Model Theory and Set Theory

PMATH 433

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Preface

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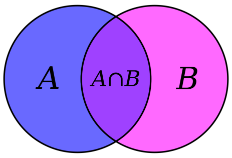

Part I:

Set Theory

Naive set theory is any of several theories of sets used in the discussion of the foundations of mathematics. Unlike axiomatic set theories, which are defined using formal logic, naive set theory is defined informally, in natural language.

Naive set theory, Wikipedia

1 Ordinals

1.1 Zermelo-Fraenkel Axioms

We use natural numbers 0, 1, 2, 3, ... to "count" finite sets. Here the word "count" has two meanings:

- enumerate, list, order. (ordinal)
- measure size. (cardinal)

Our aim is to develop ordinals and cardinals to do this work for arbitrary (possibly infinite) sets.

We start by building the set of natural numbers. We can't construct a theory without foundations. We need some undefined notions. So we start with the undefined notions: set; members in a set, $\in$.

Define

$$\begin{aligned}
0 &:= \text{the empty set, } \varnothing \\
1 &:= \text{the set whose only member is } 0, \text{denoted } \{0\} \\
2 &:= \text{the set whose only members are } 0, 1, \text{denoted } \{0, 1\} \\
3 &:= \{0, 1, 2\} \\
&\vdots \\
n + 1 = S(n) &:= n \cup \{n\}
\end{aligned}$$

where S is the successor function. But wait, why do these exist? We need some unprovable results to start with, i.e., some axioms.

Emptyset Axiom

There exists a set which has no members, it is called the empty set, and it is denoted by $\varnothing$. Or we can say: there exists a set $\varnothing$ satisfying: $\neg\exists y, y \in \varnothing$.

To produce 1 from 0 we need to know that if x is a set then so is $\{x\}$.

Pairset Axiom

Given sets x, y there is a set, denoted by $\{x, y\}$, with the property that its only members are x and y.

$$t \in \{x, y\} \iff t = x \text{ or } t = y.$$

Note:
If $x = y$ then $t \in \{x, y\} \iff t = x$. So $\{x, y\} = \{x\}$.

Extension Axiom

For any sets x, y, $x = y$ if and only if x and y have the same members.

With these axioms we have that $0 = \varnothing, 1 = \{0\}$ exist. In general from n to get that $S(n) = n \cup \{n\}$ we need the following axiom.

Unionset Axiom

Given a set x there exists a set denoted by $\bigcup x$, whose members are precisely the members of the members of x, i.e.,

$$t \in \bigcup x \iff t \in y \text{ for some } y \in x.$$

Then $S(n) = \bigcup\{n, \{n\}\}$, i.e., $t \in S(n) \iff t \in n$ or $t = n$. Hence if n exists then by Pairset axiom twice and Unionset axioms, $S(n)$ exists.

With these 4 axioms we can prove that every natural number exists. What about the set of <u>all</u> natural numbers? Why don't we add an axiom saying

"there exists a set whose elements are precisely the natural numbers"?

i.e., "There exists a set N such that $t \in N \iff t = 0$ or $t = 1$ or $t = 2$ or $\ldots$"

Problem: The RHS is not a **definite condition** on t. We will answer what is definite condition formally in the next part of the course. Now define it roughly.

definite condition

$x \in Y$, $x = y$ are definite (binary) conditions. If P, Q are definite conditions then

- "not P", denoted $\neg P$,
- "P and Q", denoted $P \wedge Q$,
- "P or Q", denoted $P \vee Q$,
- "for all x, P", denoted $\forall x, P$,
- "there exists x, P", denoted $\exists x, P$,

are all definite conditions. Only conditions arising as above in finitely many steps are definite conditions.

Note:
"If P then Q", denoted $P \to Q$ is definite as it can be expressed as $\neg P \vee Q$.

Want our existence axioms to be of the form:

"There exists a set N such that $t \in N \iff P(t)$"

where P is a definite condition.

Infinity Axiom

There exists a set I that contains 0 and is preserved by the successor function, i.e., I satisfies:

$$(0 \in I) \wedge (\forall x(x \in I \to S(x) \in I))$$

where $S(x) \in I$ means

$$\exists y((y \in I) \wedge \underbrace{\forall t((t \in y) \leftrightarrow ((t \in x) \vee (t = x)))}_{y=S(x)}).$$

definite operation

An operation $H(x)$ is **definite** if the condition $y = H(x)$ is definite.

inductive set

We call a set I **inductive** if it contains 0 and is closed under S.

We want the "smallest" inductive set. How? Let's fix an inductive set I. Consider

$$\bigcap \{J \subseteq I : J \text{ is inductive}\}$$

Exercise:
Given a non-empty set x there exists (using the first four axioms[a]) a set $\bigcap x$ satisfying

$$\forall t(t \in \bigcap x \leftrightarrow \forall y(y \in x \to t \in y)).$$

And think about what happens if $x = \varnothing$.

The proof can be found in https://math.stackexchange.com/a/949032.

[a]plus one more separation axiom stated later, from *Axiomatic Set Theory* by Suppes

Separation axiom

Let A a set, and for each $x \in A$, let $\varphi(x)$ a property pertaining x. There exists a set $C := \{x \in A : \varphi(x) \text{ is true}\}$, whose elements are precisely the elements x in A for which $\psi(x)$ is true.

$$\exists C\, \forall x\, (x \in C \iff x \in A \,\wedge\, \varphi(x)).$$

$x \subseteq y$

$x \subseteq y$ means every member of x is a member of y.

$$\forall t(t \in x \to t \in y)$$

Powerset Axiom

Give a set x there exists a set $\mathcal{P}(x)$ satisfying

$$\forall t(t \in \mathcal{P}(x) \leftrightarrow \underbrace{\forall y(y \in t \to y \in x)}_{t \subseteq x}).$$

Bounded Separation Axiom

Suppose x is a set and P is a definite condition. Then there exists a set y satisfying:

$$\forall t(t \in y \leftrightarrow ((t \in x) \wedge P(t)))$$

Notation: $y = \{t \in x | P(t)\}$, which is of course a subset of x.

Note:
P must be definite.

Bounded: must start with a set x as a domain.

Russell's Paradox

If we were allowed unbounded separation then we could consider the set $R = \{t : t \notin t\}$.

Is $R \in R$? If $R \in R$ then $R \notin R$, contradiction. If $R \notin R$ then $R \in R$, contradiction.

Problem: R does *not* exist as a set.

Recall we want to define the set of natural numbers as

$$\bigcap \{J \subseteq I : J \text{ is inductive}\}$$

where I is a fixed inductive set.

Construct the set of natural numbers, ω, as follows: Fix an inductive set I. Then

$$\omega := \bigcap \{J \in \mathcal{P}(I) | (0 \in J) \wedge \forall t(t \in J \to S(t) \in J)\}$$

Exercise:
Prove that ω does *not* depend on I.

Replacement Axiom

Suppose P is a definite binary condition such that for every set x there is a unique set y such that $P(x, y)$. Given any set A there exists a set B with the property

$$y \in B \iff \exists x((x \in A) \wedge P(x, y)).$$

In other words, image of a set under a definite operation exists as a set.

ZF = **Zermelo-Fraenkel Set Theory** is these 8 axioms plus "regularity". We won't assume regularity. From wiki,

Regularity axiom

Every non-empty set A contains an element that is disjoint from A. In first-order logic, the axiom reads

$$\forall x\,(x \neq \varnothing \to \exists y \in x\,(y \cap x = \varnothing)).$$

1.2 Classes

A **class** is a collection of sets satisfying some definite property. Another way to think about it is what you get if you apply unbounded separation. If P is a definite condition, then $[[z : P(z)]]$ is a **class**.

Remark:

1. All sets are classes. If x is a set, $x = [[t : t \in x]]$.
2. Some classes are sets. $[[z : z \in \omega]] = \omega$.
3. Not all classes are sets. $R = [[t : t \notin t]]$, this Russell class is *not* a set.

A **proper class** is a class that is not a set.

$U = [[t : t = t]]$, the universal class is not a set.

Proof:
If U is a set, then $R = \{t \in U : t \notin t\}$ would be bounded separation. □

Two classes are equal if and only if they have the same members.

Note:
Membership is a binary relation between a set and a class:

$$\underset{\text{set}}{x} \in \underset{\text{class}}{Y}$$

It does not make sense to talk about a class being a member of another class.

Given sets x, y we define the **ordered pair** $(x, y) := \{\{x\}, \{x, y\}\}$.

With this definition, we have a property: $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$.

Suppose X, Y are classes. The **cartesian product** of X and Y is defined to be

$$X \times Y := [[z : z = (x, y) \text{ for some } x \in X, y \in Y]]$$

Note that this is a definite condition since we can rewrite the definition as:

If $X = [[x : P(x)]]$, $Y = [[y : Q(y)]]$, then

$$X \times Y = [[z : \exists x \exists y (P(x) \wedge Q(y) \wedge z = \{\{x\}, \{x, y\}\})]]$$

Therefore, the cartesian product is class.

If X, Y are sets, then $X \times Y$ is a set since

$$X \times Y = \{z \in \mathcal{P}(\mathcal{P}(X \cup Y)) : z = (x, y) \text{ for some } x \in X, y \in Y\}$$

Since we have $\mathcal{P}(\mathcal{P}(X \cup Y))$ as a set, we can use bounded separation.

Given classes X, Y, by **definite operation** $f : X \to Y$, we mean a subclass $\Gamma(f) \subseteq X \times Y$ such that for all $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in \Gamma(f)$, where we read $\Gamma(f)$ as "the graph of f".

Here we are identifying the operation f with it graph $\Gamma(f)$. We often write $f(x) = y$ instead of $(x, y) \in \Gamma(f)$.

Example:

$S : \mathsf{Sets} \to \mathsf{Sets}$ where S is the successor function, and

$$\begin{aligned} \mathsf{Sets} :&= \text{class of all sets} \\ &= [[z : z = z]] \end{aligned}$$

Proof:

$\Gamma(S) = [[z \mid z = (x, y) \wedge \forall t (t \in y \leftrightarrow (t \in x) \vee (t = x))]]$

$\Gamma(S) \subseteq \mathsf{Sets} \times \mathsf{Sets}$ is a subclass that satisfies the "vertical line test". Therefore S is a definite operation. □

Remark:

Suppose X, Y are sets and $f : X \to Y$ is a definite operation. Then $\Gamma(f) \subseteq X \times Y$ is a subset. In fact, if B is a set and A is a class and $A \subseteq B$ then A is a set.

Proof:

$A = [[z|P(z)]]$ with P definite. $A = \{z \in B | P(z)\}$ uses bounded separation. □

Finally, with this language, we can restate the Replacement Axiom in a cleaner way.

Replacement Axiom (restated)

Suppose $f : \mathsf{Sets} \to \mathsf{Sets}$ is a definite operation and $A \in \mathsf{Sets}$. Then there exists a set B satisfying:

$$\forall t (t \in B \leftrightarrow \exists a ((a \in A) \wedge \underbrace{(t = f(a))}_{(a,t) \in \Gamma(f)}))$$

Remark:
Though we can look at $f(A)$, that's not what we are doing here. We are looking at the members of A, so it's the image of f applied to the elements of A.

1.3 Ordering the natural numbers

Inductive Principle

Suppose $J \subseteq \omega$ is such that $0 \in J$ and if $x \in J$ then $S(x) \in J$. Then $J = \omega$.

Proof:
$J \subseteq \omega$ by assumption. But by definition of ω we have $\omega \subseteq J$. $\square$

Lemma 1.1

Suppose $n \in \omega$.

(a) If $x \in n$ then $x \in \omega$, (i.e., every element of ω is a subset of ω).

(b) If $x \in n$ then $x \subseteq n$.

(c) $n \notin n$.

(d) $n = 0$ or $0 \in n$.

(e) If $x \in n$, then $S(x) \in n$ or $S(x) = n$.

Proof:

(a) Let $J := \{n \in \omega : n \subseteq \omega\} \subseteq \omega$. This gives a definition of a set since $n \subseteq \omega$ is definite condition and bounded by a set. We want $J = \omega$. $0 \in J$ since $\varnothing \subseteq A$ for all sets A.

Suppose $n \in J$. $S(n) = n \cup \{n\} \in \omega$. Since $n \subseteq \omega$ and $n \in \omega$ we have that $S(n) \subseteq \omega$. Thus $S(n) \in J$. By the induction principle $J = \omega$.

(b) $J := \{n \in \omega : \forall x(x \in n \to x \subseteq n)\}$. We want to prove $J = \omega$. Clearly $0 \in J$ is true vacuously. Suppose $n \in J$. Consider $S(n) = n \cup \{n\}$.

Note $n \in S(n), n \subseteq S(n)$.

Let $x \in S(n)$. If $x = n$ then $x \subseteq S(n)$. Otherwise $x \in n$, then $x \subseteq n$ since $n \in J$. Together with $n \subseteq S(n)$ then we have $x \subseteq S(n)$.

Either case, we conclude $x \subseteq S(n)$, then $S(n) \in J$. Hence by inductive principle we have $J = \omega$. $\square$

Conditions a, b and c imply that $(\omega, \in)$ is a strict poset.

strict partial ordering

A **strict partial ordering** on a set E is a binary relation R on E satisfying:

(i) Antireflexivity: $\neg xRx$ for any $x \in E$.

(ii) Antisymmetry: If xRy and yRx then $x = y$.

(iii) Transitivity: If xRy and yRz then xRz.

Proposition 1.2

$\in$ is a strict partial ordering on ω.

Proof:
Part (c) of Lemma 1.1 is precisely antireflexivity.

Suppose $n, m \in \omega$ such that $n \in m$ and $m \in n$. By part (b) of Lemma 1.1 $n \subseteq m$ and $m \subseteq n$. Thus $n = m$.

Suppose $\ell, m, n \in \omega$, where $\ell \in m$ and $m \in n$. Since $m \in n$ we have $m \subseteq n$ by part (b). Hence $\ell \in n$. □

total ordering

A strict partial ordering R on E is **linear** or **total** if whenever $x, y \in E$ we have xRy or yRx or $x = y$.

Lemma 1.3

$(\omega, \in)$ is a linear ordering.

Proof:
Fix $n \in \omega$. Consider $J := n \cup \{m \in \omega : n \in m\} \cup \{n\}$. Here J is made up of those things that are membership less than n or membership bigger than n or membership equal to n. We want to prove $J = \omega$.

Note that if $n = 0$ then $J = \omega$ from part (d) of Lemma 1.1, since every n is either 0 or contains 0. So we may assume $n \neq 0$. Hence again by (d), $0 \in n$, then $0 \in J$.

Also note that $J \subseteq \omega$ since $n \subseteq \omega$ by part (a) of Lemma 1.1.

Suppose $m \in J$.

- If $m \in n$, by part (e) of Lemma 1.1, we have $S(m) \in n$ or $S(m) = n$. Both cases imply $S(m) \in J$.
- If $n \in m$, then $n \subseteq S(m)$. So $S(m) \in J$.
- If $m = n$, then $S(m) = S(n) \ni n$, therefore $S(m) \in J$.

By inductive principle, we have $J = \omega$. □

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