Groups and Geometry: A Gentle Introduction

From Euclid to Lie

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Lecture 4: Lie Groups, Symmetries and Dynamics

Lie and Dynamics

- Consider the dynamics of an infinite plane of fluid excited by a point source after it falls into it

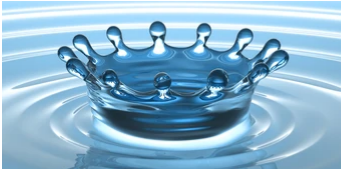


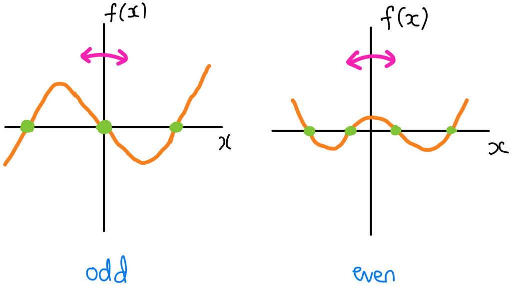
Figure: Snapshot of a water-crown with **14-fold symmetry** formed after a drop of water falls on a large container with still water

Question

If the cause has rotational symmetry - the water is still and same everywhere and the disturbance is at a point, why do we get a shape with lesser symmetry than expected?

Equations and Symmetry

- Consider an equation in one variable $f(x) = 0$ where f is smooth and odd or even. i.e. $f(x) = -f(-x)$ or $f(x) = f(-x)$.
- Symmetry group of the equation: $G = \{I, x \rightarrow -x\} = \{g_1, g_2\}$
- Let $C = \{x \mid f(x) = 0\}$. Then, $x \in C \Leftrightarrow g_i(x) \in C \; \forall i$



Effect of Symmetry

- Symmetry: brings order into the solution set - predictable distribution of solutions
- Knowing one solution $x \in C$ enables us to get many solutions $g_i(x) \in C$ where $G = \{g_i|_{i \in I\}}\}$ is the symmetry group of the equation

Dynamical Systems and Symmetry

- A vector field in a manifold (eg. an ODE $\dot{x} = f(x)$ in $\mathbb{R}^n$) can have a symmetry group as well - say G
- If $x(t)$ is a trajectory for the dynamical system and $g \in G$, then $g(x(t))$ is also a trajectory

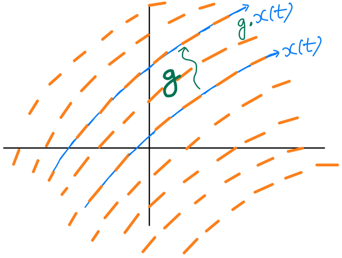


Figure: If $x(t)$ is a trajectory, acting on g - $g(x(t))$ also is a trajectory

Symmetries of Steady State Behavior

- Any question we ask about a dynamical system will have symmetric answers for a system with symmetry - consider the omega-limit set (steady state) $\Omega = cl\left(\cap_{s \in \mathbb{R}} \{\phi(x,t) \mid t > s\} \right)$ or the set of equilibria $E = \{x \in M \mid f(x) = 0\}$. Then, $x \in \Omega/E \Leftrightarrow g(x) \in \Omega/E$

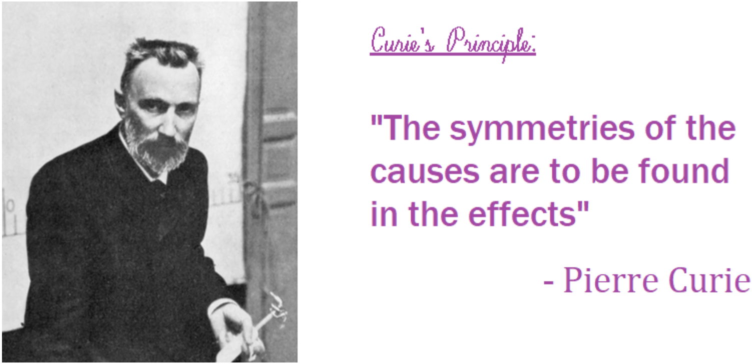


Figure: Curie's Principle

Bifurcation: An Introduction

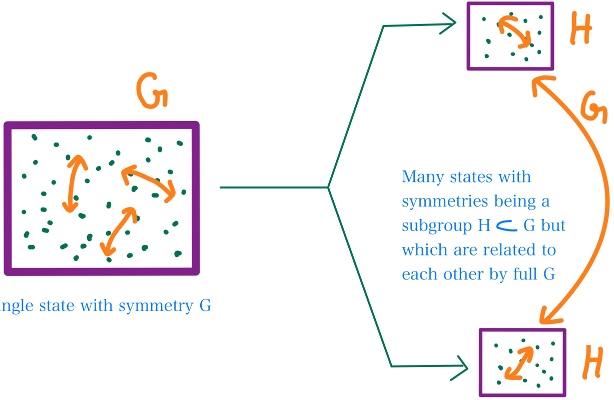
- Many a times, the behavior of a dynamical system changes as parameters drift slowly

$$\dot{x} = f(x, r)$$

- If the dynamical system retains its symmetry group under all ranges of the parameter, some interesting phenomena happened
- The change in behavior as one real valued parameter is changed in a parametrized dynamical system is called **bifurcation** .

Symmetry Breaking

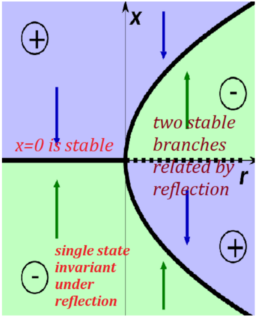
- Many a times, as a bifurcation happens, a single stable limit set (eg. steady state) possessing the full system symmetry G can become unstable and break down into a set of multiple symmetrically related stable states, each having lesser symmetry - a subgroup of G



Single state with symmetry G

Pitchfork Bifurcation as Symmetry Breaking

- Consider the system $\dot{x} = rx - x^3$ - its symmetry group is $G = \{I, -I\}$ for all r
- For $r < 0$, $x = 0$ is the only stable state - and it is invariant under all of G - $I(0) = -I(0) = 0$
- But for $r > 0$, the equilibrium $x = 0$ becomes unstable and instead two new equilibria $x = \pm\sqrt{r}$ arise that do not each have any reflectional symmetry but are reflectionally related to each other!



An Interlude: More Symmetry $\neq$ More Pattern

- We saw that symmetry is the transformations that leave a system invariant and more the symmetry - more the pattern
- But too much of symmetry also makes everything trivial


Figure: The infinite plane of the surface of the still lake has more symmetry - $SE(2)$ than the one with ripples - $SO(2)$

Corn Circles: Hoax / UFO / Electrostatics / Atmosphere ?

https://www.bbc.com/travel/article/20210822-englands-crop-circle-controversy

Although these mysterious formations have appeared worldwide, south-west England is the unlikely world capital of crop circles, baffling locals and farmers alike.

Ears of wheat prickled my shins and the sun beat down on my neck as I trudged through the tractor lines of a golden field on Wiltshire's Hackpen Hill. It was August – the height of crop circle season – and I'd been directed here by frenzied online reports of a new formation, which had appeared, as they are wont to do, overnight; apparently unseen by observers. From the ground, I could make out nothing but intersecting lines of trampled wheat – but photographed from above the pattern resembled a crosshair.

Was this the nexus for some kind of potent Earth energy? Or, terrifyingly, a target for extra-terrestrial weaponry? In this instance, something more mundane. "That's the logo of the Barge Inn down in Honeystreet," chuckled a fellow visitor, a potbellied man in a Dark Side of the Moon T-shirt. "Probably man-made, this one."

Although such formations have appeared worldwide, from California to the rice paddies of Indonesia, south-west England is the world capital of crop circles. They are particularly concentrated in the county of Wiltshire, where a treasure trove of ancient history includes the Neolithic sites of Stonehenge and Avebury – both crop circle hotspots.

We see the symmetry of the circle but ignore the greater but even blander symmetry of the field of the corn and wonder where the circles come from

Figure: Whatever is the mechanism of the formation of these mysterious corn circles, the lesson is that the circle is the last thing to worry about and very probable as a result of symmetry breaking $SE(2) \to SO(2)$

Other examples of symmetry breaking

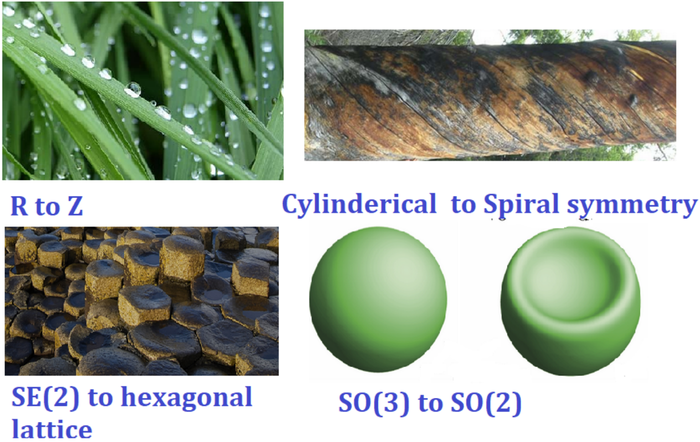
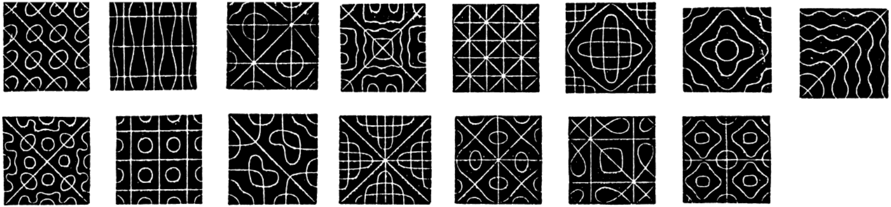


Figure: Symmetry breaking is everywhere!

Chladni Plate



A square plate vibrating under the influence of a violin with sand sprinkled on it . The nodes settle into one of these patterns - some of which have lesser symmetry than the full square

Source: Entdeckungen über die Theorie des Klanges (Discoveries in the Theory of Sound) by Ernst Chladni (1756–1827)

Bringing time into the picture

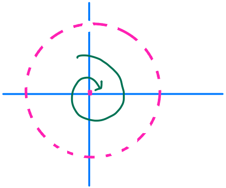
- Let us bring time also into the picture - most dynamical systems are time-invariant - i.e. they are invariant under all time translations
- A system that is perfectly still - at **equilibrium** (like a rock) has **complete time translational symmetry** - but once again our mind ignores this great bland symmetry!

Figure: A system at equilibrium has the fullest time transational symmetries

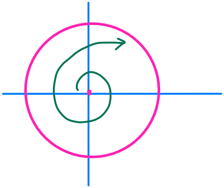
Hopf Bifurcation: when time symmetry is broken

- When the full time translational symmetry is broken, the candidate subgroup to break into is periodic symmetry
- So, by symmetry breaking, we can expect *equilibrium* $\rightarrow$ *periodic motion* which is exactly the **Hopf bifurcation** - eg. $\dot{r} = \alpha r - r^3, \dot{\theta} = 1$ when α changes sign!!

$$\dot{r} = \alpha r - r^3$$



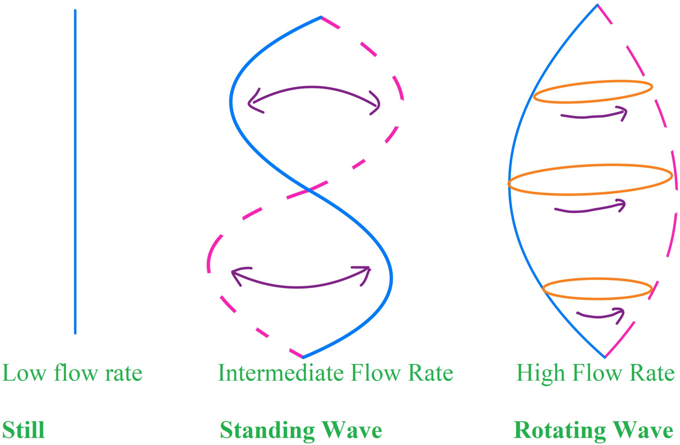
$\alpha < 0$



$\alpha > 0$

Mixing Space and Time - Hose Pipe

- Sometimes, the symmetry of space-time combined can break into a subgroup



Mixing permutations and Time - Ring of Oscillators

- Sometimes, the symmetry of permutation-time combined can break into a subgroup

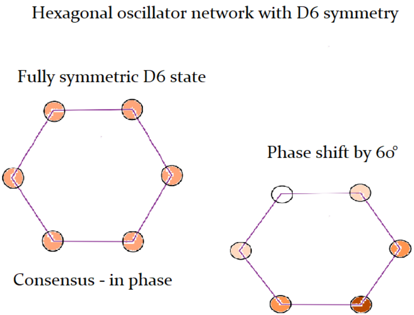


Figure: Initially synchronous and time invariant state with full $D_6 \times \mathbb{Z}_t$ symmetry can bifurcate into a rotating wave where each oscillator is out of phase by 60^0 and a time translation combined with permutation is required to leave the state of the system invariant

Symmetry and Conservation

- Most magnificient manifestations of continuous symmetries - conservation laws for conservative dynamical systems!
- Lagrangian $L = T - V$
  - T : Kinetic Energy
  - V : Potential Energy
- Emmy Noether's theorem - **if the Lagrangian is invariant under a continuous Lie group of transformations of dimension m , then**
  - **the classical system has m conserved quantities**
  - **the corresponding quantum system has energy levels with m-fold degeneracy**

Emmy Noether

Emmy Noether

Noether's theorem towers like an intellectual Mount Everest, the grand peak standing bright and clear over an impressive mountain range of powerful ideas
- Dwight Neuenschwander

David Hilbert on his struggles for the appointment of his colleague Emmy Noether as Privatdozent :

David Hilbert (1862–1943) was furious! He had been sitting in silence for quite some time, overhearing the conversation among his colleagues, but he could not hold his temper any longer. Slapping his fist onto the table, he shouted, *"I do not see that the sex of the candidate is an argument against her admission as a Privatdozent! After all, the [faculty] senate is not a bathhouse!"*

Source: "Shattered Symmetry - Group theory from the eightfold way to the periodic table" by Peter Thyssen and Arnout Ceulemans

Baby Steps: Part 1 - Rotational Symmetry

- Consider a conservative mechanical system - a particle in $\mathbb{R}^3$ with potential energy invariant under rotations - i.e. $V = V(r)$ only in polar coordinates (r, θ, ϕ).
- Such a function that depends only on distance is invariant under all translations - so its symmetry group is $G = O(3)$ with $dim(G) = 3$. So, we expect a three conserved quantites.
- The torque on such a system about the origin is zero since $\tau = \vec{r} \times \underbrace{-\vec{grad}(V)_{\text{parallel to } \vec{r}}} = 0$
- Hence the three components of angular momentum conserved!
- Rotational symmetry under $O(3)$ - $\leftrightarrow$ conservation of angular momentum

Baby Steps: Part 2 - Translational Symmetry

- Consider a system of particles interacting conservatively with potential energy invariant under all total translations of the system - i.e. $V(x_1 + a, x_2 + a, \ldots, x_m + a) = V(x_1, x_2, \ldots, x_m)$.
- Such a function is invariant under all translations and so its symmetry group is $G = \mathbb{R}^3$ with $dim(G) = 3$. So, we expect three conserved quantities.
- The total force on such a system is zero since $\vec{F}_T = -\sum_{i=1}^{m} grad_i(V) = 0$ and hence we know that the three components of the total linear momenta are conserved!!
- Translational symmetry under $\mathbb{R}^3$ - $\leftrightarrow$ conservation of total linear momentum

Baby Steps: Part 3 - Translational and Rotational Symmetry

- Consider a system of particles interacting conservatively with potential energy invariant under all total translations and rotations of the system - i.e. $V(Rx_1 + a, Rx_2 + a, \ldots, Rx_m + a) = V(x_1, x_2, \ldots, x_m)$.
- Such a function is invariant under all translations and rotations and so its symmetry group is $G = E(3)$ with $dim(G) = 6$. So, we expect three conserved quantities.
- They are nothing but total linear momentum and total angular momentum
- Translational and rotational symmetry under $E(3)$ - $\leftrightarrow$ conservation of total linear momentum and total angular momentum

Baby Steps: Part 4 - Time Translational Symmetry

- If a conservative system is time-invariant, then its potential energy is independent of time - $V(x, t) = V(x)$
- Its symmetry group is $\mathbb{R}_t$
- We expect one conserved quantity - turns out it is energy!
- Time Translational symmetry under $\mathbb{R}$ - $\leftrightarrow$ conservation of total energy