DESIGN AND ANALYSIS of ALGORITHM

Name: PRABHAT MALHAN Cl. Roll: 51

Section: ML University Roll No: 2015257

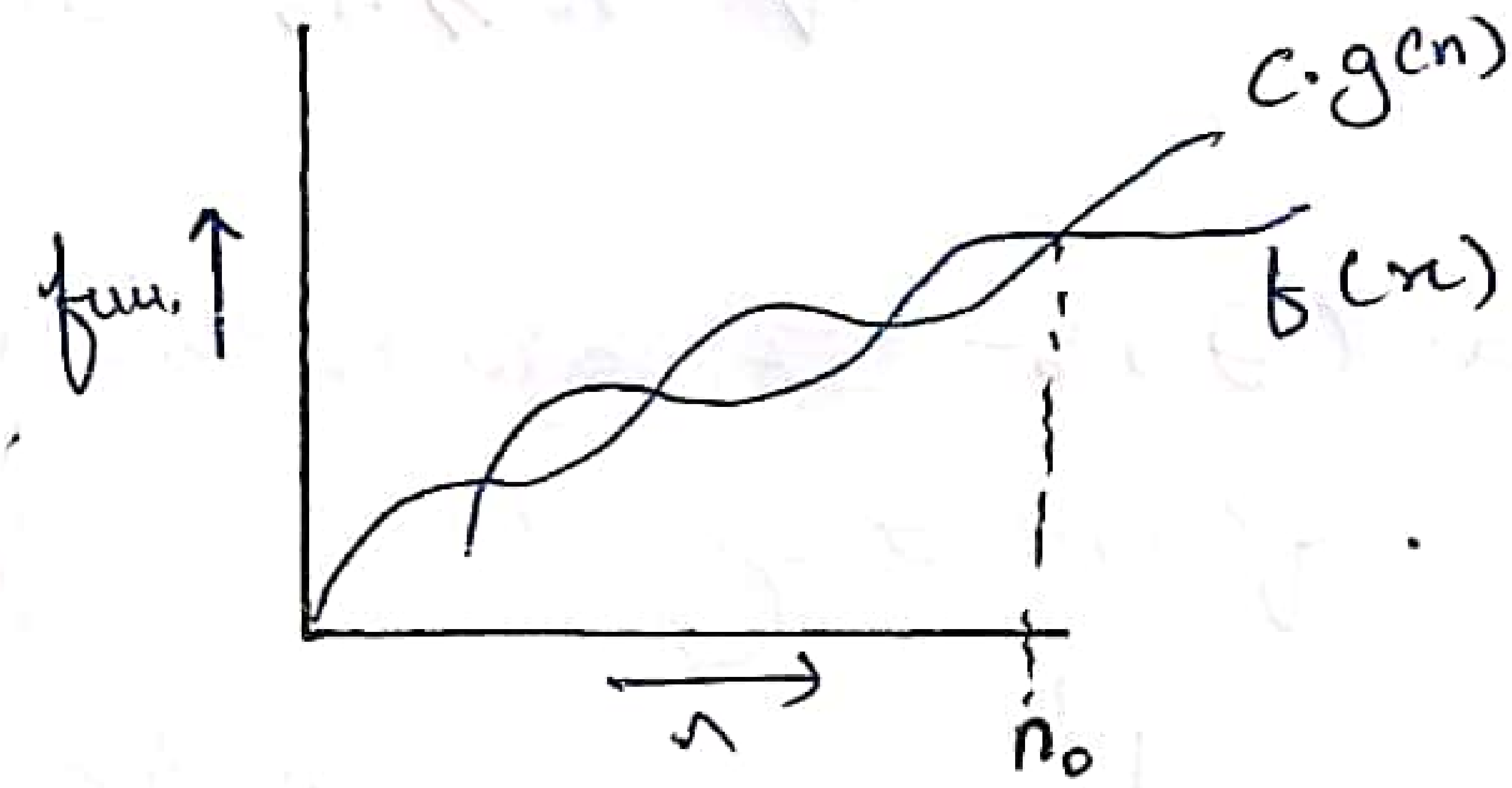
ASSIGNMENT - 1

Q1 The notations that are used to tell the complexity of an algorithm when input is very large is known as asymtotic notation.
The various types of asymtotic notation are :-

(i) Big-Oh (O) :-

$$f(n) = O(g(n))$$

g(n) is tight upper bound of



$$f(n) = O(g(n))$$

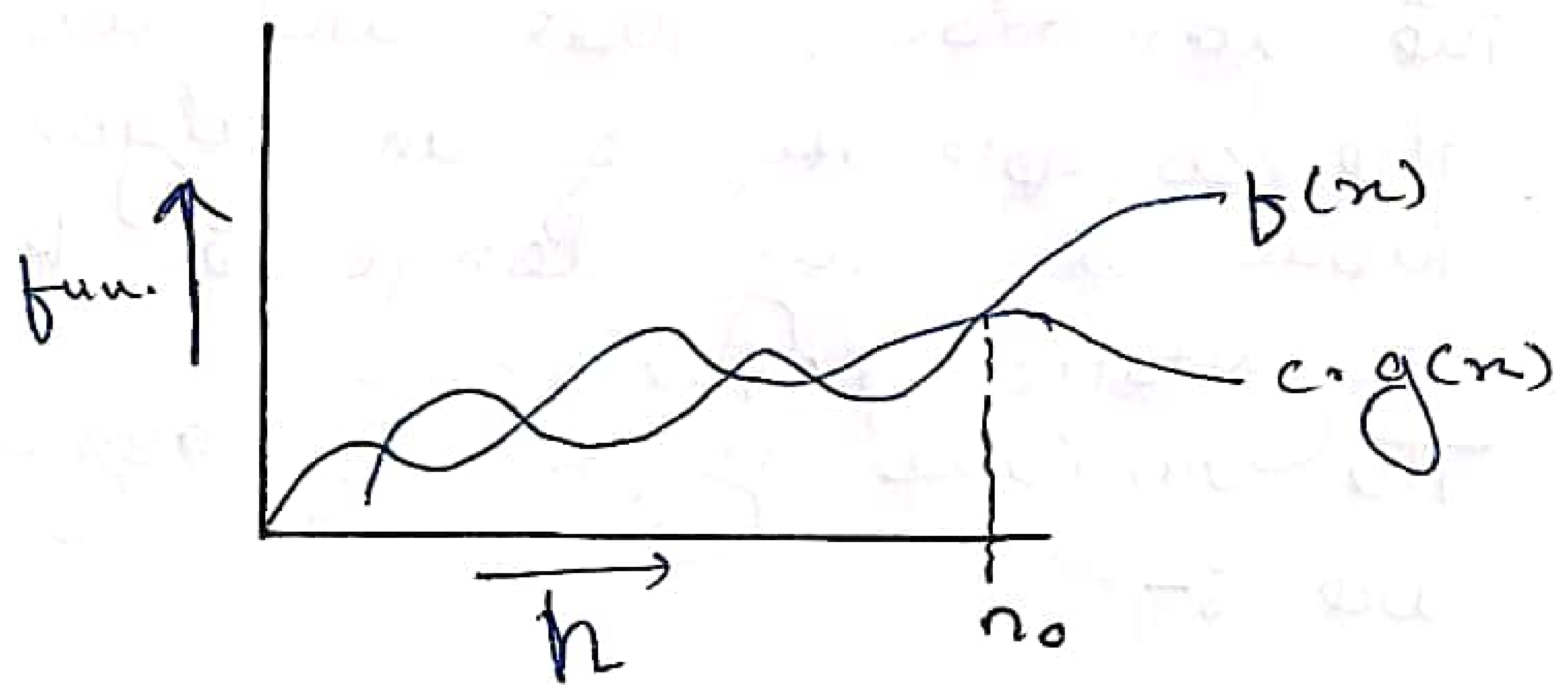
iff

$$f(n) <= c \cdot g(n)$$
$$\forall \; n >= n_0 \text{ and } c > 0.$$

(ii) Big - Omega (Ω) :-

$f(n) = \Omega(g(n))$

$g(n)$ is 'tight'. lower bound of f(



$f(x) = \Omega(g(x))$

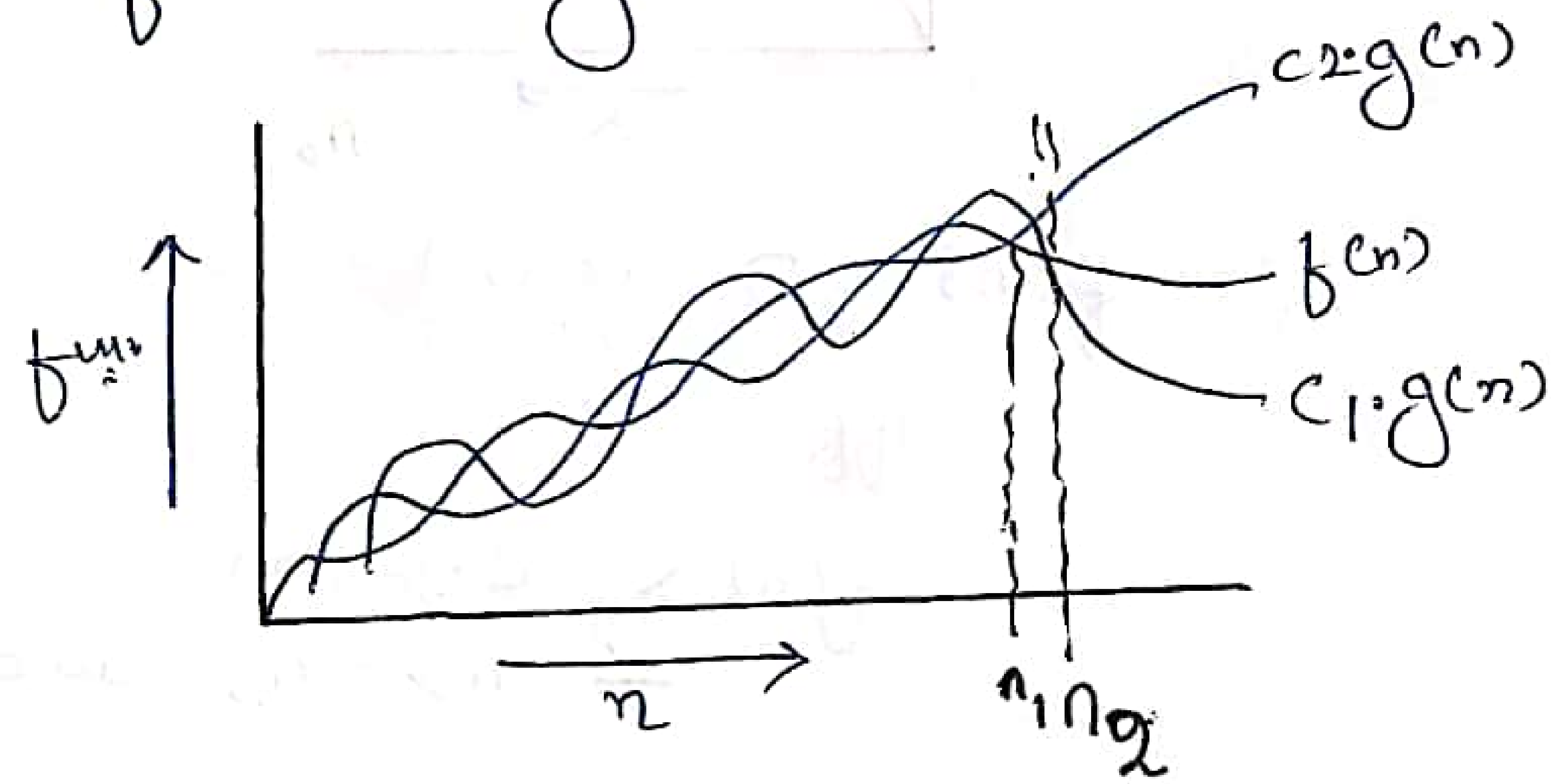
iff.

$f(x) > c \cdot g(x)$

$\forall\ x \geq n_0$ and $c > 0$.

(iii) Theta (Θ) :- It gives both upper & lower bo

$f(n) = \Theta(g(n))$



$\Theta(g(n)) = f(n)$

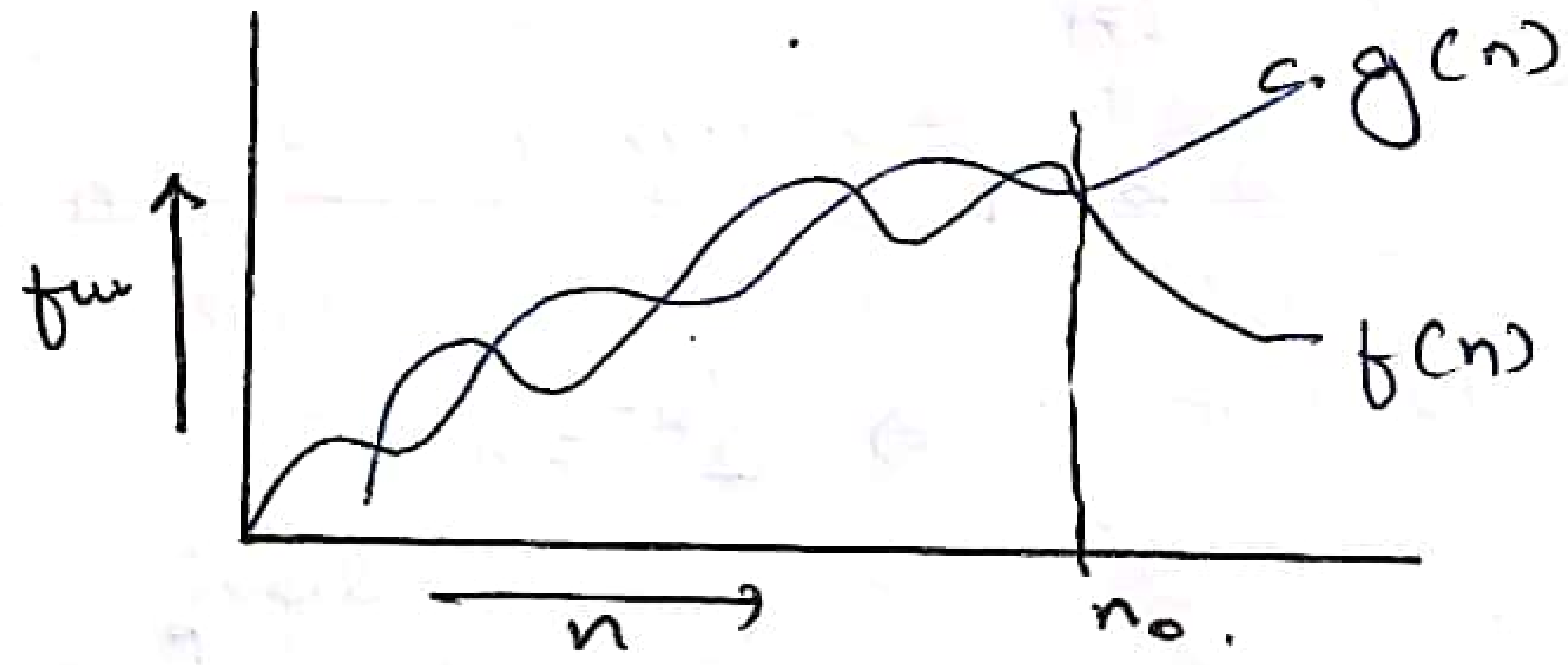
iff.

$c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$

$\forall$ ~~...~~ $c_1, c_2 > 0$. and

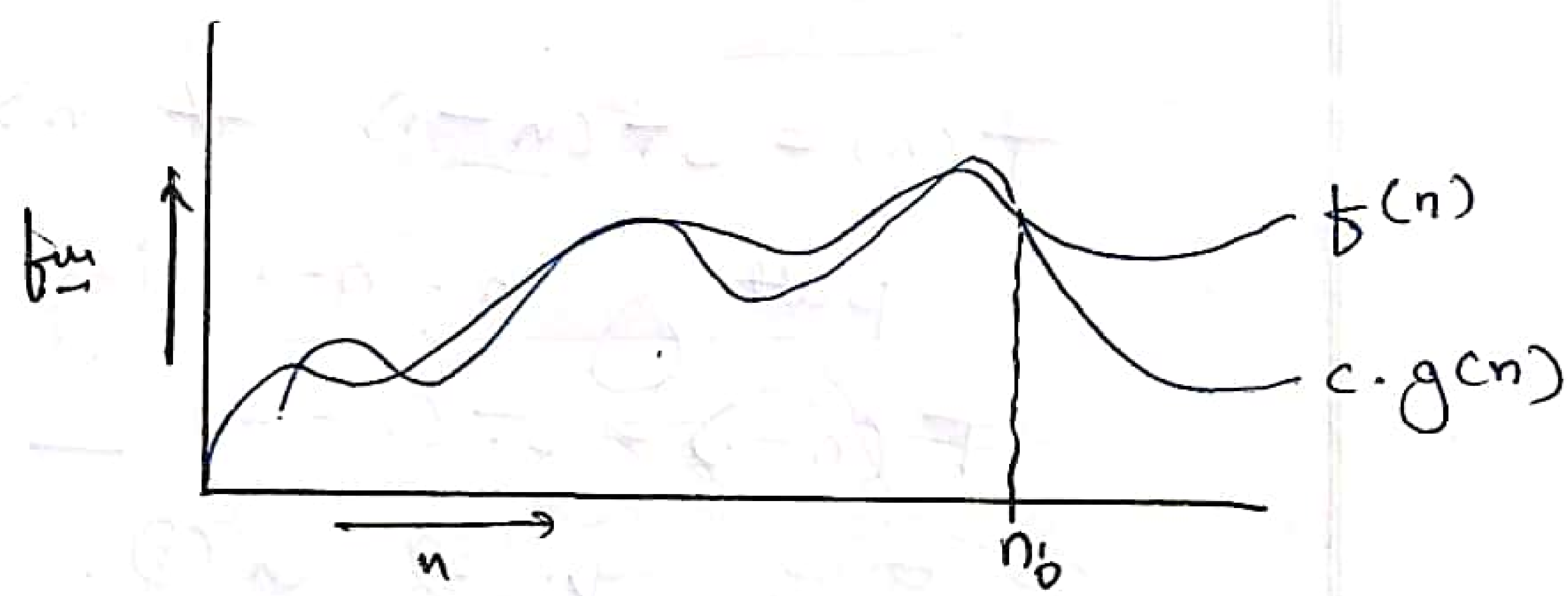
$max(n_1, n_2) \geq n$

(iv) small - oh (o) :-
It gives the upper bound of the fu



$f(n) = o(g(n))$
iff.
$f(n) < c.g(n) \;\forall\; n > n_0$ and $c > 0$.

(v) small omega (ω) aka rho :-
It gives the lower bound of the function (or algorithm).



$f(n) = \omega(g(n))$
iff

$f(n) > c.g(n)$
$\forall\; n > n_0$ and $c > 0$.

Q2 for (i=1 to n)
i = i*2

$$\sum_{i=1}^{n \text{ step} * 2} i$$

$\Rightarrow$ 1, 2, 4 ---- , n (k-terms).

$\Rightarrow 2^{(k-1)} = n.$

taking log

$k-1 = \log(n).$

$k = \log(n)+1.$

$\Rightarrow$ complexity = $O(\log n)$

Q3

$$T(n) = \begin{cases} 3T(n-1) & n>0. \\ 1 & \text{otherwise} \end{cases}$$

$T(n) = 3T(n-1) \quad \forall \; n>0.$ — (1)

putting $n = n-1$ in eq. (1).

$T(n-1) = 3T(n-2)$ — (2)

putting eq. (2) in (1).

$T(n) = 3(3T(n-2))$

$= 3^2 \cdot T(n-2)$ — (3)

putting $n = n-2$ in eq. (1).

$T(n-2) = 3T(n-3)$ — (4)

putting eq (4) in (3).

$T(n) = 3^3 \, T(n-3).$ — (5)

$\Rightarrow T(n) = 3^k T(n-k)$ — (6)

Base case

$T(0) = 1$

$\Rightarrow n - k = 0$

$n = k$

putting ~~oo~~ ~~oo~~ $k = n$ in eq (6).

$T(n) = 3^n T(n-n)$

$= 3^n T(0)$

$= 3^n$

$\Rightarrow T(n) = O(3^n)$

Q4

$$T(n) = \begin{cases} 2T(n-1)+1 & n > 0 \\ 1 & n \leq 0 \end{cases}$$

$T(n) = 2T(n-1)+1 \quad \forall\, n > 0.$ — (1).

putting $n = n-1$ in eq (1).

$T(n-1) = 2T(n-2)+1$ — (2).

putting eq (2) in (1).

$T(n) = 2(2T(n-2)+1)+1$

$= 2^2 T(n-2) + 1 + 2$ — (3).

putting $n = n-2$ in eq (1).

$T(n-2) = 2T(n-3)+1$ — (4).

putting eq (4) in eq. (3).

$T(n) = 2^2(2T(n-3)+1)+1+2$

$= 2^3 T(n-3) + (1 + 2 + 2^2)$

$\Rightarrow T(n) = 2^k T(n-k) - (1+2+2^2 - - - +2^k)$. — (5)

~~$1+2+2^2 + - - + 2^k$~~

Base case $\Rightarrow T(0) = 1$

$n-k=0$

$k=n$

putting $k=n$ in eqn (5)

$T(n) = 2^n T(n-n) - (1+2+2^2+ - - - + 2^n)$

$= 2^n - (1+2+2^2+ - - - + 2^n)$

$= 2^n - \dfrac{2^n - 1}{2-1}$

$= 2^n - (2^n - 1)$

$= \cancel{2^n} + 1 - \cancel{2^n}$

$\Rightarrow T(n) = O(1)$

5.) int i=1, s=1;

```
while (s<=n)
{
    i++; s=s+i;
    printf("#");
}
```

i	s
1	1
2	3
3	6
4	10

i ~~1~~ ~~2~~ ~~3~~ 4 s ~~1~~ ~~3~~ ~~6~~ 10

1, 3, 6, 10 --- n terms

$\Rightarrow S = 1 + 3 + 6 + 10 + - - - + k$.

$S = \quad 1 + 3 + 6 + 10 - - + k_{n-1} + k_n$

$0 = 1+2+3+4 - - - - k$

$k(k-1)$

$\Rightarrow n \approx k^2$

$k = \sqrt{n}$

$\Rightarrow T(n) = O(\sqrt{n})$

Q6

```
void fuction (int n) {
    int i, count =0;
        for(i= 1; i*i <=n; i++)
                    count ++;
}
```

$1, 2, 3, \text{——} \sqrt{n}$

$\Rightarrow T_n = O(\sqrt{n})$

Q7

```
void fuction (int n) {
    int i, j, k, count =0;
    for (int i= n/2 ; i<=n; ++i)
        for(j=1; j<=n; j=j*2)
            for (k=1; k<=n; k=k*2)
                count ++;
}.
```

$$\sum_{i=n/2}^{n} \sum_{j=1}^{n(j*2)\,\text{step}} \sum_{\substack{k=1\\ \text{step}=k*2}}^{n} 1.$$

$$\sum_{i=n/2}^{n} \sum_{\substack{j=1\\ \text{step } j*2}}^{n} \log(n).$$

$$\sum_{i=n/2}^{n} (\log(n))^2$$

$$(n/2+1)(\log(n))^2 \Rightarrow T(n) = O(n(\log

Q8

```
function (int n) {
    if (n==1) return;
    for (i=1 to n) {
        for (j=1 to n) {
            printf("*");
        }
    }
    function (n-3);
}
```

$$\sum_{i=1}^{n} \sum_{j=1}^{n} 1 = \sum_{i=1}^{n} n = n^2$$

$T(n) = T(n-3) + n^2$ — (1)

putting $n = n-3$ in eq (1)

$T(n-3) = T(n-6) + (n-3)^2$ — (2)

putting (2) in eq (1).

$T(n) = T(n-6) + n^2 + (n-3)^2$ — (3)

putting $n = n-6$ in eq (1).

$T(n-6) = T(n-9) + (n-6)^2$ — (4)

putting eq (4) in (3).

$T(n) = T(n-9) + n^2 + (n-3)^2 + (n-6)^2$

$\Rightarrow T(n) = T(n-3k) + n^2 + (n-3)^2 + \cdots + (n+3(k-1))^2$

$T(1) = 0.$

$n - 3k = 1$

$k = \frac{n-1}{3}.$

$T(n) = n^2 + (n-3)^2 + \cdots + (n-k)^2.$

$\Rightarrow T(n) = n^3$

Q9 void function (int n) {
for (i=1 to n)
for (j=1; j<=n; j=j+i)
printf("*");
}

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ Step=i}}^{n} 1$$

$n = 1+(k-1)i$

$\frac{n-1}{i}+1 = k$

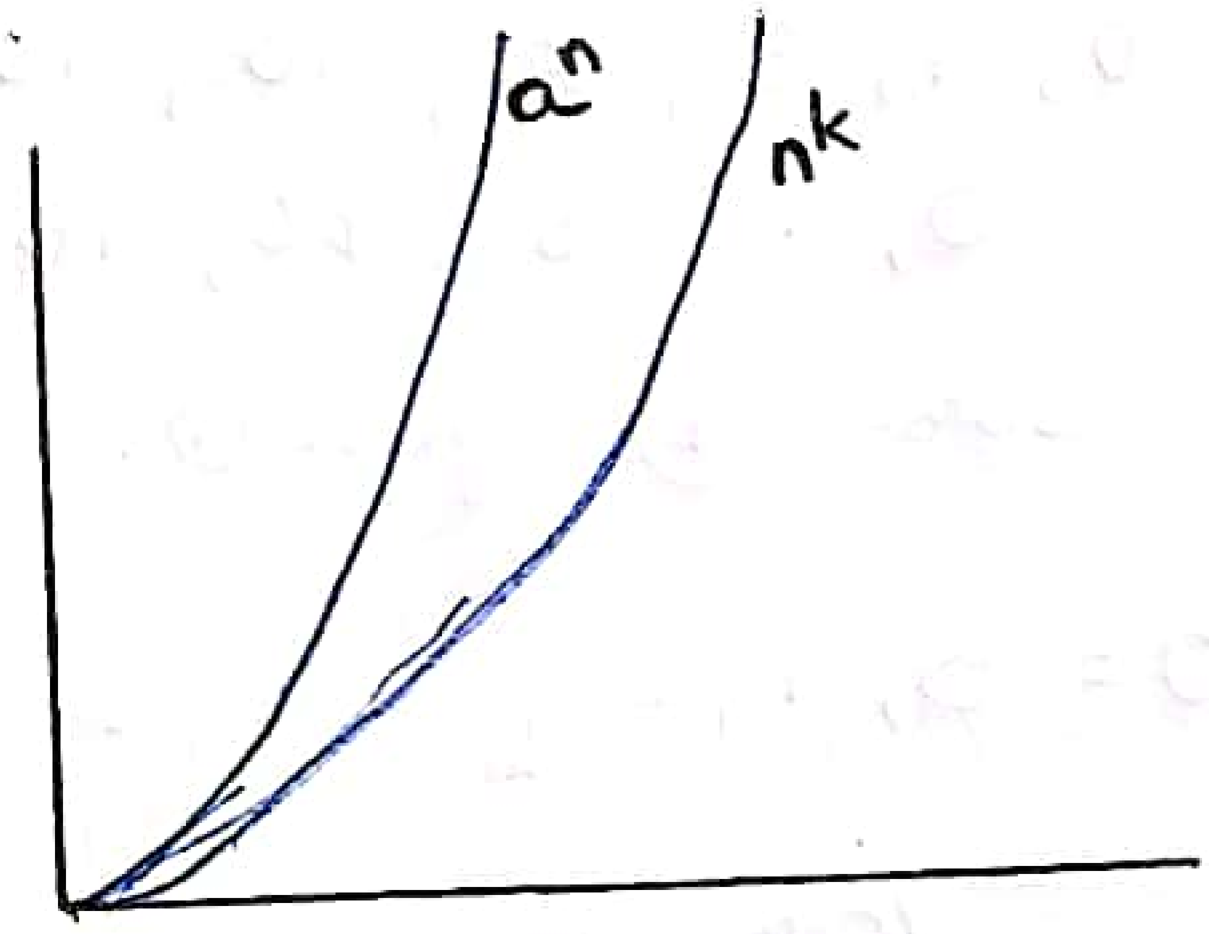
$$\sum_{i=1}^{n} \left(\frac{n-1}{i}+1\right)$$

$$(n-1) \sum_{i=1}^{n} 1/i + \sum_{i=1}^{n} 1.$$

$$(n-1).\log n + n.$$

$$\Rightarrow T(n) = O(n \log n)$$

Q10



$$\Rightarrow n^k = O(a^n)$$

$\therefore n^k \leq a^n . c \quad \forall c>0$ and $n \geq n_0$

let $n = n_0$.

$\Rightarrow n_0^k \leq c . a^{n_0}$

$n_0^3 \leq c . 3^{n_0}$. $k = a = 3$ (say)

$\Rightarrow c \geq 1$ & $n_0 \geq 1$.

Q11

```
void fun (int n) {
    int j=1, i=0;
    while (i<n) {
        i = i+j;
        j++
}
```

j	i
1	0
2	1
3	3
4	6
5	10
6	15
7	21

n $\boxed{20.}$

$S = 0, 1, 3, 6, 10, 15 \text{ --- } k$ — ①

$S = \quad 0, 1, 3, 6, 10 \text{ ------ } + k$ — ②.

Sub. ② from ①.

$0 = 0, 1, 2, 3, 4, 5 \text{ -- } +k - k$

$n = 0 + 1 + 2 \text{ ---- } k.$

$n = \frac{k(k-1)}{2}.$

$\Rightarrow n \simeq k^2$

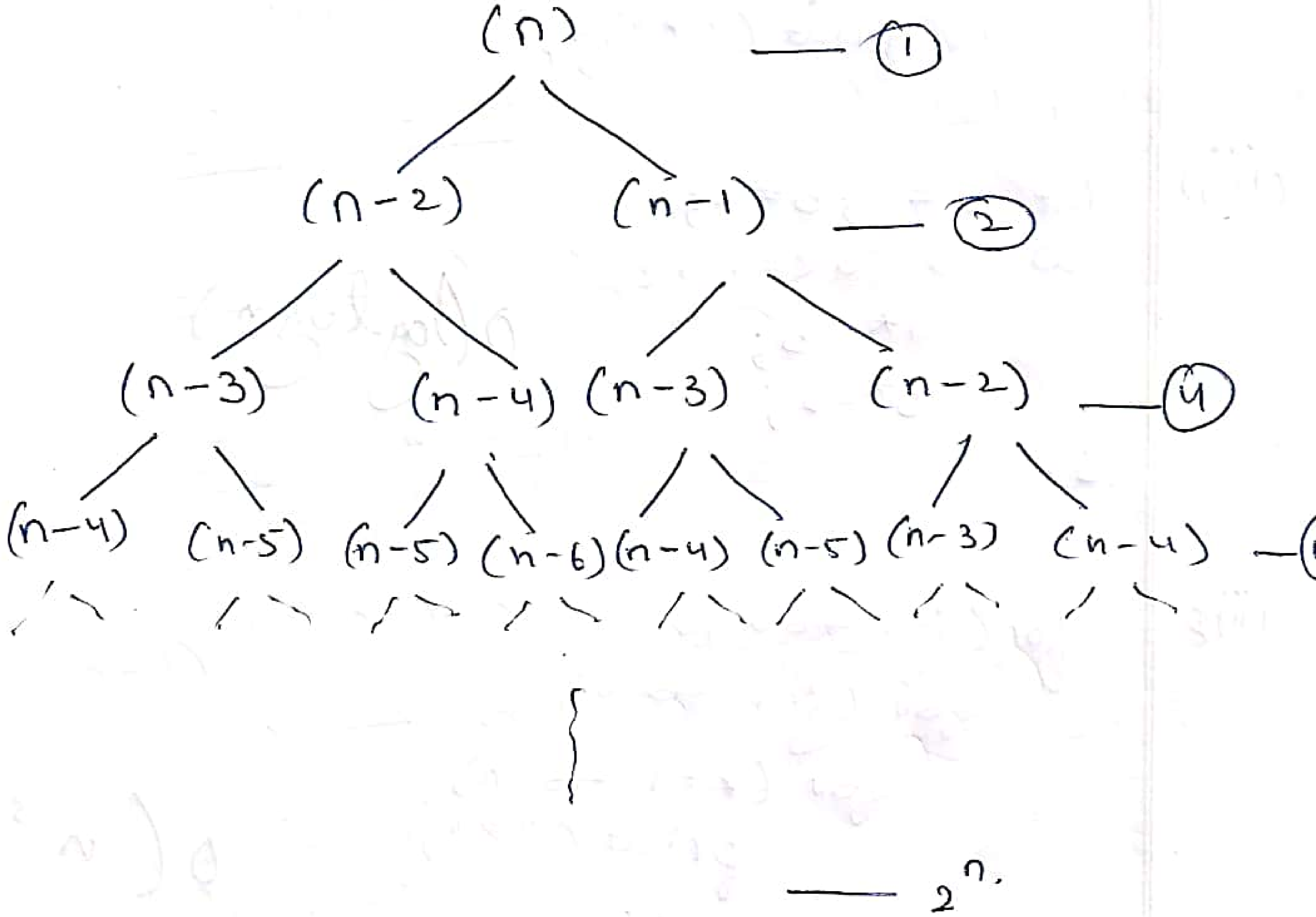
$k = \sqrt{n}$

$\Rightarrow T(n) = O(\sqrt{n})$.

Q_{12}

$0, 1, 1, 2, 3, \text{---}, T_n.$

$$T(n) = T(n-2) + T(n-1) + 1.$$



$$T = 1 + 2 + 4 + \text{---} + 2^n$$

$a = 1$

$r = 2.$

$$T = 1\,\frac{(2^{n+1} - 1)}{2 - 1}$$

$$= 2^{n+1} - 1$$

$$T(n) = O(2^n).$$

Q13

(i)

```
for (int i=0; i<=n; ++i)
    for (int j=1; j<=n; j*=2)
        print ("*");
```

O(n log n)

(ii)

```
int a=1, b=2;
while (a<=n) {
    a*=b;
    b*=2;
}.
```

O(log log n)

(iii)

```
for (i=1 to n) {
    for (j=1 to n) {
        for (k=1 to n) {
            print ("*");
        }
    }
}
```

$O(n^3)$

$$T(n) = T(n/4) + T(n/2) + cn^2$$

$T(n)$ — cn^2

$T(n/4)$ $T(n/2)$ — $cn^2 (3/2)$

$T(n/16)$ $T(n/8)$ $T(n/8)$ $T(n/4)$ — $cn^2 (3/$

$T(n/64)$ $T(n/32)$ $T(n/32)$ $T(n/16)$ — $c.(3/4)^3 n^2$

— $cn^2 (3/4)^k$

$$\frac{n}{2^k} = 1$$

$$n = 2^k$$

$$k = \log n$$

$$T(n) = cn^2 \left[1 + \left(\frac{3}{4}\right) + (3/4)^2 + \dots (3/4)^{\log n}\right]$$

$$= cn^2 \cdot (1)$$

$$= n^2$$

$$\Rightarrow T(n) = O(n^2).$$

15

```
int fun(int n) {
    for (int i=1; i<=n; ++i) {
        for (int j=1; j<n; j = j+i) {
            Some O(1) task
        }
    }
};
```

$$T(n) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ \text{step } i}}^{n-1} (1)$$

$$= \sum_{i=1}^{n} \frac{n-1}{i}$$

$$= (n-1)\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots n\right)$$

$$= (n-1)\log n$$

$$T(n) = n\log n$$

$$\Rightarrow T(n) = O(n\log n)$$

Q16

```
for (int i=2; i<=n; i=pow(i,k)) {
    Some O(1) expression
}
```

$$i = 2 \quad 2^k \quad 2^{k^2} \quad 2^{k^3} \quad \cdots$$

$$\Rightarrow 2^{k^x} = n$$

$\therefore k$ is constant

$$\log(n) = k^x \log 2$$

$$\log(n) = k^x$$

$$\log_k(\log(n)) = x \log k$$

$$\Rightarrow x = \log(\log(n))$$

$$\Rightarrow T(n) = O(\log\log(n))$$

17 $T(n) = T\left(\frac{99}{100}n\right) + n/100.$

$T(1) = 0$ —— (1)

putting $n = \frac{99}{100}n$ in eq (1).

$T(99/100\, n) = T\left(\left(\frac{99}{100}\right)^2 n\right) + n/100$ —— (2)

putting eq (2) in (1).

$T(n) = T\left(\left(\frac{99}{100}\right)^2 n\right) + 2n/100.$ —— (3)

$|$

$T(n) = T\left(\left(\frac{99}{100}\right)^k n\right) + kn/100.$ —— (4)

$\left(\frac{99}{100}\right)^k n = 1.$

$n = \left(\frac{100}{99}\right)^k$

$k = \log_{\frac{100}{99}} n.$ —— (5)

putting $k = \log_{\frac{100}{99}} n$ in eq (4).

$$T(n) = \frac{n\left(\log_{\frac{100}{99}} n\right)}{100}$$

$\Rightarrow T(n) = O(n \log n)$

18

(a) $100 < \log\log n < \log n < \sqrt{n} < n <$
$n\log n = \log(n!) < n^2 < 2^n < 2^{2^n}$
$4^n < n!$

(b) $1 < \log\log(n) < \sqrt{\log(n)} < \log(n) <$
$2n < 4n < 2(2^n) < \log(2N) <$
$2\log(n) < n < n\log n = \log(n!) <$
$< N!$

(c). $96 < \log_2(n) = \log_8(n) < n\log_6(n) = n\log$
$= \log(n!) < 5n < 8n^2 < 7n$
$< 8^{2n}.$

Q19

```
INPUT ARR[N], KEY;
for (i = 0 to n-1) {
    if (ARR[i] = key) {
        return i;
    }
}
return -1;
```

d20 (a) Iterative Insertion Sort

```
void InsertionSort (int arr[], int n)
{
    int i, temp, j;
    for (i = 1 to n-1) {
        temp = arr[i]
        j = i-1;
        while (j>=0 AND arr[j] > temp) {
            arr [j+1] = arr[j];
            j = j-1;
        }
        arr [j+1] = temp;
    }
}.
```

(b) Recursive ~~Sort~~ Insertion Sort

```
void InsertionSort (int arr[], int n)
{
    if (n<2)
        return.

    InsertionSort (arr, n-1);
    last = arr[n-1], j=n-2;
    while ( j>=0 AND arr[j] > temp {
            arr [j+1] = arr [j];
            j = j-1;
    }
    arr [j+1] = last;
}
```

Insertion sort is online algorithm because it process the element one-by-one in a serial fashion without considering the future elem. Whereas bubble sort, selection sort and merge sort are offline as they require all inputs on which they can process the data for correct output i.e these algorithms want all the input beforehand.

Q21

	Algorithm	Best Case	Avg. Case	Worst Case
1	Bubble Sort	$O(n^2)$	$O(n^2)$	$O(n^2)$
2	Selection Sort	$O(n^2)$	$O(n^2)$	$O(n^2)$
3	Insertion Sort	$O(n)$	$O(n^2)$	$O(n^2)$
4	Merge Sort	$O(n\log n)$	$O(n\log n)$	$O(n\log n)$
5	Quick Sort	$O(n\log n)$	$O(n\log n)$	$O(n^2)$
6	Heap Sort	$O(n\log n)$	$O(n\log n)$	$O(n\log n)$

Q22

	Algorithm	In-place	Stable	Online
1	Bubble Sort	✓	✓	✗
2	Selection Sort	✓	✗	✗
3	Insertion Sort	✓	✓	✓
4	Merge Sort	✗	✓	✗
5	Quick Sort	✗	✗	✗
6	Heap Sort	✓	✗	✗

Q23 Iterative Binary search

```
int BinarySearch (int arr[], int l, int r, int
{
    while (l<=r){
        int m = (l+r)/2;
        if (arr[m]= x)
            return m;
        else if (arr[m] < x)
            l= m+1;

        else
            r = m-1;
    }
    return -1;
}
```

Recursive Binary Search

```
int BinarySearch(int arr[], int l, int r, int x)
{
    if (l>r)
        return -1
    int m = (l+r)/2;
    if (arr[m]= x)
        return m;
    else if (arr[m] < x)
    return BinarySearch(arr, m+1, r, x);
    else
}       return BinarySearch (arr, l, m-1,
```

→ Iterative Binary Search

(c) Time Complexity

Best Case = $O(1)$

Average Case = $O(\log n)$

Worst Case = $O(\log n)$.

(c) Space Complexity = $O(1)$
(All cases)

→ Recursive Binary Search

(d) Time Complexity

Best Case = $O(1)$

Average Case = $O(\log n)$

Worst Case = $O(\log n)$

(d) Space complexity :

Best Case = $O(1)$

Average Case = $O(\log n)$

Worst Case = $O(\log n)$

Q4 $T(n) = T(n/2) + 1$

$T(n) = O(\log n)$.