Game Theory

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Preface

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Combinatorial games

1.1 Impartial games

> **Reference**
> - http://web.mit.edu/sp.268/www/nim.pdf
> - https://ivv5hpp.uni-muenster.de/u/baysm/teaching/3u03/notes/14-games.pdf

> **Example: Game of Nim**
>
> We are given a collection of piles of chips. Two players play alternatively. On a player's turn, they remove at least 1 chip from a pile. First player who cannot move loses the game.
>
> For example, we have three piles with 1, 1, 2 chips. Is there a winning strategy? In this case, there is one for the first player: Player I (p1) removes the pile of 2 chips. This forces p2 to move a pile of 1 chip. p1 removes the last chip. p2 has no move and loses the game. In this case, p1 has a winning strategy, so this is a **winning game** or **winning position**.
>
> Now let's look at another example with two piles of 5 chips each. Regardless of what p1 does, p2 can make the same move on the other pile. p1 loses. If p1 loses regardless of their move (i.e., p2 has a winning strategy), then this is a **losing game** or **losing position**.
>
> What if we have two piles have unequal sizes? say 5, 7. p1 moves to equalize the chip count (remove 2 from the pile of 7). p2 then loses, this is a winning game.

> **Lemma 1.1**
>
> In instances of Nim with two piles of n, m chips, it is a winning game if and only if $m \neq n$.

Solving Nim with only two piles is easy, but what about games with more than two piles?

This is more complicated.

Nim is an example of an **impartial game**. Conditions required for an impartial game:

1. There are 2 players, player I and player II.

2. There are several positions, with a starting position.

3. A player performs one of a set of allowable moves, which depends only on the current position, and not on the player whose turn it is. ("impartial") Each possible move generates an option.

4. The players move alternately.

5. There is complete information.

6. There are no chance moves.

7. The first player with no available move loses.

8. The rules guarantee that games end.
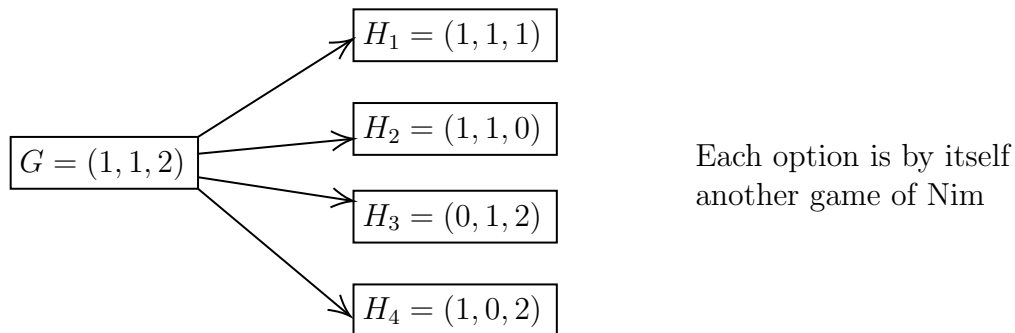
Example: Not an impartial game

Tic-tac-toe: violates 7.

Chess: violates 3, since players can only move their own pieces.

Monopoly: violates 6. Poker: violates 5.

Example:

Let $G = (1, 1, 2)$ be a Nim game. There are 4 possible moves (hence 4 possible options):

$$H_1 = (1, 1, 1)$$
$$H_2 = (1, 1, 0)$$
$$G = (1, 1, 2)$$
$$H_3 = (0, 1, 2)$$
$$H_4 = (1, 0, 2)$$

Each option is by itself another game of Nim

Note:
We can define an impartial game by its position and options recursively.

> **simpler**
>
> A game H that is reachable from game G by a sequence of allowable moves is **simpler** than G.
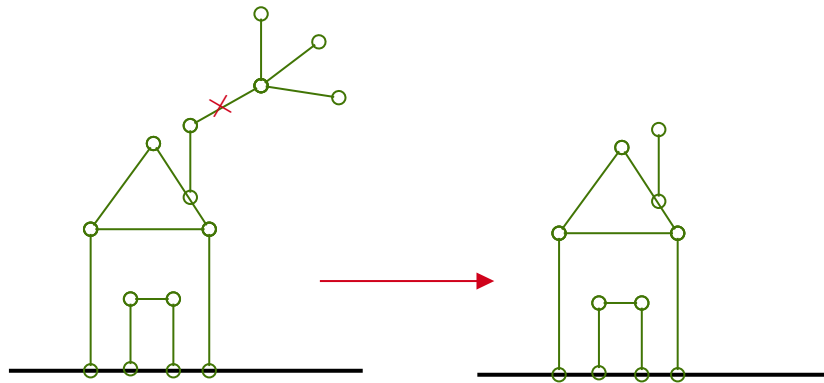
Other impartial games:

1. Subtraction game: We have one pile of n chips. A valid move is taking away 1, 2, or 3 chips. The first player who cannot move loses.

2. Rook game: We have an $m \times n$ chess board, and a rook in position (i, j). A valid move is moving the rook any number of spaces left or up. The first player who cannot move loses.
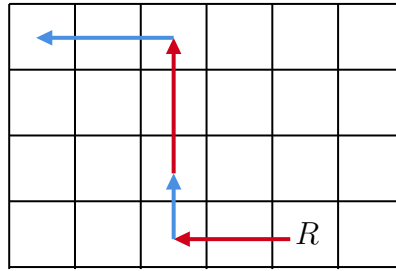
3. Green hackenbush game: We have a graph and the floor. The graph is attached to the floor at some vertices. A move consists of removing an edge of the graph, and any part of the graph not connected to the floor is removed. The first player who cannot move loses.

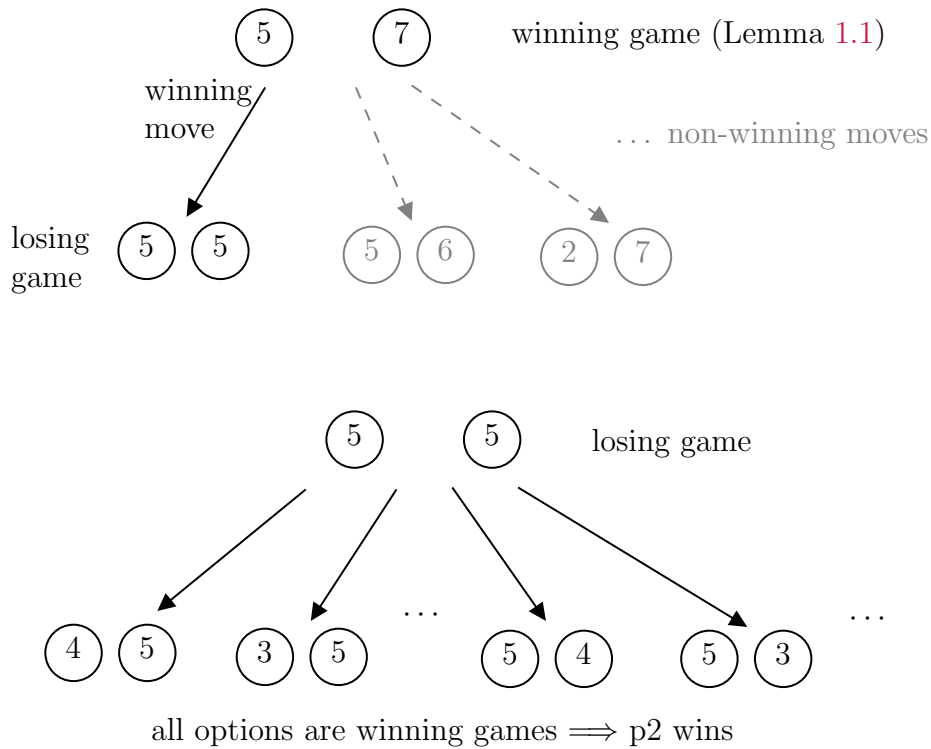**Spoiler** A main result we will prove is that all impartial games are essentially like a Nim game.

Lemma 1.2

In any impartial game G, either player I or player II has a winning strategy.

Proof:
We prove by induction on the simplicity of G. If G has no allowable moves, then p1 loses, so p2 has a winning strategy. Assume G has allowable moves and the lemma holds for games simpler than G. Among all options of G, if p1 has a winning strategy in one of them, then p1 moves to that option and wins. Otherwise, p2 has a winning strategy for all options. So regardless of p1's move, p2 wins. □

So every impartial game is either a winning game (p1 has a winning strategy) or a losing game (p2 has a winning strategy).

Example: Nim

winning game (Lemma 1.1)

winning move

... non-winning moves

losing game

losing game

all options are winning games $\implies$ p2 wins

Note:
We assume players play perfectly. If there is a winning move, then they will take it.

1.2   Equivalent games

> **game sums**
>
> Let G and and H be two games with options $G_1, \ldots, G_m$ and $H_1, \ldots, H_n$ respectively. We define $G + H$ as the games with options
>
> $$G_1 + H, \ldots, G_m + H, G + H_1, \ldots, G + H_n.$$

Example:

We denote $*n$ to be a game of Nim with one pile of n chips. Then $*1 + *1 + *2$ is the game with 3 piles of 1, 1, 2 chips.

Example:

If we denote $\#2$ to be the subtraction game with n chips, then $*5 + \#7$ is a game where a move consists of either removing at least 1 chip from the pile of 5 (Nim game), or removing 1, 2 or 3 chips from the pile of 7 (subtraction game).

Lemma 1.3

Let $\mathcal{G}$ be the set of all impartial games. Then for all $G, H, J \in \mathcal{G}$,

1. $G + H \in \mathcal{G}$ (closure)

2. $(G + H) + J = G + (H + J)$ (associative)

3. There exists an identity $0 \in \mathcal{G}$ (game with no options) where $G + 0 = 0 + G = G$

4. $G + H = H + G$ (symmetric)

Note:
This is an abelian group except the inverse element.

equivalent game

Two games G, H are **equivalent** if for any game J, $G + J$ and $H + J$ have the same outcome (i.e., either both are winning games, or both are losing games).

Notation: $G \equiv H$.

Example:
$*3 \equiv *3$ since $*3 + J$ is the same game as $*3 + J$ for any J, so they have the same outcome.

$*3 \not\equiv *4$ since $*3 + *3$ is a losing game, but $*4 + *3$ is a winning game from Lemma 1.1.

Lemma 1.4

$*n \equiv *m$ if and only if $n = m$.

Lemma 1.5

The relation $\equiv$ is an equivalence relation. That is, for all $G, H, K \in \mathcal{G}$,

1. $G \equiv G$ (reflexive)

2. $G \equiv G$ if and only if $H \equiv G$ (symmetric)

3. If $G \equiv H$ and $H \equiv K$, then $G \equiv K$ (transitive).

Exercise:
Prove that if $G \equiv H$, then $G + J \equiv H + J$ for any game J.

Note that the definition above only says they have the same outcome. To prove that they are equivalent, one needs to add another game on both sides to show they have the same outcome.

Nim with one pile $*n$ is a losing game if and only if $n = 0$.

Theorem 1.6

G is a losing game if and only if $G \equiv *0$.

Proof:

$\Leftarrow$ If $G \equiv *0$, then $G + *0$ has the same outcome as $*0 + *0$. But $*0$ is a losing game, so G is a losing game.

$\Rightarrow$ Suppose J is a losing game. (We want to show $G \equiv *0$, meaning $G + J$ and $*0 + J \equiv J$ have the same outcome.)

1. Suppose J is a losing game. (We want to show that $G + J$ is a losing game.)

   We will prove "If G and J are losing games, then $G + J$ is a losing game" by induction on the simplicity of $G + J$. When $G + J$ has no options, then G, J both have no options, so $G, J, G + J$ are all losing games.

   Suppose $G + J$ has some options. Then p1 makes a move on G or J. WLOG say p1 makes a move in G, and results in $G' + J$. Since G is a losing game, G' is a winning game. So p2 makes a winning move from G' to G'', and this results in $G'' + J$. Then G'' is a losing game, so by induction, $G'' + J$ is a losing game for p1. So p1 loses, and $G + J$ is a losing game.

2. Suppose J is a winning game. Then J has a winning move to J'. So p1 moves from $G + J$ to $G + J'$. Now both G, J' are losing games, so by case 1, $G + J'$ is a losing game. So p2 loses, meaning p1 wins, so $G + J$ is a winning game.

$\square$

Corollary 1.7

If G is a losing game, then J and $J + G$ have the same outcome for any game J.

Proof:

Since G is a losing game, $G \equiv *0$ by Theorem 1.6. Then $J + G \equiv J + *0 \equiv J$ (previous exercise + Lemma 1.3). So J and $G + J$ have the same outcome. $\square$

Example:

1. Recall $*5 + *5$ and $*7 + *7$ are losing games. Then Corollary 1.7 says $*5 + *5 + *7 + *7$ is also a losing game. (p1 moves in either $*5 + *5$ or $*7 + *7$. Then p2 makes a winning move from the same part, equalizing piles.)

2. $\underbrace{*1 + *1 + *2}_{\text{winning}} + \underbrace{*5 + *5}_{\text{losing}}$. Corollary 1.7 implies this is a winning game.

   (p1 makes a winning move in $*1 + *1 + *2$, therefore we have $\underbrace{*1 + *1}_{\text{losing}} + \underbrace{*5 + *5}_{\text{losing}}$.
   p2 loses.)

Lemma 1.8: Copycat principle

For any game G, $G + G \equiv *0$.

Proof:
Induction on the simplicity of G. When G has no options, $G + G$ has no options, so $G + G \equiv *0$ by Theorem 1.6. Suppose G has options, and WLOG suppose p1 moves from $G + G$ to $G' + G$. Then p2 can move to $G' + G'$. By induction, $G' + G' \equiv *0$, so it is a losing game for p1. Therefore, $G + G$ is a losing game, and $G + G \equiv *0$.   □

Lemma 1.9

$G \equiv H$ if and only if $G + H \equiv *0$.

Proof:

$\Rightarrow$ From $G \equiv H$, we add H to both sides to get $G + H \equiv H + H \equiv *0$ by the copycat principle.

$\Leftarrow$ From $G + H \equiv *0$, we add H to both sides to get $G + H + H \equiv *0 + H \equiv H$. But $G + G + G \equiv G + *0 \equiv G$ by the copycat principle. So $G \equiv H$.   □

Example:
$*1 + *2 + *3$ is a losing game, so $*1 + *2 + *3 \equiv *0$. By Lemma 1.9, $*1 + *2 \equiv *3$, or $*1 + *3 \equiv *2$.

Another way to prove game equivalence is by showing that they have equivalent options.

Lemma 1.10

If the options of G are equivalent to options of H, then $G \equiv H$. (More precisely: There is a bijection between options of G and H where paired options are equivalent.)

Proof:
It suffices to show that $G + H \equiv *0$ by Lemma 1.9, i.e., $G + H$ is a losing game. This is true when G, H both have no options. Suppose G, H have options, and suppose WLOG p1 moves to $G'H$. By assumption, there exists an options of H, say H', such that $H' \equiv G'$. So p2 can move to $G' + H'$. Since $G' \equiv H'$, $G' + H' \equiv *0$ by Lemma 1.9. So $G' + H'$ is a losing game for p1. Hence $G + H$ is a losing game.   □

Example:
We can show $*1 + *2 \equiv *3$ using Lemma 1.10.
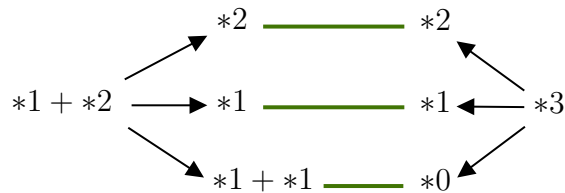
Note: The converse is false.

1.3   Nim and nimbers

Goal   Show that every Nim game is equivalent to a Nim game with a single pile.

> **nimber**
>
> If G is a game such that $G \equiv *n$ for some n, then n is the **nimber** of G,

Example:

Any losing game has nimber 0 by Theorem 1.6.

Exercise:

Show that the notion of a nimber is well-defined.

> **Theorem 1.11**
>
> Suppose $n = 2^{a_1} + 2^{a_2} + \dots$ where $a_1 > a_2 > \dots$, then $*n \equiv *2^{a_1} + *2^{a_2} + \dots$.

Example:

$11 = 2^3 + 2^1 + 2^0$, $13 = 2^3 + 2^2 + 2^0$. Using this theorem, $*11 \equiv *2^3 + *2^1 + *2^0$ and $*13 \equiv *2^3 + *2^2 + *2^0$. Then

$$
\begin{aligned}
*11 + *13 &\equiv (*2^3 + *2^1 + *2^0) + (*2^3 + *2^2 + *2^0) \\
&\equiv (*2^3 + *2^3) + *2^2 + *2^1 + (*2^0 + *2^0) \quad \text{by assoc'y and commu'y} \\
&\equiv *0 + *2^2 + *2^1 + *0 \quad \text{by copycat principle} \\
&\equiv *2^2 + *2^1 \\
&\equiv *(2^2 + 2^1) \\
&\equiv *6
\end{aligned}
$$

So the nimber of $*11 + *13$ is 6.

In general, how can we find the nimber for $*b_1 + *b_2 + \dots + *b_n$? Look for binary expansions of each b_i. Copycat principle cancels any pair of identical powers of 2. So we look for powers of 2's that appear in odd number of expansions of the b_i's.

Use binary numbers: 11 in binary is 1011, 13 in binary is 1101. Take the XOR operation.

$$
\begin{array}{r}
1011 \\
\oplus \quad 1101 \\
\hline
0110
\end{array}
\quad \text{and 0110 is 6. So } 11 \oplus 13 = 6.
$$

Example:

Consider $*25 + *21 + *11$. In binary they are $11001, 10101, 01011$.

$$
\begin{array}{r}
11001 \\
10101 \\
\oplus \quad 01011 \\
\hline
00111
\end{array}
\quad \text{and 00111 is 7. So } *25 + *21 + *11 \equiv *7. \text{ (The nimber is 7)}
$$

> **Corollary 1.12**
>
> $*b_1 + *b_2 + \dots + *b_n \equiv *(b_1 \oplus b_2 \oplus \dots \oplus b_n)$.

This shows that every Nim game has a nimber.

Winning strategy for Nim

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