

## Data Science, 2022

### Tut 4: Independent Component Analysis

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Ex. 1

Exercise: Mixing statistically independent sources

Given some scalar and statistically independent random variables (signals)  $s_i$  with zero mean, unit variance, and a value  $a_i$  for the kurtosis that lies between  $-a$  and  $+a$ , with arbitrary but fixed value of  $0 < a$ . The  $s_i$  shall be mixed like

$$x := \sum_i w_i s_i$$

with constant weights  $w_i$ .

- Which constraints do you have to impose on the weights  $w_i$  to guarantee that the mixture has unit variance as well?

Hint

$$\begin{aligned} \text{var}(x) &= \langle (x - \langle x \rangle)^2 \rangle \\ &= \langle x^2 \rangle - \langle x \rangle^2 \end{aligned}$$

1) Independent component analysis:  
 Mixing statistically independent sources  
 variance of mixture

$$\begin{aligned}
 \text{var}(x) &= \langle (x - \langle x \rangle)^2 \rangle \\
 &= \langle x^2 \rangle - (\langle x \rangle)^2 \\
 &= \langle \left( \sum_i w_i s_i \right)^2 \rangle - \left( \langle \sum_i w_i s_i \rangle \right)^2 \\
 &= \langle \left( \sum_i w_i s_i \right) \left( \sum_j w_j s_j \right) \rangle - \left( \sum_i w_i \langle s_i \rangle \right) \left( \sum_j w_j \langle s_j \rangle \right) \\
 &= \langle \sum_{i,j} w_i w_j s_i s_j \rangle - \sum_{i,j} w_i w_j \langle s_i \rangle \langle s_j \rangle \\
 &= \sum_{i,j} w_i w_j (\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle) \\
 &= \sum_{i,j} w_i w_j (\langle s_i s_j \rangle - \delta_{ij} \langle s_i \rangle \langle s_j \rangle) \\
 &= \sum_i w_i^2 (\langle s_i^2 \rangle - \langle s_i \rangle^2) + \sum_{i,j} w_i w_j (\langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle)
 \end{aligned}$$

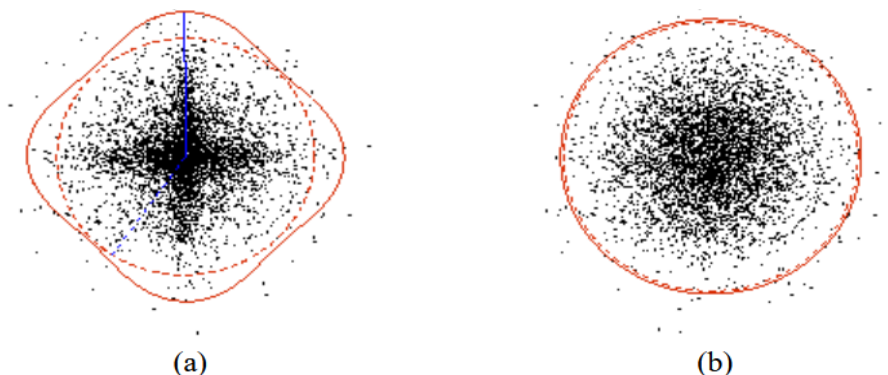
$s_i$  &  $s_j$  are statistically independent  
 for  $i \neq j$   
 $\langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle = 0$  and  $\text{var}(s_i) = 1$

$\text{var}(x) = \sum_i w_i^2$  to guarantee that  
 mixture has unit variance  $\text{var}(x) = 1$   
 $\sum_i w_i^2 = 1$

The full constraint has to be imposed  
 on  $w_i$  for the mixture to have unit  
 variance

## Ex.2

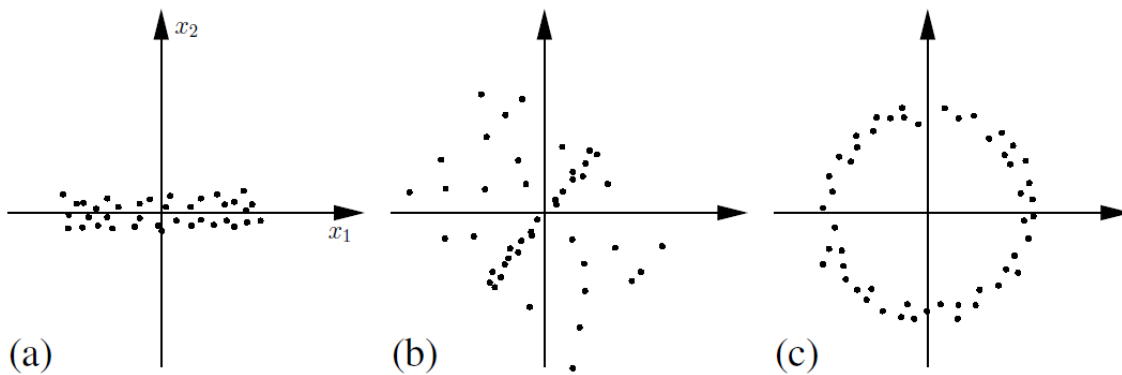
Two examples of joint probability densities are shown in following figure. One is a mixture of arbitrary non-Gaussian densities, and the other one a mixture of Gaussians. The dashed curves around the densities plot the projected variance measured in all directions. The dashed line marks the direction of maximum variance, that is, the first principal component. Similarly, the values of kurtosis are shown using solid curves and the direction of maximum kurtosis with a solid line.



Example joint probability densities. (a) For non-Gaussian densities the principal (dashed line) and independent (solid line) directions can be identified, whereas (b) for Gaussian ones the directions are all equal. The corresponding dashed and solid curves show the values of variance and kurtosis in all directions respectively.

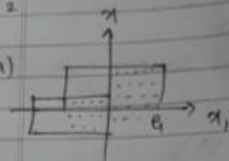
Referring to above provide the guess independent components and distributions from data

- Decide whether the following distributions can be linearly separated into independent components. If yes,
- sketch the (not necessarily orthogonal) axes onto which the data must be projected to extract the independent components. Draw these axes also the marginal distributions of the corresponding components.



Ex 2

a)



c)  $x$  - cannot be separated into independent components.



b)

