Lecture 09

So far,

chapter 1 covers quantum states: dirac (bra-ket) notation
probabilistic quantum measurements

chapter 2 covers operators
probability calculation
expectation values + uncertainty

Focus now on

chapter 3 covers schrödinger equation
time-dependent
time-independent
spin-precession

Note, there are 6 quantum postulates:

Quantum Postulates

1. The state of a quantum mechanical system, including all the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.
2. A physical observable is represented mathematically by an operator $\hat{A}$ that acts on kets.
3. The only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator $\hat{A}$.
4. The probability of obtaining the eigenvalue a_n in a measurement of the observable $\hat{A}$ on the system in the state $|\psi\rangle$ is

$$\mathcal{P}_{a_n} = |\langle a_n|\psi\rangle|^2,$$

Where $|a_n\rangle$ is the normalized eigenvector of $\hat{A}$ corresponding to the eigenvalue a_n.

Quantum Postulates

5. After a measurement of $\hat{A}$ that yields the result a_n, the quantum system is in a new state the is the normalized projection of the original system ket onto the ket (or kets) corresponding to the result of the measurement:

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}.$$

6. The time evolution of a quantum system is determined by the Hamiltonian or total energy operator $\hat{H}(t)$ through the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle.$$

Schrödinger equation
linear partial d.e that governs wave function of a quantum mechanical system

6. The time evolution of a quantum system is determined by the Hamiltonian or total energy operator $\hat{H}(t)$ through the Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle.$$

$\psi(t)$ is wave function of system
$|\psi(t)\rangle$ is quantum system state

Let's break this down.

Hamiltonian (H)

↳ total energy of the system where

$H = K.E + P.E$

kinetic energy   potential energy

classically.

$K.E = \frac{p^2}{2m} + V$

p = momentum of particle

m = mass of particle

V = potential energy

quantum mechanically,

$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$ where

$\hat{p}$ is momentum operator

$\hat{m}$ is mass of particle

$\hat{V}$ is potential energy operator

Note: Hamiltonian in QM is an operator that represents the total energy of the system.

As a hermitian operator,

hamiltonian 'eigenvalues' are real

which represent possible energy measurements

specific values if we measure energy

(observable energies are physically meaningful)

Basically, $\hat{H} = \hat{H}^\dagger$

There are two O.D.E:

Time-independent Schrödinger equation

$$\hat{H}|\psi\rangle = \left(\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right)|\psi\rangle = E|\psi\rangle$$

where E is the total energy of the quantum system and the eigenvalue of $\hat{H}$.

Time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = E|\psi(t)\rangle$$

Solution of time-independent S.E.

Let allowed energies be E_n for associated eigenvectors are $|E_n\rangle$.

Time-independent Schrödinger equation

$$\hat{H}|\psi\rangle = \left(\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right)|\psi\rangle = E|\psi\rangle$$

where E is the total energy of the quantum system and the eigenvalue of $\hat{H}$.

$\Rightarrow \hat{H}|E_n\rangle = E_n|E_n\rangle$

eigenvectors of hamiltonian form complete basis to make $\hat{H}$ the diagonal in their eigenvectors

($|E_n\rangle$ is energy basis)

meaning

eigenvectors span entire Hilbert space of system

- any state of quantum state system can be expressed as combination / superposition of eigenvectors

So.

$\therefore \sum_n |E_n\rangle\langle E_n| = 1$

or

$\langle E_k|E_n\rangle = \delta_{kn}$

Kronecker delta

$\delta_{kn} = 1$ if $k = n$

or 0 otherwise

- H operator is diagonal i.e. $\begin{bmatrix} a_1 & 0 & \cdots \\ 0 & a_2 & \cdots \\ \vdots & \vdots & \ddots & a_n \end{bmatrix}$

The general state vectors are represented as:

$$|\psi(t)\rangle = \sum_n c_n(t)\,|E_n\rangle$$

Solution of time-dependent S.E.

Time-dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = E|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

Plug in: $|\psi(t)\rangle = \sum_n c_n(t)\,|E_n\rangle$

$$i\hbar\frac{d}{dt}\left[\sum_n c_n(t)\,|E_n\rangle\right] = \hat{H}\sum_n c_n(t)\,|E_n\rangle$$

$$i\hbar\sum_n \frac{d}{dt}c_n(t)\,|E_n\rangle = \sum_n c_n(t)\,E_n|E_n\rangle$$

Apply $\langle E_k|$ to both sides,

$$i\hbar\sum_n \frac{d}{dt}c_n(t)\,\langle E_k|E_n\rangle = \sum_n c_n(t)\,E_n\,\langle E_k|E_n\rangle$$

$$i\hbar\frac{d}{dt}\big(c_k(t)\big) = E_k\,c_k(t)$$

solve this ODE basically.

So, $c_k(t) = c_k(0)\,e^{-iE_k t/\hbar}$

$$|\psi(t)\rangle = \sum_n c_n(0)\,\underbrace{e^{-iE_n t/\hbar}}_{\text{phase factor}}\,|E_n\rangle$$

Lecture 10

Time evolution of quantum state vector

i.e.

case 1 : consider a quantum system in $|\psi(0)\rangle = |E_1\rangle$ at $t=0$
(initial condition)

energy eigenstate with the energy eigenvalue

What is the system state at $t>0$?

Time-dependent; $|\psi(t)\rangle = e^{-iE_1t/\hbar}\,|E_1\rangle$ upto an overall phase factor

If we measure observable $\hat{A}$, what is the probability of measuring an eigenvalue a_j?

$$\hat{A}|a_j\rangle = a_j|a_j\rangle$$

$$P_{a_j} = |\langle a_j|\psi(t)\rangle|^2$$

substitute time evolved state

$$= |\langle a_j|\,e^{-iE_1t/\hbar}\,|E_1\rangle|^2$$

$$= |\langle a_j|E_1\rangle|^2$$

(time-independent)!

This probability is equal to the probability at the initial time.

energy eigenstates are stationary states.

$\boxed{\text{a system in an energy eigenstate remains in that state}}$

Case 2: Consider a quantum system in $|\psi(0)\rangle = c_1|E_1\rangle + c_2|E_2\rangle$.
What is the system state at $t>0$?

$$|\psi(t)\rangle = c_1e^{-iE_1t/\hbar}|E_1\rangle + c_2e^{-iE_2t/\hbar}|E_2\rangle$$

→ similar to above,
the system in an energy eigenstate remains in that state!

What is the probability of the eigenvalue E_1?

$$\mathscr{P}_{E_1} = |\langle E_1|\psi(t)\rangle|^2$$
$$= |\langle E_1|[c_1e^{-iE_1t/\hbar}|E_1\rangle + c_2e^{-iE_2t/\hbar}|E_2\rangle|^2$$
$$= |c_1|^2 \quad \text{Time-independent!}$$
$$\mathscr{P}_{E_2} = |\langle E_2|\psi(t)\rangle|^2 = |c_2|^2$$

This probability is equal to the probability at the initial time.
The energy eigenstates are **stationary states.**

Case 3: Consider a quantum system $|\psi(0)\rangle = c_1|E_1\rangle + c_2|E_2\rangle$.
What is the system state at $t>0$?

$$|\psi(t)\rangle = c_1e^{-iE_1t/\hbar}|E_1\rangle + c_2e^{-iE_2t/\hbar}|E_2\rangle$$

Now measure an operator $\hat{A}$ whose eigenvalues are time-independent.

What do you want to check first? Commutation relations of $[\hat{A},\hat{H}]$ first!

determine if there are common eigenstates.

1. $[\hat{A},\hat{H}]=0$: There are common eigenstates of $\hat{A}$ and $\hat{H}$!

This means that the probability of measurement $\hat{A}$ is equivalent to that of measuring $\hat{H}$.

$$\hat{A}|E_1\rangle = a_1|E_1\rangle, \quad \hat{A}|E_2\rangle = a_2|E_2\rangle$$

$$\mathscr{P}_{E_1} = |\langle E_1|\psi(t)\rangle|^2 = |c_1|^2, \quad \mathscr{P}_{E_2} = |\langle E_2|\psi(t)\rangle|^2 = |c_2|^2$$

Case 3: Consider a quantum system $|\psi(t)\rangle = c_1e^{-iE_1t/\hbar}|E_1\rangle + c_2e^{-iE_2t/\hbar}|E_2\rangle$.

2. $[\hat{A},\hat{H}]\neq 0$: There are no common eigenstates of $\hat{A}$ and $\hat{H}$!

$\hat{A}$ have eigenvalues a_i and eigenstates $|a_i\rangle$: Namely, $\hat{A}|a_i\rangle = a_i|a_i\rangle$.

Suppose $|a_1\rangle = \alpha_1|E_1\rangle + \alpha_2|E_2\rangle$. can be formed from basis

What is the probability of measuring the eigenvalue a_1?

$$\mathscr{P}_{a_1} = |\langle a_1|\psi(t)\rangle|^2$$
$$= |[\alpha_1^*\langle E_1| + \alpha_2^2\langle E_2|][c_1e^{-iE_1t/\hbar}|E_1\rangle + c_2e^{-iE_2t/\hbar}|E_2\rangle|^2$$
$$= |\alpha_1^*c_1e^{-iE_1t/\hbar} + \alpha_2^*c_2e^{-iE_2t/\hbar}|^2$$
$$= |e^{-iE_1t/\hbar}|^2|\alpha_1^*c_1 + \alpha_2^*c_2e^{-i(E_2-E_1)t/\hbar}|^2$$
$$= |\alpha_1|^2|c_1|^2 + |\alpha_2|^2|c_2|^2 + 2\Re(\alpha_1c_1^*\alpha_2^*c_2e^{-i(E_2-E_1)t/\hbar}) \quad \text{time-dependent!}$$

Relative phase
Bohr frequency $\hbar\omega_{21}$

Recipe for solving a time-dependent S.E with time independent Hamiltonian.

Given hamiltonian $\hat{H}$ and initial state $|\psi(0)\rangle$, the probability that the eigenvalue a_j of operator $\hat{A}$ is measured at time t is calculated according to:

1. Diagonalize $\hat{H}$ to find eigenvalues E_n and eigenvectors $|E_n\rangle$
2. Write $|\psi(0)\rangle$ in terms of energy eigenstates $|E_n\rangle$
3. Multiply each eigenstate coeff. by $e^{iE_it/\hbar}$ to get $|\psi(t)\rangle$
4. Calculate
$$P_{a_j} = |\langle a_j|\psi(t)\rangle|^2$$
to get $|\psi(t)\rangle$

Explanations for case II

Consider quantum state in $|\psi(0)\rangle = c_1|E_1\rangle + c_2|E_2\rangle$

What is the system state at $t>0$?

$$|\psi(t)\rangle = c_1 e^{-iE_1\frac{t}{\hbar}}|E_1\rangle + c_2 e^{-iE_2\frac{t}{\hbar}}|E_2\rangle$$

What is the probability of eigenvalue E_1?

$$P_{E_1} = |\langle E_1|\psi_t\rangle|^2$$
$$= \left|\langle E_1|\left(c_1 e^{-iE_1\frac{t}{\hbar}}|E_1\rangle + c_2 e^{-iE_2\frac{t}{\hbar}}|E_2\rangle\right)\rangle\right|$$
$$= |c_1|^2$$
$$P_{E_2} = |c_2|^2$$

Explanations for case III

case 3. consider a quantum system $|\psi(0)\rangle = c_1|E_1\rangle + c_2|E_2\rangle$

what is system state at $t>0$?

$$|\psi(t)\rangle = c_1 e^{-iE_1\frac{t}{\hbar}}|E_1\rangle + c_2 e^{-iE_2\frac{t}{\hbar}}|E_2\rangle$$

measure operator $\hat{A}$ whose eigenvalues are time independent.

We check commutation relations first because.

if they share eigenstates i.e. $[\hat{A},\hat{H}]=0$ then

- measurement in $\hat{A}$ does not disturb state so it does not change system's energy eigenstate.
  - probabilities remain constant.

when $[\hat{A},\hat{H}] \neq 0$ (there are no common eigenstates of $\hat{A}$ & $\hat{H}$)

↳ $\hat{A}|a_i\rangle = a_i|a_i\rangle$ and $|a_i\rangle = \alpha_1|E_1\rangle + \alpha_2|E_2\rangle$

prob. of measuring a_1

$$P_{a_1} = |\langle a_1|\psi\rangle|^2$$
$$= \left|\left[\alpha_1^*\langle E_1| + \alpha_2^*\langle E_2|\right]\left[c_1 e^{-iE_1\frac{t}{\hbar}}|E_1\rangle + c_2 e^{-iE_2\frac{t}{\hbar}}|E_2\rangle\right]\right.$$
$$= \left|\alpha_1^* c_1 e^{-iE_1 t/\hbar} + \alpha_2^* c_2 e^{-iE_2\frac{t}{\hbar}}\right|^2$$
$$= \left|\underbrace{e^{-iE_1 t/\hbar}}_{|e^{\cdot\cdot}|^2 = 1}\left(\alpha_1^* c_1 + \alpha_2^* c_2 e^{-i(E_2-E_1)\frac{t}{\hbar}}\right)\right|^2$$
$$= |\alpha_1^*|^2|c_1|^2 + |\alpha_2^*|^2|c_2|^2 + 2R\left[\alpha_1 c_1^* \alpha_2^* c_2 e^{-i(E_2-E_1)\frac{t}{\hbar}}\right]$$

Bohr frequency

$$\omega_{21} = \frac{E_2 - E_1}{\hbar}$$

* time dependent.

evolves at this rate

(diff. in energy levels)

Lecture 11 spin precession

Consider a spin magnetic dipole in a uniform magnetic field.
e.g. in silver atom

What is the Hamiltonian $\hat{H}$?
total energy of system.

Assume the dipole is at rest. Then, there is only magnetic potential energy.

magnetic dipole $\vec{\mu} = g \frac{q}{2m_e} \vec{S}$
relates to spin vector

where
g = gyromagnetic ratio
q = charge
m_e = electron mass
$\vec{S}$ = spin vector

hamiltonian $\vec{H} = -\mu \cdot \vec{B}$
energy of dipole in magnetic field $\vec{B}$

$= \frac{-g \cdot q}{2m_e} \cdot \vec{S} \cdot \vec{B}$

Let $g = 2$
$q = -e$

$= \frac{e}{m_e} \vec{S} \cdot \vec{B}$

Consider a spin magnetic dipole in a uniform magnetic field along z-axis: $\vec{B} = B_0 \hat{z}$

Hamiltonian $\vec{H} = \frac{e}{m_e} \vec{S} \cdot B_0 \hat{z}$

$= \frac{e}{m_e} \left(\vec{S_x} \hat{x} + \vec{S_y} \hat{y} + \vec{S_z} \hat{z} \right) \cdot B_0 \hat{z}$

$= \frac{e}{m_e} S_z B_0$

$= \omega_0 S_z$ where $\omega_0 = \frac{e B_0}{m_e}$ Larmor freq.

In matrix form

$= \frac{\hbar \omega_0}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

So, $[\hat{H}, \hat{S}_z] = 0$ based on diagonalization

$\hat{H}|+\rangle = \frac{\hbar\omega_0}{2} |+\rangle = E_+ |+\rangle$

$\hat{H}|-\rangle = -\frac{\hbar\omega_0}{2} |-\rangle = E_- |-\rangle$

Connecting to "time-evolution" of spin:

Case 1: Consider a quantum system in $|\psi(0)\rangle = |+\rangle$ at $t=0$ (initial condition).
What is the system state at $t>0$?

$$|\psi(t)\rangle = e^{-iE_+t/\hbar}|+\rangle = e^{-i\omega_0 t/2}|+\rangle, \text{ where } E_+ = \frac{\hbar\omega_0}{2}$$

What is the probability of measuring the spin-up along the z-axis?

$$P_+ = |\langle + | \psi \rangle|^2 = |\langle + | e^{-i\omega_0 t/2} | + \rangle|^2 = 1$$

$\therefore$ time independent

$|+\rangle$ are stationary states.

i.e. a system in an energy eigenstate remains in that state at all times.

Case 2: Consider a general case $|\psi(0)\rangle = |+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$ at $t=0$.
What is the system state at $t>0$?

$$|\psi(t)\rangle = e^{-iE_+t/\hbar}\cos\frac{\theta}{2}|+\rangle + e^{-iE\_t/\hbar}\sin\frac{\theta}{2}e^{i\phi}|-\rangle$$

In the matrix notation

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$$

$$= \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\phi} \end{bmatrix}$$

So, by S.E.

$$|\psi(t)\rangle = \underbrace{e^{-i\hat{H}t/\hbar}}\,|\psi(0)\rangle \quad \text{where } \hat{H} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\frac{\hbar\omega_0}{2}$$

$$= \begin{bmatrix} e^{-i\omega_0 t/2} & 0 \\ 0 & e^{-i\omega_0 t/2} \end{bmatrix}\begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\; e^{i\phi} \end{bmatrix}$$

$$= e^{-i\omega_0/2}\begin{bmatrix} \cos\theta/2 \\ e^{i(\theta+\omega_0 t)}\sin\theta/2 \end{bmatrix}$$

What is the probability of measuring spin-up along z-axis?

$$P_+ = |\langle + | \psi(t)\rangle|^2 = \left|(1\ \ 0)\, e^{-i\omega_0 t/2}\begin{bmatrix} \cos\theta/2 \\ e^{i(\phi+\omega_0 t)}\sin\theta/2 \end{bmatrix}\right|^2$$

$$= \cos^2\frac{\theta}{2}$$

time-independent

stationary

$[\hat{H}, \hat{S}_z] = 0$

Case 2: Consider a general case $|\psi(0)\rangle = |+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$ at $t=0$.

$$|\psi(t)\rangle = e^{-iE_+t/\hbar}\cos\frac{\theta}{2}|+\rangle + e^{-iE\_t/\hbar}\sin\frac{\theta}{2}e^{i\phi}|-\rangle = e^{-i\omega_0 t/2}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\phi+\omega_0 t)}\sin\frac{\theta}{2} \end{pmatrix}$$

What is the probability of measuring the spin-up along the x-axis?

$$\mathscr{P}_+ = |{}_x\langle + |\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1\quad 1)\, e^{-i\omega_0 t/2}\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i(\phi+\omega_0 t)}\sin\frac{\theta}{2} \end{pmatrix}\right|^2$$

$$= \frac{1}{2}\left|\cos\frac{\theta}{2} + e^{i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\right|^2 = \frac{1}{\sqrt{2}}\left[\cos^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{i(\phi+\omega_0 t)} + e^{-i(\phi+\omega_0 t)}\right) + \sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{2}|1 + \sin\theta\cos(\phi+\omega_0 t)| \qquad \because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

Time-dependent and the $\hat{S}_x$-eigenstates are not stationary. $\because [\hat{H}, \hat{S}_x] \neq 0$

The spin-precession around the z-axis !

Spin Larmor Precession Illustration

Case 2: Consider a general case $|\psi(0)\rangle = |+\rangle_n = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}e^{i\phi}|-\rangle$ **at** $t=0$.

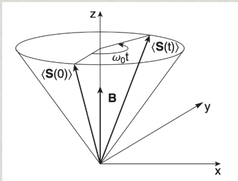
$$|\psi(t)\rangle = e^{-iE_+t/\hbar}\cos\frac{\theta}{2}|+\rangle + e^{-iE_-t/\hbar}\sin\frac{\theta}{2}e^{i\phi}|-\rangle = e^{-i\omega_0 t/2}\begin{pmatrix}\cos\frac{\theta}{2} \\ e^{i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\end{pmatrix}$$

Expectation values of $\langle \hat{S}_{x,y,z}\rangle$:

$$\langle \hat{S}_z\rangle = \frac{\hbar}{2}\cos\theta,$$

$$\langle \hat{S}_y\rangle = \frac{\hbar}{2}\sin\theta\sin(\phi+\omega_0 t)$$

$$\langle \hat{S}_x\rangle = \frac{\hbar}{2}\sin\theta\cos(\phi+\omega_0 t)$$



ω_0 = Larmor frequency

$$\langle S_z\rangle = \langle\psi(t)|\hat{S}_z|\psi(t)\rangle$$

$$= \frac{\hbar}{2}e^{i\omega_0 t/2}\begin{pmatrix}\cos\frac{\theta}{2} & e^{-i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}e^{-i\omega_0 t/2}\begin{pmatrix}\cos\theta/2 \\ e^{i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\frac{\theta}{2} & -e^{-i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}\cos\theta/2 \\ e^{i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right) = \frac{\hbar}{2}\cos\theta$$

also,

$$\langle \hat{S}_y\rangle = \langle\psi(t)|\hat{S}_y|\psi(t)\rangle = e^{i\omega_0 t/2}\left(\cos\frac{\theta}{2} \quad e^{-i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\right)\frac{\hbar}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}e^{-i\omega_0 t/2}\begin{pmatrix}\cos\frac{\theta}{2} \\ e^{i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(ie^{-i(\phi+\omega_0 t)}\sin\frac{\theta}{2} \quad -i\cos\frac{\theta}{2}\right)\begin{pmatrix}\cos\frac{\theta}{2} \\ e^{i(\phi+\omega_0)}\sin\frac{\theta}{2}\end{pmatrix} = \frac{\hbar}{2}i\left(e^{-i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\cos\frac{\theta}{2} - e^{-i(\phi+\omega_0 t)}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \frac{\hbar}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}i\left(e^{-i(\phi+\omega_0 t)} - e^{-i(\phi+\omega_0 t)}\right) = \frac{\hbar}{2}\sin\theta\sin(\phi+\omega_0 t) \quad , \langle \hat{S}_x\rangle = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle = \frac{\hbar}{2}\sin\theta\cos(\phi+\omega_0 t)$$

alternate question:

Consider a spin-1/2 particle with a magnetic moment that is subject to a uniform field $\vec{B} = B_0\hat{z}$.

$$|\psi(0)\rangle = |-\rangle_x$$

What are the Hamiltonian and $|\psi(t)\rangle$? $\hat{H} = -\mu\cdot\vec{B} = \omega_0\hat{S}_z$ **, where** $\omega_0 = \frac{eB_0}{m_e}$.

$$|\psi(0)\rangle = |-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \rightarrow \quad |\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega_0 t/2}|+\rangle - \frac{1}{\sqrt{2}}e^{+i\omega_0 t/2}|-\rangle$$

What is the probability to be found in $|+\rangle_x$?

$$\mathscr{P}_{+x} = |{}_x\langle +|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(1 \quad 1)\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\omega_0 t/2} \\ -e^{+i\omega_0 t/2}\end{pmatrix}\right|^2$$

$$= \frac{1}{4}|e^{-i\omega_0 t/2} - e^{+i\omega_0 t/2}|^2 = \frac{1}{4}|-2i\sin(\omega_0 t/2)|^2$$

$$= \sin^2(\omega_0 t/2)$$