Coding Theory

CO 331

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LaTeXed by Sibelius Peng

Preface

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Pre

Example: Replication code

source msgs		codewords
0	$\to$	0
1	$\to$	1

# of errors/codeword that be detected: 0
errors/codeword that can be corrected: 0
Rate: 1

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source msgs		codewords
0	$\to$	00
1	$\to$	11

# of errors/codeword that be detected: 1
errors/codeword that can be corrected: 0
Rate: 1/2

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source msgs		codewords
0	$\to$	000
1	$\to$	111

# of errors/codeword that be detected: 2
errors/codeword that can be corrected: 1 (nearest neighbour decoding)
Rate: 1/3

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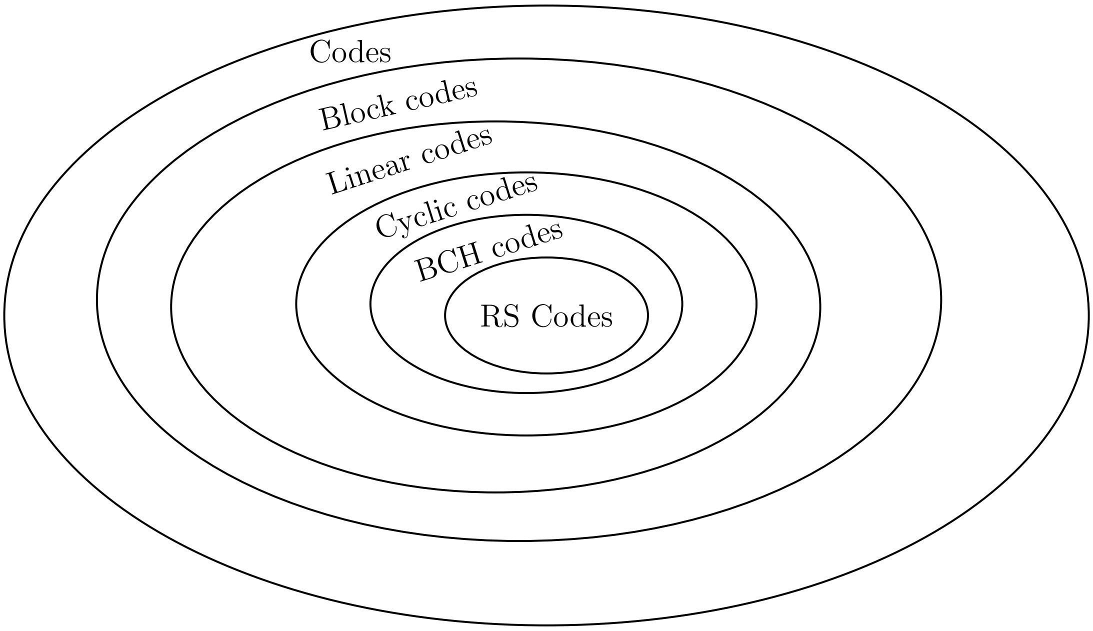
source msgs		codewords
0	$\to$	00000
1	$\to$	11111

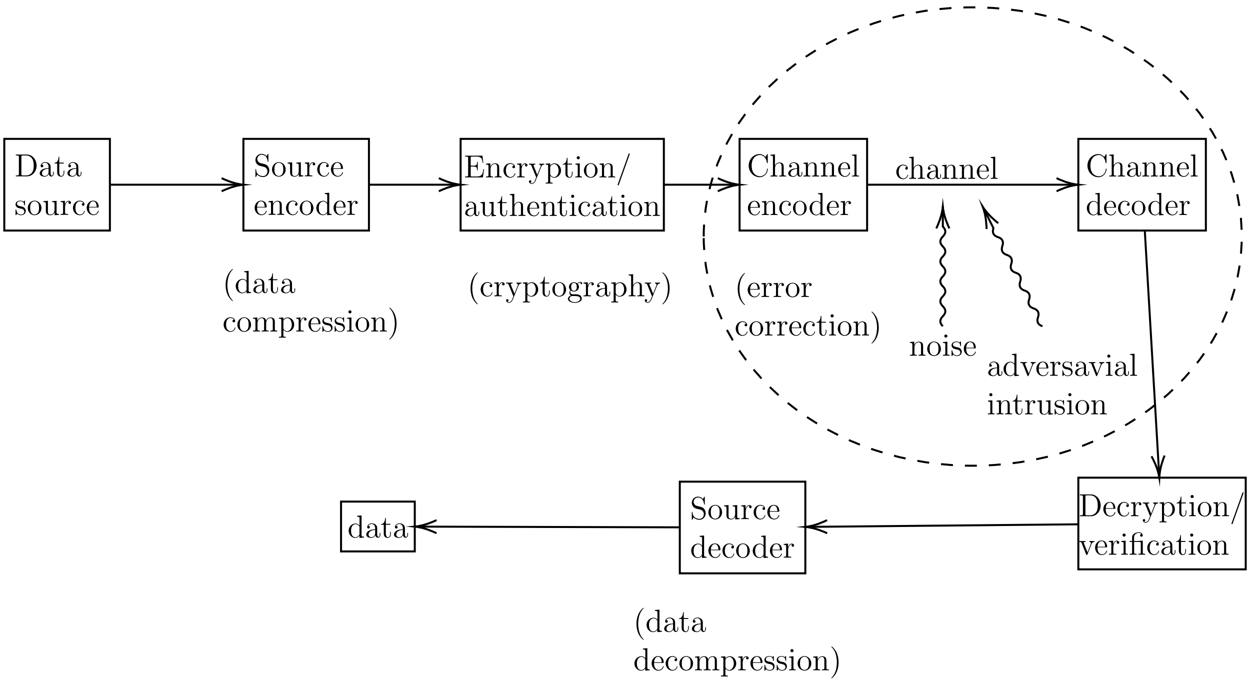
# of errors/codeword that be detected: 4
errors/codeword that can be corrected: 2 (nearest neighbour decoding)
Rate: 1/5

Goal of Coding Theory Design codes so that:

1. High information rate
2. High error-correcting capability
3. Efficient encoding & decoding algorithms



The big picture In its broadest sense, coding deals with the reliable, efficient, secure transmission of data over channels that are subject to inadvertent noise and malicious intrusion.



1

Introduction & Fundamentals

alphabet, word, length...

An *alphabet* A is a finite set of $q \geq 2$ symbols. E.g. $A = \{0, 1\}$.

A *word* is a finite sequence of symbols from A. (tuples or vectors)

The *length* of a word is the number of symbols in it.

A *code* C over A is a finite set of words over A (of size ≥ 2).

A *codeword* is a word in C.

A *block code* is a code where all codewords have the same length.

A block code C of length n containing M codewords over A is a subset $C \subseteq A^n$, with $|C| = M$. This is denoted by $[n, M]$.

Example:

$A = \{0, 1\}$. $C = \{00000, 11100, 00111, 10101\}$ is a $[5, 4]$-code over $\{0, 1\}$.

Messages		Codewords
00	$\rightarrow$	00000
10	$\rightarrow$	11100
01	$\rightarrow$	00111
11	$\rightarrow$	10101

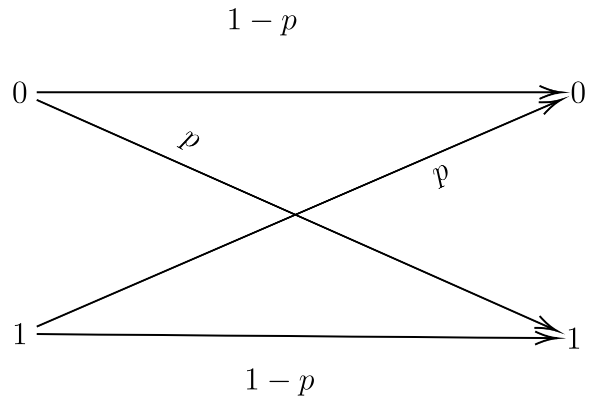
Encoding 1-1 map

The channel encoder transmits only codewords. But, what's received by the channel decoder might not be codeword.

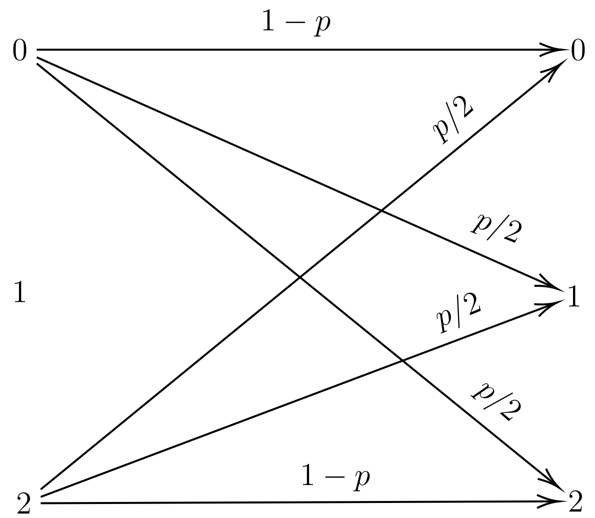
Example:

Suppose the channel decoder receives $r = 11001$. What should it do?

Example: $q = 2$ (Binary symmetric channel, BSC)



Example: $q = 3$



Assumptions about the communications channel

1) The channel only transmits symbols from A.

2) No symbols are deleted, added, or transposed.

3) (Errors are "random") Suppose the symbol transmitted are $X_1, X_2, X_3, \ldots$. Suppose the symbols received and $Y_1, Y_2, Y_3, \ldots$. Then for all $i \geq 1$, and all $i \leq j, k \leq q$,

$$Pr(Y_i = a_j | X_i = a_k) = \begin{cases} 1 - p, & \text{if } j = k \\ \frac{p}{q-1}, & \text{if } j \neq k \end{cases}$$

where $p =$ symbol error prob.

Notes about BSC

(i) If $p = 0$, the channel is perfect.

(ii) If $p = \frac{1}{2}$, the channel is useless.

(iii) If $1 \geq p > \frac{1}{2}$, then simply flip all bits that are received.

(iv) WLOG, we will assume that $0 < p < \frac{1}{2}$.

(v) Analogously, for a q-ary channel, we can assume that $0 < p < \frac{q-1}{q}$. (Optional exercise)

Hamming distance

If $x, y \in A^n$, the *Hamming distance* $d(x, y)$ is the # of coordinate positions in which x&y differ.

The *distance of a code* C is

$$d(C) = \min\{d(x, y) \in C, x \neq y\}$$

Example:

$$d(10111, 01010) = 4$$

Theorem 1.1

d is a metric. For all $x, y, z \in A^n$

(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$

(iii) $\triangle$ inequality $d(x, z) \leq d(x, y) + d(y, z)$

rate

The *rate* of an $[n, M]$-code C over A with $|A| = q$ is

$$R = \frac{\log_q M}{n}.$$

If the source messages are all k-tuples over A,

$$R = \frac{\log_q(q^k)}{n} = \frac{k}{n}.$$

Example:

$$C = \{00000, 11100, 00111, 10101\} \qquad A = \{0, 1\}$$

Here $R = \frac{2}{5}$ and $d(C) = 2$.

1.1 Decoding Strategy

Let C be an $[n, M]$-code over A of distance d. Suppose some codeword is transmitted, and $r \in A^n$ is received. The channel decoder has to decide the following:

(i) no errors have occurred, accept r.

(ii) errors have occurred, and (decode) correct r to some codeword.

(iii) errors has occurred, correction is not possible.

1.1.1 Nearest Neighbour Decoding

Incomplete Maximum Likelihood Decoding (IMLD). Correct r to the unique codeword c for which $d(r, c)$ is smallest. If c is not unique, reject r. Complete MLD (CMLD). Same as IMLD, accept ties are broken arbitrarily.

Question Is IMLD a reasonable strategy?

Theorem 1.2

IMLD selects the codeword c that maximizes $P(r|c)$ prob. that r is received given that c was sent.

Proof:
Suppose $c_1, c_2 \in C$ with $d(c_1, r) = d_1$ and $d(c_2, r) = d_2$. Suppose $d_1 > d_2$.

Now
$$P(r|c_1) = (1-p)^{n-d_1} \left(\frac{p}{q-1}\right)^{d_1}$$
and
$$P(r|c_2) = (1-p)^{n-d_2} \left(\frac{p}{q-1}\right)^{d_2}$$

So,
$$\frac{P(r|c_1)}{P(r|c_2)} = (1-p)^{d_2-d_1} \left(\frac{p}{q-1}\right)^{d_1-d_2} = \left(\frac{p}{(1-p)(q-1)}\right)^{d_1-d_2}$$
Recall
$$p < \frac{q-1}{q} \implies pq < q-1 \implies 0 < q - pq - 1$$
$$\implies p < p + q - pq - 1 \implies p < (1-p)(q-1) \implies \frac{p}{(1-p)(q-1)} < 1$$

Hence
$$\frac{P(r|c_1)}{P(r|c_2)} < 1$$
and so
$$P(r|c_1) < P(r|c_2)$$
□

The ideal strategy is to correct r to $c \in C$ that minimizes $P(c|r)$. This is Minimum error decoding (MED).

Example: (IMD is not the same as MED)

Let $C = \{\underbrace{000}_{c_1}, \underbrace{111}_{c_2}\}$. (corresponding to 0, 1).

Suppose $P(c_1) = 0.1, P(c_2) = 0.9$. Suppose $p = 1/4$ and $r = 100$.

IMLD $r \to 000$

MED

$$\begin{aligned} P(c_1|r) &= \frac{P(r|c_1) \cdot P(c_1)}{P(r)} \\ &= p(1-p)^2 \times 0.1/P(r) \\ &= \frac{9}{640 \cdot P(r)} \end{aligned}$$

Similarly

$$\begin{aligned} P(c_2|r) &= \frac{P(r|c_2) \cdot P(c_2)}{P(r)} \\ &= p(1-p)^2 \times 0.9/P(r) \\ &= \frac{27}{640 \cdot P(r)} \end{aligned}$$

So MED: $r \to 111$

Note

1. IMLD: Select c. s.t. $P(r|c)$ is maximum
   MED: Select c. s.t. $P(c|r)$ is maximum

2. MED has the drawback that it requires knowledge of $P(c_i)$, $1 \le i \le M$

3. Suppose source messages are equally likely, so $P(c_i) = \frac{1}{M}$, for each $1 \le i \le M$. Then

   $$P(r|c_i) = P(c_i|r) \cdot P(c_i)/P(r) = P(c_i|r) \cdot \underbrace{\left[\frac{1}{M \cdot P(r)}\right]}_{\text{does not depend on } i}$$

   So IMLD is the same as MED.

4. In the remainder of the course, we will use IMLD/CMLD.

1.2 Error Correcting & Detecting Capabilities of a Code

- If C is used for error correction, the strategy is IMLD/CMLD.
- If C is used for error detection (only), the strategy is:

$$\text{If } r \notin C, \text{ then reject } r; \text{ otherwise accept } r.$$

e-error correcting code

A code C is called an *e-error correcting code* if the decoding always makes the correct decision if at most e errors per codeword are introduced. (Similarly: e-error detecting code)

Example:

$C = \{0000, 1111\}$ is 1-error correcting code, but not a 2-error correcting code.

$C = \{\underbrace{0\ldots0}_{m}, \underbrace{1\ldots1}_{m}\}$ is a $\lfloor \frac{m-1}{2} \rfloor$-error correcting code.

$C = \{0000, 1111\}$ is a 3-error detecting code.

Theorem 1.3

Suppose $d(C) = d$. Then C is a $(d-1)$-error detecting code.

Proof:

Suppose $c \in C$ is transmitted and r is received.

- If no error occur, then $r = c \in C$ and the decoder accepts r.
- If ≥ 1 and $\leq (d-1)$ errors occur, then $1 \leq d(r,c) \leq d-1$. So, $r \notin C$, and hence the decoder rejects r.

□

Theorem 1.4

If $d(C) = d$, then C is not a d-error detecting code.

Proof:

Since $d(C) = d$, there exist $c_1, c_2 \in C$ with $d(c_1, c_2) = d$. If c_1 is sent, it is possible that d errors occur and c_2 is received. In this case, the decoder accepts c_2. □

Theorem 1.5

If $d(C) = d$, then C is a $\lfloor \frac{d-1}{2} \rfloor$-error correcting code.

Proof:

Suppose $c \in C$ is transmitted, at most $\frac{d-1}{2}$ errors are introduced, and r is received. Let $c_1 \in C, c_1 \neq c$.

By $\triangle$ ineq, $d(c, c_1) \leq d(c, r) + d(r, c_1)$. So

$$d(r, c_1) \geq d(c, c_1) - d(c, r) \geq d - \frac{d-1}{2} = \frac{d+1}{2} \geq \frac{d-1}{2}$$

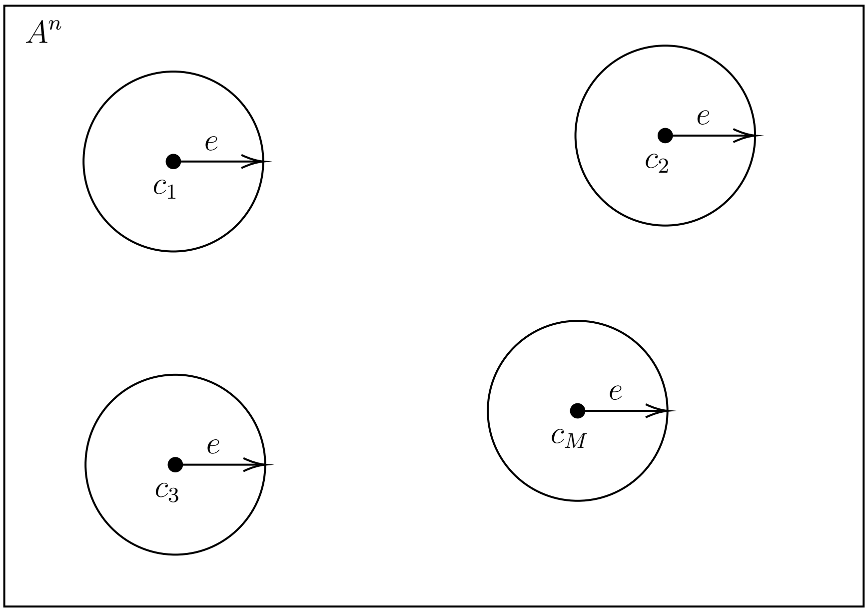
So c is the unique codeword closest to r.

So IMLD/CMLD will decode r to c. $\square$

Theorem 1.6

If $d(C) = d$, then C is not a $\left(\left\lfloor \frac{d-1}{2} \right\rfloor + 1\right)$-error correcting code.

Question Given q, n, M, d, does there exist an $[n, M]$-code C over A (with $|A| = q$), with $d(C) = d$?



$C = \{c_1, c_2, \ldots, c_M\}$. Let $e = \left\lfloor \frac{d-1}{2} \right\rfloor$. For $c \in C$, let S_c =sphere of radius e centered at $c = \{r \in A^n : d(r, c) \leq e\}$. We proved: If $c_1, c_2 \in C, c_1 \neq c_2$, then $S_{c_1} \cap S_{c_2} \neq \varnothing$. The question can be viewed as a *sphere packing problem*: Can we place M spheres of radius e in A^n (such that no 2 spheres overlap)? This is purely combinatorial problem.

Example:

Take $q = 2$, $n = 128$, $M = 2^{64}$, $d \geq 22$. Does a code with these parameters exist?

Answer YES.

Question What are the codewords?

Question How do we encode and decode efficiently?

Preview We'll view $\{0,1\}^{128}$ as a vector space of dimension 128 over $\mathbb{Z}_2$. We'll choose C to be a 64-dimensional subspace of this vector space.

2

Introduction to Finite Fields

field

A *field* $(F, +, \cdot)$ consists of a set F and two operations

$$+ : F \times F \to F$$

and

$$\cdot : F \times F \to F,$$

such that

(i) $a + (b + c) = (a + b) + c \quad \forall a, b, c \in F.$

(ii) $a + b = b + a, \quad \forall a, b \in F.$

(iii) $\exists 0 \in F$ such that $a + 0 = a$, $\forall a \in F.$

(iv) $\forall a \in F$, $\exists -a \in F$ such that $a + (-a) = 0.$

(v) $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall a, b, c \in F.$

(vi) $a \cdot b = b \cdot a, \quad \forall a, b \in F.$

(vii) $\exists 1 \in F, 1 \neq 0$, such that $a \cdot 1 = a \quad \forall a \in F.$

(viii) $\forall a \in F, a \neq 0, \exists a^{-1} \in F$ such that $a \cdot a^{-1} = 1.$

(ix) $a \cdot (b + c) = a \cdot b + b \cdot c, \quad \forall a, b, c \in F.$

infinite, finite, order

A field F is *infinite* if $|F|$ is infinite. F is *finite* if $|F|$ is finite, in which case $|F|$ is the *order* of F.

Example:
$\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are infinite fields. $\mathbb{Z}$ is *not* a field.

Q For what integers $n \geq 2$ do there exist finite fields of order n? if a field of order n exists, how do we "construct"?

Recall Let $n \geq 2$, the integers modulo n, $\mathbb{Z}_n$, is the set of all equivalent classes mod n,

$$\mathbb{Z}_n = \{[0], [1], [2], \dots, [n-1]\}$$

where $[a] + [b] = [a+b], \qquad [a] \cdot [b] = [a \cdot b]$.

More simply $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ with addition & multiplication performed mod n.

Example:
$\mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

In $\mathbb{Z}_9, 5 + 7 = 3,\ 5 \cdot 7 = 8$.

Fact $\mathbb{Z}_n$ is a *commutative ring.* (i.e. field axioms (i)-(ix) are satisfied, except possibly (viii)).

Theorem 2.1

$\mathbb{Z}_n$ is a field if and only if n is prime.

Proof:
($\Longleftarrow$) Suppose n is prime. Let $a \in \mathbb{Z}_n, a \neq 0$ (so $1 \leq a \leq n-1$). Since n is prime, $\gcd(a, n) = 1$, so $\exists s, t \in \mathbb{Z}$ such that $as + nt = 1$. Reducing both sizes $[\bmod n]$, gives

$$as \equiv 1 \ [\bmod n]$$

So $a^{-1} = s$. So (viii) is satisfied, so $\mathbb{Z}_n$ is a field (of order n).

($\Longrightarrow$) Suppose n is composite, say $n = a \cdot b$. where $2 \leq a, b \leq n-1$. Suppose a^{-1} exists, $a^{-1} = s$. Then $as \equiv 1 \ [\bmod n]$. So

$$abs \equiv b \ [\bmod n],$$

so

$$ns \equiv b \ [\bmod n],$$

so $0 \equiv b \ [\bmod n]$, so $n | b$ which is impossible.

$\therefore a^{-1}$ does not exist, so $\mathbb{Z}_n$ is not a field.

$\square$

Q Do there exist finite fields of orders 4 and 6?

characteristic

The *characteristic* of a field denoted $\operatorname{char}(F)$, is the smallest positive integer m such that

$$\underbrace{1+1+1+\ldots+1}_{m} = 0.$$

If no such m exists, then $\operatorname{char}(F) = 0$.

Example:

$\operatorname{char}(\mathbb{Q}) = 0, \quad \operatorname{char}(\mathbb{R}) = 0, \quad \operatorname{char}(\mathbb{C}) = 0.$

$\operatorname{char}(\mathbb{Z}_p) = p$ (p is prime)

Theorem 2.2

If $\operatorname{char}(F) = 0$, then F is infinite.

Proof:

Consider 1, 1+1, 1+1+1, 1+1+1+1,...

Then no 2 elements in this list are equal, because if

$$\underbrace{1+1+1+\ldots+1}_{a} = \underbrace{1+1+1+\ldots+1}_{b} \qquad \text{where } a < b$$

then $0 = \underbrace{1+1+1+\ldots+1}_{b-a}$ which contradicts $\operatorname{char}(F) = 0$.

So F is infinite. $\square$

Theorem 2.3

If F is a finite field, then $\operatorname{char}(F)$ is prime.

Proof:

Suppose $\operatorname{char}(F) = m$, which is composite. Say, $m = a \cdot b$, where $2 \leq a, b \leq m-1$. Now $(\underbrace{1+1+1+\ldots+1}_{a}) \cdot (\underbrace{1+1+1+\ldots+1}_{b}) = \underbrace{1+1+1+\ldots+1}_{m} = 0$ since $\operatorname{char}(F) = m$.

Let $\underbrace{1+\ldots+1}_{a} = s$ and $\underbrace{1+\ldots+1}_{b} = t$, so $s \cdot t = 0$.

But $s \neq 0$, and so s^{-1} exists, thus $s^{-1} \cdot s \cdot t = 0$, therefore $t = 0$, which contradicts $\operatorname{char}(F) = m$. $\square$

Next class Let F be a finite field of order n. Then $\operatorname{char}(F) = p$ (prime). Then $\mathbb{Z}_p$ is a "subfield" of F. And F is a vector space over $\mathbb{Z}_p$ say of dimension k. Then order of F is p^k.

2.1 Non-existence of finite fields

Let F be a finite field of characteristic p. Consider

$$E = \{0, 1, 1+1, 1+1+1, \ldots, \underbrace{1+1+1+\ldots+1}_{p-1}\} \subseteq F$$

Check: E is a field w.r.t the field operations of F. Also, E has order p. If we label the elements of E in a natural way

$$1+1 \leftrightarrow 2, 1+1+1 \leftrightarrow, \ldots, \underbrace{1+1+1+\ldots+1}_{p-1} \leftrightarrow p-1,$$

then E is really just $\mathbb{Z}_p$. (E is *isomorphic* to $\mathbb{Z}_p$).

Theorem 2.4

If F be a finite field of order n, then $\text{char}(F) = p$ (prime). Then $\mathbb{Z}_p$ is a "subfield" of F.

So let's identify:

$$\begin{aligned} \text{elements of } F &\leftrightarrow \text{vectors} \\ \text{elements of } \mathbb{Z}_p &\leftrightarrow \text{scalars} \\ \text{addition in } F &\leftrightarrow \text{vector addition} \\ \text{multiplication in } F &\leftrightarrow \text{scalar multiplication} \end{aligned}$$

Theorem 2.5

If F is a finite char P, then F is a vector space over $\mathbb{Z}_p$.

Proof:
Read Appendix A (of the textbook). □

Theorem 2.6

If F is a finite field of char P, then order of F is p^n for some $n \geq 1$.

Proof:
Let n be the dimension of (the vector space) F over $\mathbb{Z}_p$. Let $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis. Then every element in F can be written uniquely as

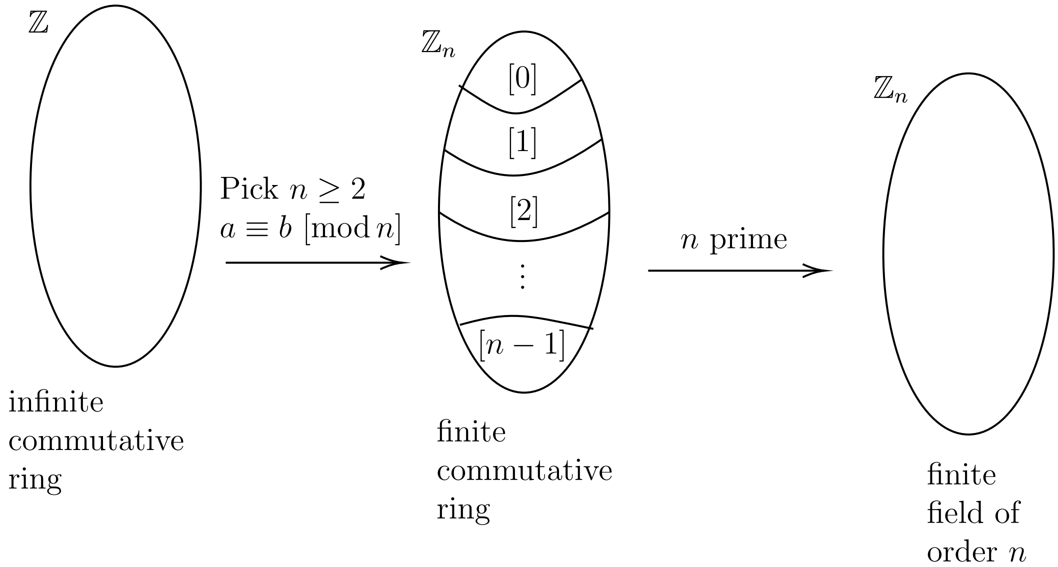
$$c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n, \tag{*}$$

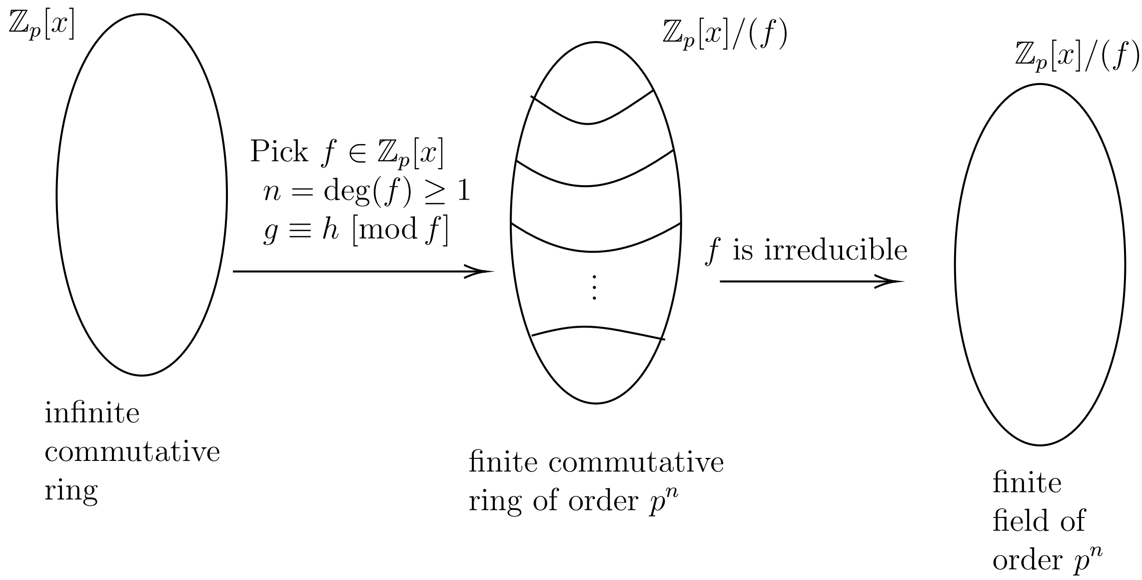
where $c_i \in \mathbb{Z}_p$.

Also every element (*) is in F. Hence $\text{ord}(F) = p^n$. □

Example:
There is no field of order 6.

Q Is there a finite field of order 4? 8? 9? Yes.





F[x]

If F is a field, then $F[x]$ is the set of all polynomials in x with coefficients from F.

Addition and multiplication is done in the usual way, with coefficient arithmetic in F.

Example:
In $\mathbb{Z}_{11}[x]$, $(2 + 5x + 6x^2) + (3 + 9x + 5x^2) = 5 + 3x$.

Theorem 2.7

$F[x]$ is an infinite commutative ring.

Some notations

Let $f \in F[x]$, $\deg(f) \geq 1$.

If $g, h \in F[x]$, we write $g \equiv h \ [\text{mod}\, f]$.

If $g - h = \ell f$ for some $\ell \in F[x]$, we write $(f|g-h)$.

Facts

1. $\equiv$ is an equivalence relation.
2. The equivalence class containing $g \in F[x]$ is
$$[g] = \{h \equiv g \ [\text{mod}\, f] : h \in F[x]\}$$
3. We define $[g_1] + [g_2] = [g_1 + g_2] \qquad [g_1] \cdot [g_2] = [g_1 \cdot g_2]$
4. The set of all equivalence classes, denoted $F[x]/(f)$ (where $f \in F[x], \deg(f) \geq 1$) is a commutative ring.
5. The polynomials in $F[x]$ of degree $< \deg(f)$ are a system of distinct representatives of the equivalence classes in $F[x]/(f)$.

Justification Let $g \in F[x]$. By division algorithm for polynomials, we can write $g = \ell f + r$ where $\deg(r) < \deg(f)$. [Convention: $\deg(0) = -\infty$]

Then $g - r = \ell f$. So $g \equiv r \ [\text{mod}\, f]$. So $[g] = [r]$.

Also if $r_1, r_2 \in F[x], r_1 \neq r_2$ and $\deg(r_1), \deg(r_2) < \deg(f)$, then $f \nmid r_1 - r_2$, so $r_1 \not\equiv r_2 \ [\text{mod}\, f]$. Hence $[r_1] \neq [r_2]$.

2.2 Constructing finite fields

We proved A system of distinct representatives for $\mathbb{Z}_p[x]/(f)$ is $[r(x)] : r \in \mathbb{Z}_p[x], \deg(r) < \deg(f)$. Therefore, $|\mathbb{Z}_p[x]/(f)| = p^n$.

irreducible

Let F be a field and $f(x) \in F[x]$ of degree $n \geq 1$. Then f is *irreducible (over F)* if f cannot be written as $f = gh$, where $g, h \in F[x]$ and $\deg(g), \deg(h) \geq 1$.

Example:

$x^2 + 1$ is irreducible over $\mathbb{R}$.

$x^2 + 1$ is reducible over $\mathbb{C}$, since $(x^2 + 1) = (x + i)(x - i)$.

$x^2 + 1$ is reducible over $\mathbb{Z}_2$, since $x^2 + 1 = (x + 1)^2$.

$x^2 + 1$ is irreducible over $\mathbb{Z}_3$.

Theorem 2.8

Let F be a field, and $f \in F[x]$ of degree $n \geq 1$. Then $F[x]/(f)$ is a field if and only if f irreducible over F.

Proof:

$F[x]/(f)$ is a commutative ring.

($\Longleftarrow$) Suppose $g \in F[x]/(f)$, $g \neq 0$, (and $\deg(g) < \deg(f)$). Then $\gcd(g, f) = 1$, and by the EEA for polynomials, there exist $s, t \in F[x]$ such that $gs + ft = 1$. Reducing both sides mod f gives $gs \equiv \ [\bmod f]$. So $g^{-1} = s$. Hence $F[x]/(f)$ is a field.

($\Longrightarrow$) Exercise.

$\square$

So, to construct a finite field of order $p^n (n \geq 2)$, we need an irreducible polynomial $f \in \mathbb{Z}_p[x]$ of degree n. Then $\mathbb{Z}_p[x]/(f)$ is a finite field of order p^n.

Fact For any prime p, integer $n \geq 2$, there exists an irreducible polynomial degree n in $\mathbb{Z}_p[x]$.

Theorem 2.9

There exists a finite field of order q iff q is a prime power.

Example: Construct a finite field of order 4.

Take $f(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$, which is irreducible over $\mathbb{Z}_2$. So, the field is $\mathbb{Z}_2[x]/(x^2 + x + 1) = \{0, 1, x, x + 1\}$.

- $x + (x + 1) = 1$.
- $x \cdot (x + 1) = x^2 + x = 1$.
- So, $x^{-1} = x + 1$.
- $1^{-1} = 1$
- $x^{-1} = x + 1$
- $(x + 1)^{-1} = x$

Example: Field of order $8 = 2^3$

We need an irreducible polynomial of degree 3 over $\mathbb{Z}_2$. Take $f(x) = x^3 + x + 1$ which is irreducible over $\mathbb{Z}_2$. Then a field of order 8 is

$$F_1 = \mathbb{Z}_2[x]/(x^3 + x + 1) = \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$$

- $x^2 + (x^2 + x + 1) = x + 1$
- $x^2 \cdot (x^2 + x + 1) = x^4 + x^3 + x^2 = 1.$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad x + 1 \\ \hline x^3 + x + 1 & x^4 + x^3 + x^2 \\ & -x^4 \quad -x^2 \quad -x \\ \hline & x^3 \quad\quad -x \\ & -x^3 \quad\quad -x - 1 \\ \hline & \quad\quad -2x - 1 \end{array}$$

- $(x^2)^{-1} = x^2 + x + 1$
- $x^{-1} = x^2 + 1$

Example: Finite field of order 8

Take $f_2(x) = f(x) = x^3 + x^2 + 1$. Then $F_2 = \mathbb{Z}_2[x]/(x^3 + x^2 + 1)$ is a finite field of order 8. Its elements are $F_2 = \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$.

- $x^{-1} = x^2 + x$.

Note

F_1 and F_2 are two different field of order 8. In fact, they are "essentially the same", i.e., they are *isomorphic*, i.e., there is a bijection $\alpha : F_1 \to F_2$ such that $\alpha(a + b) = \alpha(a) + \alpha(b)$ and $\alpha(a \cdot b) = \alpha(a) \cdot \alpha(b), \quad \forall a, b \in F$.

Fact Any two fields of order q are isomorphic.

GF(q)

We will denote *the* finite field of order q by GF(q).

We saw two different representations of GF(2^3).

Recall A finite field of order q exists iff $q = p^n$ for some prime p and $n \geq 1$. (p =characteristic)

- Also GF$(q) = \mathbb{Z}_p[x]/(f)$, where $f \in \mathbb{Z}_p[x]$ is irreducible and has degree n.

Example: Construct GF(16)

Take $f(x) = x^4 + x + 1 \in \mathbb{Z}_2[x]$.

f has no roots in $\mathbb{Z}_2$, and hence no linear factors.

Long division shows that $x^2+x+1 \nmid x^4+x+1$, so f has no irreducible quadratic factor.

$\therefore$ f is irreducible over $\mathbb{Z}_2$. So $\mathrm{GF}(16) = \mathbb{Z}_2[x]/(x^4+x+1)$.

2.3 Properties of finite fields

Theorem 2.10: Frosh's Dream

Let $\alpha, \beta \in \mathrm{GF}(q)$, where $\mathrm{char}(\mathrm{GF}(q)) = p$. Then $(\alpha+\beta)^p = \alpha^p + \beta^p$.

Proof:

$$(\alpha+\beta)^p = \alpha^p + \sum_{i=1}^{p-1} \binom{p}{i} \alpha^i \beta^{p-i} + \beta^p$$

Now, $\displaystyle\binom{p}{i} = \frac{p(p-1)\dots(p-i+1)}{1\cdot 2\cdot \ldots \cdot i} \in \mathbb{N}$.

If $1 \le i \le p-1$, then $p|$ numerator; but $p \nmid$ denominator. $\therefore p|\binom{p}{i}$. So,

$$\begin{aligned}
\binom{p}{i}\alpha^i\beta^{p-i} &= \underbrace{\alpha^i\beta^{p-i} + \ldots + \alpha^i\beta^{p-i}}_{\binom{p}{i}} \\
&= \alpha^i\beta^{p-i}(\underbrace{1+1+1+\ldots+1}_{\binom{p}{i}}) \\
&= \alpha^i\beta^{p-i}\cdot 0 \qquad \text{since char} = p \text{ and } p \left| \binom{p}{i} \right. \\
&= 0
\end{aligned}$$

□

Note
More generally,

$$(\alpha+\beta)^{p^m} = \alpha^{p^m} + \beta^{p^m}$$

for all $m \ge 1$.

Theorem 2.11

Let $\alpha \in \mathrm{GF}(q)$. Then $\alpha^q = \alpha$.

Proof:

- If $\alpha = 0$, then of course $\alpha^q = \alpha$.
- Suppose $\alpha \neq 0$. Let $\alpha_1, \dots, \alpha_{q-1}$ be the nonzero elements in $\mathrm{GF}(q)$. Consider $\alpha\alpha_1, \dots, \alpha\alpha_{q-1}$. The elements in this list are pairwise distinct because

if $\alpha\alpha_i = \alpha\alpha_j \quad (i \neq j)$, then $\alpha^{-1}\alpha\alpha_i = \alpha^{-1}\alpha\alpha_j$, so $\alpha_i = \alpha_j$. Also

$$\alpha\alpha_i \neq 0, \quad \forall 1 \leq i \leq q-1.$$

Hence

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{q-1}\} = \{\alpha\alpha_1, \ldots, \alpha\alpha_{q-1}\}$$

$\therefore \alpha_1 \ldots \alpha_{q-1} = (\alpha\alpha_1)\ldots(\alpha\alpha_{q-1})$

$\therefore \alpha^{q-1} = 1$

$\therefore \alpha^q = \alpha$

□

GF$(q)^*$

Let $\mathrm{GF}(q)^* = \mathrm{GF}(q) \setminus \{0\}$.

ord(alpha)

Let $\alpha \in \mathrm{GF}(q)^*$. The order of α, denoted $\mathrm{ord}(\alpha)$, is the smallest, positive integer t such that $\alpha^t = 1$.

Example:

How many elements of order 1 are there in GF(q)?

$\alpha = 1$

Example:

Find $\mathrm{ord}(x)$ in $\mathrm{GF}(16) = \mathbb{Z}_2[x]/(x^4 + x + 1)$.

$x^1 = 1, x^2 = x^2, x^3 = x^3, x^4 = x + 1, x^5 = x^2 + x, \ldots, x^{15} = 1.$

Since $\mathrm{ord}(x) \neq 1, 3, 5, \mathrm{ord}(x) | 15$, we have $\mathrm{ord}(x) = 15$.

Lemma 2.12

Let $\alpha \in \mathrm{GF}(q)^*, \mathrm{ord}(\alpha) = t, s \in \mathbb{Z}$. $\alpha^s = 1 \iff t|s$.

Proof:

Let $s \in \mathbb{Z}$. Long division g gives $s = \ell t + r$, where $0 \leq r \leq t - 1$.

Then $\alpha^s = \alpha^{\ell t + r} = (\alpha^t)^\ell \alpha^r = \alpha^r$.

So

$$\begin{aligned} \alpha^s = 1 &\iff \alpha^r = 1 \\ &\iff r = 0 \qquad \text{since } 0 \leq r \leq t-1 \\ &\iff t|s \end{aligned}$$

□

Corrollary 2.13

If $\alpha \in \mathrm{GF}(q)^*$, then $\mathrm{ord}(\alpha)|q-1$.

Proof:
We know that $\alpha^{q-1} = 1$. So $\mathrm{ord}(\alpha)|q-1$ by previous lemma. □

generator

An element $\alpha \in \mathrm{GF}(q)$ is a *generator of* $\mathrm{GF}(q)^*$ (primitive element in $\mathrm{GF}(q)$). If $\mathrm{ord}(\alpha) = q-1$.

Lemma 2.14

If α is a generator of $\mathrm{GF}(q)^*$ then $\{\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{q-2}\} = \mathrm{GF}(q)^*$.

Lemma 2.15

If $\alpha \in \mathrm{GF}(q)^*$ has order t, then $\alpha^0, \alpha^1, \ldots, \alpha^{t-1}$ are pairwise distinct.

Proof:
Suppose $\alpha^i = \alpha^j$, where $0 \leq i < j \leq t-1$. Then $\alpha^{j-1} = 1$ which contradicts $\mathrm{ord}(\alpha) = t$ since $1 \leq j - i \leq t-1$. □

So, if α is a generator of $\mathrm{GF}(q)^*$ then $\{\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^{q-2}\} = \mathrm{GF}(q)^*$.

Theorem 2.16

$\mathrm{GF}(q)^*$ has at least one generator.

Proof:
See LEARN (optional). □

Example:
Find a generator of $\mathrm{GF}(8) = \mathbb{Z}_2[x]/(x^3+x+1)$.

x is a generator.

3

Linear Codes

Let $F = \mathrm{GF}(q)$.

Let $V_n(F) = F \times F \times \ldots \times F = F^n$

Then $V_n(F)$ is an n-dimensional vector space over F.

We have $|V_n(F)| = q^n$.

linear (n, k)-code over F

A *linear (n, k)-code over F* is a k-dimensional subspace of $V_n(F)$.

subspace

A subspace of of a vector space V over F is a subset $S \subseteq V$ such that

(i) $S \neq \varnothing$.

(ii) $v_1 + v_2 \in S \qquad \forall v_1, v_2 \in S$.

(iii) $\lambda v \in S, \qquad \forall v \in S, \lambda \in F$.

Note
S is also a vector space over F.

$0 \in S$.

3.1 Properties of Linear Codes

Let C be an (n, k)-code over F. Let $v_1, v_2, \ldots, v_k$ be an ordered basis for C.

1) The codewords in C are precisely:

$$mv_1 + m_2v_2 + \ldots + m_kv_k,$$

where $m_i \in F$.

So $|C| = M = q^k$.

2) The rate of C is $R = \frac{\log_q M}{n} = \frac{k}{n}$,

3) Distance

weight

The (Hamming)*weight* of $v \in V_n(F)$, $\omega(v)$, is the number of nonzero coordinate positions in v.

The weight of C is $\omega(C) = \min\{\omega(c) : c \in C, c \neq 0\}$.

Theorem 3.1

If C is a linear code, then $d(C) = \omega(C)$.

Proof:

$$\begin{aligned} d(C) &= \min\{d(x,y) : x, y \in C, x \neq y\} \\ &= \min\{\omega(x-y) : x, y \in C, x \neq y\} \\ &= \min\{\omega(c) : c \in C, c \neq 0\} \\ &= \omega(C) \end{aligned}$$

□

4) Encoding.

Since $M = q^k$, there are q^k source messages. We'll assume that the source messages are elements of $V_k(F)$. A natural encoding rule is: Given $(m_1, m_2, \ldots, m_k) \in V_n(F)$. We will encode it as $c = m_1v_1 + m_2v_2 + \ldots + m_kv_k$.

Note
The encoding rule depends on the basis chosen for C.

5) Note if $m = (m_1, \ldots, m_k)$, then the encoding rule can be written as follows.

$$c = (m_1, m_2, \ldots, m_k) \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_k & - \end{bmatrix}_{k \times n}$$

$$c = mG$$

generator matrix

Let C be an (n, k) code. A *generator matrix* G for C is a $k \times n$ matrix whose rows form a basic for C.

Note
An encoding rule for C w.r.t. G is $c = mG$.

Note
Performing elementary row operations on G gives a different matrix for the same code C.

Example: Consider a binary $(5, 3)$-code C

where binary means "over $F = \mathrm{GF}(2) = \mathbb{Z}_2$. 5 is n, length of code. 3 is k, dimension.

Then $M = q^k = 2^3$ and $R = \frac{k}{n} = \frac{3}{5}$. and

$$C = \langle \underbrace{10010}_{v_1}, \underbrace{01011}_{v_2}, \underbrace{00101}_{v_3} \rangle$$

$$G = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]_{3\times 5}$$

indeed has rank 3 so G is *a* GM for C.

Encoding rule is $c = mG$.

m source msgs	$\to$	c codewords
000	$\to$	00000
001	$\to$	00101
010	$\to$	01011
011	$\to$	01110
100	$\to$	10010
101	$\to$	10111
110	$\to$	11001
111	$\to$	11100

$d(C) = 2, e = 0$

Note
Any matrix row equivalent to G is also a GM for C, but yields a different encoding rule.

systematic, standard form

Let matrix $[I_k|A]_{k\times n}$ is a GM for an (n, k)-code C. If an (n, k)-code has a GM of this form, then C is *systematic*, and the GM is in *standard form*.

Example:

$C = \langle 100011, 101010, 100110 \rangle$ is a non-systematic $(6,3)$-code. A GM for C is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Another GM for C is

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Another GM for C:

$$G_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

C is not systematic.

However, if every codeword is permuted by moving the second bit to a new fourth bit, then we get a new code C' that is linear, and has the same n, k, d as C.

equivalent

Let C be an (n,k)-code. If π is a permutation on $\{1, 2, \ldots, n\}$, Then $\pi(C)$[a] is an (n,k)-code and is said to be *equivalent* to C.

[a] i.e. apply π to each codeword

Fact

1. If C, C' are equivalent codes, then $d(C) = d(C')$.
2. Every linear code is equivalent to a systematic code.

   Proof:

   Let C be an (n,k)-code. Let G be a GM for C in row reduced form. Then one can permute to columns of G to get a matrix $G' = [I_k|A]$ in standard form.

   Then G' is a GM for a code C' that is equivalent to C. $\square$

3.2 Dual Codes

inner product

Let $x, y \in V_n(F)$. The *inner product* of x and y is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \in F$$

Properties For all $x, y, z \in V_n(F)$ and all $\lambda \in F$

1. $x \cdot y = y \cdot x$
2. $x \cdot (y + z) = x \cdot y + x \cdot z$
3. $(\lambda x) \cdot y = \lambda(x \cdot y)$
4. $x \cdot x = 0$ does **not** imply that $x = 0$.

Example:

Consider $V_2(\mathbb{Z}_2)$

Then $(1, 1) \cdot (1, 1) = 0$.

dual code

Let C be an (n, k)-code over F. The *dual code* of C is

$$C^\perp = \{x \in V_n(F) : x \cdot c = 0, \quad \forall c \in C\}$$

orthogonal

If $x, y \in V_n(F)$ and $x \cdot y = 0$, then x, y are *orthogonal.*

Theorem 3.2

If C is an (n, k)-code over F, then $C^\perp$ is an $(n, n - k)$-code over F.

Proof:
Let $v_1, v_2, \ldots, v_k$ be a basis for C.

Claim Let $x \in V_n(F)$. Then $x \in C^\perp$ iff $v_1 \cdot x = v_2 \cdot x = \ldots = v_k \cdot x = 0$.

$(\implies)$ If $x \in C^\perp$, then $x \cdot c \quad \forall c \in C$. In particular, $x \cdot v_1 = 0, \ldots, x \cdot v_k = 0$.

$(\impliedby)$ Suppose $x \cdot v_1 = x \cdot v_2 = \ldots = x \cdot v_k = 0$. Let $c \in C$. We can write

$$c = \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_k v_k, \qquad v_i \in F$$

Then $x \cdot c = \lambda_1(x \cdot v_1) + \ldots + \lambda_k(x \cdot v_k) = 0$. Hence $x \in C^\perp$.

Consider

$$G = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}_{k \times n}$$

Then $x \in C^\perp$ iff $Gx^T = 0$. So $C^\perp$ is the nullspace of G. Hence $C^\perp$ is an $(n-k)$-dimensional subspace of $V_n(F)$. □

Theorem 3.3

If C is a linear code, then $(C^\perp)^\perp = C$.

Proof:
Let C be an (n,k)-code, then $C^\perp$ is an $(n, n-k)$-code. So $(C^\perp)^\perp$ is an (n,k)-code. But $C \subseteq (C^\perp)^\perp$ by definition of $C^\perp$.

Suppose C is a code over $F = \mathrm{GF}(q)$.Then $|C| = q^k$ and $|(C^\perp)^\perp| = q^k$.

$\therefore C = (C^\perp)^\perp$. □

Theorem 3.4: Constructing a GM for $C^\perp$

Let C be an (n,k)-code with GM $G = \left[I_k | A_{k \times (n-k)}\right]_{k \times n}$. Then *a* GM for $C^\perp$ is

$$H = \left[-A^T | I_{n-k}\right]_{(n-k) \times n}$$

Proof:
$\mathrm{rank}(H) = n-k$, so H is indeed a GM for some $(n, n-k)$-code $\overline{C}$.

Now,

$$GH^T = [I_k | A] \left[\frac{-A}{I_{n-k}}\right] = -A + A = 0$$

Since $GH^T = 0$, every row of H is orthogonal to every row of G. So, every vector in the row space of H is orthogonal to every vector in the row space of G. Hence $\overline{C} \subseteq C^\perp$. Since $\dim(\overline{C}) = \dim(C^\perp)$, we have $\overline{C} = C^\perp$. □

parity-check matrix

A GM for $C^\perp$ is called a *parity-check matrix* (PCM) for C.

Example:

Consider a $(5,2)$-code C over $\mathbb{Z}_3$ with GM

$$G = \begin{bmatrix} 2 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}_{2\times 5}$$

For C: $q = 3, n = 5, k = 2, M = 3^2 = 9$.

$$C = \{00000, \underset{c_1}{20210}, \underset{2c_1}{10120}, \underset{c_2}{11001}, \underset{2c_2}{22002}, \underset{c_1+c_2}{01211}, \underset{c_1+2c_2}{12212}, \underset{2c_1+c_2}{21121}, \underset{2c_1+2c_2}{02122}\}$$

Now find a GM for $C^\perp$

$$\begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{row reductions}} \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

So,

$$H = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

is a GM for $C^\perp$ which is an $(5,3)$-code over $\mathbb{Z}_3$.

Note

Let C be an (n,k)-code over F with GM G:

1. $C^\perp$ is the nullspace of G.
2. $C^\perp$ is an $(n, n-k)$-code over F.
3. $(C^\perp)^\perp = C$
4. Let H be a GM for $C^\perp$, then H is a PCM for C (by definition).
5. G is a PCM for $C^\perp$.
6. $GH^T = 0$.
7. $\boxed{\text{For } x \in V_n(F), x \in C \text{ iff } Hx^T = 0.}$

   [C is the nullspace of H.]

Theorem 3.5

Let C be an (n,k)-code over F, and let H be a PCM for C. Then $d(C) \geq s$ iff every $s-1$ cols of H are linearly independent over F.

Proof:
Let $h_1, h_2, \ldots, h_n$ be the cols of H.

$\Longleftarrow$) Suppose $d(C) \leq s-1$, so $\omega(C) \leq s-1$. Let $c \in C$, with $1 \leq \omega(C) \leq s-1$. WLOG, suppose $c_j = 0$, $\forall s \leq j \leq n$. Since $c \in C$, we have $Hc^T = 0$.

$\therefore c_1h_1 + c_2h_2 + \ldots + c_{s-1}h_{s-1} = 0$

Since $\omega(C) \geq 1$, this is a non-trivial linear combinations of $h_1, \ldots, h_{s-1}$ that equal 0. So $h_1, \ldots, h_{s-1}$ are linear dependent over F.

$\Longrightarrow$) Suppose there are $s-1$ cols of H that are linear dependent over F, say $h_1, \ldots, h_{s-1}$. So we can write $c_1h_1 + c_2h_2 + \ldots + c_{s_1}h_{s-1}$ where $c_j \in F$, not all zero.

Let $c = (c_1, c_2, \ldots, c_{s-1}, \underbrace{0, \ldots, 0}_{n-s+1}) \in V_n(F)$.

Then $Hc^T = 0$. So $c \in C$. And $1 \leq \omega(C) \leq s-1$, so $d(C) \leq s-1$.

□

Corrollary 3.6

Let C be an (n,k)-code over F with PCM H. Then $d(C)$ is the smallest number of cols of H that are linearly dependent over F.

Example:
Recall we found a PCM

$$H = \left[\begin{array}{cc|ccc} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

for a $(5,2)$-code C over $\mathbb{Z}_3$.

Find $d(C)$

- No 0 col in $H \implies d(C) \geq 2$
- No two linearly dependent cols in H (since no repeated cols, and no col is 2 times another cols $\implies d(C) \geq 2$)

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

so columns 1, 3, 4 are linearly dependent over $\mathbb{Z}_3$. Then $d(C) \ngeq 4$, so $d(C) = 3$.

Example:
C be a binary code, with PCM H

- $d(C) = 1$ iff H has a 0 column.
- $d(C) = 2$ iff the cols of H are non-zero and two are the same.
- $d(C) = 3$ iff the cols of H are non-zero, distinct, and one column is the sum of two other (distinct) columns.

Example: Construct a $(\underset{n}{7}, \underset{k}{4}, \underset{d}{3})$-binary code C

Consider a PCM for C:

$$H = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]_{3\times 7}$$

This is a *Hamming Code of order 3 over* $\mathbb{Z}_2$.

3.3 Perfect Code

perfect code

Let C be an $[n, M]$-code C over A of distance d. Then

$$M \sum_{i=0}^{e} \binom{n}{i} (q-i)^i \leq q^n$$

where $e = \left\lfloor \frac{d-1}{2} \right\rfloor$. [Sphere packing bound]

Then C is *perfect* if

$$M \sum_{i=0}^{e} \binom{n}{i} (q-i)^i = q^n$$

Note
If C is perfect, then IMLD = CMLD.

For fixed n, q, d, a perfect code maximized $R = \frac{\log_q M}{n}$.

Example:
$C = \mathrm{GF}(q)^n$ is a (trivial) perfect code with $d = 1$.

Example:
$C = \{\underbrace{0\ldots0}_{n}, \underbrace{1\ldots1}_{n}\}$ over $\mathbb{Z}_2$ is a perfect code if n is odd. (distance $= n$).

Proof:

$$\begin{aligned} 2\left(\sum_{i=0}^{e}\binom{n}{i}\right) &= 2\left(\binom{n}{0}+\ldots+\binom{n}{e}\right) \\ &= \binom{n}{0}+\ldots+\binom{n}{e}+\binom{n}{e+1}+\ldots+\binom{n}{n} \\ &= 2^n \end{aligned}$$

□

Exercise

Prove that every perfect code must have odd distance.

Theorem 3.7: Tietäräinen, 1973

The only perfect codes are

(i) $V_n(\mathrm{GF}(q))$

(ii) The binary replication code of odd length.

(iii) The $(23, 12, 7)$-binary Golay code and all codes equivalent to it.

(iv) The $(11, 6, 5)$-ternary[a] Golay and all codes equivalent to it.

A GM is

$$G = \left[\begin{array}{c|ccccc} & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 2 & 2 & 1 \\ & 1 & 0 & 1 & 2 & 2 \\ I_6 & 2 & 1 & 0 & 1 & 2 \\ & 2 & 2 & 1 & 0 & 1 \\ & 1 & 2 & 2 & 1 & 0 \end{array}\right]_{6\times 11}$$

(v) The Hamming codes and all codes of the same $[n, M, d]$ parameters as them. $(d = 3)$.

[a] over $\mathbb{Z}^3$

Hamming code of order r over GF(q)

A Hamming code of order r over $\mathrm{GF}(q)$ is a linear code over $\mathrm{GF}(q)$ with $n = \frac{q^r-1}{q-1}$, $k = n - r$ and PCM a $r \times n$ matrix whose columns are nonzero & no two are scalar multiples of each other.

Example: A Hamming code of order $r = 3$ over $\mathrm{GF}(2)$

is a $(7, 4, 3)$-binary code with PCM

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}$$

Example: A Hamming code of order $r = 3$ over $\mathrm{GF}(3)$

is a $(13, 10, 3)$-code over $\mathrm{GF}(3)$ with PCM

$$H = \left[\begin{array}{ccc|ccc|ccc|ccc|c} 1 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 1 \end{array}\right]_{3\times 13}$$

Observations

1. For every nonzero vector $v \in V_r(\mathrm{GF}(q))$, exactly one scalar of v must be a column of a PCM (for the Hamming code of order r over $\mathrm{GF}(q)$) $n = \frac{q^r-1}{q-1}$
2. The dimension of the code is indeed k, since $\mathrm{rank}(P(M)) = r = n - k$. Since $\lambda_i e_i$ ($e_i = i^{th}$ unit vector, λ_i's are non-zero scalars) are cols of PCM.
3. The Hamming codes have distance 3. (since $\lambda_1 e_1, \lambda_2 e_2$ and $\lambda_3(e_1 + e_2)$ are cols of H for some scalar multiples $\lambda_1, \lambda_2, \lambda_3$)
4. The Hamming codes are perfect:

$$\begin{aligned} M \sum_{i=0}^{e} \binom{n}{i} (q-1)^i &= q^{n-r}\left(1 + n(q-1)\right) \\ &= q^{n-r}\left(1 + \frac{q-1}{q-1}(q-1)\right) \\ &= q^n \end{aligned}$$

3.4 Error Correction (for Hamming Codes)

error vector

Suppose $c \in C$ is transmitted. Suppose $r \in V_n(F)$ is received. The *error vector* is $e = r - c$. $(c + e = r)$

Example:

Over $\mathbb{Z}_3$, if $c = (120212)$ is sent and $r = (122102)$ is received, then $e = (00220)$.

Decoding algorithm for single-error correcting codes (e.g. Hamming codes)

Let H be a PCM for an (n, k)-code C over $\mathrm{GF}(q)$ with $d \geq 3$.

Recall $c \in C$ is sent, $r \in V_n(\mathrm{GF}(q))$ is received, the *error vector* is $e = r - c$.

Main idea $Hr^T = H(c+e)^T = Hc^T + He^T = He^T$

syndrome

If $r \in V_n(\mathrm{GF}(q))$, $s = Hr^T$ is called the *syndrome* of r.

Note

1) r and e have the same syndrome.

2) If $e = 0$, then $He^T = 0$

3) If $\omega(e) = 1$, say $e = (0, \dots, \underbrace{\alpha}_{i^{th} \text{ position}}, \dots, 0)$ where $\alpha \neq 0$.

   Then $He^T = \alpha h_i$ (nonzero), where $h_i = i^{th}$ col of H.

4) Note: The converses of 2) and 3) are false.

Algorithm 1: Decoding algorithm (for single error-correcting codes)

Given : H, r

Compute $s = Hr^T$

If $\omega(s) = 0$, then accept r. (STOP)

Compare r with the columns of H. If $s = \alpha h_i$ (where $\alpha \neq 0$), then $e = (0, \dots, \underbrace{\alpha}_{i^{th}}, \dots, 0)$, and correct r to $c = r - e$. (STOP)

Reject.] NOT NEEDED if H is a Hamming code (because it is perfect)

Claim If $\omega(e) \leq 1$, then the decoding algorithm always makes the correct decision.

Note

If H is a Hamming code & $\omega(e) \geq 2$, then this decoding algorithm always makes the wrong decision.

Example:

Consider the $(7, 4, 3)$-binary Hamming code with PCM

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}_{3\times 7}$$

Decode $r = (0111110)$.

1. Compute $s = Hr^T = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, which is the 6^{th} col of H.

2. So, $e = (0000010)$

3. Decode r to $c = (0111100)$. [Check $Hc^T = 0$]

3.5 General Decoding Problem for binary linear codes

Instance

- An $(n-k) \times n$ matrix H over GF(2) with $\text{rank}(H) = n-k$.
- $r \in V_n(\text{GF}(2))$

Find Find a vector $e \in V_n(\text{GF}(2))$ of minimum weight with $Hr^T = He^T$.

Fact This problem is **NP-hard**[1].

Decoding Linear Codes

Let C be an (n,k)-code over $F = \text{GF}(q)$ with PCM H.

$\equiv$ [mod C]

We write $x \equiv y$ [mod C], where $x, y \in V_n(F)$ if $x - y \in C$.

Note

1) $\equiv$ [mod C] is an equivalence relation. (Reflexive, Symmetric, Transitive)
2) So, the set of equivalence classes partitions $V_n(F)$.
3) The equivalence class containing $x \in V_n(F)$ is called a *coset* of $V_n(F)$.

   This class is $\{y \in V_n(F) : y \equiv x \text{ [mod } C]\} = \{x + c : c \in C\} = C + x$.

   We call $C = x$ the coset of C represented by x. (See the example below)
4) $C + 0 = C$
5) If $y \in C + x$, then $C + y = C + x$
6) Every coset has size q^k

[1] These ideas could be found in CS 341/466/666 ...

- P = problems solvable in "polynomial time" (i.e. efficiently)
- NP = a "certain" class of problems including many problems of strong practical interest which do not know to solve efficiently.
- NP-hard: If any single problem in this class of problems can be solved efficiently, then so can all problems in NP (in which, P=NP)

7) # of cosets is $q^n/q^k = q^{n-k}$.

Example:

Consider the $(5, 2)$-binary code C with GM

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The cosets of C are:

$$\begin{aligned} C + 00000 &= \{00000, 10111, 01110, 11001\} \\ = C + 10111 &= \{10111, 00000, 11001, 01110\} \\ = C + 01110 &= \dots \\ = C + 11001 &= \dots \end{aligned}$$

and

$$C + 10000 = \{10000, 00111, 11110, 01001\} = C + 00111$$

which is disjoint from the previous equivalent class. And brand new cosets:

$$\begin{aligned} C + 01000 &= \{01000, 11111, 00110, 10001\} \\ C + 00100 &= \{00100, 10011, 01010, 11101\} \\ C + 00010 &= \{00010, 10101, 01100, 11011\} \\ C + 00001 &= \{00001, 10110, 01111, 11000\} \\ C + 00011 &= \{00011, 10100, 01101, 11010\} \\ C + 11100 &= \{11100, 01011, 10010, 00101\} \end{aligned}$$

Theorem 3.8

Let $x, y \in V_n(F)$. Then $x \equiv y \ [\bmod C]$ iff $Hx^T = Hy^T$.

Proof:

$$x \equiv y \ [\bmod C] \iff H(x-y)^T = 0 \iff Hx^T = Hy^T$$

□

So, cosets are characterized by their syndromes.

Decoding $c \in C$ is send, $r \in V_n(F)$ is received. $e = r - c \in V_n(F)$. $Hr^T = He^T$.

So, r and e belong to the same coset of C.

CMLD Given r, find a vector e of smallest weight in $C+r$, or, equivalently, find a vector e of smallest weight with the same syndrome as r. Then decode r to $c = r - e$.

IMLD Find the unique vector e of smallest weight having the same syndrome as r. If no such e exists, then reject r. Otherwise, decode r to $c = r - e$.

coset leader

A vector of smallest weight is a coset of C is distinguished and called a *coset leader* (of that coset).

3.5.1 Syndrome Decoding

Algorithm 2: Syndrome Decoding Algorithm

Given : A PCM H for an (n, k)-code C over $F = \mathrm{GF}(q)$

Create a table of coset leaders and their syndromes. Given r, do

Compute $s = Hr^T$

Look up the coset leader corresponding to s, say ℓ.

Decode r to $c = r - \ell$.

Example:

$$G = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right]_{2\times 5}$$

$$H = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{array}\right]_{3\times 5}$$

$n = 5, k = 2, q = 2$

Coset Leaders		Syndromes
00000	—	000
10000	—	111
01000	—	110
00100	—	100
00010	—	010
00001	—	001
00011	—	011
10010	—	101

Suppose $r = 10111$.

Compute $s = Hr^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Coset leader is $\ell = 00000$.

So $c = r - e = 10111$.

Recall There is a 1-1 correspondence between cosets of C and syndromes. Also, $s = Hr^T = He^T$.

For a binary (n, k)-code C, the syndrome table has size $2^{n-k} \times n$, which is exponentially large.

Goal Design decoding algorithm which require very little space.

Example:

Use only the PCH, H, which is $(n-k)n$ bits.

4

The binary Golay code

4.1 The (binary) Golay code C_{23} (1949)

Let

$$\hat{B} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ \vdots \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}_{12\times 11} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

All rows below the second are left cyclic shifts of second row.

Let $\hat{G} = [I_{12}|\hat{B}]_{12\times 23}$

Then $\hat{G}$ is a GM for a $(23, 12)$-binary code called C_{23}

Facts

(i) $d(C_{23}) = 7$ (proof later) So $e = 3$

(ii) C_{23} is perfect:

$$2^{12}\left(\binom{23}{0} + \binom{23}{1} + \binom{23}{2} + \binom{23}{3}\right) = 2^{23}$$

4.2 The Extended Golay Code C_{24}

Let

$$B = \left[\begin{array}{c|cc} 0 & & \\ 1 & & \\ 1 & & \\ 1 & & \\ 1 & & \\ 1 & & \\ 1 & \hat{B} & \\ 1 & & \\ 1 & & \\ 1 & & \\ 1 & & \\ 1 & & \end{array}\right]_{12\times 12} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Let $G = [I_{12}|B]_{12\times 24}$

Note

(i) C_{24} is a $(24, 12, 8)$-binary code (so $e = 3$)

(ii) $G^T G = 0$ (Check this)

(iii) Hence $C_{24} \subseteq C_{24}^{\perp}$ (C_{24} is a self orthogonal code)

(iv) $\dim(C_{24}) = 12$ and $d(C_{24}^{\perp}) = 12$, so $C_{24} = C_{24}^{\perp}$ (C_{24} is a self dual code)

(v) B is symmetric (check this).

(vi) A PCM for C_{24} is $H = [-B^T|I_{12}] = [B|I_{12}]$

(vii) Since $C_{24} = C_{24}^{\perp}$, H is also a GM for C_{24}

(viii) G is also a PCM for $C_{24}^{\perp}$.

4.2.1 Decoding Algorithm for C_{24}

$s = Hr^T$ syndrome table has size $2^{12} \times 24 \approx 96,000$ bits.

Recall C_{24} is a $(24, 12, 8)$-binary codes with PCMs $[B|I_{12}]$ and $[I_{12}|B]$.

Decoding strategy (IMLD) Compute a syndrome of r. Find a vector e of weight ≤ 3 that has the same syndrome as r.

If no such vector e exists, then reject r; else decode r to $c = r - e$.

Let $r = (x, y), e = (e_1, e_2)$. [all are 12 bits in length][1]

There are cases 5 cases (not mutually exclusive) in the event $\omega(e) \leq 3$.

(A) $w(e_1) = 0, w(e_2) = 0$ [2]

(B) $1 \leq w(e_1) \leq 3, w(e_2) = 0$

(C) $w(e_1) = 1$ or $2, w(e_2) = 1$

(D) $w(e_1) = 0, 1 \leq w(e_2) \leq 3$

(E) $w(e_1) = 1, w(e_2) = 1$ or 2

Theorem 4.1

Let C be an (n, k, d)-code over $\mathrm{GF}(q)$. Let $x \in V_n(\mathrm{GF}(q))$ have weight $\leq \left\lfloor \frac{d-1}{2} \right\rfloor$. Then x is the unique vector of min weight in the coset of C containing x (so must be a coset leader).

Proof:
Let y be a vector in the same coset of C as x, with $y \neq x$ and

$$w(y) \leq w(x) \leq \left\lfloor \frac{d-1}{2} \right\rfloor$$

Then $y - x \neq 0$, and $x \equiv y \;[\bmod C]$, and $x - y \in C$. Now

$$\begin{aligned} w(x-y) &= w(x + (-y)) \\ &\leq w(x) + w(-y) \\ &= w(x) + w(y) \\ &\leq \left\lfloor \frac{d-1}{2} \right\rfloor + \left\lfloor \frac{d-1}{2} \right\rfloor \\ &\leq d - 1 \end{aligned}$$

contradicting $d(C) = d$. □

Recall $[I_{12}|B]$ and $[B|I_{12}]$ are both PCMs for C_{24}.

Note

$$\begin{aligned} S_1 &= [I_{12}|B]r^T \\ &= [I_{12}|B]e^T \\ &= [I_{12}|B] \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix} \\ &= e_1^T + Be_2^T \end{aligned}$$

Similarly, $s_2 = [B|I_{12}]r^T = Be_1^T + e_2^T$

[1] Different presentation from textbook
[2] Note that I use w to represent weight from now on...

Algorithm 3: Decoding Algorithm for C_{24}

Suppose: $r = (x, y)$ is received

1 Compute $s_1 = [I_{12}|B]r^T$. If $s_1 = 0$, then accept r; STOP.

2 If $1 \leq w(s_1) \leq 3$, then correct x in the positions corresponding to is in s_1; STOP.

3 Compare s_1 to the columns (or rows) of B. If ant column, say col i, differs in one position (say j) or differs in two position (j and k) from s_1, then correct r as follows:

- Correct x in the position k or position j and k
- Correct y in position i.

STOP.

4 Compute $s_2 = [B|I_{12}]r^T$. If $w(s_2) \leq 3$, then correct y in position corresponding to the pos in s_2. STOP.

5 Compute s_2 to the cols (or rows) of B. If any col, say col i, differs in one pos (say j) or 2 pos (j and k), then

- Correct y in pos j, or pos j and k
- Correct x in pos i.

STOP.

6 Reject (since $w(e) \geq 4$).

See examples in handouts on LEARN.

Note

1. If $w(e) \leq 3$, then the algorithm makes the correct decision.
2. No storage is needed:
$$s_1 = [I_{12}|B] \begin{bmatrix} x \\ y \end{bmatrix} = x + By$$
3. Algorithm is very simple and efficient (good for hardware)

4.2.2 Reliability of C_{24}

- $p =$ symbol error prob. (BSC)
- $C = \{c_1, c_2, \ldots, c_M\}$
- $w_i =$ prob. that decoding algorithm makes an incorrect decision if c_i is sent.
- Error prob of C is $P_C = \dfrac{1}{M} \sum w_i$
- $1 - P_C =$Reliability of C

p	(1) $(1-p)^{12}$	(2) 1-$P_{C_{24}}$	(3) $1-P_T$	(4) $1-P_H$
0.1	0.282429	0.7857377	0.7112056	0.5490430
0.01	0.8863848	0.99990946	0.9964298	0.9903702
0.001	0.9880657	0.9999999895	0.99996402	0.998959
Rate	1	1/2	1/3	$11/15 \approx 0.73$

(1) If no source is used, then the reliability for 12-bit messages is $(1-p)^{12}$.

(2) $w_i = 1 - [(1-p)^{24} + \binom{24}{1}p(1-p)^{23} + \binom{24}{2}p^2(1-p)^{22} + \binom{24}{3}(1-p)^{21}]$

$$P_{C_{24}} = \frac{1}{2^{12}} \sum_{i=1}^{2^{12}} w_i = w_i$$

(3) T =Triplication code

$$\underbrace{10110\ldots 0}_{12} \to \underbrace{111\ 000\ 111\ 111\ 000\ldots\ 111}_{36}$$

$$1 - P_T = [(1-p)^3 + 3p(1-p)^2]^{12}$$

(4) $(15, 11)$-binary Hamming code

$$1 - P_H = (1-p)^{15} + 15p(1-p)^{14}$$

5

Cyclic Codes

cyclic subspace

A *cyclic subspace* S of $V_n(F)$ is a subspace such that $(a_0, a_1, \ldots, a_{n-1}) \in S \implies (a_{n-1}, a_0, \ldots, a_{n-2}) \in S$.

cyclic code

A *cyclic code* is a cyclic subspace of $V_n(F)$.

Let $R = F[x]/(x^n - 1)$

Associate

$$\underbrace{(a_0, a_1, \ldots, a_{n-1})}_{\in V_n(F)} \leftrightarrow \underbrace{a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}}_{\in R}$$

Addition is preserved

$$a + b \leftrightarrow a(x) + b(x)$$

Scalar multiplication is preserved

$$\lambda a \leftrightarrow \lambda a(x)$$

Why choose $x^n - 1$?

Let $a = (a_0, \ldots, a_{n-1}) \in V_n(F)$. Let $a(x)$ be its associated polynomial in R. Then

$$\begin{aligned} x \cdot a(x) &= a_0 x + a_1 x^2 + \ldots + a_{n-2}x^{n-1} + a_{n-1}x^n \\ &\equiv a_{n-1} + a_0 x + a_1 x^2 + \ldots + a_{n-2}x^{n-1} \; [\bmod x^n - 1] \\ &\leftrightarrow (a_{n-1}, a_0, \ldots, a_{n-2}) \end{aligned}$$

So multiplying a polynomial in R by x corresponds to a (right) cyclic shift of the associated vector.

We'll define $\cdot : V_n(F) \times V_n(F) \to V_n(F) : a \cdot b \leftrightarrow a(x) \cdot b(x)\ [\bmod x^n - 1]$

$$\text{Cyclic subspaces of } V_n(F) \leftrightarrow \text{Ideals in } R \leftrightarrow \text{monic divisors of} x^n - 1$$

ideal

Let R be a commutative finite ring. A non-empty subset I of R is an *ideal* of R if

(1) For all $a, b \in I$, $a + b \in I$.

(2) For all $a \in I, b \in R, a \cdot b \in I$.

Example:
$\{0\}$ and R are (trivial) ideal of R.

Theorem 5.1

Let $S \subseteq V_n(F)$, non-empty. Let I be the associated polynomials.

Then S is a cyclic subspace of $V_n(F)$ iff I is an ideal of $R = F[x]/(x^n - 1)$.

Proof:

$\Rightarrow$) Suppose S is a cyclic subspace of $V_n(F)$.

Since S is closed under addition, so is I. Let $a(x) \in I$, $b(x) = b_0 + b_1x + \ldots + b_{n-1}x^{n-1} \in R$. Then $xa(x) \in I$ since S is a cyclic subspace.

So, $x^i a(x) \in I$, for all $0 \leq i \leq n - 1$.

Also, $b_i x^i a(x) \in I$, since S is closed under scalar multiplication.

Finally, $a(x) \cdot b(x) = a(x)(b_0 + b_1x + \ldots + b_{n-1}x^{n-1})$ which is in I, since I is closed under addition.

$\therefore$ I is an ideal.

$\Leftarrow$) Suppose I is an ideal of R. Since I is closed under addition, so is S. Since I is closed under multiplication by constant polynomials, S is closed under scalar multiplication. Since I is closed under multiplication by x, S is closed under (right) cyclic shifts.

$\therefore$ S is a cyclic subspace.

$\square$

Remark:
So, we have a 1-1 correspondence between cyclic subspaces of $V_n(F)$ and ideals of $R = F[x]/(x^n - 1)$.

ideal generated by $g(x)$

Let $g(x) \in R$. Then $\langle g(x) \rangle = \{g(x) \cdot a(x) : a(x) \in R\}$. Then $\langle g(x) \rangle$ is an ideal of R. Called the *ideal generated by* $g(x)$.

principal ideal & principal ideal ring

If I is an ideal of R, then I is *principal ideal* if $\exists g(x) \in I$ such that $I = \langle g(x) \rangle$.

R is called a *principal ideal ring* if every ideal of R is principal.

Theorem 5.2

$R = F[x]/(x^n - 1)$ is a principal ideal ring.

Proof:
Let I be an ideal of R.

Suppose first that $I = \{0\}$. Then $I = \langle 0 \rangle$ is principle.

Suppose $I \neq \{0\}$. Let $g(x)$ ne a polynomial of smallest degree in I. Let $a(x) \in I$.

Long division gives

$$a(x) = \ell(x)g(x) + r(x)$$

where $\ell, r \in F[x]$, and $\deg(r) < \deg(g)$.

But $\ell(x)g(x) \in I$. (since I is closed under multiplication by R).

And $a(x) - \ell(x)g(x) \in I$.

$\therefore$ $r(x) \in I$. Since $\deg(r) < \deg(g)$, we must have $r(x) = 0$. Hence $a(x) = \ell(x)g(x)$.

$\therefore I = \langle g(x) \rangle$

$\therefore$ R is a principal ideal ring. $\square$

Note
We can take $g(x)$ to be *monic* (i.e. $g(x) = x^\ell + g_{\ell-1}x^{\ell-1} + \ldots + g_0$)

If $g(x)$ were not monic, say $g_\ell x^\ell + \ldots + g_0$ where $g_\ell \neq 0, 1$, then

$$g_\ell^{-1} g(x) = x^\ell + \ldots + g_\ell^{-1} g_0$$

is monic and is also in I. We will call this process "making $g(x)$ monic".

the generator polynomial of I

Let I be an ideal in $R = F[x]/(x^n - 1)$. If $I = \{0\}$, then *the generator polynomial* of I is $x^n - 1$ (since $x^n - 1 \equiv 0 \; [\bmod\, x^n - 1]$).

If $I \neq \{0\}$, the monic polynomial of least degree in I is called *the generator polynomial of I*.

Theorem 5.3

Let I be a nonzero ideal in $R = F[x]/(x^n - 1)$.

(i) There is *unique* monic polynomial $g(x)$ of smallest degree in I.

(ii) $g(x) | x^n - 1$.

Proof:

(i) Suppose $g(x), h(x)$ are two monic polynomials of (the same) smallest degree in I. Then $g(x) - h(x) \in I$ and $\deg(g - h) < \deg(g)$. Hence we must have $g - h = 0$, so $g(x) = h(x)$.

(ii) We can write $x^n - 1 = \ell(x)g(x) + r(x)$ where $\ell, r \in F[x]$, and $\deg(r) < \deg(g)$. So
$$0 \equiv \ell(x)g(x) + r(x) \; [\bmod\, x^n - 1],$$
so
$$r(x) \equiv -\ell(x)g(x) \; [\bmod\, x^n - 1].$$
Since $\langle g(x) \rangle = I$, we have $r(x) \in I$. Hence $\deg(r) < \deg(g)$, we have $r(x) = 0$.

$\therefore g(x) | x^n - 1$.

$\square$

Theorem 5.4

Let $h(x)$ be a monic divisor of $x^n - 1$ in $F[x]$. Then the generator polynomial of $\langle h(x) \rangle$ is $h(x)$.

Proof:

If $h(x) = x^n - 1$, then $I = \{0\}$ and, by defn, its generator polynomial is $x^n - 1$.

If $\deg(h) < n$, so $I \neq \{0\}$, then let $g(x)$ be the monic polynomial of smallest degree in I. Since $g(x)$ is a generator of I, we can write $h(x) \equiv a(x)g(x) \; [\bmod\, x^n - 1]$.

So, $g(x) = a(x)h(x) + \ell(x)(x^n - 1)$ for some $\ell(x) \in F[x]$.

Since $h | x^n - 1$ and $h | ah$, we have $h(x) | g(x)$.

So, $\deg(h) \leq \deg(g)$. Since g is the monic polynomial of smallest degree in I, we have $\deg(g) \leq \deg(h)$, so $\deg(g) = \deg(h)$. Since g, h are both monic, we have $g(x) = h(x)$. □

Corollary 5.5

There is 1-1 correspondence between monic divisors of $x^n - 1$ in $F[x]$ and ideals in R. There is a 1-1 correspondence between monic divisors of $x^n - 1$ in $F[x]$ and cyclic subspaces of $V_n(F)$.

Example:

Find all cyclic subspaces of $V_3(\mathbb{Z}_2)$. ($n = 3$)

The complete factorization of $x^3 - 1$ over $\mathbb{Z}_2$ is:

$$x^3 - 1 = (1 + x)(1 + x + x^2) \qquad R = \mathbb{Z}_2[x]/(x^3 - 1)$$

Monic divisor of $x^3 - 1$	$\langle g_i(x) \rangle$ [a]	$\dim \langle g_i(x) \rangle$
$g_1(x) = 1$	$\underbrace{\{000, 001, \ldots, 111\}}_{8}$	3
$g_2(x) = 1 + x$	$\{000, 110, 011, 101\}$	2
$g_3(x) = 1 + x + x^2$	$\{000, 111\}$	1
$g_4(x) = 1 + x^4$	$\{0\}$	0

[a] Cyclic subspace generated by $g_i(x)$

We have one to one associations:

$$V_n(F) \leftrightarrow R = F[x]/(x^n - 1)$$

$$a = (a_0, a_1, \ldots, a_{n-1}) \in V_n(F) \leftrightarrow a_0 + a_1x^1 + \ldots + a_{n-1}x^{n-1} \in R = F[x]/(x^n - 1)$$

C : (cyclic subspace)		$g(x)$ [monic divisor of x^n-1]
$\dim C = k$		$\deg g = n - k$
GM for C in terms of $g(x)$		$h(x) = (x^n - 1)/g(x)$
Encoding: $mG \leftrightarrow m(x)g(x)$	$\leftrightarrow I$, (ideal in R) $\leftrightarrow$	$\deg(h) = k$
$C^\perp$: dual code of C is cyclic		$h_R(x)$: reciprocal poly of $h(x)$
PCM H for C:		$h^*(x) = h_R$ by making h_R monic
$s(x) \equiv r(x)$ [mod $g(x)$]		$h^*(x)$ the gen polynomial of $C^\perp$

Distance of C?

C: BCH code: $g(x)$ is specially selected to give a lower bound of C.

Lemma 5.6

Let $g(x)$ be a monic divisor of deg $n - k$ of $x^n - 1$ in $F[x]$.

Recall $\langle g(x) = \{g(x)a(x) : a(x) \in R\}\rangle$ [a]

In fact, $\langle g(x)\rangle = \{g(x)\overline{a}(x) : \deg(\overline{a}) < k\}$ [b]

[a] $a(x)$ with mod

[b] $\overline{a}(x)$ no mod

Proof:
Let $h(x) = g(x)a(x)$ [mod $x^n - 1$] where $\deg(a) < n$. So, $h(x) - g(x) = \ell(x)(x^n - 1)$ for some $\ell \in F[x]$.

$\therefore g(x) | h(x)$

So $h(x) = g(x)\overline{a}(x)$, for some $\overline{a} \in F[x]$ with $\deg(\overline{a}) \leq k - 1$. $\square$

Theorem 5.7

Let $g(x)$ be a monic divisor of $x^n - 1$ of deg $n - k$ in $F[x]$. Then the cyclic code C generated by $g(x)$ has $\dim k$.

Proof:
We'll show that

$$B = \{g(x), xg(x), \ldots, x^{k-1}g(x)\}$$

is a basis of C.

B is lin indep over F.

Suppose

$$\underbrace{\lambda_0 g(x)}_{\deg = n-k} + \underbrace{\lambda_1 x g(x)}_{\deg = n-k+1} + \ldots + \underbrace{\lambda_{k-1} x^{k-1} g(x)}_{\deg = n-k+k-1 = n-1} = 0$$

where $\lambda_i \in F$.

The coeff of x^{n-1} in the LHS is λ_{k-1}. The coeff of x^{n-1} in RHS is 0. Hence $\lambda_{k-1} = 0$. Similarly, $\lambda_0 = \lambda_1 = \ldots = \lambda_{k-2} = 0$.

Claim B spans C.

Let $h(x) \in \langle g(x) \rangle$. By Lemma, we can write

$$h(x) = \underbrace{\underbrace{g(x)}_{n-k}\underbrace{a(x)}_{k-1}}_{n-1 \text{ no mod}}$$

for some $a(x) \in F[x], \deg(a) \leq k - 1$.

Let $a(x) = \sum_{i=0}^{k-1} a_i x^i$, where $a_i \in F$.

Then $h(x) = g(x)a(x) = \sum_{i=0}^{k-1} a_i[x^i g(x)]$

Hence $\dim C = k$. $\square$

So a GM for C is

$$G = \begin{bmatrix} g(x) \\ xg(x) \\ \vdots \\ x^{k-1}g(x) \end{bmatrix}_{k\times n} = \begin{bmatrix} \boxed{g(x)} & 0 & \cdots & 0 \\ 0 & \boxed{xg(x)} & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & 0 & \boxed{x^{k-1}g(x)} \end{bmatrix}$$

Note
G is a non-systematic GM for C.

Encoding

$$\begin{aligned} c &= mG \\ &= (m_0, \cdots, m_{k-1}) \begin{bmatrix} g(x) \\ xg(x) \\ \vdots \\ x^{k-1}g(x) \end{bmatrix} \\ &= m_0 g(x) + m_{k-1}x^{k-1}g(x) \\ &= g(x)\left(m_0 + \cdots + m_{k-1}x^{k-1}\right) \\ &\implies c(x) = m(x)g(x) \end{aligned}$$

Example: Construct a cyclic $(7,4)$-code over $\mathbb{Z}_2$

We need a monic divisor of deg 3 of $x^7 - 1$ in $\mathbb{Z}_2[x]$. Table 3 in page 157 of textbook:

$$x^7 - 1 = (1+x)(1+x+x^2)(1+x^2+x^3)$$

Let's take $g(x) = 1 + x + x^3$. Then $\langle g(x)\rangle = 1 + x + x^3$. Then $\langle g(x)\rangle$ is a $(7,4)$-cyclic code over $\mathbb{Z}_2$. A GM for C is

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}_{4\times 7}$$

(hamming code cyclic)

Encode $m = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}$

$$c = mG = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$c(x) = m(x)g(x) = \left(1 + x + x^3\right)\left(1 + x + x^3\right) = \left(1 + x + \cdots + x^6\right) = c$$

5.1 Dual Code of a Cyclic Code

Let C be an (n,k)-cyclic code over F with generated polynomial $g(x)$.

Let $g(x) = \underset{\neq 0}{g_0} + g_1 x + g_2 x^2 + \ldots + \underset{=1}{g_{n-k}} x^{n-k} + \underbrace{g_{n-k+1}x^{n-k+1} + \ldots + g_{n-1}x^{n-1}}_{0}$

Let $h(x) = (x^n - 1)/g(x) = \underset{\neq 0}{h_0} + h_1 x + \ldots + h_{k-1}x^{k-1} + \underset{=1}{h_k} x^j + \underbrace{\ldots + h_{n-1}x_0^{n-1}}_{0}$

Let $a(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$

and

$$a(x) = g(x)h(x) \;[\mathrm{mod}\, x^n - 1] \qquad (*)$$

Note $a(x) = 0$. Equating coeffs of $x^i, 0 \leq i \leq n-1$, of $(*)$:

$$a_i = 0 = g_0 h_i + g_1 h_{i-1} + \ldots + g_i h_0 + g_{i+1}h_{n-1} + g_{i+1}h_{n-2} + \ldots + g_{n-1}h_{i+1}$$

Let $g = (g_0, g_1, \ldots, g_{n-1})$ and $\overline{h} = (h_{n-1}, h_{n-2}, \ldots, h_1, h_0)$

Then g is orthogonal to $\overline{h}$ and all the cyclic shifts of $\overline{h}$. So every cyclic shift of g is orthogonal to every cyclic shift of $\overline{h}$.

Recall a GM for C is

$$\begin{bmatrix} g_0 & g_1 & \ldots & g_{n-k} & 0 & \ldots & 0 & 0 \\ 0 & g_0 & g_1 & \ldots & g_{n-k} & 0 & \ldots & 0 \\ \vdots \\ 0 & \ldots & 0 & 0 & g_0 & g_1 & \ldots & g_{n-k} \end{bmatrix}_{k\times n}$$

Consider

$$H = \begin{bmatrix} h_k & h_{k-1} & \ldots & h_0 & 0 & \ldots & 0 & 0 \\ 0 & h_k & h_{k-1} & \ldots & h_0 & 0 & \ldots & 0 \\ \vdots \\ 0 & \ldots & 0 & 0 & h_k & h_{k-1} & \ldots & h_0 \end{bmatrix}_{(n-k)\times n}$$

We have observed that $GH^T = 0$. Let C' be the code spanned by the rows of H. Then $C' \subseteq C^\perp$. But $\mathrm{rank}(H) = n - k$ (since $h_k = 1$). Since $\dim(C') = n - k$, we have $C' = C^\perp$.

Hence H is a PCM for C.

reciprocal of h

Let $h(x) = h_0 + h_1 x + \ldots + h_k x^k$ be a degree k polynomial. The reciprocal of h is $h_R(x) = h_k x^0 + h_{k-1}x + \ldots + h_1 x^{k-1} + h_0 x^k$.

Note

$h_R(x) = x^k h\left(\frac{1}{k}\right)$

If $h_0 \neq 0$, then $h^*(x) = h_0^{-1} \cdot h_R(x)$.

Theorem 5.8

If C is an (n, k)-cyclic code, then $C^\perp$ is an $(n, n-k)$-cyclic code.

Proof:
$g(x)h(x) = x^n - 1$

So,

$$g\left(\frac{1}{x}\right) h\left(\frac{1}{x}\right) = \frac{1}{x^n} - 1$$

So,

$$\left(x^{n-k} g\left(\frac{1}{x}\right)\right)\left(x^k h\left(\frac{1}{x}\right)\right) = 1 - x^n$$

Then

$$g_R(x) \cdot h_R(x) = -(x^n - 1)$$

Then

$$h_R(x) | x^n - 1$$

So, $h_R(x)$ is a degree k divisor pf $x^n - 1$. So, the matrix H is a GM for the cyclic code generated by $h^*(x)$.

Hence $C^\perp$ is cyclic with generator polynomial $h^*(x)$. □

5.2 Computing Syndromes

Let's find a more convenient PCM for C.

(i) Find a GM for C of the form $[R|I_k]_{k\times n}$. (Essentially systematic)

For $0 \le i \le k-1$, long division gives:

$$x^{n-k+i} = \underbrace{\ell_i(x)}_{\deg \le k-1} \cdot \underbrace{g(x)}_{\deg = n-k} + \underbrace{r_i(x)}_{\deg \le n-k-1}$$

Then $-r_i(x) + x^{n-k+i} = \ell_i(x) \cdot g(x) \in C$

Let

$$G = \begin{bmatrix} -r_0(x) + x^{n-k} \\ -r_1(x) + x^{n-k+1} \\ \vdots \\ -r_{k-1}(x) + x^{n-1} \end{bmatrix}_{k\times n} = \begin{bmatrix} \overbrace{\boxed{-r_0(x)}}^{n-k} & 1 & 0 & \dots & 0 \\ \boxed{-r_1(x)} & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ \boxed{-r_{k-1}(x)} & 0 & 0 & \dots & 1 \end{bmatrix} = [R|I_k]$$

Then G has rank k, so G is a GM for C.

(ii) Construct a PCM for C

This is $H = [I_{n-k}| - R^T]_{(n-k)\times n}$

Then $Hr^T = r(x)\ [\bmod g(x)]$

Then

$$H^T = \begin{bmatrix} I_{n-k} \\ \hline -R \end{bmatrix}_{n\times(n-k)} = \begin{array}{l} \left[\begin{array}{c} x^0 \ [\bmod g(x)] \\ x^1 \ [\bmod g(x)] \\ \vdots \\ x^{n-k-1} \ [\bmod g(x)] \\ \hline x^{n-k} \ [\bmod g(x)] \\ \vdots \\ x^{n-1} \ [\bmod g(x)] \end{array}\right] \begin{array}{l} \\ \\ \\ \\ \longrightarrow r_0(x) \\ \\ \longrightarrow r_{k-1}(x) \end{array} \end{array}$$

So, columns of H are

$$x^0 \ [\bmod g(x)], \ldots, x^{n-1} \ [\bmod g(x)]$$

Hence if $r = (r_0, r_1, \ldots, r_{n-1}) \in V_n(F)$, then

$$\begin{aligned} s &= Hr^T \\ &= (r_0 \cdot x^0 \ [\bmod g(x)]) + \ldots + (r_{n-1} \cdot x^{n-1} \ [\bmod g(x)]) \\ &= (r_0 x^0 + r_1 x + \ldots r_{n-1} x^{n-1}) \ [\bmod g(x)] \\ &= r(x) \ [\bmod g(x)] \end{aligned}$$

Theorem 5.9

Let C be a cyclic code with g.p. $g(x)$, and $r \in V_n(F)$. Then the syndrome of r (w.r.t. the previous PCM) is

$$s(x) = r(x) \ [\bmod g(x)]$$

Example:

$g(x) = 1 + x + x^2 + x^3 + x^6$ is the g.p. for a $(15, 9)$-binary cyclic code. (Check: $g(x) | x^{15} - 1$ over GF(2))

Compute the syndrome of $r = (1110\ 1110\ 1100\ 000)$

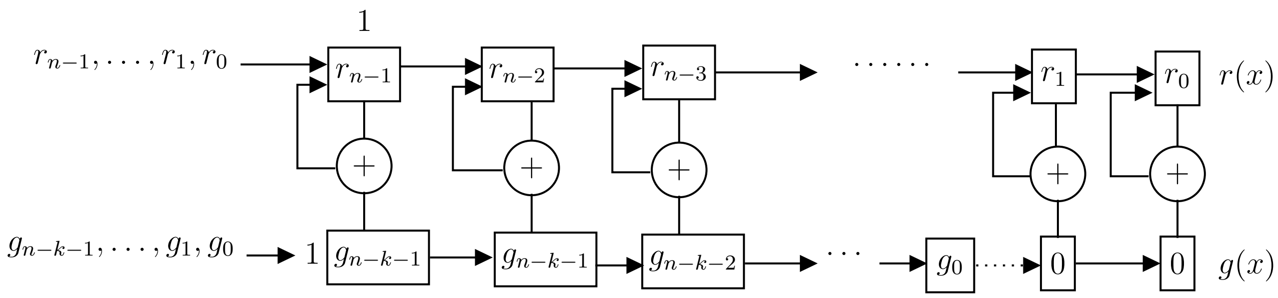
So $r(x) = 1 + x + x^2 + x^4 + x^5 + x^6 + x^8 + x^9$.

$$\begin{array}{r|l} & x^3 + x^2 \\ \hline x^6 + x^3 + x^2 + x + 1 & x^9 + x^8 + x^6 + x^5 + x^4 \qquad + x^2 + x + 1 \\ & -x^9 \qquad - x^6 - x^5 - x^4 - x^3 \\ \hline & x^8 \qquad\qquad - x^3 + x^2 \\ & -x^8 \qquad - x^5 - x^4 - x^3 - x^2 \\ \hline & - x^5 - x^4 - 2x^3 \qquad + x + 1 \end{array}$$

$s(x) = 1 + x + x^4 + x^5, \quad s = (110011)$

$$
\begin{array}{r|l}
g(x) & r(x) \\
1\,0\,0\,1\,1\,1\,1 & 1\,1\,0\,1\,1\,1\,0\,1\,1\,1 \\
 & 1\,0\,0\,1\,1\,1 \\
\hline
 & \quad 1\,0\,0\,0\,0\,1\,1\,1\,1 \\
 & \quad 1\,0\,0\,1\,1\,1\,1 \\
\hline
 & \qquad\quad 1\,1\,0\,0\,1\,1
\end{array}
$$

Hardware: $r(x)$ [mod $g(x)$]



So, $r(x)$ [mod $g(x)$] can be implemented in hardware using a very simple and fast circuit.

Note
Given the syndrome s of r, the syndromes of cyclic shifts of r can be easily computed.

Theorem 5.10

Let $r \in V_n(F)$ and $s = r(x)$ [mod $g(x)$] $= s_0 + s_1 x + \ldots + s_{n-k-1}x^{n-k-1}$.

Then the syndrome of $\underbrace{xr(x)}_{\text{cyclic shift}}$ is

(i) $xs(x)$, if $s_{n-k-1} = 0$

(ii) $xs(x) + s_{n-k-1}g(x)$, if $s_{n-k-1} \neq 0$

These two above are not cyclic shifts.

Proof:
We have $r(x) = \ell(x) + g(x) + s(x)$

Multiply by x:

$$xr(x) = x\ell(x) \underbrace{g(x)}_{\deg = n-k} + xs(x)$$

If $s_{n-k-1} = 0$, then $\deg(s) \leq n-k-2$, so $\deg(xs(x)) \leq n-k-1$.

So $xs(x)$ is the remainder upon dividing $xr(x)$ by $g(x)$.

So, $xs(x)$ is the syndrome of $r(x)$.

If $s_{n-k-1} \neq 0$, then $\deg(s) = n-k-1$. Then

$$\begin{aligned} xr(x) &= x\ell(x)g(x) + xs(x) + s_{n-k-1}g(x) - s_{n-k-1}g(x) \\ &= (x\ell(x) + s_{n-k-1})g(x) + \underbrace{[xs(x) - s_{n-k-1}g(x)]}_{\deg \leq n-k-1} \end{aligned}$$

Because

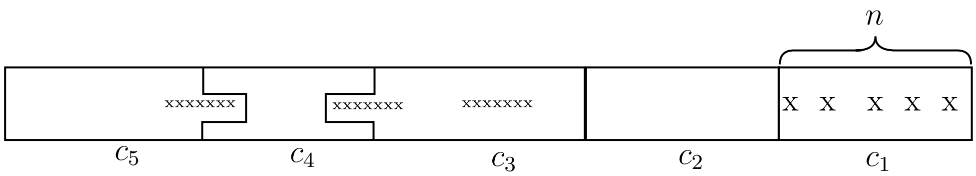
$$\begin{array}{rl} xs(x) & = s_0 x + \ldots + s_{n-k-1}x^{n-k} \\ - \quad s_{n-k-1}g(x) & = \ldots + \ldots + s_{n-k-1}x^{n-k} \\ \hline \multicolumn{2}{c}{xr(x)} \end{array}$$

The last term canceled.

So, $xs(x) - s_{n-k-1}g(x)$ is the syndrome of $xr(x)$. □

5.3 Burst Error Correcting

- "cyclic codes are good for (cyclic) burst error correcting."
- $C : (n,k,d)$-binary code, say $e = \left\lfloor \frac{d-1}{2} \right\rfloor = 5$



cyclic burst length of e

Let $e \in V_n(F)$. The *cyclic burst length of* e is the length of the smallest cyclic block that contain all the nonzero entries of e.

Example:

$e = \quad 0\;1\;0\;0\;0\;0\;0\;1$

has cyclic burst length 4. Say e is a cyclic burst error of length t if its cyclic burst length is t.

t-cyclic burst error correcting code

A linear code C is a t-cyclic burst error correcting code if every cyclic burst error of length at most t lies unique coset of C. The largest such t is called the cyclic burst error capability of C

Example:

$g(x) = 1 + x + x^2 + x^3 + x^6$ generates a $(15,9)$-binary cyclic code C that is a 3-cyclic burst error correcting code.

Note

$d(C) \leq 5$, so $e \leq 2$. Verify by checking that each cyclic burst error of length ≤ 3 has a unique syndrome.

Cyclic burst errors	Syndromes
0	000000
x^0	100000
x^1	010000
x^2	001000
$\vdots$	
x^5	000001
x^6	111100
x^7	011110
x^8	001111
x^9	111011
$\vdots$	
x^{14}	111001
$1+x$	110000
$x(1+x)$	011000
$\vdots$	
$x^{14}(1+x)$	011001
$1+x+x^2$	111000
$x(1+x+x^2)$	01100
$\vdots$	
$x^{14}(1+x+x^2)$	001001
$1+x^2$	101000
$x(1+x^2)$	010100
$\vdots$	
$x^{14}(1+x^2)$	101001

All syndromes are unique.

# of cyclic bursts of length ≤ 3 is 61. # of syndromes is 64.

Example:

$g(x) = 1 + x^4 + x^6 + x^7 + x^8$ generates a $(15, 7)$-binary cyclic code that is 4-cyclic burst error correcting. Distance ≤ 5, so $e \leq 2$.

Q How to construct codes with high cyclic burst error, correcting capability?

A 1. Use computer search. 2. RS codes. 3. Interleaving.

Theorem 5.11

Let C be an (n, k, d)-code over $\mathrm{GF}(q)$.

Let t be its cyclic burst error correcting capability.

$$\left\lfloor \frac{d-1}{2} \right\rfloor \leq t \leq n-k$$

Proof:
Every cyclic burst of length $\leq t$ has weight $\leq t$. Since every vector of weight $\leq \left\lfloor \frac{d-1}{2} \right\rfloor$ has a unique syndrome. We have $\left\lfloor \frac{d-1}{2} \right\rfloor \leq t$.

To prove the upper bound. Note that the # of cyclic burst errors where all the nonzero entries lie in the first t coordinate positions is q^t. Each has unique coset, and the total # of cosets is q^{n-k}. So $q^t \leq q^{n-k}$, $t \leq n-k$. □

Exercise

Prove that $t \leq \dfrac{n-k}{2}$.

5.4 Decoding cyclic burst errors

Let C be a t-cyclic burst e.c.c.[1] generated by $g(x)$ which is a degree-k monic divisor of $x^n - 1$ over $\mathrm{GF}(q)$.

Recall A PCM for C is $H = [I_{n-k} | -R^T]$ whose columns are x^0 [mod $g(x)$], $\ldots$, x^{n-1} [mod $g(x)$]. The syndrome of $r(x)$ is $s(x) = r(x)$ [mod $g(x)$].

Idea Suppose e is a cyclic burst of length $\leq t$.

Compute $s = Hr^T = r(x)$ [mod $g(x)$]

$e = \boxed{\text{x o —— o x x x}}$

$xe = \boxed{\text{x x o —— o x x}}$

$x^2e = \boxed{\text{x x x o —— o x}}$

$x^3e = \boxed{\text{x x x x o —— o}}$

$$\begin{aligned} s &= Hr^T = He^T \\ s_1 &= H(xr)^T = H(xe)^T \\ s_2 &= H(x^2r)^T = H(x^2e)^T \\ s_3 &= H(x^3r)^T = H(x^3e)^T \end{aligned}$$

[1] error correcting code

Idea (cont'd) Suppose e is a cyclic burst of length $\leq t$. Compute shifts of e, say $e_i = x^i$, has all its nonzero entries if the first $n-k$ positions. Then $s_i(x) = e_i(x)$ $[\bmod g(x)]$, and we can recognize such an $s_i(x)$ since it is a (non-cyclic) burst of length $\leq t$. Then $e = x^{n-i}e_i$.

How to compute $s(x)$? Recall $r = c + e$.

So $x^i r = x^i c + x^i e$, so $x^i r$ and $x^i e$ have the same syndrome.

5.5 Error trapping decoding

- Let $r(x)$ = received poly
- Let $s_i(x)$ = syndrome of $x^i r(x)$, $0 \leq i \leq n-1$

  So $s_0(x) = r(x)$ $[\bmod g(x)]$

Algorithm 4: Error trapping decoding

```
for i ← 0 to n − 1 do
    Compute s_i(x)
    if s_i(x) is a (non-cyclic) burst of length ≤ t then
        Let e_i(x) = (s_i(x), 0)
        Let e(x) = x^{n−i} e_i(x)
        Decode r(x) to r(x) − e(x)
    end
end
Reject r
```

Example:

$g(x) = 1 + x + x^2 + x^3 + x^6$ is the g.p. for $(15, 9)$-binary cyclic code with c.b.e.c capability 3. Decode $r = (1110\ \ 1110\ \ 1100\ \ 000)$

Solution Compute $s_0(x) = r(x)$ $[\bmod g(x)] = x^5 + x^4 + x + 1$

i	$s_i(x)$
0	110011
1	100101
2	101110
3	010111
4	110111
5	100111
6	101111
7	101011
8	101001
9	$\underbrace{101}_{\substack{\text{non-cyclic} \\ \text{burst of} \\ \text{length } \leq 3}}\ 000$

So $e_9 = (101000\ 000000000)$

So $e = x^6 e_9 = (000000101000000)$

So $c = r - e = (1110\ 1100\ 0100\ 000)$

Check $Hc^T = 0$, or $g(x)|c(x)$

5.6 Interleaving

Goal Improve c.b.e.c capability of a code

Suppose C is an (n, k)-code with c.b.e.c capability t.

Suppose the following codewords are transmitted.

$$\begin{aligned} v_1 &= (v_{11}, v_{12}, \ldots, v_{1n}) \in C \\ v_2 &= (v_{21}, v_{22}, \ldots, v_{2n}) \in C \\ &\vdots \\ v_s &= (v_{s1}, v_{s2}, \ldots, v_{sn}) \in C \end{aligned}$$

Suppose $v_1, v_2, \ldots, v_s$ are transmitted in that order. If a cyclic error of length $\leq t$ occurs in any codeword, that error can be corrected.

Instead we transmit the columns in order:

$$[v_{11}, v_{21}, \ldots, v_{s1}, v_{12}, v_{22}, \ldots, v_{s2}, \ldots, v_{1n}, \ldots, v_{sn}]$$

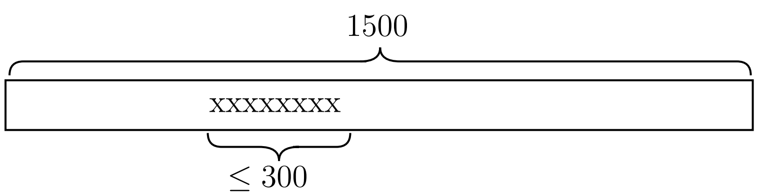
Now, if a cyclic burst error of length $\leq st$ occurs in this (flat) codeword, this means each original codeword suffered a cyclic burst error of length $\leq t$.

Theorem 5.12

Suppose C is an (n, k)-cyclic code with g.p. $g(x)$ and c.b.e.c. capability t. Then C^*, the code obtained by interleaving C to a depth s_1 is an (ns, ks)-cyclic code with g.p. $g^*(x) = g(x^s)$.

Example:

Interleave C to depth $s = 100$, to get code C^*. Then C^* is a $(1500, 900)$, 300-c.b.e.c.c, binary cyclic code with g.p. is $g(x^{100}) = 1 + x^{100} + x^{200} + x^{300} + x^{600}$,

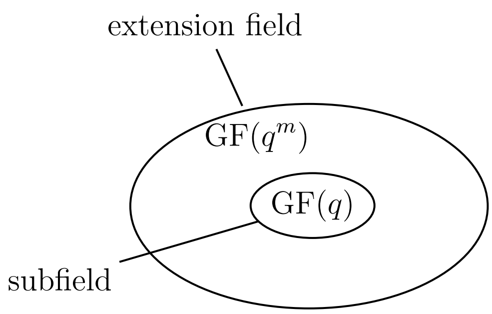


6 BCH codes

6.1 Minimal Polynomials

Recall We can view $F = \mathrm{GF}(p^m)$ as a vector space of dim m over $\mathbb{Z}_p$ and $\mathbb{Z}_p$ is a **subfield** of F.

More generally, for any prime power q, we can view the finite field $\mathrm{GF}(q^m)$ as a vector space of dim m over $\mathrm{GF}(q)$, and $\mathrm{GF}(q)$ is a subfield of $\mathrm{GF}(q^m)$.



Example:

$\mathrm{GF}(2^{16})$ is vector space of dim 16 over $\mathrm{GF}(2)$.

$\mathrm{GF}(2^{16})$ is vector space of dim 8 over $\mathrm{GF}(2^2)$.

$\mathrm{GF}(2^{16})$ is vector space of dim 4 over $\mathrm{GF}(2^4)$.

$\mathrm{GF}(2^{16})$ is vector space of dim 2 over $\mathrm{GF}(2^8)$.

$\mathrm{GF}(2^{16})$ is vector space of dim 1 over $\mathrm{GF}(2^{16})$.

minimal polynomial

Let $\alpha \in \mathrm{GF}(q^m)$. The minimal polynomial of α over $\mathrm{GF}(q)$, denoted $m_\alpha(x)$, is the monic polynomial of least degree in $\mathrm{GF}(q)[x]$ such that $m(\alpha) = 0$.

Example:

Let $0 \in \mathrm{GF}(q^m)$. Then $m_0(x) = x$.

Example:

Consider $\mathrm{GF}(2^2) = \mathbb{Z}_2[x]/(x^2+x+1) = \{0, 1, x, x+1\}$. Find min poly of $\alpha \in \mathrm{GF}(2^2)$ over $\mathrm{GF}(2)$:

$$\begin{aligned} m_0(y) &= y \\ m_1(y) &= y+1 \\ m_x(y) &= y^2+y+1 \\ m_{x+1}(y) &= y^2+y+1 \end{aligned}$$

Q Why does $m_\alpha(x)$ exist? ($\alpha \neq 0$)

The $\mathrm{ord}(\alpha) = t | (q^m - 1)$. So $\alpha^t = 1$. So, α is a root of $x^t - 1 \in \mathrm{GF}(q)[x]$. Also, if $f(x) \in \mathrm{GF}(q)[x]$ with $f(\alpha) = 0$, then if $c \in \mathrm{GF}(q)$ is the leading coeff of f, then $f' = c^{-1}f \in c^{-1}f \in \mathrm{GF}(q)[x]$ and $f'(\alpha) = 0$.

$\therefore m_\alpha(x)$ exists.

Theorem 6.1: Properties of $m_\alpha(x)$

Let $\alpha \in \mathrm{GF}(q^m)$, and $m_\alpha(x)$ ⓐ minimal poly of α over $\mathrm{GF}(q)$,
the

(1) $m_\alpha(x)$ is unique.

(2) $m_\alpha(x)$ is irreducible over $\mathrm{GF}(q)$.

(3) $\deg(m_\alpha) \leq m$

Proof:

(1) Suppose not. Let $s(x), t(x) \in \mathrm{GF}(q)[x]$ be two monic polys of (the same) least degree having α as a root. Consider $r(x) = s(x) - t(x) \in \mathrm{GF}(q)[x]$. Then $r(\alpha) = s(\alpha) - t(\alpha) = 0$ and $\deg(r) < \deg(s)$. So we must have $r(x) = 0$. Hence $s(x) = t(x)$.

(2) Suppose not. Let $m_\alpha(x) = s(x)t(x)$, where $s, t \in \mathrm{GF}(q)[x]$, and $\deg(s), \deg(t) < \deg(m_\alpha)$. So $m_\alpha(\alpha) = s(\alpha)t(\alpha) = 0$, so $s(\alpha) = 0$ or $t(\alpha) = 0$. This contradicts minimality of $\deg(m_\alpha)$.

(3) Recall $\mathrm{GF}(q^m)$ is a vector space of dim m over $\mathrm{GF}(q)$. So, $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^m$ are linearly dependent over $\mathrm{GF}(q)$. So we can write

$$a_0 + a_1\alpha + \ldots + a_m\alpha^m = 0, \qquad \text{where } a_i \in \mathrm{GF}(q), \text{ not all } 0$$

So, α is a root of $f(x) = a_0 + a_1x + \ldots + a_mx^m$ which is nonzero and $\deg(f) \leq m$.

□

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