



Introduction to General Relativity

AMATH 475



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Preface

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Some of the notes (especially special relativity part) are projected to the screen instead of using blackboards. They can be found on [professor's course page](#).

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Pre-Math

0.1 Index notation

$$A = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} \quad B = \begin{pmatrix} B^1_1 & B^1_2 \\ B^2_1 & B^2_2 \end{pmatrix}$$

$$(A \cdot B)^a_b = A^a_c B^c_b = B^c_b A^a_c \quad \text{sum over all possible } c$$

Identify followings:

$$\begin{aligned} B_\kappa^\nu A_\mu^\kappa &= A_\mu^\kappa B_\kappa^\nu = C_\mu^\nu = (A \cdot B)_\mu^\nu \\ A^\kappa_\mu B_\kappa^\nu &= D_\mu^\nu = (A^T)_\mu^\kappa B_\kappa^\nu = (A^T \cdot B)_\mu^\kappa \\ A_\kappa^\nu B_\mu^\kappa &= E_\mu^\nu = (B \cdot A)_\mu^\nu \\ A^\kappa_\mu B^\nu_\kappa &= (A^T)_\mu^\kappa (B^T)_\kappa^\nu = \left((B \cdot A)^T \right)_\mu^\nu \end{aligned}$$

$$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 \quad \{\mathbf{e}_1, \mathbf{e}_2\} \text{ Basis 1.}$$

$$\mathbf{v} = v^a \mathbf{e}_a = v'^a \mathbf{e}'_a \quad \{\mathbf{e}'_1, \mathbf{e}'_2\} \text{ Basis 2.}$$

Change of basis matrix Λ

$$\begin{aligned} \mathbf{e}'_a &= \Lambda_a^b \mathbf{e}_b \\ v'^a &= \tilde{\Lambda}^a_b v^b \end{aligned}$$

$$\begin{aligned}
v^a \mathbf{e}_a &= v'^a \mathbf{e}'_a \\
&= \tilde{\Lambda}^a_b v^b \Lambda_a^c \mathbf{e}_c \\
&= \tilde{\Lambda}^a_b \Lambda_a^c v^b \mathbf{e}_c \\
&= \underbrace{\left(\tilde{\Lambda}^T \right)_b^a}_{\delta_b^c} \Lambda_a^c v^b \mathbf{e}_c \\
&= v^b \mathbf{e}_b \\
\\
\Rightarrow \left(\tilde{\Lambda}^T \right)_b^a \Lambda_a^c &= \delta_b^c \\
\tilde{\Lambda}^T \cdot \Lambda &= \mathbb{1}
\end{aligned}$$

$\tilde{\Lambda}^T$ is the inverse transpose of Λ

covariant and contravariant object

A covariant object is an object that under change of basis transforms like the elements of a basis. Λ . (sub-indices)

A contravariant object transforms like components of vectors. $(\tilde{\Lambda} = (\Lambda^T)^{-1})$. (super-indices)

0.2 Vectors and one-forms

one-form

Let V be a vector space. A one-form is a linear map $\omega : V \rightarrow \mathbb{R}$.

or we write: $(\omega, \cdot) : V \rightarrow \mathbb{R}$ and $(\omega, \mathbf{v}) \in \mathbb{R}$.

dual vector space

The set of all one-forms on V (call V^*) is a vector space as well called the dual vector space to V .

dual basis

Let $\{\Upsilon_1, \Upsilon_2, \dots\}$ (or $\{\Upsilon_i\}$) be a basis of V so that any $\mathbf{v} \in V$ can be written as $\mathbf{v} = v^i \Upsilon_i$.

We define the dual basis (of V^*) to $\{\Upsilon_i\}$ as $\{\omega^i\}$ such that $\omega^i(\Upsilon_j) = \delta_j^i$.

For a one form ω we denote its “components of the basis Υ ” as $(\omega, \Upsilon_m) = \omega_m$

Proposition 0.1

The dual basis of V^* is actually a basis of V^* .

The action of $\omega \in V^*$ on a vector $\mathbf{v} = v^\mu \Upsilon_\mu \in V$ is

$$(\omega, \mathbf{v}) = (\omega, v^\mu \Upsilon_\mu) = v^\mu \omega_\mu$$

Let’s prove $\{\Upsilon^a\}$ is linear independent.

Proof:

A linear comb. $c_a \Upsilon^a$ acts on a vector $\mathbf{v} = v^a \Upsilon_a$

$$\begin{aligned} (c_a \Upsilon^a, \mathbf{v}) &= c_a (\Upsilon^a, \mathbf{v}) \\ &= c_a (\Upsilon^a, v^b \Upsilon_b) \\ &= c_a v^b \underbrace{(\Upsilon^a, \Upsilon_b)}_{\delta_b^a} \\ &= c_a v^b \delta_b^a = c_a v^a \end{aligned}$$

For LI,

$$\begin{aligned} c_a \Upsilon^a = 0 &\iff c_a = 0 \quad \forall a \\ c_a v^a = 0 \quad \forall \mathbf{v} &\iff c_a = 0 \end{aligned}$$

□

vectors: take one-forms $\rightarrow \mathbb{R}$ one-forms: take vectors $\rightarrow \mathbb{R}$

0.3 Tensor

type (m, n) tensor

A type (m, n) tensor is a multilinear map that

$$\mathbf{T} : V^n \otimes (V^*)^m \rightarrow \mathbb{R}$$

Components of \mathbf{T} :

$$\mathbf{T}(\Upsilon_{a_1}, \dots, \Upsilon_{a_n}, \Upsilon^{b_1}, \dots, \Upsilon^{b_m}) = T_{a_1 \dots a_n}{}^{b_1 \dots b_m}$$

1. Tensor product takes $\binom{m}{n}$ and $\binom{m'}{n'} \rightarrow \binom{m+m'}{n+n'}$ tensor
2. Contraction takes $\binom{m}{n} \rightarrow \binom{m-1}{n-1}$

Example.

$$1. T_a^b, S_c^d.$$

$$(\mathbf{T} \otimes \mathbf{S})_a^b{}_c^d = T_a^d S_c^d = P_a^b{}_c^d$$

$$2. T_a^{bc} \rightarrow c^b T_a^{ba}$$

$$v^a, w_b \begin{cases} v^a \omega_b \\ v^a \omega_a \end{cases}$$

If you have a favorite type $(2, 0)$ symmetric tensor \mathbf{g}

$$v_\mu = g_{\mu\nu} v^\nu$$

$g^{\mu\nu} :=$ components of the inverse of $\mathbf{g}_{\mu\nu}$

$$v^\nu = g^{\mu\nu}$$

then

$$g^{\mu\nu} g_{\nu\sigma} = \delta_\sigma^\mu$$

$$g_{\mu\nu} v^\mu w^\nu = v_\mu w^\mu = \mathbf{v} \cdot \mathbf{w}$$

$$||\mathbf{v}||^2 = g_{\nu\mu} v^\mu v^\nu$$

Then we can define the angle

$$\frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}|| ||\mathbf{v}||} := \cos \theta$$

$$T_\mu{}^\nu = g^{\nu\sigma} T_{\mu\sigma}$$

$$T^{\mu\nu} = g^{\nu\sigma} g^{\mu\rho} T_{\sigma\rho}$$

$$g_\mu^\nu = g^{\nu\sigma} g_{\sigma\mu} = \sigma_\mu^\nu$$

0.4 Levi-Civita symbol

Levi-Civita symbol $\epsilon^{abc\dots}, \epsilon_{abc\dots}$

- is antisymmetric

- $\epsilon^{1234\dots} = 1, \epsilon_{1234} = 1$

$$\epsilon^{123} = 1, \quad \epsilon^{213} = -1, \quad \epsilon^{312} = 1, \quad \epsilon^{113} = 0$$

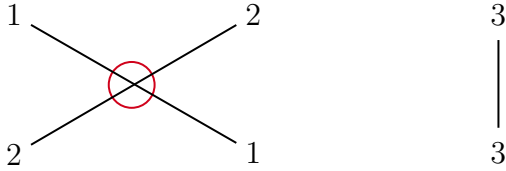
$$\epsilon^{123456} = 1, \quad \epsilon^{612453} = -1$$

Idea just see the permutations

Levi-Civita symbol

$$\varepsilon_{a_1 a_2 a_3 \dots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \dots, a_n) \text{ is an even permutation of } (1, 2, 3, \dots, n) \\ -1 & \text{if } (a_1, a_2, a_3, \dots, a_n) \text{ is an odd permutation of } (1, 2, 3, \dots, n) \\ 0 & \text{otherwise} \end{cases}$$

Here is a short-cut:



odd number crossings, so odd permutation.

Note that $\det(M) := \epsilon_{ijk\dots} M^i_1 M^j_2 M^k_3 \dots$

Exercise.

prove $\epsilon^{i_1 i_2 \dots i_n} \epsilon_{j_1 j_2 \dots j_n} = n! \delta_{i_j}^{j_j} = 1, \dots, n$

$$\begin{aligned} \epsilon^{ijk} \epsilon_{ilm} &= \delta_l^j \delta_m^k - \delta_m^j \delta_l^k \\ \epsilon^{ijmn} \epsilon_{klmn} &= 2(\delta_k^i \delta_l^j - \delta_l^i \delta_k^j) \end{aligned}$$

Prove $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

Proof.

Let $\vec{F} = \vec{A} \times (\vec{B} \times \vec{C})$ $\vec{D} = \vec{B} \times \vec{C}$

Then

$$\begin{aligned} D^k &= \epsilon^k_{ij} B^i C^j \\ F^l &= \epsilon^l_{mk} A^m D^k \implies F^l = \epsilon^l_{mk} \epsilon^k_{ij} A^m B^i C^j \end{aligned}$$

Then

$$\begin{aligned} F^l &= (\delta_i^l \delta_{mj} - \delta_j^l \delta_{mi}) A^m B^i C^j \\ &= \delta_i^l \delta_{mj} A^m B^i C^j - \delta_j^l \delta_{mi} A^m B^i C^j \\ &= B^l (A_j C^j) - C^l (A_i B^i) \end{aligned}$$

where we use

$$\vec{A} \cdot \vec{B} = A^i B_i$$

□

Special Relativity

1.1 Postulates of SR

Postulate 0

Newton's first law

Postulate 1: Principle of relativity

In the absence of gravity, all the laws of Physics are identical in all inertial reference frames.

Postulate 2

The speed of light in vacuum c is constant and the same from all inertial reference frames, regardless of their state of motion.

1.2 Lorentz Transformation

We define the spacetime interval Δs^2

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 = -c^2 (t_2 - t_1)^2 + (\mathbf{x}_2 - \mathbf{x}_1)^2$$

Assuming the following:

1. The difference between the two frames is a constant speed $\lesssim c$
2. The transformation has to be linear.

$$t' = \gamma \left(t - \frac{\mathbf{v} \cdot \mathbf{x}}{c^2} \right), \quad \mathbf{x}' = \mathbf{x} + (\gamma - 1)(\mathbf{n} \cdot \mathbf{x})\mathbf{n} - \gamma \mathbf{v}t$$

and index notation

$$t' = \gamma \left(t - \frac{v_i x^i}{c^2} \right), \quad x^i = x^i + (\gamma - 1) \frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

1.3 Line element, proper time and spacelike, time-like and null separation

1.3.1 Classification of spacetime intervals

We can classify events according to the following criterion:

- Spacelike separated, $\Delta s^2 > 0$
- Timelike separated, $\Delta s^2 < 0$
- Lightlike (null) separated, $\Delta s^2 = 0$

Given the trajectory of a physical particle moving inertially, we will call co-moving frame (inertial) or proper frame (non-inertial) to the frame S_p where the particle is at rest.

1.3.2 Proper time and line element

$$ds^2 = -c^2 dt^2 + d\mathbf{x}^2$$

We will call ds^2 the spacetime line element.

$$\mathbf{v} := \frac{d\mathbf{x}}{dt}$$

P0, P1, P2 + linearity

$$\implies t' = t \left(t - \frac{v_i x^i}{c^2} \right) \tag{1}$$

$$x'^i = x^i + (\gamma - 1) \frac{x^j v_j v^i}{v^2} - \gamma v^i t$$

Particle trajectory in a given inertial (Lab) frame $\mathbf{x}(t)$

Particle trajectory in its proper frame $\boldsymbol{\xi}(t) = 0$

Comoving frame's trajectories at each t (from lab frame) $\mathbf{x} = \mathbf{v}(t)t$.

$$d\tau = dt' = \gamma(t) \left(1 - \frac{\mathbf{v}(t)^2}{c^2} \right) dt \quad (2)$$

$$ds^2 = -c^2 dt^2 \left(1 - \frac{1}{c^2} \underbrace{\left(\frac{d\mathbf{x}}{dt} \right)^2}_{\mathbf{v}(t)} \right) = -c^2 \underbrace{\gamma^{-2} dt^2}_{d\tau^2} \implies ds^2 = -c^2 d\tau^2 \quad (3)$$

Example.

Find $\tau(t)$ for the three following trajectories.

1. $x(t) = v(t)$

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + d\mathbf{x}^2 \implies d\tau = \gamma^{-1} dt \implies \Delta\tau = \gamma^{-1} \Delta t$$

2. $x(t) = \frac{c^2}{a} \left[\sqrt{1 + \frac{a^2 t^2}{c^2}} - 1 \right]$

Then $\frac{dx}{dt} = \frac{at}{\sqrt{1 + \frac{a^2 t^2}{c^2}}}$

$$\left(\frac{d\tau}{dt} \right)^2 = 1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2$$

$$\implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2} \frac{a^2 t^2}{1 + \left(\frac{at}{c} \right)^2}}$$

$$\implies \tau(t) = \frac{c}{a} \operatorname{arcsinh} \left(\frac{at}{c} \right) \quad \text{and} \quad t(\tau) = \frac{c}{a} \sinh \left(\frac{a\tau}{c} \right)$$

3. $x(t) = L \sin(\omega t) \implies \frac{dx}{dt} = L\omega \cos(\omega t)$ with $L\omega < c$

$$\left(\frac{d\tau}{dt} \right)^2 = 1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2 \implies \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2} \left(\frac{dx}{dt} \right)^2} \implies d\tau = \sqrt{1 - \frac{L^2 \omega^2}{c^2}} dt$$

Then

$$\tau(t) = \frac{E \left(t\omega, \frac{1}{1 - \frac{L^2 \omega^2}{c^2}} \right)}{\omega \sqrt{\frac{1}{1 - \frac{L^2 \omega^2}{c^2}}}}$$

where

$$E(\phi|m) = \int_0^\phi (1 - m \sin^2 \theta)^{1/2} d\theta$$

1.4 Lorentzian Tensors

See notes for details.

A_μ transposes with Λ and it's covariant.

A^μ transposes with $\tilde{\Lambda} = (\Lambda^{-1})^T$ and it's contravariant.

1.5 Poincare group

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