Applied Real Analysis

AMATH 331

Park Heng Henry Shum

LaTeXed by Sibelius Peng

Preface

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Sibelius Peng

Contents

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Real Numbers

Refs 1 for review. 2.1-2.2, 2.9

1.1 Decimal expansions and the real number line

finite decimal expansion

A finite decimal expansion has the form

$$x = a_{0\circ}a_1a_2a_3\dots a_N$$

where a_0 is an integer (positive, negative or zero) for $1 \leq n \leq N$ $a_n \in \{0, 1, \dots, 9\}$

Example:

$$1_\circ 45$$

$$-38_\circ 298743$$

You can think of this as

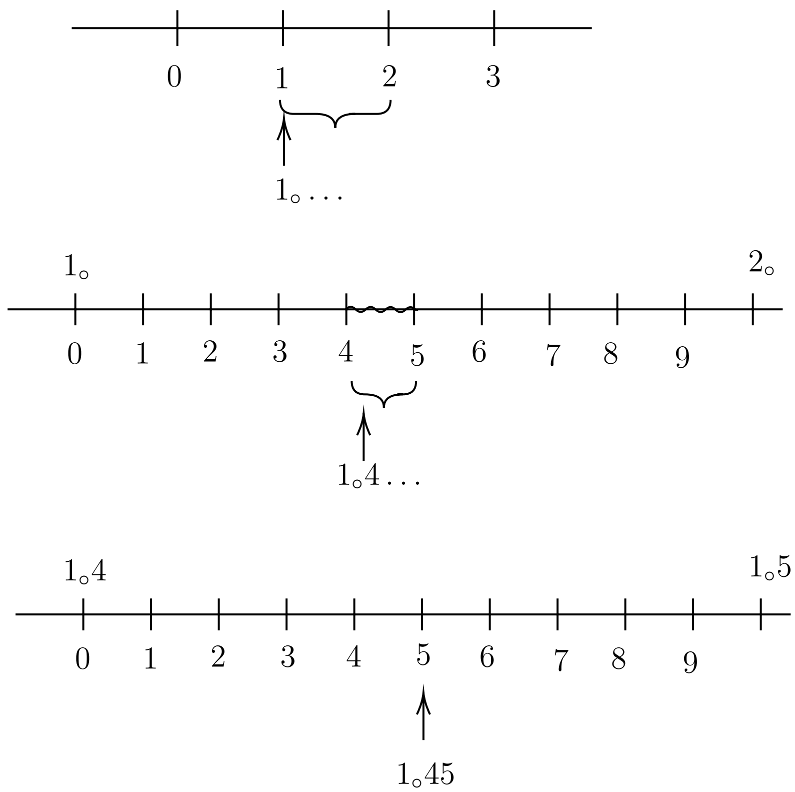
$$x = a_0 + a_1\left(\frac{1}{10}\right) + \dots + a_N\left(\frac{1}{10^N}\right)$$

Warning This looks like the usual decimal representation but it is not the same for negative numbers.

Any finite decimal expansion can be replaced on the real number line.

Example:

Where is $1_\circ 45$?



We can similarly define infinite decimal expansions

infinite decimal expansions

$$x = a_{0\circ} a_1 a_2 \ldots$$

Example:

$$1_\circ 450000000\ldots$$

$$\pi = 3_\circ 1415926535\ldots$$

Assuming the real number line has no gaps, every infinite decimal expansion x corresponds to a point on the line.

Given any positive integer k, let $y = a_{0\circ} a_1 a_2 \ldots a_k$ be the finite decimal expansion of x to the k-th decimal space. Then, x lies in the interval from y to $(y + 10^{-k})$. So, y approximates x to an accuracy of $1/10^k$. As we increase k, we improve the accuracy; in fact, the error can be made arbitrarily small.

The converse direction: given a point on the real number line, can we find its decimal expansion?

Yes!

It is possible for two decimal expansions to represent the same point. This happens precisely when one ends in an infinite string of 0's.

Example:

$$1.000\ldots \qquad \text{and} \qquad 0.999\ldots$$

$$25.300\ldots \qquad \text{and} \qquad 25.2999\ldots$$

We define the real numbers $\mathbb{R}$ as the set of all infinite decimal expansions.

1.2 Ordering of real numbers

Suppose

$$x = x_{0\circ}x_1x_2x_3\ldots, \qquad y = y_{0\circ}y_1y_2y_3\ldots$$

We say that x and y are equal and write $x = y$ if infinite decimal expansions are identical or equivalent, as discussed previously.

If x and y are not equal, then we say that x are not equal, then x is *less than* y and write $x < y$ if there exists integer $k \geq 0$ such that $x_k < y_k$ and $x_i = y_i$ for $i < k$. x is *greater than* y $(x > y)$ if ...

For any two real numbers x, y, exactly one of the following holds:

$$x = y \qquad x < y \qquad x > y$$

2

Bounds and Limits

2.1 Bounded sets of real numbers

upper bound

A set $S \subseteq \mathbb{R}$ is *bounded above* if there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. M is an *upper bound* of S.

lower bound

A set $S \subseteq \mathbb{R}$ is *bounded below* if there exists $m \in \mathbb{R}$ such that $s \geq m$ for all $s \in S$. m is an *lower bound* of S.

bounded

A set is *bounded* if it is both bounded above and bounded below.

supremum

The *supremum* or *least upper bound* of a nonempty set S that is bounded above is the upper bound L satisfies $L \leq M$ for all upper bounds M of S is written as $\sup S$.

infimum

The *infimum* or *greatest lower bound* of a nonempty set S is the lower bound ℓ satisfying $\ell \geq m$ for all lower bounds m of S. The infimum is denoted $\inf S$.

max

If there exists $M \in S$ such that $s \leq M$ for all $s \in S$, then M is called the *maximum* of S, $\max S$.

min

Analogous defn for $\min S$.

2.2 Examples

0. $S_0 = \varnothing$. Bounded above and below. No supremum or infimum.
1. $S_1 = \{n \in \mathbb{Z}^+\} = \{1, 2, 3, \ldots\}$ not bounded above, bounded below.

   1 is infimum and minimum
2. $S_2 = \{-3, -2, 0.5, 1.423\}$. Bounded above and below. Bounded. Has max, min.
3. $S_3 = \left\{1 - \frac{1}{n} : n \in \mathbb{Z}^+\right\} = \{0, \frac{1}{2}, \frac{2}{3}, \ldots\}$

   Bounded above by 1. Bounded below by 0.

   Supremum is 1, but there is no max.

2.3 Least Upper Bound Principle

Theorem 2.1: Least Upper Bound Principle

Every nonempty set S of $\mathbb{R}$ that is bounded above has a supremum. Every nonempty set that is bounded below has an infimum.

Sketch of proof for "infimum". There are only finitely many integers from m_0 to $s_0 + 2$. Choose the greatest integer lower bound $\rightarrow$ call it a_0.

$a_0 + 1$ is not a lower bound. Divide $[a_0, a_0 + 1]$ into 10, find a_1 such that $a_{0\circ}a_1$ is lower bound of S, but $a_{0\circ}a_1 + 1/10$ is not. Repeat infinitely many times to construct $L = a_{0\circ}a_1a_2a_3\ldots$.

Now, show that L is infimum.[1] $\square$

[1] See details in textbook.

3

Limits of Sequences

3.1 Sequences

An *infinite sequence of real* numbers is an infinite, enumerated list of real numbers, denoted by

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, \ldots)$$

Each $a_n \in \mathbb{R}$ is an *element* of the sequence.

We will just refer to them as sequences, and often write (a_n). Formally, a sequence is a function that maps positive integers to $\mathbb{R}$.

We say that a sequence is [bounded above/bounded below/bounded] if the set $A = \{a_n\}$ is respectively [bounded above/bounded below/bounded].

3.2 Examples

1. $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^n$ for $n \geq 1$.
2. $a_n = \frac{1}{n}$, for $n \geq 1$.
3. $(a_n) = (1, 1, 2, \frac{1}{2}, 3, \frac{1}{3}, \ldots)$

3.3 Limits of Sequences

limit

Let $(a_n)_{n=1}^{\infty}$ be a sequence. We call $L \in \mathbb{R}$ the *limit* of the sequence if for all $\epsilon > 0$, there exists an integer N such that

$$|a_n - L| < \epsilon$$

for all $n \geq N$.

If such L exists, then we say that (a_n) is convergent, and converges to L and we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$.

If a sequence does not have such a limit, then we say it *diverges*, or is *divergent*.

A sequence (a_n) *diverges to* ∞ if for all $M > 0$, there exists N such that $a_n > M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = \infty$.

A sequence (a_n) *diverges to* $-\infty$ if for all $M < 0$, there exists N such that $a_n < M$ for all $n \geq N$. We write $\lim_{n\to\infty} a_n = -\infty$.

Note
$\lim_{n\to\infty} a_n = \pm\infty$ does not mean limit exists.

3.4 Examples

1. $a_n = 1/n, \quad \lim_{n\to\infty} a_n = 0$

   For any $\epsilon > 0$, we need to show that there exists N such that $|a_n - 0| < \epsilon$ for all $n \geq N$.

   Choose N to be any integer greater than $1/\epsilon$. ($N > \frac{1}{\epsilon}$)

   For any $n \geq N$, $a_n = 1/n \leq \frac{1}{N} < \epsilon$. We also have $a_n \geq 0$

   $$\implies |a_n| < \epsilon$$

   for all $n \geq N$ as required.

3.5 Some basic properties of limits

Theorem 3.1: Squeeze Theorem

Let $(a_n), (b_n), (c_n)$ be sequences.

If $a_n \leq b_n \leq c_n$ for all $n \geq 1$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then

$$\lim_{n\to\infty} b_n = 1$$

Proof:
We want to show that for all $\epsilon > 0$, there exists N such that $|b_n - L| < \epsilon$ for all $n \geq N$.

Let $\epsilon > 0$. Since $a_n \to L$, we can find N_1 such that $|a_n - L| < \epsilon$ for all $n \geq N_1$.

Similarly, there exists N_2 s.t. $|c_n - L| < \epsilon$ for all $n \geq N_2$.

Define $N := \max\{N_1, N_2\}$. Then, for $n \geq N$, $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$.

Equivalently,

$$L - \epsilon < a_n < L + \epsilon \qquad L - \epsilon < c_n < L + \epsilon$$

Since $a_n \leq b_n \leq c_n$. $L - \epsilon < b_n < L + \epsilon$, or

$$|b_n - L| < \epsilon$$

as required. □

Proposition 3.2

If a sequence converges to a limit L, then this limit is unique.

Proof:
See PDF. □

Proposition 3.3

If a sequence (a_n) converges, then the set $A := \{a_n : n \geq 1\}$ is bounded.

Proof:
Exercises. □

Theorem 3.4

Let (a_n) and (b_n) be two convergent sequences. If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1. $\lim_{n\to\infty}(a_n + b_n) = L + M$
2. for any $\alpha \in \mathbb{R}$, $\lim_{n\to\infty}(\alpha a_n) = \alpha L$
3. $\lim_{n\to\infty}(a_n b_n) = LM$, and
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$ and $b_n \neq 0$ for all n.

4

Monotone Sequence and Applications

4.1 Monotone Sequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. it is

1. monotone increasing if $a_{n+1} \geq a_n$ for all $n \geq 1$.
2. strictly monotone increasing if $a_{n+1} > a_n$ for all $n \geq 1$.
3. monotone decreasing if $a_{n+1} \leq a_n$
4. strictly monotone decreasing if $a_{n+1} < a_n$

monotone

A sequence is monotone is *monotone* if it is either (monotone) increasing or (monotone) decreasing.

Theorem 4.1: Monotone Convergence Theorem

Monotone Convergence Theorem:

(i) Every monotone increasing sequence that is bounded above converges

(ii) Every monotone decreasing sequence that is bounded below converges

Proof:
We will first show that (i) $\implies$ (ii).

Let (a_n) be a monotone decreasing sequence that is bounded below by m.

The sequence $(-a_n)_{n=1}^{\infty}$ is monotone increasing and is bounded above by $-m$. By part (i), $(-a_n)$ must converge. Call the limit $L = \lim_{n\to\infty}(-a_n)$.

By Theorem 3.4 Part 2,

$$\lim_{n\to\infty} = \lim_{n\to\infty}[(-1)(-a_n)] = (-1)\lim_{n\to\infty}(-a_n) = -L$$

To prove Part(i) of this theorem, suppose (a_n) is monotone increasing and bounded above.

The set $A = \{a_n | n \in \mathbb{Z}^+\}$ is bounded above, and nonempty.

By LUBP(Theorem 2.1), A has a supremum, which we call $L = \sup A$. We show that L is the limit of (a_n).

Given $\epsilon > 0$, we know that $L - \epsilon$ cannot be an upper bound of A.

So there exists N such that $a_n > L - \epsilon$.

Since (a_n) is increasing, $a_n > L - \epsilon$ for all $n \geq N$. Since L is an upper bound of A, $a_n \leq L$ for all $n \geq N$.

$$\implies L - \epsilon < a_n \leq L < L + \epsilon$$

That is $|a_n - L| \leq \epsilon$ for all $n \geq N$. □

4.2 Applications: Calculate Square Roots

The square root of a real number $a > 0$ can be obtained as the limit of the sequence defined recursively by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right), \qquad \text{for } n \geq 1$$

where the starting point x_0 is any positive number.

Moreover, for any $n \geq 1$, the error in approximating $\sqrt{a}$ by x_n satisfies the bound

$$0 \leq x_n - \sqrt{a} < x_n - \frac{a}{x_n}$$

Proof:
Strategy:

1. Prove that (x_n) is bounded below.
2. Prove that (x_n) is monotone decreasing.
3. Prove that (x_n) is monotone decreasing.
4. Use MCT to prove that (x_n) converges.

5. Use properties of limits to determine that $\sqrt{a}$ is the limit.

6. Look for upper and lower bounds for error.

See PDF for full proof. □

4.3 Warning about computing limits that don't exist

Example:

$a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n^2 + 1)$ for $n \geq 1$.

If we assume (a_n) has a limit L, then we can get nonsense.

$$a_{n+1} = \frac{1}{2}(a_n^2 + 1)$$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}(a_n^2 + 1)$$
$$\implies L = \frac{1}{2}\left(\lim_{n\to\infty} a_n\right)^2 + \frac{1}{2} = \frac{1}{2}L^2 + \frac{1}{2}$$
$$L^2 - 2L + 1 = 0 \implies L = 1 \text{ is a solution}$$

However, it can be shown that (a_n) is monotone increasing. Since $a_1 = 2$, (a_n) cannot possibly converge to 1.

(In fact, it does not converge.)

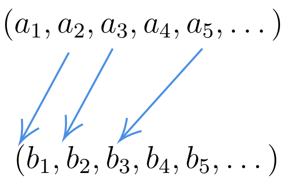
5

Subsequences

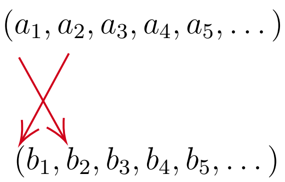
5.1 Definitions of subsequences

Let $(a_n)_{n=1}^{\infty}$ be a sequence. The sequence $(b_k)_{k=1}^{\infty}$ is a *subsequence* of (a_n) of there exist integers n_k with $1 \leq n_1 < n_2 < n_3 < \ldots$ such that $b_k = a_{n_k}$ for each $k \geq 1$.

Example:



cannot do the following:



not allowed to change order

Example:

$$(a_n)_{n=1}^{\infty} = \left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \ldots\right)$$

The sequence (b_k) with $b_k = a_k$ for all $k \geq 1$ is a subsequence of (a_n).

The sequence $\left(-1, -\frac{1}{3}, -\frac{1}{5}, \ldots\right)$ is a subsequence.

The sequence $\left(\frac{1}{2}, \frac{1}{4}, \ldots\right)$ is another subsequence.

5.2 Some properties of Subsequences

Lemma 5.1

Let n_k be integers satisfying $n_1 \geq 1$ and $n_k < n_{k+1}$ for all $k \geq 1$. Then $n_k \geq k$ for all $k \geq 1$.

Theorem 5.2

Suppose the sequence $(a_n)_{n=1}^{\infty}$ converges to the limit L. Then every subsequence of (a_n) also converges to L.

Proof:
By definition of limit, for every $\epsilon > 0$, there exists N such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Let $(b_k)_{k=1}^{\infty}$ be any subsequence of (a_n), where $b_k = a_{n_k}$ for each $k \geq 1$.

From Lemma 5.1, we know that $n_k \geq k$ for each k. Given $\epsilon > 0$, chose N as in definition of $\lim_{n\to\infty} a_n = L$. For every $k \geq N$,

$$n_k \geq k \geq N \implies |b_k - L| = |a_{n_k} - L| < \epsilon$$

□

Example:

1. From 5.1, the theorem holds just as it is.
2. Converse is not true. If a subsequence converges, we cannot conclude that the original sequence converges.

5.3 Bolzano-Weierstrass

If for every integer $n \geq 1$, we have a nonempty, closed interval $I_n = [a_n, b_n]$ such that $I_{n+1} \subseteq I_n$, then we say that (I_n) is a *nested sequence of closed, bounded intervals.*

Lemma 5.3: Nested Intervals Lemma

If (I_n) is a nested sequence of closed bounded intervals, then

$$\bigcap_{n=1}^{\infty} I_n \neq \varnothing.$$

Proof:
Exercise. □

Theorem 5.4: Bolzano-Weierstrass Theorem

Every bounded sequence of real numbers has a convergent subsequence.

Proof:
Outline.

1. Given a bounded sequence (a_n), construct a nested sequence of closed, bounded intervals I_n with lengths decreasing to zero, and such that each I_n contains infinitely many elements of the sequence (a_n).
2. Construct a subsequence (b_k) such that $b_k \in I_k$ for each $k \geq 1$.
3. Show that (b_k) converges.

$\square$

Proof:

Step 1: Suppose $(a_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers. Let m_1 be a lower bound and M_1 be an upper-bound for $A = \{a_n : n \geq 1\}$.

Define an interval $I_1 = [m_1, M_1]$. Define the point $c_1 = \frac{1}{2}(m_1 + M_1)$. Choose one smaller interval either $[m_1, c_1]$ or $[c_1, M_1]$ that contains an infinite member of elements of $(a_n) \rightarrow$ call this interval $I_2 = [m_2, M_2]$.

We repeat this process for all $k \geq 2$. This gives a sequence of intervals $(I_k)_{k=1}^{\infty}$ such that $I_{n+1} \subseteq I_n$ for all $n \geq 1$, and lengths of I_n converges to zero. Also each I_k contains an infinite number of elements of (a_n).

Step 2: Let $n_1 = 2$ so $b_1 = a_1$. Suppose we have our subsequence (b_j) up to element k. Then we have $n_i \geq 1$ for all $i = 1, 2, \ldots, k$ and $n_i < n_{i+1}$ for all $i = 1, 2, \ldots, k-1$.

Since there are an infinite number of elements of (a_n) contained in I_{k+1}, we can choose n_{k+1} such that $n_{k+1} > n_k$ and $a_{n_{k+1}} \in I_{k+1}$, i.e. $b_{k+1} \in I_{k+1}$. In this way, we inductively define (b_j) as a subsequence of (a_n).

Step 3: By Nested Intervals Lemma (Lemma 5.3), $\bigcap\limits_{k=1}^{\infty} I_k \neq \varnothing$, so there must exist a point $L \in \cap_{k=1}^{\infty} I_k$. The length of interval I_j is $\dfrac{(M_1 - m_1)}{2^{j-1}}$. For any $k \geq 1$, we have $L \in I_k$ and $b_k \in I_k$. Hence $|b_k - L| \leq \dfrac{(M_1 - m_1)}{2^{k-1}}$.

Consider sequence $(|b_k - L|)_{k=1}^{\infty}$. We can use Squeeze Theorem to show that $\lim_{n\to\infty} |b_k - L| = 0$ since

$$0 \leq |b_k - L| \leq \frac{(M_1 - m_1)}{2^{k-1}}.$$

Hence $\lim\limits_{k\to\infty} b_k = L$.

□

6 Cauchy Sequences

6.1 Definition

A sequence (a_n) is *Cauchy* if for any $\epsilon > 0$, there exists an integer N such that

$$|a_n - a_m| < \epsilon$$

for all $n, m \geq N$.

Example:

$$(a_n)_{n=1}^{\infty} = (3, 3.1, 3.14, 3.141, \dots)$$

More generally, if x is any real number with infinite decimal expression $x_{0\circ}x_1x_2x_3\dots$, then the sequence of finite truncations, i.e., a_k is the truncation of x to k decimal places, is Cauchy.

$$a_k = x_{0\circ}x_1 \dots x_k 000 \dots$$

Given $\epsilon > 0$, we can find N such that $10^{-N} < \epsilon$.

For any $n \geq 1$, we have

$$a_n \leq x \leq a_n + 10^{-n}$$

In particular,

$$a_N \leq x \leq a_N + 10^{-N}$$

Note that (a_n) is monotone increasing, so $a_N \leq a_n, a_m \leq x \leq a_N + 10^{-N}$ for any $n, m \geq N$.

So

$$|a_n - a_m| \leq \text{length of interval} = 10^{-N} < \epsilon$$

$\implies (a_n)_{n=1}^{\infty}$ is Cauchy.

7 Cauchy and Completeness

7.1 Properties of Cauchy Sequences

Proposition 7.1

If a Cauchy sequence (a_n) has a convergent subsequence, then (a_n) converges. The limit is the same as the limit of the subsequence.

Proof:
Let $\epsilon > 0$. By definition of limit of $(b_k) = (a_{n_k})$ being L, i.e., $\lim_{k\to\infty} b_{n_k} = L$, there exists K such that

$$|b_k - L| = |a_{n_k} - L| < \frac{\epsilon}{2}$$

for all $k \geq K$.

By Cauchy property of (a_n), there exists N such that

$$|a_n - a_m| < \frac{\epsilon}{2}$$

for all $n, m \geq N$.

By Lemma 5.1, $n_k \geq k$ for all $k \geq 1$, so

$$|a_n - a_{n_k}| < \frac{\epsilon}{2}$$

for all $n, k \geq N$. Choose any $k \geq \max\{K, N\}$. Then, for all $n \geq N$,

$$|a_n - L| = |a_n - a_{n_k} + a_{n_k} - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Proposition 7.2

If a sequence (a_n) is Cauchy, then the set $\{a_n : n \geq 1\}$ is bounded.

Proof:
Exercise, or see PDF. $\square$

7.2 Example of not quite Cauchy

Consider the sequence $(a_n)_{n=1}^{\infty}$, with $a_n = \log n$.

The difference between successive terms is

$$|a_{n+1} - a_n| = |\log(n+1) - \log(n)| = \left|\log\left(\frac{n+1}{n}\right)\right|$$

$\lim_{n\to\infty} \frac{n+1}{n} = 1$, so $\lim |a_{n+1} - a_n| = 0$.

(a_n) is not bounded, since $\log(n) \to \infty$, hence by Proposition 7.2, (a_n) is not Cauchy.

7.3 Cauchy, Convergent and Complete

Proposition 7.3

Every convergent sequence is Cauchy.

Proof:
(Sketch)

N, K and use $\epsilon/2$. $\square$

complete

We say that a subset X of $\mathbb{R}$ is *complete* if every Cauchy sequence in X has a limit in X.

Theorem 7.4: Completeness Theorem for Real Numbers

$\mathbb{R}$ is complete.

In other words, every Cauchy sequence of real numbers converges.

Proof:
Suppose (a_n) is any Cauchy sequence of real numbers. By Proposition 7.2, $\{a_n : n \geq 1\}$ is bounded. By Theorem 5.4, there must exist a convergent subsequence.

By Proposition 7.1, (a_n) must also converge. $\square$

Remark:

The sequence of truncated decimal expansions of x (from Lecture 6) was shown to be Cauchy. Now we know, it must converge. It can be shown that the limit is x.

Note

$\mathbb{Q}$ is not a complete subset of $\mathbb{R}$. Using sequence of finite decimal expansions, we see that sequences of rational numbers can converge to an irrational limit.

7.4 Equivalent Statements of Completeness

We showed that construction of $\mathbb{R}$ as set of infinite decimal expansions leads to Least Upper Bound Principle.

$\implies$ Monotone Convergence Theorem

$\implies$ Nested Intervals Lemma

$\implies$ Bolzano-Weierstrass Theorem

$\implies$ Completeness Theorem

It is possible to show that Completeness $\implies$ LUBP. So all of these properties describe the same "behaviour" of $\mathbb{R}$.

7.5 Application: Proving convergence by Cauchy property

Sometimes it's easier to show that a sequence is Cauchy than convergent.

Example:

Consider a sequence $a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n}$. We can show that $(a_n)_{n=1}^{\infty}$ is Cauchy. For $m > n$,

$$\begin{aligned} |a_m - a_n| &= \left| \frac{(-1)^{n+2}}{n+1} + \frac{-1^{n+3}}{n+2} + \ldots + \frac{(-1)^m}{m-1} + \frac{(-1)^{m+1}}{m} \right| \\ &= \ldots \end{aligned}$$

Suppose $m - n$ is even

$$|a_m - a_n| = \left| \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+3} - \ldots + \frac{1}{m-1} - \frac{1}{m} \right|^{a}$$

[a] Sth wrong here... corrected in the lecture notes.

8 Series

8.1 Definitions for series

If $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers, we define its *sequence of partial sums* $(S_n)_{n=1}^{\infty}$ by $S_n = \sum_{k=1}^{n} a_k$.

The (infinite) series associated with (a_n) is $\sum_{n=1}^{\infty} a_n$. If the sequence of partial sums converges to a limit $L \in \mathbb{R}$, then we say the series $\sum_{n=1}^{\infty}$ converges. In this case, we say the sum or value of the series is L.

The series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

If a series does not converge, then it diverges.

A series that converges but is not absolutely convergent, then we say it is conditionally convergent.

Example:

1. $(a_n)_{n=1}^{\infty} = (1, 1, 1, 1, 1, \ldots)$. This sequence converges to 1.

   Sequence of partial sums is $(S_n) = (1, 2, 3, 4, 5, \ldots)$ does not converge (it diverges to ∞) so the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   Note
   $S_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ forms a sequence such that

   $$S_{n+1} - S_n = \frac{1}{n+1} \to 0$$

   but (S_n) is not convergent, which means (S_n) is not Cauchy.

3. $a_n = \frac{1}{n(n+2)}$.

We will show that $\sum_{n=1}^{\infty} a_n$ converges.

Note

We can write
$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$
Then the sequence of partial sums is
$$S_n = \frac{1}{2}\sum_{k=1}^{n} \frac{2}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2}\left[\left(1+\frac{1}{2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)\right]$$
$$\lim_{n\to\infty} S_n = \frac{3}{4}$$
Hence,
$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{4}$$

4. A geometric series $\sum_{n=0}^{\infty} a_n$ is one where the elements are of the form $a_n = a_0 r^n$ for some $a_0 \in \mathbb{R}, r \in \mathbb{R}$, for each $n \geq 0$.

   If $|r| < 1$, then the series converges
$$\sum_{n=0}^{\infty} a_n = \frac{a_0}{1-r}$$
   If $|r| \geq 1$ and $a_0 \neq 0$, then the series diverges.

5. The alternating harmonic series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
   converges. It is not absolutely convergent. (See Example 2), so it is conditionally convergent.

Proposition 8.1

Every absolute convergent series is convergent.

Proof:
Trivial. □

8.2 Convergence Tests

Theorem 8.2: Cauchy criterion for series

Given a series $\sum_{n=1}^{\infty} a_n$, the following are equivalent:

1. The series converges.
2. Given $\epsilon > 0$, there exists an integer N such that

$$\left|\sum_{k=n+1}^{m} a_k\right| < \epsilon$$

for all $m > n \geq N$.

Note

If (S_n) is sequence of partial sums. Suppose $m > n$,

$$|S_m - S_n| = \left|\sum_{k=1}^{m} a_k - \sum_{k=1}^{n} a_k\right| = \left|\sum_{k=n+1}^{m} a_k\right|$$

Theorem 8.3: Comparison Test for Series

Suppose $(a_n), (b_n)$ are two sequences and $|a_n| \leq b_n$ for all $n \geq 1$.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, and

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} b_n$$

2. If $\sum_{n=1}^{n} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof:

Note that 2 follows from 1.

So, we just need to prove 1.

First, we show that

$$\sum_{n=1}^{\infty} b_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

Let $\epsilon > 0$. By Cauchy criterion, there exists N such that

$$\left|\sum_{k=n+1}^{m} b_k\right| < \epsilon \text{ for all } m > n \geq N$$

Since $b_k \geq 0$ for all k, we can ignore absolute value sign.

$$\epsilon > \sum_{k=n+1}^{m} b_k \geq \sum_{k=n+1}^{m} |a_k| \geq \left| \sum_{k=n+1}^{m} a_k \right|$$

This is the Cauchy criterion for $\sum a_n$, so $\sum a_n$ converges.

The rest of proof is left as an exercise: Show remaining inequality. $\square$

9 Rearrangements of Series

9.1 Definition

A rearrangement is a series considering of the same terms as another series but in a different order. Suppose $\pi : \mathbb{Z}^+ \to \mathbb{Z}^+$ is a permutation of the positive integers. Then, the series $\sum_{n=1}^{\infty} a_{\pi(n)}$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots$$

$$\sum_{n=1}^{\infty} a_{\pi(n)} = a_3 + a_4 + a_2 + a_1 + a_6 + \dots$$

9.2 Rearrangements of absolutely convergent series

Proposition 9.1

If an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ converges to L, then every rearrangement of $\sum_{n=1}^{\infty} a_n$ also converges to L.

Proof:
Let $\sum_{n=1} a_{\pi(n)}$ be a rearrangement. Fix $\epsilon > 0$. By absolute convergence of

$\sum_{n=1}^{\infty} a_n$, there exist N such that

$$\left|\sum_{n=1}^{N} |a_n| - \sum_{n=1}^{\infty} |a_n|\right| = \sum_{n=N+1}^{\infty} |a_n| < \frac{\epsilon}{2}$$

Since every term of the series $\sum_{n=1}^{\infty} a_n$ must appear in the rearrangement, there must exist $M \geq N$ such that $\sum_{n=1}^{M} a_{\pi(n)}$ includes all terms

$$a_1, a_2, a_3, \ldots, a_N$$

For any $m \geq M$,

$$\begin{aligned}
\left|\sum_{n=1}^{m} a_{\pi(n)} - L\right| &= \left|\sum_{n=1}^{m} a_{\pi(n)} - \sum_{n=1}^{N} a_{\pi(n)} + \sum_{n=1}^{N} a_{\pi(n)} - L\right| \\
&\leq \left|\sum_{n=1}^{m} a_{\pi(n)} - \sum_{n=1}^{N} a_{\pi(n)}\right| + \left|\sum_{n=1}^{N} a_{\pi(n)} - L\right| \\
&\leq \sum_{n=N+1}^{\infty} |a_n| + \sum_{n=N+1}^{\infty} |a_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

So $\sum_{n=1}^{\infty} a_{\pi(n)} = L$. $\square$

9.3 Rearrangements of conditionally convergent series

Lemma 9.2

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series. Then there is an infinite number of non-negative terms and an infinite number of negative terms in the series.

Proof:
Use contrapositive.

Suppose there is a finite number of negative terms.

> Remark:
> Case with finite number of non-negative terms can be proved in the same way.

There must exist integer N such that N is the largest number for which $a_N < 0$. i.e. $a_n \geq 0$ for all $n > N$.

Case (i) $\sum_{n=1}^{\infty} a_n$ diverges. Trivially, not conditionally convergent.

Case (ii) $\sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_N$$

$$\implies \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

$$= \underbrace{\sum_{n=1}^{N} |a_n|}_{\substack{\text{finite sum} \\ \implies \text{ real number}}} + \underbrace{\sum_{n=1}^{\infty} a_n}_{\text{converges}} - \underbrace{\sum_{n=1}^{N} a_n}_{\substack{\text{finite sum} \\ \implies \text{ real number}}}$$

By properties of limits, $\sum_{n=1}^{\infty} |a_n|$ converges.

$\implies \sum_{n=1}^{\infty} a_n$ is absolutely convergent.

$\implies \sum_{n=1}^{\infty} a_n$ not conditionally convergent.

□

Lemma 9.3

Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent. For each $n \geq 1$, define b_n to be the n-th non-negative and c_n is the n-th negative term in the series. Then,

1. $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 0$, and
2. $\sum_{n=1}^{\infty} b_n = \infty$ and $\sum_{n=1}^{\infty} c_n = -\infty$.

Proof:
Exercise. □

Theorem 9.4

Let $\sum_{n=1}^{\infty} a_n$ be conditionally convergent series. Then, for any $L \in \mathbb{R}$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$ that convergent to L.

Proof:
Exercise. □

10

Euclidean Space

10.1 $\mathbb{R}^n$ Euclidean inner product and norm

We define the space $\mathbb{R}^n$ to be the set of all n-vectors $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$ where $x_i \in \mathbb{R}$ for each $i = 1, 2, 3, \ldots, n$.[1]

$\mathbb{R}^n$, equipped with vector addition and scalar multiplication, is a *vector* space.

We also define the Euclidean *inner product* of two vectors x and y is

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

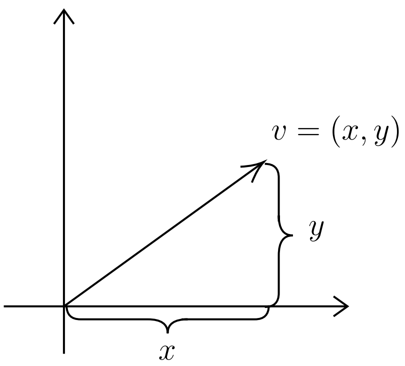
Also called dot product or scalar product.

The *Euclidean norm* of a vector $x \in \mathbb{R}^n$ is

$$\|x\| = \langle x, x \rangle^{1/2} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

Bt *Euclidean space*, we mean $\mathbb{R}^n$ with the structure imposed by the Euclidean inner product and norm.

[1] I will use $\mathbf{x}, \vec{x}$ or just x (from optimization courses) to represent vector. Readers should be clear when it is vector.



$$\|v\| = \sqrt{x^2 + y^2} = \left(\sum_{i=1}^{2} |v_i|^2 \right)^{1/2}$$

Figure 10.1: Example of $n = 2$

10.2 Properties of Euclidean inner product and norm

Proposition 10.1

Let $x, y, z\mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. The Euclidean inner product satisfies

1. $\langle x, x \rangle \geq 0$ with equality iff $x = 0$. (positive definite)
2. $\langle x, y \rangle = \langle y, x \rangle$. (symmetry)
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$. (Bilinearty)

Proposition 10.2

Let $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. The Euclidean norm satisfies:

1. $\|x\| \geq 0$ with equality iff $x = 0$. (positive definite)
2. $\|\alpha x\| = |\alpha| \, \|x\|$. (homogeneous)
3. $\|x + y\| \leq \|x\| + \|y\|$. ($\triangle$ ineq)

Proof:
1,2 $\to$ Exercise.

3 $\to$ See Theorem 10.4. $\square$

10.3 Inequalities

Theorem 10.3: Cauchy-Shwarz Inequality

For any $x, y \in \mathbb{R}^n$,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Equality holds iff x and y are linearly dependent (i.e., there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\alpha_1 x + \alpha_2 y = 0$ and it is not true that $\alpha_1 = \alpha_2 = 0$).

Proof:
First, note that the result is trivial if $x = 0$ or $y = 0$.

Suppose $x \neq 0$ and $y \neq 0$. We define the unit vectors

$$\mathbf{u} = (u_1, \dots, u_n) = \frac{x}{||x||}$$

and

$$\mathbf{v} = (v_1, \dots, v_n) = \frac{y}{||y||}$$

For each $i = 1, 2, \dots, n$,

$$0 \leq (u_i - v_i)^2 = u_i^2 - 2u_i v_i + v_i^2$$

$$u_i v_i \leq \frac{1}{2}(u_i^2 + v_i^2)$$

Adding together inequalities for all i:

$$\sum_{i=1}^{n} u_i v_i \leq \frac{1}{2} \sum_{i=1}^{n} (u_i^2 + v_i^2) \implies \langle u, v \rangle \leq \frac{1}{2}(||u||^2 + ||v||^2) = 1$$

We can do the same manipulation as above starting form

$$0 \leq (u_i + v_i)^2 = u_i + 2u_i v_i + v_i^2 \implies \langle u, v \rangle \geq -1$$

Hence $|\langle u, v \rangle| \leq 1$.

Exercise: Complete this proof. □

Theorem 10.4: Triangle Inequality

For any two vectors, $x, y \in \mathbb{R}^n$

$$\|x + y\| \leq \|x\| + \|y\|$$

Equality holds iff $x = 0$ or $y = \alpha x$ for some $\alpha \geq 0$.

Proof:

$$
\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle && \text{by bilinearity} \\
&\le \langle x,x\rangle + |\langle x,y\rangle| + |\langle x,y\rangle| + \langle y,y\rangle && \text{by properties of abs values} \\
&\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2
\end{aligned}
$$

Take square roots:

$$\|x+y\| \le \|x\| + \|y\|$$

Now prove "equality" statement.

$\Longrightarrow$) If equality holds, then

$$\langle x,y\rangle = |\langle x,y\rangle| = \|x\| \cdot \|y\|$$

The first "=": compare first introduction of inequality in proof.

The second "=": second inequality.

So we need C.S equality condition and we need $\langle x,y\rangle \ge 0$.

Case 1 $\alpha_2 \ne 0$, then $y = \alpha x$ where $\alpha = -\frac{\alpha_1}{\alpha_2}$.

Case 2 $\alpha_2 = 0$. Exercise.

$\square$

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