Program : **B.Tech**

Subject Name: **Signals and Systems**

Subject Code: **EC-402**

Semester: **4th**

Unit-2 Linear Time- Invariant Systems: Introduction, Impulse Response Representation for LTI Systems, Convolution, Properties of the Impulse Response Representation for LTI Systems, Difference Equation for LTI Systems, Block Diagram Representations (direct form-I, direct form- II, Transpose, cascade and parallel). impulse response of DT-LTI system and its properties.

Impulse response and convolution integral:

Convolution is a mathematical operation which is used to find the response of LTI system. In the LTI system analysis it relates the impulse response of the system and input signal to the output. Any signal x(t) can be represented as a continuous sum of impulse signals. The response y(t) can then be represented as sum of responses of various impulse components.

The impulse response is denoted as $h(t) = T[\delta(t)]$

Any arbitrary signal can be represented as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

The system output is given as $y(t) = T[x(t)]$

$$\therefore y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right]$$

For a linear system

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)T[\delta(t-\tau)]d\tau$$

If the system response due to impulse signal is h(t), then the response of the system due to delayed impulse signal is $h(t, \tau)$

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)d\tau$$

For a time invariant system, the output due to input delayed by τ sec is equal to the output delayed by τ sec. that is

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Properties of Convolution:

(i) Commutative Property : The commutative property of convolution state that.

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

(ii) Distributive Property : The distributive property of convolution state that.

$$x_1(t) *[x_2(t)+ x_3(t)]= x_1(t) * x_2(t)+ x_1(t) * x_3(t)$$

(iii) Associative Property : The associative property of convolution state that.

$$x_1(t) *[x_2(t)* x_3(t)]= [x_1(t) * x_2(t)] * x_3(t)$$

Example 1: For an LTI system with unit impulse response $h(t) = e^{-2t} \quad \text{for} \quad t \geq 0$, find the system response for the input signal $x(t) = A \quad \text{for} \quad 0 \leq t \leq 2$. Sketch the output signal.

Solution :

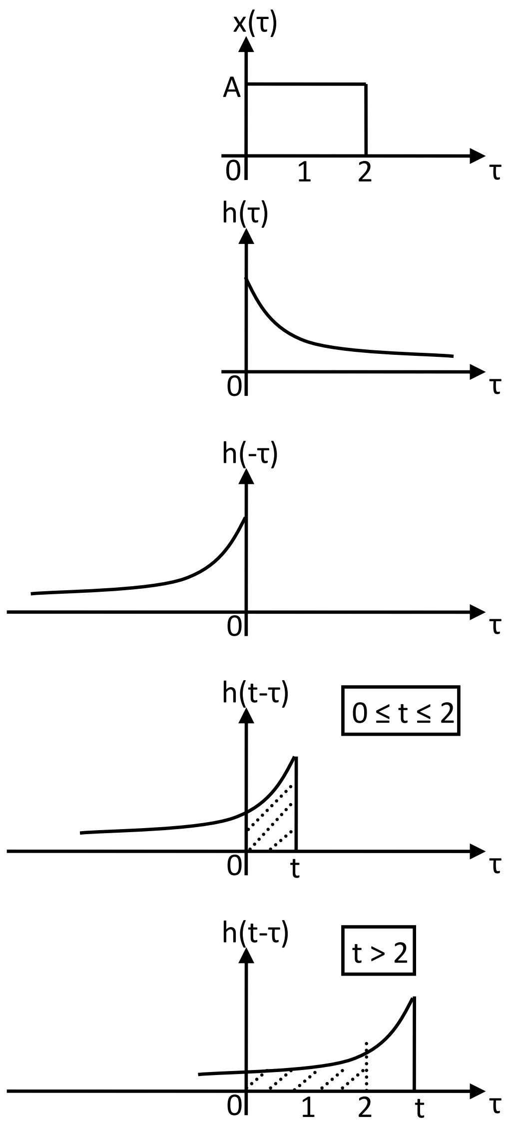


Fig.1a :Evaluation of Convolution Integral

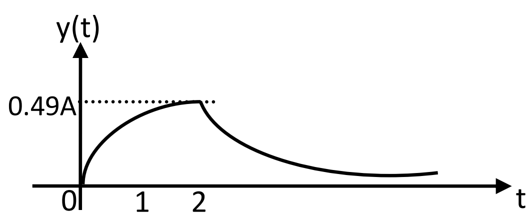


Fig.1b :Sketch of output Signal

Given signals x(t) and h(t) can be written in terms of 'τ',

$$x(\tau) = A \quad \text{for} \quad 0 \le \tau \le 2$$

and $h(\tau) = e^{-2\tau} \quad \text{for} \quad \tau \ge 0$

Now $h(t-\tau) = e^{-2(t-\tau)} \quad \text{for} \quad t-\tau \ge 0$

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig. .

The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between

Case – I : For 0 ≤ t ≤ 2

In this case, there will be partial overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig.

$$\therefore y(t) = \int_0^t x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_0^t A\, e^{-2(t-\tau)} d\tau = Ae^{-2t} \int_0^t e^{2\tau} d\tau$$

$$= Ae^{-2t}\, \tfrac{1}{2} [e^{2\tau}]_0^t = Ae^{-2t}\, \tfrac{1}{2} (e^{2t} - 1)$$

$$\therefore y(t) = \frac{A}{2}(1 - e^{-2t})$$

Case – I : For t > 2

In this case, there will be complete overlap from 0 to 2 as shown in the Fig.

$$\therefore y(t) = \int_0^2 x(\tau)\, h(t-\tau) d\tau$$

$$= \int_0^2 A\, e^{-2(t-\tau)} d\tau = Ae^{-2t} \int_0^2 e^{2\tau} d\tau$$

$$\therefore y(t) = Ae^{-2t}\, \frac{1}{2} [e^{2\tau}]_0^2 = 26.8Ae^{-2t}$$

$$\text{Thus } y(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ \frac{A}{2}(1 - e^{-2t}) & \text{for} \quad 0 \le t \le 2 \\ 26.8Ae^{-2t} & \text{for} \quad t < 0 \end{cases}$$

Example 2: Consider $x(t) = 2 \;\text{for}\; 1 \le t \le 2$
and $h(t) = 1 \;\text{for}\; 0 \le t \le 3$, find x(t) * h(t)

Solution :

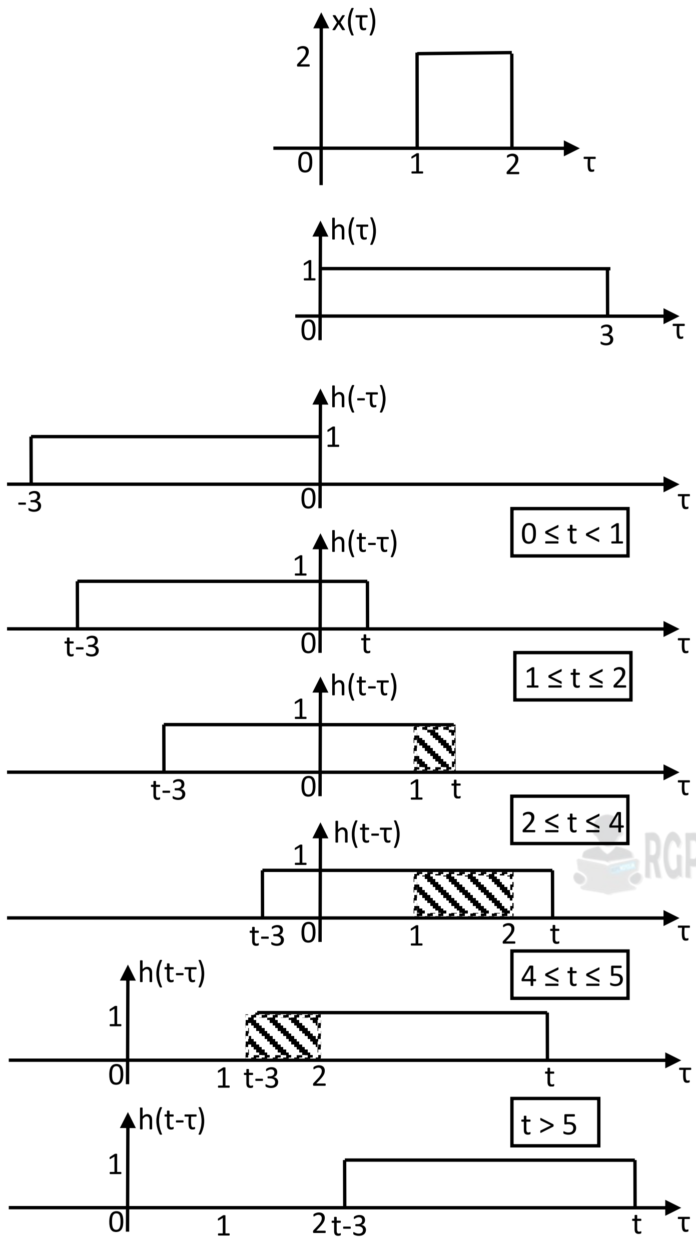


Fig.2a :Evaluation of Convolution Integral

Given signals x(t) and h(t) can be written in terms of 'τ',

$$x(\tau) = 2 \quad \text{for } 1 \le \tau \le 2$$

and $h(\tau) = 1 \quad \text{for } 0 \le \tau \le 3$

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.2a.

The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For 0 ≤ t < 1

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For 1 ≤ t ≤ 2

In this case, there is partial overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are 1 to t

$$\therefore y(t) = \int_{1}^{t} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{1}^{t} 2.1.d\tau = 2 \int_{1}^{t} d\tau = 2\,[\tau]_{1}^{t} = 2t$$

$$\boxed{\therefore y(t) = 2t}$$

Case – III : For 2 ≤ t ≤ 4

In this case, there is complete overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are 1 to 2

$$\therefore y(t) = \int_{1}^{2} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{1}^{2} 2.1.d\tau = 2 \int_{1}^{2} d\tau = 2\,[\tau]_{1}^{2} = 2$$

$$\boxed{\therefore y(t) = 2}$$

Case – IV : For 4 ≤ t ≤ 5

In this case, there is partial overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-3 to 2

$$\therefore y(t) = \int_{t-3}^{2} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-3}^{2t} 2.1.d\tau = 2 \int_{t-3}^{2t} d\tau = 2\,[\tau]_{t-3}^{2} = 2(1-t)$$

$$\boxed{\therefore y(t) = 2(1-t)}$$

Case – V : For t > 5

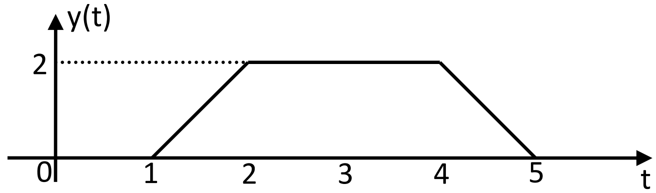


Fig.2b :Sketch of output Signal

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

<u>Example 3:</u> Consider $x(t) = u(t+2)$
and $h(t) = u(t-3)$, find x(t) * h(t)

Solution :

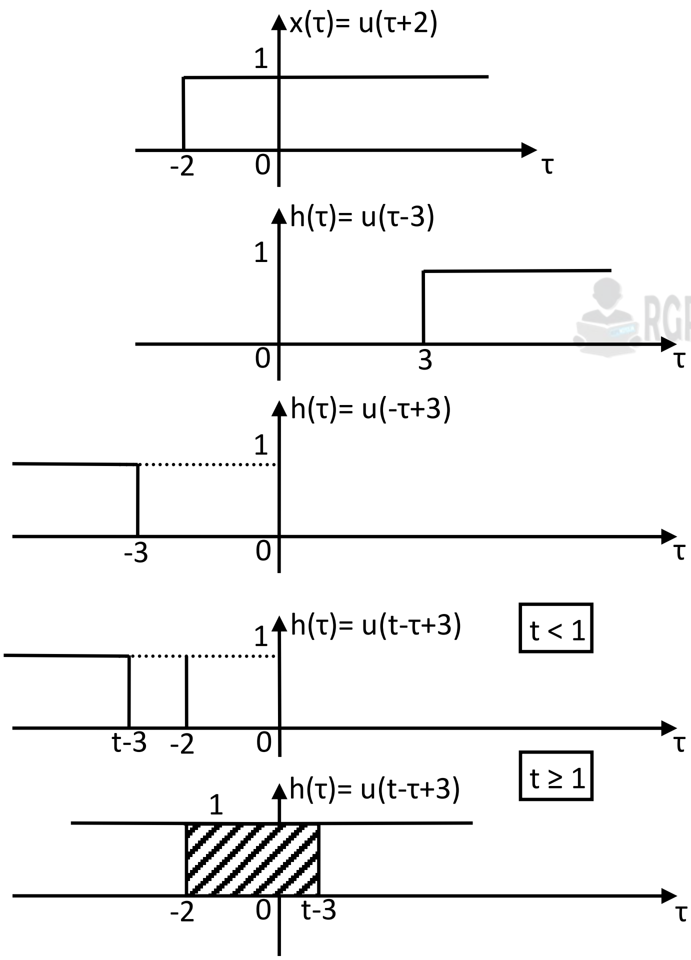


Fig.3a :Evaluation of Convolution Integral

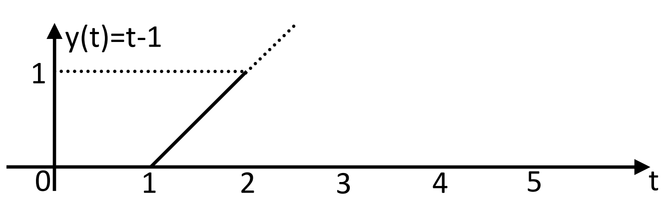


Fig.3b :Sketch of output Signal

Given signals x(t) and h(t) can be written in terms of 'τ',

$$x(\tau) = u(\tau+2)$$

and $h(\tau) = u(\tau-3)$

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.3a.
The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For t < 1
In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For t ≥ 1
In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are -2 to t-3

$$\therefore y(t) = \int_{-2}^{t-3} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{-2}^{t-3} 1.1. d\tau = \int_{1}^{t} d\tau = [\tau]_{-2}^{t-3} = (t-3+2)$$

$$\boxed{\therefore y(t) = t-1}$$

Example 4: Obtain the convolution of the following two signals

$$x(t) = \begin{cases} 1 & \text{for } -3 \le t \le 3 \\ 0 & \text{else where} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} 2 & \text{for } 0 \le t \le 3 \\ 0 & \text{else where} \end{cases}$$

Solution :

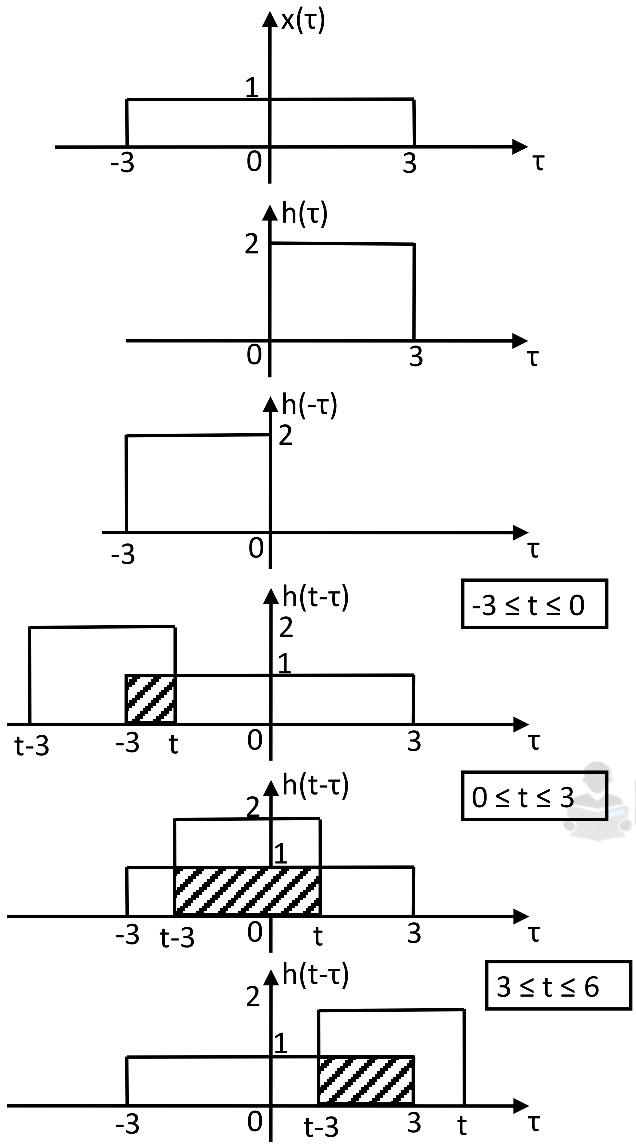


Fig.4a :Evaluation of Convolution Integral

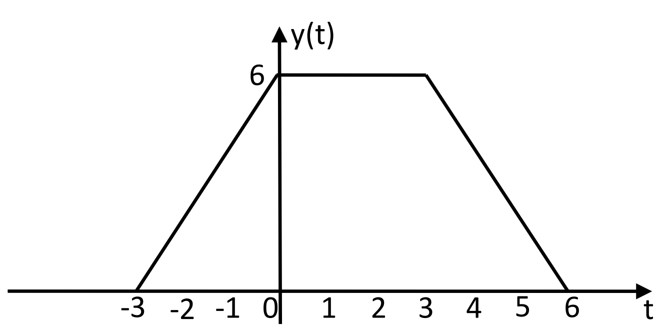


Fig.4b :Sketch of output Signal

Given signals x(t) and h(t) can be written in terms of 'τ',

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.4a.

The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For t < -3

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For -3 ≤ t ≤ 0

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are -3 to t

$$\therefore y(t) = \int_{-3}^{t} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{-3}^{t} 1.2.\, d\tau = 2\int_{-3}^{t} d\tau = 2[\tau]_{-3}^{t} = 2(t+3)$$

$$\boxed{\therefore y(t) = 2(t+3)}$$

Case – III : For 0 ≤ t ≤ 3

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-3 to t

$$\therefore y(t) = \int_{t-3}^{t} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-3}^{t} 1.2.\, d\tau = 2\int_{t-3}^{t} d\tau = 2[\tau]_{t-3}^{t} = 2(t-t+3)$$

$$\boxed{\therefore y(t) = 6}$$

Case – IV : For 3 ≤ t ≤ 6

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-3 to 3

$$\therefore y(t) = \int_{t-3}^{3} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-3}^{3} 1.2.\, d\tau = 2\int_{t-3}^{3} d\tau = 2[\tau]_{t-3}^{3} = 2(3-t+3)$$

$$\boxed{\therefore y(t) = 2(-t+6)}$$

Case – V : For t >6

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Example 5: Obtain the convolution of the following two signals shown in Fig.5a.

Solution :

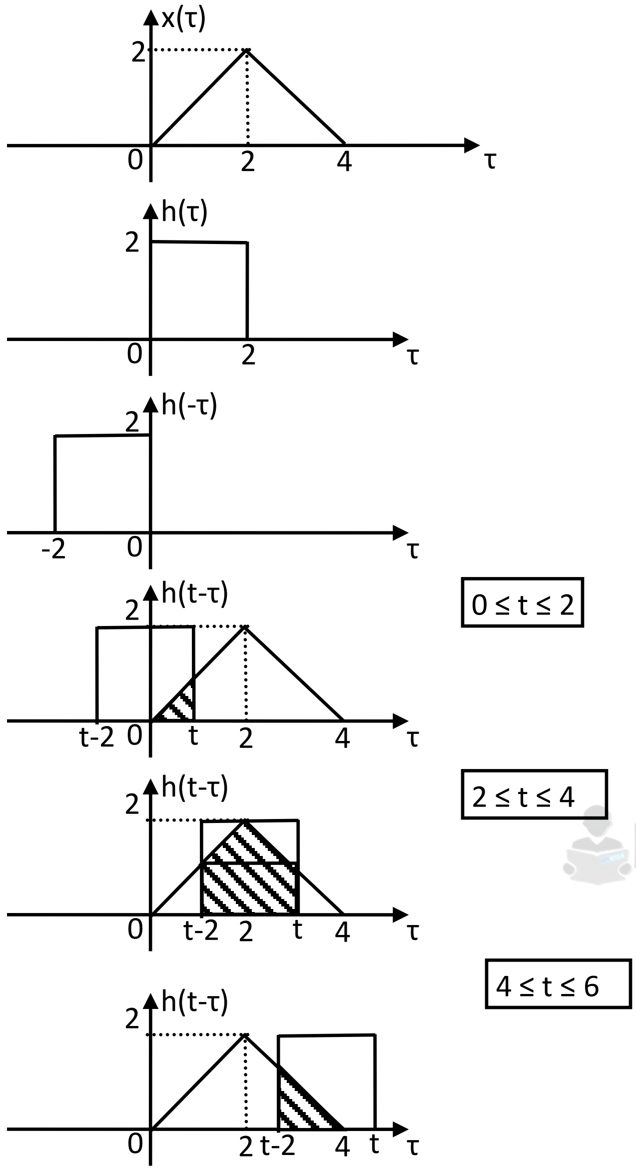


Fig.5a :Evaluation of Convolution Integral

Given signals x(t) and h(t) can be written in terms of 'τ',

The signals $x(\tau)$ and $h(t-\tau)$ are shown in Fig.5a.
The output of the system is given by the linear convolution as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) d\tau$$

The integral is non Zero for overlap between $x(\tau)$ and $h(t-\tau)$

Case – I : For t < 0

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Case – II : For 0 ≤ t ≤ 2

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are 0 to t

$$\therefore y(t) = \int_0^t x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_0^t \tau.2.d\tau = 2\int_0^t \tau d\tau = 2\left[\frac{\tau^2}{2}\right]_0^t = t^2$$

$$\boxed{\therefore y(t) = t^2}$$

Case – III : For 2 ≤ t ≤ 4

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-2 to t

$$\therefore y(t) = \int_{t-2}^{t} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-2}^{2} \tau.2.d\tau + \int_{2}^{t} (4-\tau).2.d\tau$$

$$\therefore y(t) = 2\left[\frac{\tau^2}{2}\right]_{t-2}^{2} + 2\left[\frac{(4-\tau)^2}{2}\right]_{2}^{t}$$

$$\therefore y(t) = (4-(t-2)^2) + ((4-t)^2 - 4)$$

$$\boxed{\therefore y(t) = 12 - 4t}$$

Case – IV : For 4 ≤ t ≤ 6

In this case, there is overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. The limits of integration are t-2 to 4

$$\therefore y(t) = \int_{t-2}^{4} x(\tau)\, h(t-\tau) d\tau$$

$$\therefore y(t) = \int_{t-2}^{4} (4-\tau).2.d\tau = 2\left[\frac{(4-\tau)^2}{2}\right]_{t-2}^{4}$$

$$\boxed{\therefore y(t) = -(6-t)^2}$$

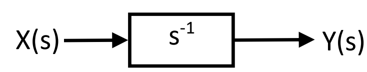
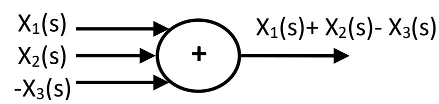
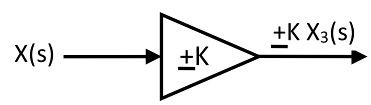
Case – V : For t >6

In this case, there is no overlap between $x(\tau)$ and $h(t-\tau)$ as shown in the Fig. $\therefore y(t) = 0$

Block Diagram Representations:

There are four types of system realisation of continuous time LTI systems. They are
(i) Direct form – I (ii) Direct form-II (iii) Cascade form (iv) Parallel form

A transfer function can be realized using either integrators or differentiators. Since the differentiator amplifies the high frequency noise, the integrators are most commonly used in the realization of system as they suppress the high frequency noise. The adder and multipliers are used along with integrator to realize the continuous time systems.

Type of Operator	Ideal Transfer Function	Block Representation
Integrator	$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s} = s^{-1}$	
Adder	$X_1(s) + X_2(s) - X_3(s)$	
Multiplier	$KX(s)$	

Note: The non-causal systems are not physically realizable. Therefore the system must be causal in this realisation.

(i) Direct Form-I : Direct form-I realization of system is the simplest form of representation of differential equation. In this separate integrators are used for input and output variables.

Example 2.1: Realize the system with the following transfer function in direct form-I:

$$H(s) = \frac{s^2 + 4s + 3}{s^2 + 2s + 5}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(1 + 4s^{-1} + 3s^{-2})}{s^2(1 + 2s^{-1} + 5s^{-2})} = \frac{1 + 4s^{-1} + 3s^{-2}}{1 + 2s^{-1} + 5s^{-2}}$$

Cross multiplying the above equation,

$$Y(s)[1 + 2s^{-1} + 5s^{-2}] = X(s)[1 + 4s^{-1} + 3s^{-2}]$$

Let $$\boxed{W(s) = X(s) + 4s^{-1}X(s) + 3s^{-2}X(s)}$$

$$\therefore Y(s)[1 + 2s^{-1} + 5s^{-2}] = W(s)$$

$$\boxed{Y(s) = W(s) - 2s^{-1}Y(s) - 5s^{-2}Y(s)}$$

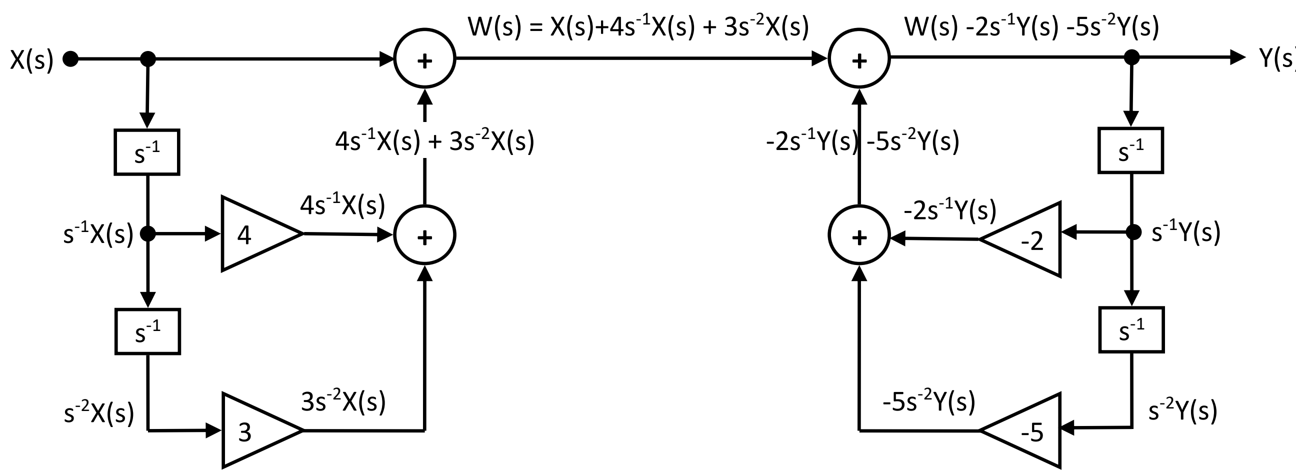


Fig.2.1 : Direct form-I realization of example (2.1)

Example 2.2: Realize the system with the following transfer function in direct form-I:

$$H(s) = \frac{s+5}{s^2 + 2s + 4}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(s^{-1} + 5s^{-2})}{s^2(1 + 2s^{-1} + 4s^{-2})} = \frac{s^{-1} + 5s^{-2}}{1 + 2s^{-1} + 4s^{-2}}$$

Cross multiplying the above equation,

$$Y(s)[1 + 2s^{-1} + 4s^{-2}] = X(s)[s^{-1} + 5s^{-2}]$$

Let $\boxed{W(s) = s^{-1}X(s) + 5s^{-2}X(s)}$

$$\therefore Y(s)[1 + 2s^{-1} + 4s^{-2}] = W(s)$$

$$\boxed{Y(s) = W(s) - 2s^{-1}Y(s) - 4s^{-2}Y(s)}$$

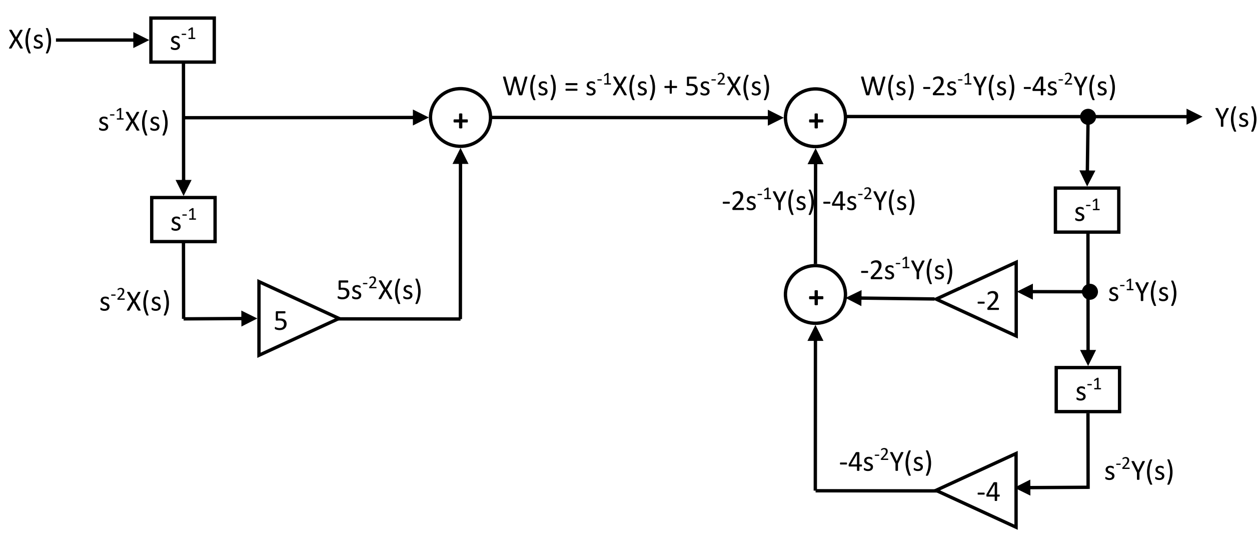


Fig.2.2 : Direct form-I realization of example (2.2)

(ii) Direct Form-II : Direct form-II realization of system uses minimum number of integrators. An intermediate variable is used to integrate the input and output variables.

Example 2.3: Realize the system with the following transfer function in direct form-II

$$H(s) = \frac{s^2 + 3s + 4}{s^2 + 5s + 2}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(1 + 3s^{-1} + 4s^{-2})}{s^2(1 + 5s^{-1} + 2s^{-2})} = \frac{1 + 3s^{-1} + 4s^{-2}}{1 + 5s^{-1} + 2s^{-2}}$$

Divide the transfer function into two parts,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)W(s)}{W(s)X(s)}$$

Where

$$\frac{W(s)}{X(s)} = \frac{1}{1 + 5s^{-1} + 2s^{-2}}$$

and

$$\frac{Y(s)}{W(s)} = 1 + 3s^{-1} + 4s^{-2}$$

Cross multiplying the above equation,

$$\boxed{\begin{aligned} X(s) &= W(s) + 5s^{-1}W(s) + 2s^{-2}W(s) \\ W(s) &= X(s) - 5s^{-1}W(s) - 2s^{-2}W(s) \\ Y(s) &= W(s) + 3s^{-1}W(s) + 4s^{-2}W(s) \end{aligned}}$$

Realize W(s) and Y(s) separately and cascade them to get H(s) as shown in Fig. 5.3(a). Since the inputs and outputs of the integrator are same, they can be commonly used as shown in Fig. 5.3(b).

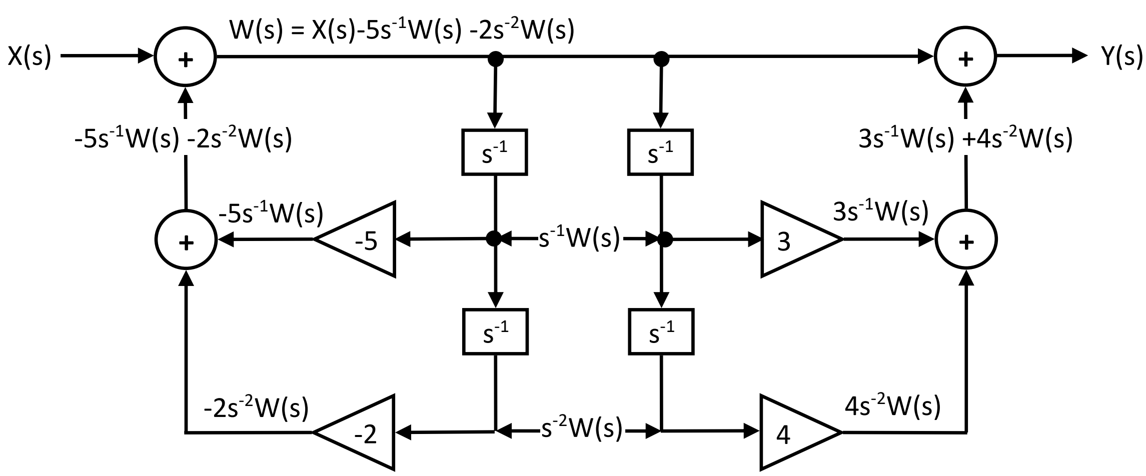


Fig.2.3 (a): W(s) and Y(s) realization of example (2.3)

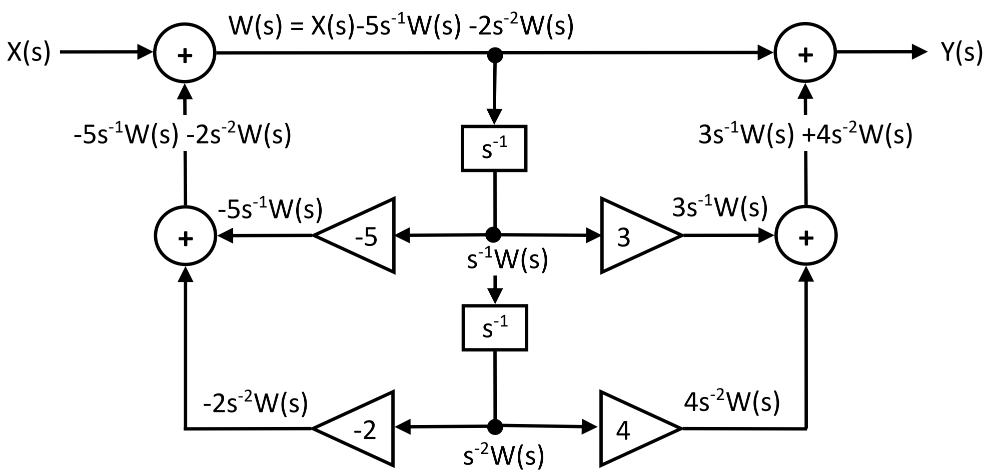


Fig.2.3 (B): Direct Form-II realization of example (2.3)

Example 2.4: Realize the system with the following transfer function in direct form-II

$$H(s) = \frac{s+4}{s^2 + 3s + 5}$$

Solution: To realize the system in direct form-I, the numerator and denominator of given transfer function has to be expressed in power of s^{-1}.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2(s^{-1} + 4s^{-2})}{s^2(1 + 3s^{-1} + 5s^{-2})} = \frac{s^{-1} + 4s^{-2}}{1 + 3s^{-1} + 5s^{-2}}$$

Divide the transfer function into two parts,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{Y(s)W(s)}{W(s)X(s)}$$

Where

$$\frac{W(s)}{X(s)} = \frac{1}{1 + 3s^{-1} + 5s^{-2}} \quad \text{and} \quad \frac{Y(s)}{W(s)} = s^{-1} + 4s^{-2}$$

Cross multiplying the above equation,

$$\boxed{\begin{aligned} X(s) &= W(s) + 3s^{-1}W(s) + 5s^{-2}W(s) \\ W(s) &= X(s) - 3s^{-1}W(s) - 5s^{-2}W(s) \\ Y(s) &= s^{-1}W(s) + 4s^{-2}W(s) \end{aligned}}$$

The given transfer function is realized in Direct Form-II as shown in Fig. 5.4

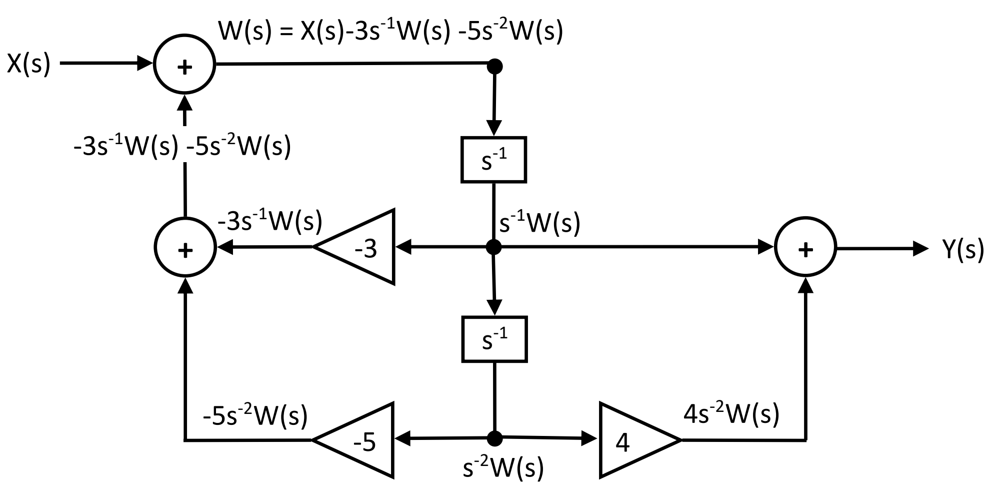


Fig.2.4: Direct Form-II realization of example (2.4)

(iii) Cascade Form : The given transfer function is represented as product of several transfer functions. These transfer functions are realized in direct form-II, and then all of them are cascaded to get the complete realization of given transfer function. The input is connected to the first block and output is taken from last transfer function realization.

Example 2.5 : Realize the system with the following transfer function in cascade form

$$H(s) = \frac{4(s^2 + 4s + 3)}{s^3 + 6.5s^2 + 11s + 4}$$

Solution: Express the given equation as product of several transfer functions, we have

$$H(s) = \frac{4(s+1)(s+3)}{(s+0.5)(s+2)(s+4)} = \frac{4}{(s+0.5)}\frac{(s+1)}{(s+2)}\frac{(s+3)}{(s+4)}$$

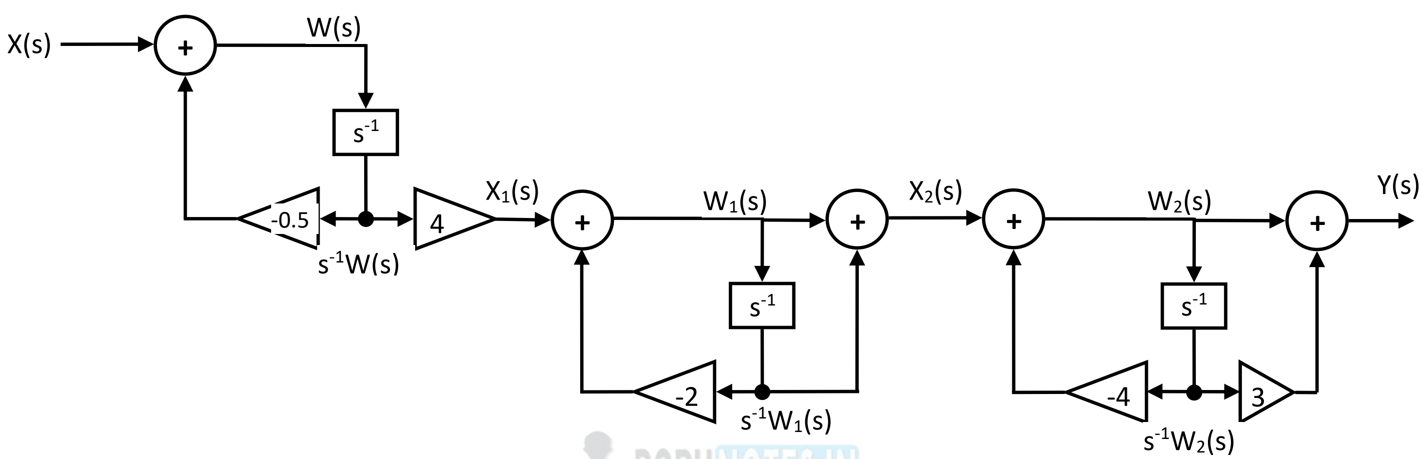


Fig.2.5: Cascade Form realization of example (2.5)

(iv) Parallel Form : The given transfer function is expressed into its partial fractions. These factors are realized in direct form-II, and then all of them are connected in parallel to get the complete realization of given transfer function. The input is connected to the each one of these blocks and outputs are added together.

Example 2.6 : Realize the system with the following transfer function in parallel form

$$H(s) = \frac{s(s+1)}{(s+2)(s+3)(s+5)}$$

Solution: Express the given function in its partial fraction form

$$H(s) = \frac{s(s+1)}{(s+2)(s+3)(s+5)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+5)}$$

$$s(s+1) = A(s+3)(s+5) + B(s+2)(s+5) + C(s+2)(s+3)$$

where the values of coefficients A ,B and C are calculated as,

$$A|_{s=-2} = \frac{-2(-2+1)}{(-2+3)(-2+5)} = \frac{2}{3}$$

$$B|_{s=-3} = \frac{-3(-3+1)}{(-3+2)(-3+5)} = -3$$

$$C|_{s=-5} = \frac{-5(-5+1)}{(-5+2)(-5+3)} = \frac{10}{3}$$

$$\therefore \mathrm{H(s)} = \frac{2/3}{(s+2)} + \frac{-3}{(s+3)} + \frac{10/3}{(s+5)}$$

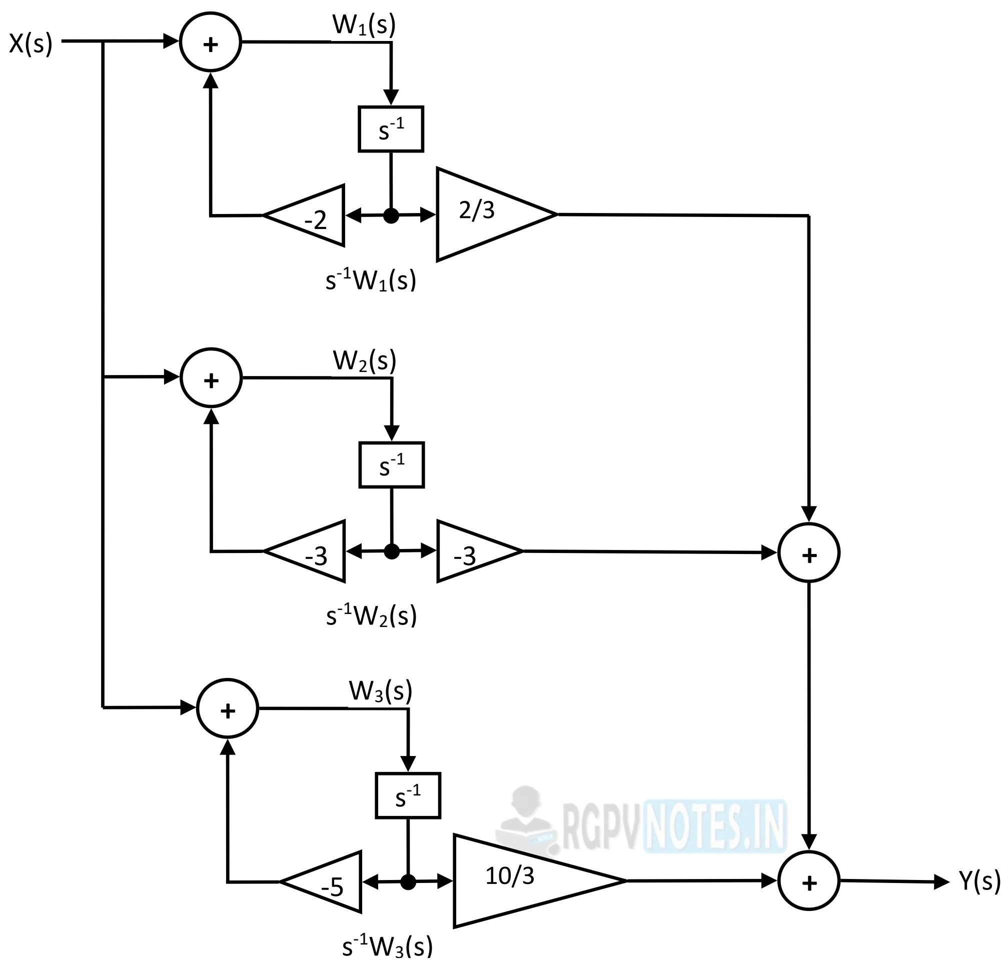


Fig.2.6: Parallel Form realization of example (2.6)

