ODE 2

AMATH 351

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Preface

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Sibelius Peng

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Introduction and Review

1.1 Definitions and Terminology

A **differential equation** is any equation involving a function and derivatives of this function.

Ordinary differential equations contain only functions of a single variable, called the independent variable, and derivatives with respect to that variable.

Partial differential equations contain a function of two or more variables and some partial derivatives of this function.

The **order** of a differential equation in the order of the highest derivative in the equation.

A general n-th order ODE has the form

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1.1}$$

where $y' = \dfrac{dy}{dx}, y'' = \dfrac{d^2y}{dx^2}$ and so on. We assume further it can be written as

$$y^{(n)} = f(x, y', \ldots, y^{(n-1)}). \tag{1.2}$$

Eq. (1.2) is said to be **linear** when f is a linear function of $y, y', \ldots, y^{(n-1)}$. In this case, Eq. (1.2) can be written as

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x). \tag{1.3}$$

A differential equation that is not linear is said to be **nonlinear**.

By a **solution** of Eq. (1.2) on an interval I we mean a function $y = \psi(x)$ such that $f(x, \psi(x), \psi'(x), \ldots, \psi^{(n-1)(x)})$ is defined for all x in I and is equal to $\psi^{(n)}(x)$ for all x in I.

A solution in which the dependent variable in expressed only in terms of the independent variable and constants is called an **explicit solution**.

A relation $G(x, y) = 0$ such that there exists at least one function $\psi(x)$ that satisfies the relation and Eq. (1.2) is called an **implicit solution**.

A solution which is free of arbitrary constants is called a **particular solution**.

A solutions that cannot be obtained by specializing any of the parameters in a family of solutions is called a **singular solution**.

Example:

Consider the DE $y' = xy^{1/2}$.

The explicit solution: $y = \left(\frac{x^2}{4} + c\right)^2$

A particular solution is $y = \frac{x^4}{16}$ obtained above for $c = 0$.

A singular solution is $y = 0$ which cannot be obtained from the explicit solution for any choice of constant c.

1.2 Initial-Value Problems

On some interval containing x_0, the problem

Solve $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$ subject to the initial conditions $y(x_0) = y_0, \dots, y^{(n-1)}(x_0) = y_{n-1}$

where $y_0, \dots, y_{n-1}$ are arbitrary specified real constants, is called an **initial-value problem** (IVP).

Consider the IVP $y' = f(x, y)$, $y(x_0) = y_0$.

Theorem 1.1: Picard

Let D be a rectangular region in the xy-plane defined by $D = \{(x, y) : a \le x \le b, c \le y \le d\}$ and $(x_0, y_0) \in D$ the interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on D, then IVP has a unique solution $y(x)$ defined in an interval I centered at x_0.

1.3 First Order ODE

Separable variables

A first order DE of the form

$$\frac{dy}{dx} = g(x)h(y) \tag{1.4}$$

is said to be **separable** or to have **separable variables**. Solution method:

$$\frac{dy}{h(y)} = g(x)dx$$

Integrate both sides

$$\int \frac{1}{h(y)} dy = \int g(x)dx + C$$

Linear equations

A first order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1.5}$$

is called a **linear equation**.

Solution method:

- Write in its **standard form**
$$\frac{dy}{dx} + p(x) = f(x)$$
- Multiply both sides by the integrating factor $\mu(x) = \exp\left(\int p(x)dx\right)$, and rearrange into the exact form $\frac{d}{dx}(\mu(x)y) = \mu(x)f(x)$

- Integrate both side with respect to x and get the general solution under the form

$$y(x) = \frac{1}{\mu(x)} \left(\int \mu(x) f(x) dx + C \right)$$

There are other type of ODEs that you learned how to solve in AMATH 251, such as homogeneous equations, exact equations, Bernouli equations.

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Theory of Second-Order Linear DEs

2.1 2nd-Order Linear ODEs

The most general 2nd order linear DE is

$$a_2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

In AMATH 251 we learned how to solve this equation where the coefficients a_2, a_1, a_0 are constants. This equation can be written in several different forms:

1. General form
$$a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{2.1}$$

2. Standard form: If $a_2(x)$ is not identically zero then we obtain
$$y'' + P(x)y' + Q(x)y = R(x) \tag{2.2}$$

3. Associated homogeneous equation: This is the same as the standard form where RHS is zero,
$$y'' + P(x)y' + Q(x)y = 0 \tag{2.3}$$

   If RHS of Eq. (2.2) is non-zero the equation is said to be non-homogeneous or inhomogeneous.

2.2 Existence and Uniqueness

Existence and Uniqueness Before we try and find solutions to the DEs it is usually a good idea to know that a solution exists and it is unique. Otherwise we could be wasting out time. We state a theorem for existence and uniqueness. The ideas of the proof will be presented later when we discuss first-order systems.

Theorem 2.1: Existence and Uniqueness

Let $P(x), Q(x)$ and $R(x)$ be continuous functions on a closed interval $[a, b]$. If x_0 is any point in $[a, b]$ and if $y(x_0)$ and $y'(x_0)$ are any numbers, then Eq. (2.2) has one and only one solution $y(x)$ on the entire interval such that the initial conditions (ICs) are satisfied.

Remark:

If we are looking for a solution to the homogeneous equation with $y(0) = 0, y'(0) = 0$ observe that the trivial solution is an allowable solution. Therefore, by the existence and uniqueness theorem, it must be the only solution.

2.3 General Solutions to 2nd-order DEs

In AMATH 251 for the case of constant coefficients, we learned that the general solution to Eq. (2.2) is a superposition of any particular solution to the non-homogeneous problem and a general solution to the homogeneous one. This also holds true in the case of non-constant coefficients. Therefore, the method of attack is as follows:

1. Find the general solution to the homogeneous problem: In the case of constant coefficients we simply sub $y = e^{rx}$ find the characteristic equation, solve for the characteristic roots, r_1, r_2 form the two independent solutions $y_1(x), y_2(x)$ and get that the general solution is,

   $$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

   where c_1, c_2 are arbitrary constants. In the case of non-constant coefficients we need to do more work to find y_1, y_2. In general we cannot find them explicitly.

2. Find a particular solution to the non-homogeneous problem. There are different methods that we can use.

The following Theorems will help us to find a unique solution of a general second-order scalar equation. First we look at the homogeneous problem and then at the more general DE.

Theorem 2.2: General solutions to 2nd-order homogeneous equations

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the homogeneous equation (Eq. (2.3)) on the interval $[a, b]$, then the general solution to the same homogeneous problem is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x),$$

for arbitrary constants c_1, c_2.

Proof:
First we can sub y_1, y_2 and their linear superposition into the homogeneous equation to verify they are solutions.

Second we need to verify that this solution can satisfy any set of conditions, say $y(0)$ and $y'(0)$ [a]. We sub in our solution and find,

$$c_1 y_1(0) + c_2 y_2(0) = y(0),$$
$$c_1 y_1'(0) + c_2 y_2'(0) = y'(0).$$

This is a system of two equations and two unknowns c_1, c_2. To be able to solve this for any initial conditions we need that the matrix is non-singular,

$$\det \begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} = y_1(0)y_2'(0) - y_2(0)y_1'(0) \neq 0$$

This motivates the definition of **Wronskian**, $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$.

To ensure that our expression is a general solution we need that the initial value of the Wronskian is nonzero, $W(y_1(0), y_2(0)) \neq 0$. □

[a] These should be replaced by $y(x_0) = y_0, y'(x_0) = y_1$ for some $x_0 \in [a, b]$

Also check the alternative proof on page 66 of `https://notes.sibeliusp.com/pdfs/1189/amath251.pdf`.

Therefore, the above tells us that if the initial value of the Wronskian of the two solutions is non-zero, we have a general solution. Next, we will show that if the Wronskian is non-zero at the initial time it is necessarily non-zero all time. The following theorem states and proves this result.

Lemma 2.3: Uniformity of the Wronskian

If $y_1(x)$ and $y_2(x)$ are any two solutions to Eq. (2.3) on the interval $[a, b]$ then their Wronskian is either identically zero or never zero on $[a, b]$.

Proof:

$$\begin{aligned} W' &= y_1 y_2'' - y_2 y_1'' \\ &= y_1 \left[-P(x)y_2' - Q(x)y_2\right] - y_2 \left[-P(x)y_1' - Q(x)y_1\right] \\ &= -P(x) \left[y_1 y_2' - y_2 y_1'\right] \\ &= -P(x)W \end{aligned}$$

Then $W = W_0 \exp\left(-\int P(x)dx\right)$ is either zero everywhere or zero nowhere, depending on its initial values. □

So far we know that the Wronskian is always zero or always non-zero for any two solutions. The next lemma shows the relation between the Wronskian and linear independence.

Lemma 2.4: Linear Dependence and the Wronskian

If $y_1(x)$ and $y_2(x)$ are two solutions of the homogeneous equation then they are linearly dependent on this interval if and only if their Wronskian is identically zero.

Proof:

⇒ Let $y_2(x) = cy_1(x)$ then calculate $W(y_1, y_2) = 0$

⇐ Assume Wronskian is zero, then

$$\det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = 0$$

If the determinant of the matrix is zero that means that the matrix is singular and that each column is a scalar multiple of the other. In particular, if we look at the first row we find that $y_2(x)$ is a linear multiple of $y_1(x)$. This is precisely the statement that the functions are linearly dependent.

□

If we combine two lemmas we deduce that if two solutions are linearly independent at the initial time (or any other time for that matter) they are necessarily linear independent for all time. The next example is to give you some practice playing with the Wronskian.

Example:

Can show that $y = c_1 \sin x + c_2 \cos x$ is the general solution to $y'' + y = 0$ on any interval.

Now that we have a grasp on how to find general solutions to homogeneous equations, we can look at solving the non-homogeneous problem.

Theorem 2.5: General solutions to 2nd-order non-homogeneous equations

If $y_h(x)$ is the general solution to the homogeneous problem, Eq. (2.3), and $y_p(x)$ is any particular solution of the non-homogeneous problem, Eq. (2.2), then their superposition, $y(x) = y_h(x)+y_p(x)$, is a general solution to the non-homogeneous problem.

Proof:

To show that we have a general solution to the non homogeneous equation we must shows two things: 1) that it is in fact a solution and 2) we can reproduce any initial condition. With the second condition we can use our uniqueness theorem to guarantee that any two solutions to the DE with the same initial conditions must be equal.

1. skipped.

2. Now we must show that we can reproduce any IC with this solution. We do this as we did before by evaluating our solution and its derivatives at the initial time. First we do as before and define $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ and then we get,

$$\begin{aligned} c_1 y_1(0) + c_2 y_2(0) &= y(0) - y_p(x) \\ c_1 y_1'(0) + c_2 y_2'(0) &= y'(0) - y_p'(x) \end{aligned}$$

   Note that the effect of the particular solution is to offset how we pick our constants c_1 and c_2. By assumption, we have that our homogeneous solution consists of two linearly independent solutions, y_1 and y_2, and so their Wronskian is non-zero and so we can invert this 2×2 system to find a unique solution. This is enough to guarantee that our solution is a general solution to the non-homogeneous system.

□

2.4 BVPs versus IVPs

ODEs can be classified as either **Boundary Value Problems** (BVPs) or **Initial Value Problems** (IVPs). The equations themselves can be the same, what differs are the conditions that are imposed to determine the unknown constants. For IVPs the two conditions are imposed at the same time, for example $y(0) = \alpha, y'(0) = \beta$.

In contrast, in BVPs the two conditions are imposed at different times, or different locations, $y(0) = \alpha, y(1) = \beta$.

As the names suggest, IVPs usually have time as the independent variable and BVPs usually have space as the independent variable.

2.5 Examples of 2nd-Order DEs with non-constant coefficients

There is a long list of DEs with non-constant coefficients. Some of them are particularly famous and have special names. The solutions usually cannot be written in terms of simple functions and we define functions to be the solutions to such equations. They are usually referred to as **special functions**. Many of them arise in looking at solutions to Laplace's equations in different co-ordinate systems. The interested reader is directed to AM 353 for more details on how to obtain these equations.

1. Bessel's equation: p is a constant integer.

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

2. Legendre's equation: p integer.

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

3. Laguerre's equation: constant a

$$xy'' + (1 - x)y' + ay = 0$$

4. Hermite's equation: constant a

$$y'' - 2xy' + 2ay = 0$$

2.6 Reduction of Order

There does not exist a general approach to find a solution for any general 2nd-order ODE. However, if we manage to find one solution to the homogeneous problem, say $y_1(x)$, there is a useful technique that allows us to find a second solution, $y_2(x)$. This is called Reduction of Order. The idea is very simple really. Look for a solution that is a product of some unknown function (that we have to determine) multiplied by the known solution, i.e.

$$y_2(x) = v(x)y_1(x)$$

some intermediate work... Sub it into DE and get

$$v' = \frac{1}{y_1^2}\exp\left(-\int P(x)dx\right)$$

Then

$$v = \int_0^x \frac{1}{y_1^2}\exp\left(-\int P(s)ds\right)\,dx$$

Therefore $y_2 = \displaystyle\int \frac{1}{y_1^2}\exp\left(-\int P(s)ds\right)\,dx$.

Then we can show y_2, y_1 are linearly independent.

Example:

$y_1 = x^2$ is an exact solution to the homogeneous DE $x^2y'' + xy' - 4y = 0$. Then by the procedure above, $v = -\frac{1}{4x^4}$. Hence $y_2 = \frac{1}{x^2}$. Thus the general solution is $y(x) = c_1x^2 + \frac{c_2}{x^2}$.

2.7 Method of Variation of Parameters

To find the particular solution to a non-homogeneous equation we can always use the method of variation of parameters.

Suppose that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions to the homogeneous problem. Next, we look for a trial solution to the non-homogeneous that is similar to the general solution to the homogeneous problem except that the constant coefficients are replaced with unknown functions, $v_1(x)$, $v_2(x)$, to be determined:

$$y = v_1y_1 + v_2y_2$$

A convenient choice to pick v_1, v_2 is

$$\boxed{v_1'y_1 + v_2'y_2 = 0}$$

Sub them into $y'' + Py' + Qy = R$ and we get in matrix form:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ R \end{bmatrix}$$

y_1, y_2 are linearly independent, as we have assumed, that the system is invertible. The solution is,

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \frac{R}{W}\begin{bmatrix} -y_2 \\ y_1 \end{bmatrix}$$

Then solution is thus

$$y = y_1\int^x \frac{-y_2R(x)}{W(y_1, y_2)}dx + y_2\int^x \frac{y_1R(x)}{W(y_1, y_2)}dx$$

In summary, given two linearly independent solutions to the homogeneous problem we can find a particular solution to the non-homogeneous equation where the inhomogeneity is $R(x)$.

Example:

Find a particular solution of $y'' + y = \csc x$.

We know homogeneous solution $y = c_1\sin x + c_2\cos x$. Using the formula above, we have

$$y = \sin x\log(\sin x) - x\cos x$$

3

Series Solutions and Special Functions

This Chapter discusses how to construct power series solutions of second-order ODEs with possibly non-constant coefficients. First we review some basic their of power series that is taught in MATH 138, then we show how to construction series solutions to various types of ODEs: ordinary points and singular points. When solving these equations we obtain special functions that arise naturally as solutions to famous equations.

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