PSB1

Given: 2 "massless" bodies orbiting about a primary in Kepler orbits.

Take Jupiter with asteroid A and C in same plane relative to Sun.

Find a) Jupiter is circular radius 5.2AU. A is in shorter period 3:2 resonance. Let C be circular in longer period 3:4 resonance. Determine $\mathbb{P}$, n of all 3 bodies.

b) Define inertial frame and plot bodies. Mark location of A and C at time when Jupiter $t = n\mathbb{P}$ $n = 0, 0.25, 0.5, 0.75, 1$. Mark motion of each body.

c) Define rotating frame for Jupiter, identify Sun as origin. Develop transformation matrix. Plot A and C in rotating frame. Again mark A and C the same times as (b).

d) Redo (b) and (c) and assume eccentricity of $e = 0.2, 0.3, 0.4$. Initial time A and C at periapsis. Mark periapsis and apoapsis in rotating frame.

Are resonance interior or exterior?

Assume periapsis 180 degrees out of phase. Repeat plot. Does relative periapsis change the motion. What conditions does A and C pass closest; are bodies located when which pass occurs?

e) Try some higher order resonances and some higher e?

Solution:

a) $$\frac{\mathbb{P}_1}{\mathbb{P}_2} = \frac{\mathbb{P}_J}{\mathbb{P}_A} = \sqrt{\frac{a_J^3}{a_A^3}} = \frac{3}{2} \quad \therefore a_A^3 = \frac{4a_J^3}{9}$$

Note $a_J = 7.7834 \times 10^8$

$$\therefore \frac{a_J^3}{a_C^3} = \frac{9}{16} \quad \therefore a_C^3 = \frac{16a_J^3}{9}$$

$$\boxed{\mathbb{P}_J = \frac{2\pi}{\sqrt{M/a_J^3}} = 3.7421 \times 10^8 s = 11.8662 \text{ years}}$$

$$\boxed{\mathbb{P}_A = \frac{2\pi}{\sqrt{M/a_A^3}} = 2.4947 \times 10^8 s = 7.9108 \text{ years}}$$

$$\boxed{\mathbb{P}_C = \frac{2\pi}{\sqrt{M/a_C^3}} = 4.9895 \times 10^8 s = 15.8216 \text{ years}}$$

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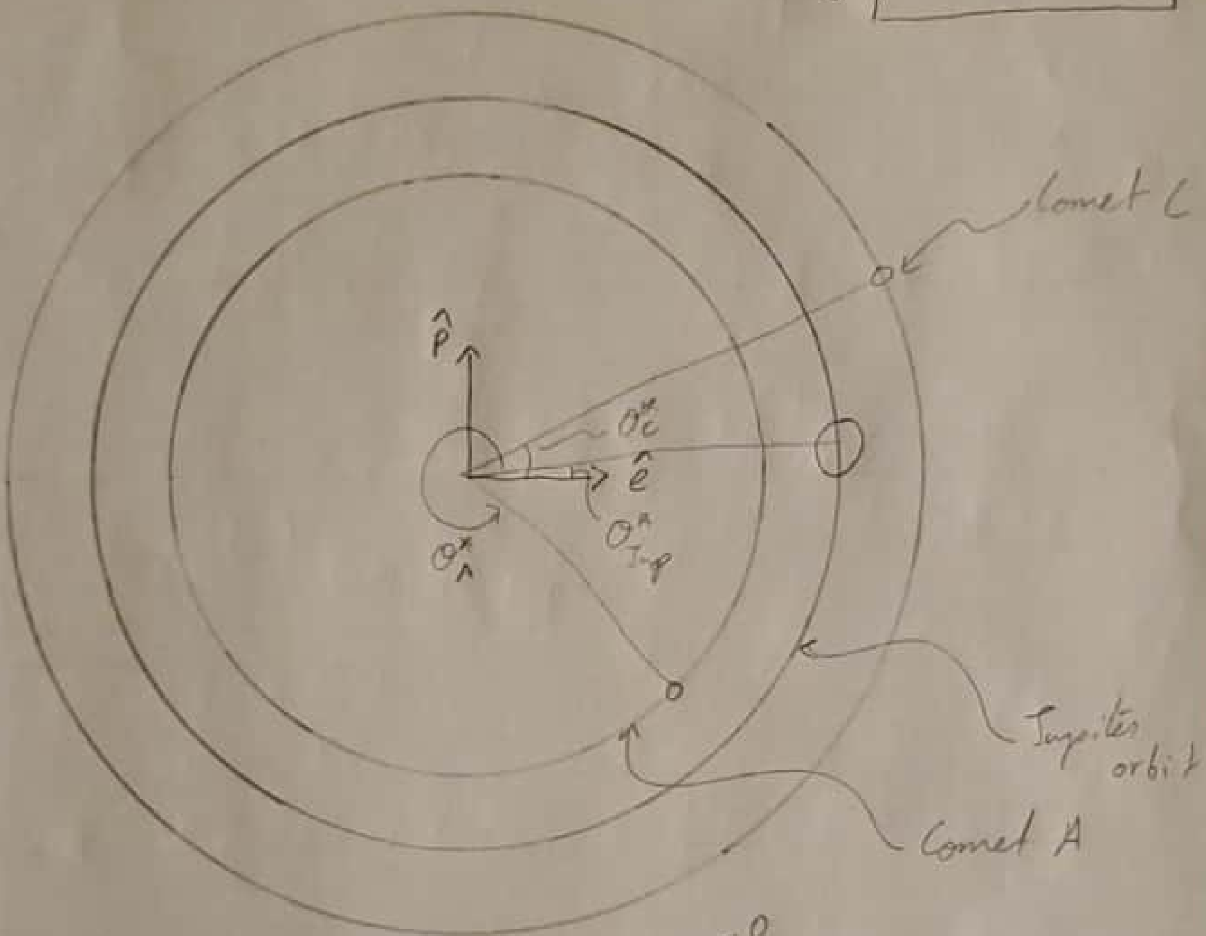
Continued...

$n = \sqrt{\frac{\mu}{a^3}}$ $\therefore n_A = \boxed{2.5186 \times 10^{-8} \frac{1}{s}}$ $n_C = \boxed{1.2593 \times 10^{-8} \frac{1}{s}}$

$n_J = \boxed{1.6790 \times 10^{-8} \, 1/s}$

6)



Steps: $t = n\mathbb{P} \Rightarrow n(t - t_p) = E - e\sin E = m$ ($t_p = 0$, $e = 0$)

once you have $E = n(t - t_p) = m$

Now you can find θ^* $\tan\frac{\theta^*}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$

$\therefore \tan\left(\frac{\theta^*}{2}\right) = \tan\left(\frac{E}{2}\right)$

$\therefore \theta^* = E$

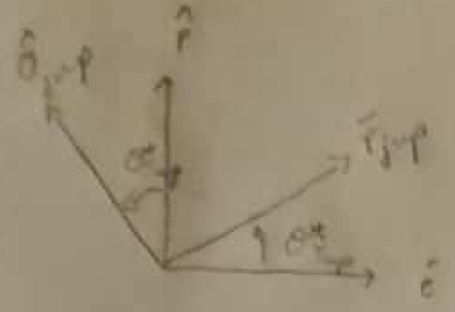
$\boxed{\text{See plots ahead Figure 1}}$

Sidereal Problem

Continued

c)

Inertial → Jupiter



$$C^{IJ} = \begin{bmatrix} \cos\theta^*_{Jup} & \sin\theta^*_{Jup} \\ -\sin\theta^*_{Jup} & \cos\theta^*_{Jup} \end{bmatrix}$$

$\boxed{\text{See plot Figure 2}}$

d) Steps for inertial frame

Not 0

1) $t = nP \Rightarrow n(t - t_p^{0}) = E - e\sin E = M$

2) Once we have E ← eccentric anomaly we can find true anomaly

3) $\therefore \tan\left(\frac{\theta^*}{2}\right) = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$

4) $r = \frac{a(1-e^2)}{1+e\cos\theta^*}$ ← radial distance in Inertial frame

5) $\bar{r}^I = [r\cos\theta^*\,\hat{e} + r\sin\theta^*\,\hat{p}]$ ← $\boxed{\text{See plot Figures 3, 5, 7}}$

↳ repeat for Jupiter $(e = 0)$, Asteroid A, comet C

Steps for rotating frame

follow till step (5) as in previous one

then

$\boxed{\bar{r}^J = C^{IJ}\,\bar{r}^I}$ ← $\boxed{\text{See plots Figure 4, 6, 8}}$

Note θ^* in C^{IJ} is $\theta^*_{Jupiter}$

$\boxed{\text{Loops on the outside for 3:2 in the Interior resonance and the loops inside for 3:4 in the Exterior resonance}}$

Continued...:

Out of Phase 180°

Same steps as in phase but for comet and Asteroid add $180° + \theta^*$ into step (4)

$$\therefore 4)\ r = \frac{a(1-e^2)}{1+\cos(180+\theta^*)} = \frac{a(1-e^2)}{1+\cos(180)\cos(\theta^*)-\sin(180)\sin(\theta^*)}$$

$$= \frac{a(1-e^2)}{1-\cos(\theta^*)}$$

See plots Figures 9, 11, 13

Out of phase rotating

Same procedure as out of phase but add in the rotation matrix

$$\bar{r}^{J} = C^{IJ}\,\bar{r}^{I}$$ ← See plots Figures 10, 12, 14

The relative shape of the orbit in the rotating frame does not change. It is just rotated by 180°.

Note, the apoapsis is now near Jupiter so Jupiter has a longer time to influence the orbit. Which may affect our assumptions and the model.

A and C are closest near their respective periapsis. The two bodies do not always align at this point but they would align every 4th orbit of C.

(e) For higher eccentricities I tried 0.6 and 0.9

With resonance of A ⇒ 4:5 and C ⇒ 5:7 ; and A ⇒ 5:3 and C ⇒ 7:4

With higher eccentricities the orbit paths overlay eachother. Thus influencing each other. This may make system unstable.

See plots ahead

Problem B1

Part (b)

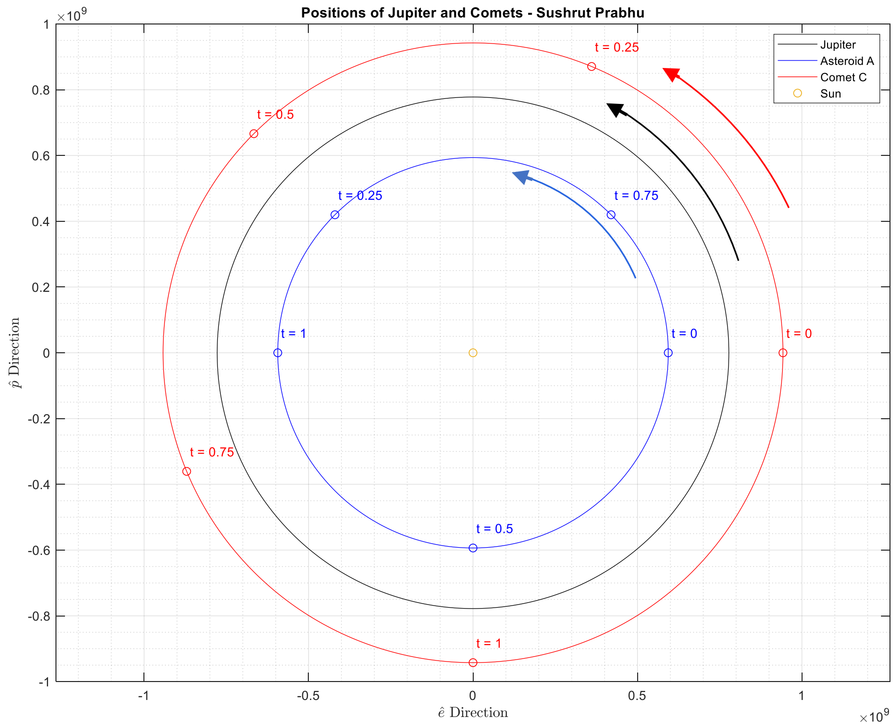


Figure 1: Circular orbit around the Sun in the inertial frame.

Part (c)

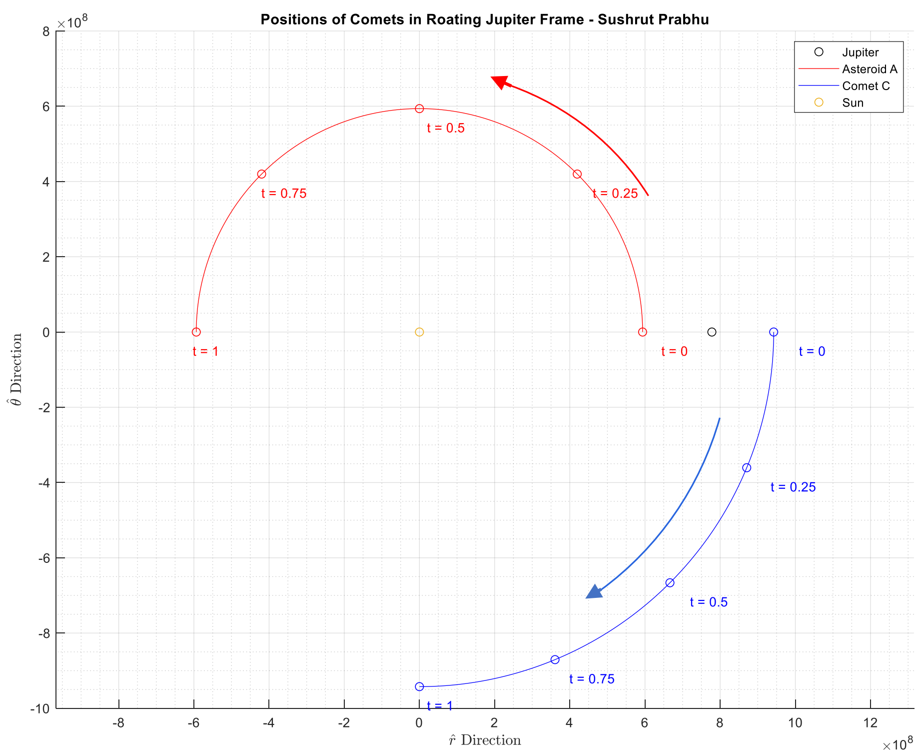


Figure 2: Circular orbit around Sun from Jupiter's rotating frame.

Part (d i)

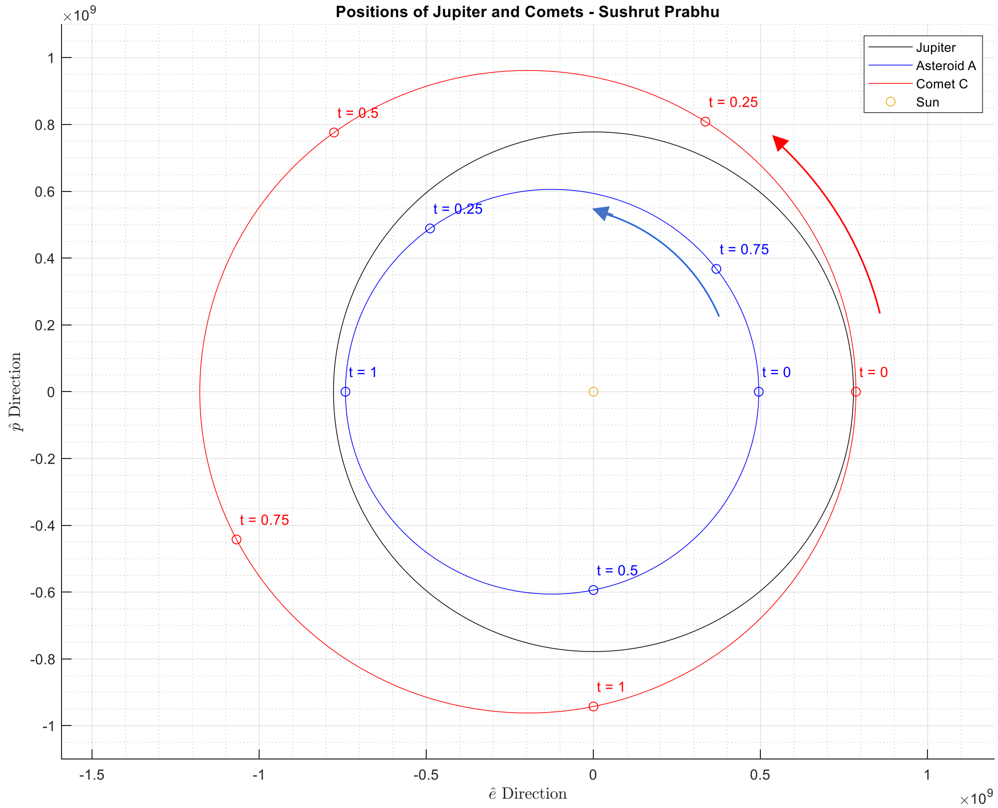


Figure 3: Eccentric orbit of 0.2 for asteroid and comet in the inertial frame.

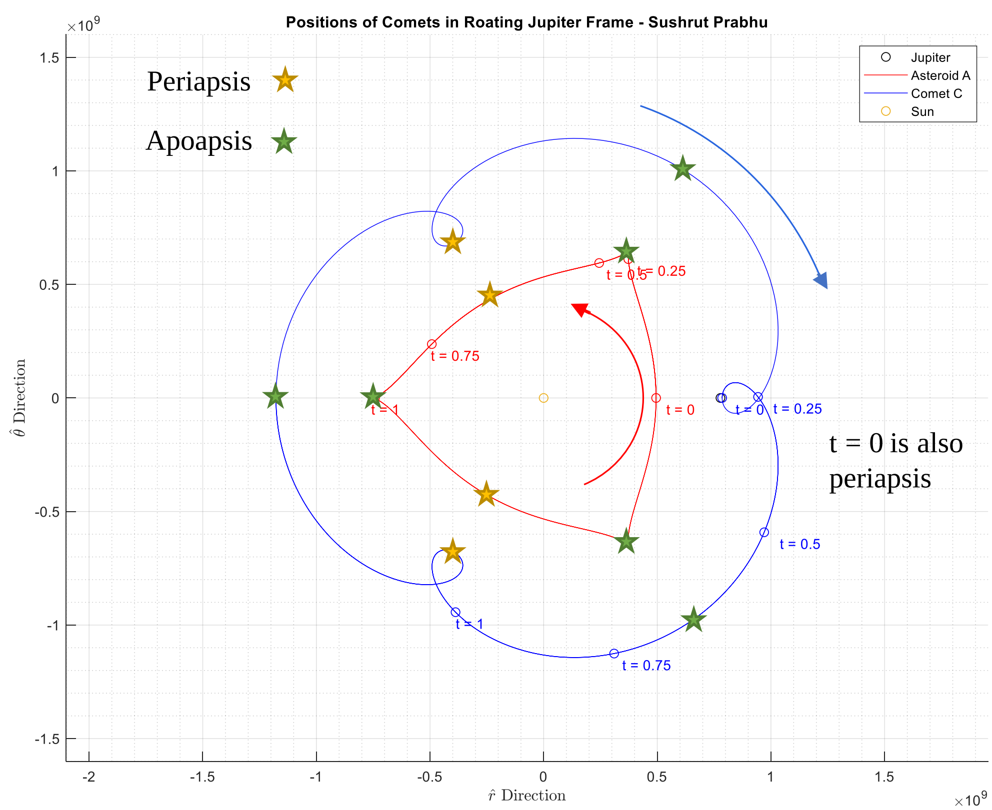


Figure 4: Eccentric orbit of 0.2 for asteroid and comet in the rotating Jupiter frame.

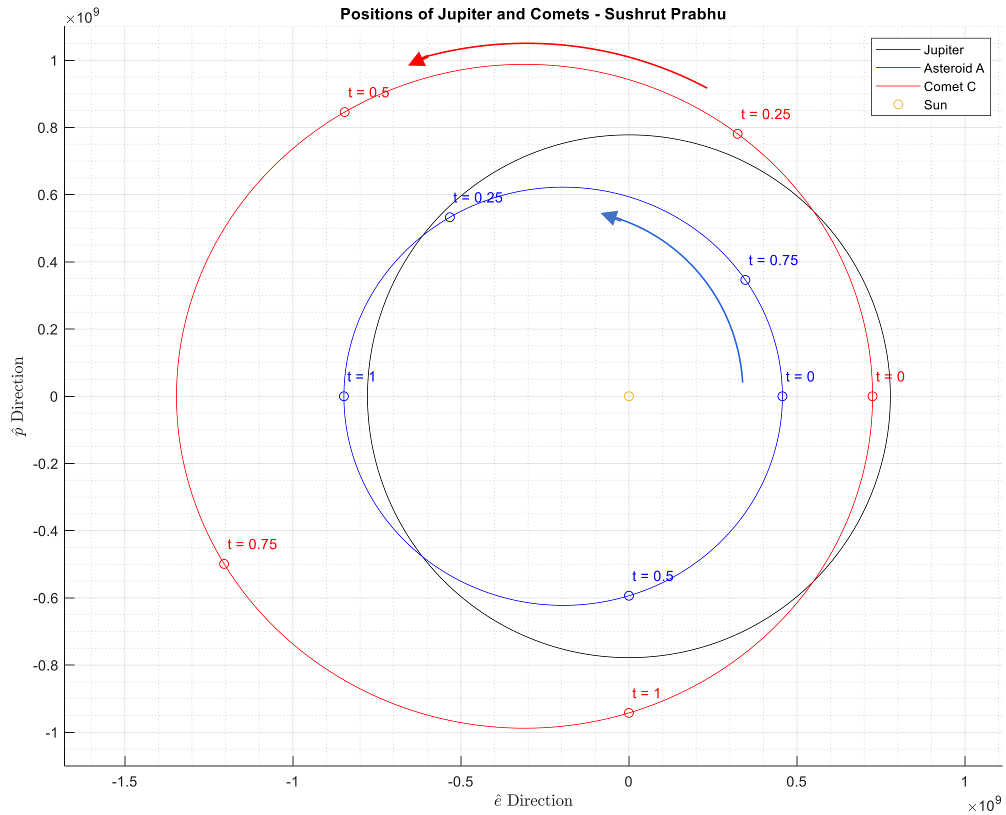


Figure 5: Eccentric orbit of 0.3 for asteroid and comet in the inertial frame.

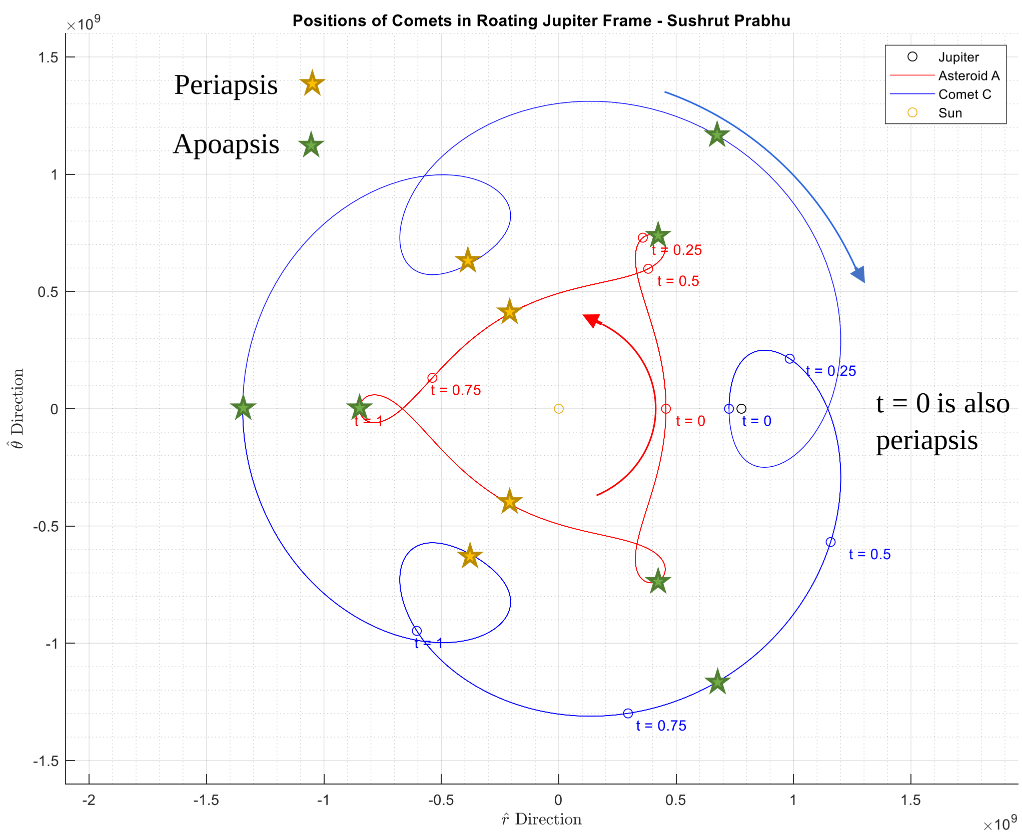


Figure 6: Eccentric orbit of 0.3 for asteroid and comet in the rotating Jupiter frame.

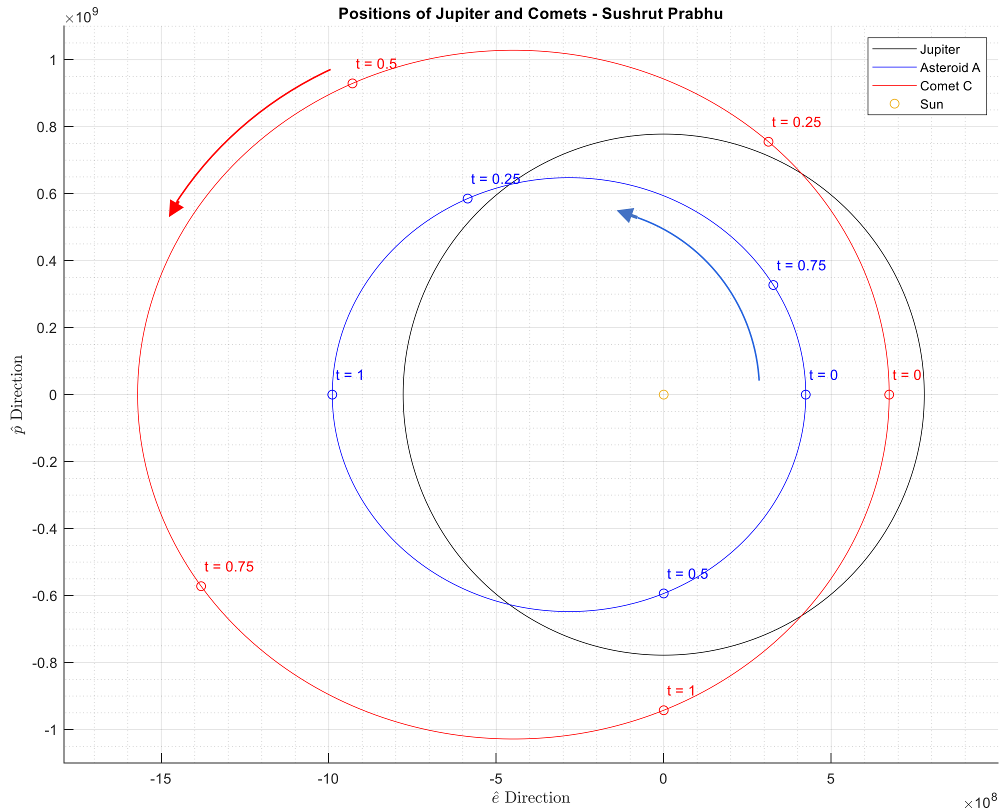


Figure 7: Eccentric orbit of 0.4 for asteroid and comet in the inertial frame.

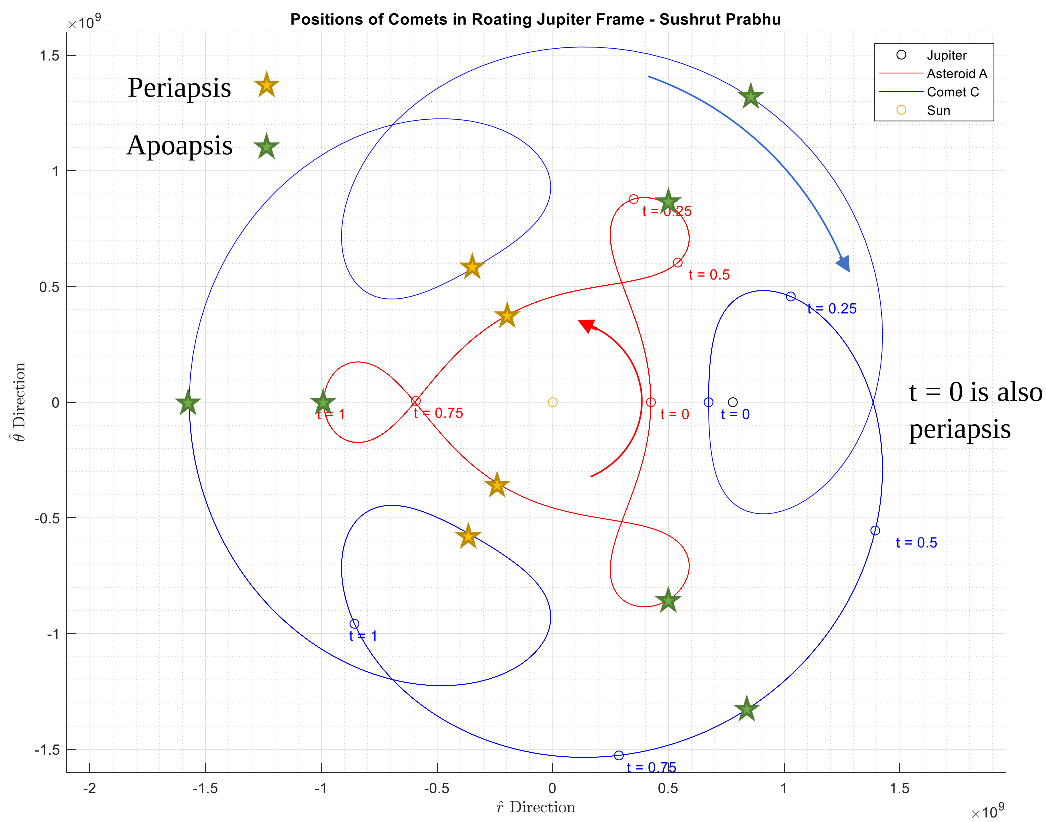


Figure 8: Eccentric orbit of 0.4 for asteroid and comet in the rotating Jupiter frame.

Part (d ii)

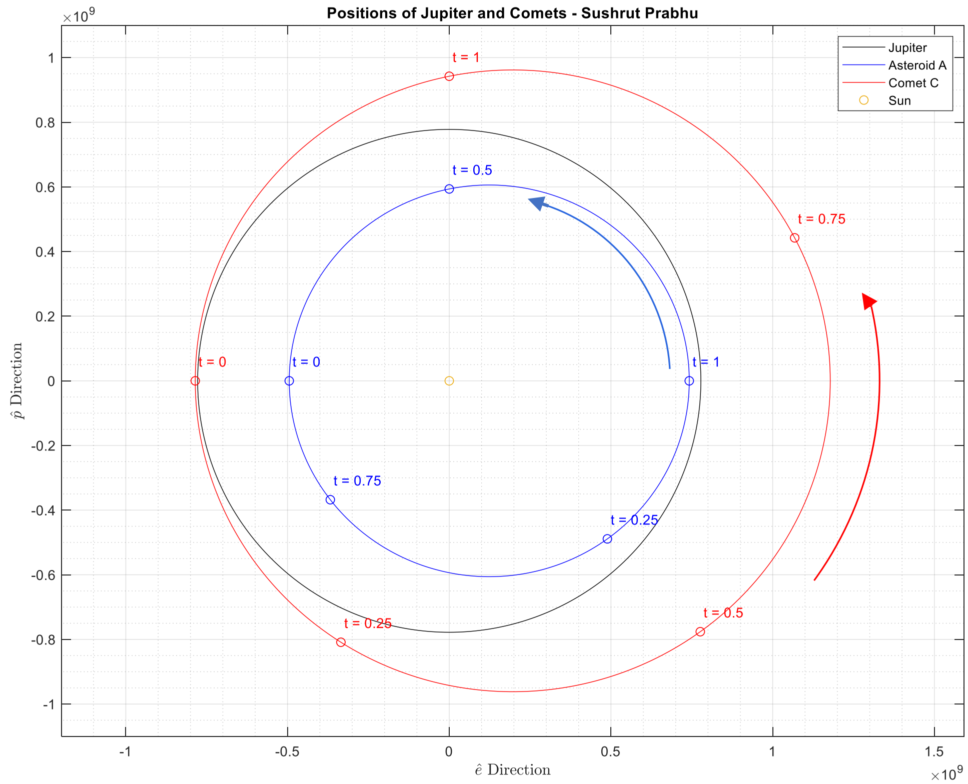


Figure 9: Eccentric orbit of 0.2 for asteroid and comet in the inertial frame 180º out of phase.

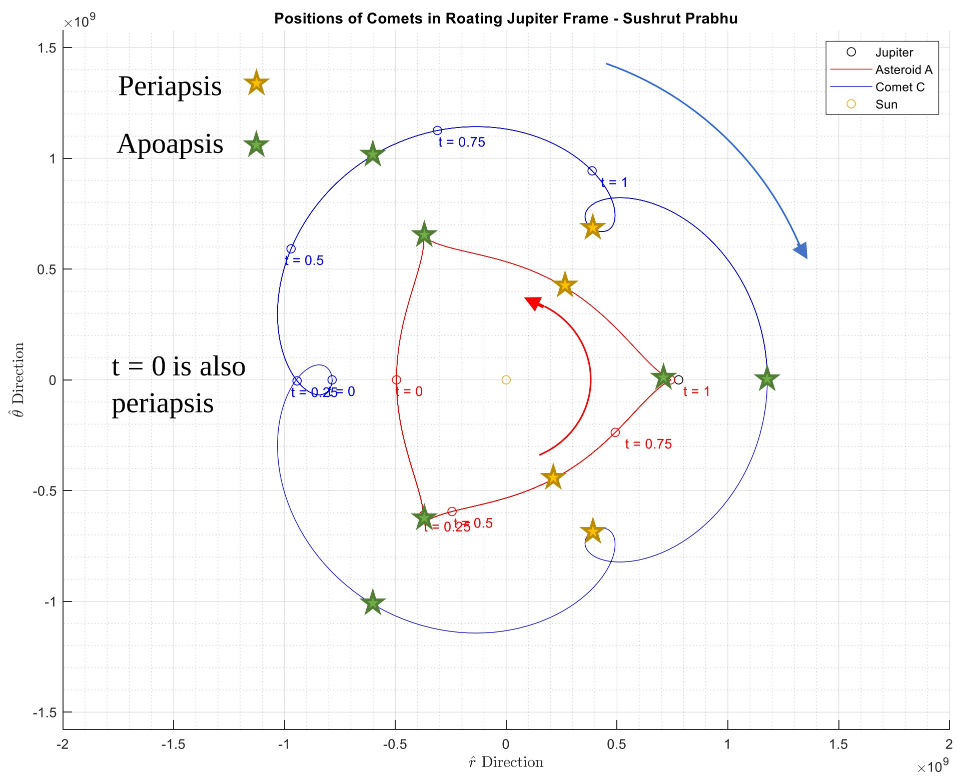


Figure 10: Eccentric orbit of 0.2 for asteroid and comet in the rotating Jupiter frame 180º out of phase.

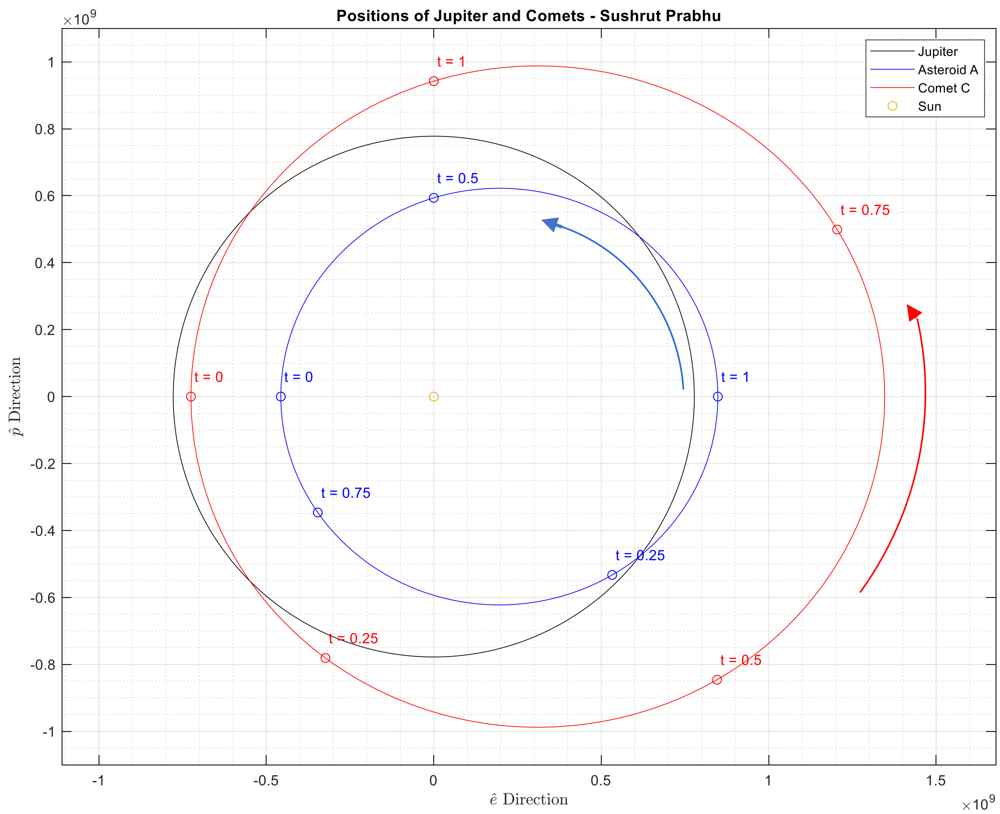


Figure 11: Eccentric orbit of 0.3 for asteroid and comet in the inertial frame 180º out of phase.

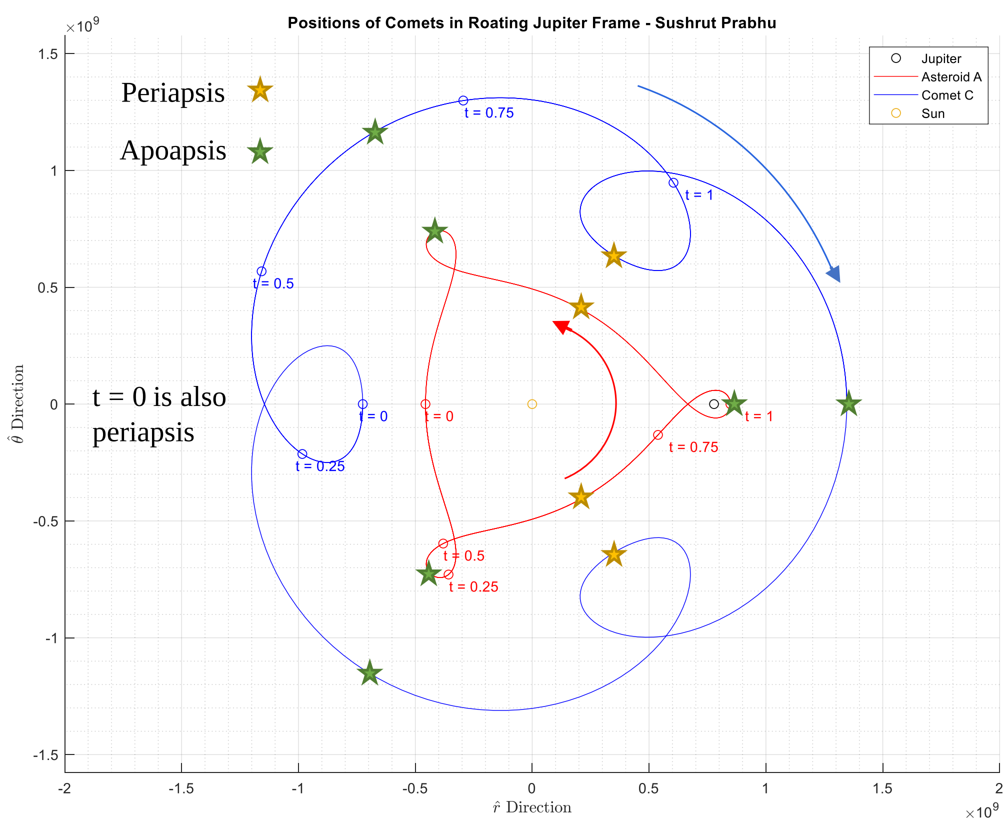


Figure 12: Eccentric orbit of 0.2 for asteroid and comet in the rotating Jupiter frame 180º out of phase.

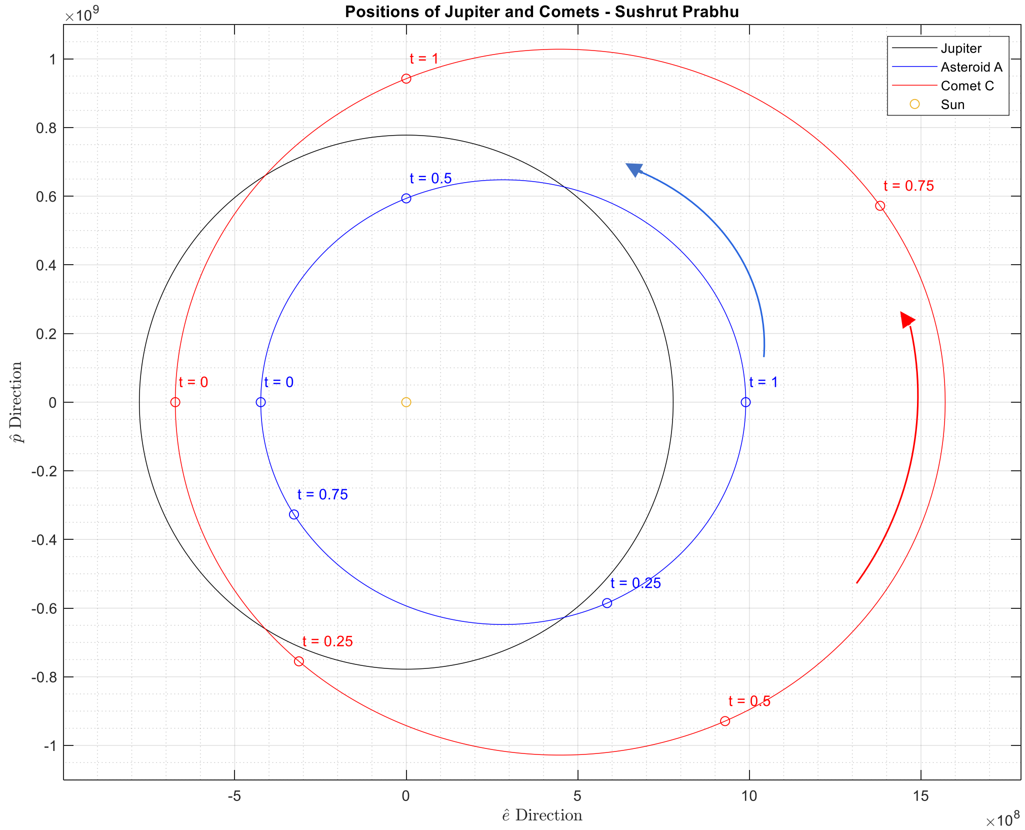


Figure 13: Eccentric orbit of 0.4 for asteroid and comet in the inertial frame 180º out of phase.

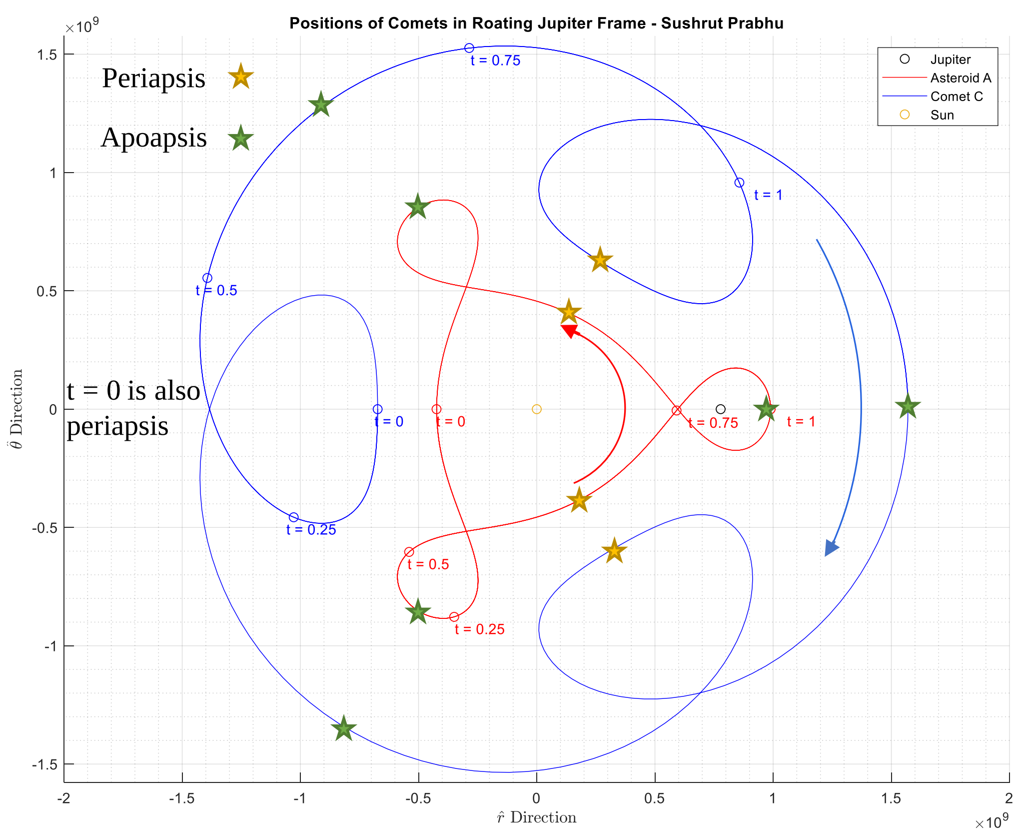


Figure 14: Eccentric orbit of 0.2 for asteroid and comet in the rotating Jupiter frame 180º out of phase.

Part (e)

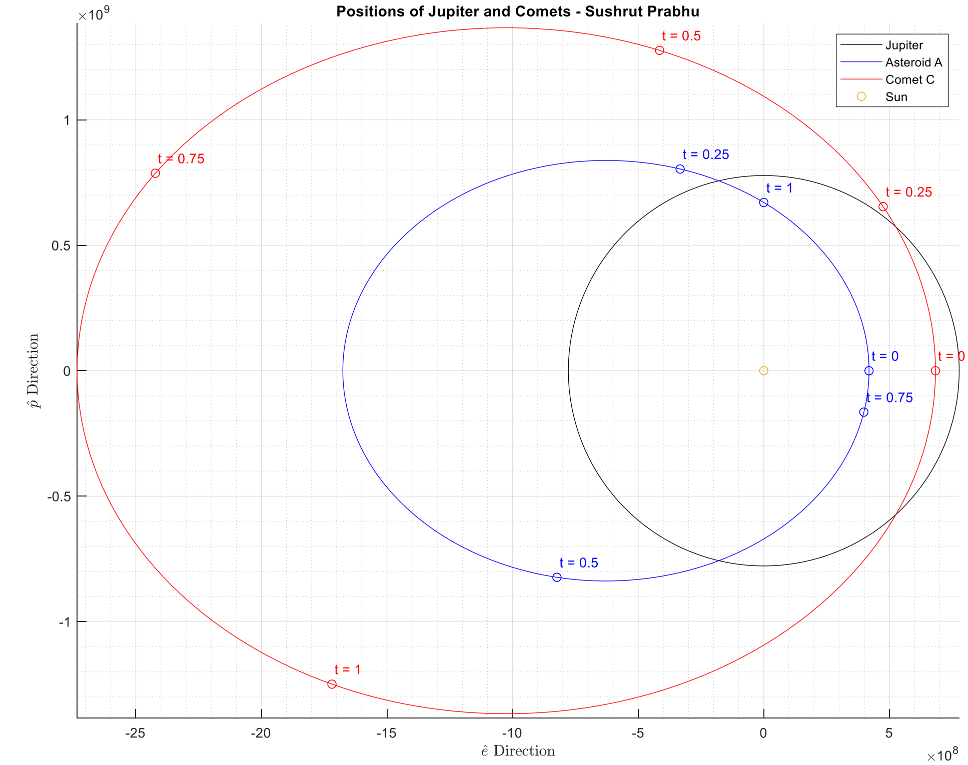


Figure 15: Eccentric orbit of __ for asteroid and comet in the inertial frame with ___ and ___ resonance.

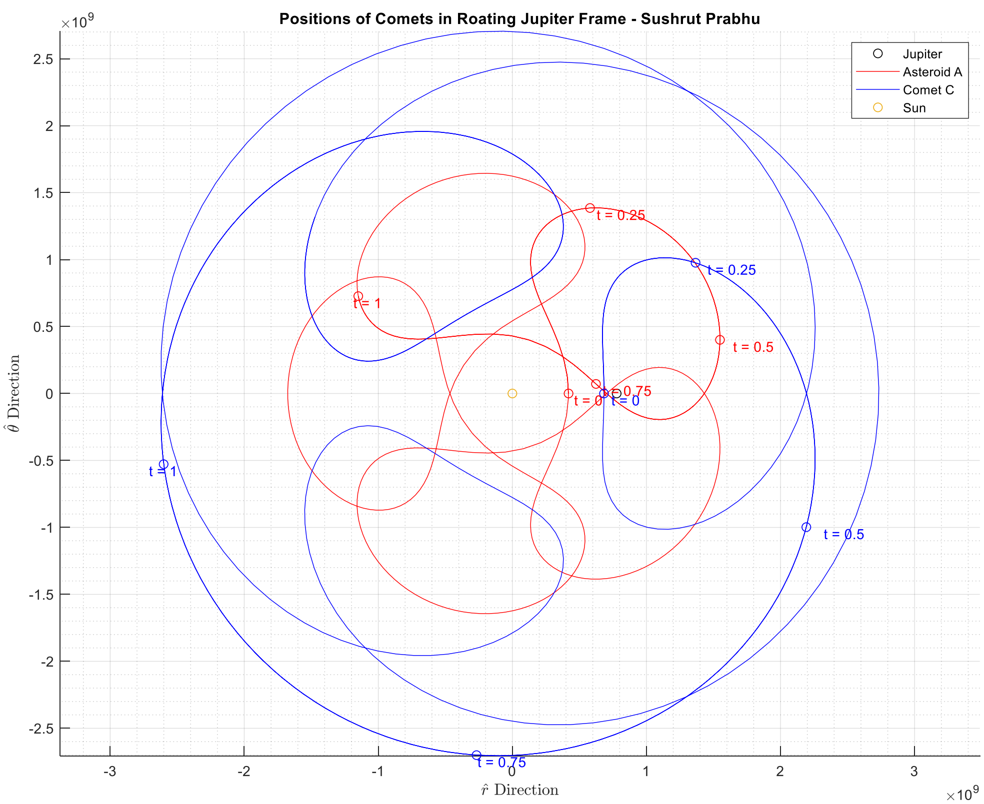


Figure 16: Eccentric orbit of __ for asteroid and comet in the rotating Jupiter frame with ___ and ___ resonance.

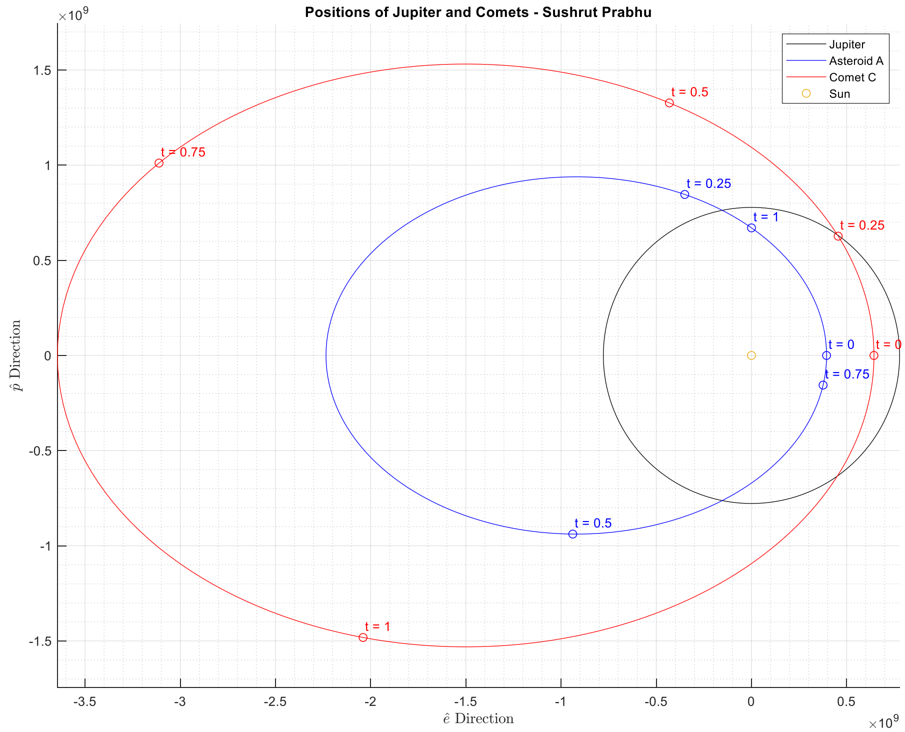


Figure 17: Eccentric orbit of 0.6 for asteroid and comet in the inertial frame with ___ and ___ resonance.

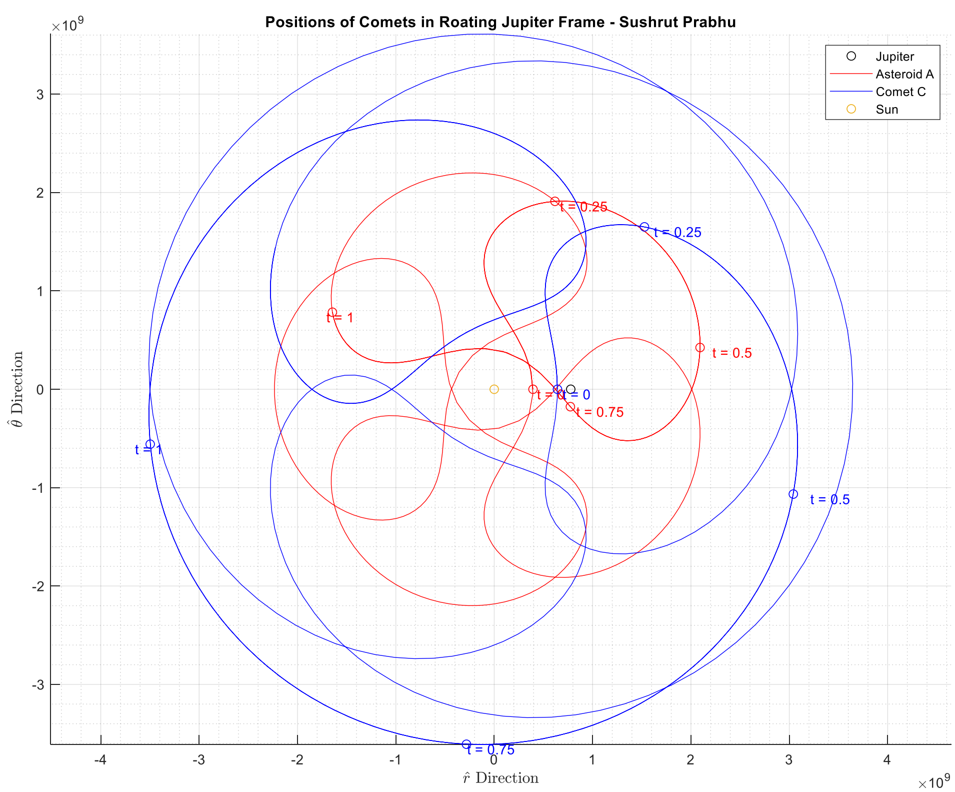


Figure 18: Eccentric orbit of __ for asteroid and comet in the rotating Jupiter frame with ___ and ___ resonance.

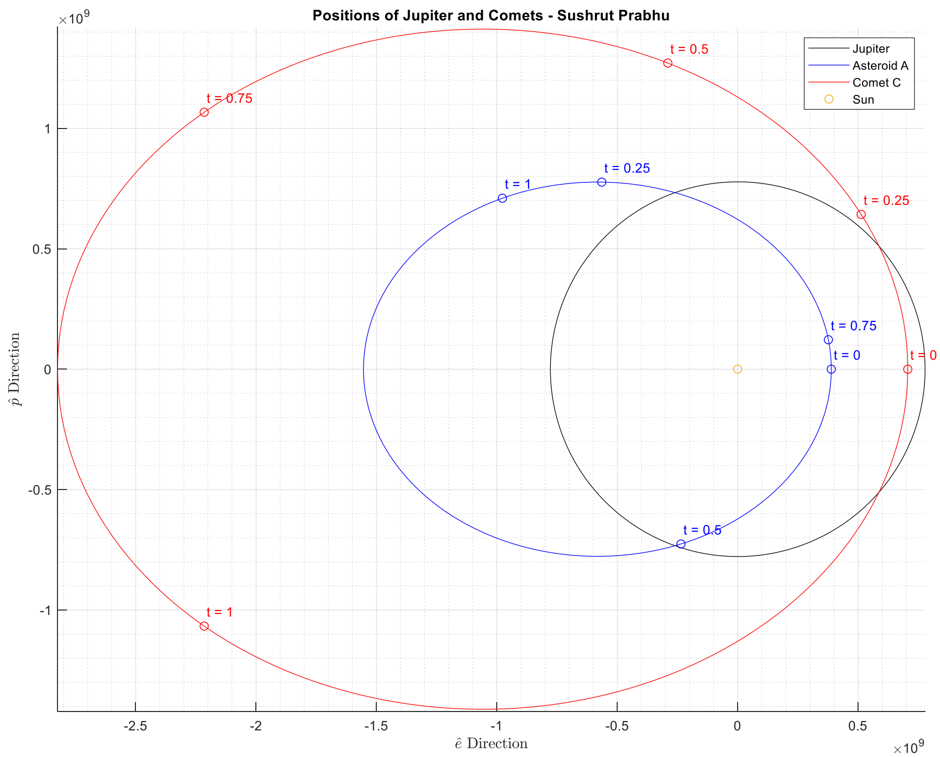


Figure 19: Eccentric orbit of 0.6 for asteroid and comet in the inertial frame with ___ and ___ resonance.

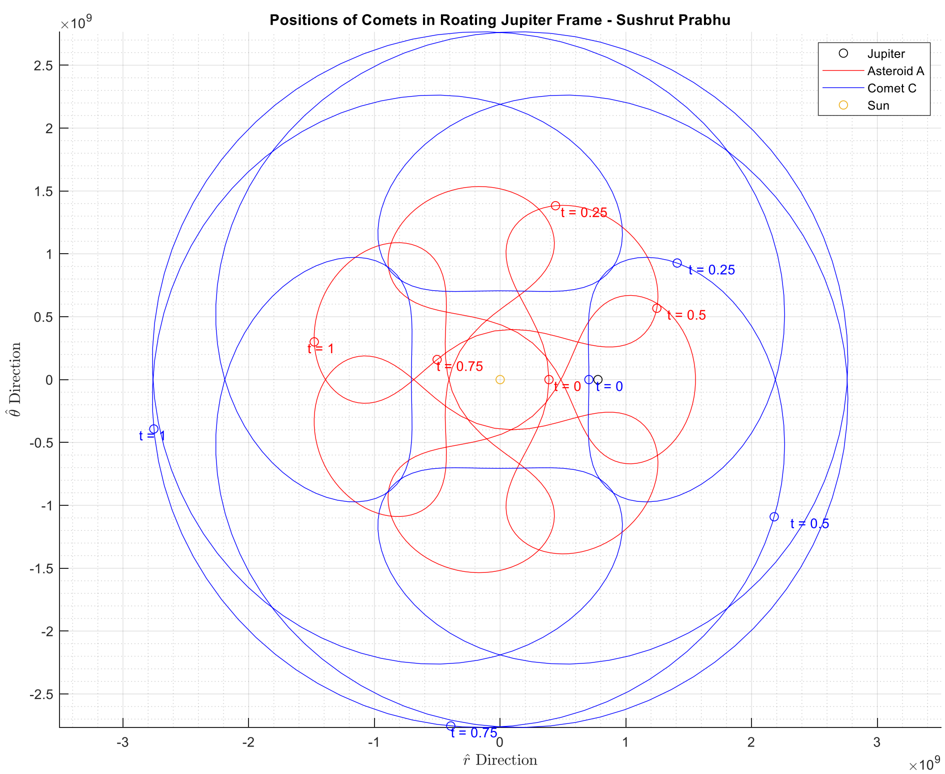


Figure 20: Eccentric orbit of __ for asteroid and comet in the rotating Jupiter frame with ___ and ___ resonance.

Table of Contents

PSB1

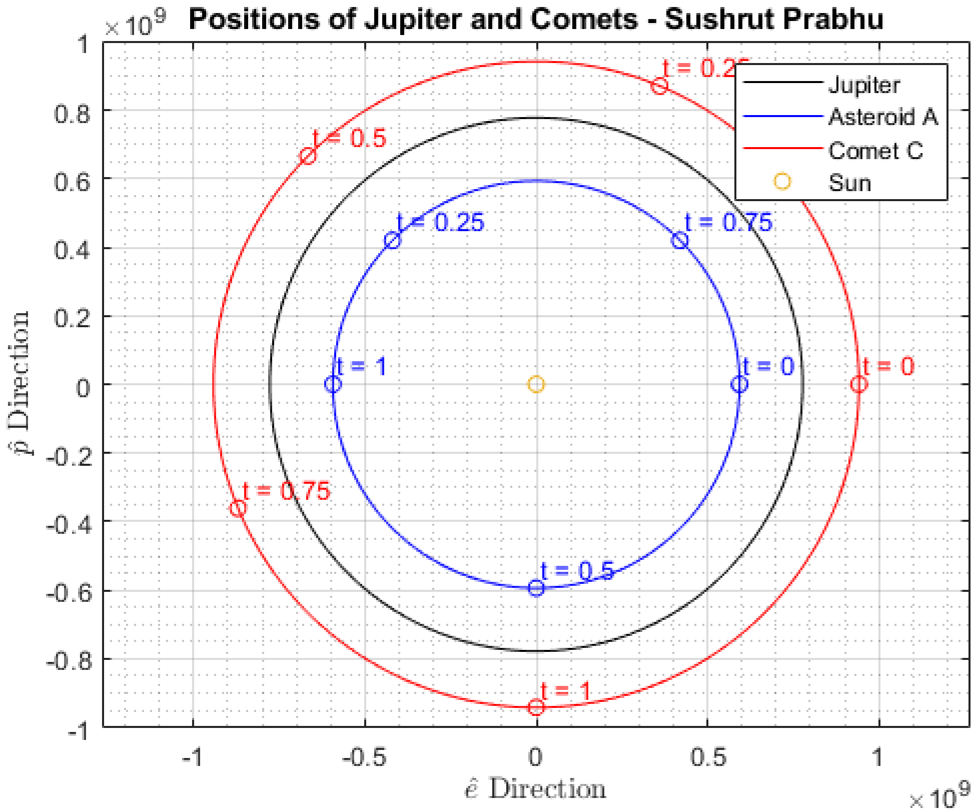
```
clear
close all
clc
```

PSB1 Part a

Constants

```
SS = SolarS;
miu_S = SS.mSun;

% Jupiter
a_jup = 5.2 * SS.dEarth;
Per_jup = 2*pi/sqrt(miu_S/a_jup^3);
n_jup = sqrt(miu_S/a_jup^3);

% Comet A
a_A = (4*a_jup^3/9)^(1/3);
Per_A = 2*pi/sqrt(miu_S/a_A^3);
Per_A_years = Per_A/3600/24/365;
n_A = sqrt(miu_S/a_A^3);

% Comet B
a_C = (16*a_jup^3/9)^(1/3);
Per_C = 2*pi/sqrt(miu_S/a_C^3);
Per_C_years = Per_C/3600/24/365;
n_C = sqrt(miu_S/a_C^3);

frac = [0, .25, .5, .75, 1];
frac2 = {"t = 0"; "t = 0.25"; "t = 0.5"; "t = 0.75"; "t = 1"};
for n = 1:length(frac)
    t_jup = Per_jup*frac(n);

    thst_A(n) = rad2deg(t_jup * n_A);
    thst_C(n) = rad2deg(t_jup * n_C);
end

trueanom = 0:360;

figure
```

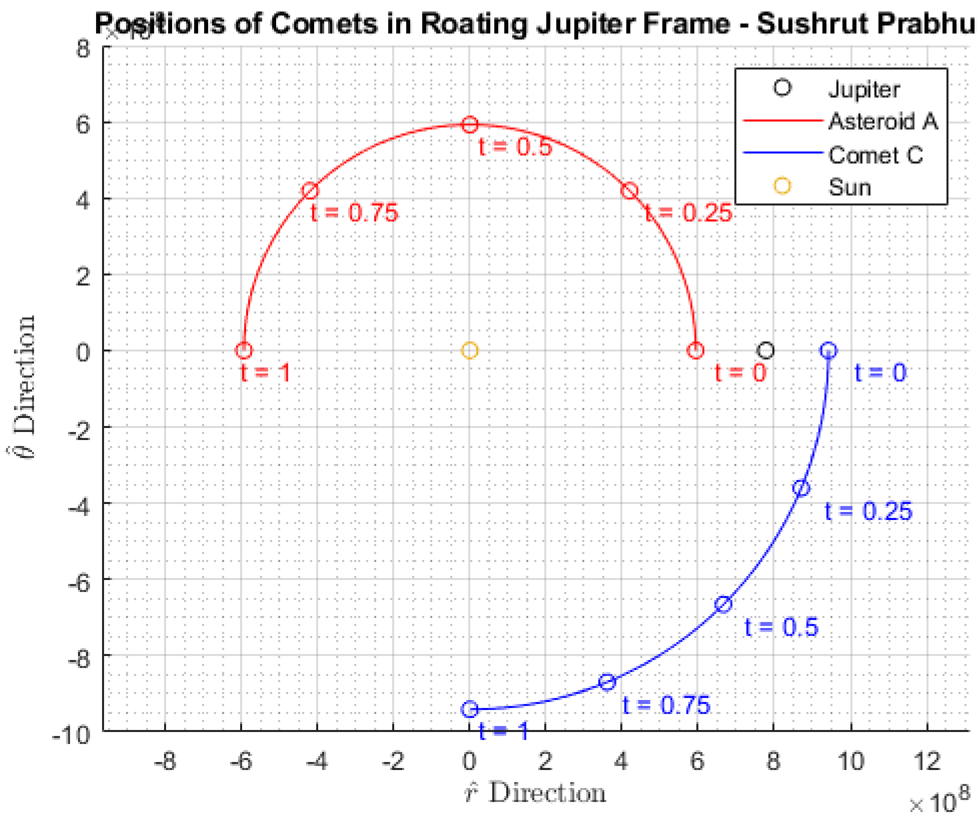
```
pl_orbit(trueanom, a_jup, 0,  "-k",1,"","",0)
hold on
pl_orbit(trueanom, a_A, 0, "-b",1,"","",0)
pl_orbit(trueanom, a_C, 0, "-r",1,"","",0)
plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
pl_orbit(thst_A, a_A, 0,  "ob",2, frac2,"Blue",0)
pl_orbit(thst_C, a_C, 0, "or",2, frac2,"Red",0)
xlabel('$$\hat{e}$$ Direction','Interpreter','Latex')
ylabel('$$\hat{p}$$ Direction','Interpreter','Latex')
title('Positions of Jupiter and Comets - Sushrut Prabhu')
grid on
grid minor
axis equal
legend('Jupiter','Asteroid A', 'Comet C','Sun')

ylim([-1 1]*10^9)
```



Part (b)

```
figure
hold on
plot(a_jup, 0, 'Ok')
pl_rot_orbit(Per_jup, n_jup, n_A, a_A, 0,'-r', 1, "", "",0)
pl_rot_orbit(Per_jup, n_jup, n_C, a_C, 0,'-b', 1, "", "",0)
plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
pl_rot_orbit(frac*Per_jup, n_jup, n_A, a_A, 0,'or', 2, frac2, "Red",0)
```

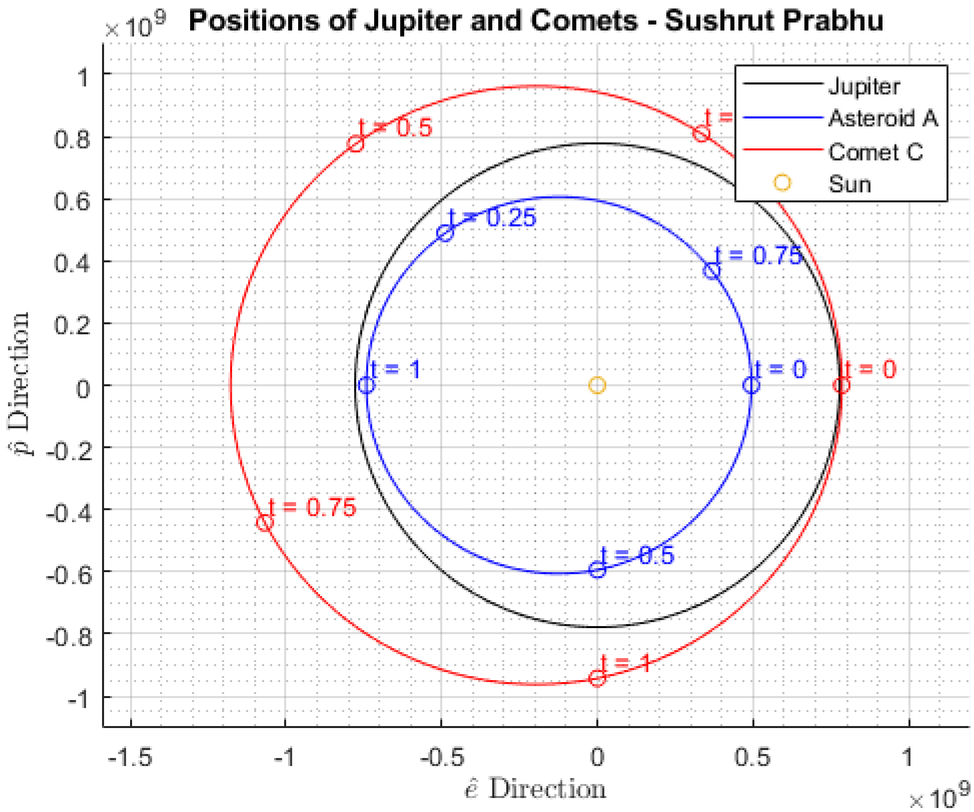
```
pl_rot_orbit(frac*Per_jup, n_jup, n_C, a_C, 0,'ob', 2,
 frac2, "Blue",0)
xlabel('$$\hat{r}$$ Direction','Interpreter','Latex')
ylabel('$$\hat{\theta}$$ Direction','Interpreter','Latex')
title('Positions of Comets in Roating Jupiter Frame - Sushrut Prabhu')
grid on
grid minor
axis equal
legend('Jupiter','Asteroid A', 'Comet C','Sun')
ylim([-1 .8]*10^9)
```

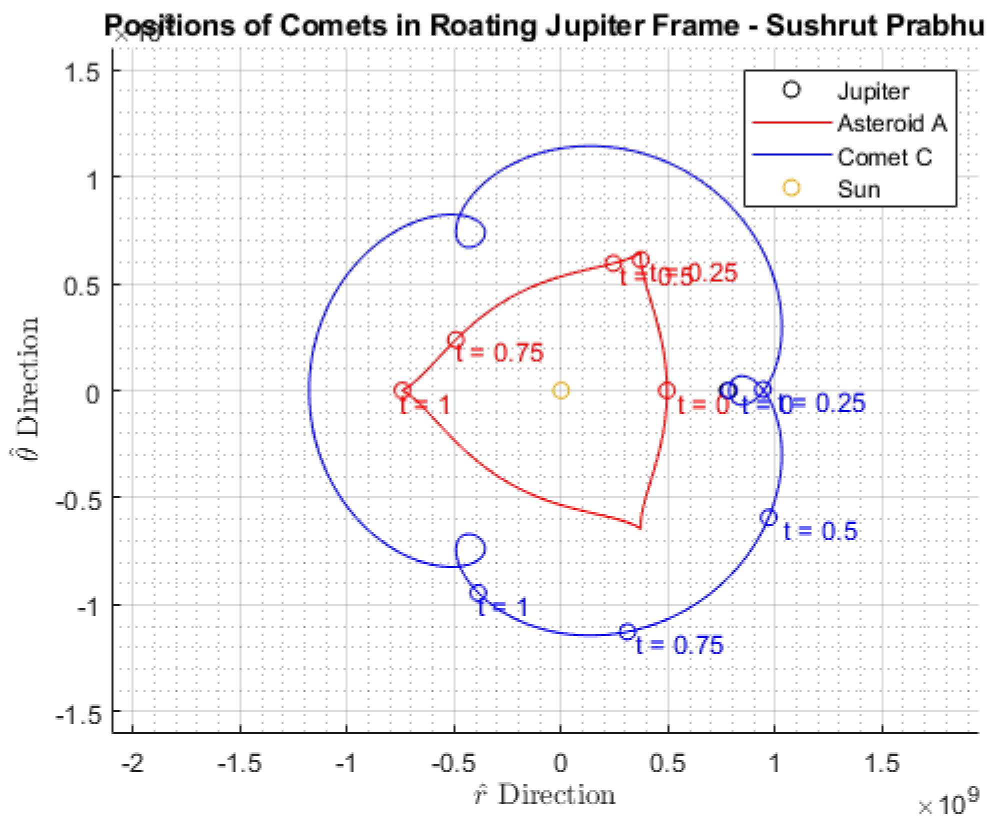


Part c

```
ecc = [0.2 0.3 0.4];

for m = 1:length(ecc)
    ecc_A = ecc(m);
    p_A = a_A*(1-ecc_A^2);

    ecc_C = ecc(m);
    p_C = a_A*(1-ecc_C^2);


    for n = 1:length(frac)
        t_jup = Per_jup*frac(n);
```

```
        thst_A(n) = rad2deg(t_jup * n_A);
        thst_C(n) = rad2deg(t_jup * n_C);
    end

    trueanom = 0:360;

    % Inertial Frame
    figure
    hold on
    pl_orbit(trueanom, a_jup, 0,  "-k",1,"","",0)
    pl_orbit(trueanom, a_A, ecc_A, "-b",1,"","",0)
    pl_orbit(trueanom, a_C, ecc_C, "-r",1,"","",0)
    plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
    pl_orbit(thst_A, a_A, ecc_A,  "ob",2, frac2,"Blue",0)
    pl_orbit(thst_C, a_C, ecc_C, "or",2, frac2,"Red",0)
    xlabel('$$\hat{e}$$ Direction','Interpreter','Latex')
    ylabel('$$\hat{p}$$ Direction','Interpreter','Latex')
    title('Positions of Jupiter and Comets - Sushrut Prabhu')
    grid on
    grid minor
    axis equal
    legend('Jupiter','Asteroid A', 'Comet C','Sun')

    ylim([-1.1 1.1]*10^9)

    % Jupiter Rotating Frame
    figure
    hold on
    plot(a_jup, 0, 'Ok')
    pl_rot_orbit(Per_jup*5, n_jup, n_A, a_A, ecc_A,'-r', 1, "", "",0)
    pl_rot_orbit(Per_jup*6, n_jup, n_C, a_C, ecc_C,'-b', 1, "", "",0)
    plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
    pl_rot_orbit(frac*Per_jup, n_jup, n_A, a_A, ecc_A,'or', 2,
 frac2, "Red",0)
    pl_rot_orbit(frac*Per_jup, n_jup, n_C, a_C, ecc_C,'ob', 2,
 frac2, "Blue",0)
    xlabel('$$\hat{r}$$ Direction','Interpreter','Latex')
    ylabel('$$\hat{\theta}$$ Direction','Interpreter','Latex')
    title('Positions of Comets in Roating Jupiter Frame - Sushrut
 Prabhu')
    grid on
    grid minor
    axis equal
    legend('Jupiter','Asteroid A', 'Comet C','Sun')

    ylim([-1.6 1.6]*10^9)
end
```


Positions of Jupiter and Comets - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
$\hat{p}$ Direction
$\hat{e}$ Direction
t = 0.5
t = 0.25
t = 0.75
t = 1
t = 0
t = 0.5
t = 0.75
t = 1
t = 0
$\times 10^9$


Positions of Comets in Roating Jupiter Frame - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
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$\hat{r}$ Direction
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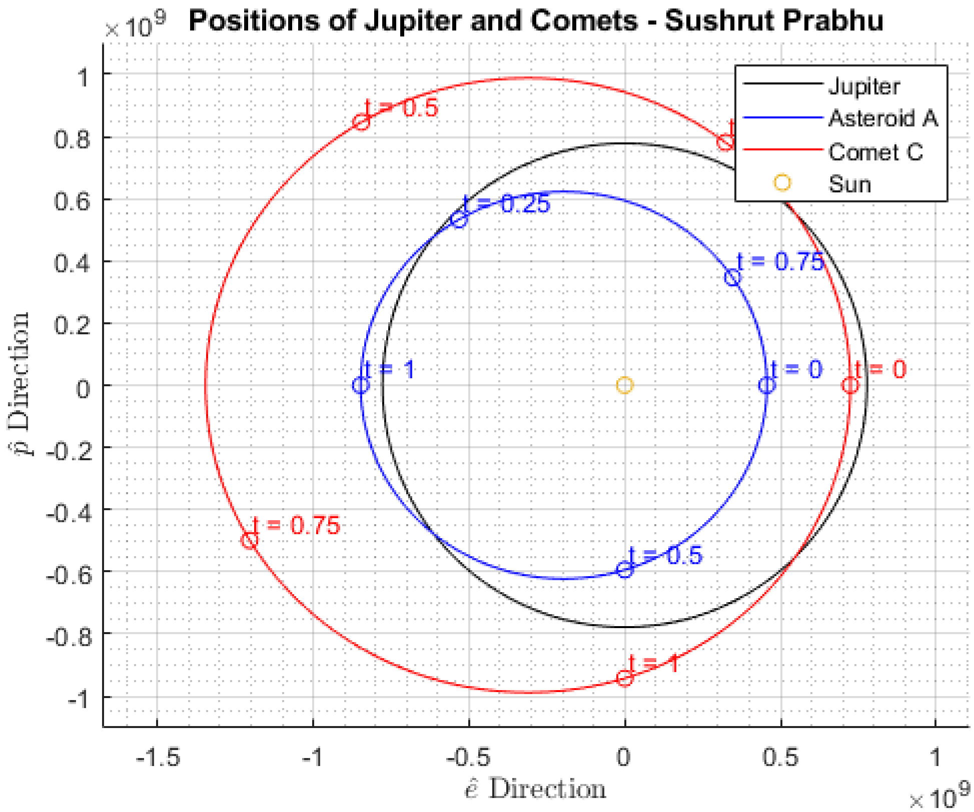
Positions of Jupiter and Comets - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
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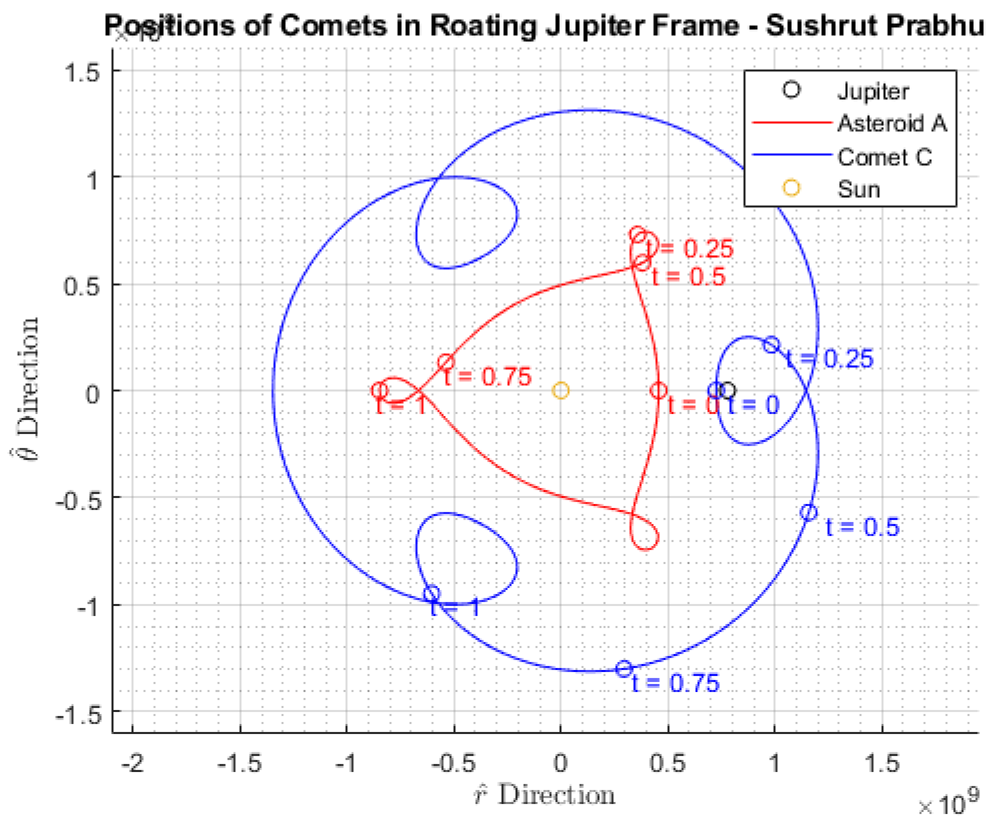
$\hat{e}$ Direction
$\times 10^9$


Positions of Comets in Roating Jupiter Frame - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
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t = 0.25
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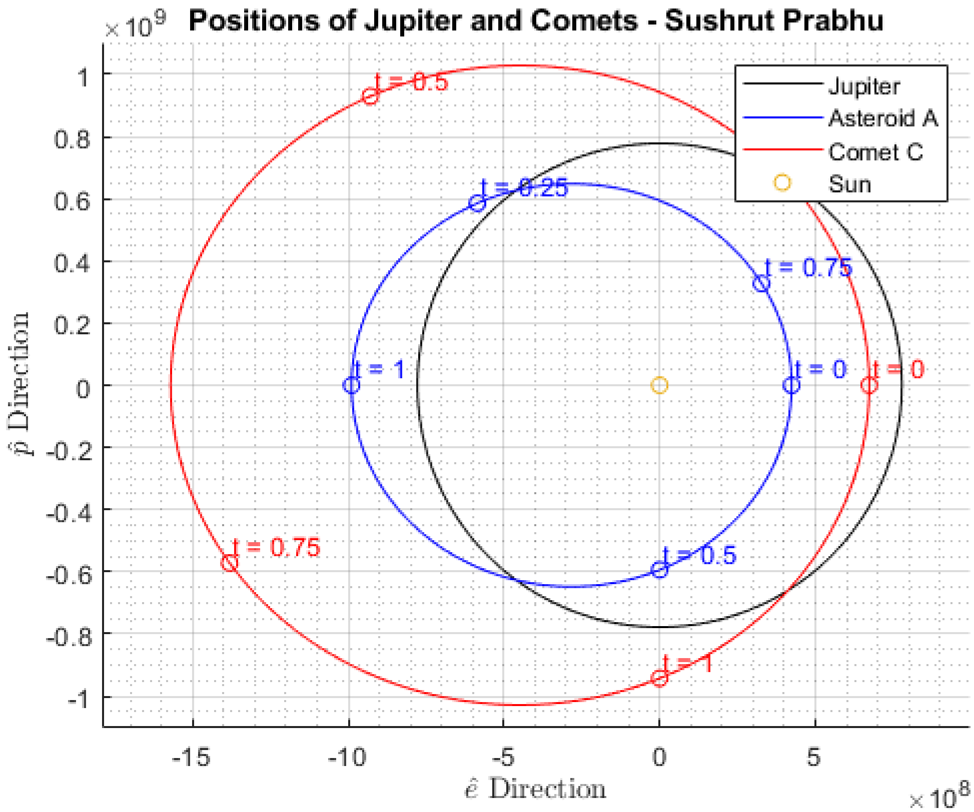
t = 1
t = 0.75
$\hat{\theta}$ Direction
$\hat{r}$ Direction
$\times 10^9$


Positions of Jupiter and Comets - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
t = 0.5
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t = 1
t = 0
t = 0
t = 0.75
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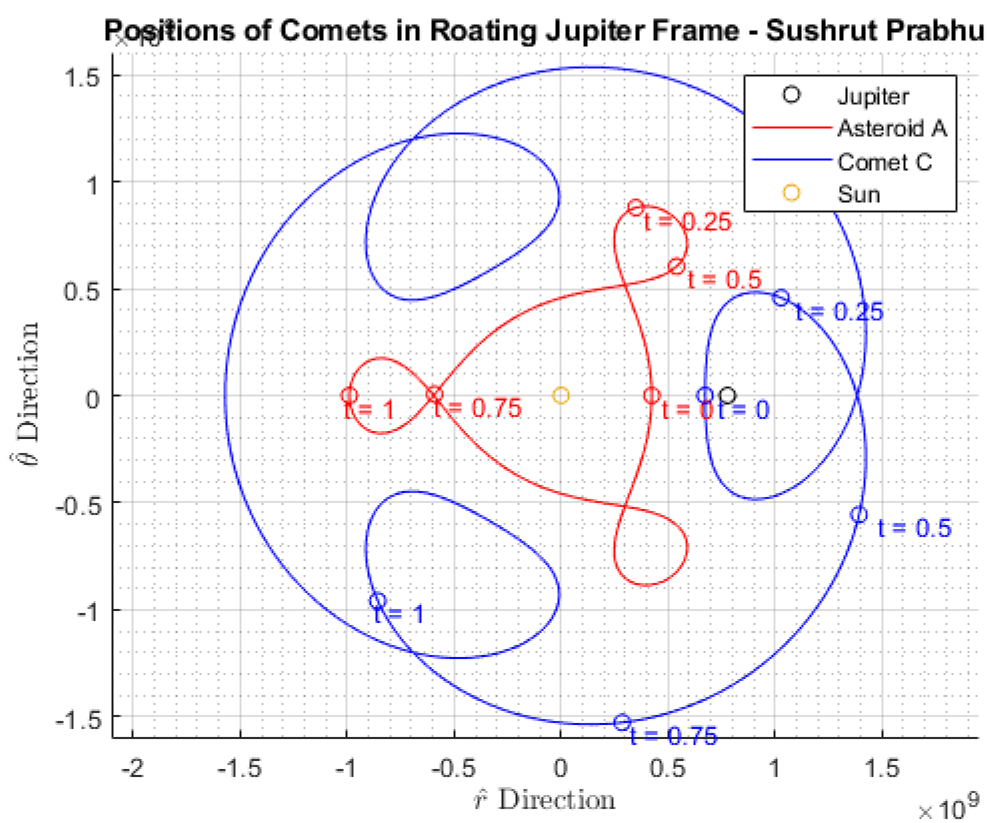
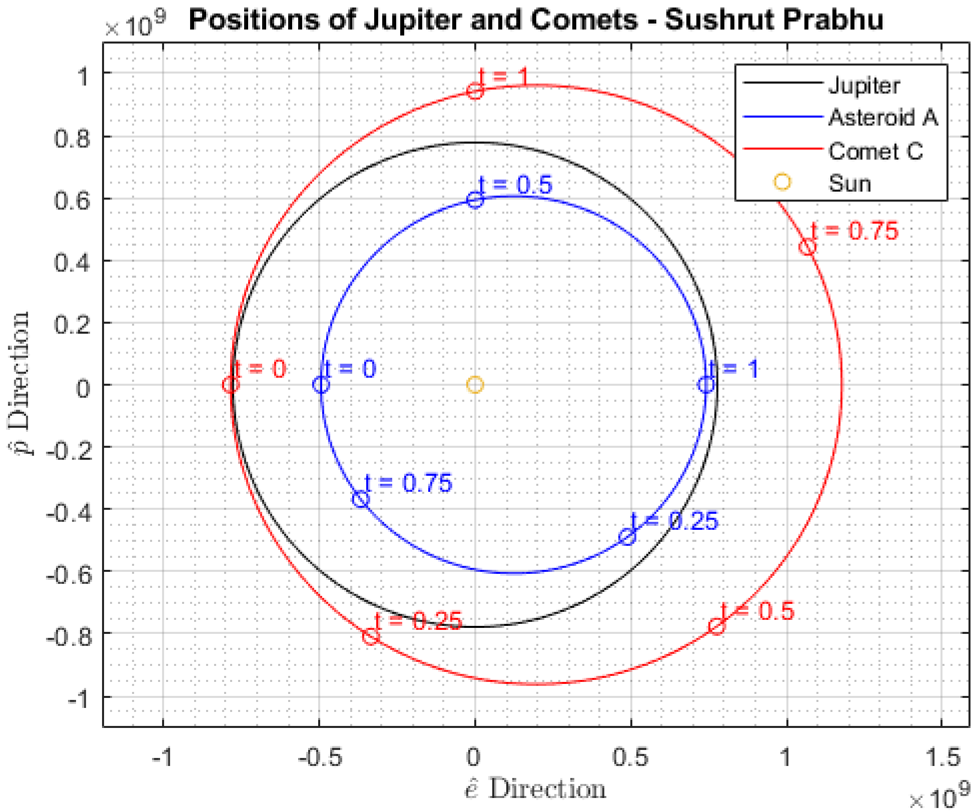
t = 1
$\hat{p}$ Direction
$\hat{e}$ Direction
$\times 10^9$
$\times 10^8$

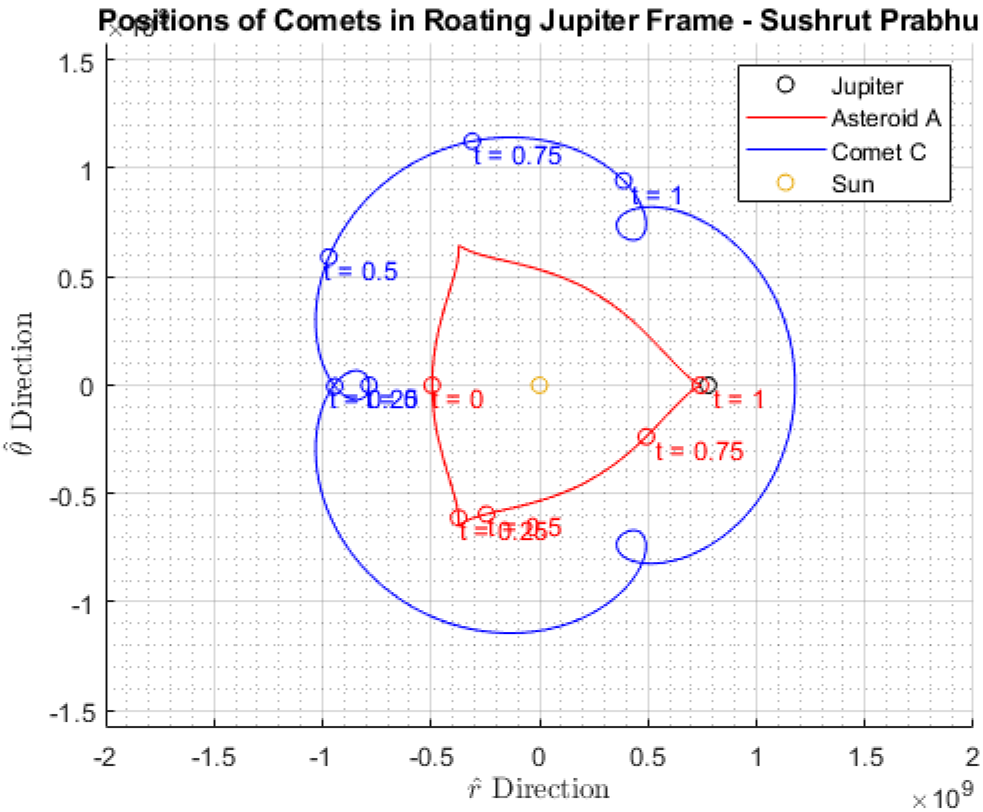

Positions of Comets in Roating Jupiter Frame - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
t = 0.25
t = 0.5
t = 0.25
t = 1
t = 0.75
t = 0
t = 0
t = 0.5
t = 1
t = 0.75
$\hat{\theta}$ Direction
$\hat{r}$ Direction
$\times 10^9$

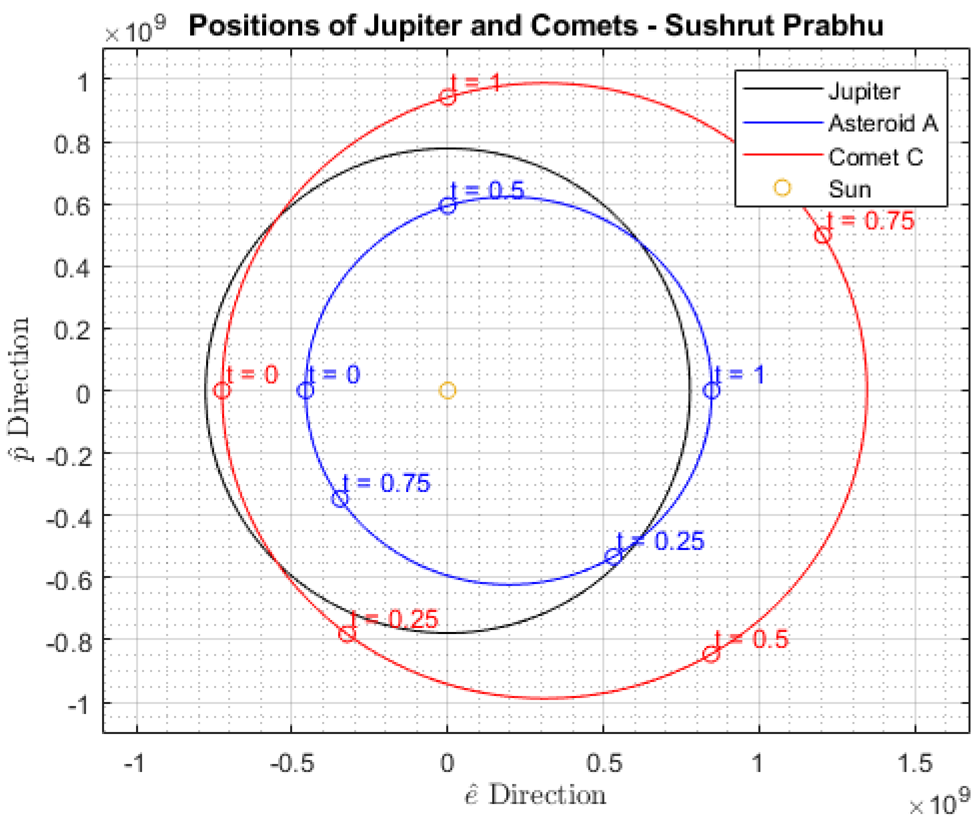
Out of phase

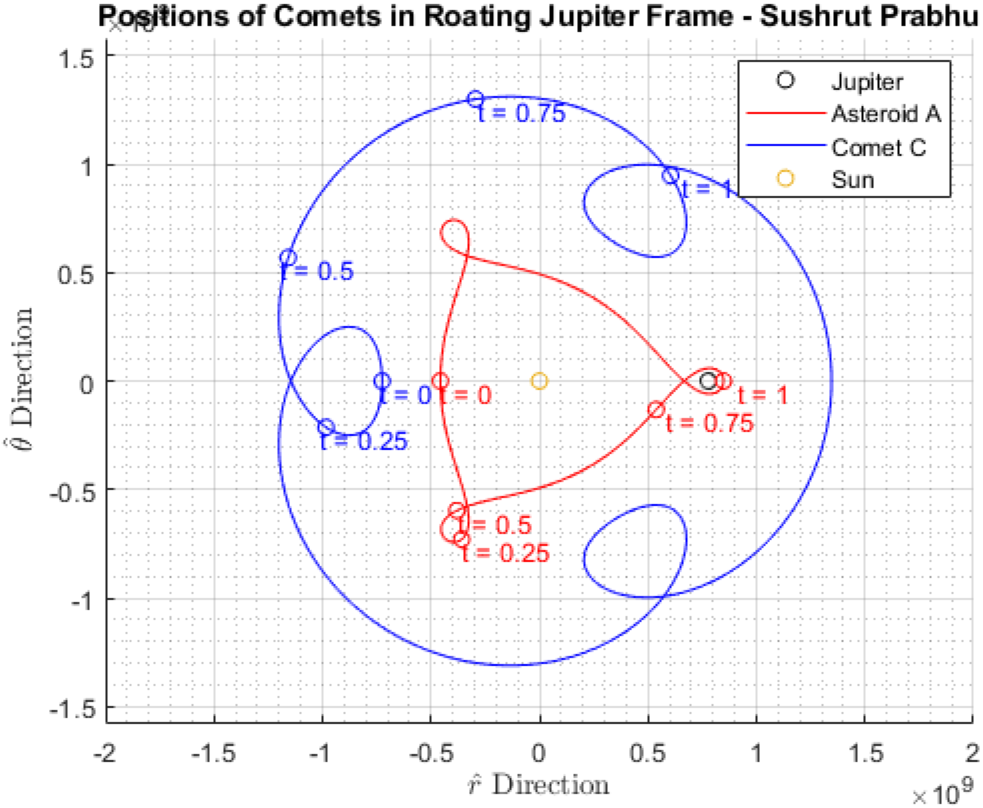
```
for m = 1:length(ecc)
    ecc_A = ecc(m);
    p_A = a_A*(1-ecc_A^2);

    ecc_C = ecc(m);
    p_C = a_A*(1-ecc_C^2);


    for n = 1:length(frac)
        t_jup = Per_jup*frac(n);

        thst_A(n) = rad2deg(t_jup * n_A);
        thst_C(n) = rad2deg(t_jup * n_C);
    end

    trueanom = 0:360;

    % Inertial Frame
    figure
    pl_orbit(trueanom, a_jup, 0,  "-k",1,"","",0)
    hold on
    pl_orbit(-180:180, a_A, ecc_A, "-b",1,"","",180)
    pl_orbit(-180:180, a_C, ecc_C, "-r",1,"","",180)
    plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
    pl_orbit(thst_A, a_A, ecc_A,  "ob",2, frac2,"Blue",180)
    pl_orbit(thst_C, a_C, ecc_C, "or",2, frac2,"Red",180)
    xlabel('$$\hat{e}$$ Direction','Interpreter','Latex')
    ylabel('$$\hat{p}$$ Direction','Interpreter','Latex')
    title('Positions of Jupiter and Comets - Sushrut Prabhu')
    grid on
    grid minor
    axis equal
    legend('Jupiter','Asteroid A', 'Comet C','Sun')

    ylim([-1.1 1.1]*10^9)

    % Jupiter Rotating Frame
    figure
    hold on
    plot(a_jup, 0, 'Ok')
    pl_rot_orbit(Per_jup*5, n_jup, n_A, a_A, ecc_A,'-r',
 1, "", "",180)
    pl_rot_orbit(Per_jup*6, n_jup, n_C, a_C, ecc_C,'-b',
 1, "", "",180)
    plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
    pl_rot_orbit(frac*Per_jup, n_jup, n_A, a_A, ecc_A,'or', 2,
 frac2, "Red",180)
    pl_rot_orbit(frac*Per_jup, n_jup, n_C, a_C, ecc_C,'ob', 2,
 frac2, "Blue",180)
    xlabel('$$\hat{r}$$ Direction','Interpreter','Latex')
    ylabel('$$\hat{\theta}$$ Direction','Interpreter','Latex')
```

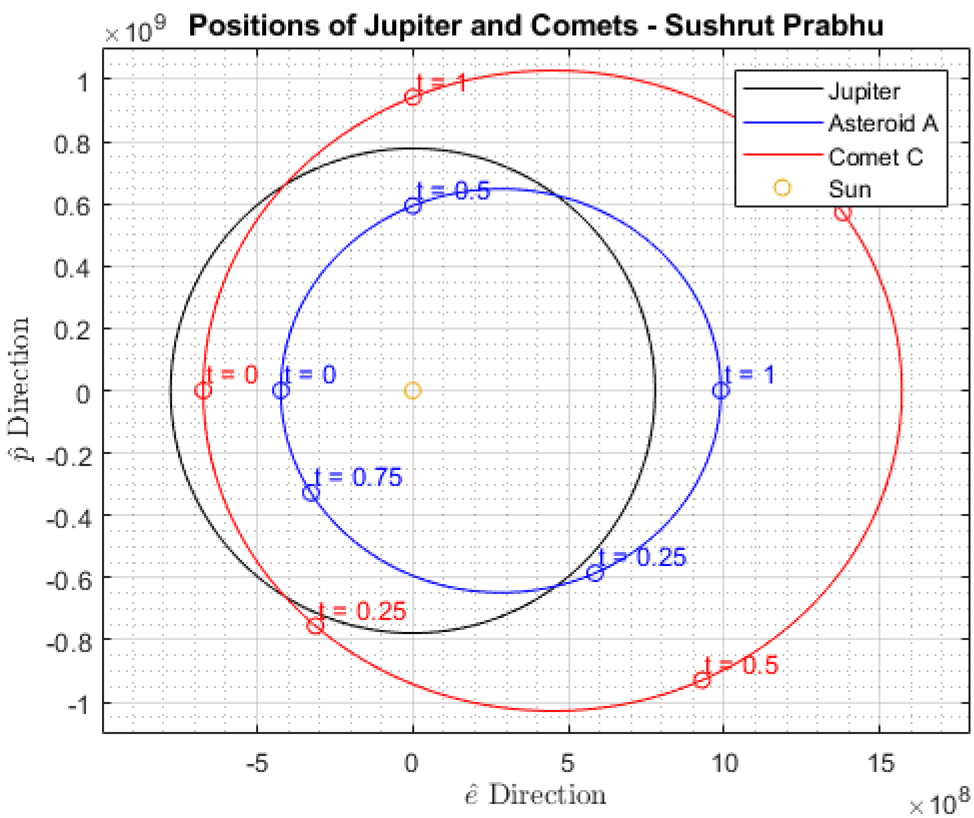
```
    title('Positions of Comets in Roating Jupiter Frame - Sushrut
 Prabhu')
    grid on
    grid minor
    axis equal
    legend('Jupiter','Asteroid A', 'Comet C','Sun')

    xlim([-2 2]*10^9)
end
```

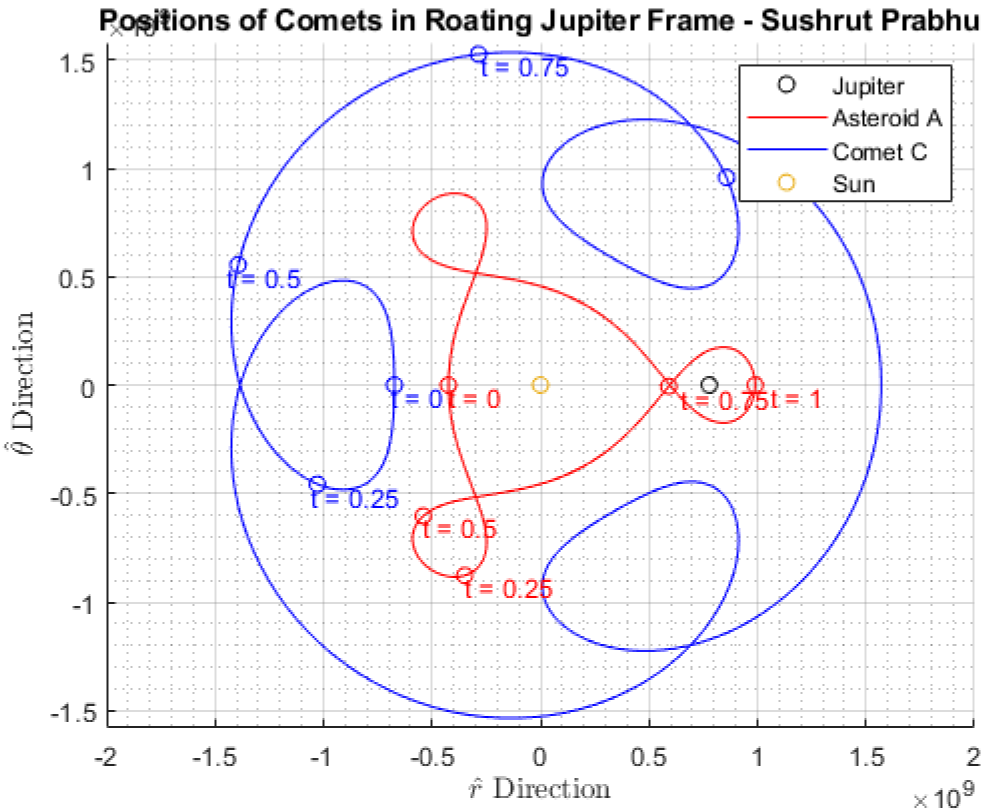



Positions of Comets in Roating Jupiter Frame - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
t = 0.75
t = 1
t = 0.5
t = 0
t = 0.75
$\hat{r}$ Direction
$\hat{\theta}$ Direction
$\times 10^9$


Positions of Jupiter and Comets - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
t = 1
t = 0.5
t = 0.75
t = 0
t = 0
t = 1
t = 0.75
t = 0.25
t = 0.25
t = 0.5
$\hat{e}$ Direction
$\hat{p}$ Direction
$\times 10^9$


Positions of Comets in Roating Jupiter Frame - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
t = 0.75
t = 1
t = 0.5
t = 0
t = 0
t = 0.25
t = 1
t = 0.75
t = 0.5
t = 0.25
$\hat{\theta}$ Direction
$\hat{r}$ Direction
$\times 10^9$

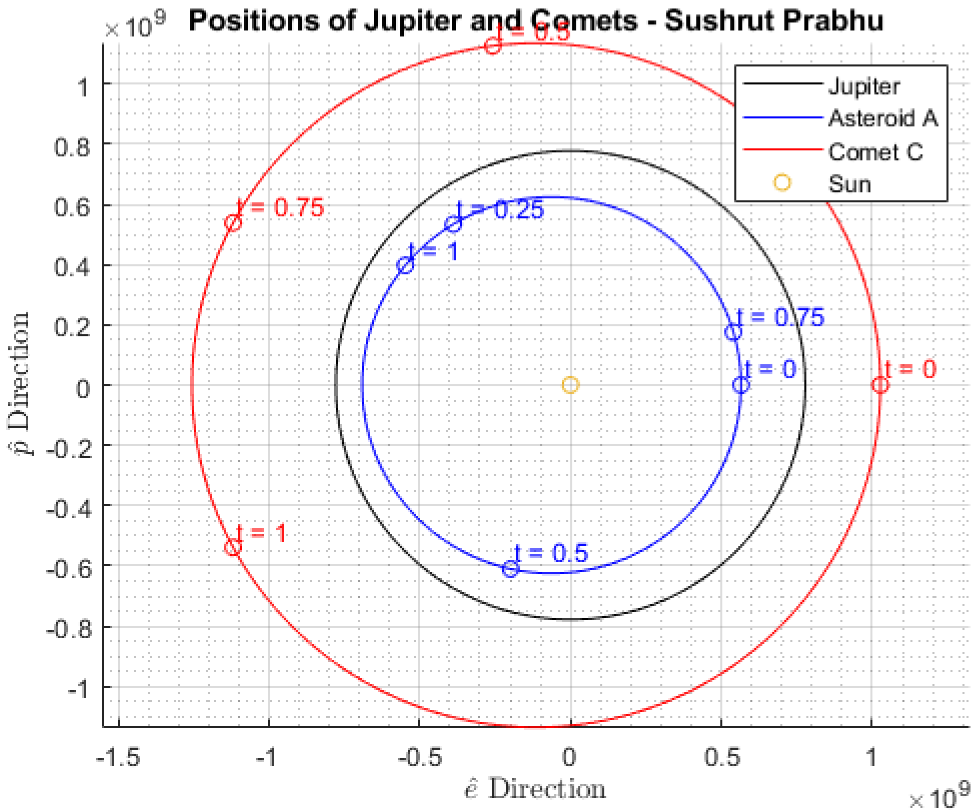

Positions of Jupiter and Comets - Sushrut Prabhu
$\times 10^9$
Jupiter
Asteroid A
Comet C
Sun
t = 1
t = 0.5
t = 0
t = 0
t = 1
t = 0.75
t = 0.25
t = 0.25
t = 0.5
$\hat{p}$ Direction
$\hat{e}$ Direction
$\times 10^8$

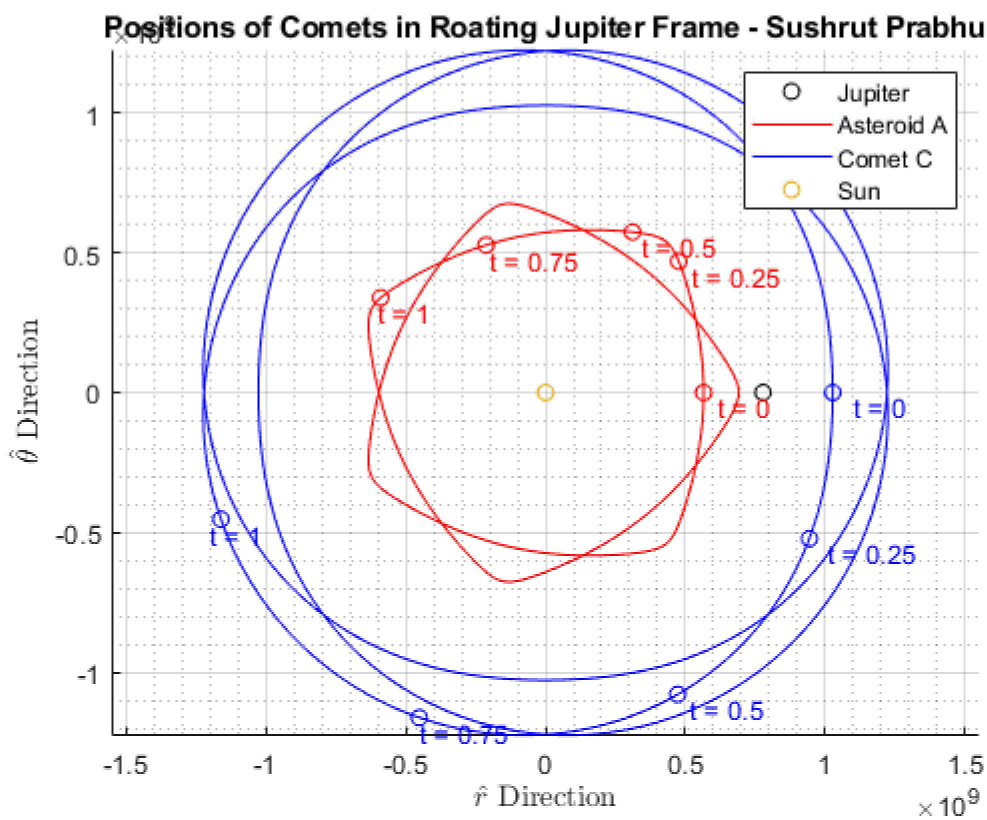


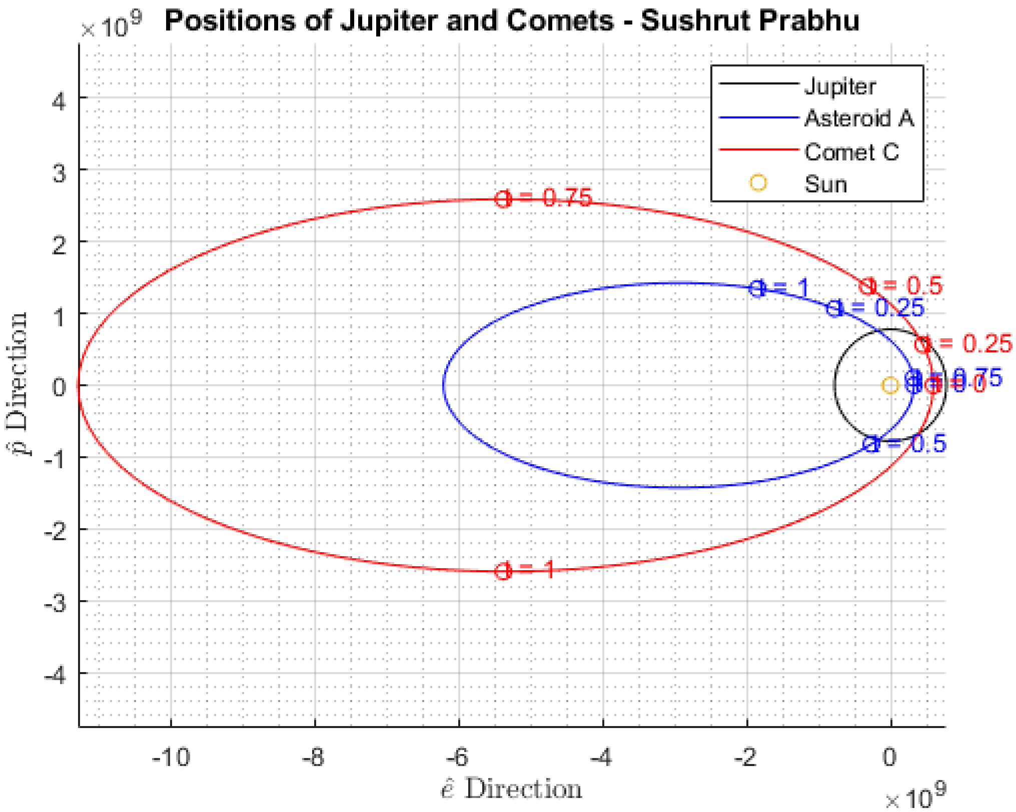
Part (e)

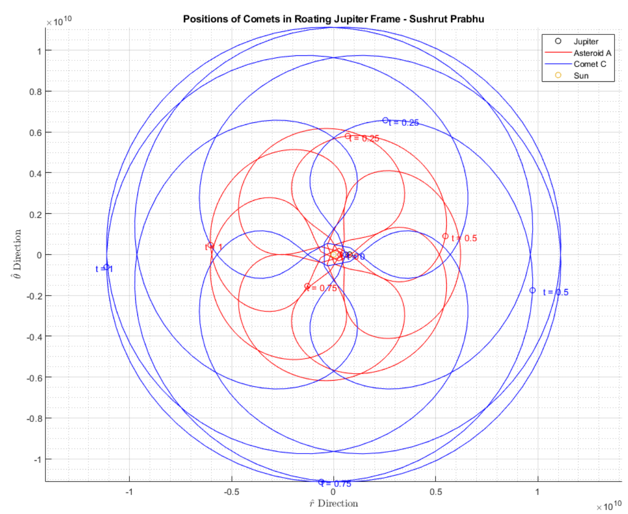
```
ecc = [0.1 0.9];

res1 = 5/7;
res2 = 7/4;

% Comet A
a_A = (res1^2 *a_jup^3)^(1/3);
Per_A = 2*pi/sqrt(miu_S/a_A^3);
Per_A_years = Per_A/3600/24/365;
n_A = sqrt(miu_S/a_A^3);

% Comet B
a_C = (res2^2 *a_jup^3)^(1/3);
Per_C = 2*pi/sqrt(miu_S/a_C^3);
Per_C_years = Per_C/3600/24/365;
n_C = sqrt(miu_S/a_C^3);


for m = 1:length(ecc)
    ecc_A = ecc(m);
    p_A = a_A*(1-ecc_A^2);

    ecc_C = ecc(m);
```

```
    p_C = a_A*(1-ecc_C^2);


    for n = 1:length(frac)
        t_jup = Per_jup*frac(n);

        thst_A(n) = rad2deg(t_jup * n_A);
        thst_C(n) = rad2deg(t_jup * n_C);
    end

    trueanom = 0:360;

    % Inertial Frame
    figure
    hold on
    pl_orbit(trueanom, a_jup, 0,  "-k",1,"","",0)
    pl_orbit(trueanom, a_A, ecc_A, "-b",1,"","",0)
    pl_orbit(trueanom, a_C, ecc_C, "-r",1,"","",0)
    plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
    pl_orbit(thst_A, a_A, ecc_A,  "ob",2, frac2,"Blue",0)
    pl_orbit(thst_C, a_C, ecc_C, "or",2, frac2,"Red",0)
    xlabel('$$\hat{e}$$ Direction','Interpreter','Latex')
    ylabel('$$\hat{p}$$ Direction','Interpreter','Latex')
    title('Positions of Jupiter and Comets - Sushrut Prabhu')
    grid on
    grid minor
    axis equal
    legend('Jupiter','Asteroid A', 'Comet C','Sun')

%     ylim([-1.1 1.1]*10^9)

    % Jupiter Rotating Frame
    figure
    hold on
    plot(a_jup, 0, 'Ok')
    pl_rot_orbit(Per_jup*5, n_jup, n_A, a_A, ecc_A,'-r', 1, "", "",0)
    pl_rot_orbit(Per_jup*7, n_jup, n_C, a_C, ecc_C,'-b', 1, "", "",0)
    plot(0, 0, 'O','Color',[0.9290, 0.6940, 0.1250])
    pl_rot_orbit(frac*Per_jup, n_jup, n_A, a_A, ecc_A,'or', 2,
 frac2, "Red",0)
    pl_rot_orbit(frac*Per_jup, n_jup, n_C, a_C, ecc_C,'ob', 2,
 frac2, "Blue",0)
    xlabel('$$\hat{r}$$ Direction','Interpreter','Latex')
    ylabel('$$\hat{\theta}$$ Direction','Interpreter','Latex')
    title('Positions of Comets in Roating Jupiter Frame - Sushrut
 Prabhu')
    grid on
    grid minor
    axis equal
    legend('Jupiter','Asteroid A', 'Comet C','Sun')

%     ylim([-1.6 1.6]*10^9)
end
```


Positions of Jupiter and Comets - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
t = 0
t = 0.25
t = 0.5
t = 0.75
t = 1
$\hat{p}$ Direction
$\hat{e}$ Direction
$\times 10^9$


Positions of Comets in Roating Jupiter Frame - Sushrut Prabhu
Jupiter
Asteroid A
Comet C
Sun
t = 0
t = 0.25
t = 0.5
t = 0.75
t = 1
$\hat{\theta}$ Direction
$\hat{r}$ Direction
$\times 10^9$





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Solar Systems Constants

Constants

```
classdef SolarS
    properties
        % Distances from Sun/ planet to planet/ Moon
        dMercury = 57909226.542
        dVenus = 108209474.537
        dEarth = 149597870.7
        dMars = 227943816.693
        dJupiter = 778340816.693
        dSaturn = 1426666414.180
        dM_E = 384400
        dMoon = 384400+149597870.7
        dPluto = 5906440596.5288
        dTitan_S = 1221865
        dPhobos_M = 9376;
        dEuropa_J = 671100
        dOberon_U = 583500
        dTriton_N = 354759
        dCharon_P = 17536

        % Mass of Planet/Star/ Moon (G*m)
        mSun = 132712440017.99
        mVenus = 324858.5988
        mEarth = 398600.4415
        mMars = 42828.3142
        mJupiter = 126712767.8578
        mSaturn = 37940626.0611
        mUranus = 5794549.0070719
        mNeptune = 6836534.06387
        mPluto = 981.600887707;

        mMoon = 4902.8011
        mTitan = 8979.766
        mPhobos = 7.11328968e-04
        mEuropa = 3203.31978
        mOberon = 192.4249
        mTriton = 1427.8589
        mCharon = 103.2187

        % Radius of Moon/Planet
        rEarth = 6378.136;
        rMars = 3397.00;
        rMoon = 1738.10;
        rSaturn = 60268.00;
        rJupiter = 71492.00;
        % Eccentricity of Planets
        eEarth = 0.01671022
        eSaturn = 0.05386179
```

```
    end
end
```

```
ans =

  SolarS with properties:

       dMercury: 5.7909e+07
         dVenus: 1.0821e+08
         dEarth: 1.4960e+08
          dMars: 2.2794e+08
       dJupiter: 7.7834e+08
        dSaturn: 1.4267e+09
           dM_E: 384400
          dMoon: 1.4998e+08
         dPluto: 5.9064e+09
       dTitan_S: 1221865
      dPhobos_M: 9376
      dEuropa_J: 671100
      dOberon_U: 583500
      dTriton_N: 354759
      dCharon_P: 17536
           mSun: 1.3271e+11
         mVenus: 3.2486e+05
         mEarth: 3.9860e+05
          mMars: 4.2828e+04
       mJupiter: 1.2671e+08
        mSaturn: 3.7941e+07
        mUranus: 5.7945e+06
       mNeptune: 6.8365e+06
         mPluto: 981.6009
          mMoon: 4.9028e+03
         mTitan: 8.9798e+03
        mPhobos: 7.1133e-04
        mEuropa: 3.2033e+03
        mOberon: 192.4249
        mTriton: 1.4279e+03
        mCharon: 103.2187
         rEarth: 6.3781e+03
          rMars: 3397
          rMoon: 1.7381e+03
        rSaturn: 60268
       rJupiter: 71492
         eEarth: 0.0167
        eSaturn: 0.0539
```

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Mean Anomaly to Eccentric Anomaly

```
function [E_n,n]= M2E_NRm(M,ecc,err_tol)

err = 1;
E_n = M;    % Initial guess for eccentric anomaly guess
n = 1;

% Newton Rhapson method to solve for Eccentric anomaly
while err> err_tol
    E_n1 =  E_n - (E_n-ecc*sin(E_n)-M)/(1-ecc*cos(E_n));
    err = abs(E_n1-E_n);
    E_n = E_n1;
    n = n+1;
end

E_n = rad2deg(E_n);
end
```

```
Not enough input arguments.

Error in M2E_NRm (line 5)
E_n = M;    % Initial guess for eccentric anomaly guess
```

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```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Plot Inertial Frame
%
% Input:
% trueanom - array of true anaomaly 0->360
% p - Semi-latus Rectum
% ecc - eccentricity
% rE - Radius of Earth
% title_n - Plot title you desire
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function pl_orbit(trueanom, p, ecc, style, type, text_inp, style2,
 shift)
```

Plot Orbit

```
r = p ./ (1 + ecc*cosd(trueanom));
epos = r.*cosd(trueanom);
ppos = r.*sind(trueanom);

if shift > 0
    ep_new = cosd(shift)*epos + sind(shift)*ppos;
    pp_new = -sind(shift)*epos + cosd(shift)*ppos;

    epos = ep_new;
    ppos = pp_new;
end

plot(epos,ppos, style)

if type == 2
    for n = 1:length(epos)
        text(epos(n)+1*10^7,ppos(n)+6*10^7, text_inp{n}, 'Color',
 style2)
    end
end
```

```
Not enough input arguments.

Error in pl_orbit (line 15)
r = p ./ (1 + ecc*cosd(trueanom));
```

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Plot rotating frame

```
function pl_rot_orbit(time_end, n_rot, n, p, ecc, style, type,
 text_inp, style2, shift)
```

Plot Orbit

```
if type == 2
    time = time_end;
else
    time = linspace(0,time_end,500);
end

thst_rot = n_rot*time*180/pi;

% True anomaly of Comet
for m = 1:length(time)
    M = n*time(m);
    E = M2E_NRm(M,ecc,10^-12);
    thst(m) = 2*atand( sqrt((1+ecc)/(1-ecc)) * tand(E/2));
end

r = p ./ (1+ecc*cosd(thst));              % Radial distance to comet

epos = r.*cosd(thst);                    % Intertial e position to
 Comet
ppos = r.*sind(thst);                    % Intertial p position to
 Comet

% 180 degree shift if rquired
if shift > 0
    ep_new = cosd(shift)*epos + sind(shift)*ppos;
    pp_new = -sind(shift)*epos + cosd(shift)*ppos;

    epos = ep_new;
    ppos = pp_new;
end

% DCM to change from inertial to rotating
for n = 1:length(epos)
    rth_rot(:,n) = [cosd(thst_rot(n)) sind(thst_rot(n)); -
sind(thst_rot(n)) cosd(thst_rot(n))] * [epos(n); ppos(n)];
end

plot(rth_rot(1,:),rth_rot(2,:), style)
if type == 2
    for n = 1:length(epos)
        text(rth_rot(1,n)*1.05+2*10^7,rth_rot(2,n)-5*10^7,
 text_inp{n}, 'Color', style2)
    end
end
```

```
Not enough input arguments.

Error in pl_rot_orbit (line 5)
if type == 2
```

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Sushrut Prabhu | AAE 632 | PS8 2

Problem P3

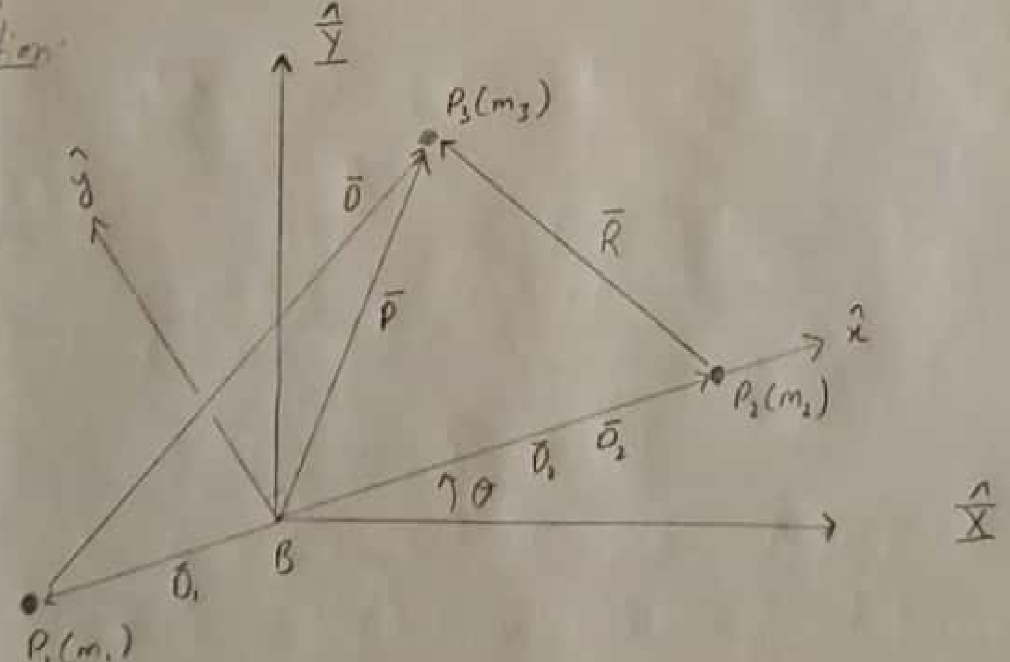
Given: A relative positions of 3 masses with spherically symmetric gravity fields are represented in following figure.

Find: a) Assumptions for restricted 3-body problem, derive differential equations that govern P_3 with respect to barycenter.

b) Compare dimensional values for characteristic quantities corresponding to distance, mass, and time in following systems:
i) Sun - Earth (ii) Earth - Moon (iii) Sun - Jupiter (iv) Saturn - Titan
(v) Sun Mars (vi) Mars - Phobos (vii) Jupiter - Europa (viii) Uranus - Oberon
(ix) Neptune - Triton (x) Pluto - Charon
Do any of the characteristic quantities reflect physical properties?

c) Take the relationships from (a) and derive classical set of non-dimensional equations. Does your set match the given in class?

Solution



i) The mass of body 3 is much less than body 1, 2. Which means that we can assume that the mass P_3 doesn't influence the motion of P_1 or P_2.

Continued...

2) $\boxed{\text{Due to the infetisimal mass of } P_3 \text{ compared } P_1 \text{ and } P_2 \text{ we can assume } P_1 \text{ and } P_2 \text{ has a conic solution}}$

3) $\boxed{\text{In most cases we can assume a circular orbit, so let us assume a circular orbit for } P_1 \text{ and } P_2}$

These assumptions result to:

$\dot{\theta}$ is constant due to circular orbit of the primaries

$$m_3 {}^I\ddot{\bar{r}}_3 = -G\frac{m_3 m_1}{|\bar{D}|^3}\bar{D} - G\frac{m_3 m_2}{|\bar{R}|^3}\bar{R}$$ ← Law of Gravitation

Let us derive the $\ddot{\bar{r}}_3$ Note: $\ddot{\theta}=0$ because orbit is circular

$$\bar{r} = \bar{\rho} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\frac{\partial \bar{r}}{\partial t} = \dot{\bar{\rho}} + \dot{\theta}\hat{z}\times\bar{\rho} = \dot{\theta}\hat{z}\times(x\hat{x} + y\hat{y} + z\hat{z})$$

$$= \dot{\bar{\rho}} + \dot{\theta}x\hat{y} - \dot{\theta}y\hat{x}$$

$$\therefore \frac{\partial^2 r}{\partial t^2} = \ddot{\bar{\rho}} + \ddot{\theta}x\hat{y} + \dot{\theta}\dot{x}\hat{y} - \ddot{\theta}y\hat{x} - \dot{\theta}\dot{y}\hat{x} + \dot{\theta}\hat{z}\times\frac{\partial \bar{r}}{\partial t}$$

$$= \ddot{\bar{\rho}} + \ddot{\theta}x\hat{y} + \dot{\theta}\dot{x}\hat{y} - \ddot{\theta}y\hat{x} - \dot{\theta}\dot{y}\hat{x}$$
$$+ \dot{\theta}\hat{z}\times(\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} + \dot{\theta}x\hat{y} - \dot{\theta}y\hat{x})$$

$$= \ddot{\bar{\rho}} + \cancel{\ddot{\theta}x\hat{y}}^{0} + \dot{\theta}\dot{x}\hat{y} - \cancel{\ddot{\theta}y\hat{x}}^{0} - \dot{\theta}\dot{y}\hat{x} + \dot{\theta}\dot{x}\hat{y} + \dot{\theta}\dot{y}\hat{z}$$
$$-\dot{\theta}^2 x\hat{x} - \dot{\theta}^2 y\hat{y}$$

$$\boxed{\therefore \frac{\partial^2 r}{\partial t} = \ddot{\bar{\rho}} + 2\dot{\theta}\dot{x}\hat{y} - 2\dot{\theta}\dot{y}\hat{x} + \dot{\theta}^2 x\hat{x} - \dot{\theta}^2 y\hat{y}}$$

See next page for break down

Continued ...

$$\boxed{\begin{aligned} \hat{x} &\Rightarrow \ddot{x} - 2\dot{\theta}\dot{y} - \dot{\theta}^2 x \\ \hat{y} &\Rightarrow \ddot{y} + 2\dot{\theta}\dot{x} - \dot{\theta}^2 y \\ \hat{z} &\Rightarrow \ddot{z} \end{aligned}}$$

Now let us equate kinematics and dynamics

$$\therefore \cancel{m_3} \frac{\partial^2 \bar{r}}{\partial t^2} = -\frac{G \cancel{m_3} m_1}{|\bar{D}|^3} \bar{D} - \frac{G \cancel{m_3} m_2}{|\bar{R}|^3} \bar{R}$$

$$\boxed{\begin{aligned} &\therefore \ddot{x} - 2\dot{\theta}\dot{y} - \dot{\theta}^2 x = -\frac{Gm_1}{|\bar{D}|^3} \bar{D}\cdot\hat{x} - \frac{Gm_2}{|\bar{R}|^3} \bar{R}\cdot\hat{x} \\ &\therefore \ddot{y} + 2\dot{\theta}\dot{x} - \dot{\theta}^2 y = -\frac{Gm_1}{|\bar{D}|^3} \bar{D}\cdot\hat{y} - \frac{Gm_2}{|\bar{R}|^3} \bar{R}\cdot\hat{y} \\ &\therefore \ddot{z} = -\frac{Gm_1}{|\bar{D}|^3} \bar{D}\cdot\hat{z} - \frac{Gm_2}{|\bar{R}|} \bar{R}\cdot\hat{z} \end{aligned}}$$

On RHS we only take $\hat{x}$, $\hat{y}$, or $\hat{z}$ components

b) Characteristic Quantities

$$l^* = D_1 + D_2 \qquad m^* = m_1 + m_2 \qquad t^* = \left[\frac{(D_1 + D_2)^3}{\underbrace{\tilde{G}}_{\text{Gravitational constant}}(m_1 + m_2)}\right]^{1/2}$$

	l^* (km)	m^* (kg)	t^* (s)
1) Sun-Earth	1.496×10^8	1.9886×10^{30}	5.0226×10^6
2) Earth-Moon	3.844×10^5	6.0461×10^{24}	3.7519×10^5
3) Sun-Jupiter	7.7834×10^8	1.9905×10^{30}	5.9577×10^7
4) Saturn-Titan	1.2219×10^6	5.6864×10^{26}	2.1952×10^5
5) Sun-Mars	2.2974×10^8	1.9886×10^{30}	9.4468×10^6

Continued ...

	l^* (km)	m^* (kg)	t^* (s)
6) Mars - Phobos	9.376×10^3	6.4174×10^{23}	4.3869×10^3
7) Jupiter - Europa	6.711×10^5	1.8987×10^{27}	4.8839×10^4
8) Uranus - Oberon	5.835×10^5	8.6828×10^{25}	1.8516×10^5
9) Neptune - Triton	3.54759×10^5	1.0246×10^{26}	8.0805×10^4
10) Pluto - Charon	1.7536×10^4	1.6255×10^{22}	7.0505×10^4

t^* is close to the formulation of Period. So $2\pi t^*$ gives the orbital period of the these systems.
l^* is just the radial distance or the semimajor axis between two bodies. $r = a$ because we assume motion to be circular
m^* is the M we used in our previous notation

c) Now let us go back to the differential equations we derived. But first we need to define some non-dimensional quantities

let $\frac{m_2}{m^*} = \mu$ ① Then $m_1 + m_2 = m_1 + \mu m^* = m^*$

$\therefore \frac{m_1}{m^*} = 1 - \mu$ ②

$\bar{d}_i = \frac{\bar{D}_i}{l^*}$ where $i = 1, 2$ Note: $\bar{d}_i$ is not the same as $\bar{d}$

$\bar{r} = \frac{\bar{R}}{l^*}$, $\bar{d} = \frac{\bar{D}}{l^*}$

$a = \frac{\bar{a}}{l^*} = 1$ because of a circular orbit

a is the nondimensional → semi-major axis

continued...

$$G = \frac{\tilde{G}}{(l^*)^3} m^*(t^*)^2 = 1$$ non dimensional Gravitational constant

$$\dot{\theta} = N = \left[\frac{\tilde{G}(m_2+m_1)}{\tilde{a}^3}\right]^{1/2} = \left[\frac{\tilde{G}m^*}{(a\,l^*)^3}\right]^{1/2} = \left[\frac{\tilde{G}m^*}{(l^*)^3}\right]^{1/2}$$

$n = Nt^* = 1$ Nondimensional mean motion

$$\bar{\rho} = \frac{\bar{R}}{l^*} \qquad \tau = \frac{t}{t^*} \qquad \therefore \dot{\bar{\rho}} = \frac{\dot{\bar{R}}}{l^*}t^* \qquad \therefore \ddot{\bar{\rho}} = \frac{\ddot{\bar{R}}}{l^*}(t^*)^2$$

$$\therefore \bar{\rho}'' = -\frac{\tilde{G}m_1}{D^3}\bar{D} - \frac{\tilde{G}m_2}{R^3}\bar{R}$$

$$RHS \to -\frac{\left[\frac{(l^*)^3}{m^*(t^*)^2}\right]\left[(1-\mu)m^*\right]\left[\bar{d}l^*\right]}{(|\bar{d}|l^*)^3} - \frac{\left[\frac{(l^*)^3}{m^*(t^*)^2}\right]\left[\mu m^*\right]\left[\bar{r}l^*\right]}{(|\bar{r}|l^*)^3}$$

$$= \frac{(1-\mu)l^*}{(t^*)^2}\frac{\bar{d}}{|\bar{d}|^3} - \frac{\mu l^*}{(t^*)^2}\frac{\bar{r}}{|\bar{r}|^3}$$

Now recall we derived LHS in part (a)

$$\bar{\rho}'' + 2\dot{\theta}\dot{x}\hat{y} - 2\dot{\theta}\dot{y}\hat{x} - \dot{\theta}^2 x\hat{x} - \dot{\theta}^2 y\hat{y}$$

$$\doteq \left[\frac{\ddot{\bar{\rho}}l^*}{(t^*)^2}\right] + 2\left[\frac{n}{t^*}\right]\left[\frac{\dot{x}}{t^*}\right]\hat{y} - 2\left[\frac{n}{t^*}\right]\left[\frac{\dot{y}}{t^*}\right]\hat{x} - \left[\frac{n}{t^*}\right]^2[x]\hat{x}$$
$$-\left[\frac{n}{t^*}\right]^2[y]\hat{y}$$

$$= \frac{\ddot{\bar{\rho}}l^*}{(t^*)^2} + \frac{2n\dot{x}l^*}{(t^*)^2}\hat{y} - \frac{2n\dot{y}l^*\hat{x}}{(t^*)^2} - \frac{n^2 x l^*\hat{x}}{(t^*)^2} - \frac{n^2 y l^*}{(t^*)^2}\hat{y}$$

Continue...

Combine LHS and RHS, note $(t^*)^2$ can be cancelled on both sides and so can l^*:

$$\boxed{\ddot{\bar{\rho}} + 2n\dot{x}\hat{y} - 2n\dot{y}\hat{x} - n^2x\hat{x} - n^2y\hat{y} = -\frac{(1-\mu)\bar{d}}{|d|^3} - \frac{\mu\bar{r}}{|\bar{r}|^3}}$$

Now let us define $\bar{d}$ and $\bar{r}$

definition of c.m. $\Rightarrow m_1\bar{D}_1 + m_2\bar{D}_2 = 0$

$$\frac{m_1}{m^*}\bar{D}_1 + \frac{m_2}{m^*}\bar{D}_2 = 0 \rightarrow \therefore (1-\mu)\bar{D}_1 = -\mu\bar{D}_2$$

$$\therefore \frac{\bar{D}_1}{\bar{D}_2} = \frac{\mu}{(\mu-1)} \qquad \therefore \bar{D}_1 = \mu \qquad \bar{D}_2 = \mu-1$$

$$\bar{d} = \bar{d}_1 + \bar{D} = (x+\mu)\hat{x} + y\hat{y} + z\hat{z}$$

$$\bar{r} = \bar{d}_2 + \bar{R} = (x+\mu-1)\hat{x} + y\hat{y} + z\hat{z}$$

Now let us split everything into individual components

$$\ddot{x} - 2n\dot{y}\hat{x} - n^2x\hat{x} = -\frac{(1-\mu)(x+\mu)}{d^3} - \frac{\mu(x-1+\mu)}{r^3}$$

$$\ddot{y} + 2n\dot{x} - n^2y = -\frac{(1-\mu)y}{d^3} - \frac{\mu y}{r^3}$$

$$\ddot{z} = -\frac{(1-\mu)z}{d^3} - \frac{\mu z}{r^3}$$

Note $d = |\bar{d}| = ((x+\mu)^2 + y^2 + z^2)^{1/2}$

$r = |\bar{r}| = ((x+\mu-1)^2 + y^2 + z^2)^{1/2}$

Equation is exactly the same as notes

PSB2

```
clear
close all
clc

SS = SolarS;
systems = {'Sun-Earth', 'Earth-Moon', 'Sun-Jupiter', 'Saturn-
Titan', 'Sun-Mars', 'Mars-Phobos', 'Jupiter-Europa', 'Uranus-
Oberon', 'Netune-Triton', 'Pluto-Charon'};
param = {'-','l* (km)','m* (kg)','t* (s)'};
G = 6.6738*10^-20;

dim_vals = num2cell(zeros(10,3));
dim_vals = [param; systems',dim_vals];

[dim_vals{2,2}, dim_vals{2,3}, dim_vals{2,4}] =
 charE(SS.dEarth,0,SS.mSun/G,SS.mEarth/G);
[dim_vals{3,2}, dim_vals{3,3}, dim_vals{3,4}] =
 charE(SS.dM_E,0,SS.mMoon/G,SS.mEarth/G);
[dim_vals{4,2}, dim_vals{4,3}, dim_vals{4,4}] =
 charE(SS.dJupiter,0,SS.mSun/G,SS.mJupiter/G);
[dim_vals{5,2}, dim_vals{5,3}, dim_vals{5,4}] =
 charE(SS.dTitan_S,0,SS.mSaturn/G,SS.mTitan/G);
[dim_vals{6,2}, dim_vals{6,3}, dim_vals{6,4}] =
 charE(SS.dMars,0,SS.mSun/G,SS.mMars/G);
[dim_vals{7,2}, dim_vals{7,3}, dim_vals{7,4}] =
 charE(SS.dPhobos_M,0,SS.mPhobos/G,SS.mMars/G);
[dim_vals{8,2}, dim_vals{8,3}, dim_vals{8,4}] =
 charE(SS.dEuropa_J,0,SS.mJupiter/G,SS.mEuropa/G);
[dim_vals{9,2}, dim_vals{9,3}, dim_vals{9,4}] =
 charE(SS.dOberon_U,0,SS.mUranus/G,SS.mOberon/G);
[dim_vals{10,2}, dim_vals{10,3}, dim_vals{10,4}] =
 charE(SS.dTriton_N,0,SS.mNeptune/G,SS.mTriton/G);
[dim_vals{11,2}, dim_vals{11,3}, dim_vals{11,4}] =
 charE(SS.dCharon_P,0,SS.mPluto/G,SS.mCharon/G);


dim_vals
```

```
dim_vals =

  11×4 cell array

  Columns 1 through 2

    {'-'              }    {'l* (km)'              }
    {'Sun-Earth'      }    {[1.495978707000000e+08]}
    {'Earth-Moon'     }    {[               384400]}
    {'Sun-Jupiter'    }    {[7.783408166930000e+08]}
    {'Saturn-Titan'   }    {[              1221865]}
    {'Sun-Mars'       }    {[2.279438166930000e+08]}
```

```
    {'Mars-Phobos'   }    {[                 9376]}
    {'Jupiter-Europa'}    {[               671100]}
    {'Uranus-Oberon' }    {[               583500]}
    {'Netune-Triton' }    {[               354759]}
    {'Pluto-Charon'  }    {[                17536]}

  Columns 3 through 4

    {'m* (kg)'               }    {'t* (s)'                }
    {[1.988564814924503e+30]}    {[5.022635348655218e+06]}
    {[6.046079334112500e+24]}    {[3.751902586518695e+05]}
    {[1.990457502260298e+30]}    {[5.957878742866884e+07]}
    {[5.686356472639275e+26]}    {[2.192457595655987e+05]}
    {[1.988559484046633e+30]}    {[9.446824268941369e+06]}
    {[6.417380639415171e+23]}    {[4.386932017240478e+03]}
    {[1.898707950157032e+27]}    {[4.883882198292967e+04]}
    {[8.682821528921902e+25]}    {[1.851587582503429e+05]}
    {[1.024597968589110e+26]}    {[8.080473196142506e+04]}
    {[1.625490107145854e+22]}    {[7.050453393398008e+04]}
```

Published with MATLAB® R2018a

Sushrut Prabhu | AAE 632 PS B3

Problem B3:

Given Locations of the equilibrium points in any 3-body system
$\bar{\nabla} U^* = \bar{0}$

Find: a) Show that the coordinates of the equilibrium point L_2 on the far side of smaller primary is

$$x_2 = 1-\mu+\gamma_2 \quad , \quad y_2 = z_2 = 0$$

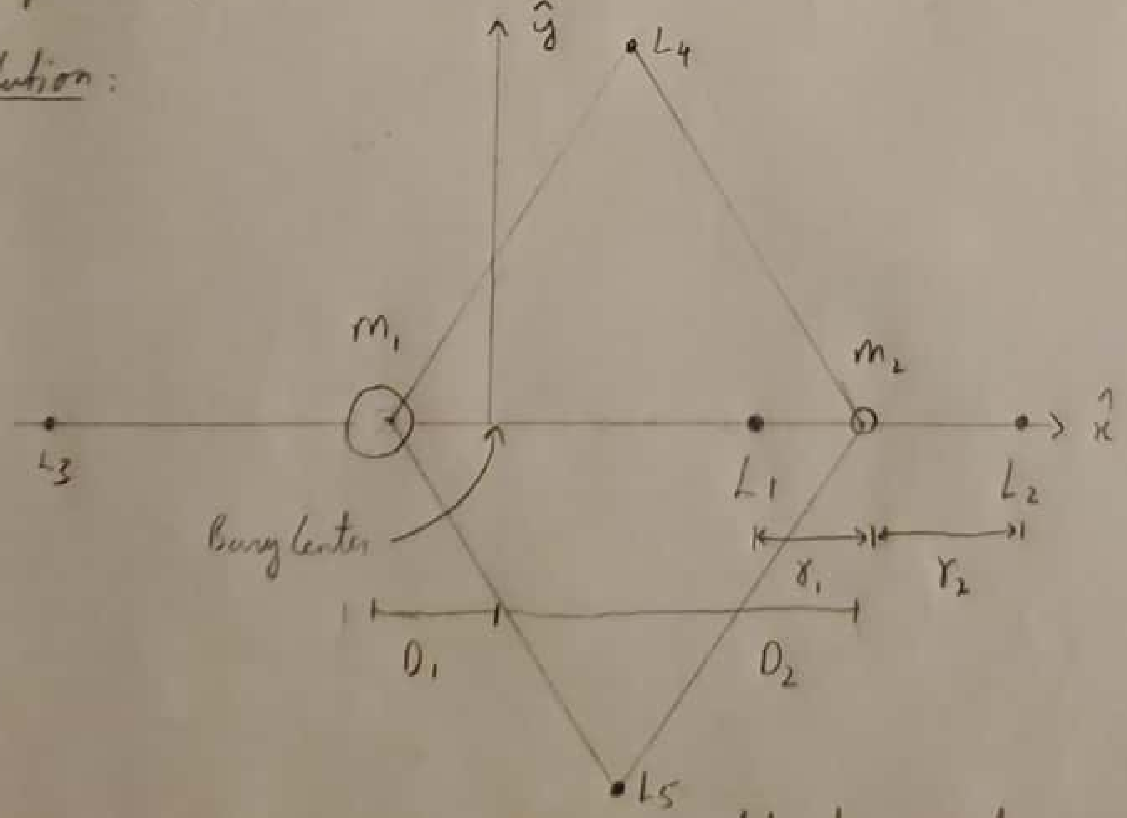
and $\gamma_2 \Rightarrow$ $$\frac{1-\mu}{(1+\gamma_2)} + \frac{\mu}{\gamma_2^2} = 1-\mu+\gamma_2$$

b) Solve equation using Newton Raphson and derive update equations. What is L_2 for Earth-Moon, Sun-Earth, Jupiter-Europa, Saturn-Titan, Pluto-Charon. (L_2 and γ_2 in dim and non-dim)

c) For liberation points L_1 and L_3, find γ_1 and γ_3. Derive new update equation.

Solution:

a)



Note in the previous problem we defined non-dimensional from barycenter to m_1 and m_2

Continued...

$-D_1 m_1 + D_2 m_2 = 0$ ← Definition of C.M., careful of direction

$\therefore D_1 \frac{m_1}{m^*} + D_2 \frac{m_2}{m^*} = 0$

Note when deriving 3-body EOM we defined $\frac{m_1}{m_*} = 1-\mu$ and that $\frac{m_2}{m_*} = \mu$

$\therefore D_1(1-\mu) = +D_2 \mu$

$\therefore \frac{D_1}{D_2} = \frac{+\mu}{1-\mu}$

Thus point L_2 on the previous figure can be written as follows:

$x_2 = 1 - \mu + \gamma_2$ and $y = z = 0$

Now we also define U^* in the notes as the pseudo potential

$d = [(x+\mu)^2 + y^2 + z^2]^{1/2}$

$r = [(x-1+\mu)^2 + y^2 + z^2]^{1/2}$

$$U^* = \frac{(1-\mu)}{d} + \frac{\mu}{r} + \frac{1}{2} \overset{n=1}{n^2}(x^2+y^2)$$

$$\therefore \bar{\nabla} U^* = \frac{\partial U^*}{\partial x}\hat{x} + \frac{\partial U^*}{\partial y}\hat{y} + \frac{\partial U^*}{\partial z}\hat{z} = \bar{0}$$

$$\hat{x} \Rightarrow \frac{\partial U}{\partial x} = -\frac{(1-\mu)(x+\mu)}{d^3} - \frac{\mu(x-1+\mu)}{r^3} + n^2 x = 0$$

$$\hat{y} \Rightarrow \frac{\partial U}{\partial y} = -\frac{(1-\mu)y}{d^3} - \frac{\mu y}{r^3} + n^2 y = 0$$

$$\hat{z} \Rightarrow \frac{\partial U}{\partial z} = -\frac{(1-\mu)z}{d^3} - \frac{\mu z}{r^3} = 0$$

Continued...

$\therefore y = z = 0$ $\qquad \therefore \hat{x} \Rightarrow \frac{(1-\mu)(x+\mu)}{d^3} - \frac{\mu(x-1+\mu)}{r^3} + 1 - \mu + \gamma_2 = 0$

$x_2 = 1 - \mu + \gamma_2$

$$\therefore -\frac{(1-\mu)(1-\mu+\gamma_2+\mu)}{\left(\left[(1-\mu+\gamma_2+\mu)^2\right]^{1/2}\right)^3} - \frac{\mu(1-\mu+\gamma_2-1+\mu)}{\left(\left[(1-\mu+\gamma_2-1+\mu)^2\right]^{1/2}\right)^3}$$

$$+ 1 - \mu + \gamma_2 = 0$$

$$\therefore -\frac{(1-\mu)\cancel{(1+\gamma_2)}}{(1+\gamma_2)^{3-2}} - \frac{\mu \cancel{\gamma_2}}{\gamma_2^{3-2}} + 1 - \mu + \gamma_2 = 0$$

$$\therefore \boxed{\frac{(1-\mu)}{(1+\gamma_2)^2} + \frac{\mu}{\gamma_2^2} = 1 - \mu + \gamma_2}$$

$$\boxed{x_2 = 1 - \mu + \gamma_2}$$

b) Let us start with the γ_2 relationship. We can turn it into a polynomical, which is easier to solve and plot.

$$\frac{(1-\mu)}{(1+\gamma_2)^2} + \frac{\mu}{\gamma_2^2} = 1 - \mu + \gamma_2$$

$$\therefore (1-\mu)\gamma_2^2 + \mu(1+\gamma_2)^2 = (1-\mu+\gamma_2)(1+\gamma_2)^2 \gamma_2^2$$

LHS: $\gamma_2^2 + 2\mu\gamma_2 + \mu$

RHS: $\gamma_2^5 + \gamma_2^4(3-\mu) + \gamma_2^3(3-2\mu) + \gamma_2^2(1-\mu)$

$\therefore$ Combine RHS and LHS

$$\therefore 0 = \gamma_2^5 + \gamma_2^4(3-\mu) + \gamma_2^3(3-2\mu) - \mu\gamma_2^2 - 2\gamma_2\mu - \mu = f(\gamma_2)$$

Now let solve for $f(\gamma_2) = 0$ using Newton Raphson

Continued...

$$\therefore f'(\gamma_2) = 5\gamma_2^4 + 4\gamma_2^3(3-\mu) + 3\gamma_2^2(3-2\mu) - 2\mu\gamma_2 - 2\mu$$

Newton Rhaphson:

$$\gamma_{2,n+1} = \gamma_n - \frac{f(\gamma_{2,n})}{f'(\gamma_{2,n})}$$

Once we have γ_2 we can get x_2, and then dimensionalize them using L^*

Note: The $f(\gamma_2)$ is a 5^{th} order polynomial, so there are 5 solutions. We want the real solution. It is smart to plot the equations to check to confirm the number of roots as the Newton-Rhapson method may not localy converge.

Secondly: it is smart to check for x_0 you cannot use.

Set $f'(\gamma_2) = 0$ $\therefore$ the γ_2 which is a root of $f'(\gamma_2)$ will cause a division error

Note x_0 also cannot be $0 = \gamma_n - \frac{f(\gamma_{2,n})}{f'(\gamma_{2,n})}$

this x_0 will only produce a trivial solution.

	Sun Earth	Earth Moon	Saturn Titan	Jupiter Europa	Pluto Charon
γ_2 (-)	0.010037	0.167833	0.043492	0.0204864	0.3530021
L_2 (-)	1.01003	1.155682	1.043255	1.0204612	1.257854
γ_2 (km)	1.5015×10^6	6.45149×10^4	5.314144×10^4	1.37484×10^4	6.19024×10^3
L_2 (km)	1.51099×10^8	4.44244×10^5	1.274717×10^6	6.54832×10^5	2.10577×10^4

Continued...

c) We can use the same method for γ_1 and γ_3
Note for both have $y = z = 0$, so we only worry about direction in x-direction

$\therefore x_1 = 1 - \mu - \gamma_1$

$$\frac{\partial U}{\partial x} = 0 = -\frac{(1-\mu)(x+\mu)}{[[(x+\mu)^2]^{1/2}]^3} - \frac{\mu(x-1+\mu)}{[[(x-1+\mu)^2]^{1/2}]^3} + n^2 x$$

$$\therefore \frac{(1-\mu)(1-\cancel{\mu}-\gamma_1 \cancel{+\mu})}{(1-\cancel{\mu}-\gamma_1 \cancel{+\mu})^3} + \frac{\mu(\cancel{1}-\cancel{\mu}-\gamma_1 \cancel{-1} \cancel{+\mu})}{(\cancel{1}-\cancel{\mu}-\gamma_1 \cancel{-1} \cancel{+\mu})^3} = 1 - \mu - \gamma_1$$

$$\therefore \frac{(1-\mu)\cancel{(1-\gamma_1)}}{(1-\gamma_1)^{\cancel{3}\,2}} + \frac{\mu \cancel{(-\gamma_1)}}{(-\gamma_1)^{\cancel{3}\,2}} = 1 - \mu - \gamma_1$$

$$\therefore \frac{(1-\mu)}{(1-\gamma_1)^2} + \frac{\mu}{\gamma_1^2} = 1 - \mu - \gamma_1$$ Now we have the same form as before

Let us get the $f(\gamma_1) = 0$

$\therefore (1-\mu)\gamma_1^2 + \mu(1-\gamma_1)^2 = (1-\mu-\gamma_1)(1-\gamma_1)^2\gamma_1^2$

$\therefore$ LHS: $\gamma_1^2 - 2\mu\gamma_1 + \mu$

RHS: $-\gamma_1^5 + \gamma_1^4(3-\mu) + \gamma_1^3(2\mu-3) + \gamma_1^2(1-\mu)$

$$f(\gamma_1) = -\gamma_1^5 + \gamma_1^4(3-\mu) + \gamma_1^3(2\mu-3) - \mu\gamma_1^2 + 2\mu\gamma_1 - \mu$$
$$f'(\gamma_1) = -5\gamma_1^4 + 4\gamma_1^3(3-\mu) + 3\gamma_1^2(2\mu-3) - 2\mu\gamma_1 + 2\mu$$

Now we can use the Newton-Raphson on the equations

$$\gamma_{1,n+1} = \gamma_{1,n} - \frac{f(\gamma_{1,n})}{f'(\gamma_{1,n})}$$

Scanned with CamScanner

Continued...

	Sun / Earth	Earth / Moon	Saturn / Titan	Jupiter / Europa	Pluto / Charon
γ_1 (-)	0.010037	0.167833	0.043492	0.0204864	0.355002
L_1 (-)	0.98996	0.8200167	0.9562713	0.9794882	0.551849
γ_1 (km)	1.50153×10^6	6.45149×10^4	5.314144×10^4	1.374848×10^4	6.19025×10^3
L_1 (km)	1.480959×10^8	3.15214×10^5	1.68434×10^6	6.57334×10^5	9.67723×10^3

we follow the same sort of steps for γ_3

$x_3 = -\mu - \gamma_3$ again $y = z = 0$

$$\therefore \frac{\partial U}{\partial x} = 0 = \frac{-(1-\mu)(x+\mu)}{[[(x+\mu)^2]^{1/2}]^3} - \frac{\mu(x-1+\mu)}{[[(x-1+\mu)^2]^{1/2}]^3} + n^2 x$$

$$\therefore \frac{-(1-\mu)(\cancel{-\mu}-\gamma_3\cancel{+\mu})}{(\cancel{-\mu}-\gamma_3\cancel{+\mu})^3} - \frac{\mu(\cancel{-\mu}-\gamma_3-1\cancel{+\mu})}{(\cancel{-\mu}-\gamma_3-1\cancel{+\mu})^3} = \mu + \gamma_3$$

$$\therefore -\frac{(1-\mu)}{\gamma_3^2} - \frac{\mu}{(\gamma_3+1)^2} = \mu + \gamma_3$$

Now turn the equation above into a polynomial $f(\gamma_3)$

$\therefore -(1-\mu)(\gamma_3+1)^2 - \mu\gamma_3^2 = (\mu+\gamma_3)(\gamma_3+1)^2\gamma_3^2$

LHS: $-[\gamma_3^2 + \gamma_3(2-2\mu) - \mu + 1]$

RHS: $\gamma_3^5 + \gamma_3^4(\mu+2) + \gamma_3^3(2\mu+1) + \mu\gamma_3^2$

$$\therefore \text{RHS} - \text{LHS} = f(\gamma_3) = \gamma_3^5 + \gamma_3^4(\mu+2) + \gamma_3^3(2\mu+1) + \gamma_3^2(\mu+1) + \gamma_3(2-2\mu) - \mu + 1$$

$$f'(\gamma_3) = 5\gamma_3^4 + 4\gamma_3^3(\mu+2) + 3\gamma_3^2(2\mu+1) + 2\gamma_3(\mu+1) + 2 - 2\mu$$

Continued...

Turn the previous 2 equations into Newton-Rhapson method

$$\therefore \gamma_{3,n+1} = \gamma_{3,n} - \frac{f(\gamma_{3,n})}{f'(\gamma_{3,n})}$$

	Sun Earth	Earth Moon	Saturn Titan	Jupiter Europa	Pluto Charon
γ_3 (-)	1.0100371	1.67832755	1.04349207	1.0204865	1.3530021
L_3 (-)	-1.010040	-1.1799833	-1.0437287	-1.0205118	-1.448150
γ_3 (km)	1.51099×10^{8}	4.489149×10^{5}	1.27501×10^{6}	6.84848×10^{6}	2.3762×10^{4}
L_3 (km)	-1.51100×10^{8}	-4.53586×10^{5}	-1.275296×10^{6}	-6.84865×10^{6}	-2.5395×10^{4}

Note: So equation did blow up at times due to initial condition chosen.

PSB3

```
clear
close all
clc
```

PSB3 Part b

```
SS = SolarS;
systems = {'-','Sun-Earth', 'Earth-Moon',  'Saturn-Titan', 'Jupiter-
Europa', 'Pluto-Charon'};
param = {'l* (km)','m* (kg)','miu' ,'gamma_2 (-)','L_2 (-)', 'gamma_2
 (km)','L_2 (km)', 'gamma_1 (-)', 'L_1 (-)', 'gamma_1 (km)', 'L_1
 (km)','gamma_3 (-)', 'L_3 (-)', 'gamma_3 (km)', 'L_3 (km)' };
G = 6.6738*10^-20;

dim_vals = num2cell(zeros(15,5));
dim_vals = [systems; param',dim_vals];

% Solution

[dim_vals{2,2}, dim_vals{3,2}, ~] = charE(SS.dEarth,0,SS.mSun/
G,SS.mEarth/G);       % Earth Sun
[dim_vals{2,3}, dim_vals{3,3}, ~] = charE(SS.dM_E,0,SS.mMoon/
G,SS.mEarth/G);        % Earth Moon
[dim_vals{2,4}, dim_vals{3,4}, ~] = charE(SS.dTitan_S,0,SS.mSaturn/
G,SS.mTitan/G);  % Saturn Titan
[dim_vals{2,5}, dim_vals{3,5}, ~] = charE(SS.dEuropa_J,0,SS.mJupiter/
G,SS.mEuropa/G);    % Jupiter Europa
[dim_vals{2,6}, dim_vals{3,6}, ~] = charE(SS.dCharon_P,0,SS.mPluto/
G,SS.mCharon/G);    % Pluto Charon

dim_vals{4,2} = SS.mEarth/dim_vals{3,2}/G;
dim_vals{4,3} = SS.mMoon/dim_vals{3,3}/G;
dim_vals{4,4} = SS.mTitan/dim_vals{3,4}/G;
dim_vals{4,5} = SS.mEuropa/dim_vals{3,5}/G;
dim_vals{4,6} = SS.mCharon/dim_vals{3,6}/G;


for n = 2:6
    % Lagrange Point 2
    dim_vals{5,n} = abs(L2_NRmethod(dim_vals{4,n}*1.1,dim_vals{4,n},
 10^-8));
    dim_vals{6,n} = 1-dim_vals{4,n} + dim_vals{5,n};
    dim_vals{7,n} = dim_vals{5,n}*dim_vals{2,n};
    dim_vals{8,n} = dim_vals{6,n}*dim_vals{2,n};

    % Lagrange Point 1
    dim_vals{9,n} = abs(L1_NRmethod(dim_vals{4,n}*.7,dim_vals{4,n},
 10^-8));
    dim_vals{10,n} = 1-dim_vals{4,n} + dim_vals{9,n};
    dim_vals{11,n} = dim_vals{9,n}*dim_vals{2,n};
```

```
    dim_vals{12,n} = dim_vals{10,n}*dim_vals{2,n};

    % Lagrange Point 3
    dim_vals{13,n} = abs(L3_NRmethod(-.9,dim_vals{4,n}, 10^-8,""));
    dim_vals{14,n} = dim_vals{4,n} - dim_vals{13,n};
    dim_vals{15,n} = dim_vals{13,n}*dim_vals{2,n};
    dim_vals{16,n} = dim_vals{14,n}*dim_vals{2,n};
end


dim_vals
```

```
dim_vals =

  16×6 cell array

  Columns 1 through 4

    {'-'           }    {'Sun-Earth'  }    {'Earth-Moon' }
 {'Saturn-Titan'}
    {'l* (km)'     }    {[ 1.4960e+08]}    {[     384400]}    {[
  1221865]}
    {'m* (kg)'     }    {[ 1.9886e+30]}    {[ 6.0461e+24]}    {[
  5.6864e+26]}
    {'miu'         }    {[ 3.0035e-06]}    {[     0.0122]}    {[
  2.3662e-04]}
    {'gamma_2 (-)' }    {[     0.0100]}    {[     0.1678]}    {[
  0.0435]}
    {'L_2 (-)'     }    {[     1.0100]}    {[     1.1557]}    {[
  1.0433]}
    {'gamma_2 (km)'}    {[ 1.5015e+06]}    {[ 6.4515e+04]}    {[
  5.3141e+04]}
    {'L_2 (km)'    }    {[ 1.5110e+08]}    {[ 4.4424e+05]}    {[
  1.2747e+06]}
    {'gamma_1 (-)' }    {[     0.0100]}    {[     0.1678]}    {[
  0.0435]}
    {'L_1 (-)'     }    {[     1.0100]}    {[     1.1557]}    {[
  1.0433]}
    {'gamma_1 (km)'}    {[ 1.5015e+06]}    {[ 6.4515e+04]}    {[
  5.3141e+04]}
    {'L_1 (km)'    }    {[ 1.5110e+08]}    {[ 4.4424e+05]}    {[
  1.2747e+06]}
    {'gamma_3 (-)' }    {[     1.0100]}    {[     1.1678]}    {[
  1.0435]}
    {'L_3 (-)'     }    {[    -1.0100]}    {[    -1.1557]}    {[
  -1.0433]}
    {'gamma_3 (km)'}    {[ 1.5110e+08]}    {[ 4.4891e+05]}    {[
  1.2750e+06]}
    {'L_3 (km)'    }    {[-1.5110e+08]}    {[-4.4424e+05]}
 {[ -1.2747e+06]}

  Columns 5 through 6
```

```
{'Jupiter-Europa'}    {'Pluto-Charon'}
{[         671100]}    {[         17536]}
{[     1.8987e+27]}    {[    1.6255e+22]}
{[     2.5280e-05]}    {[        0.0951]}
{[         0.0205]}    {[        0.3530]}
{[         1.0205]}    {[        1.2579]}
{[     1.3748e+04]}    {[    6.1902e+03]}
{[     6.8483e+05]}    {[    2.2058e+04]}
{[         0.0205]}    {[        0.3530]}
{[         1.0205]}    {[        1.2579]}
{[     1.3748e+04]}    {[    6.1902e+03]}
{[     6.8483e+05]}    {[    2.2058e+04]}
{[         1.0205]}    {[        1.3530]}
{[        -1.0205]}    {[       -1.2579]}
{[     6.8485e+05]}    {[    2.3726e+04]}
{[    -6.8483e+05]}    {[   -2.2058e+04]}
```

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Gamma1 Newton Rhapson

```
function g1_n1 = L1_NRmethod(g1_n, miu, acc)

err = acc*2;
n = 1;

while (err > acc)

    fn = -g1_n^5 + g1_n^4 *(3-miu) + g1_n^3 * (2*miu-3) + g1_n^2 *(-
miu) + 2*miu*g1_n - miu;
    fn_p = -5*g1_n^4 + 4*g1_n^3 *(3-miu) + 3*g1_n^2 * (2*miu-3) +
 2*g1_n*(-miu) + 2*miu;

    g1_n1 = g1_n - fn/fn_p;

    err = abs(g1_n1-g1_n)/abs(g1_n);
    g1_n = g1_n1;
    n = n+1;

end

end
```

```
Not enough input arguments.

Error in L1_NRmethod (line 4)
err = acc*2;
```

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Gamma2 Newton Rhapson

```
function g2_n1 = L2_NRmethod(g2_n, miu, acc)

err = acc*2;
n = 1;

while (err > acc)

    fn = g2_n^5 + g2_n^4 *(3-miu) + g2_n^3 * (3-2*miu) - miu*g2_n^2 -
 2*miu*g2_n - miu;
    fn_p = 5*g2_n^4 + 4*g2_n^3 *(3-miu) + 3*g2_n^2 * (3-2*miu) -
 2*g2_n*miu - 2*miu;

    g2_n1 = g2_n - fn/fn_p;

    err = abs(g2_n1-g2_n)/abs(g2_n);
    g2_n = g2_n1;
    n = n+1;

end

end
```

```
Not enough input arguments.

Error in L2_NRmethod (line 4)
err = acc*2;
```

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Gamma3 Newton Rhapson

```
function g3_n1 = L3_NRmethod(g3_n, miu, acc, pl)

err = [acc*2];
n = 1;

while (err(end) > acc)

    fn = g3_n^5 + g3_n^4 *(miu+2) + g3_n^3 * (2*miu + 1) + g3_n^2
 *(miu + 1) + g3_n*(2-2*miu) - miu + 1;
    fn_p = 5*g3_n^4 + 4*g3_n^3 *(miu+2) + 3*g3_n^2 * (2*miu + 1) +
 2*g3_n*(miu+1) - 2*miu + 2;


    g3_n1 = g3_n - fn/fn_p;

    err(n) = abs(g3_n1-g3_n)/abs(g3_n);

    if pl == "plot"
        g3_nvec(n) = g3_n;
    end
    g3_n = g3_n1;
    n = n+1;

    if n > 400
        err(end) = acc/10;
    end
end

if pl == "plot"
    figure
    plot(1:n-1,g3_nvec,'-k')
    grid on
    grid minor
    title('\gamma value for each Iteration of Newton Rhapson')
    xlabel('n')
    ylabel('\gamma')

    figure
    plot(1:n-2,err(1:end-1),'-r')
    grid on
    grid minor
    title('Error for each Iteration of Newton Rhapson')
    xlabel('n')
    ylabel('Error')
end

end
```

Not enough input arguments.

```
Error in L3_NRmethod (line 4)
err = [acc*2];
```

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L3 Plot

To check for other convergences

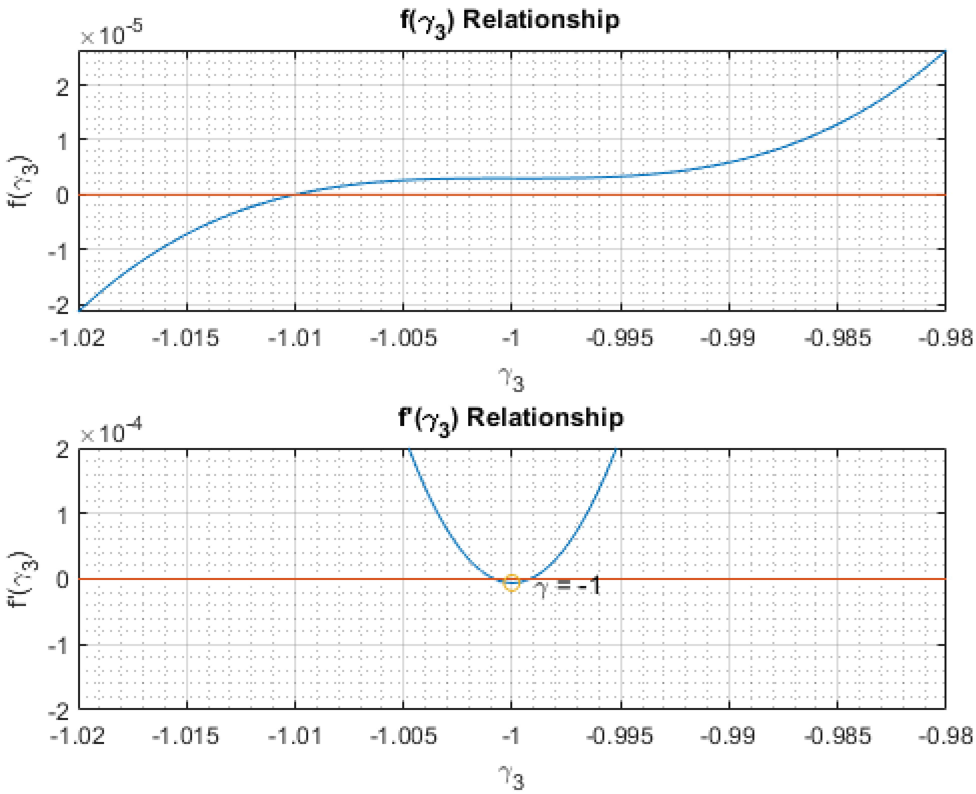
```
clear
close all
clc

miu = 3.0035e-06;

g3 = linspace(-1.02,-.98,1000);
fn = g3.^5 + g3.^4 *(miu+2) + g3.^3 * (2*miu + 1) + g3.^2 *(miu + 1) +
 g3*(2-2*miu) - miu + 1;
fn_p = 5*g3.^4 + 4*g3.^3 *(miu+2) + 3*g3.^2 * (2*miu + 1) + 2*g3*(miu
+1) - 2*miu + 2;


subplot(2,1,1)
plot(g3,fn)
hold on
plot(g3,zeros(size(g3)))
xlim([-1.02, -.98])
title('f(\gamma_3) Relationship')
xlabel('\gamma_3')
ylabel('f(\gamma_3)')
grid on
grid minor

subplot(2,1,2)
plot(g3,fn_p)
hold on
plot(g3,zeros(size(g3)))
plot(g3(find(min(fn_p)==fn_p)),min(fn_p),'o')
ylim([-.2 .2]*10^-3)
xlim([-1.02, -.98])
title('f''(\gamma_3) Relationship')
text(g3(find(min(fn_p)==fn_p))*.999,min(fn_p)*.9,['\gamma = ' ,
 num2str(g3(find(min(fn_p)==fn_p)))])
xlabel('\gamma_3')
ylabel('f''(\gamma_3)')
grid on
grid minor
```



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