Dot products

in Overleaf:
\mathbb{R}

$\mathbb{R}$: real numbers

Vectors

n: there are n real numbers in our vector.

- A **vector** in $\mathbb{R}^n$ is an **ordered collection of n numbers**.
- We use lower-case letters with an arrow on top to represent vectors, and we usually write vectors as **columns**.

$$\vec{v} = \begin{bmatrix} 8 \\ 3 \\ -2 \\ 5 \end{bmatrix}$$

transpose

- Another way of writing the above vector is $\vec{v} = [8, 3, -2, 5]^T$.
- Since $\vec{v}$ has four **components**, we say $\vec{v} \in \mathbb{R}^4$.

"elements"

"in"

The geometric interpretation of a vector

$\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

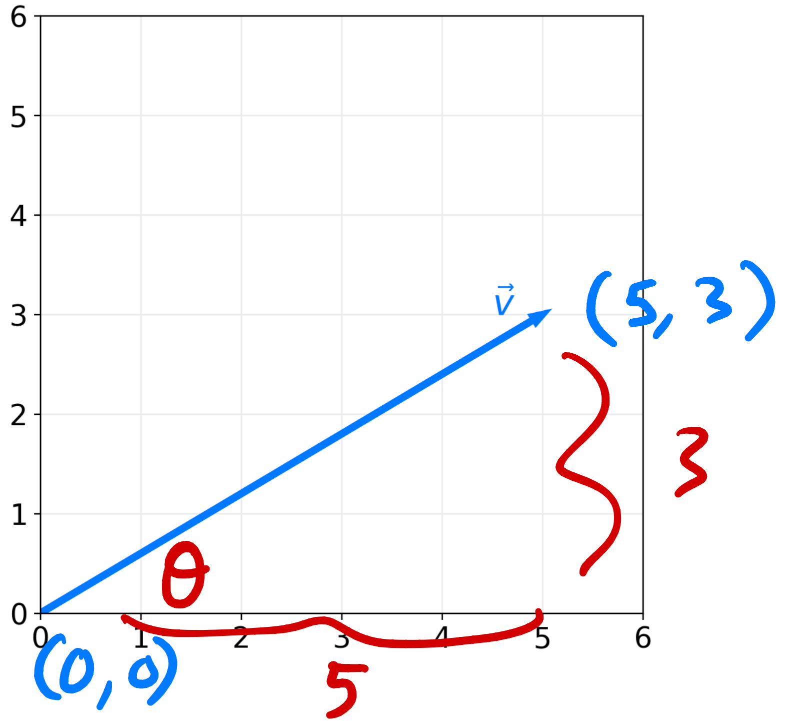
$\|\vec{v}\| = \sqrt{5^2 + 3^2} = \sqrt{34}$

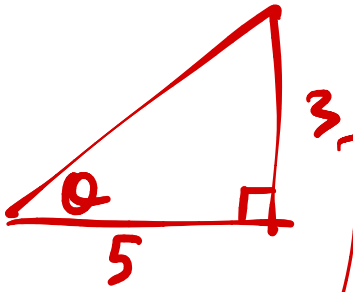
- A vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is an arrow to the point $(v_1, v_2, \ldots, v_n)$ from the origin.
- The **length**, or L_2 **norm**, of $\vec{v}$ is:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

multi-dimensional Pythagorean theorem

- A vector is sometimes described as an object with a **magnitude/length** and **direction**.





$\tan\theta = \frac{3}{5}$

$\boxed{\theta = \tan^{-1}\left(\frac{3}{5}\right)}$

both have the same number of elements

$\Rightarrow$ the same dimension

Dot product: coordinate definition

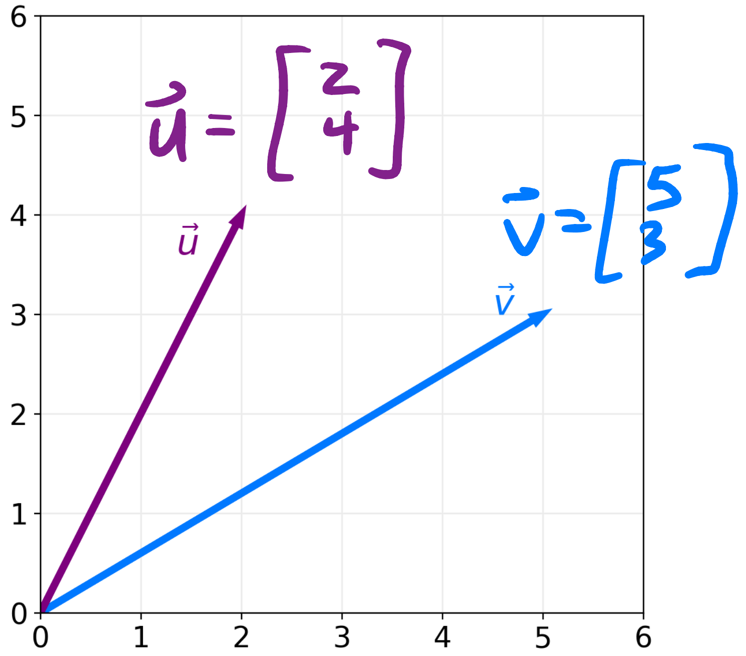
- The **dot product** of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is written as:

$$\vec{u} \cdot \vec{v} = \vec{u}^{\mathsf{T}} \vec{v}$$

- The computational definition of the dot product:

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The result is a **scalar**, i.e. a single number.



$$\vec{u} \cdot \vec{v} = (2)(5) + (4)(3) = 10 + 12 = \boxed{22}$$

scalar! just one number!!!

$$\vec{u}^{T}\vec{v} = \begin{bmatrix} 2 & 4 \end{bmatrix}_{1\times 2} \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{2\times 1} = 22$$

no dot!

match!

Question 🤔

Answer at q.dsc40a.com

Which of these is another expression for the length of $\vec{v}$?

- A. $\vec{v} \cdot \vec{v}$
- B. $\cancel{\sqrt{\vec{v}^2}}$
- C. $\sqrt{\vec{v} \cdot \vec{v}}$
- D. $\cancel{\vec{v}^2}$
- E. More than one of the above.

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$\vec{v}^2$ is undefined!

$\vec{v}_{n \times 1} \quad \vec{v}_{n \times 1}$

inner dimensions must match! but, $1 \neq n$.

$$\sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = \|\vec{v}\|$$

$\sqrt{20}\sqrt{34}\cos\theta = 22 \Rightarrow \cos\theta = \dfrac{22}{\sqrt{20}\sqrt{34}}$

$\cos\theta = \dfrac{\cancel{22}^{\,11}}{\cancel{2}\sqrt{5}\sqrt{34}}$

Dot product: geometric definition

- The computational definition of the dot product:

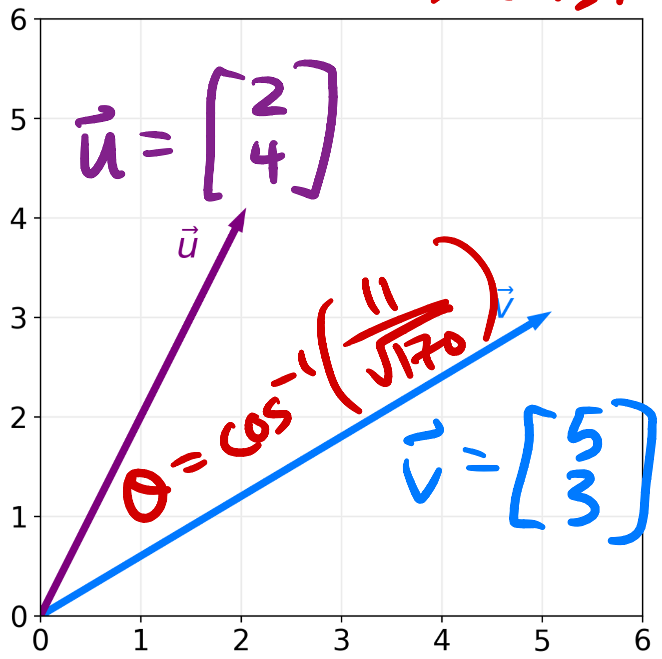
$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

- The geometric definition of the dot product:

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

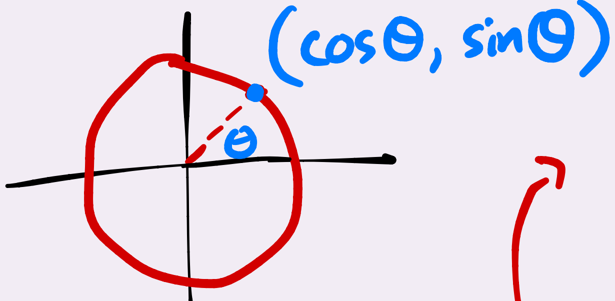
- The two definitions are equivalent! This equivalence allows us to find the angle θ between two vectors.



equal!

$\vec{u}\cdot\vec{v} = (2)(5) + (4)(3) = 22$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta = \sqrt{2^2+4^2}\sqrt{5^2+3^2}\cos\theta = \sqrt{20}\sqrt{34}\cos\theta$



Question 🤔

Answer at q.dsc40a.com

What is the value of θ in the plot to the right?

$\vec{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

① $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$= 4(-2) + 1(-4) = -12$

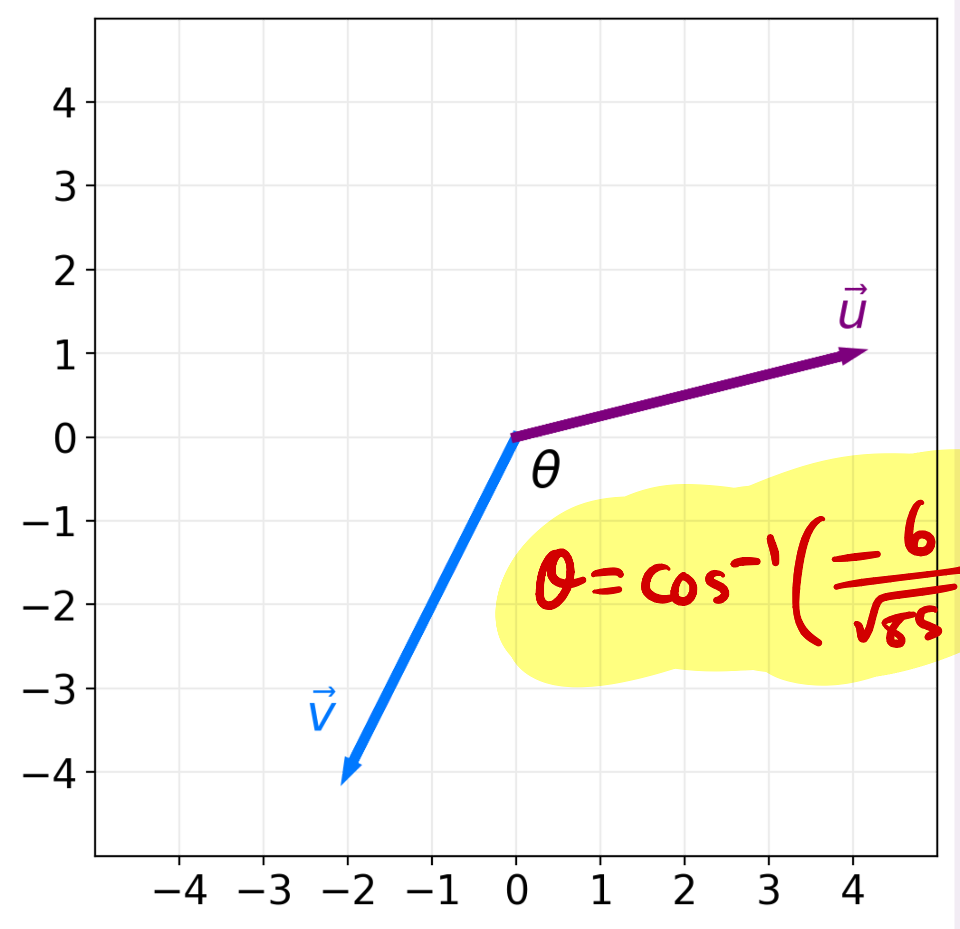
② $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

$= \sqrt{4^2+1^2}\sqrt{(-2)^2+(-4)^2} \cos\theta$

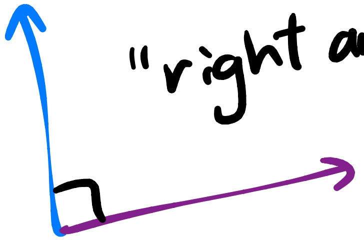
$= \sqrt{17}\sqrt{20}\cos\theta$

$\sqrt{17}\sqrt{20}\cos\theta = -12$

$\Rightarrow \cos\theta = \dfrac{-12}{\sqrt{17}\sqrt{20}} = \dfrac{-6}{\sqrt{17}\sqrt{5}} = \dfrac{-6}{\sqrt{85}}$



$\theta = \cos^{-1}\left(\dfrac{-6}{\sqrt{85}}\right)$



"right angle" "perpendicular"

Orthogonal vectors

- Recall: $\cos 90^{\circ} = 0$.
- Since $\vec{u} \cdot \vec{v} = ||\vec{u}||\,||\vec{v}|| \cos\theta$, if the angle between two vectors is 90°, their dot product is $||\vec{u}||\,||\vec{v}|| \cos 90^{\circ} = 0$.
- If the angle between two vectors is 90°, we say they are perpendicular, or more generally, **orthogonal**.
- **Key idea**:

$$\boxed{\text{two vectors are } \textbf{orthogonal} \iff \vec{u} \cdot \vec{v} = 0}$$

"if and only if"
bidirectional statement

Exercise

Find a non-zero vector in $\mathbb{R}^3$ orthogonal to:

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \\ -8 \end{bmatrix}$$

could find solutions to

$2u_1 + 5u_2 - 8u_3 = 0$

Infinitely many possibilities!

$\begin{bmatrix} 2 \\ 12 \\ 8 \end{bmatrix}$: $(2)(2) + (12)(5) + (8)(-8)$

$= 4 + 60 - 64$

$= 0$

$\begin{bmatrix} 0 \\ 8 \\ 5 \end{bmatrix}$: $(0)(2) + (8)(5) + (5)(-8)$

$= 40 - 40$

$= 0$

Spans and projections

$$\vec{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \Rightarrow -2\vec{u} = \begin{bmatrix} -4 \\ -8 \end{bmatrix}$$

Adding and scaling vectors

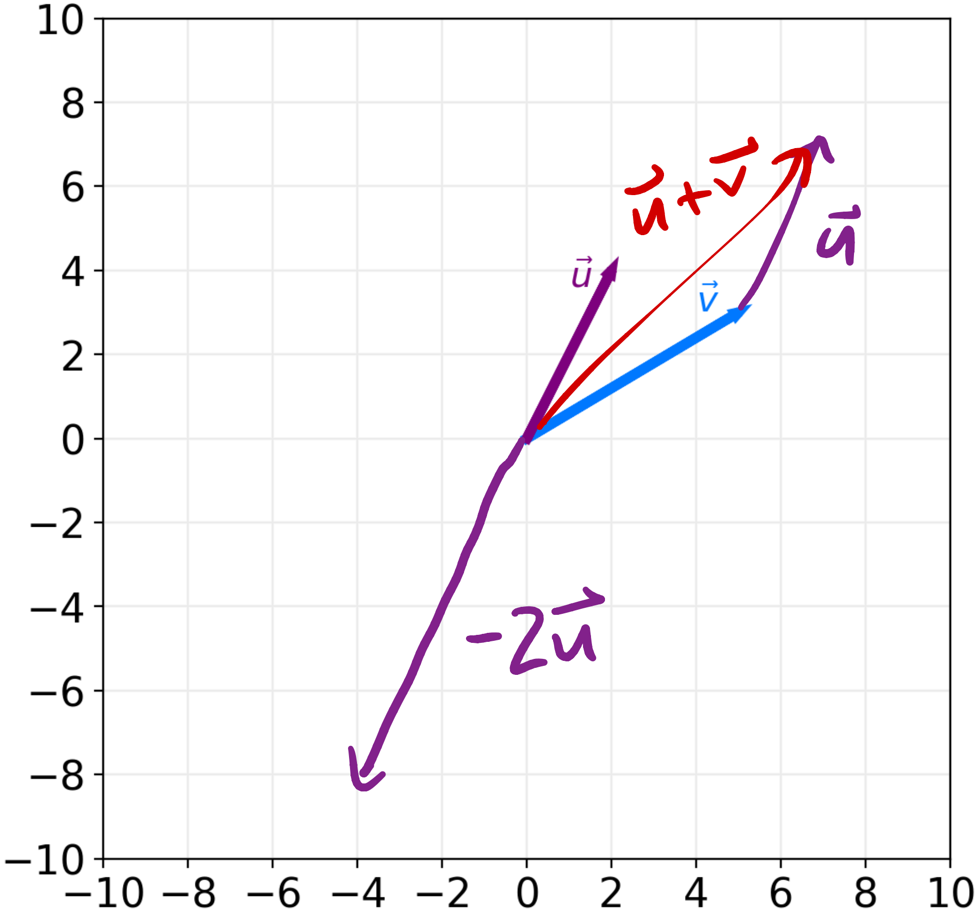
- The sum of two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the **element-wise sum** of their components:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

also a vector!

- If c is a scalar, then:

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$



d vectors,
each has n components

Linear combinations

- Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ all be vectors in $\mathbb{R}^n$.
- A **linear combination** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ is any vector of the form:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d$$

where $a_1, a_2, \ldots, a_d$ are all scalars.

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$$

Examples

$$2\vec{v}_1 + \vec{v}_2 + \frac{1}{9}\vec{v}_3 = \begin{bmatrix} \_ \\ \_ \end{bmatrix}$$ a vector in $\mathbb{R}^2$!

$$0\vec{v}_1 + \vec{v}_2 - \vec{v}_3$$

$\vdots$

Span

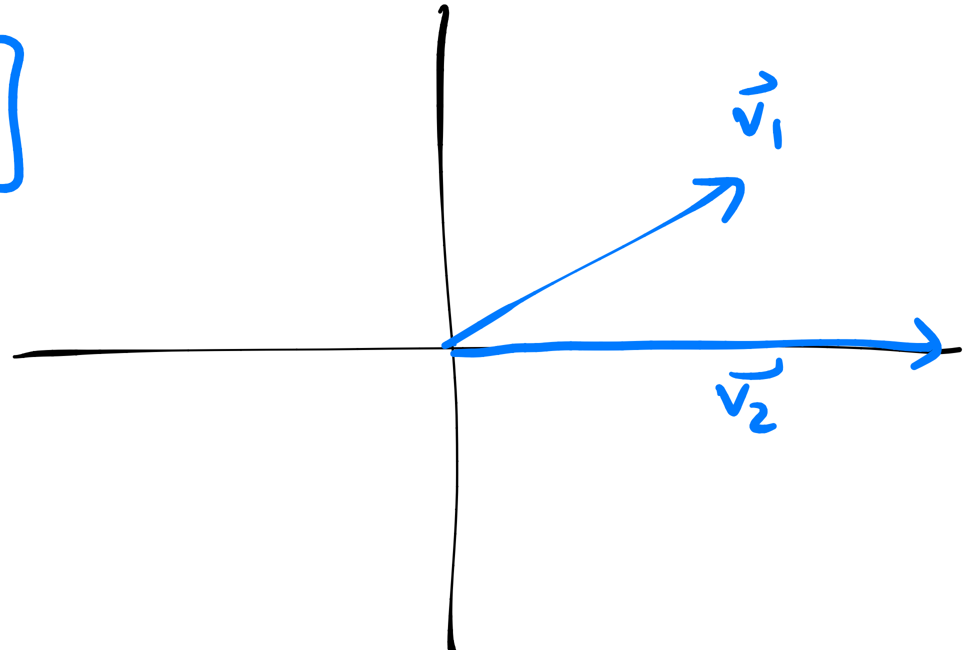
- Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ all be vectors in $\mathbb{R}^n$.
- The **span** of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ is the set of all vectors that can be created using linear combinations of those vectors.
- Formal definition:

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d) = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + a_d\vec{v}_d : a_1, a_2, \ldots, a_d \in \mathbb{R}\}$$

Example

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

$\vec{v}_1$ and $\vec{v}_2$ span all of $\mathbb{R}^2$!



We can! $\vec{v}_1$ and $\vec{v}_2$ aren't scalar multiples of each other: they point in diff. directions

Exercise

Let $\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and let $\vec{v}_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Is $\vec{y} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2)$?

If so, write $\vec{y}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

$$w_1 \vec{v}_1 + w_2 \vec{v}_2 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2w_1 \\ -3w_1 \end{bmatrix} + \begin{bmatrix} -w_2 \\ 4w_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} 2w_1 - w_2 = 9 \\ -3w_1 + 4w_2 = 1 \end{array} \longrightarrow \text{solve for } w_1, w_2.$$