Friedrich-Alexander-Universität
Technische Fakultät

# Machine Learning for Time Series

## (MLTS or MLTS-Deluxe Lectures)

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# Topics overview

- Time series fundamentals and definitions (2 lectures)
- Bayesian Inference (1 lecture)
- Gussian processes (2 lectures)
- State space models (2 lectures) ←
- Autoregressive models (1 lecture)
- Data mining on time series (1 lecture)
- Deep learning on time series (4 lectures)
- Domain adaptation (1 lecture)

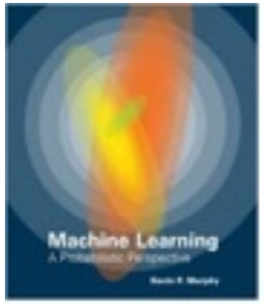
# In this lecture…

1. **State Space Models (SSMs)**
2. **Kalman Filtering (KF)**
3. **Real-world example with KF**
4. **Extended Kalman Filter (EKF)**
5. **Unscented Kalman Filter (UKF)**

# References

## Machine learning: A Probabilistic Perspective,

by Kevin Murphy (2012)




**<u>Additional references:</u>**

1. Gala, A. A. et al. (2005). Fundamentals of Kalman Filtering: A Practical Approach.
2. Faragher, R. (2012). Understanding the basis of the kalman filter via a simple and intuitive derivation [lecture notes]. IEEE Signal processing magazine, 29(5), 128-132.

# State Space Models (SSMs) and Kalman Filtering (KF)

## State Space Models (SSMs)

# State space models

State Space Models

State Space Models (SSM) are commonly used in a wide range of applications:

- Object tracking (e.g., pedestrians or vehicles in self driving cars)
- Navigation (e.g., GPS)
- Aerospace engineering
- Remote surveillance
- Finance

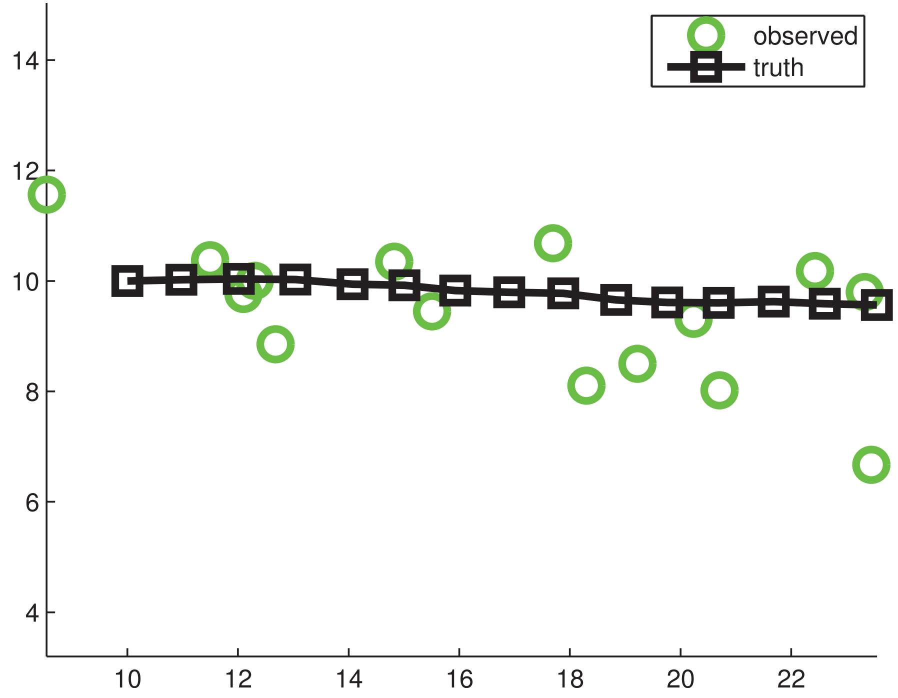
## State Space Models

deterministic
stochastic dynamic systems.
(time varying)

The SSM provides a general framework to describe deterministic and stochastic dynamical systems (i.e., **time varying systems)** which are indirectly observed through a stochastic process (i.e., **noisy measurements**).

→ Noisy measurements

Noisy measurements




It describes a probabilistic dependence between latent state variables and the observed measurements.

🔍 The term "state space" originated in the area of control engineering (Kalman, 1960).

Probablistic dependence
b/w → latent space variables ;
observed measurements.

# State Space Models

We denote with $z_n \in \mathbb{R}^D$ a continuous state variable at time $n$, and with $y_n \in \mathbb{R}^d$ the associated observation.

$u_n$ – deterministic variable

The state space model can be written in the **generic form:**

$$z_n = f(z_{n-1}, u_{n-1}, r_n) \quad \leftarrow \text{transition model}$$

$$y_n = h(z_n, u_n, q_n) \quad \leftarrow \text{measurement model}$$

Continuous state → system noise

Associated observation → observation noise.

where $u_n$ is a deterministic (optional) variable, $r_n$ is the system noise, and $q_n$ is the observation noise.

We use SSM to **recursively** estimate the belief state and an initial state $z_1$ needs to be specified.

(Recursion type.)

# Linear-Gaussian State Space Models

Linear-Gaussian state space models (LG-SSM), also called linear dynamical systems, is an important special case of an SSM where we assume:

LG SSM.

- The transition and the observation models are linear functions
  - $f(z_{n-1}, r_n) = Fz_{n-1} + r_n, F \in \mathbb{R}^{D\times D}$ ← transition model
  - $h(z_n, q_n) = Hz_n + q_n, H \in \mathbb{R}^{d\times D}$ ← observation model
- The system and observation noise processes are Gaussian
  - $r_n \sim \mathcal{N}(0, R)$
  - $q_n \sim \mathcal{N}(0, Q)$

We assume $f$, $h$ and the noise processes to be known.

## Linear-Gaussian State Space Models

The LG-SSM can be reformulated as:

from Model to density.

Transition density: $p(z_n|z_{n-1}) = \mathcal{N}(Fz_{n-1}, R)$

Observation density: $p(y_n|z_n) = \mathcal{N}(Hz_n, Q)$

A general formulation of our problem:

- We are interested to have an estimation of our hidden state at time $n$.
- We estimate hidden states by a density.

estimation by density.

Kalman filters → linear functions; Gaussian Noises.

We can analytically compute Kalman filtering for linear functions and Gaussian noises.

## Linear-Gaussian State Space Models

The conditional mean is a good candidate for estimating the state $z_n$:

Conditional mean.

$$\bar{\mu}_n = \mathbb{E}[z_n | y_{1:k}]$$

A suitable measure for the uncertainty of the hidden state $z_n$ is, then, given by the conditional covariance:

conditional covariance.

$$\bar{\Sigma}_n = \mathbb{E}[(z_n - \bar{\mu}_n)(z_n - \bar{\mu}_n)^T | y_{1:k}]$$

Depending on the value of $k$, we call the problem:

- Prediction, if $k < n$
- Filtering, if $k = n$
- Smoothing, if $k > n$

C.S.

# State Space Models (SSMs) and Kalman Filtering (KF)

## Kalman Filtering (KF)

## Review concept: the Markov property

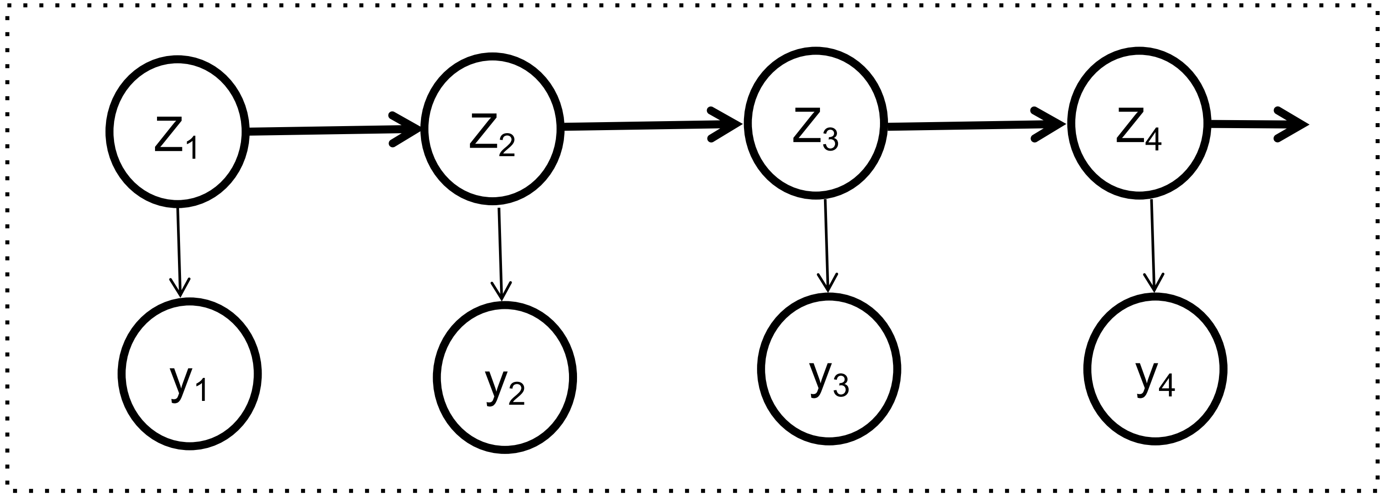
When dealing with sequential data, the Markov property ensures that each data point **depends *only* on the previous data point** (and not to older instances!).

markov property;
only last data point
(not older instances)

In formulas:

$$p(z_n|z_{n-1}, y_{1:n-1}) = p(z_n|z_{n-1})$$

$$p(y_n|z_n, y_{1:n-1}) = p(y_n|z_n)$$



# Kalman filtering (KF)

P

C.

The Kalman filtering is an algorithm for exact Bayesian filtering for linear-Gaussian state space models.

- In other words, we recursively estimate the state of a dynamical system
- E.g., indirect measurements of a rocket thruster temperature

recursively estimate state

**It consists of two steps:**

1. **Prediction:** Given an initial state we leverage our knowledge of the process to produce an estimate of the current state, along with its uncertainty
2. **Correction (or Filtering):** We update the current belief based on new measurements (sensory information)

Since everything is Gaussian, we can perform the prediction and update steps in closed form.

closed form.

## Kalman filtering (KF)

The predictive density (or prior) is given by:

$$p(z_n|y_{1:n-1}) = \int p(z_n, z_{n-1}|y_{1:n-1})\, dz_{n-1}$$

$$= \int p(z_n|z_{n-1}, y_{1:n-1})\, p(z_{n-1}|y_{1:n-1})\, dz_{n-1}$$

Markov property

$$= \int \underbrace{p(z_n|z_{n-1})}_{\text{transition density}}\ \underbrace{p(z_{n-1}|y_{1:n-1})}_{\text{filtering density}}\ dz_{n-1}$$

We can compute the filtering density (or posterior) using the Bayes rule:

$$p(z_n|y_{1:n}) \propto p(y_n|z_n, y_{1:n-1})\, p(z_n|y_{1:n-1})$$

$$\propto \underbrace{p(y_n|z_n)}_{\text{likelihood}}\, p(z_n|y_{1:n-1})$$

Markov property

Integrals are analytically-tractable for Kalman filtering for an LG-SSM.

# Kalman filtering (KF)

The Kalman filter is only concerned with propagating the first two moments (mean and variance) of the filtering density.

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We assume the filtering density at time $n-1$ is given by

$$p(z_n|y_{1:n-1}) = \mathcal{N}(\bar{\mu}_{n-1}, \bar{\Sigma}_{n-1})$$

## Kalman filtering (KF)

The Kalman filter is only concerned with propagating the first two moments (mean and variance) of the filtering density.

We assume the filtering density at time $n-1$ is given by

$$p(z_{n-1}|y_{1:n-1}) = \mathcal{N}(\bar{\mu}_{n-1}, \bar{\Sigma}_{n-1})$$

Then, the predictive density is Gaussian:

$$p(z_n|y_{1:n-1}) \quad = \int \underbrace{\mathcal{N}(Fz_{n-1}, R)}_{\text{transition density}} \underbrace{\mathcal{N}(\bar{\mu}_{n-1}, \bar{\Sigma}_{n-1})}_{\text{filtering density}} \, dz_{n-1}$$

$$= \mathcal{N}(F\bar{\mu}_{n-1}, R + F\bar{\Sigma}_{n-1}F^T)$$

$$= \mathcal{N}(\hat{\mu}_n, \hat{\Sigma}_n)$$

# Kalman filtering (KF)

The new filtering density is also Gaussian:

$$p(z_n|y_{1:n}) \propto \underbrace{\mathcal{N}(Hz_n, Q)}_{\text{likelihood}}\ \underbrace{\mathcal{N}(\hat{\mu}_n, \hat{\Sigma}_n)}_{\text{predictive density}}$$

$$= \mathcal{N}(\hat{\mu}_n + K_n(y_n - H\hat{\mu}_n), (\mathrm{I} - \mathrm{K_n H})\hat{\Sigma}_n)$$

$$= \mathcal{N}(\bar{\mu}_n, \bar{\Sigma}_n)$$

Where $\mathrm{K_n}$ is the Kalman gain matrix:

$$\mathrm{K_n} = \hat{\Sigma}_n \mathrm{H}^T S_n^{-1}$$

$$S_n = H\,\hat{\Sigma}_n H^T + Q_n$$

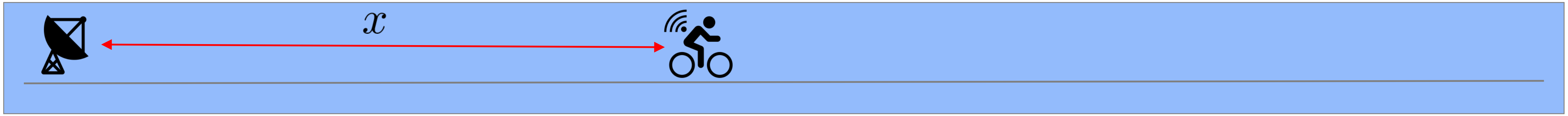
(c.s)

We see that filtering and predictive densities in KF are Gaussian at any time

# State Space Models (SSMs) and Kalman Filtering (KF)

## Real-world example with KF

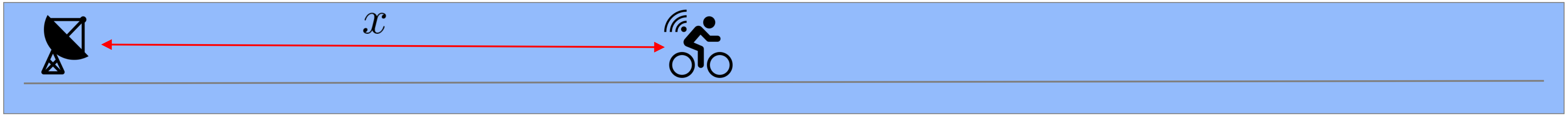
# KF example: biker position estimation




| | | | |
|---|---|---|---|
| $x$ | is the position of a rider | $z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ | is the system's state |
| $\dot{x}$ | is the rider's velocity | | |

- We measure the distance of a biker from an antenna on a 1-dimensional plane.
- From the antenna we get noisy obervations about the position and the velocity of the biker.

# KF example: biker position estimation




$x$ is the position of a rider

$\dot{x}$ is the rider's velocity

$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ is the system's state

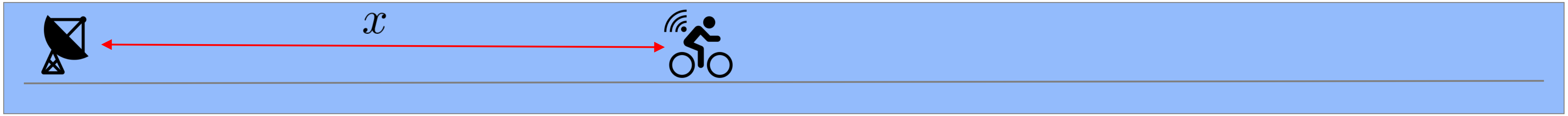
Transition model:

$$x_n = x_{n-1} + \dot{x}_{n-1}\Delta t + \frac{1}{2}\frac{f}{m}(\Delta t)^2 + r_{1_n}$$
$$\dot{x}_n = \dot{x}_{n-1} + \frac{f}{m}\Delta t + r_{2_n}$$

$s = ut + \frac{1}{2}at^2$

$v = at$

# KF example: biker position estimation




$x$ is the position of a rider

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Transition model:

$$x_n = x_{n-1} + \dot{x}_{n-1}\Delta t + \frac{1}{2}\frac{f}{m}(\Delta t)^2 + r_{1_n}$$

$$\dot{x}_n = \dot{x}_{n-1} + \frac{f}{m}\Delta t + r_{2_n}$$

$$\begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ \dot{x}_{n-1} \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} f + \begin{bmatrix} r_{1_n} \\ r_{2_n} \end{bmatrix}$$

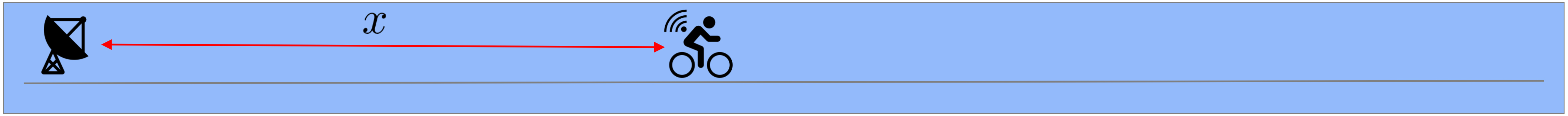
$$z_n = F z_{n-1} + B u_n + r_n$$

$r_n$ Gaussian Noise

$F$ State Transition Matrix

$B$ Additional Information

# KF example: biker position estimation




$x$ is the position of a rider

$\dot{x}$ is the rider‘s velocity

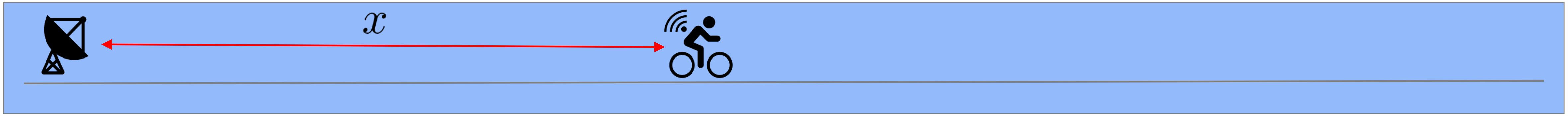
$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ is the system‘s state

Measurement model:

$$y_n = x_n + q_n$$

$$y_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} q_{1_n} \\ q_{2_n} \end{bmatrix}$$

# KF example: biker position estimation

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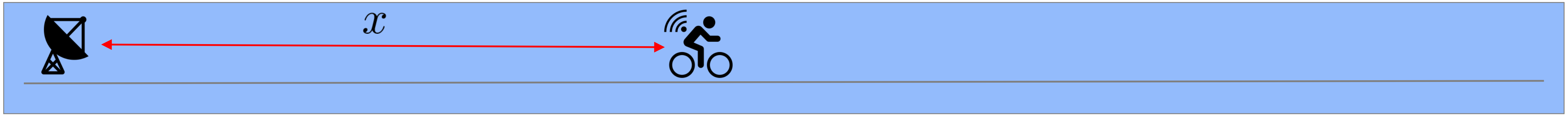
$$y_n = H z_n + q_n$$

# KF example: biker position estimation




$x$ is the position of a rider

$\dot{x}$ is the rider's velocity

$z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$ is the system's state

Initial conditions:

$$z_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \qquad \Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We also need to define covariance matrices associated with r and q:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

initial search

Notice: In practice: we perform a search over these parameters.

# KF: An algorithmic view

**Prediction step (time update):**

$$\bar{z}_n = F_n z_{n-1} + B_n u_n$$

$$\bar{\Sigma}_n = F_n \Sigma_{n-1} F_n^T + R_n$$

## KF: An algorithmic view

**Prediction step (time update):**

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$$\bar{\Sigma}_n = F_n \Sigma_{n-1} F_n^T + R_n$$

**Filtering step (Measurement update):**

$$K_n = \bar{\Sigma}_n H_n^T \left( H_n \bar{\Sigma}_n H_n^T + Q_n \right)^{-1}$$

$$z_n = \bar{z}_n + K_n \left( y_n - H_n \bar{z}_n \right)$$

$$\Sigma_n = \left( 1 - K_n H_n \right) \bar{\Sigma}_n$$

## KF: An algorithmic view

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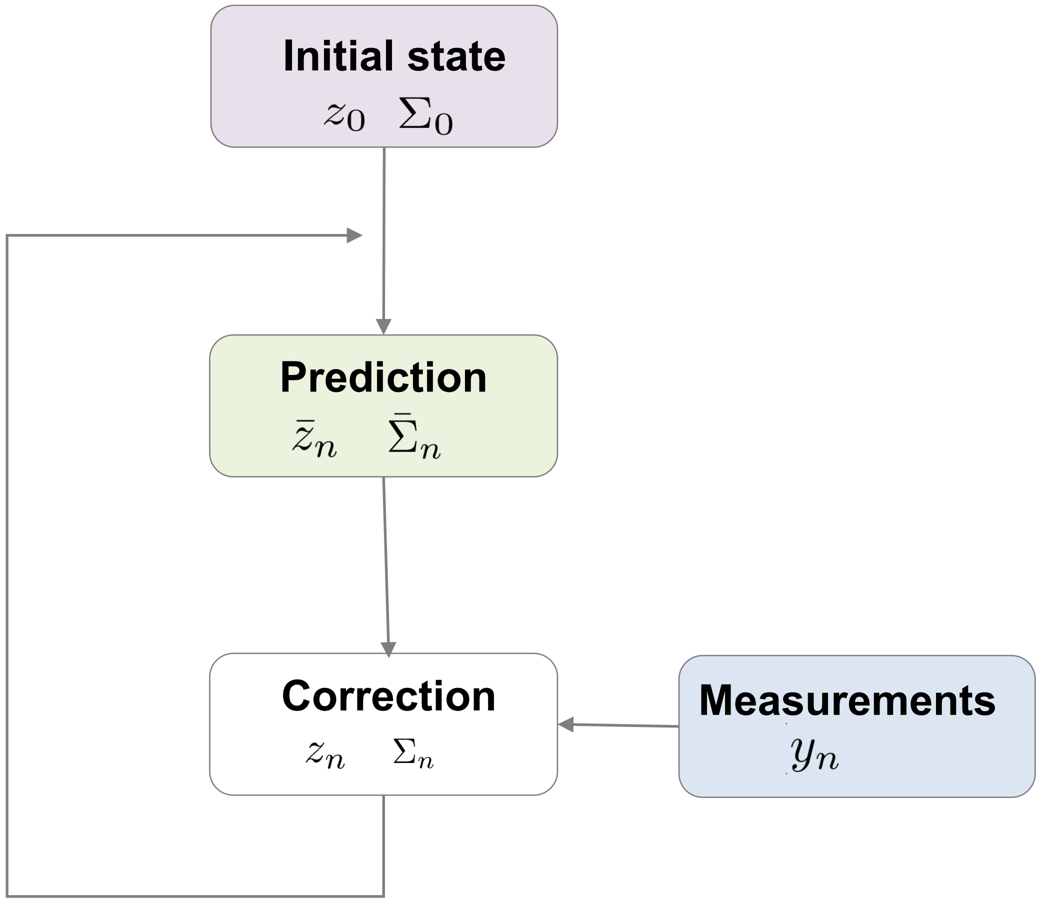


# State Space Models (SSMs) and Kalman Filtering (KF)

## Extended Kalman Filter (EKF)

# Motivations

Recall KF assumptions:

- Linear state transition model
- Linear measurement model
- Gaussian noise

If these assumptions do not hold, we need apply other methods!

# Linearized Dynamical Systems

When the transition model $f$ and/or the measurement model $h$ are not linear, then:

- the transition probability $p(z_n|z_{n-1})$ is non-Gaussian
- the predictive distribution $p(z_n|y_{1:n-1})$ is, in general, intractable

A possible approach is to consider the **linearized** dynamical system (constructed using the **Taylor expansion**) around the estimate of the current state:

$$z_n \approx f(\bar{\mu}_{n-1}) + \bar{F}_{n-1}(z_{n-1} - \bar{\mu}_{n-1}) + \cdots + r_n$$

$$y_n \approx h(\hat{\mu}_{n-1}) + \hat{H}_n(z_n - \hat{\mu}_n) + \cdots + q_n$$

Jacobian w.r.t to $z_n$

where $\bar{F}$ and $\hat{H}$ are the Jacobian of $f$ and $h$ respectively, w.r.t $z$.

# Linearized Dynamical Systems

Given a linearized system:

$$z_n \approx f(\bar{\mu}_{n-1}) + \bar{F}_{n-1}(z_{n-1} - \bar{\mu}_{n-1}) + \cdots + r_n$$

$$y_n \approx h(\hat{\mu}_{n-1}) + \widehat{H}_n(z_n - \hat{\mu}_n) + \cdots + q_n$$

If we use the linear term in the Taylor expansion and discard the higher order parts, the approximated transition density and likelihood are again Gaussian:

$$q(\mathbf{z}_n|\mathbf{z}_{n-1}) = \mathcal{N}(\mathbf{f}(\bar{\boldsymbol{\mu}}_{n-1}) + \bar{\mathbf{F}}_{n-1}(\mathbf{z}_{n-1} - \bar{\boldsymbol{\mu}}_{n-1}), \mathbf{R}),$$
$$q(\mathbf{y}_n|\mathbf{z}_n) = \mathcal{N}(\mathbf{h}(\hat{\boldsymbol{\mu}}_n) + \hat{\mathbf{H}}_n(\mathbf{z}_n - \hat{\boldsymbol{\mu}}_n), \mathbf{Q}).$$

This idea is the basis for the so called Extended Kalman Filter (EKF).

## Extended Kalman Filter (EKF)

Let's assume the filtering density is equal to $\mathcal{N}(\bar{\mu}_{n-1}, \Sigma_{n-1})$ at time $n-1$.

The approximated predictive density is Gaussian:

$$
\begin{aligned}
p(\mathbf{z}_n|\mathbf{y}_{1:n-1}) &= \int q(\mathbf{z}_n|\mathbf{z}_{n-1})p(\mathbf{z}_{n-1}|\mathbf{y}_{1:n-1})d\mathbf{z}_{n-1} \\
&= \mathcal{N}(\underbrace{\mathbf{f}(\bar{\boldsymbol{\mu}}_{n-1})}_{=\hat{\boldsymbol{\mu}}_n}, \underbrace{\bar{\mathbf{F}}_{n-1}\bar{\boldsymbol{\Sigma}}_{n-1}\bar{\mathbf{F}}_{n-1}^T + \mathbf{R}}_{=\hat{\boldsymbol{\Sigma}}_n}).
\end{aligned}
$$

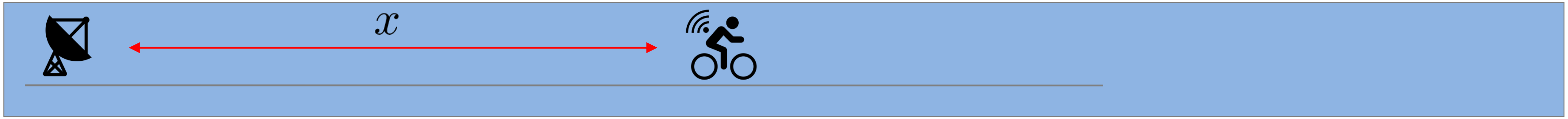
(cs)

The approximated filtering density is also Gaussian:

$$
\begin{aligned}
p(\mathbf{z}_n|\mathbf{y}_{1:n}) &\propto q(\mathbf{y}_n|\mathbf{z}_n)p(\mathbf{z}_n|\mathbf{y}_{1:n-1}) \\
&= \mathcal{N}(\bar{\boldsymbol{\mu}}_n, \bar{\boldsymbol{\Sigma}}_n),
\end{aligned}
$$

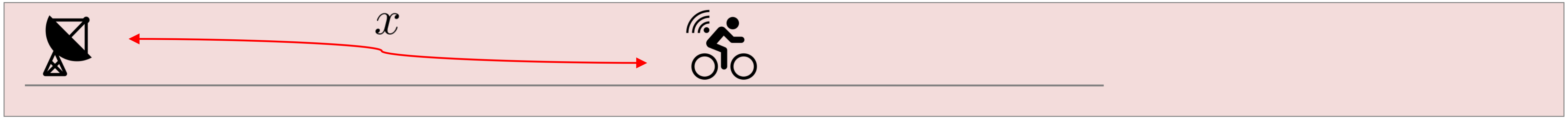
$$
\begin{aligned}
\bar{\boldsymbol{\mu}}_n &= \hat{\boldsymbol{\mu}}_n + \mathbf{K}_n(\mathbf{y}_n - \mathbf{h}(\hat{\boldsymbol{\mu}}_n)), \\
\bar{\boldsymbol{\Sigma}}_n &= (\mathbf{I} - \mathbf{K}_n\hat{\mathbf{H}}_n)\hat{\boldsymbol{\Sigma}}_n, \\
\mathbf{K}_n &= \hat{\boldsymbol{\Sigma}}_n\hat{\mathbf{H}}_n^T(\hat{\mathbf{H}}_n\hat{\boldsymbol{\Sigma}}_n\hat{\mathbf{H}}_n^T + \mathbf{Q}_n)^{-1}.
\end{aligned}
$$

# Example: EKF




$$z_n = F z_{n-1} + B u_n + r_n$$

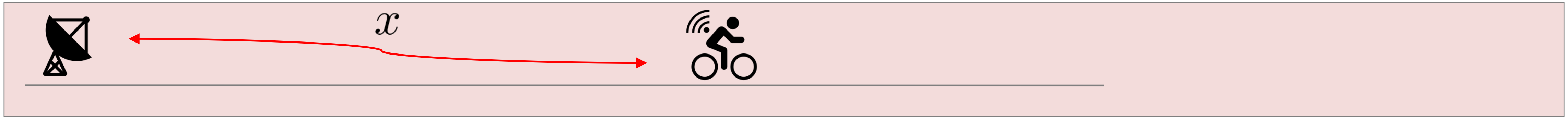
$$y_n = H z_n + q_n$$




$$z_n = f(z_{n-1}, u_n) + r_n$$

$$y_n = h(z_n) + q_n$$

# Example: EKF




$$z_n = f(z_{n-1}, u_n) + r_n$$
$$y_n = h(z_n) + q_n$$

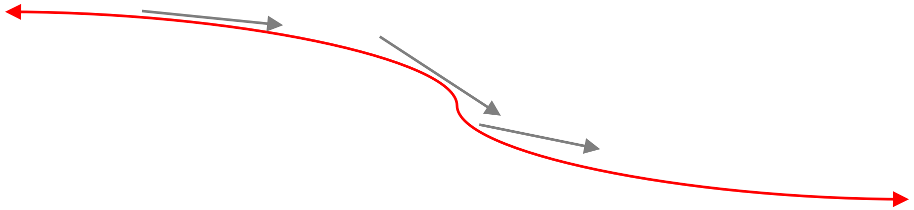
→ Non-linear but differentiable.

**Assumption**: non-linear (but differentiable) transition and/or measurement models.

→ We apply first-order Taylor expansion:

$$J^f_{n-1} = \nabla f|_{z_{n-1}, u_n}$$

$$J^h_n = \nabla h|_{z_n}$$




This approach works if the functions are "sufficiently" linear (or locally linear).

## EKF: An algorithmic view

**Prediction step (Temporal update):**

$$\bar{z}_n = f(z_{n-1}, u_n)$$

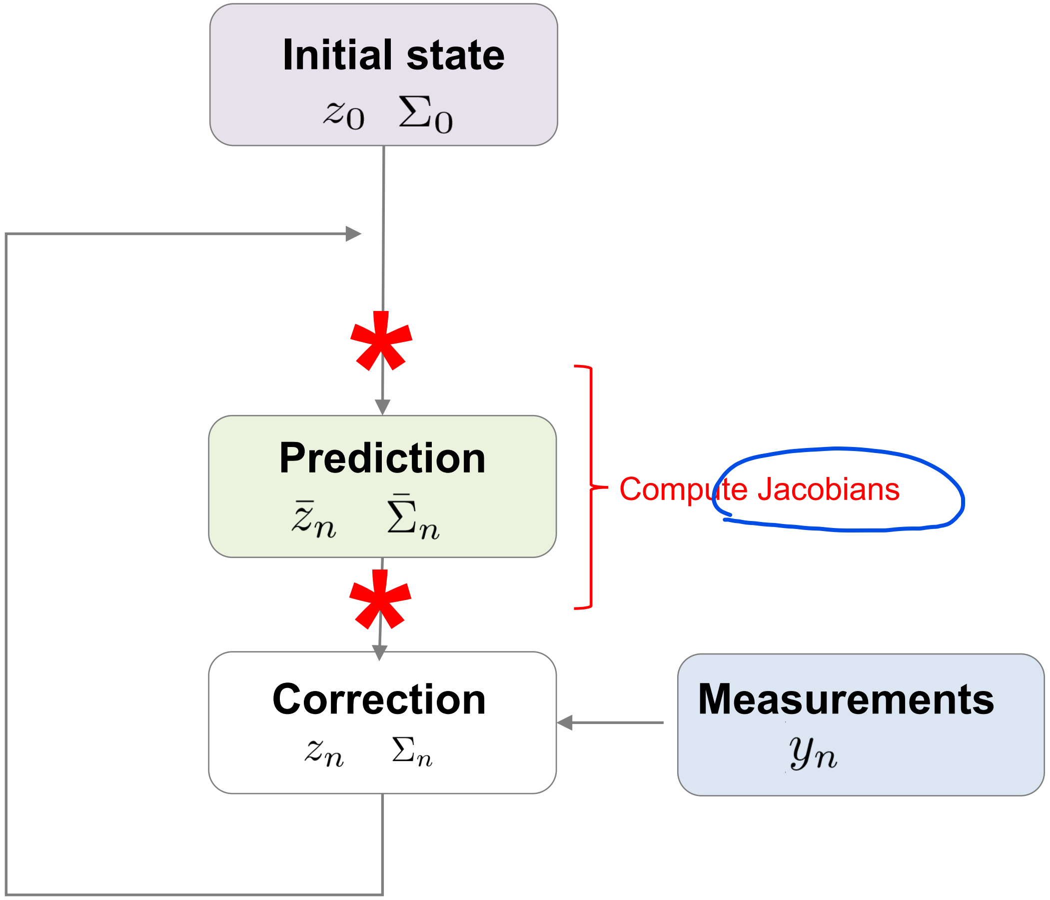
$$\bar{\Sigma}_n = J_n^f \Sigma_{n-1} {J_n^f}^T + R_n$$

**Filtering step (Measurement update):**

$$K_n = \bar{\Sigma}_n {J_n^h}^T \left( J_n^h \bar{\Sigma}_n {J_n^h}^T + Q_n \right)^{-1}$$

$$z_n = \bar{z}_n + K_n \left( y_n - h(\bar{z}_n) \right)$$

$$\Sigma_n = \left( 1 - K_n J_n^h \right) \bar{\Sigma}_n$$



# State Space Models (SSMs) and Kalman Filtering (KF)

## Unscented Kalman Filter (UKF)

## Motivations

There are two cases in which both KF and EKF perform poorly:

1. When the covariance is large. → large Covariance.
2. When the transition and/or measurement functions are highly non-linear.

To overcome these limitations, we can use Unscented Kalman Filter which is based on the concept of sigma points.

Unscented KF

sigma points

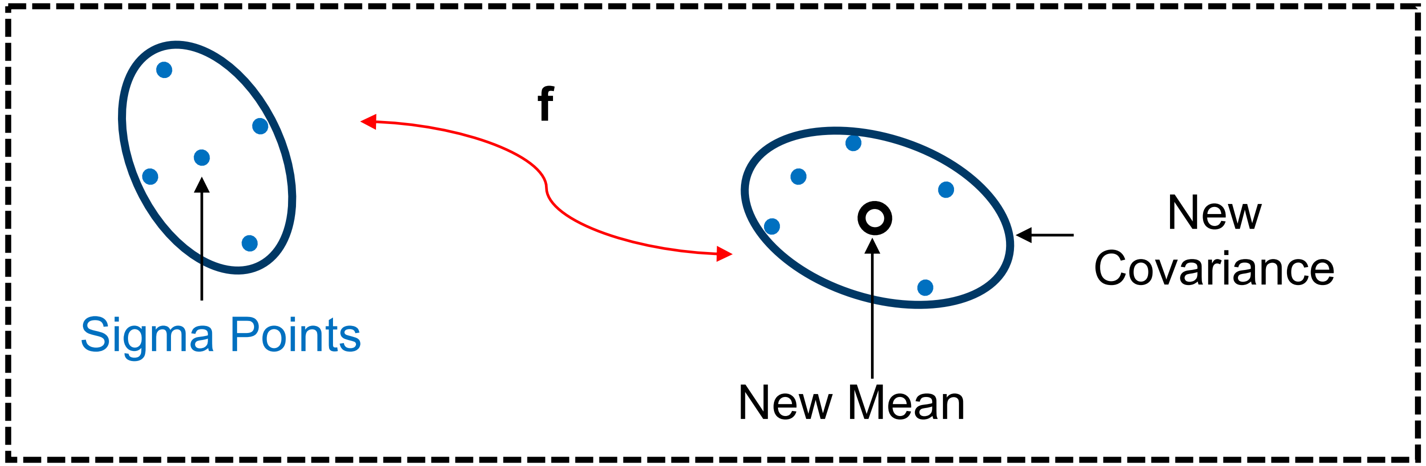
## Unscent Kalman Filter (UKF): the basic idea

The Unscent Kalman Filter makes use of the deterministic sampling technique, namely the **unscent transformation**

→ Pick up minimal set of sigma points

Then, sigma points are propagated through a non-linear function **f**

→ **We obtain new mean and covariance estimates**




**Unscented Transformation**

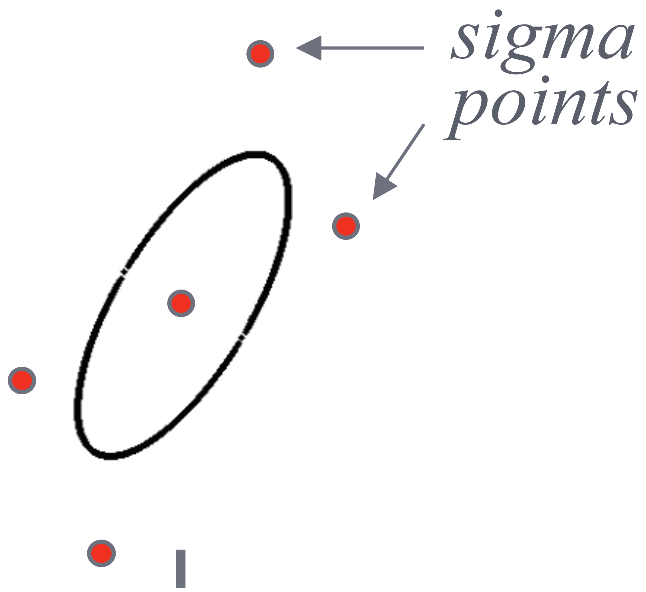
## Sigma points

Let's call sigma points a set of weighted points $\{z_i\}_{i=0}^{L}$ chosen deterministically.

We assume these points capture the mean and covariance of the random variable $z$, i.e.,

$$\boldsymbol{\mu} \approx \sum_{l=0}^{L} w_l \mathbf{z}_l,$$

$$\boldsymbol{\Sigma} \approx \sum_{l=0}^{L} w_l (\mathbf{z}_l - \boldsymbol{\mu}_n)(\mathbf{z}_l - \boldsymbol{\mu}_n)^{\prime}$$




where $\{w\}_{i=0}^{L}$ is a set of weights, with $\sum_i w_i = 1$

Compared to the EKF, we do not approximate a non-linear function but we estimate a Gaussian distribution.

## Sigma points

Let $\mu$ and $\Sigma$ be the mean and the covariance of $z$.

The *2D + 1* sigma points and weights are defined as follows:

$$
\begin{aligned}
\mathbf{z}_0 &= \boldsymbol{\mu}, & w_0 &= \frac{\kappa}{D+\kappa}, & l &= 0,\\
\mathbf{z}_l &= \boldsymbol{\mu} + \left[\sqrt{(D+\kappa)\boldsymbol{\Sigma}}\right]_l, & w_l &= \frac{1}{2(D+\kappa)}, & l &= 1, \ldots, D,\\
\mathbf{z}_l &= \boldsymbol{\mu} - \left[\sqrt{(D+\kappa)\boldsymbol{\Sigma}}\right]_l, & w_l &= \frac{1}{2(D+\kappa)}, & l &= D+1, \ldots, 2D,
\end{aligned}
$$

where $\kappa$ is a scale parameter (determining the radius of the sigma points from the mean).

- The sigma points capture the mean and covariance of **z**.
- When propagated through any nonlinear system, the transformed sigma points capture the predictive and filtering mean and covariance.

## Unscented Kalman Filter (UKF)

The Unscented Kalman Filter is simply two applications of the unscented transformation,

- one to compute the predictive density, i.e., $p(z_n|y_{1:n-1})$
- and another to compute the filtering density, i.e., $p(z_n|y_{1:n})$.

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**In the first step**, the old state $\mathcal{N}(\mu_{n-1}, \Sigma_{n-1})$ is passed through the transition function $f$ in order to approximate the predictive densitiy $\mathcal{N}(\bar{\mu}_n, \bar{\Sigma}_n)$.

Let $z_{n-1}^0 = \{z_i\}_{i=0}^L$ be a set of sigma points; we pass them through the function $f$ and obtain:

$$z_{n-1}^{*i} = f(z_{n-1}^{0i})$$

and the mean and covariance of the new points are:

$$\bar{\boldsymbol{\mu}}_n = \sum_{i=0}^{2D} w_i \mathbf{z}_n^{*i} \qquad \bar{\boldsymbol{\Sigma}}_n = \sum_{i=0}^{2D} w_i (\mathbf{z}_n^{*i} - \bar{\boldsymbol{\mu}}_n)(\mathbf{z}_n^{*i} - \bar{\boldsymbol{\mu}}_n)^T + \mathbf{R}_n$$

# Unscented Kalman Filter (UKF)

**In the second step**, we approximate the likelihood $p(y_n, |z_n)$ by passing the predictive density $\mathcal{N}(\bar{\mu}_n, \bar{\Sigma}_n)$ through the observation function $h$.

Passing the sigma points through the function $h$ we obtain:

$$\bar{\boldsymbol{y}}_{\boldsymbol{n}}^{*\boldsymbol{i}} = h(z_n^{0i})$$

Again we compute mean and covariance:

$$\hat{\boldsymbol{y}}_n = \sum_{i=0}^{2D} w_i \bar{\boldsymbol{y}}_n^{*i}$$

$$\boldsymbol{S}_n = \sum_{i=0}^{2D} w_i (\bar{\boldsymbol{y}}_n^{*i} - \hat{\boldsymbol{y}}_n)(\bar{\boldsymbol{y}}_n^{*i} - \hat{\boldsymbol{y}}_n)^T + \mathbf{Q}_n$$

# Unscented Kalman Filter (UKF)

**Finally**, we can use the Bayes rule for Gaussian to get the filtering density (or posterior) $p(z_n|y_{1:n})$.

We use the following formulas to compute the covariance between $z$ and $y$

$$\bar{\boldsymbol{\Sigma}}_n^{z,y} = \sum_{i=0}^{2D} w_i(\bar{\boldsymbol{z}}_n^{*i} - \bar{\boldsymbol{\mu}}_n)(\bar{\boldsymbol{y}}_n^{*i} - \hat{\boldsymbol{y}}_n)^T$$

the Kalman gain

$$\mathbf{K}_n = \bar{\boldsymbol{\Sigma}}_n^{z,y} \boldsymbol{S}_n^{-1}$$

[C-S]

And estimating mean and covariance of the filtering density

$$\boldsymbol{\mu}_n = \bar{\boldsymbol{\mu}}_n + \mathbf{K}_n(\mathbf{y} - \hat{\boldsymbol{y}}) \qquad \boldsymbol{\Sigma}_n = \bar{\boldsymbol{\Sigma}}_n - \mathbf{K}_n \boldsymbol{S}_n \mathbf{K}_n^T$$

## UKF: An algorithmic view

**The simplest choice for sigma points:**

$$\{s^0, \dots, s^{2D}\}_{n-1}$$

$$s^0_{n-1} = z_{n-1}$$

$$s^i_{n-1} = z_{n-1} + \sqrt{\frac{D}{1-w_0}} A_i, i = 1, \dots, D$$

$$s^{D+i}_{n-1} = z_{n-1} - \sqrt{\frac{D}{1-w_0}} A_i, i = 1, \dots, D$$

$$w_i = \frac{1-w_0}{2D}, i = 1, \dots, 2D$$

$$AA^T = \Sigma_{n-1}$$

# UKF: An algorithmic view

## Prediction step (time update):

$$\{s^0, ..., s^{2D}\}_{n-1} \qquad \bar{z}_n = \sum_{i=0}^{2D} w_i f(s^i_{n-1}) \qquad \bar{\Sigma}_n = \sum_{i=0}^{2D} w_i \left(f(s^i_{n-1}) - \bar{z}_n\right)\left(f(s^i_{n-1}) - \bar{z}_n\right)^T + R_n$$

## Filtering step (Measurement update):

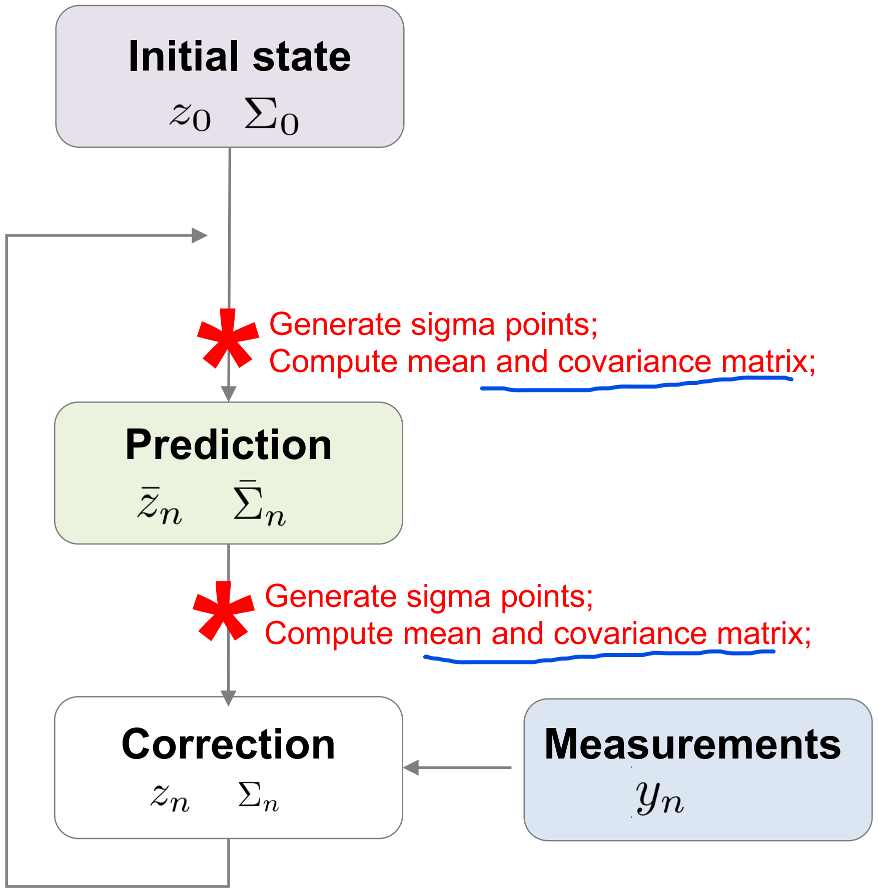
$$\{\bar{s}^0, ..., \bar{s}^{2D}\}_{n-1} \qquad \bar{y}_n = \sum_{i=0}^{2D} w_i h(\bar{s}^i_{n-1}) \qquad \bar{S}_n = \sum_{i=0}^{2D} w_i \left(h(\bar{s}^i_{n-1}) - \bar{y}_n\right)\left(h(\bar{s}^i_{n-1}) - \bar{y}_n\right)^T + Q_n$$

$$\bar{\Sigma}^{z,y}_n = \sum_{i=0}^{2D} w_i \left(\bar{s}^i_{n-1} - \bar{z}_n\right)\left(h(\bar{s}^i) - \bar{y}_n\right)^T$$

$$K_n = \bar{\Sigma}^{z,y}_n \bar{S}^{-1}_n$$

$$z_n = \bar{z}_n + K_n\left(y_n - \bar{y}_n\right) \qquad \Sigma_n = \bar{\Sigma}_n - K_n \bar{S}_n K^T_n$$

# UKF: An algorithmic view



# State Space Models (SSMs) and Kalman Filtering (KF)

Recap

# Recap

- State space models
- Kalman Filtering
- Extended Kalman Filter
- Unscented Kalman Filter

# Recap

## Critical comparison

| Estimator | State-transition / Measurement models assumptions | Assumed noise distribution | Computational cost |
|---|---|---|---|
| Kalman Filter | Linear | Gaussian | Low |
| Extended Kalman Filter | Non-linear (but locally linear) | Gaussian | Low / Medium (depending on the difficulty of computing the Jacobian) |
| Unscented Kalman Filter | Non-linear | Gaussian | Medium |