Analysis of a planar quadrotor

Group 2

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3-DOF Planar Quadrotor Model

System dynamics:

$$m\ddot{x} = -F\sin\theta$$
$$m\ddot{z} = F\cos\theta - mg$$
$$J\ddot{\theta} = M$$

where:

- (x, z) — position coordinates
- θ — attitude angle
- F — total thrust force
- M — control torque

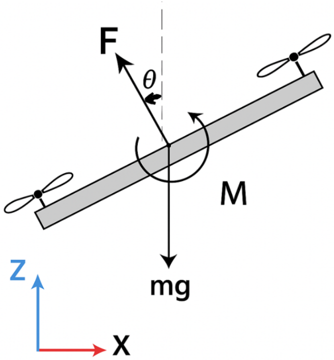


Fig 1 Quadrotor diagram

Key insight: Underactuated system where horizontal motion is achieved indirectly through attitude changes.

State-Space Representation

The 6-dimensional state vector includes the pose of the quadrotor (x, z, θ) and its time derivative $(\dot{x}, \dot{z}, \dot{\theta})$. The control-affine form:

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x \\ z \\ \theta \\ \dot{x} \\ \dot{z} \\ \dot{\theta} \end{pmatrix}, \qquad u = \begin{pmatrix} F \\ M \end{pmatrix}, \qquad f(x) = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ 0 \\ -g \\ 0 \end{pmatrix}, \qquad g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\frac{\sin(x_3)}{m} & 0 \\ \frac{\cos(x_3)}{m} & 0 \\ 0 & \frac{1}{J} \end{pmatrix}$$

Nominal System Analysis

To analyze the stability and existence of a unique solution without a stabilizing controller, we let $u = (mg,\ 0)^T$. The system (1) becomes:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ -g\sin(x_3) \\ g(\cos(x_3) - 1) \\ 0 \end{pmatrix} \tag{2}$$

Nominal System Analysis

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g\cos(x_3) & 0 & 0 & 0 \\ 0 & 0 & -g\sin(x_3) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\frac{\partial f}{\partial x}$ is uniformly bounded on $\mathbb{R}^6$ (using the matrix norm $\left\|\frac{\partial f}{\partial x}\right\|_\infty \leq g$), f is globally Lipschitz. Then, the system (2) has a unique solution over $[t_0, t_1]$, where t_1 can be arbitrarily large.

Nominal System Analysis

Intuitively, we know the system (2) is unstable around any hover equilibrium $x = (d,\ h,\ 0,\ 0,\ 0,\ 0)^T$. Without loss of generality, consider the equilibrium $x = 0$, the solution with initial condition $x(0) = (x_{10},\ x_{20},\ x_{30},\ x_{40},\ x_{50},\ 0)^T$ is:

$$
\begin{aligned}
x_1(t) &= x_{10} + x_{40}t - \frac{g\sin(x_{30})}{2}t^2 \\
x_2(t) &= x_{20} + x_{50}t + \frac{g\big(\cos(x_{30}) - 1\big)}{2}t^2 \\
x_3(t) &= x_{30}
\end{aligned}
\qquad
\begin{aligned}
x_4(t) &= x_{40} - g\sin(x_{30})t \\
x_5(t) &= x_{50} + g\big(\cos(x_{30}) - 1\big)t \\
x_6(t) &= 0
\end{aligned}
$$

Even with $x_{10} = x_{20} = x_{40} = x_{50} = 0$ but $x_{30} \neq 0$, we have $\lim_{t\to\infty} \|x(t)\| = \infty$, i.e., there exists no ball around 0 in which if the system starts, it always stays in another ball.

Feedback Linearization Control

Feedback Linearization Control

Since the position and altitude do not directly influence each other, we break the system (1) into two subsystems (Li 2023): an **altitude** subsystem controlled by thrust F, and a **position** subsystem controlled by torque M. The subsystem dynamics and corresponding control laws are:

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \\ \frac{F\cos(x_3)}{m} - g \end{pmatrix} \qquad F = \frac{m(v_1 + g)}{\cos(x_3)}$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_6 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_6 \\ -\frac{F\sin(x_3)}{m} \\ \frac{M}{J} \end{pmatrix} \qquad M = -\frac{mJ}{F\cos(x_3)}\left(v_2 - \frac{F\sin(x_3)x_6^2}{m}\right) \text{ for } F \neq 0$$

It is clear that after substituting in the control F, the first subsystem becomes a second-order integrator with a new input v_1.

Feedback Linearization Control

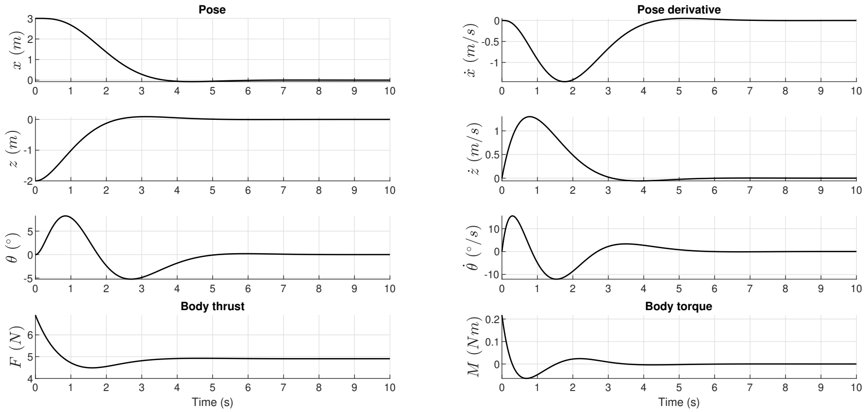
For the second subsystem, we need to define a diffeomorphic coordinate transformation:

$$z = T(x) = \begin{pmatrix} x_1 \\ x_4 \\ -\frac{F\sin(x_3)}{m} \\ -\frac{F\cos(x_3)x_6}{m} \end{pmatrix} \rightarrow \dot{z} = \begin{pmatrix} x_4 \\ -\frac{F\sin(x_3)}{m} \\ -\frac{F\cos(x_3)x_6}{m} \\ \frac{F\sin(x_3)x_6^2 - F\cos(x_3)M/J}{m} \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ v_2 \end{pmatrix}$$

This feedback linearization control scheme and diffeomorphic coordinate transformation is only valid on $D = \{x \in \mathbb{R}^6 \mid |x_3| < \frac{\pi}{2}\}$. The new control inputs v_1, v_2 can be designed on the feedback linearized systems using conventional linear system techniques (e.g., pole placement).

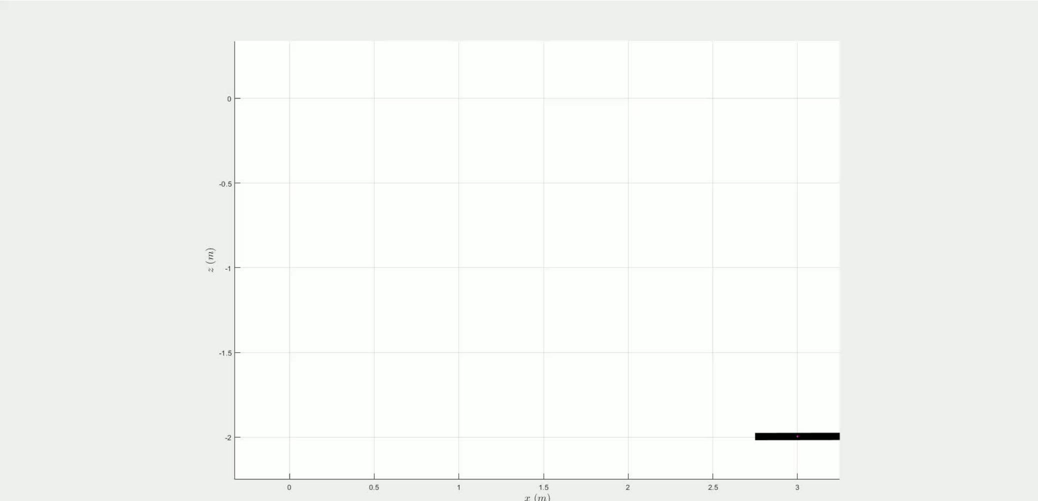
Feedback Linearization Control

MATLAB simulation showing stabilization to the origin from initial state $x(0) = (3,\ -2,\ 0,\ 0,\ 0,\ 0)^T$:



Feedback Linearization Control

MATLAB simulation showing stabilization to the origin from initial state $x(0) = (3,\ -2,\ 0,\ 0,\ 0,\ 0)^T$:



Feedback Linearization Control - Stability Analysis

The first subsystem requires no coordinate transformation and is straightforward. We concentrate on the second subsystem, its feedback linearization is:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ v_2 \end{pmatrix} \text{ where } z = T(x) = \begin{pmatrix} x_1 \\ x_4 \\ -\frac{F\sin(x_3)}{m} \\ -\frac{F\cos(x_3)x_6}{m} \end{pmatrix}$$

which is a fourth-order integrator. The new control input v_2 can be designed such that the closed-loop system is asymptotically stable (make A_{cl} Hurwitz). Then, there exists a $P \succ 0$ satisfying $PA_{cl} + A_{cl}^T P = -I$, such that $V(z) = z^T Pz$ and $\dot{V}(z) = -\|z\|^2$, or in the original coordinate: $V(x) = T(x)^T PT(x)$ positive definite and $\dot{V}(x) = -\|T(x)\|^2$ negative definite. Then, 0 is a **asymptotically stable** equilibrium for the second subsystem.

Feedback Linearization Control - Stability Analysis

- The region of attraction (RoA) can be estimated by the sublevel set

$$\Omega_{cz} = \{z \in \mathbb{R}^4 \mid z^T P z \leq c_F\},$$

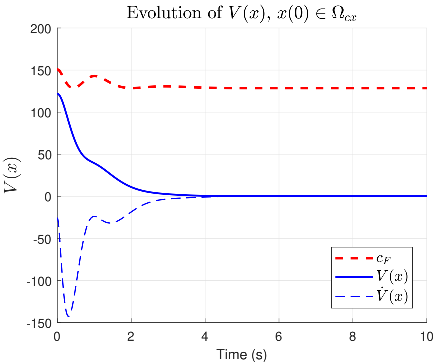
where c_F is determined so that its preimage $\Omega_{cx} := \{x \in \mathbb{R}^4 \mid z^T P z = T(x)^T P T(x) \leq c_F\}$ lies entirely in $D = \{x \in \mathbb{R}^4 \mid -\frac{\pi}{2} < x_3 < \frac{\pi}{2}\}$ (where $\dot{V}(x) \leq 0$, and the feedback linearization scheme is valid).

- To find c_F, we consider the boundary of D where $x_3 = \pm\frac{\pi}{2}$, which corresponds to $z_3 = \pm\frac{F}{m}$, $z_4 = 0$. Then, we find the smallest level set that intersects the boundary of D,

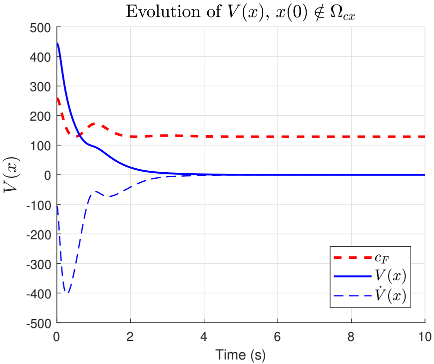
$$\bar{c}_F = \min_{z_3 = \pm\frac{F}{m}, z_4 = 0} z^T P z,$$

and c_F can be chosen to be slightly smaller than $\bar{c}_F$.

This estimate of RoA is conservative: it guarantees that trajectories starting in Ω_{cx} will converge to 0, but does not guarantee that trajectories outside Ω_{cx} will not converge.



(a) $x(0) = (3,\ 0,\ -\pi/8,\ 0,\ 0,\ 0)^T \in \Omega_{cx}$



(b) $x(0) = (3,\ 0,\ -\pi/4,\ 0,\ 0,\ 0)^T \notin \Omega_{cx}$

Feedback Linearization Control - Ultimate Bound

For the first subsystem:

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \\ \frac{F\cos(x_3)}{m} - g \end{pmatrix}, \qquad F = \frac{m(v_1 + g)}{\cos(x_3)}$$

Suppose we design $v_1 = -2x_2 - 2x_5$, and use the Lyapunov function $V(x) = \begin{pmatrix} x_2 & x_5 \end{pmatrix} \begin{pmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{pmatrix} \begin{pmatrix} x_2 \\ x_5 \end{pmatrix}$, then $\dot{V}(x) = -\left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\|^2$. Now, suppose F is under a disturbance, $F = \frac{m(v_1+g)}{\cos(x_3)} + A\sin(\omega t)$, $A > 0$, the closed-loop system becomes:

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} x_5 \\ -2x_2 - 2x_5 + \frac{A\cos(x_3)\sin(\omega t)}{m} \end{pmatrix}$$

Feedback Linearization Control - Ultimate Bound

Using the same $V(x)$, compute its derivative along the system trajectories:

$$\begin{aligned}
\dot{V}(t,x) &= -\left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\|^2 + \frac{A\cos(x_3)\sin(\omega t)}{m}\left(\frac{1}{2}x_2 + \frac{3}{4}x_5\right) \\
&\leq -\left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\|^2 + \frac{A}{m}\left(\sqrt{\frac{13}{16}}\left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\|\right) \\
&= -(1-\eta)\left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\|^2 - \left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\|\left(\eta\left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\| - \sqrt{\frac{13}{16}}\frac{A}{m}\right), \qquad \eta \in (0,1) \\
&\leq -(1-\eta)\left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\|^2 \quad \text{for all } \left\| \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \right\| \geq \sqrt{\frac{13}{16}}\frac{A}{\eta m} := \mu
\end{aligned}$$

Feedback Linearization Control - Ultimate Bound

Since $P = \begin{pmatrix} 5/4 & 1/4 \\ 1/4 & 3/8 \end{pmatrix}$ is positive definite, we find 2 class $\mathcal{K}$ functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1\left(\left\|\begin{pmatrix} x_2 \\ x_5 \end{pmatrix}\right\|\right) = \lambda_{min}(P)\left\|\begin{pmatrix} x_2 \\ x_5 \end{pmatrix}\right\|^2 \leq V(x) \leq \lambda_{max}(P)\left\|\begin{pmatrix} x_2 \\ x_5 \end{pmatrix}\right\|^2 = \alpha_2\left(\left\|\begin{pmatrix} x_2 \\ x_5 \end{pmatrix}\right\|\right)$$

So, the solution satisfies:

$$\left\|\begin{pmatrix} x_2(t) \\ x_5(t) \end{pmatrix}\right\| \leq \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{max}(P)\mu^2}{\lambda_{min}(P)}}$$

where $\lambda_{max}(P) = \frac{13+\sqrt{65}}{16}$, $\lambda_{min}(P) = \frac{13-\sqrt{65}}{16}$, and $\mu = \sqrt{\frac{13}{16}\frac{A}{\eta m}}$, $\eta \in (0,1)$.

References

- Li, Y.-R., Chen, C.-C., & Peng, C.-C. (2023). Integral Backstepping Control Algorithm for a Quadrotor Positioning Flight Task: A Design Issue Discussion. *Algorithms*, 16(2), 122. https://doi.org/10.3390/algorithms16020122
- Khalil, H. K. (2002). *Nonlinear Systems* (3rd ed.). Prentice Hall.

Thank you for listening