Applied Real Analysis

AMATH 331

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LaTeXed by Sibelius Peng

Preface

Disclaimer Much of the information on this set of notes is transcribed directly/indirectly from the lectures of AMATH 331 during Winter 2020 as well as other related resources. I do not make any warranties about the completeness, reliability and accuracy of this set of notes. Use at your own risk.

I gave up using definition blocks gradually since the professor uses a subsection to give all definition...

Also, I am not following the numbering convention in professor's lecture notes: Instead of setting the counter within the section (Theorem 13.1.1), I am using the counter within the chapter/lecture (Theorem 13.1).

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Sibelius Peng

Contents

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Real Numbers

Refs 1 for review. 2.1-2.2, 2.9

1.1 Decimal expansions and the real number line

finite decimal expansion

A finite decimal expansion has the form

$$x = a_{0\circ}a_1a_2a_3\dots a_N$$

where a_0 is an integer (positive, negative or zero) for $1 \leq n \leq N$ $a_n \in \{0, 1, \dots, 9\}$

Example:

$$1_\circ 45$$

$$-38_\circ 298743$$

You can think of this as

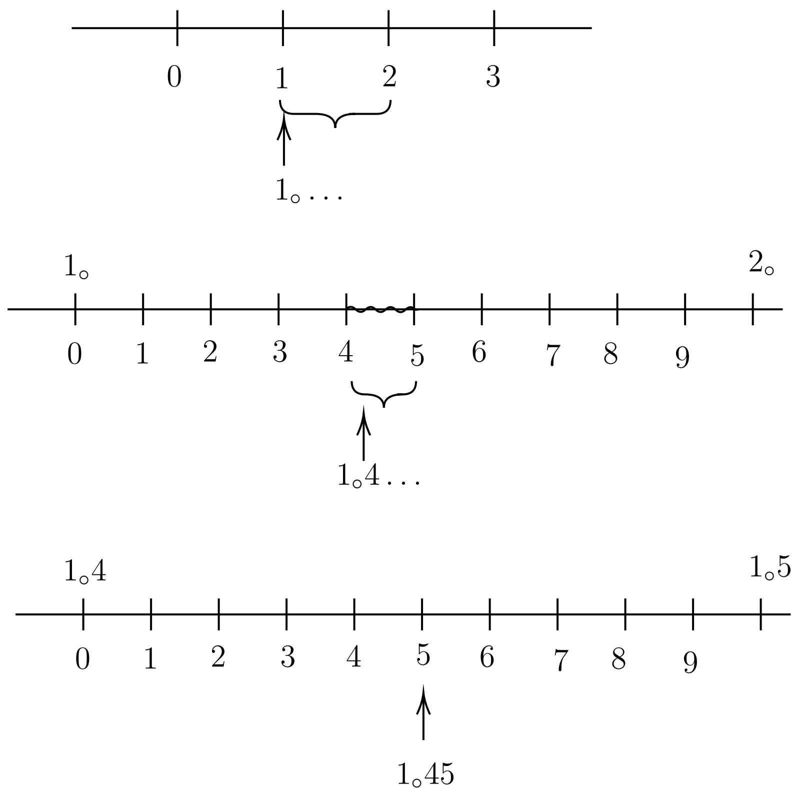
$$x = a_0 + a_1\left(\frac{1}{10}\right) + \dots + a_N\left(\frac{1}{10^N}\right)$$

Warning This looks like the usual decimal representation but it is not the same for negative numbers.

Any finite decimal expansion can be replaced on the real number line.

Example:

Where is $1_\circ 45$?



We can similarly define infinite decimal expansions

infinite decimal expansions

$$x = a_{0\circ} a_1 a_2 \ldots$$

Example:

$$1_\circ 450000000\ldots$$

$$\pi = 3_\circ 1415926535\ldots$$

Assuming the real number line has no gaps, every infinite decimal expansion x corresponds to a point on the line.

Given any positive integer k, let $y = a_{0\circ} a_1 a_2 \ldots a_k$ be the finite decimal expansion of x to the k-th decimal space. Then, x lies in the interval from y to $(y + 10^{-k})$. So, y approximates x to an accuracy of $1/10^k$. As we increase k, we improve the accuracy; in fact, the error can be made arbitrarily small.

The converse direction: given a point on the real number line, can we find its decimal expansion?

Yes!

It is possible for two decimal expansions to represent the same point. This happens precisely when one ends in an infinite string of 0's.

Example:

$$1.000\ldots \qquad \text{and} \qquad 0.999\ldots$$

$$25.300\ldots \qquad \text{and} \qquad 25.2999\ldots$$

We define the real numbers $\mathbb{R}$ as the set of all infinite decimal expansions.

1.2 Ordering of real numbers

Suppose

$$x = x_{0\circ}x_1x_2x_3\ldots, \qquad y = y_{0\circ}y_1y_2y_3\ldots$$

We say that x and y are equal and write $x = y$ if infinite decimal expansions are identical or equivalent, as discussed previously.

If x and y are not equal, then we say that x are not equal, then x is *less than* y and write $x < y$ if there exists integer $k \geq 0$ such that $x_k < y_k$ and $x_i = y_i$ for $i < k$. x is *greater than* y $(x > y)$ if ...

For any two real numbers x, y, exactly one of the following holds:

$$x = y \qquad x < y \qquad x > y$$

2

Bounds and Limits

2.1 Bounded sets of real numbers

upper bound

A set $S \subseteq \mathbb{R}$ is *bounded above* if there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. M is an *upper bound* of S.

lower bound

A set $S \subseteq \mathbb{R}$ is *bounded below* if there exists $m \in \mathbb{R}$ such that $s \geq m$ for all $s \in S$. m is an *lower bound* of S.

bounded

A set is *bounded* if it is both bounded above and bounded below.

supremum

The *supremum* or *least upper bound* of a nonempty set S that is bounded above is the upper bound L satisfies $L \leq M$ for all upper bounds M of S is written as $\sup S$.

infimum

The *infimum* or *greatest lower bound* of a nonempty set S is the lower bound ℓ satisfying $\ell \geq m$ for all lower bounds m of S. The infimum is denoted $\inf S$.

max

If there exists $M \in S$ such that $s \leq M$ for all $s \in S$, then M is called the *maximum* of S, $\max S$.

min

Analogous defn for $\min S$.

2.2 Examples

0. $S_0 = \varnothing$. Bounded above and below. No supremum or infimum.
1. $S_1 = \{n \in \mathbb{Z}^+\} = \{1, 2, 3, \ldots\}$ not bounded above, bounded below.

   1 is infimum and minimum
2. $S_2 = \{-3, -2, 0.5, 1.423\}$. Bounded above and below. Bounded. Has max, min.
3. $S_3 = \left\{1 - \frac{1}{n} : n \in \mathbb{Z}^+\right\} = \{0, \frac{1}{2}, \frac{2}{3}, \ldots\}$

   Bounded above by 1. Bounded below by 0.

   Supremum is 1, but there is no max.

2.3 Least Upper Bound Principle

Theorem 2.1: Least Upper Bound Principle

Every nonempty set S of $\mathbb{R}$ that is bounded above has a supremum. Every nonempty set that is bounded below has an infimum.

Sketch of proof for "infimum". There are only finitely many integers from m_0 to $s_0 + 2$. Choose the greatest integer lower bound $\to$ call it a_0.

$a_0 + 1$ is not a lower bound. Divide $[a_0, a_0 + 1]$ into 10, find a_1 such that $a_{0\circ}a_1$ is lower bound of S, but $a_{0\circ}a_1 + 1/10$ is not. Repeat infinitely many times to construct $L = a_{0\circ}a_1a_2a_3\ldots$.

Now, show that L is infimum.[1] $\square$

[1] See details in textbook.

3

Limits of Sequences

3.1 Sequences

An *infinite sequence of real* numbers is an infinite, enumerated list of real numbers, denoted by

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, \dots)$$

Each $a_n \in \mathbb{R}$ is an *element* of the sequence.

We will just refer to them as sequences, and often write (a_n). Formally, a sequence is a function that maps positive integers to $\mathbb{R}$.

We say that a sequence is [bounded above/bounded below/bounded] if the set $A = \{a_n\}$ is respectively [bounded above/bounded below/bounded].

3.2 Examples

1. $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^n$ for $n \geq 1$.
2. $a_n = \frac{1}{n}$, for $n \geq 1$.
3. $(a_n) = (1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \dots)$

3.3 Limits of Sequences

limit

Let $(a_n)_{n=1}^{\infty}$ be a sequence. We call $L \in \mathbb{R}$ the *limit* of the sequence if for all $\epsilon > 0$, there exists an integer N such that

$$|a_n - L| < \epsilon$$

for all $n \geq N$.

If such L exists, then we say that (a_n) is convergent, and converges to L and we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$.

If a sequence does not have such a limit, then we say it *diverges*, or is *divergent*.

A sequence (a_n) *diverges to* ∞ if for all $M > 0$, there exists N such that $a_n > M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = \infty$.

A sequence (a_n) *diverges to* $-\infty$ if for all $M < 0$, there exists N such that $a_n < M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = -\infty$.

Note
$\lim_{n\to\infty} a_n = \pm\infty$ does not mean limit exists.

3.4 Examples

1. $a_n = 1/n, \quad \lim_{n\to\infty} a_n = 0$

   For any $\epsilon > 0$, we need to show that there exists N such that $|a_n - 0| < \epsilon$ for all $n \geq N$.

   Choose N to be any integer greater than $1/\epsilon$. $(N > \frac{1}{\epsilon})$

   For any $n \geq N$, $a_n = 1/n \leq \frac{1}{N} < \epsilon$. We also have $a_n \geq 0$

   $$\implies |a_n| < \epsilon$$

   for all $n \geq N$ as required.

3.5 Some basic properties of limits

Theorem 3.1: Squeeze Theorem

Let $(a_n), (b_n), (c_n)$ be sequences.

If $a_n \leq b_n \leq c_n$ for all $n \geq 1$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = 1$$

Proof:
We want to show that for all $\epsilon > 0$, there exists N such that $|b_n - L| < \epsilon$ for all $n \geq N$.

Let $\epsilon > 0$. Since $a_n \to L$, we can find N_1 such that $|a_n - L| < \epsilon$ for all $n \geq N_1$.

Similarly, there exists N_2 s.t. $|c_n - L| < \epsilon$ for all $n \geq N_2$.

Define $N := \max\{N_1, N_2\}$. Then, for $n \geq N$, $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$.

Equivalently,

$$L - \epsilon < a_n < L + \epsilon \qquad L - \epsilon < c_n < L + \epsilon$$

Since $a_n \leq b_n \leq c_n$. $L - \epsilon < b_n < L + \epsilon$, or

$$|b_n - L| < \epsilon$$

as required. □

Proposition 3.2

If a sequence converges to a limit L, then this limit is unique.

Proof:
See PDF. □

Proposition 3.3

If a sequence (a_n) converges, then the set $A := \{a_n : n \geq 1\}$ is bounded.

Proof:
Exercises. □

Theorem 3.4

Let (a_n) and (b_n) be two convergent sequences. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. $\lim_{n\to\infty}(a_n + b_n) = L + M$
2. for any $\alpha \in \mathbb{R}$, $\lim_{n\to\infty}(\alpha a_n) = \alpha L$
3. $\lim_{n\to\infty}(a_n b_n) = LM$, and
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$ and $b_n \neq 0$ for all n.

4 Monotone Sequence and Applications

4.1 Monotone Sequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. it is

1. monotone increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.
2. strictly monotone increasing if $a_{n+1} > a_n$ for all $n \geq 1$.
3. monotone decreasing if $a_{n+1} \leq a_n$
4. strictly monotone decreasing if $a_{n+1} < a_n$

monotone

A sequence is monotone is *monotone* if it is either (monotone) increasing or (monotone) decreasing.

Theorem 4.1: Monotone Convergence Theorem

Monotone Convergence Theorem:

(i) Every monotone increasing sequence that is bounded above converges

(ii) Every monotone decreasing sequence that is bounded below converges

Proof:
We will first show that (i) $\implies$ (ii).

Let (a_n) be a monotone decreasing sequence that is bounded below by m.

The sequence $(-a_n)_{n=1}^{\infty}$ is monotone increasing and is bounded above by $-m$. By part (i), $(-a_n)$ must converge. Call the limit $L = \lim_{n\to\infty}(-a_n)$.

By Theorem 3.4 Part 2,

$$\lim_{n\to\infty} = \lim_{n\to\infty}[(-1)(-a_n)] = (-1)\lim_{n\to\infty}(-a_n) = -L$$

To prove Part(i) of this theorem, suppose (a_n) is monotone increasing and bounded above.

The set $A = \{a_n | n \in \mathbb{Z}^+\}$ is bounded above, and nonempty.

By LUBP(Theorem 2.1), A has a supremum, which we call $L = \sup A$. We show that L is the limit of (a_n).

Given $\epsilon > 0$, we know that $L - \epsilon$ cannot be an upper bound of A.

So there exists N such that $a_n > L - \epsilon$.

Since (a_n) is increasing, $a_n > L - \epsilon$ for all $n \geq N$. Since L is an upper bound of A, $a_n \leq L$ for all $n \geq N$.

$$\implies L - \epsilon < a_n \leq L < L + \epsilon$$

That is $|a_n - L| \leq \epsilon$ for all $n \geq N$. □

4.2 Applications: Calculate Square Roots

The square root of a real number $a > 0$ can be obtained as the limit of the sequence defined recursively by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right), \qquad \text{for } n \geq 1$$

where the starting point x_0 is any positive number.

Moreover, for any $n \geq 1$, the error in approximating $\sqrt{a}$ by x_n satisfies the bound

$$0 \leq x_n - \sqrt{a} < x_n - \frac{a}{x_n}$$

Proof:
Strategy:

1. Prove that (x_n) is bounded below.
2. Prove that (x_n) is monotone decreasing.
3. Prove that (x_n) is monotone decreasing.
4. Use MCT to prove that (x_n) converges.

5. Use properties of limits to determine that $\sqrt{a}$ is the limit.

6. Look for upper and lower bounds for error.

See PDF for full proof. $\square$

4.3 Warning about computing limits that don't exist

Example:

$a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$ for $n \geq 1$.

If we assume (a_n) has a limit L, then we can get nonsense.

$$a_{n+1} = \frac{1}{2}(a_n^2 + 1)$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n^2 + 1)$$
$$\implies L = \frac{1}{2}\left(\lim_{n\to\infty} a_n\right)^2 + \frac{1}{2} = \frac{1}{2}L^2 + \frac{1}{2}$$
$$L^2 - 2L + 1 = 0 \implies L = 1 \text{ is a solution}$$

However, it can be shown that (a_n) is monotone increasing. Since $a_1 = 2$, (a_n) cannot possibly converge to 1.

(In fact, it does not converge.)

5

Subsequences

5.1 Definitions of subsequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence. The sequence $(b_k)_{k=1}^{\infty}$ is a *subsequence* of (a_n) of there exist integers n_k with $1 \leq n_1 < n_2 < n_3 < \ldots$ such that $b_k = a_{n_k}$ for each $k \geq 1$.

Example:

$$(a_1, a_2, a_3, a_4, a_5, \ldots)$$

$$(b_1, b_2, b_3, b_4, b_5, \ldots)$$

cannot do the following:

$$(a_1, a_2, a_3, a_4, a_5, \ldots)$$

$$(b_1, b_2, b_3, b_4, b_5, \ldots)$$

not allowed to change order

Example:

$$(a_n)_{n=1}^{\infty} = \left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \ldots\right)$$

The sequence (b_k) with $b_k = a_k$ for all $k \geq 1$ is a subsequence of (a_n).

The sequence $\left(-1, -\frac{1}{3}, -\frac{1}{5}, \ldots\right)$ is a subsequence.

The sequence $\left(\frac{1}{2}, \frac{1}{4}, \ldots\right)$ is another subsequence.

5.2 Some properties of Subsequences

Lemma 5.1

Let n_k be integers satisfying $n_1 \geq 1$ and $n_k < n_{k+1}$ for all $k \geq 1$. Then $n_k \geq k$ for all $k \geq 1$.

Theorem 5.2

Suppose the sequence $(a_n)_{n=1}^{\infty}$ converges to the limit L. Then every subsequence of (a_n) also converges to L.

Proof:
By definition of limit, for every $\epsilon > 0$, there exists N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Let $(b_k)_{k=1}^{\infty}$ be any subsequence of (a_n), where $b_k = a_{n_k}$ for each $k \geq 1$.

From Lemma 5.1, we know that $n_k \geq k$ for each k. Given $\epsilon > 0$, chose N as in definition of $\lim_{n\to\infty} a_n = L$. For every $k \geq N$,

$$n_k \geq k \geq N \implies |b_k - L| = |a_{n_k} - L| < \epsilon$$

□

Example:

1. From 5.1, the theorem holds just as it is.
2. Converse is not true. If a subsequence converges, we cannot conclude that the original sequence converges.

5.3 Bolzano-Weierstrass

If for every integer $n \geq 1$, we have a nonempty, closed interval $I_n = [a_n, b_n]$ such that $I_{n+1} \subseteq I_n$, then we say that (I_n) is a *nested sequence of closed, bounded intervals.*

Lemma 5.3: Nested Intervals Lemma

If (I_n) is a nested sequence of closed bounded intervals, then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof:
Exercise. □

Theorem 5.4: Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Proof:
Outline.

1. Given a bounded sequence (a_n), construct a nested sequence of closed, bounded intervals I_n with lengths decreasing to zero, and such that each I_n contains infinitely many elements of the sequence (a_n).
2. Construct a subsequence (b_k) such that $b_k \in I_k$ for each $k \geq 1$.
3. Show that (b_k) converges.

$\square$

Proof:

Step 1: Suppose $(a_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers. Let m_1 be a lower bound and M_1 be an upper-bound for $A = \{a_n : n \geq 1\}$.

Define an interval $I_1 = [m_1, M_1]$. Define the point $c_1 = \frac{1}{2}(m_1 + M_1)$. Choose one smaller interval either $[m_1, c_1]$ or $[c_1, M_1]$ that contains an infinite member of elements of $(a_n) \to$ call this interval $I_2 = [m_2, M_2]$.

We repeat this process for all $k \geq 2$. This gives a sequence of intervals $(I_k)_{k=1}^{\infty}$ such that $I_{n+1} \subseteq I_n$ for all $n \geq 1$, and lengths of I_n converges to zero. Also each I_k contains an infinite number of elements of (a_n).

Step 2: Let $n_1 = 2$ so $b_1 = a_1$. Suppose we have our subsequence (b_j) up to element k. Then we have $n_i \geq 1$ for all $i = 1, 2, \ldots, k$ and $n_i < n_{i+1}$ for all $i = 1, 2, \ldots, k-1$.

Since there are an infinite number of elements of (a_n) contained in I_{k+1}, we can choose n_{k+1} such that $n_{k+1} > n_k$ and $a_{n_{k+1}} \in I_{k+1}$, i.e. $b_{k+1} \in I_{k+1}$. In this way, we inductively define (b_j) as a subsequence of (a_n).

Step 3: By Nested Intervals Lemma (Lemma 5.3), $\bigcap_{k=1}^{\infty} I_k \neq \varnothing$, so there must exist a point $L \in \cap_{k=1}^{\infty} I_k$. The length of interval I_j is $\dfrac{(M_1 - m_1)}{2^{j-1}}$. For any $k \geq 1$, we have $L \in I_k$ and $b_k \in I_k$. Hence $|b_k - L| \leq \dfrac{(M_1 - m_1)}{2^{k-1}}$.

Consider sequence $(|b_k - L|)_{k=1}^{\infty}$. We can use Squeeze Theorem to show that $\lim_{n\to\infty} |b_k - L| = 0$ since

$$0 \leq |b_k - L| \leq \frac{(M_1 - m_1)}{2^{k-1}}.$$

Hence $\lim\limits_{k\to\infty} b_k = L$.

□

6

Cauchy Sequences

6.1 Definition

A sequence (a_n) is *Cauchy* if for any $\epsilon > 0$, there exists an integer N such that

$$|a_n - a_m| < \epsilon$$

for all $n, m \geq N$.

Example:

$$(a_n)_{n=1}^{\infty} = (3, 3.1, 3.14, 3.141, \dots)$$

More generally, if x is any real number with infinite decimal expression $x_{0\circ}x_1x_2x_3\dots$, then the sequence of finite truncations, i.e., a_k is the truncation of x to k decimal places, is Cauchy.

$$a_k = x_{0\circ}x_1 \dots x_k000\dots$$

Given $\epsilon > 0$, we can find N such that $10^{-N} < \epsilon$.

For any $n \geq 1$, we have

$$a_n \leq x \leq a_n + 10^{-n}$$

In particular,

$$a_N \leq x \leq a_N + 10^{-N}$$

Note that (a_n) is monotone increasing, so $a_N \leq a_n, a_m \leq x \leq a_N + 10^{-N}$ for any $n, m \geq N$.

So

$$|a_n - a_m| \leq \text{length of interval} = 10^{-N} < \epsilon$$

$\implies (a_n)_{n=1}^{\infty}$ is Cauchy.

7 Cauchy and Completeness

7.1 Properties of Cauchy Sequences

Proposition 7.1

If a Cauchy sequence (a_n) has a convergent subsequence, then (a_n) converges. The limit is the same as the limit of the subsequence.

Proof:
Let $\epsilon > 0$. By definition of limit of $(b_k) = (a_{n_k})$ being L, i.e., $\lim_{k\to\infty} b_{n_k} = L$, there exists K such that

$$|b_k - L| = |a_{n_k} - L| < \frac{\epsilon}{2}$$

for all $k \geq K$.

By Cauchy property of (a_n), there exists N such that

$$|a_n - a_m| < \frac{\epsilon}{2}$$

for all $n, m \geq N$.

By Lemma 5.1, $n_k \geq k$ for all $k \geq 1$, so

$$|a_n - a_{n_k}| < \frac{\epsilon}{2}$$

for all $n, k \geq N$. Choose any $k \geq \max\{K, N\}$. Then, for all $n \geq N$,

$$|a_n - L| = |a_n - a_{n_k} + a_{n_k} - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Proposition 7.2

If a sequence (a_n) is Cauchy, then the set $\{a_n : n \geq 1\}$ is bounded.

Proof:
Exercise, or see PDF. $\square$

7.2 Example of not quite Cauchy

Consider the sequence $(a_n)_{n=1}^{\infty}$, with $a_n = \log n$.

The difference between successive terms is

$$|a_{n+1} - a_n| = |\log(n+1) - \log(n)| = \left|\log\left(\frac{n+1}{n}\right)\right|$$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$, so $\lim |a_{n+1} - a_n| = 0$.

(a_n) is not bounded, since $\log(n) \to \infty$, hence by Proposition 7.2, (a_n) is not Cauchy.

7.3 Cauchy, Convergent and Complete

Proposition 7.3

Every convergent sequence is Cauchy.

Proof:
(Sketch)

N, K and use $\epsilon/2$. $\square$

complete

We say that a subset X of $\mathbb{R}$ is *complete* if every Cauchy sequence in X has a limit in X.

Theorem 7.4: Completeness Theorem for Real Numbers

$\mathbb{R}$ is complete.

In other words, every Cauchy sequence of real numbers converges.

Proof:
Suppose (a_n) is any Cauchy sequence of real numbers. By Proposition 7.2, $\{a_n : n \geq 1\}$ is bounded. By Theorem 5.4, there must exist a convergent subsequence.

By Proposition 7.1, (a_n) must also converge. $\square$

Remark:
The sequence of truncated decimal expansions of x (from Lecture 6) was shown to be Cauchy. Now we know, it must converge. It can be shown that the limit is x.

Note
$\mathbb{Q}$ is not a complete subset of $\mathbb{R}$. Using sequence of finite decimal expansions, we see that sequences of rational numbers can converge to an irrational limit.

7.4 Equivalent Statements of Completeness

We showed that construction of $\mathbb{R}$ as set of infinite decimal expansions leads to Least Upper Bound Principle.

$\implies$ Monotone Convergence Theorem

$\implies$ Nested Intervals Lemma

$\implies$ Bolzano-Weierstrass Theorem

$\implies$ Completeness Theorem

It is possible to show that Completeness $\implies$ LUBP. So all of these properties describe the same "behaviour" of $\mathbb{R}$.

7.5 Application: Proving convergence by Cauchy property

Sometimes it's easier to show that a sequence is Cauchy than convergent.

Example:
Consider a sequence $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n}$. We can show that $(a_n)_{n=1}^{\infty}$ is Cauchy. For $m > n$,

$$\begin{aligned} |a_m - a_n| &= \left| \frac{(-1)^{n+2}}{n+1} + \frac{-1^{n+3}}{n+2} + \ldots + \frac{(-1)^m}{m-1} + \frac{(-1)^{m+1}}{m} \right| \\ &= \ldots \end{aligned}$$

Suppose $m - n$ is even

$$|a_m - a_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \ldots + \frac{1}{m-1} - \frac{1}{m} \right|^{a}$$

[a] Sth wrong here... corrected in the lecture notes.

8 Series

8.1 Definitions for series

If $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers, we define its *sequence of partial sums* $(S_n)_{n=1}^{\infty}$ by $S_n = \sum_{k=1}^{n} a_k$.

The (infinite) series associated with (a_n) is $\sum_{n=1}^{\infty} a_n$. If the sequence of partial sums converges to a limit $L \in \mathbb{R}$, then we say the series $\sum_{n=1}^{\infty}$ converges. In this case, we say the sum or value of the series is L.

The series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

If a series does not converge, then it diverges.

A series that converges but is not absolutely convergent, then we say it is conditionally convergent.

Example:

1. $(a_n)_{n=1}^{\infty} = (1, 1, 1, 1, 1, \ldots)$. This sequence converges to 1.

   Sequence of partial sums is $(S_n) = (1, 2, 3, 4, 5, \ldots)$ does not converge (it diverges to ∞) so the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   Note
   $S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ forms a sequence such that

   $$S_{n+1} - S_n = \frac{1}{n+1} \to 0$$

   but (S_n) is not convergent, which means (S_n) is not Cauchy.

3. $a_n = \frac{1}{n(n+2)}$.

We will show that $\sum_{n=1}^{\infty} a_n$ converges.

Note
We can write
$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$
Then the sequence of partial sums is
$$S_n = \frac{1}{2}\sum_{k=1}^{n} \frac{2}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\left[\left(1+\frac{1}{2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)\right]$$
$$\lim_{n\to\infty} S_n = \frac{3}{4}$$
Hence,
$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$$

4. A geometric series $\sum_{n=0}^{\infty} a_n$ is one where the elements are of the form $a_n = a_0 r^n$ for some $a_0 \in \mathbb{R}, r \in \mathbb{R}$, for each $n \geq 0$.

   If $|r| < 1$, then the series converges
   $$\sum_{n=0}^{\infty} a_n = \frac{a_0}{1-r}$$
   If $|r| \geq 1$ and $a_0 \neq 0$, then the series diverges.

5. The alternating harmonic series
   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
   converges. It is not absolutely convergent. (See Example 2), so it is conditionally convergent.

Proposition 8.1

Every absolute convergent series is convergent.

Proof:
Trivial. $\square$

8.2 Convergence Tests

Theorem 8.2: Cauchy criterion for series

Given a series $\sum_{n=1}^{\infty} a_n$, the following are equivalent:

1. The series converges.
2. Given $\epsilon > 0$, there exists an integer N such that

$$\left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$$

for all $m > n \geq N$.

Note

If (S_n) is sequence of partial sums. Suppose $m > n$,

$$|S_m - S_n| = \left| \sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k \right| = \left| \sum_{k=n+1}^{m} a_k \right|$$

Theorem 8.3: Comparison Test for Series

Suppose $(a_n), (b_n)$ are two sequences and $|a_n| \leq b_n$ for all $n \geq 1$.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, and

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} b_n$$

2. If $\sum_{n=1}^{n} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof:

Note that 2 follows from 1.

So, we just need to prove 1.

First, we show that

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

Let $\epsilon > 0$. By Cauchy criterion, there exists N such that

$$\left| \sum_{k=n+1}^{m} b_k \right| < \epsilon \text{ for all } m > n \geq N$$

Since $b_k \geq 0$ for all k, we can ignore absolute value sign.

$$\epsilon > \sum_{k=n+1}^{m} b_k \geq \sum_{k=n+1}^{m} |a_k| \geq \left| \sum_{k=n+1}^{m} a_k \right|$$

This is the Cauchy criterion for $\sum a_n$, so $\sum a_n$ converges.

The rest of proof is left as an exercise: Show remaining inequality. $\square$

9 Rearrangements of Series

9.1 Definition

A rearrangement is a series considering of the same terms as another series but in a different order. Suppose $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a permutation of the positive integers. Then, the series $\sum_{n=1}^{\infty} a_{\pi(n)}$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots$$

$$\sum_{n=1}^{\infty} a_{\pi(n)} = a_3 + a_4 + a_2 + a_1 + a_6 + \dots$$

9.2 Rearrangements of absolutely convergent series

Proposition 9.1

If an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ converges to L, then every rearrangement of $\sum_{n=1}^{\infty} a_n$ also converges to L.

Proof:
Let $\sum_{n=1} a_{\pi(n)}$ be a rearrangement. Fix $\epsilon > 0$. By absolute convergence of

$\sum_{n=1}^{\infty} a_n$, there exist N such that

$$\left|\sum_{n=1}^{N} |a_n| - \sum_{n=1}^{\infty} |a_n|\right| = \sum_{n=N+1}^{\infty} |a_n| < \frac{\epsilon}{2}$$

Since every term of the series $\sum_{n=1}^{\infty} a_n$ must appear in the rearrangement, there must exist $M \geq N$ such that $\sum_{n=1}^{M} a_{\pi(n)}$ includes all terms

$$a_1, a_2, a_3, \ldots, a_N$$

For any $m \geq M$,

$$\begin{aligned}
\left|\sum_{n=1}^{m} a_{\pi(n)} - L\right| &= \left|\sum_{n=1}^{m} a_{\pi(n)} - \sum_{n=1}^{N} a_{\pi(n)} + \sum_{n=1}^{N} a_{\pi(n)} - L\right| \\
&\leq \left|\sum_{n=1}^{m} a_{\pi(n)} - \sum_{n=1}^{N} a_{\pi(n)}\right| + \left|\sum_{n=1}^{N} a_{\pi(n)} - L\right| \\
&\leq \sum_{n=N+1}^{\infty} |a_n| + \sum_{n=N+1}^{\infty} |a_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

So $\sum_{n=1}^{\infty} a_{\pi(n)} = L$. □

9.3 Rearrangements of conditionally convergent series

Lemma 9.2

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series. Then there is an infinite number of non-negative terms and an infinite number of negative terms in the series.

Proof:
Use contrapositive.

Suppose there is a finite number of negative terms.

> Remark:
> Case with finite number of non-negative terms can be proved in the same way.

There must exist integer N such that N is the largest number for which $a_N < 0$. i.e. $a_n \geq 0$ for all $n > N$.

Case (i) $\sum_{n=1}^{\infty} a_n$ diverges. Trivially, not conditionally convergent.

Case (ii) $\sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_N$$

$$\implies \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

$$= \underbrace{\sum_{n=1}^{N} |a_n|}_{\substack{\text{finite sum} \\ \implies \text{real number}}} + \underbrace{\sum_{n=1}^{\infty} a_n}_{\text{converges}} - \underbrace{\sum_{n=1}^{N} a_n}_{\substack{\text{finite sum} \\ \implies \text{real number}}}$$

By properties of limits, $\sum_{n=1}^{\infty} |a_n|$ converges.

$\implies$ $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$\implies$ $\sum_{n=1}^{\infty} a_n$ not conditionally convergent.

□

Lemma 9.3

Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. For each $n \geq 1$, define b_n to be the n-th non-negative and c_n is the n-th negative term in the series. Then,

1. $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$, and
2. $\sum_{n=1}^{\infty} b_n = \infty$ and $\sum_{n=1}^{\infty} c_n = -\infty$.

Proof:
Exercise. □

Theorem 9.4

Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent series. Then, for any $L \in \mathbb{R}$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that convergent to L.

Proof:
Exercise. □

10

Euclidean Space

10.1 $\mathbb{R}^n$ Euclidean inner product and norm

We define the space $\mathbb{R}^n$ to be the set of all n-vectors $\mathbf{x} = (x_1, x_2, x_3, \dots, x_n)$ where $x_i \in \mathbb{R}$ for each $i = 1, 2, 3, \dots, n$.[1]

$\mathbb{R}^n$, equipped with vector addition and scalar multiplication, is a *vector* space.

We also define the Euclidean *inner product* of two vectors x and y is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

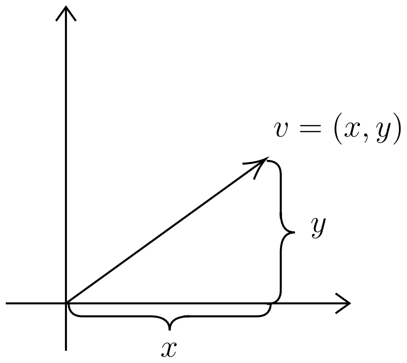
Also called dot product or scalar product.

The *Euclidean norm* of a vector $x \in \mathbb{R}^n$ is

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

Bt *Euclidean space*, we mean $\mathbb{R}^n$ with the structure imposed by the Euclidean inner product and norm.

[1] I will use $\mathbf{x}, \vec{x}$ or just x (from optimization courses) to represent vector. Readers should be clear when it is vector.



$$\|v\| = \sqrt{x^2 + y^2} = \left(\sum_{i=1}^{2} |v_i|^2\right)^{1/2}$$

Figure 10.1: Example of $n = 2$

10.2 Properties of Euclidean inner product and norm

Proposition 10.1

Let $x, y, z\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. The Euclidean inner product satisfies

1. $\langle x, x \rangle \geq 0$ with equality iff $x = 0$. (positive definite)
2. $\langle x, y \rangle = \langle y, x \rangle$. (symmetry)
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$. (Bilinearty)

Proposition 10.2

Let $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. The Euclidean norm satisfies:

1. $\|x\| \geq 0$ with equality iff $x = 0$. (positive definite)
2. $\|\alpha x\| = |\alpha| \, \|x\|$. (homogeneous)
3. $\|x + y\| \leq \|x\| + \|y\|$. ($\triangle$ ineq)

Proof:
1,2 $\to$ Exercise.

3 $\to$ See Theorem 10.4. $\square$

10.3 Inequalities

Theorem 10.3: Cauchy-Shwarz Inequality

For any $x, y \in \mathbb{R}^n$,

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

Equality holds iff x and y are linearly dependent (i.e., there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 x + \alpha_2 y = 0$ and it is not true that $\alpha_1 = \alpha_2 = 0$).

Proof:
First, note that the result is trivial if $x = 0$ or $y = 0$.

Suppose $x \neq 0$ and $y \neq 0$. We define the unit vectors

$$\mathbf{u} = (u_1, \ldots, u_n) = \frac{x}{||x||}$$

and

$$\mathbf{v} = (v_1, \ldots, v_n) = \frac{y}{||y||}$$

For each $i = 1, 2, \ldots, n$,

$$0 \leq (u_i - v_i)^2 = u_i^2 - 2u_i v_i + v_i^2$$

$$u_i v_i \leq \frac{1}{2}(u_i^2 + v_i^2)$$

Adding together inequalities for all i:

$$\sum_{i=1}^{n} u_i v_i \leq \frac{1}{2}\sum_{i=1}^{n}(u_i^2 + v_i^2) \implies \langle u, v\rangle \leq \frac{1}{2}(||u||^2 + ||v||^2) = 1$$

We can do the same manipulation as above starting form

$$0 \leq (u_i + v_i)^2 = u_i + 2u_i v_i + v_i^2 \implies \langle u, v\rangle \geq -1$$

Hence $|\langle u, v\rangle| \leq 1$.

Exercise: Complete this proof. □

Theorem 10.4: Triangle Inequality

For any two vectors, $x, y \in \mathbb{R}^n$

$$\|x + y\| \leq \|x\| + \|y\|$$

Equality holds iff $x = 0$ or $y = \alpha x$ for some $\alpha \geq 0$.

Proof:

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle && \text{by bilinearity} \\
&\le \langle x,x\rangle + |\langle x,y\rangle| + |\langle x,y\rangle| + \langle y,y\rangle && \text{by properties of abs values} \\
&\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

Take square roots:

$$\|x+y\| \le \|x\| + \|y\|$$

Now prove "equality" statement.

$\Longrightarrow$) If equality holds, then

$$\langle x,y\rangle = |\langle x,y\rangle| = \|x\| \cdot \|y\|$$

The first "=": compare first introduction of inequality in proof.

The second "=": second inequality.

So we need C.S equality condition and we need $\langle x,y\rangle \ge 0$.

Case 1 $\alpha_2 \neq 0$, then $y = \alpha x$ where $\alpha = -\frac{\alpha_1}{\alpha_2}$.

Case 2 $\alpha_2 = 0$. Exercise.

□

11

Convergence and Completeness in $\mathbb{R}^n$

11.1 Definitions of sequences and convergence in $\mathbb{R}^n$

An (infinite) sequence of vectors or points in $\mathbb{R}^n$ is an infinite enumerated list $(x_k)_{k=1}^{\infty} = (x_1, x_2, \ldots)$ where each $x_k \in \mathbb{R}^n$ for $k \geq 1$.

A sequence $(x_k)_{k=1}^{\infty}$ converges to a point $a \in \mathbb{R}^n$ if

Given $\epsilon > 0$, there exists N such that $\|x_k - a\| < \epsilon$ for all $k \geq N$.

If this holds, then a is called the limit of the sequence, and we write

$$\lim_{k\to\infty} x_k = a$$

Lemma 11.1

Let $(x_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. Then,

$$\lim_{k\to\infty} x_k = a \iff \lim_{k\to\infty} \|x_k - a\| = 0$$

Each $x_k = (\underbrace{x_{k,1}}_{\in\mathbb{R}}, \ldots, x_{k,n}) \in \mathbb{R}^n$.

Lemma 11.2

Let x_k be a sequence in $\mathbb{R}^n$. Then $\lim\limits_{k\to\infty} x_k = a = (a_1, a_2, \ldots, a_n)$ if and only if

$$\lim_{k\to\infty} x_{k,j} = a_j \quad \text{for } 1 \leq j \leq n$$

Proof:

$\Longrightarrow$) Suppose $\lim_{k\to\infty} x_k = a$. We must show that for each $j \in \{1, 2, \ldots, n\}$ and for all $\epsilon > 0$, we can find N_j such that

$$|x_{k,j} - a_j| < \epsilon \quad \text{for all } k \geq N_j$$

Fix (arbitrary) $j \in \{1, \ldots, n\}$ and let $\epsilon > 0$. By definition of $\lim_{k\to\infty} x = a$, there exists N such that $\|x_k - a\| < \epsilon$ for all $k \geq N$. By definition or norm,

$$\|x_k - a\|^2 = \sum_{i=1}^{n} |x_{k,i} - a_i|^2 \geq |x_{k,j} - a_j|^2$$

Hence, for $N_j := N$, we have

$$|x_{k,j} - a_j| < \epsilon \quad \text{for all } k \geq N$$

as required.

$\Longleftarrow$) Let $\epsilon > 0$. By convergence of components for each $j \in \{1, \ldots, n\}$, there exists N_j such that

$$|x_{k,j} - a_j| < \overline{\epsilon} := \frac{\epsilon}{\sqrt{n}} \quad \text{for all } k \geq N_j$$

Define $N = \max\{N_j\}$. Then for all $k \geq N$

$$|x_{k,j} - a_j| < \overline{\epsilon} \quad \text{for all } j \in \{1, \ldots, n\}$$

Then

$$\|x_k - a\|^2 = \sum_{j=1}^{n} |x_{k,j} - a_j|^2 < n\overline{\epsilon}^2 = \epsilon^2$$

So $\|x_k - a\| < \epsilon$ for all $k \geq N$, as required.

□

11.2 Cauchy sequences

A sequence $(x_k)_{k=1}^{\infty}$ is Cauchy if ...

Lemma 11.3

Let (x_k) be a sequence of points in $\mathbb{R}^n$. Then (x_k) is Cauchy if and only if $(x_{k,j})_{k=1}^{\infty}$ is Cauchy for all $j \in \{1, 2, \ldots, n\}$.

11.3 Completeness

A subset S of $\mathbb{R}^n$ is complete if every Cauchy sequence in S converges to a limit in S.

Proposition 11.4

Every convergent sequence in $\mathbb{R}^n$ is Cauchy.

Theorem 11.5: Completeness Theorem for $\mathbb{R}^n$

$\mathbb{R}^n$ is complete.

Proof:

$$\begin{gathered}(x_k)_{k=1}^{\infty} \text{ is Cauchy} \\ \Big\Updownarrow \text{Lemma 11.3} \\ (x_{k,j})_{k=1}^{\infty} \text{ is Cauchy for all } j \in \{1, \ldots, n\} \\ \Big\Updownarrow \text{Theorem 11.5} \\ (x_{k,j})_{k=1}^{\infty} \text{ is convergent for each } j \\ \Big\Updownarrow \text{Lemma 11.2} \\ (x_k)_{k=1}^{\infty} \text{ is convergent}\end{gathered}$$

Actually, we have shown iff. □

11.4 Closed subsets of $\mathbb{R}^n$

Let $X \subset \mathbb{R}^n$. We define a *limit point* of X as a point $a \in \mathbb{R}^n$ for which there exists a sequence in X converging to a.

Example:

$(0, 2] = X$

1, 2 is a limit point of X. 3 is not a limit point. 0 is a limit point, but not in X.

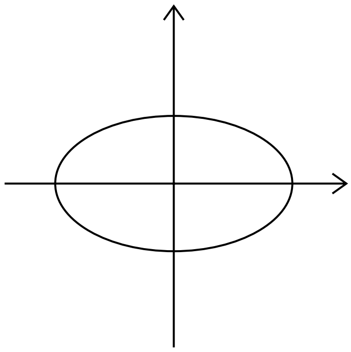
If X contains all of its limit points, we say it is *closed*.

So $(0, 2]$ is not closed.

Given any subset $X \subseteq \mathbb{R}^n$, we define the closure of X, denoted $\overline{X}$, as the set of all limit points of X.

Example:

1. $\varnothing$ is closed since it contains all limit points.
2. $\mathbb{R}^n$ is closed.

3. The open interval $(0,1) \subseteq \mathbb{R}$. It is not closed.
4. The closure of $(0,1)$ is $[0,1]$ which is closed.
5. 

$$X = \left\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + y^2 < 1\right\}$$

   X is not closed. Consider

$$x_k = \left(\left(0, 1 - \frac{1}{k}\right)\right)_{k=1}^{\infty} \to (0,1) \notin X$$

Proposition 11.6

A subset $S \in \mathbb{R}^n$ is closed if and only if it is complete.

Remark:
Not always equivalent - relies on completeness of $\mathbb{R}^n$.

12

Closed Subsets in $\mathbb{R}^n$

12.1 Properties of closed sets

Proposition 12.1

For any subset $X \subseteq \mathbb{R}^n$, the closure of X is closed, and is in fact, the smallest closed set that contains X.

Proof:
Exercise. $\square$

Proposition 12.2

If A and B are closed subsets of $\mathbb{R}^n$, then $A \cup B$ is closed.

Proof:
First note that $A \cup B$ is trivially closed if $A = B = \varnothing$. Now, suppose A and B are closed, and not both empty.

Let x be a limit point of $A \cup B$. We must show that $x \in A \cup B$.

By definition, there must be a sequence $(x_k)_{k=1}^{\infty}$ in $A \cup B$ converging to x. Either A or B (or both) must contain infinitely many terms of the sequence.

Suppose WLOG that A contains infinitely many terms in the sequence. Then, we can make a subsequence $(x_{k_j})_{j=1}^{\infty}$ in A.

Since this is a subsequence of a convergent sequence, it must converge, and its limit is x.

By closeness of A, $x \in A \implies x \in A \cup B \implies A \cup B$ is closed. $\square$

Remark:

By induction, $\bigcup_{i=1}^{N} X_i$ is closed for any integer N if X_i is closed for $1 \leq i \leq N$. But this does not extent to infinite unions.

Suppose $X_i = \left[0, 1 - \frac{1}{i+1}\right], \quad i = 1, 2, \ldots$

Proposition 12.3

If $A_i \subseteq \mathbb{R}^n$ is closed for each i in an arbitrary (possibly infinite) indexing set I, then $\bigcap_{i \in I} A_i$ is closed.

Proof:

Let $X = \bigcap_{i \in I} A_i$. If $X = \varnothing$, then it is closed. Now, suppose $X \neq \varnothing$. Let x be a limit point of X. Then, there is a sequence $(x_k)_{k=1}^{\infty}$ in X converges to x.

By definition of X, $x_k \in A_j$ for all $k \geq 1$, and for all $j \in I$. Hence $x \in A_j$ for all $j \in I$ (by closed property of A_j) $\implies x \in X = \bigcap_{j \in I} A_j$. $\square$

12.2 Closedness and Boundaries

We define the open ball of radius $r > 0$ about a point $a \in \mathbb{R}^n$ as the set

$$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

The complement of a set $X \subseteq \mathbb{R}^n$ is

$$X' = \mathbb{R}^n \setminus X = \{x \in \mathbb{R}^n : x \notin X\}$$

This can also be denoted by X^C.

A point $a \in \mathbb{R}^n$ is a boundary point of $S \subseteq \mathbb{R}^n$ if for every $r > 0$, the open ball $B_r(a)$ contains a point in S and a point in S'.

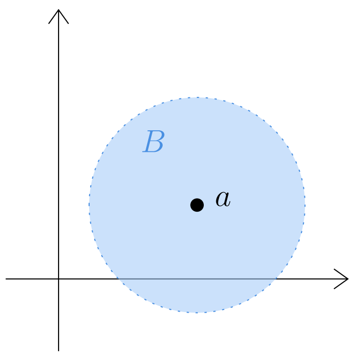


Figure 12.1: An example of open ball

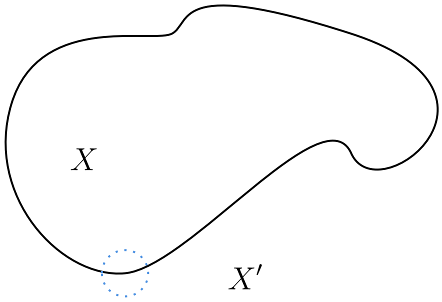


Figure 12.2: An example of boundary point

The boundary of a set $S \subseteq \mathbb{R}^n$ is the set of all boundary points of S, where we denote it by ∂S.

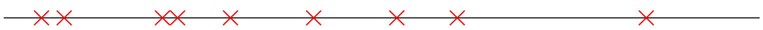
Example:

1. Let's take a look at $(0,1)$ and $[0,1)$. 0 and 1 are the (only) BPs for both cases. BPs may or may not in S.

2. $[0,0.5) \cap (0.5,1]$. 0.5 is also a boundary point.

   Note
   boundary point can be "in the middle" of a set.

3. X is a finite set.

   X is the boundary of itself, i.e., $\partial X = X$

4. $X = \{s \in \mathbb{Q} : |s| < 1\}$. 0,1, still BPs. Every number in $[0,1]$ is BP.

Proposition 12.4

A set $S \subseteq \mathbb{R}^n$ is closed if and only if it contains all of its boundary points.

Proof:
Exercise. □

13

Open and Compact Subsets of $\mathbb{R}^n$

13.1 Open subsets in $\mathbb{R}^n$

As defined in the last lecture, the open ball of readius r about a point a in $\mathbb{R}^n$ is the set

$$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

A subset $U \subseteq \mathbb{R}^n$ is open if for all $a \in U$, there exists some $r > 0$ such that $B_r(a) \subseteq U$.

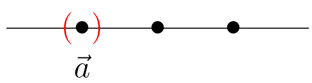
If U is an open set containing a point a, then we say that U is an open neighbourhood of a.

An interior point of a set $X \subseteq \mathbb{R}^n$ is a point $x \in X$ such that $B_r(x) \subseteq X$ for some $r > 0$.

The interior of a set $X \subseteq \mathbb{R}^n$ is the set of all interior points of X. It is denoted by $\text{int}(X)$. If $\text{int}(X)$ is empty, then we say X has empty interior. Otherwise, it has nonempty interior.

Example:

1. $X = \{1, 2, 3\}$



$B_r(a) = (a - r, a + r) \subseteq X$? No $\implies$ X not open.

2. $\varnothing$ is open

3. $\mathbb{R}^n$ is open

Note
The only subsets in $\mathbb{R}^n$ that are closed an open is $\mathbb{R}^n$ and $\varnothing$.

4. The open interval (a, b) is open
5. The close interval $[a, b]$ is not open.

   $\text{int}([a, b]) = (a, b)$
6. $(a, b]$, $[a, b)$ is not open and not closed.
7. $Br(a)$ is open for any $r > 0$ $a \in \mathbb{R}^n$
8. $X = \{s \in \mathbb{Q}, |s| < 1\}$. For any $r > 0$, $B_r(0) = (-r, r)$ contains irrational points. So it is not contained in X. In fact, it has empty interior.

   In fact, $\text{int}(X) = \varnothing$.

   Note
   "interior" doesn't always coincide with what you think of as the "inside" of a set.
9. $X = (-1, 1) \setminus \{0\}$. 0 is not an interior point.

13.2 Properties of open subsets

Proposition 13.1

If U and V are open subsets of $\mathbb{R}^n$, then $U \cap V$ is open.

Proof:
Exercise. $\square$

Proposition 13.2

If U_i is an open subset of $\mathbb{R}^n$ for each i in an arbitrary (possible infinite) indexing set I, then $X = \bigcap_{i \in I}$ is open.

Proof:
Exercise. $\square$

Note
Cannot take intersection of infinitely many open set and expect it to be open.

Example:

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

Theorem 13.3

A set $X \subseteq \mathbb{R}^n$ is open if and only if its complement is closed.

Proof:

$\Longrightarrow$) Let X be an open subset of $\mathbb{R}^n$ and suppose that a is a limit point of X'. Suppose for contradiction that $a \in X$. Since X is open, there exists an open ball $B_r(a) \subseteq X$. Then there is no point y in X' with $\|y - a\| < r$. No sequence in X' can converge to a, contradicting the assumption that a is a limit point of X'. Therefore, all limit points of X' must be in X', i.e., X' is closed.

$\Longleftarrow$) Suppose that X is not open. Then there must be a point $x \in X$ such that for every $r > 0$, the open ball $B_r(x)$ contains a point in X'. Construct a sequence $(a_k)_{k=1}^{\infty}$ in X' such that $a_k \in B_{r=1/k}(x)$ for each $k \geq 1$. Then, $\lim_{k\to\infty} a_k = x \in X$, which means that there is a limit point of X' that is not in X'. This proves that X' is not closed.

$\square$

Proposition 13.4

A set $X \subseteq \mathbb{R}^n$ is open if and only if it contains none of its boundary points.

Proof:

Exercise. $\square$

13.3 Bounded sequences and subsets in $\mathbb{R}^n$

We say that a sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ is bounded if there exists a real number R such that $\|x_k\| < R$ for all k.

We say a set $X \subseteq \mathbb{R}^n$ is bounded if there exists a real number R such that $\|x\| < R$ for all $x \in X$.

Theorem 13.5: Bolzano-Weierstrass in $\mathbb{R}^n$

Every bounded sequence $(x_k)_{k=1}^{\infty}$ in $\mathbb{R}^n$ has a convergent subsequence $(x_{k_l})_{l=1}^{\infty}$.

Proof:

Immediate. $\square$

Corrollary 13.6

If S is a closed and bounded subset of $\mathbb{R}^n$, then every sequence of points in S has a subsequence that converges in S.

Proof:

Trivial. $\square$

13.4 Compact sets

Let K be a subset of $\mathbb{R}^n$. We say that K is compact if every sequence of points in K has a convergent subsequence with a limit in K.

Theorem 13.7: The Heine–Borel Theorem

A subset of $\mathbb{R}^n$ is compact if and only if it closed and bounded.

Proof:
The "if" part is Corollary 13.6. We now show that K is compact only if it is closed and bounded. Let K be a compact subset of $\mathbb{R}^n$.

To show that K is closed, suppose x is a limit point of K. Then there is a sequence of points in K such that $\lim_{k\to\infty} x_k = x$. Since K is compact, there is a subsequence of this sequence that converges to a point in K. But every subsequence of a convergent sequence must converge to limit of the sequence. Hence $x \in K$.

We show that K is bounded by contradiction. Suppose that K is not bounded. Then, we can construct a sequence $(x_k)_{k=1}^{\infty}$ such that $\|x_k\| > k$ for each $k \geq 1$. If K is compact, then there must be a subsequence $(x_{k_j})_{j=1}^{\infty}$ that converges; denote the limit $x = \lim_{j\to\infty} x_{k_j}$.

Choose $\epsilon = 1$ in the definition of convergence. There exists an integer N such that $\left\|x_{k_j} - x\right\| < \epsilon = 1$ for all $j \geq N$. By the Reverse $\triangle$ Ineq,

$$\left|\left\|x_{k_j}\right\| - \|x\|\right| \leq \left\|x_{k_j} - x\right\| < 1 \qquad \forall j \geq N$$

$$\implies \left\|x_{k_j}\right\| < \|x\| + 1 \qquad \forall j \geq N$$

But for any point $x \in \mathbb{R}^n$, there exists an integer $M > \|x\| + 1$. For sufficiently large j, we will have $k_j > M$ and by construction, $\left\|x_{k_j}\right\| > k + j > M > \|x\| + 1$. This contradicts the statement above, proving that K must be bounded. □

Proposition 13.8

If K is a compact subset of $\mathbb{R}^n$ and C is a closed subset of K, then C is compact.

Proof:
If $C \subseteq K$ and K is bounded, then C must be bounded. By assumption, C is closed. Hence, by Theorem 13.5, C is compact. □

13.4.1 Examples and nonexamples

1. $\varnothing$ is compact

2. $\mathbb{R}^n$ is not compact (not bounded)

3. $(0, 1]$ not compact

4. Intervals $[a, b]$ are compact for $a, b \in \mathbb{R}$

5. $[a, b]^n$ is compact in $\mathbb{R}^n$

Proof:
Sketch in $\mathbb{R}^2$.

Suppose $X = [a, b] \times [a, b]$.

Let $\boldsymbol{x}_k = (x_k, y_k)$ for $k \geq 1$ such that $(\boldsymbol{x}_k)_{k=1}^{\infty}$ is in X. Need to find subsequence that converges to $\boldsymbol{x} = (x, y) \in X$.

Consider real sequence $(x_k)_{k=1}^{\infty}$ in $[a, b]$. Bounded, so we can apply Theorem 13.5. $\rightarrow (x_{k_j})_{j=1}^{\infty}$ that converges to x. Since $[a, b]$ is closed, $x \in [a, b]$.

Now $(y_{k_j})_{j=1}^{\infty}$ is a sequence in $[a, b]$.

By Theorem 13.5 and closed property of $[a, b]$, there is a subsequence $\left(y_{k_{j_l}}\right)_{l=1}^{\infty}$ that converges to $y \in [a, b]$.

$\left(\boldsymbol{x}_{k_{j_l}}\right)_{l=1}^{\infty}$ must converge to (x, y)

[convergence of each component to component of limit, + subsequence of (x_{k_j}) must converge to limit of (x_{k_j})]

Since $x \in [a, b]$ and $y \in [a, b]$, $\boldsymbol{x} \in X$. Hence X is compact. $\square$

14

Limits and Continuity of Functions

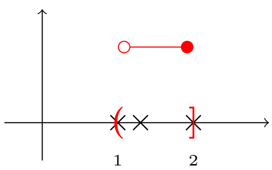
14.0 Preliminaries

Suppose $S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$. $\lim_{x \to a} f(x) = L$?

For all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x such that $0 < |x - a| < \delta$.

Where does the notion of limit make sense?

Consider $f(x) = 1$ for all $x \in S$.



We want to be able to talk about limit at $x = 0$.

14.1 The limit of a function

Let $S \subseteq \mathbb{R}^n$. We say that $\boldsymbol{a} \in \mathbb{R}^n$ is an *accumulation point* of S if it is a limit point of $S \setminus \{\boldsymbol{a}\}$.

The set of all accumulation points of S is denoted by S^a.

A point $\boldsymbol{a} \in S \setminus S^a$ is called an *isolated point* of S.

Let $f : S \to \mathbb{R}^m$ be a function. Let $\boldsymbol{a} \in S^a$. The vector $\boldsymbol{v} \in \mathbb{R}^m$ is the limit of f at $\boldsymbol{a}$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\boldsymbol{x}) - \boldsymbol{v}\| < \epsilon$ for all $\boldsymbol{x} \in S$ satisfying $0 < \|\boldsymbol{x} - \boldsymbol{a}\| < \delta$.

If this holds, we write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = \boldsymbol{v}$.

Example:

1.
$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ |x-1|, & \text{otherwise} \end{cases}$$

   Then $\lim_{x\to 1} f(x) = 0$

2. $f(x,y) = \frac{x^3}{x^2+y^4}$ on $\mathbb{R}^2 \setminus \{0\}$

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} f(\boldsymbol{x}) = 0$$

   Note that this is equivalent to

$$\lim_{\boldsymbol{x}\to\boldsymbol{0}} |f(\boldsymbol{x})| = 0$$

   We can do as follows

$$|f(\boldsymbol{x})| = \frac{x^2}{x^2+y^4}|x| \leq |x|$$

   Show definition of limit holds (use δ, ϵ).

14.2 Infinite limits

Suppose $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$, $\boldsymbol{a} \in S^a$. The limit of f at $\boldsymbol{a}$ is $+\infty$ if for all $N \geq 1$, there exists $\delta > 0$ such that $f(\boldsymbol{x}) > N$ for all $\boldsymbol{x} \in S$ satisfying $0 < \|\boldsymbol{x} - \boldsymbol{a}\| < \delta$.

14.3 Continuity

Let $S \subseteq \mathbb{R}^n$. We say that a function $f : S \to \mathbb{R}^m$ is continuous at the point $\boldsymbol{a} \in S$ if:

> For all $\epsilon > 0$, there exists $\delta > 0$ such that $\|f(\boldsymbol{x}) - f(\boldsymbol{a})\| < \epsilon$ for all $\boldsymbol{x} \in S$ satisfying $\|\boldsymbol{x} - \boldsymbol{a}\| < \delta$.

If f is continuous at every $\boldsymbol{a} \in S$ then we say that it is continuous (on S). If f is not continuous at $\boldsymbol{a} \in S$, then we say it is discontinuous at $\boldsymbol{a}$.

Proposition 14.1

Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}^m$. For every $\boldsymbol{a} \in S \cap S^a$, the function is continuous at $\boldsymbol{a}$ iff

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$$

Proof:
Exercise. $\square$

Example:
Show that $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous on its domain.

Proof:
First, choose arbitrary $a > 0$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ for all $x \in (a - \delta, a + \delta)$.

For any $x > 0$,

$$\begin{aligned} |f(x) - f(a)| &= \left|\frac{1}{x} - \frac{1}{a}\right| \\ &= \left|\frac{a - x}{ax}\right| = \frac{1}{ax}|x - a| \end{aligned}$$

Note
We *cannot* choose $\delta = \epsilon a x$ so that

$$|x - a| < \delta \implies |f(x) - f(a)| < \frac{1}{ax} \cdot \epsilon a x = \epsilon$$

δ is not allowed to depend on x since it must work for all x.

First, suppose $|x - a| < \frac{a}{2}$.

Then $x > \dfrac{a}{2} \implies ax > \dfrac{a^2}{2} \iff \dfrac{1}{ax} < \dfrac{2}{a^2}$.

So we have $|f(x) - f(a)| < \dfrac{2}{a^2}|x - a|$.

Pick $\delta = \min\{\dfrac{\epsilon a^2}{2}, \dfrac{a}{2}\}$ so that

$$|x - a| < \delta \implies |f(x) - f(a)| < \frac{2}{a^2}|x - a| < \frac{2}{a^2}\delta < \epsilon$$

$\square$

15

Discontinuous Functions

15.1 Examples

1. $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$

   This is discontinuous at $x = 0$, continuous everywhere else.

   This kind of discontinuity is called a *removable* discontinuity because you can remove it by changing the value of the function at $x = 0$ to $f(0) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

   Discontinuous at $x = 0$, not removable.

3. The Heaviside function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   Discontinuity at $x = 0$, not removable.

15.2 One-sided limits

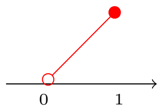
Let $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$. Consider a point ~~$a \in S$~~[a] a that is a limit point of $S \cap (a, \infty)$. We say that the *limit of f as x approaches a from the right* exists and is equal to L if given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in S$ satisfying $a < x < a + \delta$.

If this holds, then we write $\lim_{x \to a^+} f(x) = L$.

The limit from the left is analogous: $\lim_{x \to a^-} f(x) = L$.

Example:

1. Heaviside function: $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^-} H(x) = 1$.
2. $f(x) = x$ on $(0, 1]$



15.3 Jump discontinuities and piecewise continuity

We say that a function f has a jump discontinuity at a point $a \in \mathbb{R}$ if the limits of f as x approaches a from the left and from the right both exist but are not equal. E.g. Heaviside function at $x = 0$.

A function on an interval is piecewise continuous if every finite subinterval contains a finite number of jump discontinuity and no other types of discontinuities.

Example: Thomae's function

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \notin Q \\ 1/q & \text{if } x = p/q \text{ in lowest terms, with } q > 0 \end{cases}$$

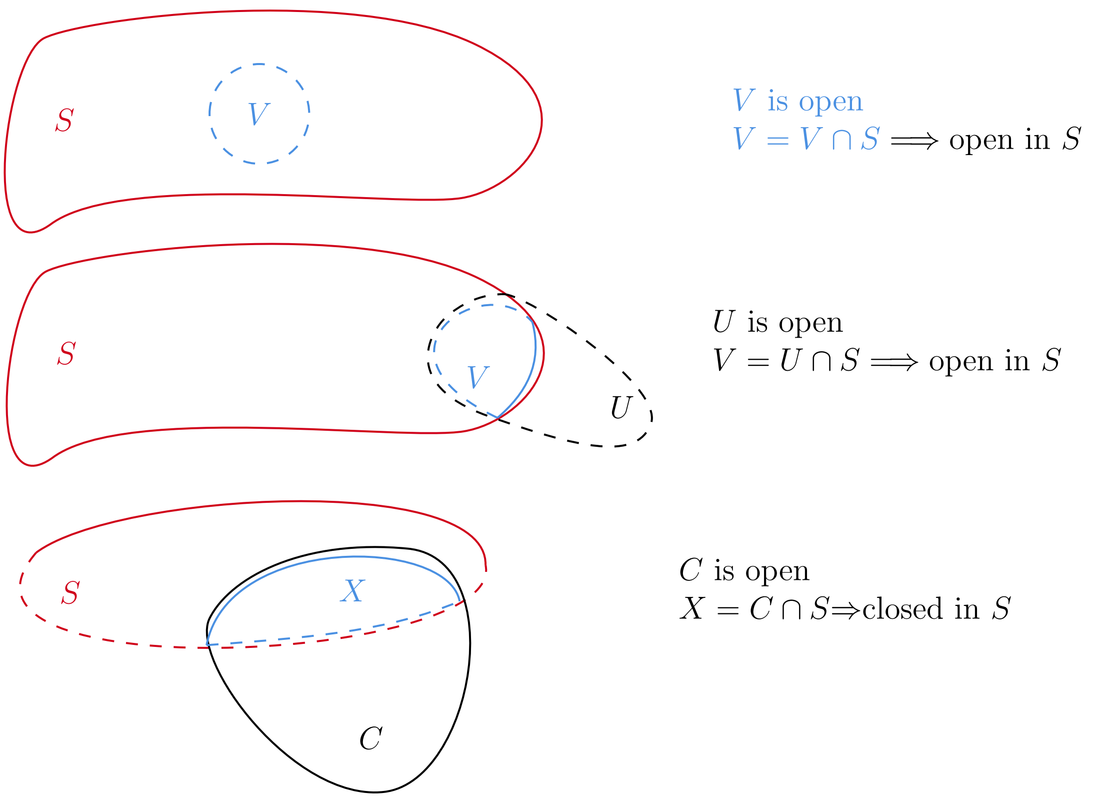
It can be shown that $\lim_{x \to a} f(x) = 0$ at any point $a \in \mathbb{R}$. f is continuous at every irrational point and has a removable discontinuity at every rational point.

16

Properties of Continuous functions

16.1 Equivalent statements of continuity

We say that a subset V of a subset $S \subseteq \mathbb{R}^n$ is open in S, or relatively open with respect to S, if there exists an open set U such that $V = U \cap S$. Also, $X \subseteq S$ is closed in S if there exists a closed set C such that $X = C \cap S$.



Proposition 16.1

A subset $V \subseteq S$ is open in $S \subseteq \mathbb{R}^n$ if and only if the following holds:

For every $x \in V$, there exists $\delta > 0$ such that $B_\delta(x) \cap S \subseteq V$.

Theorem 16.2

For a function $f : S \to \mathbb{R}^m$ where $S \subseteq \mathbb{R}^n$, the following are equivalent:

1. f is continuous on S.
2. For any sequence $(x_k)_{k=1}^\infty$ in S that converges to a limit $a \in S$. then $\lim_{k\to\infty} f(x_k) = f(a)$
3. If U is an open subset of $\mathbb{R}^m$, then the preimage set $f^{-1}(U) = \{x \in S : f(x) \in U\}$ is open in S.

Remark:

- Statement 2 is called "sequential continuity".
- Statement 3 is "topological continuity".
- From the proof (below). Statements 1 and 2 can also be applied at each point a (pointwise) i.e., continuity at a is equivalent to sequential continuity at a.

Proof:

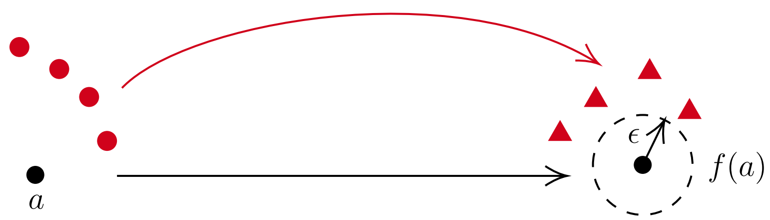
$1 \implies 2$ Continuity of f at every $a \in S$ means: fix $a \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$ such that $\|f(x) - f(a)\| < \epsilon$ for all $x \in S$ satisfying $\|x - a\| < \delta$.

Suppose $\lim_{k\to\infty} x_k = a$ for (x_k) is S.

There exists N such that $\|x_k - a\| < \delta$ for all $k \geq N$.

$$\implies \|f(x_k) - f(a)\| < \epsilon \text{ for all } k \geq N$$
$$\implies \lim_{k\to\infty} f(x_k) = f(a)$$

$\neg 1 \implies \neg 2$ If f is not continuous at $a \in S$, then there exists $\epsilon > 0$ such that for all $\delta > 0$, $\|x - a\| < \delta$ and $\|f(x) - f(a)\| \geq \epsilon$ for some $x \in S$.



For each integer $k \geq 1$, choose $\delta_k = \frac{1}{k}$ and choose $x_k \in S$ such that

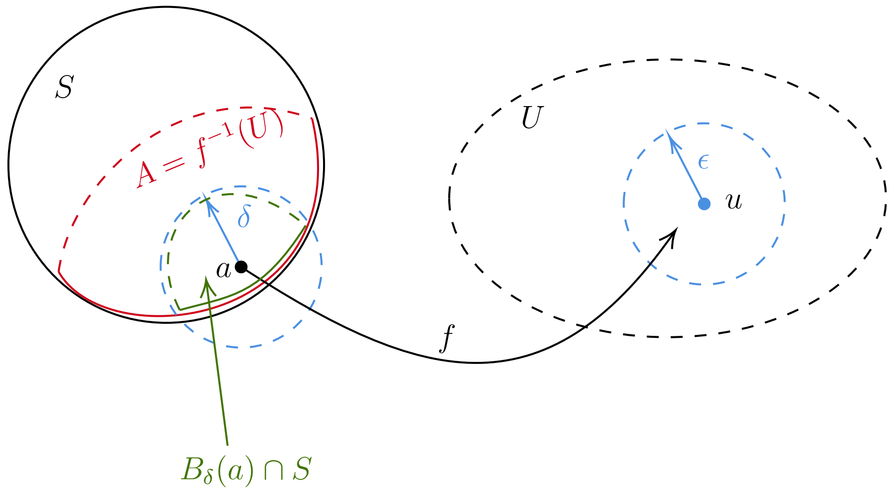
$\|x_k - a\| < \delta_k$ and $\|f(x_k) - f(a)\| \geq \epsilon$.

So, $\lim_{k\to\infty} f(x_k) \neq f(a)$ [It may not exist]

$1 \implies 3$ Suppose f is continuous on S. Let $U \subseteq \mathbb{R}^m$ be an arbitrary open set.

The pre-image $f^{-1}(U)$ is either empty or nonempty. If empty, then open, therefore open in S.

Otherwise, if $f^{-1}(U)$ is non-empty, then there exists a point $a \in A := f^{-1}(U)$.



Define $u = f(a) \in U$.

Since U is open, there exists $\epsilon > 0$ such that $B_\epsilon(u) \subseteq U$.

By continuity of f, there exists $\delta > 0$ such that $\|f(x) - f(a)\| < \epsilon$ for all $x \in S$ satisfying $\|x - a\| < \delta$.

Hence, $f(B_\delta(a) \cap S) \subseteq B_\epsilon(u) \subseteq U$. That is, $B_\delta(a) \cap S \subseteq f^{-1}(U) = A$.

From Proposition 16.1, A is open in S.

$3 \implies 1$ Let a be an arbitrary point in S and define $u = f(a)$. For arbitrary $\epsilon > 0$, the open ball $B_\epsilon(u)$ is open in $\mathbb{R}^m$ so Statement 3 $\implies f^{-1}(B_\epsilon(u))$ is open in S.

Note that $a \in f^{-1}(B_\epsilon(u))$. By Proposition 16.1, there exists $\delta > 0$ such that $B_\delta(a) \cap S = \{x \in S : \|x - a\| < \delta\}$ is a subset of A.

Equivalently, $\|f(x) - f(a)\| < \epsilon$ for all $x \in S$ satisfying $\|x - a\| < \delta$.

□

Example:

Find the limit of the sequence $a_n = \cos\left(\frac{1}{n}\right)$.

The function $f(x) = \cos(x)$ is continuous on $\mathbb{R}$. Since $\left(x_n = \frac{1}{n}\right)_{n=1}^{\infty}$ converges to

0, sequential continuity $\implies$

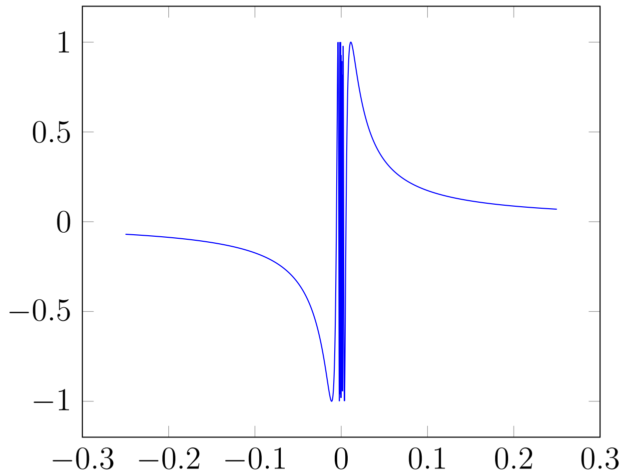
$$\lim_{n\to\infty} a_n = \cos\left(\lim_{n\to\infty}\frac{1}{n}\right) = \cos 0 = 1$$

Note
We only need continuity of f at $x = \lim_{n\to\infty} x_n$.

We can also use this theorem to disprove continuity.

Example:

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



16.2 Combining Limits

Theorem 16.3

Let f and g be two functions from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Let $a \in S$ and $u, v \in \mathbb{R}^m$ such that

$$\lim_{x\to a} f(x) = u \text{ and } \lim_{x\to a} g(x) = v$$

Then,

1. $\lim_{x\to a}(f+g)(x) = u + v$
2. $\lim_{x\to a} \alpha f(x) = \alpha u$ for any $\alpha \in \mathbb{R}$.

   In addition, if $m = 1$, we have

3. $\lim_{x\to a} f(x)g(x) = uv$, and
4. $\lim_{x\to a} f(x)/g(x) = u/v$, provided $v \neq 0$.

16.3 Combining continuous functions

Theorem 16.4

Let f and g be two functions from $S \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. If there is a point $a \in S$ such that f and g continuous at a, then

1. $f + g$ is continuous at a,
2. αf is continuous at a,

   In addition, for $m = 1$

3. fg is continuous at a, and
4. f/g is continuous at a, provided $g(a) \neq 0$.

Theorem 16.5

Let $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^m$. Suppose we have functions $f : S \to T$ and $g : T \to \mathbb{R}^\ell$. If f is continuous at $a \in S$ and g is continuous at $f(a) \in T$, then the composition $g \circ f$ is continuous at a.

16.4 Examples

Every polynomial is continuous on $\mathbb{R}$.