \exists y \text{ so that } (x, y) \in S\}$$

is the (orthogonal) *projection* if S onto x.



We will find if $P = \varnothing$ by looking at $\operatorname{proj}_{x_1, \dots, x_{n-1}}$ (P)

2.2 Fourier-Motzkin Elimination

Call a_{ij} entries of A. Let

$$\begin{aligned} M &:= \{1, 2, \dots, m\} \\ M^+ &:= \{i \in M : a_{in} > 0\} \\ M^- &:= \{i \in M : a_{in} < 0\} \\ M^0 &:= \{i \in M : a_{in} = 0\} \end{aligned}$$

For $i \in M^+$:

$$a_i^T x \le b_i \iff \sum_{j=1}^{n} a_{ij} x_j \le b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j + x_n \le \frac{b_i}{a_{in}}, \qquad \forall i \in M^+ \tag{1}$$

For $i \in M^-$

$$a_i^T x \le b_i \iff \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} x_j - x_n \le \frac{b_i}{-a_{in}}, \qquad \forall i \in M^- \tag{2}$$

For $i \in M^0$

$$a_i^T x \leq b_i \iff \sum_{j=1}^{n-1} a_{ij} x_j \leq b_i, \qquad \forall i \in M^0 \tag{3}$$

$$P = \{x \in \mathbb{R}^n : (1)(2)(3)\}$$

Define

$$\sum_{j=1}^{n-1} \left(\frac{a_{ij}}{a_{in}} - \frac{a_{kj}}{a_{kn}} \right) x_j \leq \frac{b_i}{a_{in}} - \frac{b_i}{a_{kn}}, \qquad \forall i \in M^+, \forall k \in M^- \tag{4}$$

Theorem 2.1

$$(\overline{x}_1, \ldots, \overline{x}_{n-1}) \text{ satisfies (3), (4)} \iff \exists \overline{x}_n : (\overline{x}_1, \ldots, \overline{x}_n) \in P$$

Proof:

$\Longleftarrow$ If $(\overline{x}_1, \ldots, \overline{x}_n)$ satisfies (1), (2), (3) then $(\overline{x}_1, \ldots, \overline{x}_{n-1})$ satisfies (3) and adding (1), (2) $\Longrightarrow$ $(\overline{x}_1, \ldots, \overline{x}_{n-1})$ satisfies (4)

$\Longrightarrow$ If $(\overline{x}_1, \ldots, \overline{x}_{n-1})$ satisfies (4)

$$\sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \overline{x}_j - \frac{b_i}{a_{in}} \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \overline{x}_j - \frac{b_k}{a_{kn}}, \quad \forall i \in M^+, k \in M^-$$

Let

$$\overline{x}_n := \max_{i \in M^+} \left\{ \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \overline{x}_j - \frac{b_i}{a_{in}} \right\}$$

$$\Longrightarrow \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \overline{x}_j - \frac{b_i}{a_{in}} \leq -\overline{x}_n, \quad \forall i \in M^+$$

and

$$-\overline{x}_n \leq \sum_{j=1}^{n-1} \frac{a_{kj}}{a_{kn}} \overline{x}_j - \frac{b_k}{a_{kn}}, \qquad \forall k \in M^-$$

$$\Longrightarrow (\overline{x}_1, \ldots, \overline{x}_n) \in P$$

□

Note:
Proof assumes M^+, M^- are nonempty. But statement holds regardless.

(if M^+ or $M^- = \varnothing$ then (4) yields no constraints)

Algorithm 1: Fourier-Motzkin

1. $A^n = A, b^n = b$
2. given A^i, b^i obtain A^{i-1}, b^{i-1} (A^{i-1} has one less column than A^i column than A^i) by applying the steps described

$$P_i := \{x \in \mathbb{R}^i : A^i x \leq b^i\}$$

then

$$P_{i-1} = \text{proj}_{x_1, \ldots, x_{i-1}} P_i$$

3. Keep applying projection until $i = 1$.

$$P_0 = \varnothing \iff P_n = P = \varnothing$$

Let

$$P_i^n = P_i \times \mathbb{R}^{n-i} = \{x \in \mathbb{R}^n (A^i, 0)x \leq b^i\}$$

not hard to see $P_i^n = \varnothing \iff P_i = \varnothing$

Notice that

$$P_0 = \varnothing \iff P_0^n = \varnothing, P_0^n = \{0 \leq b^0\}$$

Example:

$$P_2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{ccc} x_1 & +2x_2 & \leq 1 \\ -x_1 & & \leq 0 \\ & -x_2 & \leq -2 \\ -3x_1 & -3x_2 & \leq -6 \end{array} \right\}$$

draw the graph, clearly empty

M^+: $\frac{1}{2}x_1 + x_2 \leq \frac{1}{2}$

M^-: $-x_2 \leq -2 \qquad -x_1 - x_2 \leq -2$

M^0: $-x_1 \leq 0$

$$P_1 = \left\{ x_1 \in \mathbb{R} : \begin{array}{cc} -x_1 & \leq 0 \\ \frac{1}{2}x_1 & \leq -\frac{3}{2} \\ -\frac{1}{2}x_1 & \leq -\frac{3}{2} \end{array} \right\}$$

M^+: $x_1 \leq -3$

M^-: $-x_1 \leq 0$ and $-x_1 \leq -3$

$$P_0^2 = \left\{ x \in \mathbb{R}^2 : \begin{array}{c} 0 \leq -3 \\ 0 \leq -6 \end{array} \right\} = \varnothing$$

Here $b^0 = \binom{-3}{-6}$

Remark:

Inequality in P_i^n:

- All inequalities are obtained by a nonnegative combination of inequality in P_{i+1}^n $\implies$ all nonnegative combination of inequalities in P.
- If all A, b are rational then so are all A^i, b^i
- If $b = 0, b_i = 0, \forall i$

Theorem 2.2: Farkas' Lemma

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} = \varnothing \iff \exists u \in \mathbb{R}^m : \begin{array}{l} u^T A = 0 \\ u^T b < 0 \\ u \geq 0 \end{array}$$

Proof:

$(\impliedby)$ Suppose $\overline{x}$ satisfies $A\overline{x} \leq b$.

$$0 = u^T A\overline{x} \leq u^T b < 0$$

which is impossible.

($\implies$) If $P = \varnothing$. Apply Fourier-Motzkin until we get

$$P_0^n = \varnothing = \{x \in \mathbb{R}^n : 0x \leq b^0\}$$

i.e. there exists j for which $b_j^0 < 0$.

If we look at corresponding constraint in P_0^n is

$$0^T x \leq b_j^0$$

which can be obtained by a vector u such that $u^T A = 0, u^T b = b_j^0, u \geq 0$.

□

Farkas' Lemma (alternate statement)

Exactly one of the following has a solution:

a) $Ax \leq b$

b) $u^T A = 0$
$u^T b < 0$
$u \geq 0$

Farkas' Lemma (Different Form)

Exactly one of the following has a solution:

a) $\begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned}$

b) $\begin{aligned} u^T A &\geq 0 \\ u^T b &< 0 \end{aligned}$

Proof:
(Sketch)

$$P = \left\{ x : \begin{aligned} Ax &= b \\ x &\geq 0 \end{aligned} \right\} = \left\{ x : \underbrace{\begin{pmatrix} A \\ -A \\ -I \end{pmatrix}}_{A'} x \leq \underbrace{\begin{pmatrix} b \\ -b \\ -0 \end{pmatrix}}_{b'} \right\}$$

Apply original Farkas' Lemma to get $P = \varnothing \iff \exists u_1 \in \mathbb{R}^m, u_2 \in \mathbb{R}^m, v \in \mathbb{R}^n$:

$$\begin{aligned} u_1^T A - u_2^T A - v &= 0 \\ u_1^T b - u_2^T b &< 0 \\ u_1, u_2, v &\geq 0 \end{aligned}$$

Let $u = (u_2 - u_2)$

$$u^T A - v = 0 \implies u^T A \geq 0, \quad u^T b < 0$$

□

Consider a linear programming (LP):

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \tag{LP}$$

Theorem 2.3: Fundamental Theorem of Linear Programming

(LP) has exactly one of 3 outcomes:

a) Infeasible

b) Unbounded

c) There exists an optimal solution.

Proof:
Let's assume a), b) don't hold.

If $n = 1$, then (LP) has an optimal solution. (Why?)

Else, define

$$\begin{array}{ll} \max & z \\ \text{s.t.} & \begin{array}{l} z - c^T x \le 0 \\ Ax \le b \end{array} \end{array} \tag{LP'}$$

(LP') is also not in case a) or b). (Why?)

Also if (x^*, z^*) is an optimal solution to (LP'), then x^* is an optimal solution to (LP). (Why?)

Apply Fourier-Motzkin to

$$\left\{ (x, z) : \begin{array}{r} z - c^T x \le 0 \\ Ax \le b \end{array} \right\}$$

Until we are left with a polyhedron

$$\{z \in \mathbb{R} : A' z \le b'\}$$

Now $\begin{array}{ll} \max & z \\ \text{s.t.} & A'z \le b' \end{array}$ is not cases a) or b). (Why?)

$\rightarrow$ can get an optimal solution z^* to such problem. Apply Fourier-Motzkin back to get (x^*, z^*) optimal solution to (LP'). (Why?) $\square$

2.3 Certifying Optimality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \end{array} \tag{LP}$$

and let $\overline{x} \in P = \{x : Ax \le b\}$

Question Can we certify that $\overline{x}$ is optimal?

Example:

$$\begin{array}{rr} \max & 2x_1 + x_2 \\ & x_1 + 2x_2 \le 2 \\ \text{s.t.} & x_1 + x_2 \le 2 \\ & x_1 - x_2 \le 0.5 \end{array}$$

Consider $\overline{x} = (0, 1)^T$ is clearly NOT optimal.

$x^* = (1, 0.5)^T$ and $c^T x^* = 2.5$. Any feasible solution satisfies

$$\begin{array}{rlr} x_1 + 2x_2 & \le 2 & \times 1/3 \\ x_1 + x_2 & \le 2 & \times 1 \\ + \quad x_1 - x_2 & \le 0.5 & \times 2/3 \\ \hline 2x_1 + x_2 & \le 3 & \end{array}$$

Instead do $1 \times 1st$ constraint $+ 1 \times 3rd$ constraint $\implies 2x_1 + x_2 \leq 2.5$

In general:

$$
\begin{array}{rll}
x_1 + 2x_2 & \leq 2 & \times y_1 \\
x_1 + x_2 & \leq 2 & \times y_2 \\
+ \quad x_1 - x_2 & \leq 0.5 & \times y_3 \\
\hline
\end{array}
$$

$$(y_1 + y_2 + y_3)x_1 + (2y_1 + y_2 - y_3)x_2 \leq 2y_1 + 2y_2 + 0.5y_3$$

As long as $y_1, y_2, y_3 \geq 0$ and

$$
\begin{aligned}
y_1 + y_2 + y_3 &= 2 \\
2y_1 + y_2 - y_3 &= 1
\end{aligned}
$$

This leads to the following linear program:

$$
\begin{aligned}
\min \ 2y_1 + 2y_2 + 0.5y_3 \\
y_1 + y_2 + y_3 = 2 \\
s.t. \ \ 2y_1 + y_2 - y_3 = 1 \\
y_1, y_2, y_3 \geq 0
\end{aligned}
$$

This is called the dual LP.

In general:

$$
\begin{array}{ll}
\max & c^T x \\
\text{s.t.} & Ax \leq b
\end{array}
\tag{P}
$$

Dual of (P)

$$
\begin{array}{ll}
\min & b^T y \\
\text{s.t.} & \begin{array}{l} y^T A = c^T \\ y \geq 0 \end{array}
\end{array}
\tag{D}
$$

Remark:
We call (P) primal LP.

Theorem 2.4: Weak Duality

Let $\overline{x}$ feasible for (P), $\overline{y}$ feasible for (D). Then $c^T x \leq b^T y$.

Proof:

$$c^T \overline{x} = \overline{y}^T (A\overline{x}) \leq \overline{y}^T b$$

where we used $A\overline{x} \leq b$ and $\overline{y} \geq 0$. □

Corollary 2.5

Several results:

- If (P) is unbounded then (D) is infeasible.
- If (D) is unbounded then (P) is infeasible.

Note:
(P) and (D) can both be infeasible.

- If $\overline{x}$ is feasible for (P) $\overline{y}$ feasible for (D) $c^T \overline{x} = b^T \overline{y}$, then $\overline{x}$ optimal for (P), $\overline{y}$ optimal for (D).

Theorem 2.6: Strong Duality

x^* is optimal for (P) $\iff$ $\exists y^*$ feasible for (D) such that $c^Tx^* = b^Ty^*$.

Proof:

$(\impliedby)$ ✓

$(\implies)$ Is (D) infeasible?

Suppose $\left\{ y \in \mathbb{R}^n : \begin{array}{r} A^Ty = c \\ y \geq 0 \end{array} \right\} = \varnothing$

(Alternate version of Farkas' Lemma) $\exists u : \begin{array}{r} u^TA^T \geq 0 \\ u^Tc < 0 \end{array} \iff \exists d : \begin{array}{r} Ad \leq 0 \\ c^Td > 0 \end{array}$

Take look at $x' = x^* + d$, then

$$\begin{aligned} Ax' &= Ax^* + Ad \leq b \\ c^Tx' &= c^Tx^* + c^Td > c^Tx^* \end{aligned}$$

Contradiction. Thus (D) has an optimal solution y^*.

Now let $\gamma = b^Ty^*$, and let $\theta := \left\{ x \in \mathbb{R}^n : \begin{array}{r} Ax \leq b \\ -c^Tx \leq -\gamma \end{array} \right\}$.

If $\theta = \varnothing$, by Farkas'

$$\exists \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix} : \begin{cases} \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix}^T \begin{pmatrix} A \\ -c^T \end{pmatrix} = 0 \\ \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix}^T \begin{pmatrix} b \\ -\gamma \end{pmatrix} < 0 \\ \begin{pmatrix} \overline{y} \\ \overline{\lambda} \end{pmatrix} \geq 0 \end{cases} \iff \begin{array}{l} A^T\overline{y} = c\overline{\lambda} \\ b^T\overline{y} < \gamma\overline{\lambda} \\ \overline{y} \geq 0 \\ \overline{\lambda} \geq 0 \end{array}$$

<u>Case 1:</u> $\overline{\lambda} > 0$.

Let $y' = \frac{\overline{y}}{\overline{\lambda}}$. Then we have

$$A^Ty' = A^T\frac{\overline{y}}{\overline{\lambda}} = c \quad \text{and} \quad b^Ty' = b^T\frac{\overline{y}}{\overline{\lambda}} < \gamma \quad \text{and} \quad y' = \frac{\overline{y}}{\overline{\lambda}} \geq 0$$

Contradicts optimality of y^*.

<u>Case 2:</u> $\overline{\lambda} = 0$. Then $\begin{array}{l} A^Ty = 0 \\ b^Ty < 0 \\ \overline{y} \geq 0 \end{array}$

Now we can do the same thing previously. Let $y' = y^* + \overline{y}$, then

$$A^Ty' = A^Ty^* + A^T\overline{y} = c$$

and

$$\begin{aligned} y' &= y^* + \overline{y} \geq 0 \\ b^Ty' &= b^Ty^* + b^T\overline{y} < b^Ty^* \end{aligned}$$

Contradicts optimality of y^*.

Thus $\theta \neq \varnothing$.

Let $\overline{x} \in \theta$,

$$c^T x^* \underbrace{\leq}_{\text{weak duality}} b^T y^* = \gamma \underbrace{\leq}_{\overline{x}\in\theta} c^T \overline{x} \leq c^T x^*$$

where the last inequality is because $\overline{x}$ feasible for (P), x^* optimal for (P).

$\square$

2.4 Possible Outcomes

See here.

2.5 Duals of generic LPs

$$\begin{array}{ll} \max & 2x_1 + 3x_2 - 4x_3 \\ \text{s.t.} & \begin{array}{llll} x_1 & & +7x_3 & \leq 5 \\ & 2x_2 & -x_3 & \geq 3 \\ x_1 & & +x_3 & = 8 \\ & x_2 & & \leq 6 \\ x_1 & & & \geq 0 \\ & x_2 & & \leq 0 \end{array} \end{array}$$

$$\begin{array}{ll} \max & (2,3,-4)x \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 7 \\ 0 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} x \leq \begin{pmatrix} 5 \\ -3 \\ 8 \\ -8 \\ 6 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

and dual

$$\begin{array}{ll} \min & (5,-3,8,-8,6,0,0)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0 \end{array} \tag{D_1}$$

$$\begin{array}{ll} \min & (5,-3,8,-8,6)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 & 1 \\ 7 & 1 & 1 & -1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y \geq 0 \end{array} \tag{D_2}$$

Claim $(y_1^*, \ldots, y_5^*)$ is optimal for (D_2) $\iff$ $(y_1^*, \ldots, y_5^*, y_6^*, y_7^*)$ optimal for (D_1) with

$$y_6^* = y_1^* + y_3^* - y_4^* - 2$$
$$y_7^* = 3 - (-2y_2^* + y_5^*)$$

$$\begin{array}{ll} \min & (5,3,8,6)y \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 7 & -1 & 1 & 0 \end{pmatrix} y \begin{matrix} \geq \\ \leq \\ = \end{matrix} \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \text{and} \quad y_1 \geq 0, y_2 \leq 0 \qquad y_4 \geq 0 \end{array} \tag{D_3}$$

Claim Opt value of (D_2) and (D_3) are same.

In general

$$\begin{array}{llc|lll} \max & c^T x & & \min & b^T y & \\ \text{s.t.} & Ax?b & (P) & \text{s.t.} & A^T y ? c & (D) \\ & x?0 & & & y?0 & \end{array}$$

2.5.1 Cheat Sheet

Here or

Primal (max)		Dual (min)	
Constraint	$\leq$	≥ 0	Variable
	$\geq$	≤ 0	
	$=$	free	
Variable	$\geq$	≥ 0	Constraint
	$\leq$	≤ 0	
	free	$=$	

Remark:
This is not symmetric... The way you can remember it is by thinking natural variables in real life, like you cannot have negative number of cars and so on...

Q What if you start with a minimization LP as primal?

Example:

$$\begin{array}{ll} \min & x_1 - x_2 \\ \text{s.t.} & 2x_1 + 3x_2 \leq 5 \\ & x_1 - x_2 \geq 3 \\ & x_1 + 5x_2 = 7 \\ & x_1 \geq 0, x_2 \leq 0 \end{array} \tag{P}$$

Rewrite as:

$$-1 \times \begin{pmatrix} \max & -x_1 + x_2 \\ \downarrow & \\ \text{s.t.} & \dots \end{pmatrix}$$

Will lead to finding dual:

$$\begin{array}{ll} \max & 5y_1 + 3y_2 + 7y_3 \\ \downarrow & \\ & 2y_1 + y_2 \leq 1 \\ \text{s.t.} & 3y_1 - y_2 + 5y_3 \geq -1 \\ & y_1 \leq 0, y_2 \geq 0, y_3 \text{ free} \end{array}$$

Also

- Weak duality holds.
  If $\overline{x}$ feasible for (P), $\overline{y}$ feasible for (D), then $c^T\overline{x} \geq b^T\overline{y}$.
- Strong duality holds

Note:
The dual of the dual of (P) is (P).

Example:
Given a simple undirected graph $G = (V, E)$. $M \subseteq E$ is a *matching* if every vertex $v \in V$ is incident to ≤ 1 edge in M.

See examples of matching in CO 342 or MATH 249.

Max cardinality matching

Find matching M with largest $|M|$.

Define $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{otherwise} \end{cases}$.

$$\begin{array}{ll} \max & \sum_{e \in E} x_e \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall v \in V \\ 0 \leq x_e, \quad \forall e \in E \end{array} \end{array}$$

where $\delta(v) =$ set of edges in E incident to v.

$$\begin{array}{ll} \min & \sum_{v \in V} y_v \\ \downarrow & \\ \text{s.t.} & \begin{array}{ll} y_u + y_v \geq 1, & \forall e = uv \in E \\ y \geq 0 & \end{array} \end{array}$$

2.6 Other interpretations of dual

Example:

			Resources	
		Per unit Profit	Per unit consumption A	B
Product	1	5	2	3
	2	3	4	1
Available Resources			15	10

$$\begin{array}{ll} \max & 5x_1 + 3x_2 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} 2x_1 + 4x_2 \leq 15 \\ 3x_1 + x_2 \leq 10 \\ x \geq 0 \end{array} \end{array}$$

Suppose somebody wants to buy A, B from me. What is the lowest price I should ask?

Let y_A, y_B be prices:

$$\begin{array}{ll} \min & 15y_A + 10y_B \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} 2y_A + 3y_B \geq 5 \\ 4y_A + y_B \geq 3 \\ y \geq 0 \end{array} \end{array}$$

Example: Zero-Sum

Alice, Bob play game. A: m choices. B: n choices. Alice play i, Bob plays j, Bob pays Alice M_{ij} dollars.

		Alice R	P	S
	R	0	1	-1
Bob	P	-1	0	1
	S	1	-1	0

Zero-sum: Amount won by Alice - Amount won by Bob = 0

Let $y \in \mathbb{R}^m_+$, Alice's probability distribution.
Let $x \in \mathbb{R}^n_+$, Bob's probability distribution.

Expected Amount Bob pays Alice:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_i M_{ij} x_j = y^T M_x$$

$$P = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \sum x_j = 1 \\ x \geq 0 \end{array} \right\}$$

$$Q = \left\{ y \in \mathbb{R}^m : \begin{array}{l} \sum y_i = 1 \\ y \geq 0 \end{array} \right\}$$

Alice wants $\max\limits_{y\in Q}\left\{\min\limits_{x\in P}\ y^T M_x\right\}$. Bob wants $\min\limits_{x\in P}\left\{\max\limits_{y\in Q}\ y^T M_x\right\}$.

Suppose $\overline{y} \in Q$ is fixed. Bob's problem is

$$\min_{x\in P}\ \overline{y}^T M_x = \begin{array}{ll} \min & \sum_{j=1}^{n}\left(\sum_{i=1}^{m} M_{ij}\overline{y}_i\right) x_j \\ \downarrow & \\ \text{s.t.} & \sum_{j=1}^{n} x_j = 1 \\ & x \geq 0 \end{array}$$

This is equivalent to picking smallest number in

$$\left\{\sum_{i=1}^{m} M_{ij}\overline{y}_i\right\}_{j=1}^{n}$$

$$\implies \max_{y\in Q}\min_{x\in P} y^T M_x = \max_{y\in Q}\left\{\begin{array}{ll} \max & u \\ \downarrow & \\ \text{s.t.} & u \leq y^T M e_j, \quad \forall j = 1,\ldots,n \end{array}\right\}$$

$$= \begin{array}{ll} \max & u \\ \downarrow & \\ \text{s.t.} & u \leq y^T M e_j, \quad \forall j = 1,\ldots,n \\ & y^T = 1 \\ & y \geq 0 \end{array}$$

Similarly Bob's problem:

$$\begin{array}{ll} \min & v \\ \downarrow & \\ \text{s.t.} & v \geq e_i^T M_x, \quad \forall i = 1,\ldots,m \\ & x^T = 1 \\ & x \geq 0 \end{array}$$

There are x^*, y^* for which strategy values match $\rightarrow$ Nash's Equilibrium.

Now get back to Farkas' Lemma Theorem 2.2. [1]

[1] Rephrase it a little bit: Exactly one of the two has a solution (i) $Ax \leq b$ (ii) $u^T \ldots$

Proof:

$$\begin{array}{ll} \max & 0^T x \\ \downarrow & \\ \text{s.t.} & Ax \le b \end{array} \tag{P}$$

$$\begin{array}{ll} \min & b^T u \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} u^T A = 0 \\ u \ge 0 \end{array} \end{array} \tag{D}$$

(D) is always feasible ($u = 0$).

If $\exists \overline{x} : A\overline{x} \le b$, $\overline{x}$ optimal for (P) $\implies$ optimal for (D) has value 0.
$\implies \nexists u$ satisfying (ii).

And the converse is also true. □

2.7 Complementary Slackness (C.S.)

Let x^*, y^* be feasible for primal and dual respectively.

Complementary Slackness

Abbreviated as C.S.

i) Either $x_j^* = 0$ or corresponding dual constraint is tight at y^*, $\forall j = 1, \ldots, n$.

ii) Either $y_i^* = 0$ or corresponding primal constraint is tight at x^*, $\forall i = 1, \ldots, m$.

Example:

$$\begin{array}{ll} \min & x_1 - x_2 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} 2x_1 + 3x_2 \le 5 \\ x_1 - x_2 \ge 3 \\ x_1 + 5x_2 = 7 \\ x_1 \ge 0, x_2 \le 0 \end{array} \end{array} \tag{P}$$

$$\begin{array}{ll} \max & 5y_1 + 3y_2 + 7y_3 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} 2y_1 + y_2 + y_3 \le 1 \\ 3y_1 - y_2 + 5y_3 \ge -1 \\ y_1 \le 0, y_2 \ge 0 \end{array} \end{array} \tag{D}$$

i) $x_1^* = 0$ OR $2y_1^* + y_2^* + y_3^* = 1$
$x_2^* = 0$ OR $3y_1^* - y_2^* + 5y_3^* = -1$

ii) $y_1^* = 0$ OR $2x_1^* + 3x_2^* = 5$
$y_2^* = 0$ OR $x_1^* - x_2^* = 3$
$y_3^* = 0$ OR $x_1^* + 5x_2^* = 7$

Theorem 2.7

Let x^*, y^* be feasible for primal/dual respectively. TFAE[a]

a) x^* opt for primal AND y^* opt. for dual

b) Obj. value of x^* = Obj. value of y^*

c) x^*, y^* satisfy C.S.

[a] the following are equivalent

Proof:

a) $\iff$ b) done.

b) $\iff$ c) Proof for

$$\begin{array}{lll} \max & c^T x \\ \downarrow \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \min & b^T y \\ \downarrow \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{array}$$

Note:

$$A^T y \geq c \iff \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad \forall j = 1, \ldots, n$$

$$\begin{aligned} c^T x^* &= \sum_{j=1}^{n} c_j x^* \\ &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i^* \right) x_j^* \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_i^* \right) y_i^* \\ &\leq \sum_{i=1}^{m} b_i y_i^* = b^T y^* \end{aligned}$$

where first and second inequalities come from $x \geq 0, y \geq 0$ respectively.

(b) $c^T x^* = b^T y^* \iff$ C.S. holds. (Just play with some strict inequality conditions)

□

Example:

$$\begin{array}{ll} \max & x_1 + x_2 \\ \downarrow \\ \text{s.t.} & x_1 + x_2 \leq 1 \end{array} \qquad \begin{array}{ll} \min & y \\ \downarrow \\ & y = 1 \\ \text{s.t.} & y = 1 \\ & y \geq 0 \end{array}$$

Consider a pair $x^* = (0, 0), y^* = 1$ which violates CS.

2.7.1 Geometric Interpretation of C.S.

$$\begin{array}{ll} \max & c^T x \\ \downarrow \\ \text{s.t.} & Ax \leq b \end{array} \qquad \begin{array}{ll} \min & c^T y \\ \downarrow \\ \text{s.t.} & A^T y = c \\ & y \geq 0 \end{array}$$

$$A = \begin{pmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{pmatrix}$$

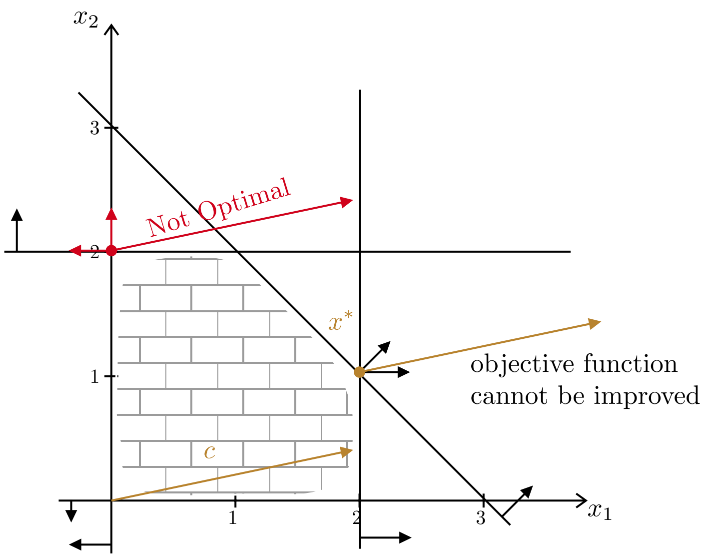
C.S says $a_i^T x^* = b_i$ or $y_i^* = 0$.

$$A^T y = c \implies \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_m \\ | & | & & | \end{pmatrix} y = c \implies \sum_{i=1}^{m} a_i y_i = c$$

C.S. says c is a nonnegative combination of tight constraint at x^*.

Example:

$$\begin{array}{ll} \max & 2x_1 + 0.5x_2 \\ \downarrow & \\ & x_1 \leq 2 \\ \text{s.t.} & x_2 \leq 2 \\ & x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$



Theorem 2.8

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \leq b \end{array} \tag{P}$$

is unbounded iff (P) is feasible and $\exists d \in \mathbb{R}^n : \begin{array}{l} c^T d > 0 \\ Ad \leq 0 \end{array}$.

Proof:

$\Longrightarrow$) Let $\overline{x}$ feasible for (P), $\overline{x} + \lambda d$ is also feasible for (P) $\forall \lambda \geq 0$.

$c^T(\overline{x} + \lambda d)$ can be made arbitrary large.

$\Longleftarrow$) Hard exercise but doable.

$\square$

2.8 Geometry of Polyhedra

line segment

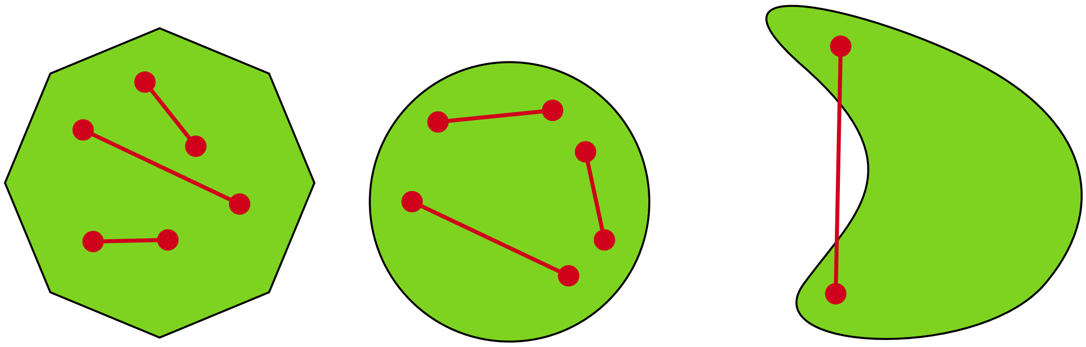
$\overline{x}, \overline{y} \in \mathbb{R}^n$ the line segment between $\overline{x}, \overline{y}$ is

$$\left\{ x \in \mathbb{R}^n : \begin{array}{l} x = \lambda \overline{x} + (1-\lambda)\overline{y} \\ \text{for some } \lambda \in [0,1] \end{array} \right\}$$

convex set

S is a convex set if $\forall x, y \in S$, line segment between x, y is contained in S.

Example:



NOT a convex set

Polyhedra are convex sets. $P = \{x : Ax \leq b\}$. $\overline{x}, \overline{y} \in P$ then

$$A(\underbrace{\lambda}_{\geq 0} \overline{x} + \underbrace{(1-\lambda)}_{\geq 0} \overline{y}) \leq \lambda b + (1-\lambda) b = b$$

convex combination

Given $x^1, \ldots, x^k \in \mathbb{R}^n$. We say $\overline{x}$ is a convex combination of $x^1, \ldots, x^k$ if $\exists \lambda$:

$$\overline{x} = \sum_{i=1}^{k} \lambda_i x^i$$
$$1 = \sum_{i=1}^{k} \lambda_i$$
$$\lambda \geq 0$$

Optimal solution seems to be happen at "corners".

Let P be a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$.

vertex

$\overline{x}$ is a vertex of P if $\exists c$: $\overline{x}$ is unique optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array}$$

extreme point

$\overline{x}$ is an extreme point of P if $\nexists u, v \in P \setminus \{\overline{x}\}$ such that $\overline{x}$ is in line segment between u, v.

basic feasible solution

$\overline{x} \in P$ is a basic feasible solution of P if there are n linearly independent tight constraints at $\overline{x}$.

Note:
Constraints

$$a_i^T x \leq b_i, \quad \forall i = 1, \ldots, m$$

are linearly independent if $\{a_i\}_{i=1}^m$ are linearly independent.

Theorem 2.9

Let $\overline{x} \in P$. TFAE:

a) $\overline{x}$ is a vertex of P.

b) $\overline{x}$ is a basic feasible solution of P.

c) $\overline{x}$ is a extreme point of P.

Proof:

a) $\implies$ c) Suppose $\exists u, v \in P \setminus \{\overline{x}\}$ such that

$$\overline{x} = \lambda u + (1-\lambda)v$$

for some $\lambda \in (0,1)$. Consider c for which $\overline{x}$ is an optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$$

$$\implies \begin{array}{l} c^T\overline{x} \geq c^T u \\ c^T\overline{x} \geq c^T v \end{array}$$

and

$$c^T\overline{x} = \underbrace{\lambda}_{\geq 0} c^T u + \underbrace{(1-\lambda)}_{\geq 0} c^T v \leq \lambda c^T\overline{x} + (1-\lambda)c^T\overline{x} = c^T\overline{x}$$

$$\implies c^T u = c^T v = c^T\overline{x}$$

$\implies$ $\overline{x}$ NOT a vertex.

c) $\implies$ b) Suppose $\overline{x}$ is not a BFS. Let $I \subseteq \{1, \ldots, m\}$ be the index set of tight constraint at $\overline{x}$. Consider

$$a_i^T d = 0, \quad \forall i \in I \tag{$*$}$$

But since $\overline{x}$ not BFS, $\exists \overline{d} \neq 0$ satisfying ($*$).[a]

$$x(\epsilon) = \overline{x} + \epsilon\overline{d}$$

$$a_i^T x(\epsilon) = a_i^T\overline{x} \leq b_i, \quad \forall i \in I$$

$$a_i^T x(\epsilon) = \underbrace{a_i^T\overline{x}}_{<b_i} + \epsilon a_i^T d \leq b_i, \quad \forall i \notin I$$

which is satisfied if $|\epsilon|$ is small enough.

$x(\epsilon) \in P$ if $|\epsilon|$ is small enough.

But then

$$\overline{x} = \frac{1}{2}x(\epsilon) + \frac{1}{2}x(-\epsilon)$$

b) $\implies$ a) Let $I \subseteq \{1, \ldots, m\}$ index set of tight constraint at $\overline{x}$.

Define

$$c := \sum_{i \in I} a_i$$

Then $\forall x \in P$

$$c^T x = \sum_{i \in I} a_i^T x \leq \sum_{i \in I} b_i$$

And

$$c^T\overline{x} = \sum_{i \in I} a_i^T\overline{x} = \sum_{i \in I} b_i$$

$\implies \overline{x}$ is optimal solution to

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array} \tag{$**$}$$

If $x' \in P$ is optimal solution to ($**$), then

$$a_i^T x' = b_i, \qquad \forall i \in I \tag{$***$}$$

But since there are n linear independent constraints in I, $\overline{x}$ is unique solution to ($***$). $\implies x' = \overline{x}$.

□

[a] by Rank-Nullity Theorem.

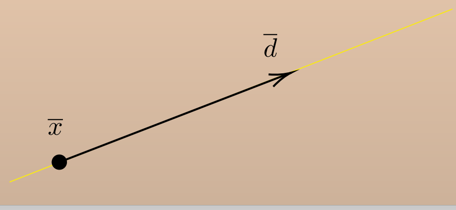
Q When does P have extreme points?

line

Let $\overline{x}, \overline{d} \in \mathbb{R}^n$, $\overline{d} \neq 0$. The set

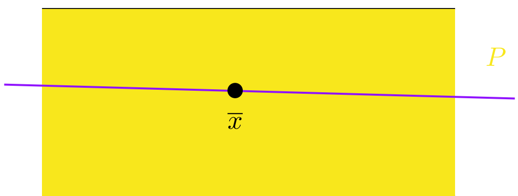
$$\{x \in \mathbb{R}^n : x = \overline{x} + \lambda d \text{ for some } \lambda \in \mathbb{R}\}$$

is called a line.



We say a polyhedron P has a line if $\exists \overline{x}, \overline{d}$ has a line if $\exists \overline{x}, \overline{d}$ s.t. $\overline{x} \in P, \overline{d} \neq 0$ and

$$\{x \in \mathbb{R} : x = \overline{x} + \lambda \overline{d} \text{ for some } \lambda \in \mathbb{R}\} \subseteq P$$



Proposition 2.10

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has a line iff $P \neq \varnothing$ and $\exists \overline{d} \neq 0$ such that $A\overline{d} = 0$

$\iff P \neq \varnothing$ and $\operatorname{rank}(A) < n$

Proof:
Exercise. □

Theorem 2.11

$P = \{x \in \mathbb{R}^n : Ax \leq b\}$ has an extreme point

$\iff P \neq \varnothing$ and P has no lines.

Proof:
Exercise. □

pointed polyhedron

A non-empty polyhedron is called pointed if it has no lines.

Note:
not pointed does not imply bounded. For example, in $\mathbb{R}^2$, $x \geq 0$ and $y \geq 0$.

Theorem 2.12

Let $P \neq \varnothing$ pointed polyhedron. If $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$ (LP) has an optimal solution, it has an optimal solution that is an extreme point.

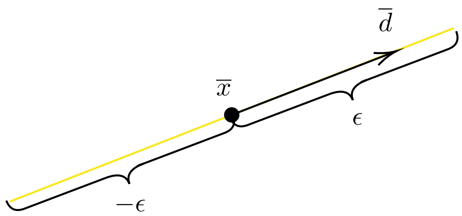
Proof:
Let $\overline{x}$ be an optimal solution to (LP) with largest number of linear independent tight constraints.

Suppose there are $\leq n-1$ linear independent tight constraints at $\overline{x}$.

Pick $\overline{d} \neq 0$ such that $a_i^T \overline{d} = 0, \forall i \in I$, where I is the index set of tight constraints. By the exact same argument as before, $\overline{x} \pm \epsilon \overline{d} \in P$ for ϵ small enough. But

$$c^T(\overline{x} \pm \epsilon \overline{d}) = c^T \overline{x} \pm \epsilon c^T \overline{d}$$

$\implies c^T \overline{d} = 0$
$\implies c^T d(\overline{x} \pm \epsilon d) = c^T \overline{x}$



Since P is pointed, $\exists \overline{\epsilon}$ for which

$$\overline{x} \pm \overline{\epsilon} \overline{d} \in P$$

and one of them not in P if $|\epsilon| > \overline{\epsilon}$. That can only happen if

$$a_k^T(\overline{x} + \overline{\epsilon}\overline{d}) = b_k \qquad \text{or} \qquad a_k^T(\overline{x} - \overline{\epsilon}\overline{d}) = b_k$$

for some $k \notin I$.

$\implies a_k^T \overline{d} \neq 0$, $\implies a_k$ is linear independent from $\{a_i\}_{i \in I}$ since non-zero cannot be linear combination of zeros. Contradiction to choice of $\overline{x}$. $\square$

2.9 Simplex Algorithm

Standard Equality Form

A linear program is in Standard Equality Form (SEF) if it is of the form

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

Proposition 2.13

Given any linear program, there exists an equivalent LP in SEF.

Example:

$$\begin{array}{ll} \max & x_1 + 2x_2 + x_3 \\ \downarrow & \\ & 3x_1 + x_2 \leq 5 \\ \text{s.t.} & -x_1 + x_3 \geq 6 \\ & x_1 \leq 0, x_3 \geq 0 \end{array} \tag{P1}$$

$x_1' = -x_1 \geq 0$ and
$x_2 = x_2^+ - x_2^-$ where $x_2^+ \geq 0, x_2^- \geq 0$

We introduce

$$s_1 = 5 - 3x_1 - x_2 \geq 0, \qquad s_2 = -x_1 + x_3 - 6 \geq 0$$

Then

$$\begin{array}{ll} \max & -x_1' + 2x_2^+ - 2x_2^- + x_3 \\ \downarrow & \\ & -3x_1' + 2x_2^+ - x_2^- + s_1 = 5 \\ \text{s.t.} & x_1' + x_3 - s_2 = 6 \\ & x_1', x_2^+, x_2^-, x_3, s_1, s_2 \geq 0 \end{array} \tag{P2}$$

x feasible for (P1) $\iff$ $(x_1', x_2^+, x_2^-, x_3, s_1, s_2)$ feasible for (P2) and they have same cost.

Assumption $A \in \mathbb{R}^{m \times n} \to \operatorname{rank}(A) = m$. This is WLOG. Since if

$$a_i = \sum_{k \neq i} \lambda_k a_k$$

Either

$$b_i \neq \sum_{k \neq i} \lambda_k b_k$$

in which case (SEF) is infeasible. Or $a_i^T x = b_i$ is redundant. So it can be removed from (SEF).

Note:
$\{x : Ax = b, x \geq 0\}$ is *pointed* polyhedron (if nonempty).

Structure of BFS Any feasible solution has m linear independent tight constraints $(n-m)$ extra tight constraint must come from $x_j \geq 0$.

Let $B \subseteq \{1, \ldots, n\}$ such that $|B| = m$ and A_B[2] is invertible.

$N = \{1, \ldots, n\} \setminus B$. $x_N = 0$, i.e. $x_j = 0, \forall j \in N$.

Feasible solutions obtained this way are precisely BFS.

Example:

$$\begin{array}{ll} \max & \begin{pmatrix} 3 & 2 & 1 & 4 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \\ & x \geq 0 \end{array}$$

[2] A_B is submatrix obtained by picking columns of A indexed by B. Such B is called a <u>basis</u>.

If we pick

$$\begin{aligned} B &= \{1,2\} & A_B &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \\ N &= \{3,4\} & A_N &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ C_B &= (3 \quad 2)^T & C_N &= (1 \quad 4)^T \end{aligned}$$

$x_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad x_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$

$B = \{1,3\}, B = \{2,4\}, A_B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, A_N = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$

$C_B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, C_N = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x_B = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}, x_N = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$

If we set $x_N = 0$ (for $B = \{1,3\}$) we are left with

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

This has a unique solution $x_1 = 3.5, x_3 = -1.5$, but not feasible.

If we pick $B = \{1,2\}$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$\underbrace{x_3 = x_4}_{x_N} = 0,\ x_1 = 3, x_2 = 1$, which is feasible.

In general,

$$Ax = b \iff A_B x_B + \cancelto{0}{A_N x_N} = b$$

has unique solution $x_b = A_B^{-1} b$.

For any basis B, the corresponding *basic solution* is

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1} b \\ 0 \end{pmatrix}$$

If $A_B^{-1} b \geq 0$, then it is a *BFS*.

2.9.1 Canonical Form

Let B be a feasible basis (i.e. corresponding basis solution is feasible).

$$\begin{aligned} Ax = b &\iff A_B x_B + A_N x_N = b \\ &\iff x_B + A_B^{-1} A_N x_N = A_B^{-1} b \end{aligned}$$

Now let's take a look at objective.

$$\begin{aligned} c^T x &= c_B^T x_B + c_N^T x_N - c_B^T (x_B + A_B^{-1} A_N x_N - A_B^{-1} b) \\ &= (c_N^T - c_B^T A_B^{-1} A_N) x_N + c_B^T A_B^{-1} b \end{aligned}$$

Thus (SEF) is said to be in canonical form for B if it is written as

$$\begin{array}{ll} \max & \overbrace{(c_N^T - c_B^T A_B^{-1} A_N)}^{\bar{c}_N^T \to \text{Reduced costs}} x_N + c_B^T A_B^{-1} b \\ \downarrow & \\ \text{s.t.} & x_B + A_B^{-1} A_N x_N = A_B^{-1} b \\ & x_B, x_N \geq 0 \end{array}$$

Example:

Back to our previous example...

$B = \{1, 2\}$. Rewriting in canonical form for B:

$$A_B^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}$$

$$A_B A = \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix}$$

$$c_B^T A_B^{-1} A_N = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \end{pmatrix} = \begin{pmatrix} 7/3 & -8/3 \end{pmatrix}$$

$$c_N^T - c_B^T A_B^{-1} A_N = \begin{pmatrix} -4/3 & 4/3 \end{pmatrix}$$

Then

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -4/3 & 4/3 \end{pmatrix} x + 11 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 1/3 & -2/3 \\ 0 & 1 & 2/3 & -1/3 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{array}$$

is in canonical form for $B = \{1, 2\}$.

Example:

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 3 & -2 & 0 & 0 \end{pmatrix} x \underbrace{+0}_{\text{obj. value}} \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ & x \geq 0 \end{array} \tag{LP}$$

Canonical form for $B = \{4, 5\}$.

Corresponding BFS $\begin{array}{l} x_4 = 4 \\ x_5 = 1 \end{array}$, $\quad x_j = 0, \forall j \in N$

$x = \begin{pmatrix} 0 & 0 & 0 & 4 & 1 \end{pmatrix}^T$

Objective value $= 0$

If increase x_1 or x_2. Objective function increases.

Let's try to increase x_1 from $0 \to \theta$. (Keep $x_2 = x_3 = 0$)

$$\begin{aligned} \theta + x_4 = 4 &\iff x_4 = 4 - \theta \\ \theta + x_5 = 1 &\iff x_5 = 1 - \theta \end{aligned}$$

New objective: $0 + \theta$. However, we have

$$\left.\begin{array}{l} x_4 \geq 0 \implies \theta \leq 4 \\ x_5 \geq 0 \implies \theta \leq 1 \end{array}\right. \implies \text{Increase } x_1 \text{ by } 1$$

x_5 will be $0 \to \begin{array}{l} x_1 \text{ enters basis} \\ x_5 \text{ leaves basis} \end{array}$. Then new basis $B = \{1, 4\}$.

Rewriting (LP) in canonical form for $B = \{1, 4\}$.

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 4 & -5 & 0 & -1 \end{pmatrix} x + \underbrace{1}_{\text{obj. value}} \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & -1 & 3 & 0 & 1 \\ 0 & 2 & -2 & 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array}$$

Corresponding BFS:

$$x = \begin{pmatrix} 1 & 0 & 0 & 3 & 0 \end{pmatrix}^T$$

Obj. value $= 1$

Pick $j \in N$: $\bar{c}_j > 0$ $(j = 2)$

Increase x_2 to θ, keep $x_3 = x_5 = 0$

$$\begin{aligned} x_1 - \theta = 1 &\iff x_1 = 1 + \theta \\ x_4 + 2\theta = 3 &\iff x_4 = 3 - 2\theta \end{aligned}$$

and

$$\begin{aligned} x_1 \geq 0 &\implies \theta \geq -1 \\ x_4 \geq 0 &\implies \theta \leq \frac{3}{2} \end{aligned}$$

Set $\theta \leftarrow \frac{3}{2} \rightarrow \begin{array}{l} x_2 \text{ enters basis} \\ x_4 \text{ leaves basis} \end{array}$

New basis $B = \{1, 2\}$.

(LP) in canonical form for $B = \{1, 2\}$.

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & -1 & -2 & 1 \end{pmatrix} x + 7 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & 2 & 0.5 & 0.5 \\ 0 & 1 & -1 & 0.5 & -0.5 \end{pmatrix} x = \begin{pmatrix} 2.5 \\ 1.5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

Corresponding BFS:

$$x = \begin{pmatrix} 2.5 & 1.5 & 0 & 0 & 0 \end{pmatrix}^T$$

Obj. value $= 7$

Find $j \in N$, $\bar{c}_j > 0$ $(j = 5)$

$$\begin{array}{l} x_1 = 2.5 - 0.5\theta \geq 0 \\ x_2 = 1.5 + 0.5\theta \geq 0 \end{array} \implies \begin{array}{l} \theta \leq 5 \\ \theta \geq -3 \end{array} \rightarrow \begin{array}{l} x_1 \text{ leaves basis} \\ x_5 \text{ enters basis} \end{array}$$

New basis $B = \{2, 5\}$

(LP) in canonical form for $B = \{2, 5\}$

$$\begin{array}{ll} \max & \begin{pmatrix} -2 & 0 & -5 & -3 & 0 \end{pmatrix} x + 12 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

$$\left.\begin{array}{l} \text{BFS } x = \begin{pmatrix} 0 & 4 & 0 & 0 & 5 \end{pmatrix}^T \\ \text{Obj. value} = 12. \end{array}\right\} \text{Optimal Solution}$$

2.9.2 Iteration of simplex

Algorithm 2: Iteration of simplex

---

1 Start with feasible basis B
2 Rewrite LP in canonical form for B
3 Pick $j \in N : \bar{c}_j > 0$ (x_j enters basis)
4 Let $\bar{b} = A_B^{-1}b$, $\overline{A}_N = A_B^{-1}A_N$
Find largest θ so that $\bar{b} - \theta\overline{A}_j \geq 0$.
Corresponding basic variable that becomes 0 (say x_k) leaves basis.
5 $B \leftarrow B \setminus \{k\} \cup \{j\}$. Iterate.

If problem has optimal solution AND θ is always > 0, simplex finishes.

> **Note:**
> If at current BFS we have a basic variable $= 0$, we may have $\theta = 0$. $\to$ May lead to cycling. (i.e. return to current basis in future iteration)

Bland's Rule

If there are multiple choices of entering or leaving variables, always pick lowest index variable.

Using Bland's Rule avoids cycling

Observations If $\bar{c}_N \leq 0$, then the (LP) obj. value in canonical form is

$$\underbrace{\bar{c}_N^T}_{\leq 0}\underbrace{x_N}_{\geq 0} + c_B^T A_B^{-1}b \leq c_B^T A_B^{-1}b$$

For any feasible solution $\implies$ Current BFS is optimal

Original LP

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \end{array}$$

Dual

$$\begin{array}{ll} \min & b^T y \\ \downarrow & \\ \text{s.t.} & A^T y \geq c \end{array} \iff \begin{array}{ll} \min & y^T b \\ \downarrow & \\ \text{s.t.} & y^T A \geq c^T \end{array}$$

$$\iff \begin{array}{ll} \min & y^T b \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} y^T A_B \geq c_B^T \\ y^T A_N \geq c_N^T \end{array} \end{array}$$

If satisfies C.S with BFS corresponding to B

$$\implies \begin{array}{l} y^T A_B = c_B^T \\ \quad y^T = c_B^T A_B^{-1} \\ y_T A_N \geq c_N^T \end{array} \iff c_B^T A_B^{-1} A_N \geq c_N^T \iff \bar{c}_N \leq 0$$

2.9.3 Mechanics of Simplex

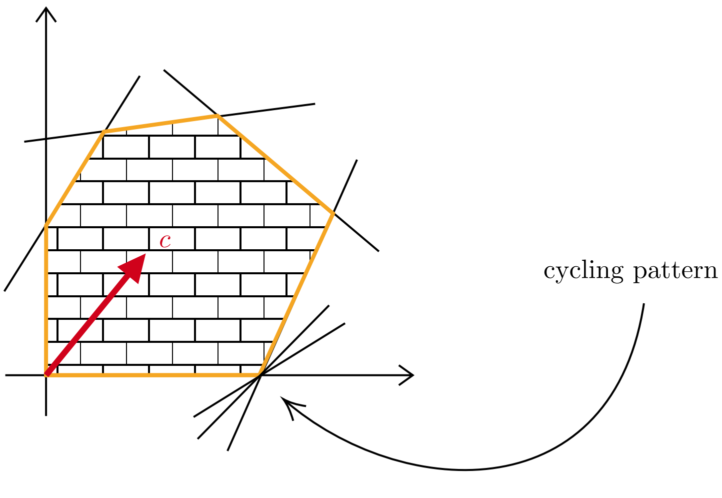


Figure 2.1: Simplex method

Example: 1

$$
\begin{array}{ll}
\max & \begin{array}{c} \begin{array}{ccccc} \text{enters basis} & & & j & \\ \uparrow & & & \uparrow & \end{array} \\ \begin{pmatrix}1 & 3 & -2 & 0 & 0\end{pmatrix} x \end{array} \\
\downarrow & \\
\text{s.t.} & \text{pivot} \begin{pmatrix}1 & 1 & 1 & 1 & 0\\ 1 & -1 & 3 & 0 & 1\end{pmatrix} x = \begin{pmatrix}4\\1\end{pmatrix} \quad \begin{array}{l} \\ \text{row } \ell \end{array} \\
 & x \geq 0
\end{array}
$$

For θ

$$\theta\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}x_4\\x_5\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix}$$

and we have

$$\begin{pmatrix}x_4\\x_5\end{pmatrix} = \begin{pmatrix}4-\theta\\1-\theta\end{pmatrix} \geq 0 \implies \begin{array}{l}\theta \leq 4\\ \boxed{\theta \leq 1}\end{array}$$

We are actually picking $\min\left\{\frac{4}{1}, \frac{1}{1}\right\}$

Pick, out of all rows $\min\left\{\frac{\bar{b}_i}{\bar{a}_{ij}}\right\}$ where j is entering variable.

Then now in row ℓ (second row here). Make row operations so that pivot element become 1, all others in col j becomes 0.

→ Row 2 ×1

→ Subtract tow 2 from row 1

→ subtract row 2 from objective function (with RHS multiplied by -1)

$$
\begin{array}{ll}
\max & \begin{array}{c} \begin{array}{ccccc} & j & & & \\ & \uparrow & & & \end{array} \\ \begin{pmatrix}0 & 4 & -5 & 0 & -1\end{pmatrix} x + 1 \end{array} \\
\downarrow & \quad \text{pivot} \\
\text{s.t.} & \begin{pmatrix}0 & 2 & -2 & 1 & -1\\ 1 & -1 & 3 & 0 & 1\end{pmatrix} x = \begin{pmatrix}3\\1\end{pmatrix} \quad \begin{array}{l} \text{row } \ell \\ \end{array} \\
 & x \geq 0
\end{array}
$$

$$2\theta + x_4 = 3 \iff x_4 = 3 - 2\theta \geq 0 \implies \theta \leq \frac{3}{2}$$
$$-\theta + x_1 = 1 \iff x_1 = \theta + 1 \geq 0 \implies \theta \geq -1$$

where we are finding $\min\limits_{\bar{a}_{ij}>0}\left\{\dfrac{\bar{b}_i}{\bar{a}_{ij}}\right\}$. Now follow the similar procedure, we have

$$\begin{aligned} &\max \quad \begin{pmatrix}0 & 0 & -1 & -2 & 1\end{pmatrix} x + 7 \\ &\downarrow \\ &\text{s.t.} \quad \begin{pmatrix}0 & 1 & -1 & 0.5 & -0.5 \\ 1 & 0 & 2 & 0.5 & 0.5\end{pmatrix} x = \begin{pmatrix}1.5 \\ 2.5\end{pmatrix} \end{aligned}$$

In general Pick $j \in N : \bar{c}_j > 0$.

Let $\ell = \operatorname*{argmin}\limits_{\bar{a}_{ij}>0}\left\{\dfrac{\bar{b}_i}{\bar{a}_{ij}}\right\}$ (Ratio Test)

- Multiply row ℓ by $\dfrac{1}{\bar{a}_{\ell j}}$
- Add $-\dfrac{\bar{a}_{ij}}{\bar{a}_{\ell j}}$ times row ℓ to row $i \neq \ell$.
- Add $-\dfrac{\bar{c}_j \cdot \bar{a}_{\ell k}}{\bar{a}_{\ell j}}$ to variable coeff in objective. $\forall k \in 1, \ldots, n$
- Add $\dfrac{\bar{b}_\ell \cdot \bar{c}_j}{\bar{a}_{ij}}$ to objective value in objective function

Example: 2

$$\begin{aligned} &\max \quad \begin{array}{c} \begin{array}{ccccc} j & & & & \\ \uparrow & & & & \end{array} \\ \begin{pmatrix}2 & 1 & 1 & 0 & 0\end{pmatrix} \end{array} x \\ &\downarrow \\ &\text{s.t.} \quad \text{pivot} \quad \begin{pmatrix}1 & 2 & -1 & 1 & 0 \\ 2 & -2 & -1 & 0 & 1\end{pmatrix} x = \begin{pmatrix}2 \\ 3\end{pmatrix} \qquad \text{row } \ell \\ &\qquad\quad x \geq 0 \end{aligned}$$

Ratio Test $\min\left\{\dfrac{2}{1}, \dfrac{3}{2}\right\} = 1.5$. $\ell = 2$. (x_2 enters, x_5 leaves)

$$\begin{aligned} &\max \quad \begin{array}{c} \begin{array}{ccccc} & & j & & \\ & & \uparrow & & \end{array} \\ \begin{pmatrix}0 & 3 & 2 & 0 & -1\end{pmatrix} \end{array} x + 3 \\ &\downarrow \\ &\text{s.t.} \quad \begin{pmatrix}0 & 3 & -0.5 & 1 & -0.5 \\ 1 & -1 & -0.5 & 0 & 0.5\end{pmatrix} x = \begin{pmatrix}0.5 \\ 1.5\end{pmatrix} \\ &\qquad\quad x \geq 0 \end{aligned}$$

If we increase $x_3 \to \theta$ and keep $x_2 = x_5 = 0$

$$\begin{aligned} -0.5\theta + x_4 &= 0.5 \\ -0.5\theta + x_1 &= 1.5 \end{aligned} \implies \begin{aligned} x_1 &= 1.5 + 0.5\theta \\ x_4 &= 0.5 + 0.5\theta \end{aligned} \to \text{ Problem is unbounded!}$$

In general Let B be a basis

$$\begin{aligned} &\max \quad \bar{c}_N^T x_N \\ &\downarrow \\ &\text{s.t.} \quad x_B + \bar{A}_N x_N = \bar{b} \\ &\qquad\quad x_B, x_N \geq 0 \end{aligned}$$

Found $j : \bar{c}_j > 0$ AND $\bar{A}_j \leq 0$.

Construct $d \in \mathbb{R}^n$ to reflect what we are trying to do when we increase $x_j \to \theta$.

Right now, we are at BFS:

$$\begin{pmatrix}x_B \\ x_N\end{pmatrix} = \begin{pmatrix}A_B^{-1}b \\ 0\end{pmatrix}$$

We want:

$$\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} A_B^{-1}b \\ 0 \end{pmatrix} + \theta \begin{pmatrix} d_B \\ d_N \end{pmatrix}$$

where $d_N = \begin{array}{cc} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} & \begin{matrix} \\ \\ \\ j \\ \\ \\ \\ \end{matrix} \end{array} = e_j$ and $d_B = -\overline{A}_j = -A_B^{-1}A_j$.

Found d: $d \geq 0$, then

$$Ad = A_B d_B + A_N d_N = -A_B A_B^{-1} A_j + A_j = 0$$

and

$$c^T d = c_B^T d_B + c_N^T d_N = -c_B^T A_B^{-1} A_j + c_j = \overline{c}_j > 0$$

i.e.,

$$\left.\begin{aligned} c^T d &> 0 \\ Ad &= 0 \\ d &\geq 0 \end{aligned}\right. \implies \text{Problem is unbounded}$$

But wait, how to find an initial BFS?

Given

$$\begin{aligned} &\max\downarrow && c^T x \\ &\text{s.t.} && Ax = b \\ & && x \geq 0 \end{aligned} \tag{LP}$$

where $b \geq 0$.

Construct auxiliary

$$\begin{aligned} &\max\downarrow && -e^T w \\ &\text{s.t.} && Ax + Iw = b \\ & && x, w \geq 0 \end{aligned} \tag{AUX}$$

Note:

- (AUX) is feasible ($x = 0, w = b$)
- (AUX) is bounded $-e^T w \leq 0$

So (AUX) has an optimal solution.

Proposition 2.14

(AUX) has optimal value 0 iff (LP) is feasible.

Proof:

If optimal solution (x^*, w^*) has value 0, then $w^* = 0$ so $Ax^* + I0 = b$

$\implies x^*$ is feasible for (LP)

If x is feasible for (LP) then $(x, 0)$ has value 0 in (AUX).

Moreover, if optimal value of (AUX) is < 0, then we can use the dual for a certificate.

$$\begin{array}{ll} \min & y^T b \\ \downarrow & \\ \text{s.t.} & y^T A \geq 0 \\ & y \geq -e \end{array} \tag{DAUX}$$

y^* optimal ${y^*}^T b < 0$ and ${y^*}^T A \geq 0$

$\implies y^*$ satisfies $\{x : Ax = b,\ x \geq 0\} = \varnothing$ $\square$

2.9.4 Two Stage Simplex

Phase 1

- write (AUX)
- solve (AUX) with BFS corresponding to w
- if opt value < 0, get certificate y^* (LP) is infeasible
- opt value 0, BFS x where $w = 0$

Phase 2

- simplex with x as initial BFS

Example: 1

$$\begin{array}{ll} \max & \begin{pmatrix} 2 & 1 & 3 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} x \begin{matrix} \leq \\ \geq \end{matrix} \begin{matrix} -1 \\ 3 \end{matrix} \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 2 & 1 & 3 & 0 & 0 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 \\ 1 & 1 & 2 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array} \tag{SEF}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array} \tag{AUX}$$

canonical form: $B = \{6, 7\}$

$$\begin{array}{ll} \max & \begin{pmatrix} -1 & 0 & 2 & -1 & -1 & 0 & 0 \end{pmatrix} x - 4 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & -1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array}$$

add 3 to the basis

$\min \left(\frac{b_i}{a_{i3}} \right) = \frac{3}{2}$

7 leaves the basis.

canonical form for $B = \{3, 6\}$

$$\begin{array}{ll} \max & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 0 & -1 \end{pmatrix} x - 1 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} -2 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1/2 & 1/2 & 1 & 0 & -1/2 & 0 & 1/2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} \end{array}$$

$x^* = \begin{pmatrix} 0 & 0 & \frac{3}{2} & 0 & 0 & 1 & 0 \end{pmatrix}$

certificate of infeasibility

$$\begin{aligned} y^T &= c_B^T A_B^{-1} \\ &= \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1/2 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \end{pmatrix} \end{aligned}$$

Example: 2

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -2 \end{pmatrix} x = \begin{pmatrix} 7 \\ -5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

in SEF.

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \end{array}$$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \end{array} \tag{AUX}$$

canonical form $B = \{4, 5\}$

$$\begin{array}{ll} \max & \begin{pmatrix} 3 & 2 & 3 & 0 & 0 \end{pmatrix} x - 12 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

1 enters basis $x + \theta d \quad d = \begin{pmatrix} 1 & 0 & 0 & -2 & -1 \end{pmatrix}^T$

$\min \left(\frac{b_i}{a_{i1}} \right) = \frac{7}{2}$

4 leaves the basis

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 1/2 & 3/2 & -3/2 & 0 \end{pmatrix} x - 3/2 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & -1/2 & 1 \end{pmatrix} x = \begin{pmatrix} 7/2 \\ 3/2 \end{pmatrix} \\ & x \geq 0 \end{array}$$

2 enters the basis

$\min \left(\frac{b_i}{a_{i2}} \right) = \frac{3/2}{1/2}$

5 leaves the basis

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 0 & -1 & -1 \end{pmatrix} x + 0 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 3 & -1 & 2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ & x \geq 0 \end{array}$$

Thus $x = \begin{pmatrix} 2 & 3 & 0 & 0 & 0 \end{pmatrix}$ is optimal for (AUX)

Forget (AUX). Start Simplex with $x = \begin{pmatrix} 2 & 3 & 0 \end{pmatrix}$ as initial BFS.

Now return to SEF.

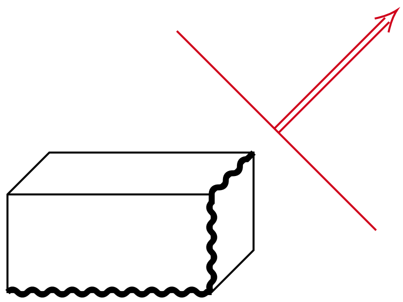
$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 0 & 2 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} x = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \\ & x \geq 0 \end{array} \tag{SEF}$$

canonical form for $B = \{1, 2\}$

$$\begin{array}{ll} \max & \begin{pmatrix} 0 & 0 & 3 \end{pmatrix} x + 2 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \end{array}$$

How long does simplex take?

At each pivot, we move from an extreme point to another.



Every pivot rule has a bad example.

Sprelman & Teng (2001): bad examples are pathological. Small changes become good examples.

Polynomial Hirsch Conjecture

~~Polynomially many vertex for bounded Polyhedral.~~

Let G be the graph of a d-polytope with n facets. Then the diameter of G is bounded above by a polynomial of d and n.

or

The (combinatorial) diameter of a polytope of dimension d with n facets cannot be greater than $n - d$.

Remark:

Here we call combinatorial diameter of a polytope the maximum number of steps needed to go from one vertex to another, where a step consists in traversing an edge.

What this conjecture tells us is that it will take only finitely many edges from initial BFS to optimal one.

There's one counterexample: 43-dimensional polytope with 86 facets and diameter (at least) 44.

2.10 Ellipsoid Algorithm

Feasibility Given polyhedron P, find $\overline{x} \in P$ or show $P = \varnothing$.

Fourier-Motzkin & simplex solve this problem.

Aside Given an algorithm an input I to it,

$$\text{size}(I) = \#\text{ of bits needed to represent } I.$$

> Example:
>
> $$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & Ax \le b \end{array}$$
>
> Assume $c \in \mathbb{Q}^n, A \in \mathbb{Q}^{m\times n}, b \in \mathbb{Q}^n$.
>
> By scaling, we may assume $c \in \mathbb{Z}^n, A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$.
>
> Let $\alpha = \max\{\|c\|_\infty, \|A\|_\infty, \|b\|_\infty\}$.
>
> Size of input to LP $\approx (n + n, m + m)\log(\alpha)$

Efficient Algorithm # of operations to solve an instance of size k are bounded by a polynomial on k.

Thus Simplex & FM NOT Efficient.

Goal Derive an efficient alg.

If you have an efficient algorithm to solve feasibility for any polyhedron P, can be used to solve LP.

Option 1

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \end{array}$$

Assume I know $L \le \text{OPT} \le U$.

Algorithm 3: Option 1

```
while Repeat do
    V = (L + U)/2
    P' = { x : Ax ≤ b, c^T x ≥ V }
    if P' == ∅ then
        U ← V
    else
        L ← V
```

Option 2

Is the following nonempty?

$$\left\{ x, y : \begin{array}{l} Ax \le b \\ y^T A = c^T \\ y \ge 0 \\ c^T x = b^T y \end{array} \right\}$$

2.10.1 Ellipsoid

Ball $B(z, R) := \{x \in \mathbb{R}^n : \|x - z\| \leq R\}$

Unit Ball $B := B(0, 1)$

Apply an affine map to B.

$f(x) = A(x - b)$ where $b \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$ invertible

$$f(B) := \{x \in \mathbb{R}^n : \|f(x)\| \leq 1\} = \{x \in \mathbb{R}^n : \|A(x - b)\| \leq 1\}$$

Sets of this form are **Ellipsoid**. Denoted $E(A, b)$.

Idea

- Suppose I know $P \subseteq B(0, R)$
- Also, suppose either $P = \varnothing$ OR $\mathrm{Vol}\, P \geq \epsilon > 0$.

Algorithm 4: Ellipsoid Algorithm

```
E ← E(M, z), where P ⊆ E(M, z).
while Vol(E) ≥ ϵ do
    if z ∈ P then
        STOP
    else
        • Find αᵀx ≤ α₀ so that αᵀx ≤ α₀, ∀x ∈ P and αᵀz > α₀
        • Find E(M′, z′) such that E ∩ {x : αᵀx ≤ α₀} ⊆ E(M′, z′) and volume of E(M′, z′) is
          much lower than E
        • E ← E(M′, z′)
```

> **Note:**
> At any point $P \subseteq E$.
>
> The reason why we choose ellipsoid instead of ball is that it can actually shrink "thinner" than ball.

Lemma 2.15

There exists $E(M', z')$ that can be computed in polynomial time such that

$$\frac{\mathrm{Vol}(E(M', z'))}{\mathrm{Vol}(E(M, z))} \leq e^{-\frac{1}{2n+2}}$$

Number of While Loop Iterations

If $B(0, R)$ initial ellipsoid, then $\mathrm{Vol}(B(0, R)) \leq (2R)^n$. After $k(2n + 2)$ iterations, $\mathrm{Vol}(E) \leq e^{-k}(2R)^n$.

We want

$$e^{-k}(2R)^n < \epsilon \implies -k + n\ln(2R) < \ln(\epsilon) \implies k \geq \lceil n\ln(2R) - \ln(\epsilon)\rceil$$

Alg stops after $\lceil n\ln(2R) - \ln(\epsilon)\rceil(2n + 2)$ iterations.

We only used that

$$z \notin P \iff \begin{array}{l} \exists \alpha^T x \leq \alpha_0 \text{ such that} \\ \quad \alpha^T \overline{x} \leq \alpha_0, \forall \overline{x} \in P \\ \quad \alpha^T z > \alpha_0 \end{array}$$

Theorem 2.16: Separating Hyperplane

Let C be a closed, convex set, $z \in \mathbb{R}^n$. Then $z \notin C \iff \exists$ a hyperplane $\alpha^T x \leq \alpha_0$ separating z and C.

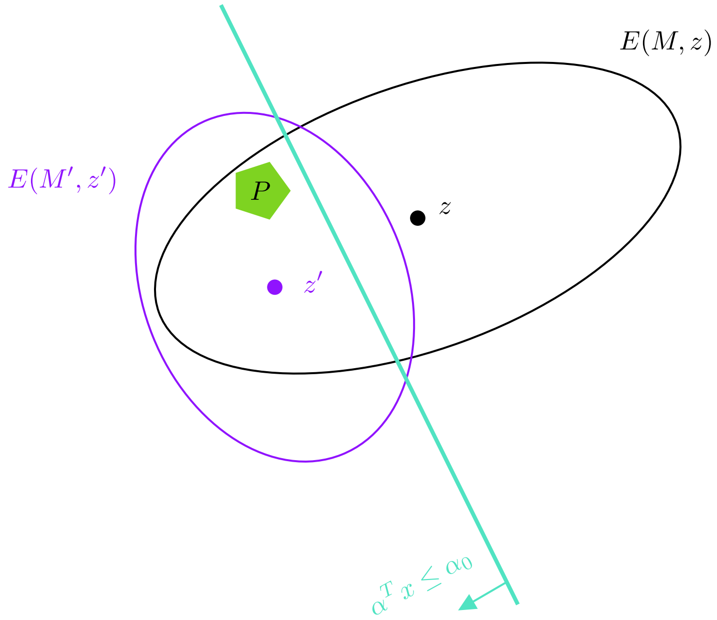


Figure 2.2: Ellipsoid Algorithm

Is runtime polynomial?

- $\ln(R)$ is polynomial in input size $\rightarrow$ NOT a problem
- Finding a separating hyperplane: can be done in polynomial time.

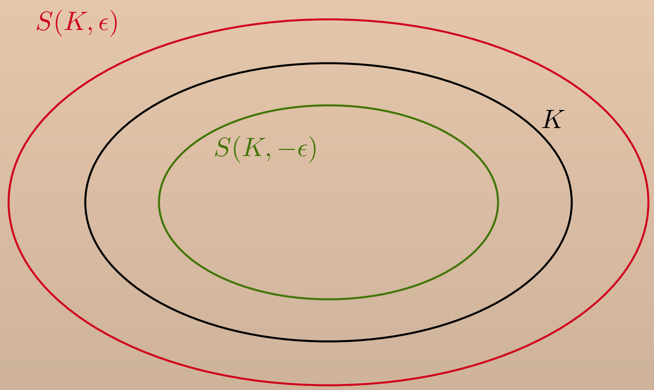
2.11 Grötchel-Lovász-Schrijver (GLS)

$S(K, \pm\epsilon)$

Let $K \subseteq \mathbb{R}^n$ be closed bounded convex set.

$$S(K, \epsilon) := \{x : \|x - y\| \leq \epsilon, \text{ for some } y \in K\}$$

$$S(K, -\epsilon) := \{x : S(x, \epsilon) \subseteq K\}$$



2.11.1 3 problems

- OPTIMIZATION

  Given $K \subseteq \mathbb{R}^n$, $c \in \mathbb{Q}^n$.

Find $x^* \in K$ such that
$$c^T x^* \geq c^T x, \forall x \in K$$
or determine $K = \varnothing$.

- SEPARATION

  Given $K \subseteq \mathbb{R}^n$, $w \in \mathbb{R}^n$.

  Determine if $w \in K$ or find α:
$$\|\alpha\|_\infty = 1 \qquad \alpha^T x < \alpha^T w, \forall x \in K$$

- FEASIBILITY

  Given $K \subseteq \mathbb{R}^n$.

  Find $\overline{x} \in K$ or determine $K = \varnothing$.

Feas $\leq_p$ Opt. (i.e. if we can solve opt efficiently, we can solve feas efficiently)

Weaker version...

- WEAK OPTIMIZATION

  Give $K \subseteq \mathbb{R}^n, c \in \mathbb{Q}^n, \epsilon > 0$

  Find $x^* \in S(K, \epsilon)$ such that
$$c^T x \leq c^T x^* + \epsilon, \qquad \forall x \in S(K, -\epsilon)$$
  or determine $S(K, -\epsilon) = \varnothing$

- WEAK SEPARATION

  Given $K \subseteq \mathbb{R}^n, w \in \mathbb{R}^n, \epsilon > 0$.

  Determine if $w \in S(K, \epsilon)$ or find α:
$$\|\alpha\|_\infty = 1 \qquad \alpha^T x < \alpha^T w + \epsilon, \forall x \in S(K, -\epsilon)$$

- WEAK FEASIBILITY

  Given $K \subseteq \mathbb{R}^n$.

  Determine $S(K, -\epsilon) = \epsilon$ or find $\overline{x} \in S(K, \epsilon)$

W-Feas $\leq_p$ W-Opt.

Ellipsoid gives us: W-Feas $\leq_p$ W-Sep.

- Grötchel-Lovász-Schrijver (GLS) have shown that

  W-SEP, W-Feas, W-OPT are polynomially equivalent.

In particular, for rational polyhedra[3] (even unbounded) then OPT, FEAS, SEP are polynomially equivalent.

Khachiyan ('80) used ellipsoid to give polytime algorithm for LPs.

2.11.2 Consequence of GLS

Example TSP: **complete** graph $G = (V, E)$

Edge costs $c_e, \forall e \in E$.

Find a tour visiting every vertex exactly once of min cost.

[3] $\{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$

IP formulation $x_e = \begin{cases} 1, & \text{if } e \text{ is in tour} \\ 0, & \text{otherwise} \end{cases}$

$$\begin{array}{ll} \min & \sum_{e \in E} c_e x_e \\ \downarrow & \\ \text{s.t.} & \sum_{e \in \delta(v)} x_e = 2, \quad \forall v \in V \end{array}$$

In general, $\delta(S) = \left\{ uv \in E : \begin{array}{c} u \in S \\ v \notin S \end{array} \right\}$ where $S \subseteq V$.

Subtour elimination $\displaystyle\sum_{e \in \delta(S)} x_e \geq 2, \quad \forall \varnothing \subsetneq S \subsetneq V$

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ \downarrow & & \\ & \sum_{e \in \delta(v)} x_e = 2, & \forall v \in V \\ \text{s.t.} & \sum_{e \in \delta(S)} x_e \geq 2, & \forall \varnothing \subsetneq S \subsetneq V \\ & x_e \in \{0,1\}, & \forall e \in E \end{array}$$

LP-relaxation Replace $x_e \in \{0,1\}$ by $0 \leq x_e \leq 1, \forall e \in E$.

Can I solve the LP in polynomial time on # vertices/edges?

Separation/Feasibility Given $\overline{x}_e, \forall e \in E$. Can I know if $\overline{x}_e$ if feasible for LP in time polynomial in # vertices?

If YES, GLS tells we can also solve OPT.

In polytime (in # vertices) I can check $\begin{cases} \sum_{e \in \delta(v)} \overline{x}_e = 2, & \forall v \in V \\ 0 \leq \overline{x}_e \leq 1, & \forall e \in E \end{cases}$

Min-Cut problem Given $G = (V, E), w_e \geq 0$. Find $\displaystyle\sum_{e \in \delta(S)} w_e$

Problem can be solved in polytime in # vertices.

Then we solve mincut with $w_e = \overline{x}_e$. If optimal value is ≥ 2, then $\overline{x}$ feasible for LP. Otherwise found $S : \displaystyle\sum_{e \in \delta(S)} \overline{x}_e < 2$.

3 Integer Programming

An integer program is a problem of the form:

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax \leq b \\ x_i \in \mathbb{Z}, \forall j \in I \end{array} \end{array}$$

where $\varnothing \neq I \subseteq \{1, \dots, n\}$.

If $I = \{1, \dots, n\}$, it's pure IP. Otherwise, Mixed IP (MIP).

If all variables are constrained to be in $\{0, 1\}$, it's a Binary IP.

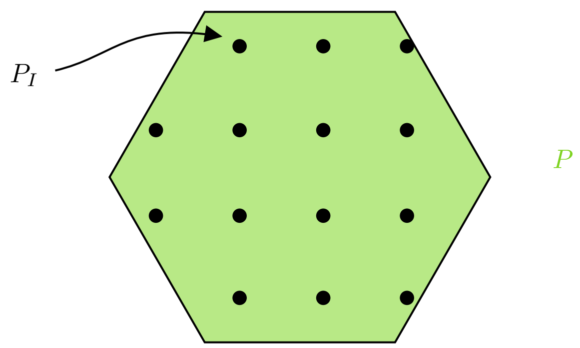
Key Assumption: All data is rational ($A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$) i.e, $Ax \leq b$ is a rational polyhedron.

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$, $P_I = P \cap \{x_j \in \mathbb{Z} : j \in I\}$.

Theorem 3.1

$\text{conv}(P_I)$ is a polyhedron.

From now on, assume we have a pure IP.



recession cone

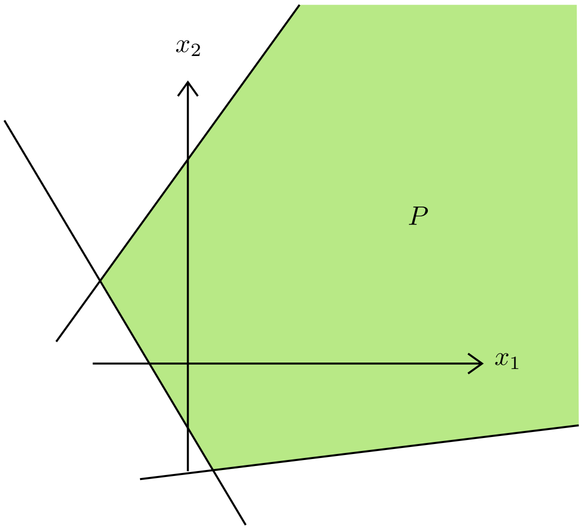
Let P be a polyhedron. Its recession cone is

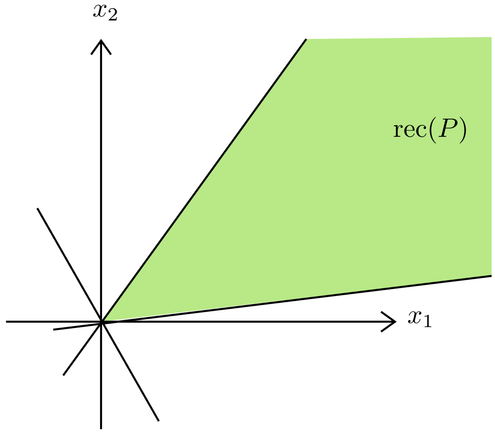
$$\text{rec}(P) := \left\{ r \in \mathbb{R}^n : \begin{array}{l} \forall \overline{x} \in P \\ \forall \lambda \geq 0 \\ \overline{x} + \lambda r \in P \end{array} \right\}$$

Lemma 3.2

Let $P = \{x \in \mathbb{R}^n : Ax \le b\} \neq \varnothing$ then

$$\underbrace{\operatorname{rec}(P)}_{R_1} = \underbrace{r \in \mathbb{R}^n : Ar \le 0}_{R_2}$$





Proof:

$R_2 \subseteq R_1$) Let $\overline{x} \in P, \lambda \ge 0, r \in R_2$

$$A(\overline{x} + \lambda r) = A\overline{x} + \lambda A r \le b \implies \overline{x} + \lambda r \in P \implies r \in R_1$$

$R_1 \subseteq R_2$) Let $r \notin R_2$, i.e., $\exists i : a_i^T r > 0$

Let $\overline{x} \in P$, it is clear $\exists \lambda > 0 : a_i^T(\overline{x} + \lambda r) > b_i \implies r \notin R_1$.

□

Theorem 3.3

$P \neq \varnothing$ is a bounded polyhedron

$\iff P = \operatorname{conv}(x^1, \ldots, x^k)$ for some vectors $x^1, \ldots, x^k \in \mathbb{R}^n$.

$conv(x^1, \ldots, x^k)$ is smallest convex set containing $x^1, \ldots, x^k \iff$ set of all finite combinations of $x^1, \ldots, x^k$.

Proof:

$\Leftarrow$) $P = \left\{ x \in \mathbb{R}^n : \begin{array}{r} x = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \ge 0 \end{array} \right\}$

$P' = \left\{ (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^k : \begin{array}{r} x = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \ge 0 \end{array} \right\}$ is a bounded polyhedron.

$P = \operatorname{proj}_x P'$ which is a bounded polyhedron.

$\Rightarrow$) P bounded $\implies$ P has no lines.

Let $x^1, \ldots, x^k$ be extreme points. Want to show $P = conv(x^1, \ldots, x^k)$

$P \supseteq conv(x^1, \dots, x^k)$ follows since P is a convex set containing $x^1, \dots, x^k$.

Suppose $\exists \overline{x} \in P \setminus conv(x^1, \dots, x^k)$

Consider

$$\begin{array}{lllll} \min & 0^T\lambda & & & \\ \downarrow & & & & \\ & \sum_{i=1}^k \lambda_i x^i & = \overline{x} & \alpha \in \mathbb{R}^n & \\ \text{s.t.} & \sum_{i=1}^k \lambda_i & = 1 & \alpha_0 \in \mathbb{R} & \\ & \lambda & \geq 0 & & \end{array} \tag{1}$$

and its dual

$$\begin{array}{ll} \max & \alpha^T \overline{x} + \alpha_0 \\ \text{s.t.} & \alpha^T x^i + \alpha_0 \leq 0, \quad \forall i = 1, \dots, k \end{array} \tag{2}$$

$(\alpha, \alpha_0) = (0, 0)$ feasible for (2). By assumption, (1) is infeasible.

Let $(\overline{\alpha}, \overline{\alpha}_0)$ be such that $\overline{\alpha}^T \overline{x} + \overline{\alpha}_0 > 0$

Now consider

$$\begin{array}{ll} \max & \overline{\alpha}^T x + \overline{\alpha}_0 \\ \text{s.t.} & x \in P \end{array} \tag{3}$$

(3) has optimal solution since $P \neq \varnothing$ bounded and its has an optimal extreme point, i.e., $\overline{\alpha}^T x^i + \overline{\alpha}_0$ is optimal value. But by (2)

$$\overline{\alpha}^T x^i + \overline{\alpha}_0 \leq 0 < \overline{\alpha}^T \overline{x} + \overline{\alpha}_0$$

Contradiction.

□

Back to IP...

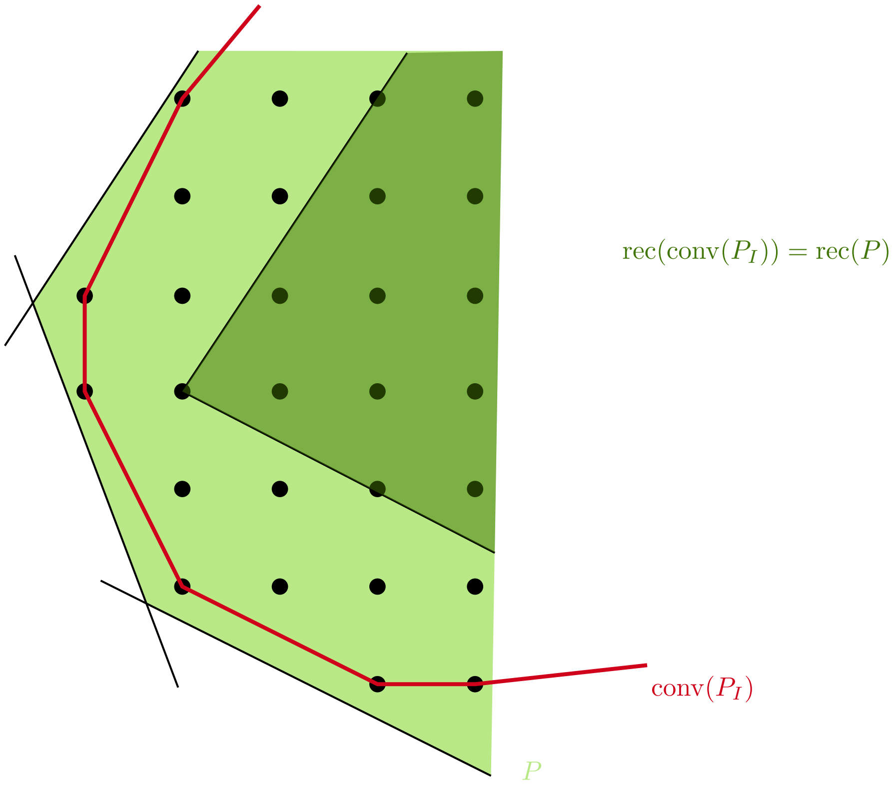
Theorem 3.4

If P is a rational polyhedron, then $\text{conv}(P_I)$ is also a rational polyhedron ($P_I = P \cap \mathbb{Z}^n$). Moreover, if $P_I \neq \varnothing$, $\text{rec}(\text{conv}(P_I)) = \text{rec}(P)$.

Proof:

Done if P is bounded ($\{0\}$).

Skipped for unbounded P.



□

Theorem 3.5

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I \end{array} = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & \text{conv}(P_I) \end{array}$$

Note:

1. Using Fund Thm of LP. I know IP is either infeas., unbounded, or $\exists$ opt. sol.

2. If $P_I \neq \varnothing$, then unboundedness can be detected by checking if $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$ is unbounded. Since $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$ unbounded iff $P \neq \varnothing$ and $\exists r : \begin{array}{l} c^T r > 0 \\ Ar \leq 0 \end{array}$.

   $P_I \neq \varnothing \implies P \neq \varnothing$. But then this implies $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in conv(P_I) \end{array}$ unbounded.

Proof:

WMA (we may assume) $P_I \neq \varnothing$.

Let $z_1 = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I \end{array}$, $z_2 = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in conv(P_I) \end{array}$.

Since $P_I \subseteq conv(P_I) \implies z_1 \leq z_2$.

Now let $x^* \in conv(P_I) \implies \begin{array}{r} x^* = \sum_{i=1}^k \lambda_i x^i \\ \sum_{i=1}^k \lambda_i = 1 \\ \lambda \geq 0 \end{array}$ for $x^1, \ldots, x^k \in P_I$.

$\implies \exists i : c^T x^i \geq c^T x^*$ since otherwise

$$c^T x^* = \sum_{i=1}^{k} \lambda_i (c^T x^*) > \sum_{i=1}^{k} \lambda_i (c^T x^i) = c^T \left(\sum_{i=1}^{k} \lambda_i x^i \right) = c^T x^*$$

contradiction $\implies z_1 \geq z_2$. □

Corollary 3.6

If $P \neq \varnothing$ and pointed. Then $\mathrm{conv}(P_I)$ is pointed and any extreme point of $\mathrm{conv}(P_I)$ is integral.

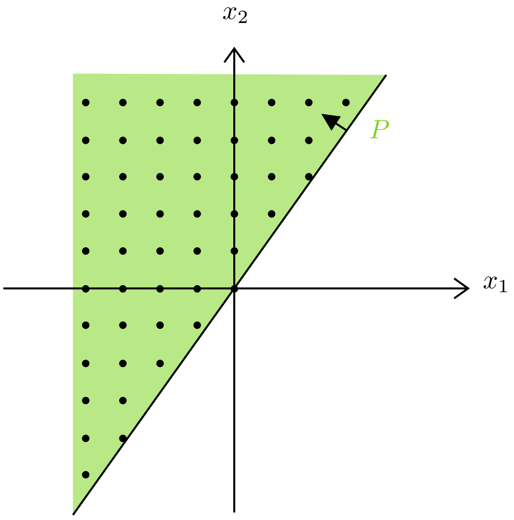
Proof:
$\mathrm{rec}(P) = \mathrm{rec}(conv(P_I))$ implies $conv(P_I)$ pointed.

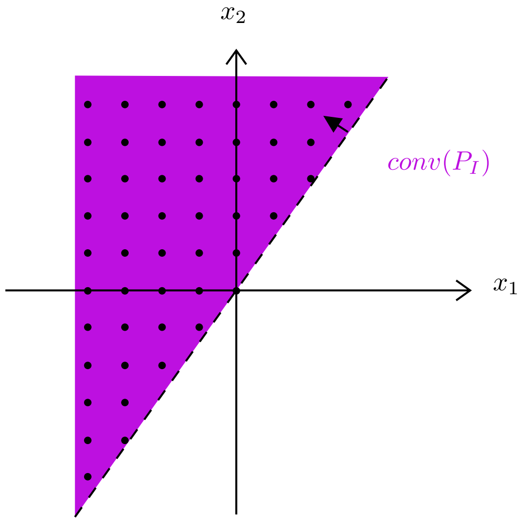
Let x^* be extreme point of $conv(P_I)$. Let c be such that x^* is unique optimal solution to $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in conv(P_I) \end{array}$.

By theorem, $\exists \overline{x} \in P_I : c^T \overline{x} = c^T x^*$.

By uniqueness of x^*, $\overline{x} = x^*$, then x^* is integral. □

Note:
$P = \{x \in \mathbb{R}^2 : x_2 \geq \sqrt{2} x_1\}$





$conv(P_I)$ is not even closed (dotted line plus $(0,0)$), NOT a polyhedron.

3.1 Cutting Plane Algorithm

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I := P \cap \mathbb{Z}^n \end{array} \tag{IP}$$

where P is rational polyhedron.

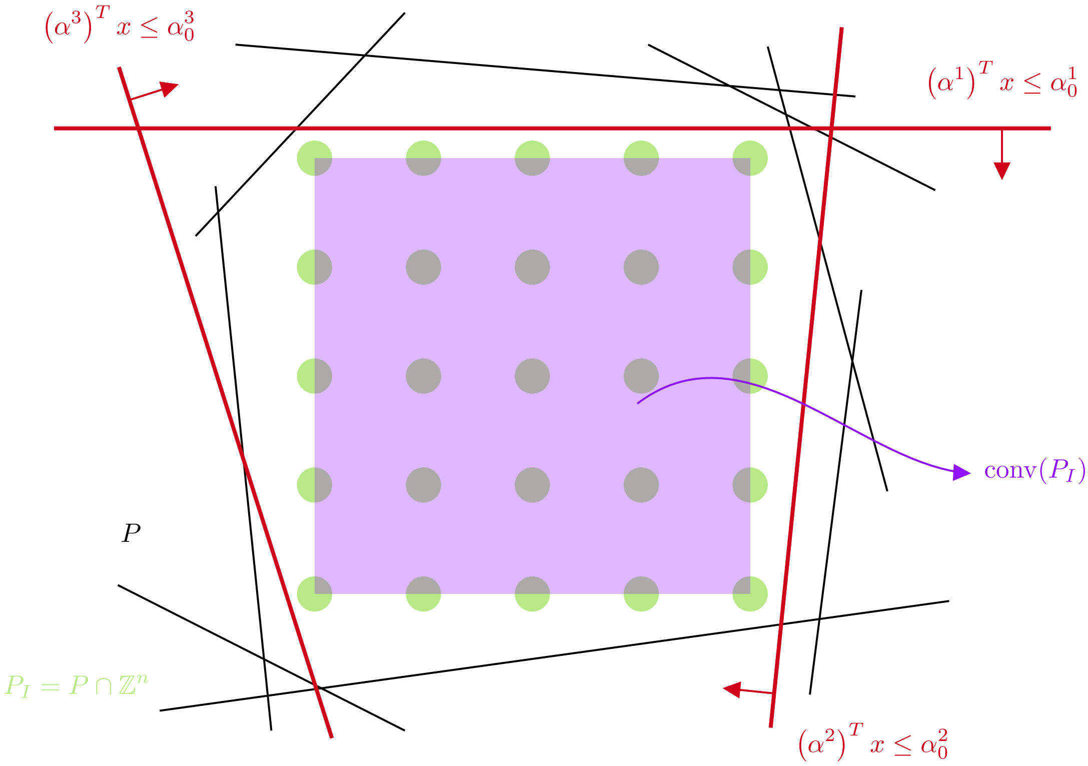
We know it can be solved by solving $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & \mathrm{conv}(P_I) \end{array}$

Problem Hard to compute $\mathrm{conv}(P_I)$.

$\mathrm{conv}(P_I)$ is smallest convex set containing P_I. P is **a** convex set containing P_I.

Idea

- Start with P
- Iteratively make P "closer" to $\text{conv}(P_I)$



Idea 2 Want to know only part of $\text{conv}(P_I)$ that is in the "direction I am optimizing".

LP relaxation

The LP you obtain from (IP) after dropping integrality, i.e.,

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P \end{array}$$

valid ineq

An ineq $\alpha^T x \leq \alpha_0$ is valid for $S \subseteq \mathbb{R}^n$ if $\forall \overline{x} \in S$: $\alpha^T \overline{x} \leq \alpha_0$.

Assumption LP relaxation has an optimal solution.

If $P = \varnothing$, then $P_I = \varnothing$. If LP relaxation is unbounded, either $P_I = \varnothing$ or (IP) is unbounded.

Algorithm 5: Cutting Plane Algorithm

$R \leftarrow P$
do
    Let x^* be optimal solution to $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in R \end{array}$
    if x^* *is integral* **then**
        `STOP // ` x^* ` is opt sol for (IP)`
    else
        Find valid ineq $\alpha^T x \leq \alpha_0$ for $\text{conv}(P_I)$ s.t. $\alpha^T x^* > \alpha_0$
        $R \leftarrow R \cap \{x : \alpha^T x \leq \alpha_0\}$
while $R \neq \varnothing$;
Declare (IP) infeasible

Issues...

1. α, α_0 must be rational
2. Finiteness?
3. How to find α, α_0?

Note:
Any any point $P_I \subseteq \text{conv}(P_I) \subseteq R \subseteq P$.

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I \end{array} \leq \begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in R \end{array}$$

If $x^* \in \mathbb{Z}^n$, then $x^* \in P_I$.

$$\implies \begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in P_I \end{array} \geq c^T x^* \implies x^* \text{ is optimal for } P_I$$

To solve the issues, impose x^* being an opt. BFS of $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in R \end{array}$

Proposition 3.7

Let R be a pointed rational polyhedron such that $R \cap \mathbb{Z}^n = P_I$. Let x^* be a BFS of R.

Then x^* is integral $\iff$ $x^* \in \text{conv}(P_I)$

Proof:
Exercise. $\square$

How to find valid ineq for $\text{conv}(P_I)$ $\alpha_T x \leq \alpha_0$ s.t. $\alpha^T x^* > \alpha_0$?

Call such ineq. a **CUTTING PLANE** or a **CUT** separating $\text{conv}(P_I)$ and x^*.

Assumption $R = \left\{ x \in \mathbb{R}^n : \begin{array}{c} Ax = b \\ x \geq 0 \end{array} \right\}$.

$$\begin{array}{ll} \max & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \end{array} \tag{1}$$

Let B be opt. basis.

$$(1) \quad \iff \quad \begin{array}{ll} \max & \bar{c}_N^T x_N + c_B^T A_B^{-1} b \\ \downarrow & \\ \text{s.t.} & x_B + \overbrace{A_B^{-1} A_N}^{\overline{A}_N} x_N = \overbrace{A_B^{-1} b}^{\bar{b}} \\ & x \geq 0 \end{array}$$

$$x^* \text{ is integral } \iff A_B^{-1} b \in \mathbb{Z}^m$$

If x^* is not integral, then $\exists i \in \{1, \dots, m\} : (A_B^{-1} b)_i \notin \mathbb{Z}$.

Look at constraint

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i$$

is valid for P_I since it is valid for R.

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \bar{b}_i$$

is valid for P_I since it is valid for R.

Since $\lfloor \bar{a}_{ij} \rfloor \leq \bar{a}_{ij}$ and $x_j \geq 0 \implies \lfloor \bar{a}_{ij} \rfloor x_j \leq \bar{a}_{ij} x_j$.

Since LHS is integer $\forall x \in P_I$,

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor \tag{$\star$}$$

is valid for P_I.

> **Note:**
> For x^*, $\quad x_j^* = 0, \forall j \in N$ $x_i^* = \bar{b}_i$.
>
> Thus
> $$x_i^* + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j^* = \bar{b}_i > \lfloor \bar{b}_i \rfloor$$

($\star$) is the cut we wanted. Called a Chvátal-Gomory (CG) cut.

Algorithm 6: Cutting Plane Algorithm (Correct)

$R \leftarrow P$ // `(P pointed)`
do
  Let x^* be optimal BFS solution to $\begin{array}{ll} \max & c^T x \\ \text{s.t.} & x \in R \end{array}$
  if x^* *is integral* **then**
    STOP `// x* is opt sol for (IP)`
  else
    Find valid ineq $\alpha^T x \leq \alpha_0$ for $\text{conv}(P_I)$ s.t. $\alpha^T x^* > \alpha_0$
    $R \leftarrow R \cap \{x : \alpha^T x \leq \alpha_0\}$
while $R \neq \varnothing$;
Declare (IP) infeasible

Theorem 3.8

The cutting plane algorithm using CG cuts terminates in finitely many iterations (for pure IPs).

Proof:
SKIPPED. $\square$

Example:

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 3 & -2 & 0 & 0 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ & x \geq 0, \qquad x \in \mathbb{Z}^5 \end{array}$$

Opt basis for LP relaxation: $B = \{2, 5\}$.

In canonical form:

$$\begin{array}{ll} \max & \begin{pmatrix} -0.5 & 0 & -3.5 & -1.5 & 0 \end{pmatrix} x + 4.5 \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 0.5 & 1 & 0.5 & 0.5 & 0 \\ 1.5 & 0 & 3.5 & 0.5 & 1 \end{pmatrix} x = \begin{pmatrix} 1.5 \\ 2.5 \end{pmatrix} \\ & x \geq 0 \end{array}$$

and $x^* = \begin{pmatrix} 0 & 1.5 & 0 & 0 & 2.5 \end{pmatrix}^T$

CG-cut:

$$\begin{array}{rl} 0x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \leq 1 \iff x_2 \leq 1 & \text{From 1st constraint} \\ x_1 + 3x_3 + x_5 \leq 2 & \text{CG-cut from 2nd constraint} \end{array}$$

Can add both to R.

New LP

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 3 & -2 & 0 & 0 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 \end{pmatrix} x \begin{matrix} = \\ = \\ \leq \\ \leq \end{matrix} \begin{pmatrix} 3 \\ 1 \\ 1 \\ 2 \end{pmatrix} \\ & x \geq 0 \end{array}$$

Add $x_6, x_7 \geq 0$ convert to SEF, where

$$x_2 + x_6 = 1, \qquad x_1 + 3x_3 + x_5 + x_7 = 2$$

If $x_1, \ldots, x_5 \in \mathbb{Z}$, then $x_6, x_7 \in \mathbb{Z}$.

New Opt for LP:

$$x^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So opt sol to original LP is $\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \end{pmatrix}$.

3.2 Total Unimodularity

totally unimodular

A matrix U is called totally unimodular (TU) if all its square submatrices have determinant in $\{-1, 0, 1\}$.

Example:

$\begin{pmatrix} \boxed{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is not TU.

$\begin{pmatrix} \boxed{1} & 1 & \boxed{-1} & 0 \\ 0 & 0 & 0 & 0 \\ \boxed{1} & 0 & \boxed{1} & 1 \end{pmatrix}$ is NOT TU.

Note:
Square submatrices are obtained by deleting rows/columns.

$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ is TU.

Theorem 3.9

If $A \in \mathbb{Z}^{m \times n}$ is TU and $b \in \mathbb{Z}^m$ then every BFS of $P = \left\{ x \in \mathbb{R}^n : \begin{array}{c} Ax = b \\ x \geq 0 \end{array} \right\}$ is integral.

Recall

Cramer's Rule

If D is $n \times n$ invertible, then unique solution to $Dx = b$ is given by

$$x_i = \frac{\det D(i)}{\det D}$$

where $D(i)$ is D replacing i-th column with b.

Example:

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solution

$$x_1 = \frac{\det \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}} = \frac{7}{3}, \qquad x_2 = \frac{\det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}}{\det \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}} = \frac{1}{3}$$

Proof:

Let x^* be a BFS of $\left\{ x : \begin{array}{c} Ax = b \\ x \geq 0 \end{array} \right\}$, B corresponding basis.

Then $x^*_B = A_B^{-1} b, x^*_N = 0$

Note x^*_B is unique solution to $A_B x_B = b$

$\implies$ By Cramer's rule,

$$x^*_i = \frac{\det A_B(i)}{\det A_B} \in \mathbb{Z}$$

since $\det A_B(i) \in \mathbb{Z}$ and by TU, $\det A_B \in \{1, -1\}$ which cannot be 0 since invertible. $\square$

Note:

Result remains true if $P = \{x : Ax \leq b\}$ or $P = \left\{ x : \begin{array}{c} Ax \leq b \\ x \geq 0 \end{array} \right\}$

integral

We say a polyhedron is integral if all its extreme points are integral.

Lemma 3.10

P is an integral polyhedron iff $P = \text{conv}(P \cap \mathbb{Z}^n)$.

Proof:
Exercise. □

Lemma 3.11

Let $A \in \mathbb{Z}^{m\times n}$ TU.

Then applying any of the following operations on A yields a TU matrix.

a) Delete row/column

b) Multiply row/column by -1

c) Permute rows/columns

d) Transpose

e) Duplicate row/column

f) Add a row/column with at most one nonzero entry, which is in $\{+1,-1\}$.

Proof:

a) ✓

b)-d) Potentially changes signs of det.

e) Only can create new submatrices if row and its duplicate are in it. But that has det $= 0$.

f) Recall

Laplace formula

D square:

$$D = \begin{pmatrix} & | & \\ -- & d_{ij} & -- \\ & | & \end{pmatrix}$$

Let M_{ij} be the matrix obtained by deleting row i, column j.

Then for any row i of D:

$$\det(D) = \sum_j (-1)^{i+j} d_{ij} \det(M_{ij})$$

For any column j:

$$\det(D) = \sum_i (-1)^{i+j} d_{ij} \det(M_{ij})$$

$$A' = \left(\begin{array}{c|c} 1 & \\ 0 & \\ 0 & \\ \vdots & A \\ \vdots & \\ \vdots & \\ 0 & \end{array}\right)$$

Let D be square submatrix of A'. If D does not contain first col, then $\det(D) \in \{\pm 1, 0\}$ since A is TU.

If D does not contain first row, but contains first column, then $\det(D) = 0$.

Else,

$$D = \left(\begin{array}{c|ccccc} 1 & \times & \times & \times & \times & \times \\ \hline 0 & & & & & \\ \vdots & & & \overline{D} & & \\ 0 & & & & & \\ 0 & & & & & \end{array}\right)$$

By Laplace formula: $|\det(D)| = |\det(\overline{D})| \in \{0, 1\}$.

$\square$

Application 1 Suppose A is TU $\in \mathbb{Z}^{m\times n}$. If $b \in \mathbb{Z}^m$ and $\ell, u \in \mathbb{Z}^n$, then

$$P = \left\{ x \in \mathbb{R} : \begin{array}{l} Ax \leq b \\ \ell \leq x \leq u \end{array} \right\}$$

is integer polyhedron.

$$P = \left\{ x \in \mathbb{R}^n : \underbrace{\begin{pmatrix} A \\ I \\ -I \end{pmatrix}}_{A'} x \leq \underbrace{\begin{pmatrix} b \\ u \\ -\ell \end{pmatrix}}_{b'} \right\}$$

b' integral, A' TU $\implies$ P is integral

Application 2 $A \in \mathbb{Z}^{m\times n}$ TU, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, then

$$\begin{array}{ll|ll} \max & c^T x & \min & b^T y \\ \downarrow & & \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax \leq b \\ x \geq 0 \end{array} & \text{s.t.} & \begin{array}{l} A^T y \geq c \\ y \geq 0 \end{array} \end{array}$$

have integral opt solutions (if both are feasible).

3.3 Sufficient condition for TU

Lemma 3.12

Let $A \in \mathbb{Z}^{m\times n}$ with entries $\{-1, 0, 1\}$. If A has:

- At most two nonzeros per column, AND
- There exists a partition I_1, I_2 of its rows such that, for every column:
  - i) Nonzero entries of same sign lie in different partitions
  - ii) Nonzero entries of opposite signs lie in same partition.

Then A is TU.

Example:

$$A = \left(\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

above the line: I_1; below: I_2. A is TU.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Line 1 and line 3: I_1; Line 2 and 4: I_2. A is TU.

Proof:

Suppose Lemma is False. Let M be a minimal counterexample, i.e.,

- M is not TU,
- M satisfies conditions of Lemma,
- Any submatrix of M is TU.

Then M itself is a square matrix with $\det(M) \notin \{-1, 0, 1\}$ and all its submatrix have det $\in \{-1, 0, 1\}$.

If M has ≤ 1 nonzero in a column, then M is obtained by adding a column with at most 1 nonzero to a TU matrix $\implies$ M is TU (By Lemma 3.11).

Thus, we may assume all columns of M has exactly two nonzero elements.

$$M = \begin{pmatrix} - & M_1^T & - \\ & \vdots & \\ - & M_m^T & - \end{pmatrix}$$

Consider:

$$\sum_{i \in I_1} M_i - \sum_{i \in I_2} M_i = 0$$

since i) and ii) hold. Then this means $\{M_i\}_{i=1}^m$ are **not** linearly independent, which implies $\det(M) = 0$. $\square$

Example:

Given $G = (V, E)$ undirected simple graph.

G is bipartite if $V = \underbrace{V_1 \dot\cup V_2}_{\text{disjoint union}}$ and $\forall u, v \in E$ has $u \in V_1, v \in V_2$.

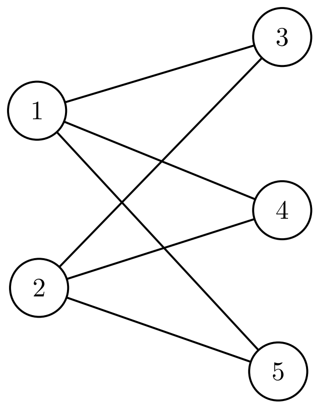
$M \subseteq E$ is a matching if $|M \cap \delta(v)| \leq 1, \forall v \in V$ where $\delta(v) := \{e \in E : v \text{ is an endpoint of } e\}$.

Given G bipartite. **Goal**: Find max carnality matching.

Let $x_e \in \{0, 1\}$ and $x_e = \begin{cases} 1, & \text{if } e \in M \\ 0, & \text{if } e \notin M \end{cases}$.

$$\begin{array}{ll} \max & \sum_{e \in E} x_e \\ \downarrow & \\ \text{s.t.} & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall c \in V \\ & x \in \{0,1\}^E \end{array} \tag{1}$$

Let's now take a look at example.



$x = \begin{pmatrix} x_{13} & x_{14} & x_{15} & x_{23} & x_{24} & x_{25} \end{pmatrix}^T$

$$\begin{array}{ll} \max & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} x \\ \downarrow & \\ \text{s.t.} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \text{vertex} \end{array} \\ & x \in \{0,1\}^E \end{array}$$

In general:

- $I_1 \to$ constraints correspond to V_1
- $I_2 \to$ constraints correspond to V_2

If we look at a column x_{uv}, it will have a 1 in row of u a 1 in row of v, 0 everywhere else.

$\to$ Bipartite $\implies$ Lemma is satisfied $\implies$ (1) can be solved via LP.

Let (2) be LP relaxation of (1) without $x_e \leq 1, \forall e \in E$, otherwise the first constraint is violated.

$$\begin{array}{ll} \max & \sum_{e \in E} x_e \\ \downarrow & \\ \text{s.t.} & \sum_{e \in \delta(v)} x_e \leq 1, \quad \forall c \in V \\ & x \geq 0 \end{array} \tag{2}$$

Let us write the dual of (2)

$$\begin{array}{ll} \min & \sum_{v \in V} y_v \\ \downarrow & \\ \text{s.t.} & y_u + y_v \geq 1, \quad \forall uv \in E \\ & y \geq 0 \end{array} \tag{3}$$

and add integral constraints,

$$\begin{array}{ll} \min & \sum_{v \in V} y_v \\ \downarrow & \\ \text{s.t.} & y_u + y_v \geq 1, \quad \forall uv \in E \\ & y \in \{0,1\}^V \end{array} \tag{4}$$

Let z_i be the optimal value for (i) then

$$z_1 \leq z_2 = z_3 \leq z_4$$

G bipartite $\implies \begin{array}{l} z_1 = z_2 \\ z_3 = z_4 \end{array}$

Vertex Cover: such that $\forall e \in E, |e \cap U| \geq 1$. **Problem**: Finding smallest vertex cover.

König's Theorem

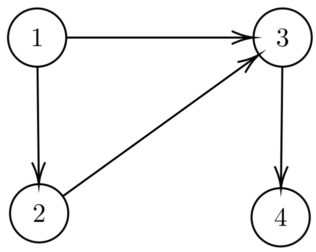
In bipartite graph G, size of largest matching = size of smallest vertex cover.

Example:

Consider a directed graph $D = (V, A)$.

Incidence matrix of D has one row per vertex, one column per arc.

For $v \in V$, $(w, y) \in A$, then $a_{ve} = \begin{cases} -1, & \text{if } v = w \\ 1, & \text{if } v = y \\ 0, & \text{otherwise} \end{cases}$



$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{pmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

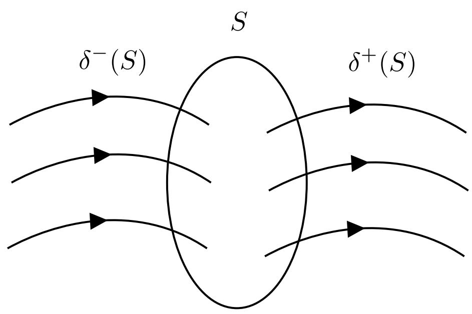
$I_1 =$ everything, $I_2 = \varnothing \implies$ Matrix is TU

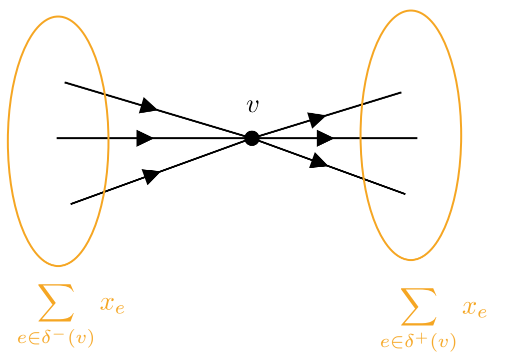
Max Flow: Given $D = (V, A)$, $\quad s, t \in V (s \neq t)$. An s-t flow is a nonnegative vector $x \in \mathbb{R}^A$, where

$$\sum_{e \in \delta^-(v)} x_e - \sum_{e \in \delta^+(v)} x_e = 0, \quad \forall v \in V \setminus \{s, t\}$$

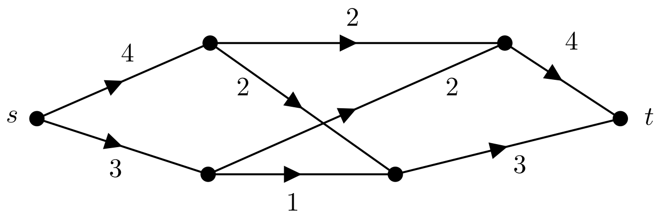
where

$$\delta^-(S) = \left\{ (u, v) \in A : \begin{array}{c} u \notin S \\ v \in S \end{array} \right\} \qquad \text{and} \qquad \delta^+(S) = \left\{ (u, v) \in A : \begin{array}{c} u \in S \\ v \notin S \end{array} \right\}$$





Goal: Find a flow maximizing $\sum_{e \in \delta^+(S)} x_e$



also $0 \leq x_e \leq c_e, \forall e \in A$ where c_e is some capacity constraint.

TU $\implies$ max flow is integral if $c_e \in \mathbb{Z}, \forall e \in A$.

Theorem 3.13

An $m \times n$ integral matrix A is TU iff for every subset $R \subseteq \{1, \ldots, m\}$, there exists a partition of R into R_1, R_2 (that is, $R_1 \cup R_2 = R$ and $R_1 \cap R_2 = \varnothing$) such that

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\}, \forall j = 1, \ldots, n$$

Note:
Careful that in the previous result that we had seen, we just needed to partition the original rows into two such sets.

This result says that if I pick ANY SUBSET of rows, I must be able to do the same.

Skipped branch-and-bound, Minimum Cost Perfect Matching in Bipartite Graphs... due to one week suspension

4 Nonlinear Programming

The general form: Let $f, g_1, \dots, g_m : \mathbb{R}^m \to \mathbb{R}$.

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \le 0, \quad \forall i = 1, \dots, m \end{array} \tag{NLP}$$

Note that this is minimization problem with "$\le$" constraints.

Example: Linear Programs

$f(x) := c^T x$ and $g_i(x) := a_i^T x - b_i$. These give us

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & a_i^T x \le b_i, \quad \forall i = 1, \dots, m \end{array}$$

Example: Binary integer program

Let $f(x) := c^T x$, $g_1(x) := x_1(1 - x_1)$ and $g_2(x) := -x_1(1 - x_1)$. These give us

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x_1(1 - x_1) = 0 \end{array}$$

where the constraint is equivalent to $x_1 \in \{0, 1\}$. Extend it to

$$\begin{array}{ll} \min & c^T x \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} Ax \le b \\ x \in \{0,1\}^n \end{array} \end{array}$$

4.1 Convex functions

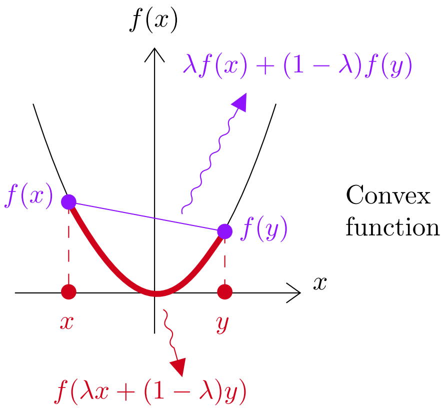
convex functions

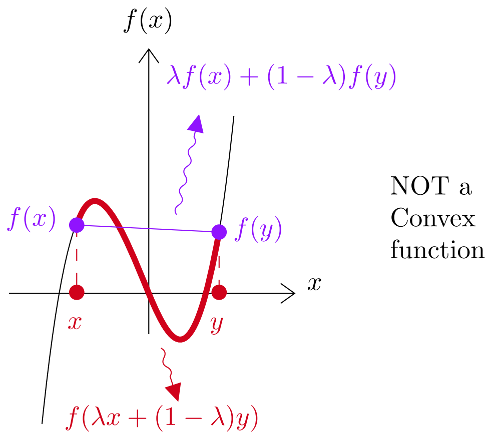
Let $S \subseteq \mathbb{R}^n$ be a convex set. The function $f : S \to \mathbb{R}^n$ is a convex function if $\forall x, y \in S, \forall \lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda) y) \le \lambda f(x) + (1 - \lambda) f(y)$$

Example:

Here we let $S = \mathbb{R}$.





A **convex NLP** is one of the form:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{array} \tag{CVX}$$

where $f, g_1, \dots, g_m$: $\mathbb{R}^n \to \mathbb{R}$ are convex functions.

Note:
It is important that constraints are $\leq$ and that the objective is a minimization problem.

Proposition 4.1

If $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function, then $S = \{x \in \mathbb{R}^n : g(x) \leq 0\}$ is a convex set.

Proof:
Let $x, y \in S$, i.e., $g(x) \leq 0, g(y) \leq 0$. Now we want to prove $\lambda x + (1-\lambda)y \in S$.

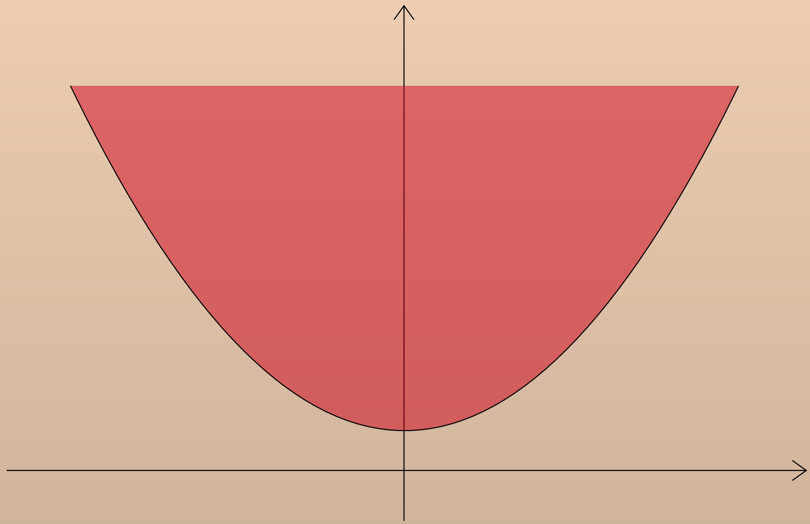
$$\begin{aligned} g(\lambda x + (1-\lambda)y) &\leq \lambda g(x) + (1-\lambda) g(y) \quad \text{since } g \text{ is a convex function} \\ &\leq 0 \end{aligned}$$

where the last ineq is from $\begin{array}{c} g(x) \leq 0, \lambda \geq 0 \\ g(y) \leq 0, (1-\lambda) \geq 0 \end{array}$.

This implies $\lambda x + (1-\lambda)y \in S, \quad \forall \lambda \in [0,1]$. $\square$

epigraph

$$\text{epi}(f) = \{(x, y) : y \geq f(x)\}$$



f is convex $\iff$ epi(f) is convex.

4.2 Gradients & Hessian

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function.

The **gradient** of f at $\overline{x}$ is the vector

$$\nabla f(\overline{x}) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

The **Hessian** of f at $\overline{x}$ is the $n \times n$ symmetric matrix

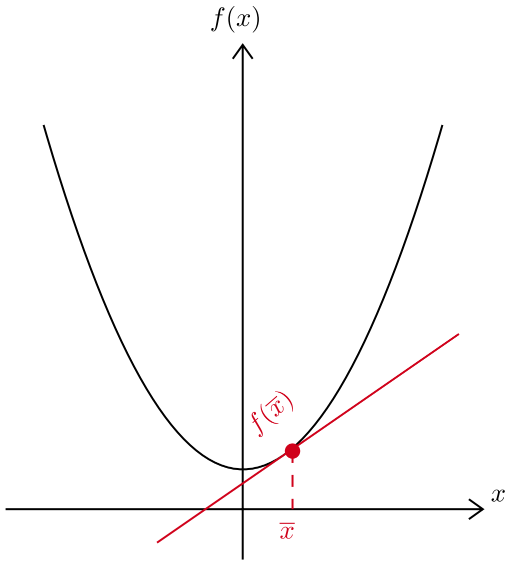
$$\nabla^2 f(\overline{x})$$

where the element is defined as

$$\left[\nabla^2 f(\overline{x})\right]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_i}$$

> Example:
>
> $f(x) = x_1^2 x_2 + 2x_1 + 3$. Then
>
> $$\nabla f(x) = \begin{pmatrix} 2x_1x_2 + 2 \\ x_1^2 \end{pmatrix} \quad \text{and} \quad \nabla^2 f(x) = \begin{pmatrix} 2x_2 & 2x_1 \\ 2x_1 & 0 \end{pmatrix}$$

Now looking at 1-D convex functions, two key properties stand out:



- second derivative is ≥ 0 (at any point $\overline{x}$)
- value of f is above tangent line at $\overline{x}$

Translating:

- $f''(x) \geq 0, \forall x$
- $f(x) \geq f(\overline{x}) + f'(\overline{x})(x - \overline{x}), \forall x, \overline{x}$

Theorem 4.2

Let $S \subseteq \mathbb{R}$ be a convex set. Let $S \to \mathbb{R}$ be twice differentiable. TFAE:

a) f is convex on S

b) $f(x) \geq f(\overline{x}) + f'(\overline{x})(x - \overline{x})$, $\forall x, \overline{x} \in S$

c) $(f'(x) - f'(\overline{x}))(x - \overline{x}) \geq 0$, $\forall x, \overline{x} \in S$

d) $f''(x) \geq 0$, $\forall x \in S$.

What is the generalization of b), c), d) to $f : \mathbb{R}^n \to \mathbb{R}$?

b): $f(x) \geq f(\overline{x}) + \nabla f(\overline{x})^T(x - \overline{x}), \quad \forall x, \overline{x} \in S.$

c): $(\nabla f(x) - \nabla f(\overline{x}))^T(x - \overline{x}) \geq 0, \quad \forall x, \overline{x} \in S.$

d): $\nabla^2 f(x)$ is Positive Semidefinite (PSD), $\forall x \in S$.

Note:
A symmetric $n \times n$ matrix Q is said to be **positive semidefinite** if $\forall y \in \mathbb{R}^n$,

$$y^T Q y \geq 0$$

Denoted as $Q \succeq 0$.

Q is said to be **positive definite** (PD) if $\forall y \in \mathbb{R}^n, y \neq 0$,

$$y^T Q y > 0$$

Denoted as $Q \succ 0$.

Theorem 4.3

Let $S \subseteq \mathbb{R}^n$ be a convex set. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous twice differentiable function. TFAE:

a) f is convex on S

b) $f(x) \geq f(\overline{x}) + \nabla f(\overline{x})^T(x - \overline{x}), \quad \forall x, \overline{x} \in S$

c) $(\nabla f(x) - \nabla f(\overline{x}))^T(x - \overline{x}) \geq 0, \quad \forall x, \overline{x} \in S$

d) $\nabla^2 f(x) \succeq 0, \forall x \in S$.

Example:

$f(x) = \|x\|^2 = \sum_{j=1}^{n} x_j^2$

$$\nabla f(x) = \begin{pmatrix} 2x_1 \\ \vdots \\ 2x_n \end{pmatrix} \qquad \text{and} \qquad \nabla^2 f(x) = 2I$$

Now

$$y^T \nabla^2 f(x) y = 2y^T I y = 2y^T y = 2\|y\|^2 \geq 0$$

$\implies \nabla^2 f(x) \succeq 0, \forall x \implies f(x)$ is convex.

Example:

$f(x) = \frac{1}{2}x^T x Q x + d^T x + p$ where Q is PSD.

$$f(x) = \sum_{j=1}^{n} \frac{x_j^2}{2} g_{jj} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j>i} 2x_i x_j q_{ij} + \sum_{j=1}^{n} x_j d_j + p$$

$$\nabla f(x) = \begin{pmatrix} \frac{2x_1}{2} q_{11} + \sum_{j=2}^{n} x_j q_{ij} + d_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} x_j q_{ij} + d_1 \\ \vdots \end{pmatrix} = Qx + d$$

$\nabla^2 f(x) = Q \succeq 0 \implies f$ is convex.

4.3 Local vs. Global optimality

Consider an NLP

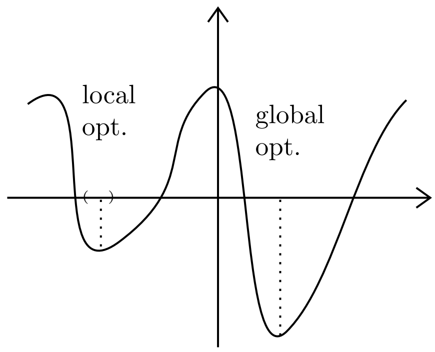
$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \ldots, m \end{array} \tag{NLP}$$

Let S be its feasible region. $x^* \in S$ is said to be a **local optimum** if $\exists R > 0$ so that

$$f(x^*) \leq f(x), \quad \forall x \in B(x^*, R) \cap S.$$

x^* is said to be a **global optimum** if

$$f(x^*) \leq f(x), \quad \forall x \in S.$$



Proposition 4.4

If (NLP) is a convex program, then

$$x^* \text{ is a local optimum} \iff x^* \text{ is a global optimum.}$$

Proof:

($\Leftarrow$) Trivial.

($\Rightarrow$) Suppose x^* is a local optimum. But suppose $\exists \overline{x} \in S$: $f(x^*) > f(\overline{x})$.

Consider $x(\lambda) = \lambda\overline{x} + (1-\lambda)x^*$.

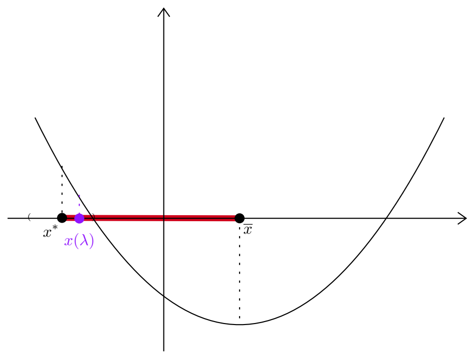
Since (NLP) is a convex program, S is a convex set, therefore $x(\lambda) \in S, \forall \lambda \in [0,1]$. Since f is a convex function, we have

$$f(x(\lambda)) = f(\lambda\overline{x} + (1-\lambda)x^*) \leq \lambda f(\overline{x}) + (1-\lambda)f(x^*)$$

Also, for any $\lambda > 0$, we have $\lambda f(\overline{x}) < \lambda f(x^*)$. Therefore,

$$f(x(\lambda)) < \lambda f(x^*) + (1-\lambda)f(x^*) = f(x^*), \ \forall \lambda \in (0,1]$$

Therefore, $\forall R > 0, \exists \lambda$ such that $x(\lambda) \in B(x^*, R) \cap S$. Contradicts local optimality of x^*.



□

Note:
This does not require differentiability.

4.3.1 Characterizing Optimality

The previous proposition suggests that only local information is needed for determining optimality.

Can we characterize optimality based on local info?

Proposition 4.5

Consider a convex optimization problem where f is differentiable. Let S be the feasible set. The x^* is global optimal iff

$$\nabla f(x^*)^T(x - x^*) \geq 0, \quad \forall x \in S.$$

Proof:

($\Leftarrow$) From convexity of f

$$f(x) \geq f(x^*) + \underbrace{\nabla f(x^*)^T(x - x^*)}_{\geq 0} \geq f(x^*), \qquad \forall x \in S$$

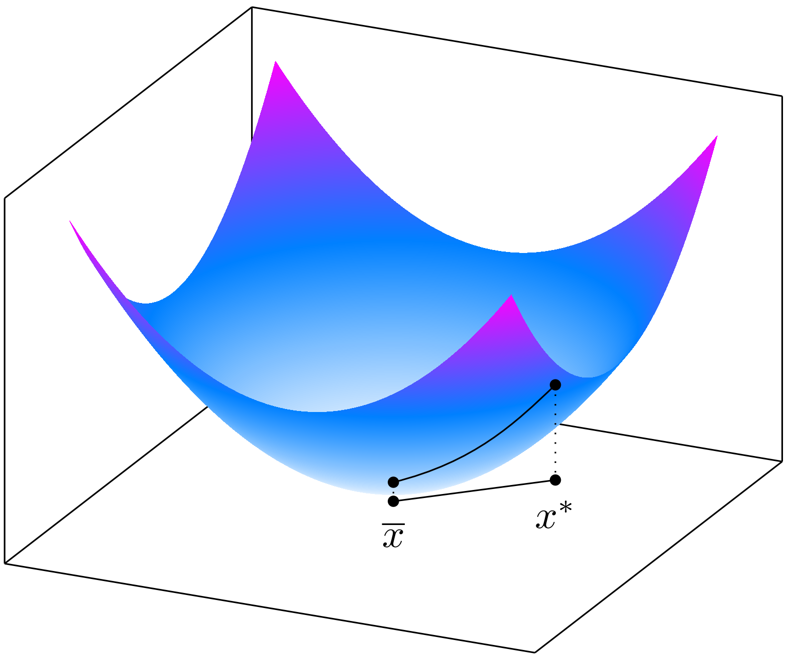
($\Rightarrow$) Sketch idea:

Suppose $\exists \overline{x} \in S : \nabla f(x^*)^T < 0$

Define $g(\lambda) := f(\lambda\overline{x} + (1 - \lambda)x^*)$

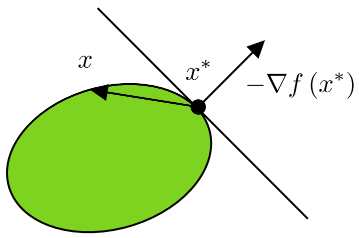
Can be argued that $g'(0) = \nabla f(x^*)^T(\overline{x} - x^*) < 0$.

For small λ, $g(\lambda) < g(0) = f(x^*)$. Therefore, x^* is not optimal.



$\square$

Intuition Going from x^* in the direction towards another x feasible takes us in the opposite direction that we want to go (opposite to the gradient).



Corollary 4.6

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, differentiable then x^* is optimal to

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathbb{R}^n \end{array}$$

iff $\nabla f(x^*) = 0$.

Proof:

($\Leftarrow$) Follows from previous proposition.

($\Rightarrow$) Suppose $\nabla f(x^*) \neq 0$. Let $y = -\nabla f(x^*) + x^*$.

$$\nabla f(x^*)^T (y - x^*) = -\nabla f(x^*)^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 \leq 0$$

$\implies$ x^* is not optimal from previous proposition.

$\square$

4.4 Lagrangian Duality

Consider a general NLP

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \ldots, m \end{array} \tag{NLP}$$

(that is NOT necessarily convex)

Lagrangian

The Lagrangian of (NLP) is the following function $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

$$L(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

λ_i are called **Lagrangian multipliers** associated to g_i constraints.

Intuitively, we associate a penalty term λ_i that would steer us away from points with $g_i \gg 0$, if we try to minimize $L(x, \lambda)$. We can restate the previous result as a generalization of LP weak duality.

Proposition 4.7

If $\overline{x} \in S$ and $\lambda \geq 0$, then $L(\overline{x}, \lambda) \leq f(\overline{x})$.

Proof:

$$L(\overline{x}, \lambda) = f(\overline{x}) + \overbrace{\sum_{i=1}^{m} \underbrace{\lambda_i}_{\geq 0} \underbrace{g_i(\overline{x})}_{\leq 0}}^{\leq 0} \leq f(\overline{x})$$

$\square$

Now let $\ell(\lambda) = \min_{x \in \mathbb{R}^n} L(x, \lambda)$.

It follows that, $\forall \lambda \geq 0$, $\ell(\lambda) \leq z^*$ where x^* is optimal value of (NLP).

Thus we get a lower bound for any $\lambda \geq 0$.

As in LP duality, we are interested in the best possible lower bound.

So we want

$$\begin{array}{ll} \max & \ell(\lambda) \\ \text{s.t.} & \lambda \geq 0 \end{array} \tag{LD}$$

This is called the **Lagrangian dual** problem.

Proposition 4.8: Weak duality

If $\overline{x} \in S$ and $\lambda \geq 0$, then $\ell(\lambda) \leq f(\overline{x})$.

Example:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \leq b \iff Ax - b \leq 0 \end{array}$$

Then $f(x) = c^T x, g_i(x) = a_i^T x - b_i, \quad \forall i = 1, \dots m$

$$\begin{aligned} L(x, \lambda) &= f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \\ &= c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i) \\ &= \left(c^T + \sum_{i=1}^{m} \lambda_i a_i^T \right) x - \sum_{i=1}^{m} \lambda_i b_i \end{aligned}$$

Then

$$\begin{aligned} \ell(\lambda) &= \min_{x \in \mathbb{R}^n} L(x, \lambda) \\ &= \begin{array}{ll} \min & (c^T + \sum_{i=1}^m \lambda_i a_i^T) x - \sum_{i=1}^m \lambda_i b_i \\ \text{s.t.} & x \in \mathbb{R}^n \end{array} \\ &= \begin{cases} -\infty, & \text{if } \left(c^T + \sum_{i=1}^m \lambda_i a_i^T\right) \neq 0 \\ -\sum_{i=1}^m \lambda_i b_i, & \text{if } \left(c^T + \sum_{i=1}^m \lambda_i a_i^T\right) = 0 \end{cases} \end{aligned}$$

Then

$$\begin{array}{ll} \max & \ell(\lambda) \\ \downarrow & \\ \text{s.t.} & \lambda \geq 0 \end{array} = \begin{array}{ll} \max & -\sum_{i=1}^m \lambda_i b_i \\ \downarrow & \\ \text{s.t.} & c^T + \sum_{i=1}^m \lambda_i a_i^T = 0 \\ & \lambda \geq 0 \end{array} \overset{y=-\lambda}{=} \begin{array}{ll} \max & b^T y \\ \downarrow & \\ \text{s.t.} & y^T A = c^T \\ & y \leq 0 \end{array}$$

Example:

$$\begin{array}{ll} \min & (x_1 - 1)^2 + (x_2 - 1)^2 \\ \downarrow & \\ \text{s.t.} & x_1 + 2x_2 - 1 \leq 0 \\ & 2x_1 + x_2 - 1 \leq 0 \end{array}$$

$$L(x, \lambda) = (x_1 - 1)^2 + (x_2 - 1)^2 + \lambda_1(x_1 + 2x_2 - 1) + \lambda_2(2x_1 + x_2 - 1)$$

Check: $L(x, \lambda)$ is a convex function (for a fixed λ it is a convex function of x)

Now for $\ell(\lambda) = \min_{x \in \mathbb{R}^n} L(x, \lambda)$ is achieved when $\nabla_x L(x, \lambda) = 0$

$$\begin{pmatrix} 2(x_1 - 1) + \lambda_1 + 2\lambda_2 \\ 2(x_2 - 1) + 2\lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{array}{l} x_1^* = \dfrac{-\lambda_1 - 2\lambda_2}{2} + 1 \\ x_2^* = \dfrac{-2\lambda_1 - \lambda_2}{2} + 1 \end{array}$$

$$\begin{aligned} L(x^*,\lambda) &= \left(\frac{-\lambda_1-2\lambda_2}{2}\right)^2 + \left(\frac{-2\lambda_1-\lambda_2}{2}\right)^2 + \lambda_1\left(\frac{-\lambda_1-2\lambda_2}{2}+1-2\lambda_1-\lambda_2+2-1\right) \\ &\qquad + \lambda_2\left(-\lambda_1-2\lambda_2+2+\frac{(-2\lambda_1-\lambda_2)}{2}+1-1\right) \\ &= -1.25\lambda_1^2 - 1.25\lambda_2^2 - 2\lambda_1\lambda_2 + 2\lambda_1 + 2\lambda_2 \\ &=: \ell(\lambda) \end{aligned}$$

$$\begin{array}{ll} \max & \ell(\lambda) \\ \text{s.t.} & \lambda \geq 0 \end{array} = \begin{array}{ll} \max & L(x^*,\lambda) \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

If we set $\nabla_\lambda L(x^*,\lambda) = 0$, we get $\lambda^* = \left(\frac{4}{9}, \frac{4}{9}\right)$ with objective value

$$\ell(\lambda^*) = -2.5 \times \left(\frac{4}{9}\right)^2 - 2\left(\frac{4}{9}\right)^2 + 4 \times \frac{4}{9} = \frac{8}{9}$$

And note that $x^* = \left(\frac{1}{3}, \frac{1}{3}\right)$ gives $f(x^*) = \frac{8}{9}$, which gives optimal solution.

4.5 Karush-Kuhn-Tucker Optimality Conditions

Lagrangean dual for problems with equality constraints

For problems of the form,

$$\begin{array}{lll} \min & f(x) & \\ \downarrow & & \\ \text{s.t.} & g_i(x) \leq 0, & \forall i = 1,\ldots,m \\ & h_i(x) = 0, & \forall i = 1,\ldots,p \end{array} \tag{NLP}$$

We can define

$$L(x,\lambda,\nu) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{i=1}^p \nu_i h_i(x)$$

Here the Lagrangean dual:

$$\begin{array}{ll} \max & \ell(\lambda,\nu) \\ \text{s.t.} & \lambda \geq 0, \nu \in \mathbb{R}^p \end{array}$$

where $\ell(\lambda,\nu) = \min_{x\in\mathbb{R}^n} L(x,\lambda,\nu)$. Weak duality still holds for $\lambda \geq 0, \nu \in \mathbb{R}^p$.

Note:
If f, g_i are convex, $\forall i = 1,\ldots,m$ and $h_i(x)$ are affine functions, then (NLP) is a convex program.

Note:
Weak Duality holds regardless if g_i, h_i are convex.

Example: Least square solutions of linear equations

Suppose we want to find, out of all possible solutions to $Ax = b$, the one with smallest norm.

$$\begin{array}{ll} \min & x^Tx \\ \text{s.t.} & Ax = b \end{array}$$

Lagrangian: $L(x,\nu) = x^Tx + \nu^T(Ax-b)$.

Then $\ell(\nu) = \min_{x\in\mathbb{R}^n} L(x,\nu)$.

$$\nabla_x L(x,\nu) = 0 \implies 2x + A^T\nu = 0 \implies x = -\frac{A^T\nu}{2}$$

$$\begin{aligned} \implies \ell(\nu) &= \frac{\nu^T AA^T \nu}{4} - \frac{\nu^T AA^T \nu}{2} - b^T \nu \\ &= -\frac{\nu^T AA^T \nu}{4} - b^T \nu \\ &\leq \begin{array}{ll} \min & x^T x \\ \text{s.t.} & Ax = b \end{array} \end{aligned}$$

When does Strong Duality Hold?

This is hard to characterize in general, but there are some easily checkable sufficient conditions.

Let

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{array} \tag{CVX}$$

where f, g_i are convex $\forall i = 1, \dots, m$.

Slater's Condition

$\exists \overline{x} : g_i(\overline{x}) < 0, \quad \forall i = 1, \dots, m.$

That is, there exists a point in the relative interior of the feasible region.

Theorem 4.9

If Slater's condition holds for (CVX), then $\exists \lambda^* \geq 0$ such that

$$\ell(\lambda^*) = \min_{x \in \mathbb{R}^n} L(x, \lambda^*) = \boxed{\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{array}} \longrightarrow$$

Recall that this was abuse of notation and it is not clear that $\exists x^*$ achieving inf.

i.e.,

$$\max_{\lambda \geq 0} \ell(\lambda) = \begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{array}$$

and the max is attained at λ^*.

For example: $\min\{e^{-x} : -x \leq 0\} = 0$, but $\nexists x^* : e^{-x^*} = 0$.

Proof:
SKIPPED. ☐

To derive optimality conditions, suppose we have λ^*, x^* opti. for dual/primal.

$$\ell(\lambda^*) = \min_{x \in \mathbb{R}^n} f(x) + \sum_{i=1}^m \lambda_i^* g_i(x) \leq f(x^*) + \sum_{i=1}^m \lambda_i^* g_i(x^*) \underset{\substack{\uparrow \\ \lambda^* \geq 0,\, g_i(x^*) \leq 0}}{\leq} f(x^*)$$

Now if we want strong duality to hold, i.e., we want $\ell(\lambda^*) = f(x^*)$ then all above inequalities must hold at equality.

The first inequality holding as equality implies x^* is a minimizer of $L(x, \lambda^*)$ for all $x \in \mathbb{R}^n$.

$$L(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x) \implies \nabla_x L(x^*, \lambda^*) = 0 \implies \nabla f(x^*) + \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0$$

The second inequality holding as equality means a complementary slackness-type condition, i.e., $\lambda_i^* g_i(x^*) = 0 \iff \lambda_i^* = 0 \quad \text{or} \quad g_i(x^*) = 0$.

Formally, these are the so-called **Karush-Kuhn-Tucker (KKT)** optimality conditions:

KKT conditions

i) $g_i(x^*) \leq 0, \ \forall i = 1, \dots, m$

ii) $\lambda^* \geq 0$

iii) $\lambda_i^* g_i(x^*) = 0, \ \forall i = 1, \dots, m$

iv) $\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* g_i(x^*) = 0$

Theorem 4.10: Necessary opt. conditions

Consider

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{array} \tag{NLP}$$

where f, g_i are differentiable, $\forall i = 1, \dots, m$.

If x^*, λ^* are optimal to the (NLP) and its Lagrangean dual, respectively, such that $f(x^*) = L(x^*, \lambda^*) = \ell(\lambda^*)$, then KKT conditions hold.

Proof:
Follows from above discussion. □

Theorem 4.11: Sufficient opt. conditions

Assume that, in addition, the functions g_i are convex, $\forall i = 1, \dots, m$, f is convex. Then if x^*, λ^* satisfy KKT conditions, x^*, λ^* are optimal for (NLP) and its Lagrangean dual, and $f(x^*) = \ell(\lambda^*) = L(x^*, \lambda^*)$.

Proof:
Follows similar to necessity proof, using the fact that $L(x, \lambda)$ is a convex function and thus $\nabla_x L(x^*, \lambda^*) = 0 \implies x^*$ is a minimizer of $L(x, \lambda^*)$ over $x \in \mathbb{R}^n$. □

Note:
For problems of the form:

$$\begin{array}{ll} \min & f(x) \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} g_i(x) \leq, \quad \forall i = 1, \dots, m \\ h_i(x) = 0, \quad \forall i = 1, \dots, p \end{array} \end{array} \tag{NLP-EQ}$$

the KKT conditions are:

KKT

i) $g_i(x^*) \leq 0, \ \forall i = 1, \dots, m$

ii) $h_i(x^*) = 0, \ \forall i = 1, \dots, p$

iii) $\lambda^* \geq 0$

iv) $\lambda_i^* g_i(x^*) = 0, \ \forall i = 1, \dots, m$

v) $\nabla f(x^*) + \sum_{i=1}^m \lambda_i^* g_i(x^*) + \sum_{i=1}^p \nu_i \nabla h_i(x^*) = 0$

With equality constraint:

- If x^* opt for (NLP-EQ), (λ^*, ν^*) opt for its lag. dual and $f(x^*) = \ell(\lambda^*, \nu^*)$ then KKT holds.
- If $f, g_1, \dots, g_m$ are convex and $h_1, \dots, h_p$ are affine functions, then x^*, λ^*, ν^* satisfying KKT $\implies x^*$ opt for (NLP-EQ), λ^*, ν^* opt for its Lag. dual and $f(x^*) = \ell(\lambda^*, \nu^*)$.

Where is Slater's condition needed in convex programs?

Example:

$$\begin{array}{ll} \min & x \\ \text{s.t.} & x^2 \leq 0 \end{array}$$

is a convex program with unique feasible solution $x = 0 \implies$ Slater's condition does not hold.

Now $x = 0$ is optimal. But $\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) = 1 + 0 = 1 \neq 0$.

Note:
$L(x, \lambda) = x + \lambda x^2$ and

$$\ell(\lambda) = \min_{x \in \mathbb{R}} x + \lambda x^2 = \begin{cases} -\infty, & \text{if } \lambda = 0 \\ -\frac{1}{2\lambda}, & \text{if } \lambda > 0 \end{cases}$$

This problem violates Slater's condition and $\nexists x^*, \lambda^*$ achieving strong duality.

Example:

$$\begin{array}{ll} \min & x^2 + 1 \\ \text{s.t.} & (x-2)(x-4) \leq 0 \end{array}$$

is a convex program (CHECK) and Slater's condition holds. ($x = 3$ satisfies it). Let us try and find KKT points.

$\nabla f(x) = 2x$, $\nabla g_1(x) = 2x - 6$, $\nabla f(x) + \lambda_1 \nabla g_1(x) = 2x + (2x - 6) = 0$

- $\lambda_1 = \frac{2x}{6-2x}$
- $\lambda_1(x-2)(x-4)$

$$\implies \begin{array}{l} x = 2, \lambda_1 = 2 \\ x = 4, \lambda_1 = -2 \quad ✗ \\ \lambda = 0 \quad \begin{array}{l} \text{(i.e., } x = 0\text{), but} \\ \text{then } (x-2)(x-4) = 8 > 0 \end{array} \quad ✗ \end{array}$$

Thus point $x = 2, \lambda_1 = 2$ satisfies KKT $\implies$ primal/dual optimal.

When does primal admit an opt. sol?

If feasible region is closed and bounded and f is continuous, then primal has optimal solution.

Coerciveness

f is coercive if $\{x : f(x) \leq \alpha\}$ is bounded $\forall \alpha \in \mathbb{R}$.

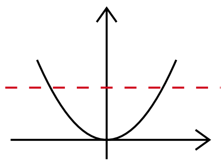
Lemma 4.12

TFAE

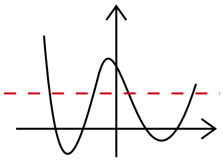
a) f is coercive

b) $f(x) \to \infty$ as $\|x\| \to \infty$

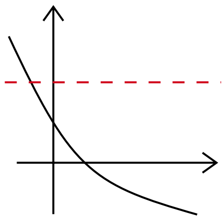
Proof:
SKIPPED. □



Coercive & Convex



Coercive & Not convex



Convex & Not coercive

Theorem 4.13

If $S \to \mathbb{R}^n$ is nonempty and closed, $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and coercive, then

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in S \end{array}$$

has a minimizer.

Proof:
SKIPPED. $\square$

4.6 Summary of NLP results

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0, \quad \forall i = 1, \ldots, m \end{array}$$

	Generic NLP	Generic & diff.	Convex	Convex & diff.
Weak duality. $\overline{\lambda}$ feas. dual, $\overline{x}$ feas. primal. $\implies \ell(\overline{\lambda}) \leq f(\overline{x})$	✓	✓	✓	✓
Slater $\implies \exists$ sol. dual matching the inf of primal	✗	✗	✓	✓
If $\exists$ opt. sol to primal & Dual w/ equal values $\implies$ KKT holds	✗	✓	✗	✓
If x, λ satisfy KKT $\implies f(x^*) = \ell(\lambda^*)$	✗	✗	✗	✓

4.7 Algorithms for convex NLPs

Unconstrained case

$$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & x \in \mathbb{R}^n \end{array}$$

f_0 convex, differentiable.

Assumption Opt. Sol exists. $\to$ Goal: find x^* so that $\nabla f_0(x^*) = 0$

4.7.1 Descent methods for unconstrained

Iterative methods that start from a feasible point x^0 and move from x^k to $x^{k+1} \leftarrow x^k + t^k d^k$ for some search direction $d^k \in \mathbb{R}^m$, step length $t^k \in \mathbb{R}_+$.

Want: $f_0(x^{k+1}) < f_0(x^k)$.

Now if we move from x to y then $d = y - x$.

Now if $\nabla f(x^k)^T(y - x^k) \geq 0, \forall y \implies x^k$ optimal.
So goal is to pick descent $d : \nabla f(x^k)^T d < 0$.

Algorithm 7: General Descent Method

1 $x^0 \in \mathbb{R}^n$
2 **while** *STOPPING CRITERION NOT SATISFIED* **do**
3   Find descent direction d^k
4   Choose step size t^k
5     $x^{k+1} \leftarrow x^k + t^k d^k$

Choosing a step size Several options exist. Here are two common.

a) **Exact line search**: Solve the 1-D convex minimization problem

$$t = \underset{s \geq 0}{\operatorname{argmin}} \left\{ f_0(x^k + sd^k) \right\}$$

b) **Backtracking**

Algorithm 8: Backtracking

1 Let $\alpha \in (0, 0.5)$ and $\beta \in (0, 1)$
2 $t \leftarrow 1$
3 **while** $f_0(x^k + td^k) > f_0(x^k) + \alpha t \nabla f_0(x^k)^T d^k$ **do**
4     $t \leftarrow \beta t$

Note for t small

$$f(x^k + td^k) \approx f(x^k) + t\nabla f(x^k)^T d^k < f(x^k) + t\alpha \nabla f(x^k)^T d^k < f(x^k)$$

So the method terminates with the desired t.

Choosing a descent direction

a) **gradient descent** $d^k = -\nabla f(x^k)$

> **Note:**
> Using exact line search, or backtracking
>
> $$f(x^k) - p^* \leq c^k (f(x^0) - p^*)$$
>
> where p^* is opt. value and c is a constant in $(0, 1)$. (we will not prove this)

b) **Newton method**

If $\nabla^2 f_0(x)$ is positive definite, $\lambda^k = -\nabla^2 f_0(x^k)^{-1} \nabla f_0(x^k)$

> **Note:**
>
> $$\nabla f_0(x^k)^T d^k = -\nabla f_0(x^k)^T \nabla^2 f_0(x^k)^{-1} \nabla f_0(x^k) < 0$$

> Remark:
> M is positive definite $\implies$ M is invertible and M^{-1} is positive definite

$\rightarrow$ Faster convergence

These are just two examples. There are lots of other variations/methods, each with pros/cons.

4.7.2 Methods for constrained problems

Consider

$$z^* = \begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0, \quad \forall i = 1, \dots, m \end{array} \tag{CVX}$$

where f_i are convex, twice differentiable, $\forall i = 0, \dots, m$

Assumptions

- $\exists$ an opt. sol. to (CVX)
- Slater's condition holds

Idea (CVX) is equivalent to:

$$\min f_0(x) + \sum_{i=1}^{m} I_-(f_i(x))$$

where $I_i : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$

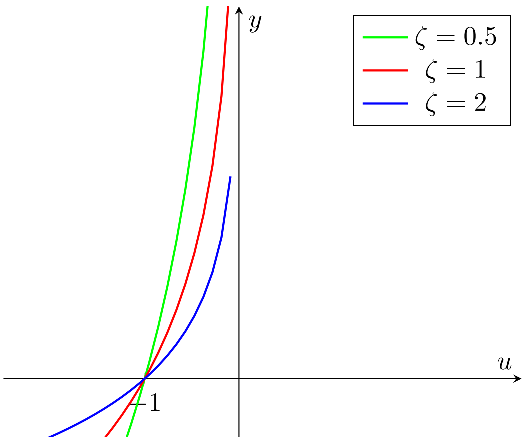
$$I_-(u) = \begin{cases} 0, & u \leq 0 \\ +\infty, & u > 0 \end{cases}$$

Problem I_- is non differentiable & highly intractable.

Consider

$$-\left(\frac{1}{\zeta}\right) \log(-u), \quad \text{for } \zeta > 0$$

which is a convex function (check!)



This function tries to approximate I_-, but has the advantage of being differentiable & convex. $\to$ Solve unconstrained min:

$$\min f_0(x) + \sum_{i=1}^{m} -\left(\frac{1}{\zeta}\right) \log(-f_i(x))$$

Solving this problem for $\zeta > 0$ ensures that we get a feasible point since obj, fct. goes to $+\infty$ as we approach $f_i(x) = 0$.

> **Note:**
> Unconstrained method can be made to work over the domain of the function.

Define $\phi(x) := -\sum_{i=1}^{m} \log(-f_i(x))$ which is called the **log-barrier** function.

We will solve $\min \zeta f_0(x) + \phi(x)$ for increasing values of ζ.

> **Note:**
> In principle, one can just solve $\min \zeta f_0(x) + \phi(x)$ for one vert large ζ. $\to$ Computationally is bad $\to$ Numerical issues!

Note:
We are using the scaled version of the objective function, for later convenience.

Algorithm 9: Barrier Method

Let x^0 be such that $f_i(x^0) < 0, \quad \forall i = 1, \ldots, m$
Let $\zeta^0 > 0$. $\mu > 1, \epsilon > 0$
$k \leftarrow 1$
while *Stopping criterion not satisfied* **do**
  Let $x^*(\zeta^k) \leftarrow \operatorname{argmin} \zeta^k f_0(x) + \phi(x)$ `// can be computed by descent method starting at` x^{k-1}
  $x^k \leftarrow x^*(\zeta)$
  $\zeta^k \leftarrow \mu \zeta^{k-1}$

Central path

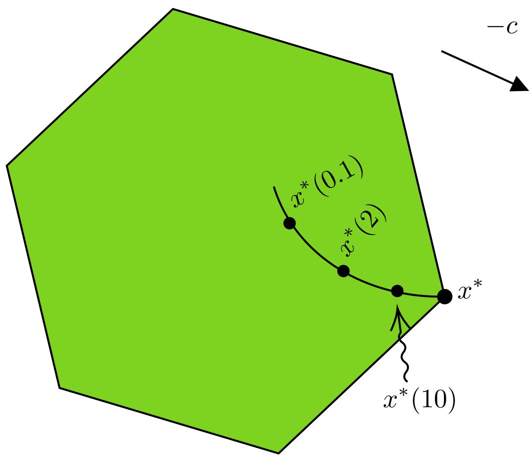
Consider, for $\zeta > 0$.

$$x^*(\zeta) \leftarrow \operatorname{argmin} \zeta f_0(x) + \phi(x)$$

We call the set of points $x^*(\zeta) : \zeta > 0$ the *central path.*

Intuition As $\zeta \to 0$, it starts becoming more important to be as far away from $f_i(x) = 0$ as possible. So points tend to go towards the "center" of feasible region.

As $\zeta \to \infty$, it starts becoming more important to minimize f_0 and $x^*(\zeta)$ tends to get closer to opt. sol.



What are properties of $x^*(\zeta)$?

- $f_i(x^*(\zeta)) < 0, \quad \forall i = 1, \ldots, m$
- $\zeta \nabla f_0(x^*(\zeta)) + \nabla \phi(x^*(\zeta)) = 0$
  $\iff \zeta \nabla f_0(x^*(\zeta)) + \sum_{i=1}^{m} \frac{1}{-f_i(x^*(\zeta))} \nabla f_i(x^*(\zeta)) = 0$

Now define $\lambda_i^*(\zeta) = -\frac{1}{\zeta f_i(x^*(\zeta))}, \quad \forall i = 1, \ldots, m$

Note $\lambda^*(\zeta) \geq 0$. Then

$$\nabla f_0(x^*(\zeta)) + \sum_{i=1}^{m} \lambda_i^*(\zeta) \nabla f_i(x^*(\zeta)) = 0$$

$\implies x^*(\zeta)$ is a minimizer of $L(x, \lambda^*(\zeta)) = f_0(x) + \sum_{i=1}^{m} \lambda_i^*(\zeta) f_i(x)$
$\implies g(\lambda^*(\zeta)) = f_0(x^*(\zeta)) - \frac{m}{\zeta}$

In other words: $f_0(x^*(\zeta)) - g(\lambda^*(\zeta)) = \dfrac{m}{\zeta}$ and since $g(\lambda^*) \leq z^*$

$$\implies f(x^*(\zeta)) - z^* \leq f(x^*(\zeta)) - g(\lambda^*(\zeta)) = \frac{m}{\zeta}$$

i.e., $x^*(\zeta)$ is not too far from optimal and as $\zeta \to \infty$, $x^*(\zeta)$ converges to the optimal solution.

Interpretation as KKT

Note that $x^*(\zeta)$ and $\lambda^*(\zeta)$ satisfy:

i) $f_i(x^*(\zeta)) \leq 0, \quad \forall i = 1, \ldots, m$

ii) $\lambda^*(\zeta) \geq 0$

iii) $-\lambda_i^*(\zeta) f_i(x^*(\zeta)) = \frac{1}{\zeta}, \quad \forall i = 1, \ldots, m$

iv) $\nabla f_0(x^*(\zeta)) + \sum_{i=1}^m \lambda_i^*(\zeta) \nabla f_i(x^*(\zeta)) = 0$

which are almost KKT conditions and as $\zeta \to \infty$, become KKT.

Note:

- This method can be adapted to deal with affine constraints $Ax = b$.
- It can be used for LPs. In particular, it performs reasonably well, outperforming simplex in dense LPs.
- Drawback
  $\to$ Does not give BFS. (Bad for cutting plane)
  $\to$ Gives usually dense solutions.

5

Conic Optimization

Let K be a closed convex cone. We will consider the following optimization problem

$$\begin{array}{ll} \min\limits_{\downarrow} & c^T x \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \in K \end{array} \end{array} \tag{Con}$$

Sometimes also represented as:

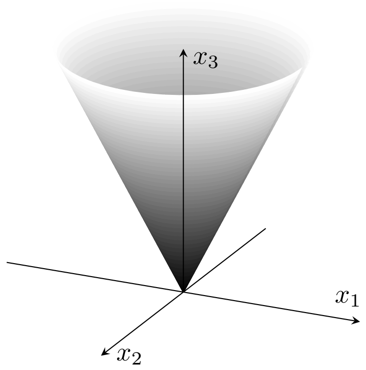
$$\begin{array}{ll} \min\limits_{\downarrow} & c^T x \\ \text{s.t.} & \begin{array}{l} Ax = b \\ x \succeq_K 0 \end{array} \end{array}$$

It is trivial to see (Con) is a convex optimization problem, i.e., the feasible region is convex and also the objective function.

Now for $K = \{x : x \geq 0\}$, i.e., non-negative orthant[1] (Con) is just LP.

Other cones:

- **Second-order cone**: $K = \left\{x : x_1 \geq \sqrt{x_2^2 + \ldots + x_n^2}\right\}$



  (Con) is called Second-Order cone program.

- **Semidefinite cone**.

[1]From wiki: In geometry, an orthant or hyperoctant is the analogue in n-dimensional Euclidean space of a quadrant in the plane or an octant in three dimensions.

Let $M(x)$ be the symmetric $k \times k$ matrix whose upper triangular submatrix is

$$\begin{bmatrix} x_1 & x_2 & \dots & x_k \\ & x_{k+1} & \dots & x_{2k-1} \\ & & \ddots & \vdots \\ & & & x_n \end{bmatrix}$$

$K = \{x : M(x) \text{ is PSD}\}$ i.e., $y^T M(x) y \geq 0, \forall y \in \mathbb{R}^k$

$\rightarrow$ This assumes n has a certain dimension, w.r.t. k.
(Con) is called a semi-definite program.

Example:

$$\begin{array}{ll} \min & 2x_1 + x_2 + x_3 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} x_1 + x_2 + x_3 = 1 \\ x \geq 0 \end{array} \end{array} \tag{LP}$$

$$\begin{array}{ll} \min & 2x_1 + x_2 + x_3 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} x_1 + x_2 + x_3 = 1 \\ x_1 \geq \sqrt{x_2^2 + x_3^2} \end{array} \end{array} \tag{SOCP}$$

$$\begin{array}{ll} \min & 2x_1 + x_2 + x_3 \\ \downarrow & \\ \text{s.t.} & \begin{array}{l} x_1 + x_2 + x_3 = 1 \\ \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} \succeq 0 \end{array} \end{array} \tag{SDP}$$

Dual cone

Given $K \subseteq \mathbb{R}^n$, a closed convex cone. The dual cone is

$$K^* := \{y \in \mathbb{R}^n : y^T x \geq 0, \forall x \in K\}$$

Note:
All cones mentioned above are self dual, i.e., $K = K^*$. (we will not prove this)

5.1 Lagrangian

Lagrangian: $L(x, y, \mu) = c^T x y^T (b - Ax) - \mu^T x$

$$g(y, \mu) = \min_x L(x, y, \mu) = \begin{cases} y^T b, & \text{if } c - A^T y - \mu = 0 \\ -\infty, & \text{otherwise} \end{cases}$$

Now, $\forall y \in \mathbb{R}^m$, $\forall \mu \in K^*$, $\overline{x}$ feasible for (Con).

$$g(y, \mu) \leq c^T \overline{x} + y^T (b - A\overline{x}) - \mu^T \overline{x} \underset{\substack{\uparrow \\ \text{Weak duality}}}{\leq} c^T \overline{x}$$

Lagrange dual:

$$\max_{y, M \in K^*} g(y, \mu) = \begin{array}{ll} \max & y^T b \\ \text{s.t.} & \mu = c - A^T y \\ & \mu \in K^* \end{array} \Leftrightarrow \begin{array}{ll} \max & y^T b \\ \text{s.t.} & c - A^T y \in K^* \end{array} \tag{D}$$

Note that writing KKT using $L(x, y, \mu)$, we get:

i) $x \in K, Ax = b$ Primal feas.

ii) $\mu \in K^*$ Dual feas.

iii) $\mu^T x = 0$ Complementary slackness $\iff (c - A^T y)^T x = 0$

iv) $\nabla_x L(x, y, \mu) = 0 \iff c^T - y^T A - \mu^T = 0 \iff \mu = c - A^T y$ Dual feas.

Theorem 5.1

Let

$$z^* = \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in K \end{array} \quad , \quad d^* = \begin{array}{ll} \max & b^T y \\ \text{s.t.} & c - A^T y \in K^* \end{array}$$

then $d^* \leq z^*$ and if both are strictly feasible, then:

- $d^* = z^*$ and both values are attained.
- (x, y) are primal/dual opt $\iff$ KKT conditions hold.

Proof:
SKIPPED. □

Note:
Strict feasible:

- Primal: $\exists \overline{x} : A\overline{x} = b, \overline{x} \in \text{int}(K)$
- Dual: $\exists \overline{y} : c - A^T \overline{y} \in \text{int}(K^*)$

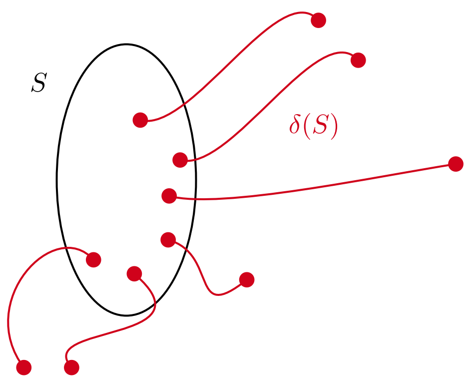
This is yet another way to generalize LPs. Leads to algorithms to solve (Con).

5.2 Connections to IP

SDP relaxations of some IPs.

5.2.1 Max-cut problem

Give $G = (V, E), c_e, \forall e \in E$. Find $\varnothing \neq S \subsetneq V$ maximizing $\sum_{e \in \delta(S)} c_e$.



We can formulate as:

$$\begin{array}{lll} \max & \sum_{e \in E} c_e x_e & \\ \downarrow & & \\ & y_u + y_v \leq 2 - x_{uv}, & \forall uv \in E \\ \text{s.t.} & (1 - y_u) + (1 - y_v) \leq 2 - x_{uv}, & \forall uv \in E \\ & y_v \in \{0, 1\}, & \forall v \in E \\ & x_e \in \{0, 1\}, & \forall e \in E \end{array}$$

Above, $y_v = \begin{cases} 1 & \text{represents } v \in S \\ 0 & \text{represents } v \notin S \end{cases}$ and $x_e = 1 \iff e \in \delta(S)$

Alternative:

$$y_v = \begin{cases} 1, & \text{if } v \in S \\ -1, & \text{if } v \notin S \end{cases}$$

Then $\begin{aligned} y_u y_v = -1 &\implies uv \in \delta(S) \\ y_u y_v = 1 &\implies uv \notin \delta(S) \end{aligned}$

$$\sum_{e \in \delta(S)} c_e = \sum_{\substack{u,v \in V \\ u \neq v}} \frac{1 - y_u y_v}{2} \cdot c_{uv}$$

So to get max-cut, it suffices to solve

$$\begin{array}{ll} \min & \sum_{\substack{u,v \in V \\ u \neq v}} y_u y_v c_{uv} \\ \text{s.t.} & y_u \in \{-1, 1\}, \quad \forall u \in V \end{array}$$

Defining $c_{uu} = 0$, we get

$$\begin{array}{ll} \min & \sum_{u,v \in V} y_u y_v c_{uv} \\ \text{s.t.} & y_u^2 = 1, \quad \forall u \in V \end{array}$$

This is NP-Hard to solve, but we can relax asa follows:

Consider $Y = yy^T \in \mathbb{R}^{v \times v}$.

Note $Y_{uu} = y_u^2$ and $Y_{uv} = y_u y_v$. And note $\forall w \in \mathbb{R}^v$,

$$w^T Y w = (w^T y)(y^T w) = (w^T y)^2 \geq 0 \implies Y \succeq 0$$

So we can write equivalently.

$$\begin{array}{lll} \min & \sum_{u \in V} \sum_{v \in V} c_{uv} x_{uv} & \\ \text{s.t.} & x_{uu} = 1, & \forall u \in V \\ & x_{uv} = x_{vu}, & \forall u, v \in V \end{array}$$

$$\begin{array}{cc} u \rightarrow & \begin{pmatrix} & & \\ & x_{uv} & \\ & & \end{pmatrix} \succeq 0 \\ & \begin{array}{c} \uparrow \\ v \end{array} \end{array}$$

$$\begin{cases} x_{uv} = y_u y_v, & \forall u, v \in V \\ y_v \in \{-1, 1\} \end{cases}$$

Eliminating the last two constraints gives an SDP which is a relaxation $\rightarrow$ gives a lower bound for MAX-CUT.

> **Note:**
> Geomans & Williamson gave an SDP-based randomized that gives the best approx. alg. for Max-Cut (≈ 0.87)

$\rightarrow$ gives rise to alternative approaches to solve NP-Hard optimization problems.