Groups and Geometry: A Gentle Introduction

From Euclid to Lie

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Lecture 3: Matrix Lie Groups and Lie Algebras

Story So Far - $SO(3)$

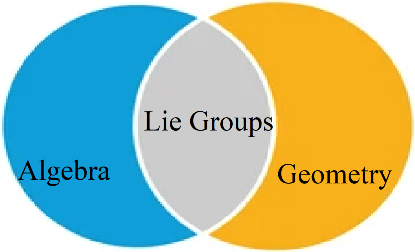
- $SO(3)$: Rotation matrices
- $\mathfrak{so}(3)$: Tangent space of $SO(3)$ at identity I
- $R\mathfrak{so}(3) = \mathfrak{so}(3)R$: Tangent space of $SO(3)$ at a point R
- Conjugation: Two rotations R_1, R_2 conjugate if $R_1 = RR_2R^{-1}$ for some rotation R
- $\hat{\Omega}_B := R^{-1}\dot{R} \in \mathfrak{so}(3)$: Body angular velocity of a curve $R(t)$
- $\hat{\Omega}_S = \dot{R}R^{-1} \in \mathfrak{so}(3)$: Space angular velocity of a curve $R(t)$
- $\dot{R} = R\hat{\Omega}$ and $R(t) = R(0)\exp(\hat{\Omega}t)$ - kinematics with constant body angular velocity $\hat{\Omega} \in \mathfrak{so}(3)$

Matrix Lie Groups: A Formal Definition

Matrix Lie Group

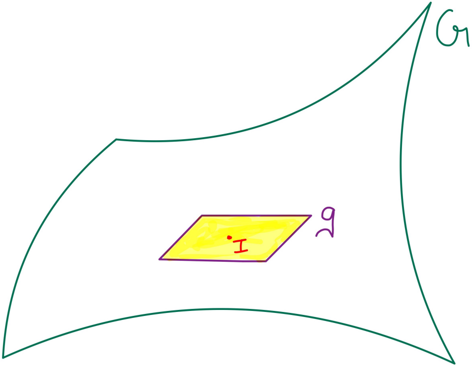
A matrix Lie Group G is a subset of $\mathbb{R}^{n \times n}$ having the following properties

- G is a **smooth manifold** of some dimension m
- G is a **group** under matrix multiplication -
  - The $n \times n$ identity matrix I is in G
  - Product of any two matrices in G stays in G
  - Inverse of any matrix in G is also in G



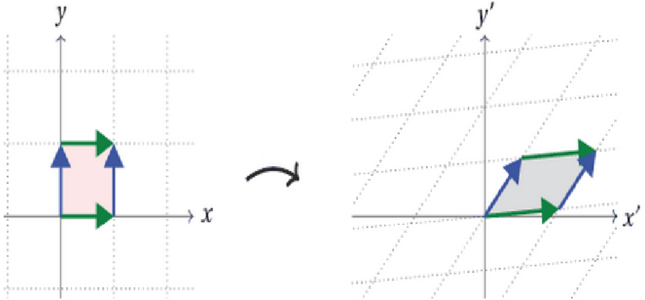
Lie Algebra

- The tangent space of a Lie group G at the identity, $T_e G$ is called the **Lie Algebra** of the Lie group G and is denoted by $\mathfrak{g}$.
- Example: $G = SO(3)$, $\mathfrak{g} = \mathfrak{so}(3)$



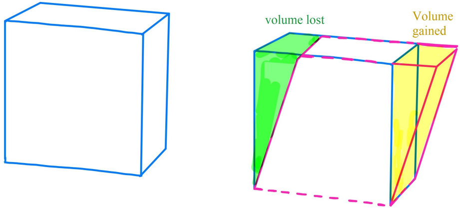
$GL_+(n, \mathbb{R})$

- $GL_+(n, \mathbb{R}) = \{M \in \mathbb{R}^{n \times n} \mid det(M) > 0\}$ - orientation preserving linear transformations in $\mathbb{R}^n$
- $dim\ GL_+(n, \mathbb{R}) = n^2$
- $\mathfrak{gl}_+(n, \mathbb{R}) = \mathbb{R}^{n \times n}$



$SL(n, \mathbb{R})$ - Special Linear Group

- $SL(n, \mathbb{R}) = \{M \in \mathbb{R}^{n \times n} \mid det(M) = 1\}$ - oriented volume preserving linear transformations in $\mathbb{R}^n$
- $dim\ SL(n, \mathbb{R}) = n^2 - 1$
- $\mathfrak{sl}(n, \mathbb{R}) = \{Q \in \mathbb{R}^{n \times n} \mid trace(Q) = 0\}$ - traceless matrices



$SO(n)$ - Special Orthogonal Group

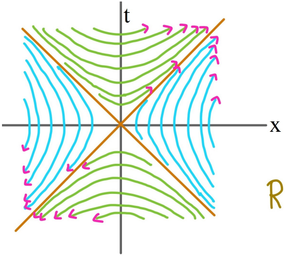
- $SO(n) = \{R \in \mathbb{R}^{n\times n} \mid R^T R = I \, , det(R) = 1\}$ - rotations in $\mathbb{R}^n$
- $dim\ SO(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$
- $\mathfrak{so}(n) = \{\hat{\Omega} \in \mathbb{R}^{n\times n} \mid \hat{\Omega}^T = -\hat{\Omega}\}$ - skew symmetric matrices

$SE(n)$ - Special Euclidean Group

- $SE(n) = \{\begin{bmatrix} R_{n\times n} & p_{n\times 1} \\ 0_{1\times n} & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \mid R^T R = I \, , det(R) = 1\}$ - rotations and translations in $\mathbb{R}^n$
- $\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ 1 \end{bmatrix} = \begin{bmatrix} Rr + p \\ 1 \end{bmatrix}$
- $dim\ SE(n) = \frac{n(n+1)}{2}$
- $\mathfrak{se}(n) = \{\begin{bmatrix} \hat{\hat{\Omega}}_{n\times n} & p_{n\times 1} \\ 0_{1\times n} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)} \mid \hat{\Omega}^T = -\hat{\Omega}\}$

$SO_+(1,3)$ - Lorentz Group

- **Physical significance**: All parity-preserving and orientation-preserving Lorentz transformations of 4 dimensional space-time
- $SO_+(1,3) = \{\gamma \begin{bmatrix} R & -v \\ -v & 1 \end{bmatrix} \Big| \gamma = \frac{1}{\sqrt{1-||v||^2}}, R \in SO(3),\ v \in \mathbb{R}^3, \}$
- $\mathfrak{so}_+(1,3) = \{\gamma \begin{bmatrix} \hat{\Omega} & -v \\ -v & 0 \end{bmatrix} \Big| \gamma = \frac{1}{\sqrt{1-||v||^2}}, \hat{\Omega} \in \mathfrak{so}(3),\ v \in \mathbb{R}^3, \}$



$x' = \gamma(x - vt)$

$t' = \gamma(t - vx)$

$\gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\theta$

$R = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$

Figure: Lorentz Transformation in 2-dimensional space-time

Groups from Quadratic Forms

- Let P any $n \times n$ matrix in $\mathbb{R}^n$. This can be used to define a subgroup of $GL(n, \mathbb{R})$ as follows

$$G_M = \{ \; M \in GL(n, \mathbb{R}), \Bigg| \; M^T P M = P \}$$

- This is basically all invertible linear transformations that preserve the quadratic form P - i.e. for all $x \in \mathbb{R}^n$, $x^T P x = (Mx)^T P (Mx)$

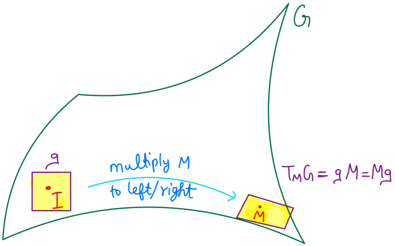
SU(n) - Special Unitary Group

- **Physical Signifiance:** Rotations in a complex vector space $\mathbb{C}^n$
- $SU(n) = \{R \in \mathbb{C}^{n \times n} \mid R^{T*}R = I \, , det(R) = 1\}$
- $dim \; SU(n) = n^2 - 1$
- $\mathfrak{su}(n) = \{\hat{\Omega} \in \mathbb{C}^{n \times n} \mid \hat{\Omega}^T = -\hat{\Omega}^* \, , \; trace(\hat{\Omega}) = 0\}$ - skew Hermitian traceless matrices

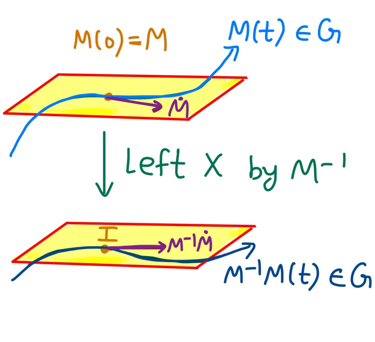
From Lie algebra to other tangent spaces

- In $SO(3)$, we saw that the tangent space at other points is related to the tangent space at identity $\mathfrak{so}(3)$ as $T_R SO(3) = \{R\hat{\Omega} \mid \hat{\Omega} \in \mathfrak{so}(3)\} = \{\hat{\Omega}R \mid \hat{\Omega} \in \mathfrak{so}(3)\}$
- Turns out it holds true in general as well in any matrix Lie group

$$T_M G = \{Mv \mid v \in \mathfrak{g}\} = \{vM \mid v \in \mathfrak{g}\}$$



Proof



- If $M(t) \in G$ with $M(0) = M$, then the curves $M^{-1}M(t)$ and $M(t)M^{-1}$ also are in G
- Moreover, they start at identity - $M^{-1}M(0) = M(0)M^{-1} = I$. So, their velocites $M^{-1}\dot{M} = v_L$ and $\dot{M}M^{-1} = v_R$ are in $\mathfrak{g}$
- Hence $\dot{M} = Mv_L = v_R M$ and the result follows

Left and Right Velocities

Nomenclature

- $M^{-1}\dot{M} \rightarrow$ left velocity of curve $M(t)$ in $\mathfrak{g}$ -
  Analog of body fixed angular velocity $R^{-1}\dot{R}$ of a curve $R(t)$ on $SO(3)$
- $\dot{M}M^{-1} \rightarrow$ right velocity of th curve $M(t)$ in $\mathfrak{g}$ -
  Analog of space fixed angular velocity $\dot{R}R^{-1}$ of a curve $R(t)$ on $SO(3)$

Motion under constant left velocity

- Given a curve $M(t) \in G$, let $v_L(t) = M^{-1}(t)\dot{M}(t)$ be its left velocity in $\mathfrak{g}$.
- If v_L is constant, we have the ODE

$$\dot{M} = Mv_L$$

whose solution is

$$M(t) = M(0)exp(v_L t)$$

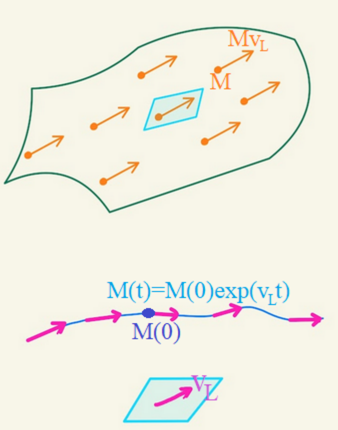


Figure: Exponential evolution under constant left velocity

Adjoint: Mismatch between left and right

- In general, the left velocity $v_L = M^{-1}\dot{M}$ and the right velocity $v_R = \dot{M}M^{-1}$ are not the same unless the group is commutative
- They are related by

$$v_R = \dot{M}M^{-1} = Mv_LM^{-1}$$

  a similarity transformation! - confirms our intuition that v_R is v_L looked in another coordinate system
- The map $v \to MvM^{-1}$ is called the Adjoint of v by M - $Ad_M(v)$

Thinking of ODEs as fluid flows

- Any ODE can be thought of as a vector field - prescribing at each point, a velocity vector that gives the magnitude and direction of the flow at that point
- The solutions to an ODE can be thought of as flows under the vector field - the family of curves got by following the velocity prescribed by the vector field
- **Notation:** $\phi_V(t, x)$ - starting at x and flowing for time t under the vector field V - where will I go?

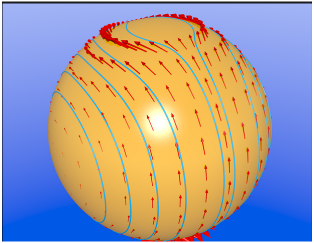


Figure: A vector field (red) and its flow (blue) on $\mathbb{S}^2$

Do flows commute?

- Consider two vector fields V_1, V_2 and these two scenarios
  - first, flow under 1 for time t_1, then flow under 2 for time t_2
    $\phi_2(t_2, .) \circ \phi_1(t_1, .)$
  - first, flow under 2 for time t_2, then flow under 1 for time t_1
    $\phi_1(t_1, .) \circ \phi_2(t_2, .)$

  Will I reach the same point? Mathematically, is

$$\phi_2(t_2, \phi_1(x, t_1)) = \phi_1(t_1, \phi_2(x, t_2))?$$

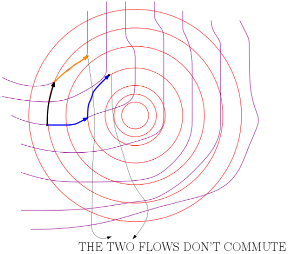


Figure: Flows in general do not commute - the vector fields are updated and re-evaluated at every different points

Recap: Constant Velocity Flows

- The flows on a Lie group G under constant left velocity / right velocity v respectively are given by

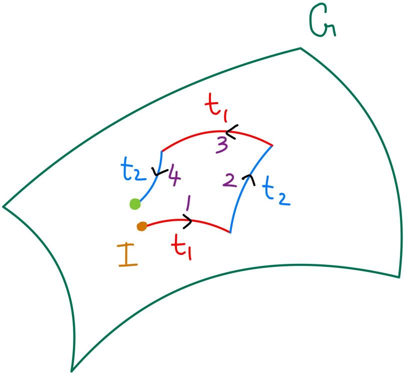
$$\text{Constant left velocity } v\text{: } \phi_v^L(t,x) = x.exp(vt)$$
$$\text{Constant right velocity } v\text{: } \phi_v^R(t,x) = exp(vt).x$$

- **Note:** These curves are not identical unless G is commutative

Commutativity of constant velocity flows

- Let v_1, v_2 be two velocities in $\mathfrak{g}$
- Let $\Phi_1(t_1, x) = x.exp(v_1 t_1)$ and $\Phi_2(t_2, x) = x.exp(v_2 t_2)$ be the flows under constant left velocities v_1 and v_2 respectively
- Do the flows commute?? It is the same as asking if

$$exp(v_1 t_1).exp(v_2 t_2).exp(-v_1 t_1).exp(-v_2 t_2) = I?$$



Alternative Formulation

- The commutativity question can further be reformulated, by asking if

$$exp(v_1 t_1).exp(v_2 t_2).exp(-v_1 t_1) = exp(v_2 t_2)?$$

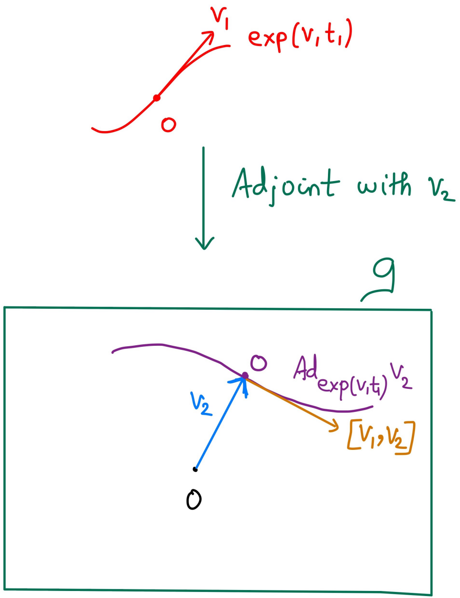
- Since we know that for matrices, $A.B = B.A$ if and only if $e^A.B = B.e^A$, we can formulate the question by asking if

$$exp(v_1 t_1).v_2.exp(-v_1 t_1) = v_2$$

- Consider now, the curve in $\mathfrak{g}$ described by

$$t_1 \to exp(v_1 t_1).v_2.exp(-v_1 t_1) = Ad_{exp(v_1 t_1)} v_2$$

Visualizing the Adjoint Curve



Velocity of the adjoint curve

- Since the adjoint curve $Ad_{exp(v_1 t_1)} v_2$ in $\mathfrak{g}$ initially has value v_2 at time $t_1 = 0$, asking for the curve to remain v_2 for all time t_1 is same as asking its velocity to vanish. So,

$$exp(v_1 t_1).v_2.exp(-v_1 t_1) = v_2 \Leftrightarrow \frac{d}{dt_1}(exp(v_1 t_1).v_2.exp(-v_1 t_1)) = 0$$

$$\begin{aligned} &\frac{d}{dt_1}(exp(v_1 t_1).v_2.exp(-v_1 t_1)) \\ &= exp(v_1 t_1).v_1.v_2.exp(-v_1 t_1) - exp(v_1 t_1).v_2.v_1.exp(-v_1 t_1) \\ &= exp(-v_1 t_1)(v_1.v_2 - v_2.v_1).exp(-v_1 t_1) = 0 \end{aligned}$$

- So, we have that for the flows to commute

$$\overbrace{[v_1, v_2]}^{\in \mathfrak{g}} := v_1.v_2 - v_2.v_1 = 0$$

(the matrices v_1, v_2 must commute!)

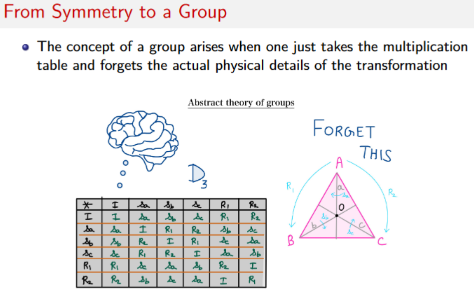
Commutator/Lie Bracket

- The commutator/ Lie bracket of two Lie algebraic vectors v_1, v_2 in $\mathfrak{g}$, denoted by $[v_1, v_2] \in \mathfrak{g}$ as it is the velocity of a curve $Ad_{exp(v_1 t_1)} v_2$ in $\mathfrak{g}$ at time $t_1 = 0$. So, any Lie algebra is closed under commutation -

$$v_1, v_2 \in \mathfrak{g} \Longrightarrow [v_1, v_2] \in \mathfrak{g}$$

- It is easy to verify that the set of traceless matrices, set of skew-symmetric matrices, and the other Lie algebras seen in the examples are closed under commutation

Group Action: From abstract to concrete



- Do you remember this slide when we forgot the action and just remembered the table as a group?
- **Group Action:** Group action, intuitively is, when you put back the action for a group - converting the group elements as transformations (active/passive) of some space which combine exactly as dictated by the group operation

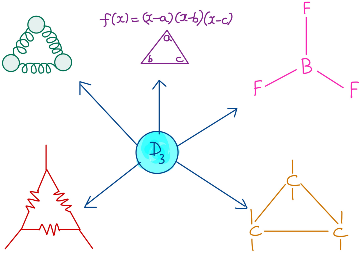
Group Action: Formal Definition

Action of a group G on a set X

A group G is said to act on a set X if there is a transformation of X parametrized by G denoted by $f : G \times X \to X$ such that

- $f(e, x) = x$ - the identity element e corresponds to the identity transformation
- $f(g * h, x) = f(g, f(h, x))$ - composing two transformations is dictated by the group operation

NOTE: One group can have many actions - remember this?



Matrix Lie Group Action

- When a group G is a matrix Lie group in $\mathbb{R}^{n\times n}$, there is a natural action of the group on $\mathbb{R}^n$ given by

$$f(R,x) = R.x$$

- Assume now that a particle in $\mathbb{R}^n$ is subject to action by a group with constant left velocity starting from identity - $R(t) = exp(vt)$ - So,

$$f(R,x) = exp(vt).x$$

- This satisfies the ODE

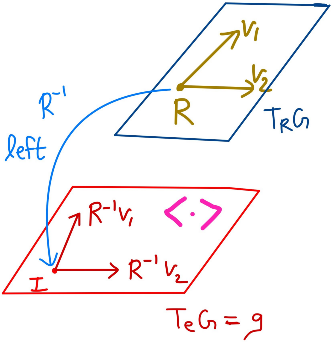
$$\dot{x} = vx$$

- **Hence, each velocity v when maintained constant, in a matrix Lie algebra $\mathfrak{g}$ defines a linear flow on $\mathbb{R}^n$**

Metric in a Lie Group

- Lie groups - smooth manifolds: need a metric on each tangent space to do geometry
- An important such metric - **left invariant metric**. Let $\langle , \rangle_{\mathfrak{g}}$ be an inner product on the Lie algebra $\mathfrak{g}$. Let $v_1, v_2 \in T_R G$. Then, the left-invariant metric $\langle , \rangle_L$ defined as follows:

$$\langle v_1, v_2 \rangle_L := \langle R^{-1}.v_1 \ , \ R^{-1}.v_2 \rangle_{\mathfrak{g}}$$



The Kinetic Energy of a Rigid Body

- Consider a rigid body moving in a trajectory $R(t)$ with left (body-frame) angular velocity $\dot{R} = R(t)\hat{\Omega}(t)$ with moment of inertia tensor $\mathbb{I}$. Let $\vec{\hat{\Omega}}$ be the vector associated with the skew-symmetric body/left angular velocity $\hat{\Omega}$.
- Define an inner product in the Lie algebra as

$$\langle \vec{\hat{\Omega}}_1, \vec{\hat{\Omega}}_2 \rangle := \vec{\hat{\Omega}}_1^T \mathbb{I} \vec{\hat{\Omega}}_2$$

  and define a corresponding left metric on $G = SO(3)$
- Then the kinetic energy of a rigid body is given by

$$K = \vec{\hat{\Omega}}^T \mathbb{I} \vec{\hat{\Omega}} = (R^{-1}\dot{R})^T \mathbb{I} (R^{-1}\dot{R}) = \langle \dot{R}, \dot{R} \rangle_L$$

- So, the kinetic energy of a rigid body defined by the inertia tensor $\mathbb{I}$ is nothing but the norm of the velocity of a rigid body under the induced left-invariant metric!

Acceleration and ∇

- metric $\rightarrow$ identifying neighboring tangent spaces $\rightarrow$ ∇ (covariant differentiation)
- We can express all of the vectors in $\mathfrak{g}$ by left multiplying by R^{-1}.
- Angular Acceleration $\vec{a}$ of a rigid body $= \nabla_{\dot{R}(t)} R(t)\hat{\Omega}(t)$

$$(\nabla_{\dot{R}}\dot{R}) = R\left(\frac{d\vec{\Omega}}{dt} - \mathbb{I}^{-1}(\mathbb{I}\vec{\Omega} \times \vec{\Omega})\right)_{matrix}$$

- Extra red term: quadratic in velocity! - curvature - physically due to non-inertial body-fixed frame! [1]

[1] This gives the Newton-Euler equations of a rotating rigid body!