$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \text{ as } n \to \infty$$

such that BVP (4.5.11) has a nontrivial solution if and only if $\lambda = \lambda_n, n = 1, 2, \cdots$. The solution y_{λ_n} is unique except for an arbitrary constant factor, and y_{λ_n} has exactly $n-1$ zeros in the open interval (a, b).

Remark. $\lambda_n, y_{\lambda_n}, n = 1, 2, 3 \cdots$, are the eigenvalues and eigenfunctions of BVP (**??**), respectively.

4.6 General Sturm-Lioville problems

Let us return briefly to the general Sturm-Lioville BVPs of the form

$$\frac{d}{dx}\Big[p(x)\frac{dy}{dx}\Big] + \Big[\lambda q(x) + r(x)\Big]y = 0 \tag{4.6.1}$$

$$c_1 y(a) + c_2 y'(a) = 0, d_1 y(b) + d_2 y'(b) = 0 \tag{4.6.2}$$

where $p(x), q(x)$ and $r(x)$ are continuous on $[a, b]$, $p(x) > 0, q(x) > 0, \forall x \in [a, b], c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$. Note that a special case of BVP (4.6.1) - (4.6.2) was discussed in Section 4.5. In the general case, the following result can be proved.

Theorem 4.6.1. There exist real numbers

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \text{ as } n \to \infty \tag{4.6.3}$$

such that BVP (4.6.1) - (4.6.2) has a nonzero solution iff $\lambda = \lambda_n$, $n = 1, 2, \cdots$, and eigenfunctions

$$y_1(x), y_2(x), \cdots, y_n(x), \cdots \tag{4.6.4}$$

are orthogonal on $[a, b]$ with respect to the weight function $q(x)$, i.e.

$$\int_a^b q(x) y_m(x) y_n(x)\, dx = \begin{cases} 0 \text{ if } m \neq n, \\ \alpha_n \neq 0 \text{ if } m = n. \end{cases} \tag{4.6.5}$$

Proof. We shall only prove (4.6.5).
Let

$$m(x) = y_m(x)y_n'(x) - y_n(x)y_m'(x) = det\begin{bmatrix} y_m(x) & y_m'(x) \\ y_n(x) & y_n'(x) \end{bmatrix}.$$

Then from (4.6.2) we have

$$\left\{ \begin{array}{c} c_1 y_m(a) + c_2 y_m'(a) = 0, \\ c_2 y_n(a) + c_2 y_n'(a) = 0, \end{array} \right\} \tag{4.6.6}$$

and

$$\left\{ \begin{array}{c} d_1 y_m(b) + d_2 y_m'(b) = 0, \\ d_a y_n(b) + d_2 y_n'(b) = 0. \end{array} \right\} \tag{4.6.7}$$

Since $c_1^2 + c_2^2 \neq 0$ and $d_1^2 + d_2^2 \neq 0$, it follows from (4.6.6) and (4.6.7) that

$$m(a) = 0 \text{ and } m(b) = 0. \tag{4.6.8}$$

Now consider

$$\begin{array}{rl} & y_n\left\{\dfrac{d}{dx}\left[p\dfrac{dy_m}{dx}\right] + \left[\lambda_m q + r\right]y_m\right\} = 0 \\ - & y_m\left\{\dfrac{d}{dx}\left[p\dfrac{dy_n}{dx}\right] + \left[\lambda_n q + r\right]y_n\right\} = 0 \\ \hline \end{array}$$
$$y_n\frac{d}{dx}\left[p\frac{dy_m}{dx}\right] - y_m\frac{d}{dx}\left[p\frac{dy_n}{dx}\right] + (\lambda_m - \lambda_n)q y_m y_n = 0$$

which implies

$$d(\lambda_m - \lambda_n)q y_m y_n = y_m(p y_n')' - y_n(p y_m')'.$$

Integrating the above equation from a to b and using integration by parts, we obtain

$$
\begin{aligned}
(\lambda_m - \lambda_n)\int_a^b q y_m y_n \, dx &= \int_a^b y_m (p y_n')' \, dx - \int_a^b y_n (p y_m')' \, dx \\
&= \left[y_m (p y_n')\right]_a^b - \int_a^b y_m' (p y -_n) \, dx - \left[y_n (p y_m')\right]_a^b + \int y_n' (p y_m') \, dx \\
&= y_m(b) p(b) y_n'(b) - y_m(a) p(a) y_n'(a) - y_n(b) p(b) y_m'(b) + y_n(a) p(a) y_m'(a)
\end{aligned}
$$

$$
\begin{aligned}
&= p(b)[y_m(b) - y_n(b) y_m'(b)] - p(a)[y_m(b) y_n'(a) - y_m(a) y_m'(a)] \\
&= p(b) m(b) - p(a) m(a).
\end{aligned}
$$

Thus from (4.6.8)

$$
(\lambda_m - \lambda_n \int_a^b q y_m y_n \, dx = 0
$$

which implies (4.6.5).

The significance of property (4.6.5) of the eigenfunctions is that we can obtain expansions of the functions $f(x)$ in terms of the eigenfunctions given by (4.6.4). if we assume

(4.6.9)
$$
f(x) = \sum_{n=1}^{\infty} a_n y_n(x),
$$

then multiplying both sides of (4.6.9) by $q(x) y_m(x)$ and integrating term by term from a to b yields

$$
\int_a^b f(x) q(x) y_m(x) \, dx = a_m \int_a^b q(x) y_m^2(x) \, dx
$$

(4.6.1)

which implies

(4.6.10)
$$
a_m = \frac{1}{\alpha_m} \int_a^b f(x) q(x) y_m(x) dx, \quad m = 1, 2, \cdots
$$

Formula (4.6.9) with a_m given in (4.6.10) is called an eigenfunction expansion of $f(x)$.

Remarks:

1. We didn't address the convergence of the series (4.6.9) whose study is beyond the scope of this course.

2. We call the BVP (4.6.1)-(4.6.2) a regular Sturm-Liouville problem because the interval $[a, b]$ is finite and the functions $p(x)$ and $q(x)$ are positive and continuous on $[a, b]$. Otherwise, it is called singular, which is considerably more difficult, and therefore not covered by our discussion here.