Groups and Geometry: A Gentle Introduction

From Euclid to Lie

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Lecture 1: Introduction to Geometry

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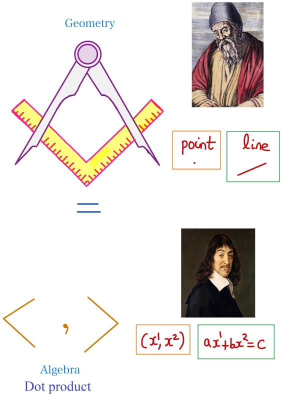
Story so far: The Euclidean Space

- A single coordinate system for the entire space - geometry is taken over by algebra of coordinates



A particle in a plane: Plane Polar coordinates

- Consider a particle moving in a plane with plane polar coordinates r, θ related to the Cartesian coordinates by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} \frac{y}{x}$$

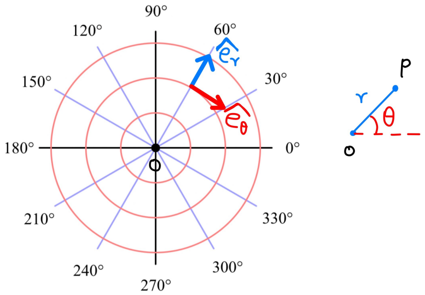


Figure: Polar Coordinates and Unit Vectors

Acceleration in Polar Coordinates

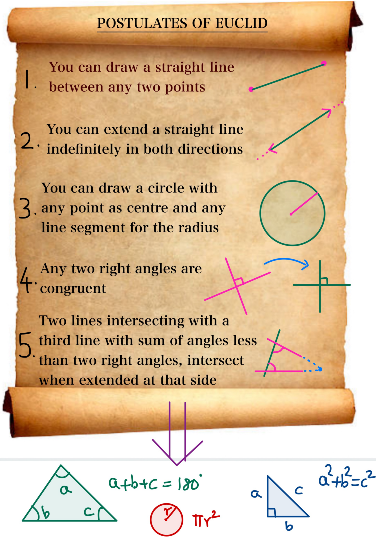
- Acceleration of a particle in polar coordinates:

$$\vec{a} = (\ddot{r}\hat{e}_r + r\ddot{\theta}\hat{e}_\theta) + (-r\dot{\theta}^2\hat{e}_r + 2\dot{r}\dot{\theta}\hat{e}_\theta) \tag{1}$$

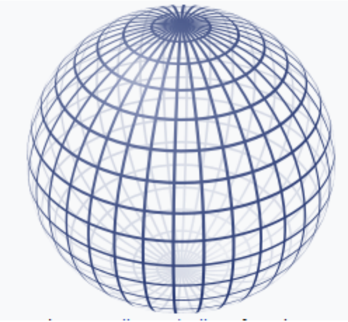
- Not simply $\vec{a} = \ddot{r}\hat{e}_r + \ddot{\theta}\hat{e}_\theta$ like Cartesian $\vec{a} = \ddot{x}\hat{e}_x + \ddot{y}\hat{e}_y$.
- Why the extra terms?? - 'price to pay' for choosing a *"curvi-linear non-orthonormal"* coordinate system rather than an *"orthonormal, linear"* (Cartesian) coordinate system
- Various interpretations possible for the extra terms: Centrifugal, Coriolis, Euler ...
- **NOTE:** Eqn (1) is the same old $\vec{a}$ - just in another "inconvenient" coordinate system

Is Euclid the only option?

- Is the Euclidean plane, the only surface worthy of a geometry?

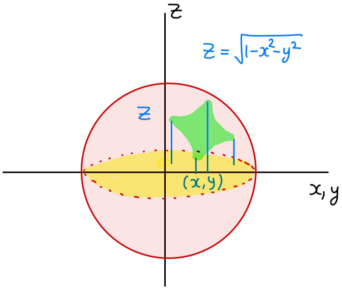


How about a sphere?



- How about the surface of the sphere $\mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 \text{ such that } x^2 + y^2 + z^2 = 1\}$?

Locally, Descartes works



Local Coordinates possible on $\mathbb{S}^2$

- In any small neighborhood of $\mathbb{S}^2$, two coordinates can be chosen independent with the third coordinate as a function of those two - e.g. $z = \sqrt{1 - x^2 - y^2}$

Question: Can a constraint like $x^2 + y^2 - z^2 - 1 = 0$ always be used to locally eliminate one variable and write it as a smooth function of the other variables like $z = \pm\sqrt{1 - x^2 - y^2}$?

$$x^2 + y^2 + z^2 - 1 = 0 \text{ (implicit) } \rightarrow \; z = \sqrt{1 - x^2 - y^2} \text{ (explicit)}$$

Mathematical Detour: Implicit Function Theorem

Implicit Function Theorem

- Let $f_i|_{1 \leq i \leq p} : \mathbb{R}^m \to \mathbb{R}$ be p smooth functions in $\mathbb{R}^m$
- Consider the set G defined by p constraints - $f_i(z) = 0$
- Consider the matrix of partial derivatives $J = \frac{\partial f_i}{\partial x_j}(z_0)|_{1 \leq i \leq p, 1 \leq j \leq m}$
- If the matrix J is of maximum rank p, then locally around x_0, p variables $(y_1, y_2, \ldots, y_p)$ in $\mathbb{R}^m$ can be eliminated and expressed in terms of $n = p - m$ other variables $(x_1, x_2, \ldots, x_n)$ using the constraints. i.e.

$$f(x_1, \ldots, x_n, y_1, \ldots, y_p) = 0 \Leftrightarrow y_i = y_i(x_1, x_2, \ldots, x_n) \ : 1 \leq i \leq p$$

Example: $\mathbb{S}^2 \subseteq \mathbb{R}^3$: $m = 3, p = 1, n = p - m = 2$ and $f_1 = x_1^2 + x_2^2 + x_3^2 - 1 = 0$

But globally, Descartes fails

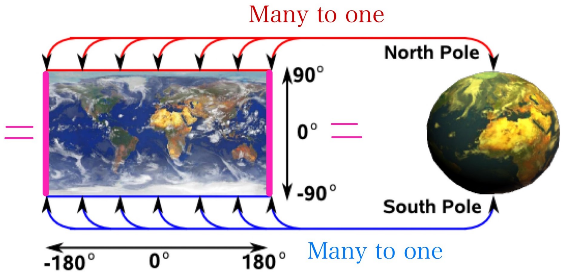


Figure: No matter whatever one tries, one cannot smoothly map a sphere ($\mathbb{S}^2$) into a bunch of two real numbers ($\mathbb{R}^2$) - this figure shows the failure of the standard Mercator Map

Smooth Manifold: General Definition

- **Sphere**: "2"-dimensional "surface" in "$\mathbb{R}^3$" , one smooth constraint $(x^2 + y^2 + z^2 - 1) = 0$
- **Smooth Manifold**: *abstraction* - a "n" dimensional "entity" in "$\mathbb{R}^m$" - $m - n$ smooth constraints

Smooth Manifold G

A smooth manifold G is a subset of $\mathbb{R}^m$ defined by p smooth constraints

$$G = \{x \in \mathbb{R}^m \ : \ f_i(x) = 0\} \ \ f_i : \mathbb{R}^m \rightarrow \mathbb{R}(i = 1, \ldots, p)$$

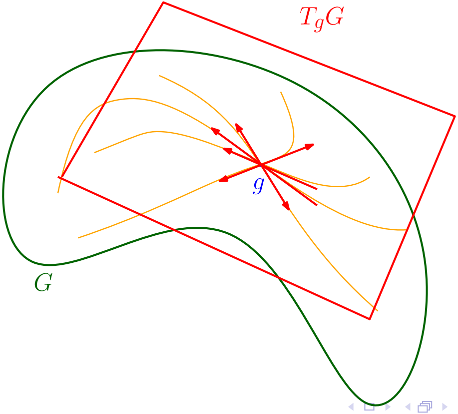
where the matrix $\{\frac{\partial f_i}{\partial x_j}\}$ is full rank $n = m - p$ at all points in G[a]

[a]The full rank comes in so that we can apply implicit function theorem and write some m coordinates as a function of the remaining $p - m$ coordinates locally so that we have available, a ready-made local coordinate system around every point

- Dimension of a smooth manifold n: number of independent local coordinates - $n = m - p$

Velocities: Tangent Space

- Imagine a particle constrained to move only on the surface of a sphere (or on a manifold) - what are its possible velocities at a point g?
- The tangent space of a manifold G at a point g, denoted by $T_g G$ is defined as the set of all velocities that a particle can have when passing through g and staying in G



Computing $T_g G$

- Let $G \subset \mathbb{R}^p$ be a n dimensional manifold defined by m smooth constraints $f_i = 0$ for $i = 1, 2, \ldots, m$ where $n = p - m$
- By implicit function theorem, it has $n-$independent local coordinates and hence we have that $T_g G$ is $n-$dimensional.
-

$$\left.\frac{df_i}{dt}\right|_{v \in T_g G} = \underbrace{\sum_{j=1}^{p} \frac{\partial f_i}{\partial x_j} v_j}_{\text{by chain rule}} = \langle grad(f_i), v \rangle = 0 \tag{2}$$

- So, $T_g G \subseteq \cap_{i=1}^{m} grad(f_i)^{\perp}$ whose dimension is also $p - m = n$
- So, $T_g G = \cap_{i=1}^{m} grad(f_i)^{\perp}$

The Tangent Space as a local approximation

- $T_g G$: Euclidean space that best approximations G around g
- G: approximated arbitrarily by gluing small sections of tangent spaces

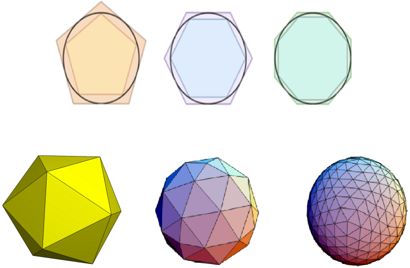


Figure: Polygonal approximations of a circle and polyhedral approximations of a sphere by using tangent lines and tangent planes respectively

How to do geometry in a manifold?

- Euclidean space: Geometry by inner product $\langle , \rangle$.
- Manifold G: assign an inner product $\langle , \rangle$ in each $T_g G$ - i.e. by assigning speeds to curves and angles between them
- Once speed of a curve $\gamma : [a, b] \to G$ is defined, its length is

$$Length[\gamma] := \int_a^b ||\dot{\gamma}(t)|| dt$$

- Once, lengths are gotten, distance between two points is simply the length of that curve which is of the shortest possible length between the two points

Straight Lines and Parallelism in a Manifold
Step 1: Approximate by Tangent Spaces

- How does one extend the notions of a straight line and parallelism to manifolds?

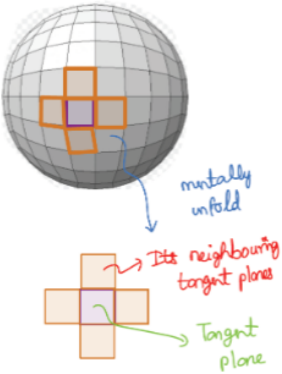
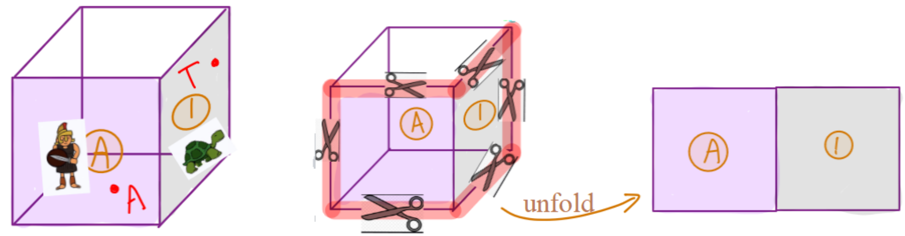


Figure: **Step 1:** - Approximate the manifold locally by various tangent spaces

Straight Lines and Parallelism in a Manifold
Step 2: Unfold and Identify



Consider two neighboring tangent spaces
A - *for Achilles!*
T - *for Tortoise!*

Flatten and identify the neighboring tangent spaces

Figure: **Step 2:** - Unfold and Identify the neighboring tangent spaces

Straight Lines and Parallelism in a Manifold

Step 3: Keep straight

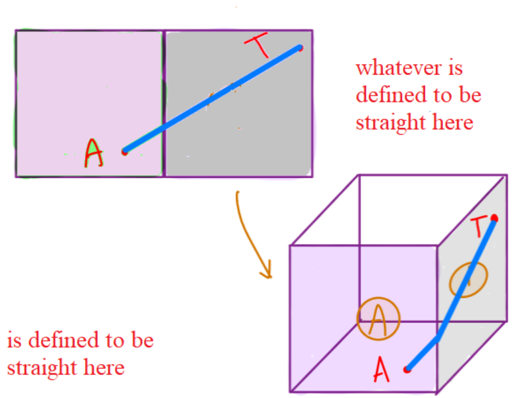
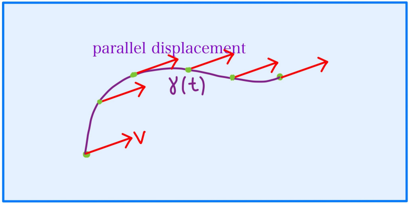


Figure: **Step 3:** - Keep straight under this identification

Parallelism: In the plane



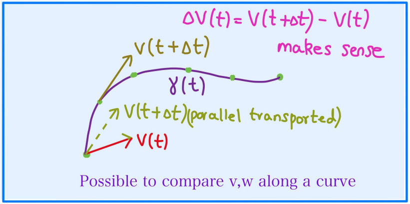
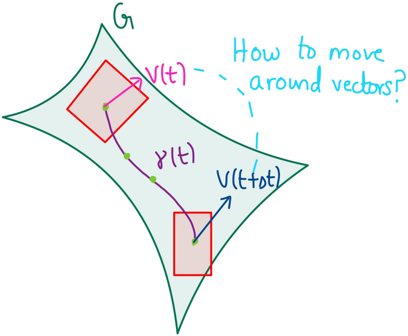


Figure: In the plane, a vector can be moved from one point to another along any curve and hence vectors at two different points can be compared after parallel transport

In a manifold: How to compare vectors at different places?



Requirements:

- Intrinsic - not allowed to look outside G in the process of moving around

Straight Lines and Parallelism in a Manifold
Step 4: Keep parallel

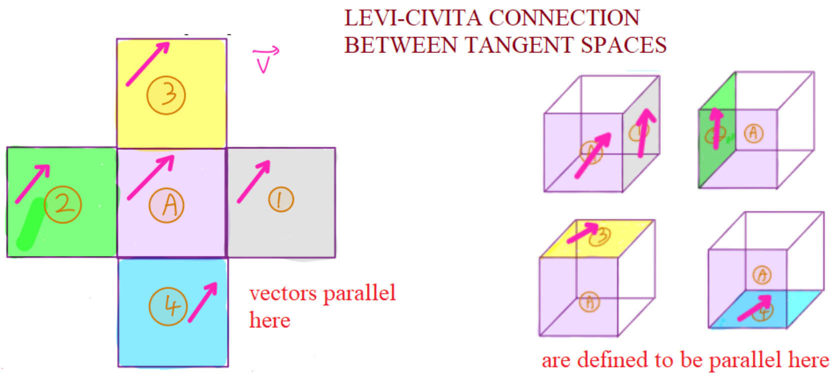


Figure: **Step 4:** - Two vectors in neighboring tangent spaces are parallel if they are parallel under this identification

Parallel Transport of Vectors Along a curve

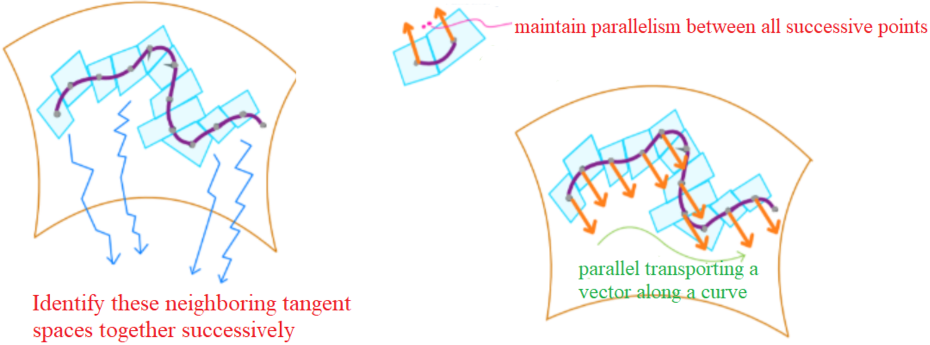


Figure: Parallel Transport of a vector along a curve

Straight lines ↔ Geodesics

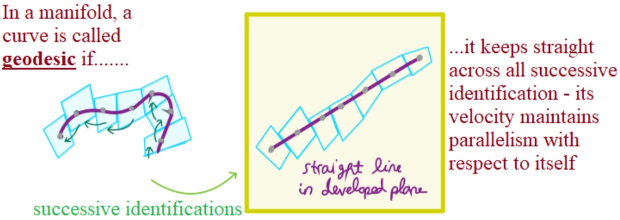


Figure: Geodesics - counterpart of straight lines in manifolds

- **Newton's first law in a manifold**: A free particle follows a geodesic path (its velocity remains constant when parallel transported along its path!)

The Covariant Derivative ∇

- For acceleration of a curve $\gamma(t)$, one has to subtract velocity vectors at two different points - $\dot{\gamma}_{\gamma(t)}$ and $\dot{\gamma}_{\gamma(t+\Delta t)}$. But we just saw how to identify neighboring tangent spaces $T_{\gamma(t)}$ and $T_{\gamma(t+\Delta t)}$

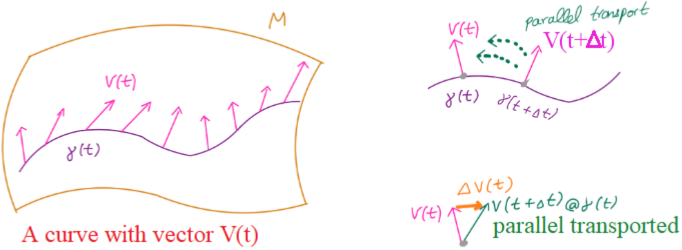


Figure: Defining the covariant derivative $\nabla_{\dot{\gamma}(t)} V(t)$

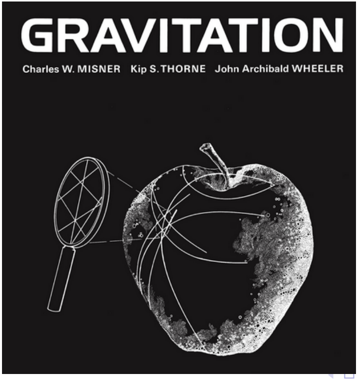
$$(\nabla_{\dot{\gamma}(t)} V)(t) := lim_{\Delta t \to 0} \frac{\Delta V(t)}{\Delta t}$$

Acceleration of a curve: Using ∇

- Acceleration of a curve is simply the rate of change of velocity of a curve along itself . So putting $V = \dot{\gamma}$, we have

$$Acceleration[\gamma](t) := (\nabla_{\dot{\gamma}(t)}\dot{\gamma})(t)$$

- A straight line is simply a curve with constant velocity or zero acceleration - i.e. $\nabla_{\dot{\gamma}(t)}\dot{\gamma} = 0$ for all time t



Euclid in a Manifold: The Genesis of Curvature

- Having approximated the manifold locally by a tangent space, is the manifold "**really**" locally Euclidean? - i.e. is it true that
  - Area of a small circle of radius r is $2\pi r^2$
  - Sides a, b, c of a small enough right angled triangle satisfy $c^2 = a^2 + b^2$
  - Angles of all small triangles add up to two right angles and so on ...

Figure: The Geometer God Plans his Universe - William Blake

Can we have Cartesian coordinates in a manifold?

- **Mathematics Question:** Does Euclidean geometry hold locally in some small region?
- **Physics Question:** Can one choose some local coordinate system say (w_1, w_2) in G such that the acceleration expressed in that coordinate system will just be

$$\vec{a} = (.)\ddot{w}_1\hat{e}_{w_1} + (.)\ddot{w}_2\hat{e}_{w_2} + \text{no extra terms quadratic in velocities } \dot{w}_1, \dot{w}_2$$

- One can choose a coordinate system so that the red term vanishes at ONE point - the next step is can the red terms be killed in an entire region?

An Experiment with the Cube

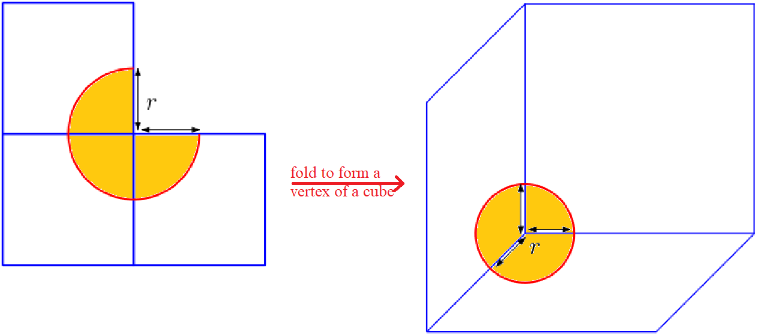


Figure: Area of any circle, no matter how small, around a vertex, is not πr^2 but $0.75\pi r^2$ around the vertex of a cube

Lines that start parallel...do not stay parallel

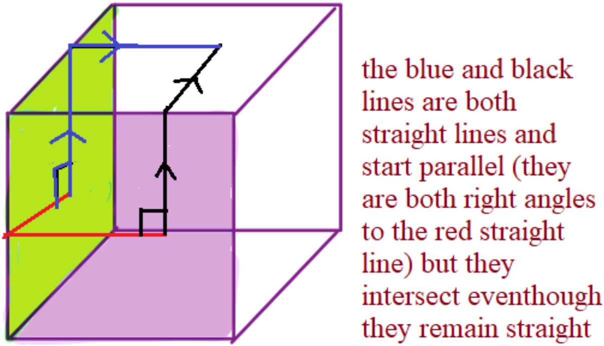
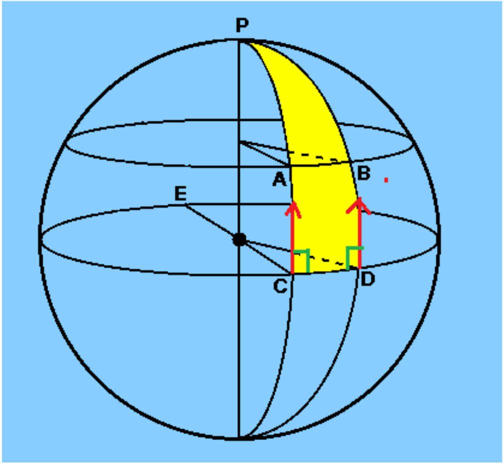


Figure: Parallel lines need not stay parallel

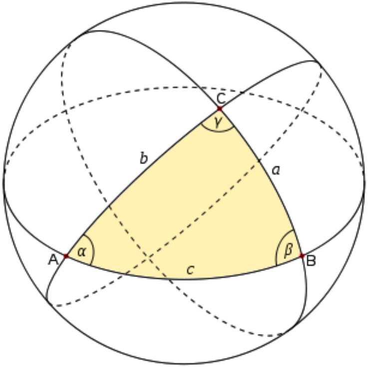
What about a sphere??

- In a sphere, the geodesics are great circles (they are the length minimizing curves) but they intersect even though they start parallel



Two great circles on a sphere starting parallel but intersecting each other

Area of a spherical triangle



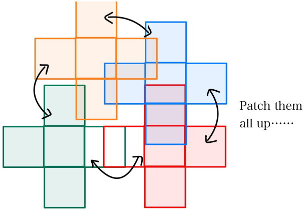
<u>Girard's theorem for spherical triangles:</u>

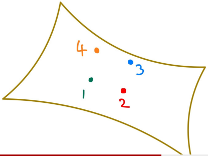
$(\alpha+\beta+\gamma)-\pi = \text{area of } \Delta$

Figure: The angle excess of a spherical triangle is equal to its area in a unit sphere - means larger the triangle, more the deviation from Euclidean postulates!

The Problem

- What is happening? - Locally, around every point, neighboring tangent spaces can be flattened and identified - but there is no way one can glue all these identifications around all these points together to form a single system that works for all points - one will face issues like partial arcs closing to form a circle during patching up as one saw in the cube





One coordinate system that patches all these identifications in a neighborhood and respecting the geometry of the space.............................. hindered when the space is "curved"

A leap: From Euclid to Einstein

- So we see that curvature can cause trajectories of free particles that start parallel, to converge/diverge eventhough locally they maintain their velocity parallel from their own previous instants
- The effect of curvature is same for all particles - any free particle has the same trajectory when starting from a given point and at a given velocity and continues moving straight
- **Einstein's Idea:** The effects of gravitational force is also independent of the mass of the particle - $ma = mg$ (m cancels out) - the equivalence principle. So, gravity = curvature of space-time