Groups and Geometry: A Gentle Introduction

From Euclid to Lie

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Lecture 2: Theory of Groups

Beginnings: Euclid

The Third Postulate of Euclidean Geometry

All right angles are congruent

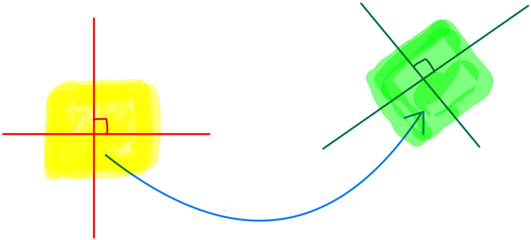


Figure: The third postulate - what does it really mean?

- What Euclid has said in modern terms is that 'given any two right angles at any two points, there is a transformation taking one to the other, preserving all the geometrical properties of the plane'.

Symmetry: Beginnings of Group Theory

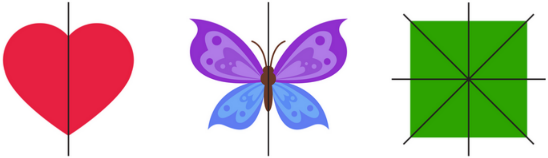


Figure: What exactly is symmetry?
'If no one asks me, I know. But if I wish to explain it to one that asketh, I know not' - St Augustine

- Etymology:
  Greek *syn* (same/together) + *metre* (measure)

En arkhêi ên ho lógos, kaì ho lógos ên pròs tòn theón, kaì theòs ên ho lógos
(In the beginning was the Word, and the Word was with God, and the Word was God)

The Gospel of John, 1:1

What was there in the beginning? In the beginning, was symmetry!

Werner Heisenberg

Defining Symmetry

Symmetry of an object

A symmetry of an object, is a **"transformation"** that leaves it **"unchanged"**

- Note: Transformation - you do something to an object
- Note: Unchanged - some property of it remains unchanged after the transformation
- So, the definition of symmetry depends on what transformations are allowed and what properties one is concerned about

Starting Simple: Equilateral Triangle

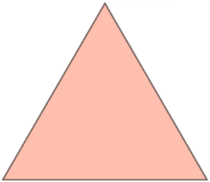


Figure: What are the symmetries of an equilateral triangle?

Property = geometric property

Symmetry Transformations of an Eq Triangle

- It turns out that the following are the symmetry operations of an equilateral triangle:

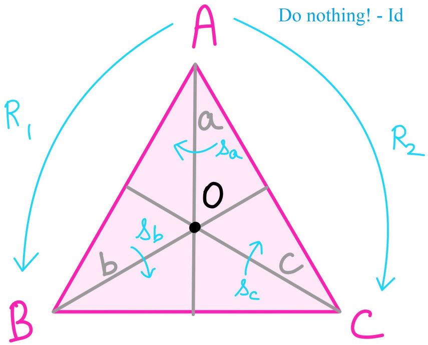


Figure: Symmetry transformations of an equilateral triangle

- I, R_1, R_2 : Rotations through $0^0, 120^0, 240^0$ respectively
- s_a, s_b, s_c : Reflections through medians a, b, c respectively

Symmetries Combine

- Note that 'doing nothing' or 'identity transformation' always is a symmetry transformation as doing nothing changes nothing and hence does not alter any property
- If S and T are two symmetry transformations that leave something unchanged, then so does $S \circ T$
- If S is a symmetry transformation that leaves something unchanged, then the same holds true for performing its inverse transformation S^{-1} as well
- Symmetries have an algebra - they combine with one another

Symmetries Combine: Example

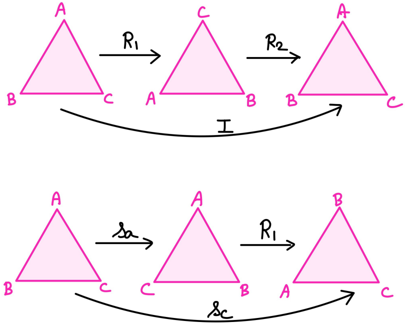


Figure: Combining two symmetry operation gives another symmetry operation

- $Id = R_2 \circ R_1$
- $s_c = R_1 \circ s_a$

A Multiplication Table for Symmetries

- One can build a table for symmetries by seeing the result of any two combinations

$*$	I	S_a	S_b	S_c	R_1	R_2
I	I	S_a	S_b	S_c	R_1	R_2
S_a	S_a	I	R_1	R_2	S_b	S_c
S_b	S_b	R_2	I	R_1	S_c	S_a
S_c	S_c	R_1	R_2	I	S_a	S_b
R_1	R_1	S_c	S_a	S_b	R_2	I
R_2	R_2	S_b	S_c	S_a	I	R_1

Figure: Multiplication Table for the symmetry transformations of an equilateral triangle

- Once this table is known, one can calculate the result of any sequence of symmetry transformations and their inverses

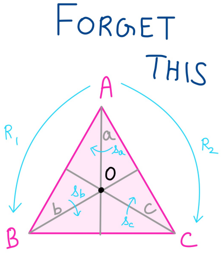
From Symmetry to a Group

- The concept of a group arises when one just takes the multiplication table and forgets the actual physical details of the transformation

Abstract theory of groups

D_3

$*$	I	S_a	S_b	S_c	R_1	R_2
I	I	S_a	S_b	S_c	R_1	R_2
S_a	S_a	I	R_1	R_2	S_b	S_c
S_b	S_b	R_2	I	R_1	S_c	S_a
S_c	S_c	R_1	R_2	I	S_a	S_b
R_1	R_1	S_c	S_a	S_b	R_2	I
R_2	R_2	S_b	S_c	S_a	I	R_1



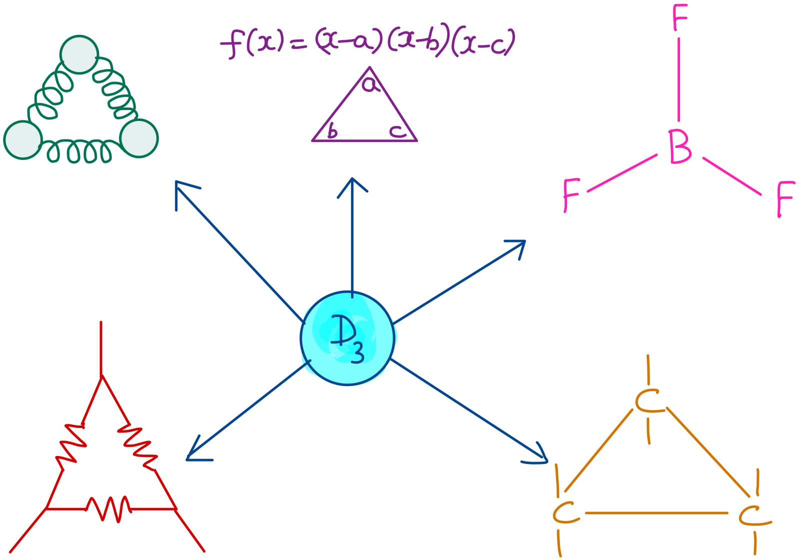
- D_3 - dihedral group of order 3

Groups versus Symmetries: A Comparison

- Combining two symmetry transformations ($\circ$) gives another symmetry transformation
- The identity (doing nothing) is a symmetry transformation always
- The inverse of a symmetry transformation is another symmetry operation

- A group G is a set with a binary operation ($*$) that takes two elements of the set to give another element in that set itself with the following two properties:
- There is an identity element $e \in G$ which has the property that $g * e = e * g = g$ for all $g \in G$
- For every $g \in G$, there is an element g^{-1} that satisfies $g^{-1} * g = g * g^{-1} = e$

Why this abstraction?



Subgroups

- A subset H of a group G is said to be a subgroup if it forms a group on its own - the group operation and inverse of elements in H stay in H

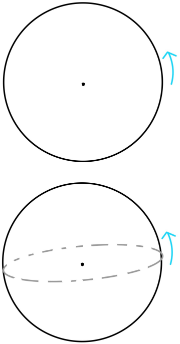
$*$	I	S_a	S_b	S_c	R_1	R_2
I	I	S_a	S_b	S_c	R_1	R_2
S_a	S_a	I	R_1	R_2	S_b	S_c
S_b	S_b	R_2	I	R_1	S_c	S_a
S_c	S_c	R_1	R_2	I	S_a	S_b
R_1	R_1	S_c	S_a	S_b	R_2	I
R_2	R_2	S_b	S_c	S_a	I	R_1

Figure: The rotations of D_3 is a subgroup of D_3

- An object A can now said to be more symmetrical than another object B if the symmetry group of B is a subgroup of the symmetry group of A

From Discrete to Continuous Groups

- So far, the dihedral group D_3 is discrete in the sense that its elements can be enumerated by natural numbers (in particular - finitely many natural numbers).
- But things are really interesting when we deal with groups that got to be specified by continuous parameters



any
continuous
rotation
is a
symmetry

Figure: What are the symmetry groups for a circle and a sphere?

$SO(3)$: A bridge to generic "Lie" Groups

- Continuous groups (technically called "Lie groups" are dificult to handle - multiplication table cannot be specified for each element separately!
- So, efficient ways of studying the group properties needed
- Hence, we start by studying a particular example of a continuous group $SO(3)$ which happens to be the symmetry group of a sphere in $\mathbb{R}^3$

$SO(3)$: All possible rotations

- We now analyze the rotational symmetries of a sphere in $\mathbb{R}^3$ - a sphere is invariant under any rotation

The Special Orthogonal Group

$SO(3) := \{R \in \mathbb{R}^{3\times 3}$ such that for all $x, y \in \mathbb{R}^3$, $\langle Rx, Ry\rangle = \langle x, y\rangle$ and $\det(R) = 1\}$
(this definition immediately yields that $SO(3)$ is a group under matrix multiplication - no surprise)

- w.r.t standard inner product,

$$SO(3) = \{R \in \mathbb{R}^{3\times 3} \text{ such that },\ R^T R = I,\ det(R) = 1\}$$

- it is a continuous group - rotation about any continuous angle and in any continuous direction possible

$SO(3)$: Also a Smooth Manifold!

- $SO(3)$ is a subset of the Euclidean space $\mathbb{R}^{3\times 3}$, which is a 9-dimensional Euclidean space
- It is defined by the constraint $R^T R = I$ which amounts to six independent quadratic constraints [1]
- The gradient condition can be verified
- Hence, turns out $SO(3)$ is not just a group - but a smooth manifold as well!
- Its dimension $n =$no. of entries - no. of constraints = 9-6=3

[1]($R^T R$ is symmetric and hence only 6 of its 9 entries are independent)

Viewing a Symmetry Transformation

- There are two ways to view any transformation - **active** and **passive**.
- **Active:** Move the object, keep the surrounding space (coordinates) fixed
- **Passive:** Keep the object fixed, move the surrounding space (coordinates)
- All symmetry requires is relative movement between the object and the space - any one can be moved keeping the other fixed

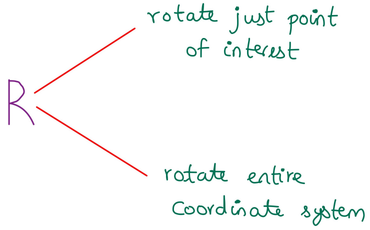
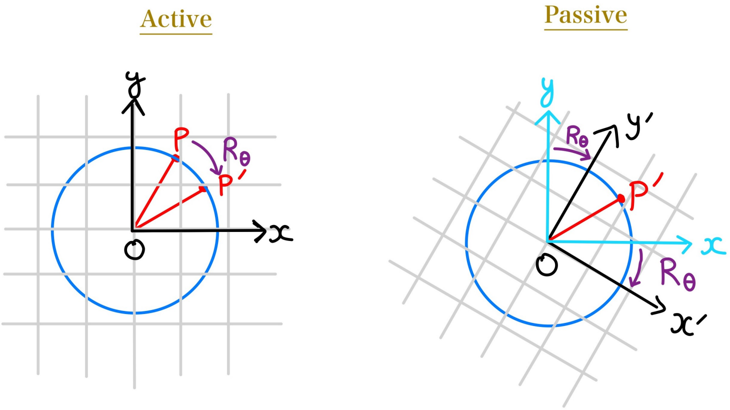


Figure: Rotations - Active and Passive

Rotation: Active and Passive View



Point moves

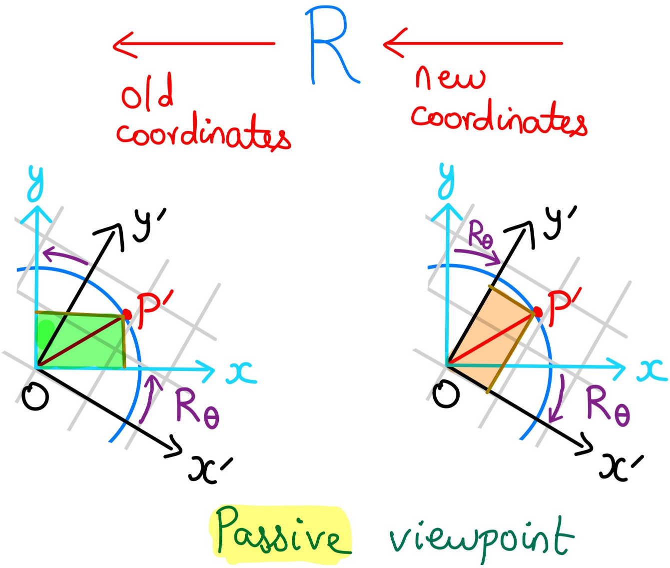
Coordinate system fixed

Point fixed

Coordinate system moves

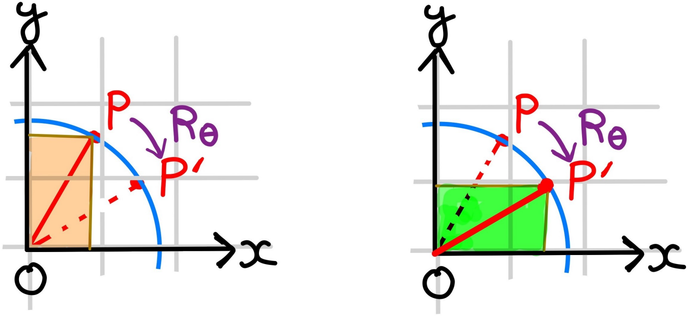
Figure: Active and Passive View of a rotation in the plane R_θ

Rotation: Passive Role



Rotation: Active Role

old point $\longrightarrow$ R $\longrightarrow$ new point



Active viewpoint

Conjugation: A Motivation

- A rotation of 12^0 about $x-$axis is 'essentially' same as a rotation of 12^0 about $y-$axis - it is just a relabelling of the axes. How do we formalize this idea?

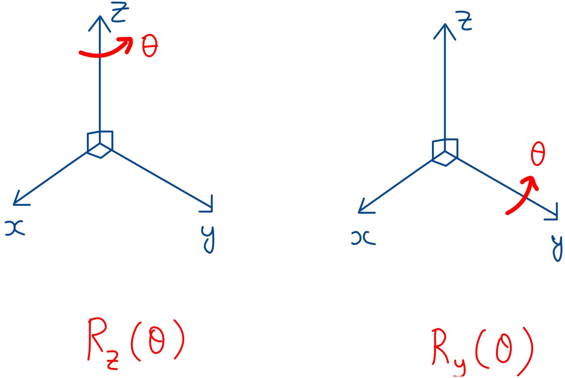


Figure: A rotation of angle θ about $x-$ axis is 'equivalent' to rotation of angle θ about y-axis - same transformation in a rotated coordinate system

Conjugation

$$\underbrace{R^{\substack{z\\\uparrow\\y}}\ \overset{\text{ACTIVE}}{R_y(\theta)}\ R^{\substack{y\\\uparrow\\z}}}$$

$$\underbrace{R^{-1}\ \overset{\text{ACTIVE}}{R_y(\theta)}\ R}$$

$$= \overset{\text{ACTIVE}}{R_z(\theta)} = R^{-1} R_y(\theta) R$$

Figure: Mathematical equivalence between $R_z(\theta)$ and $R_y(\theta)$

Conjugation: Abstract Definition

Conjugacy Relation

Two elements g_1, g_2 in a group G are said to be conjugate if there exists another transformation in the group $g' \in G$ such that

$$g_1 = g' * g_2 * g'^{-1}$$

- Conjugacy is an equivalence relation
- The equivalence class of an element g under the conjugacy relation is the set of all group elements equivalent to g
- Physically, g_1, g_2 are said to be conjugate in the group, if g_1 looks like g_2 in another coordinate system which is achieved passively through an element g' in the group itself

Geometry begins: Evaluating Tangent Space of $SO(3)$

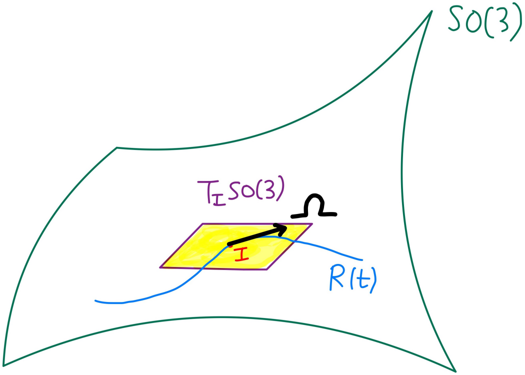


Figure: Determining $T_I SO(3)$

$T_I SO(3) := \mathfrak{so}(3)$

- Let us now consider a curve $R(t)$ in $SO(3)$ through I - $R(0) = I$. Let Ω be its velocity at time $t = 0$ - $\dot{R}(0) = \Omega$
- $$0 = \frac{d}{dt}\bigg|_{t=0}(R^T R) = R(0)^T\Omega + \Omega^T R(0) = \Omega + \Omega^T$$
- So we get that any curve in $SO(3)$ passing through velocity can have velocity Ω that should satisfy

$$\Omega = -\Omega^T \ \ \text{(skew-symmetric)}$$

- So, $T_I SO(3) := \mathfrak{so}(3)$:=set of all 3×3 skew-symmetric matrices

Tangent Space at other points

- Consider a curve with $R(0) = R$ and velocity $\dot{R}(0) = R\Omega$. Then,

$$0 = \frac{d}{dt}\bigg|_{t=0} (R^T R) = R(0)^T [R\Omega] + [\Omega^T R^T] R(0)\Omega + \Omega^T$$

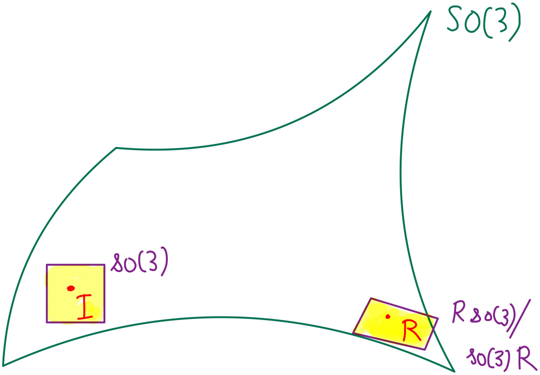
- Repeat with $\dot{R} = \Omega R$ with

$$0 = \frac{d}{dt}\bigg|_{t=0} (RR^T) = R(0)[R^T \Omega^T] + [\Omega R] R(0)^T$$

$T_R SO(3)$

$$T_R SO(3) = \{R\Omega \mid \Omega \in \mathfrak{so}(3)\} = \{\Omega R \mid \Omega \in \mathfrak{so}(3)\}$$

Visualizing $SO(3)$, $\mathfrak{so}(3)$ and $T_R SO(3)$



Euler's Theorem: Every rotation has an axis

- **Euler's Theorem** - Every rotation R is equivalent to a rotation about an axis $\hat{\omega}$ by an angle θ.

Proof of Euler's Theorem

- An odd matrix has a real eigen vector with real eigen value with remaining two eigen values either real or in complex conjugate pairs
- Since R is a rotation matrix, the magnitude of all eigen values is 1 - so the real eigen values possible are just ± 1.
- The eigen vector corresponding to the real eigen value $+1$ is the axis *omêga* of rotation
- The plane $\hat{\omega}^{\perp}$ which is preserved by R is a rotation in a plane which has an angle θ
- Since $det(R) = +1$, there has to be an eigen value $+1$ (this can be proved easily by contradiction)

$\mathfrak{so}(3)$ - Angular velocity

- Consider a rigid body with an attached body coordinate (B) that is rotated by a rotation $R(t)$

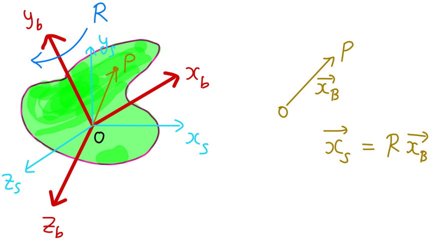


Figure: A rotating rigid body with an attached coordinate system

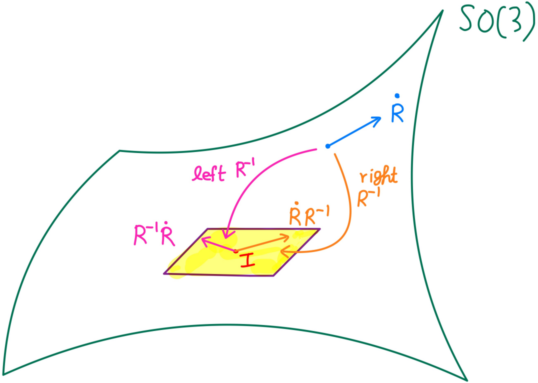
$$\dot{x}_S = \overbrace{\left(\dot{R}R^{-1}\right)}^{\in\,\mathfrak{so}(3)} x_S$$

$$\dot{x}_B := R^{-1}\dot{x}_S = \underbrace{\left(R^{-1}\dot{R}\right)}_{\in\,\mathfrak{so}(3)} .x_B$$

Angular Velocities

Nomenclature

- $R^{-1}\dot{R}$ - angular velocity in the body frame
- $\dot{R}R^{-1}$ - angular velocity in the space frame



Motion under constant angular velocity

- When a rigid body rotates with a constant body angular velocity Ω_B, we have

$$\Omega_B = R^{-1}\dot{R}$$
$$\implies \dot{R} = R\Omega_B$$

- This is a linear ODE whose solution is an exponential given by

$$R(t) = R(0)exp(\Omega_B t)$$

Cross Products and $\mathfrak{so}(3)$

- Observe that any 3×3 skew symmetric matrix acting on a vector is equivalent to a cross product in $\mathbb{R}^3$ -

$$\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Considering the equation

$$\dot{x}_B = \Omega_B x_B = \vec{\Omega}_B \times x_B$$

generalizes the familiar relation

$$v = \omega r$$

from planar rotation

Story So Far

- $SO(3)$: Rotation matrices
- $\mathfrak{so}(3)$: Tangent space of $SO(3)$ at identity I
- $R\mathfrak{so}(3) = \mathfrak{so}(3)R$: Tangent space of $SO(3)$ at a point R
- Conjugation: Two rotations R_1, R_2 conjugate if $R_1 = RR_2R'^{-1}$ for some rotation R
- $R^{-1}\dot{R} \in \mathfrak{so}(3)$: Body angular velocity of a curve $R(t)$
- $\dot{R}R^{-1} \in \mathfrak{so}(3)$: Space angular velocity of a curve $R(t)$
- $\dot{R} = R\Omega$ and $R(t) = R(0)\exp(\Omega t)$ - kinematics with constant body angular velocity $\Omega \in \mathfrak{so}(3)$

Next Time: Generalize all these concepts to arbitrary Lie groups