Introduction to General Relativity

AMATH 475

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Preface

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Pre-Math

0.1 Index notation

$$A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \qquad B = \begin{pmatrix} B^1{}_1 & B^1{}_2 \\ B^2{}_1 & B^2{}_2 \end{pmatrix}$$

$$(A \cdot B)^a{}_b = A^a{}_c B^c{}_b = B^c{}_b A^a{}_c \qquad \text{sum over all possible } c$$

Identify followings:

$$B_\kappa{}^\nu A_\mu{}^\kappa = A_\mu{}^\kappa B_\kappa{}^\nu = C_\mu{}^\nu = (A \cdot B)_\mu{}^\nu$$

$$A^\kappa{}_\mu B_\kappa{}^\nu = D_\mu{}^\nu = (A^T)_\mu{}^\kappa B_\kappa{}^\nu = (A^T \cdot B)_\mu{}^\kappa$$

$$A_\kappa{}^\nu B_\mu{}^\kappa = E_\mu{}^\nu = (B \cdot A)_\mu{}^\nu$$

$$A^\kappa{}_\mu B^\nu{}_\kappa = (A^T)_\mu{}^\kappa (B^T)_\kappa{}^\nu = \Big((B \cdot A)^T\Big)_\mu{}^\nu$$

$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 \qquad \{\mathbf{e}_1, \mathbf{e}_2\}$ Basis 1.

$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a \qquad \{\mathbf{e}'_1, \mathbf{e}'_2\}$ Basis 2.

Change of basis matrix Λ

$$\mathbf{e}'_a = \Lambda_a{}^b \mathbf{e}_b$$

$$v'^a = \tilde{\Lambda}^a{}_b v^b$$

$$
\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a{}_b v^b \Lambda_a{}^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a{}_b \Lambda_a{}^c v^b \mathbf{e}_c \\
&= \underbrace{\left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c}_{\delta^c_b} v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b
\end{aligned}
$$

$$
\begin{aligned}
\implies \left(\tilde{\Lambda}^T\right)_b{}^a \Lambda_a{}^c &= \delta^c_b \\
\tilde{\Lambda}^T \cdot \Lambda &= \mathbb{1}
\end{aligned}
$$

$\tilde{\Lambda}^T$ is the inverse transpose of Λ

covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis. Λ. (sub-indices)

A contravariant object transforms like components of vectors. ($\tilde{\Lambda} = (\Lambda^T)^{-1}$). (super-indices)

0.2 Vectors and one-forms

dual vector space

Let V be a vector space. A one-form is a linear map $\boldsymbol{\omega} : V \to \mathbb{R}$.

The set of all one-forms on V (call V^*) is a vector space as well called the dual vector space to V.

dual basis

Let $\{\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2, \ldots\}$ (or $\{\boldsymbol{\Upsilon}_i\}$) be a basis of V so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = v^i \boldsymbol{\Upsilon}_i$.

We define the dual basis (of V^*) to $\{\boldsymbol{\Upsilon}_i\}$ as $\{\boldsymbol{\omega}^i\}$ such that $\boldsymbol{\omega}^i(\boldsymbol{\Upsilon}_j) = \delta^i_j$.

Proposition 0.1

The dual basis of V^* is actually a basis of V^*.

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