Lecture 12

Loss Functions and Simple Linear Regression

EECS 398: Practical Data Science, Winter 2025

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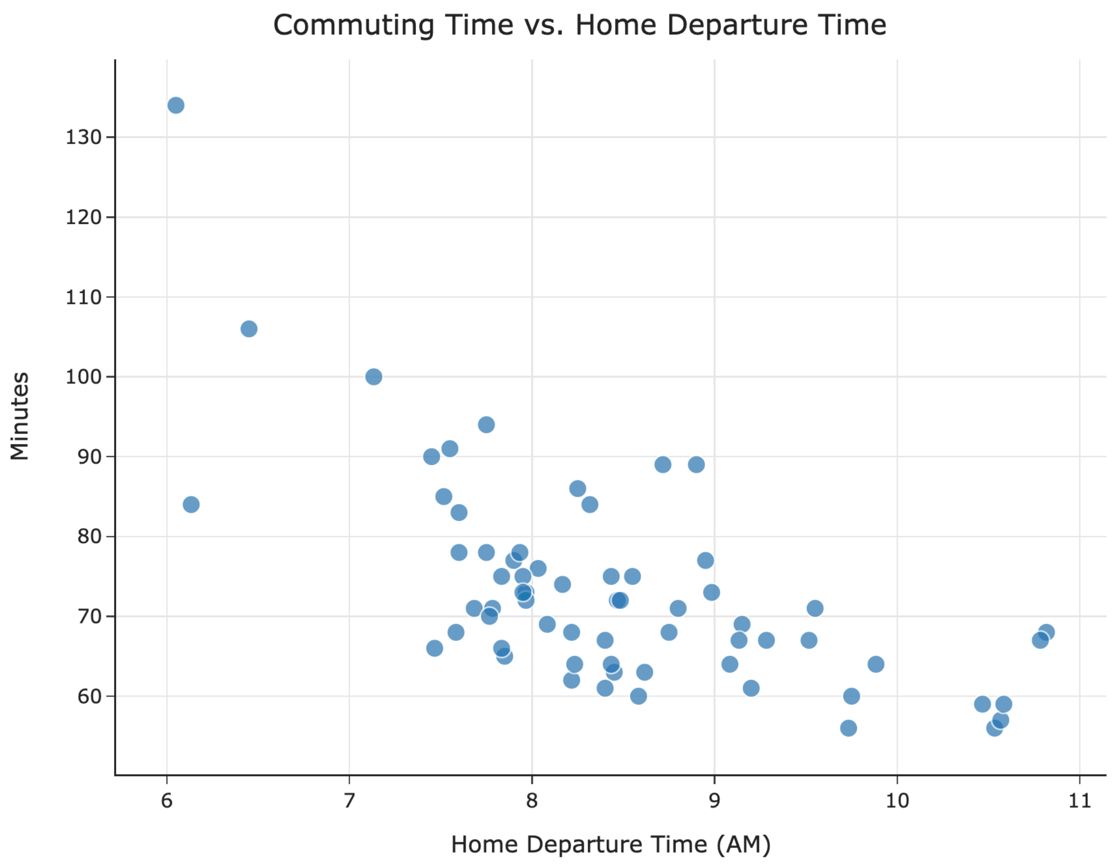
Agenda 📆

- Recap: Models and loss functions.
- Another loss function.
- Towards simple linear regression.
- Minimizing mean squared error for the simple linear model.
- Correlation.
- Interpreting the formulas.

There are several important videos for Lectures 11 and 12; they are all in this YouTube playlist.

Recap: Models and loss functions

Overview



- We started by introducing the idea of a hypothesis function, $H(x_i)$.
- We looked at two possible models:
  - The constant model, $H(x_i) = h$.
  - The simple linear regression model, $H(x_i) = w_0 + w_1 x_i$.
- We decided to find the **best constant prediction** to use for predicting commute times, in minutes.

Recap: Mean squared error

- Let's suppose we have just a smaller dataset of just five historical commute times in minutes.

$$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 92$$

- The **mean squared error** of the constant prediction h is:

$$R_{\text{sq}}(h) = \frac{1}{5}\left((72 - h)^2 + (90 - h)^2 + (61 - h)^2 + (85 - h)^2 + (92 - h)^2\right)$$

- For example, if we predict $h = 100$, then:

$$\begin{aligned} R_{\text{sq}}(100) &= \frac{1}{5}\left((72 - 100)^2 + (90 - 100)^2 + (61 - 100)^2 + (85 - 100)^2 + (92 - 100)^2\right) \\ &= \boxed{538.8} \end{aligned}$$

- **We can pick any h as a prediction, but the smaller $R_{\text{sq}}(h)$ is, the better h is!**

The mean minimizes mean squared error!

- The problem we set out to solve was, find the h^* that minimizes:

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2$$

- The answer is:

$$h^* = \text{Mean}(y_1, y_2, \ldots, y_n)$$

- The **best constant prediction**, in terms of mean squared error, is always the **mean**.
- We call h^* our **optimal model parameter**, for when we use:
  - the constant model, $H(x_i) = h$, and
  - the squared loss function, $L_{\text{sq}}(y_i, h) = (y_i - h)^2$.
- Review the derivation steps from Lecture 11's slides, and watch the video we posted.

The modeling recipe

- We've implicitly introduced a three-step process for finding optimal model parameters (like h^*) that we can use for making predictions:

    1. Choose a model.

    2. Choose a loss function.

    3. Minimize average loss to find optimal model parameters.

- Most modern machine learning methods today, including neural networks, follow this recipe, and we'll see it repeatedly this semester!

Question 🤔

Answer at practicaldsc.org/q

What questions do you have?

Another loss function

Another loss function

- We started by computing the **error** for each of our predictions, but ran into the issue that some errors were positive and some were negative.

$$e_i = y_i - H(x_i)$$

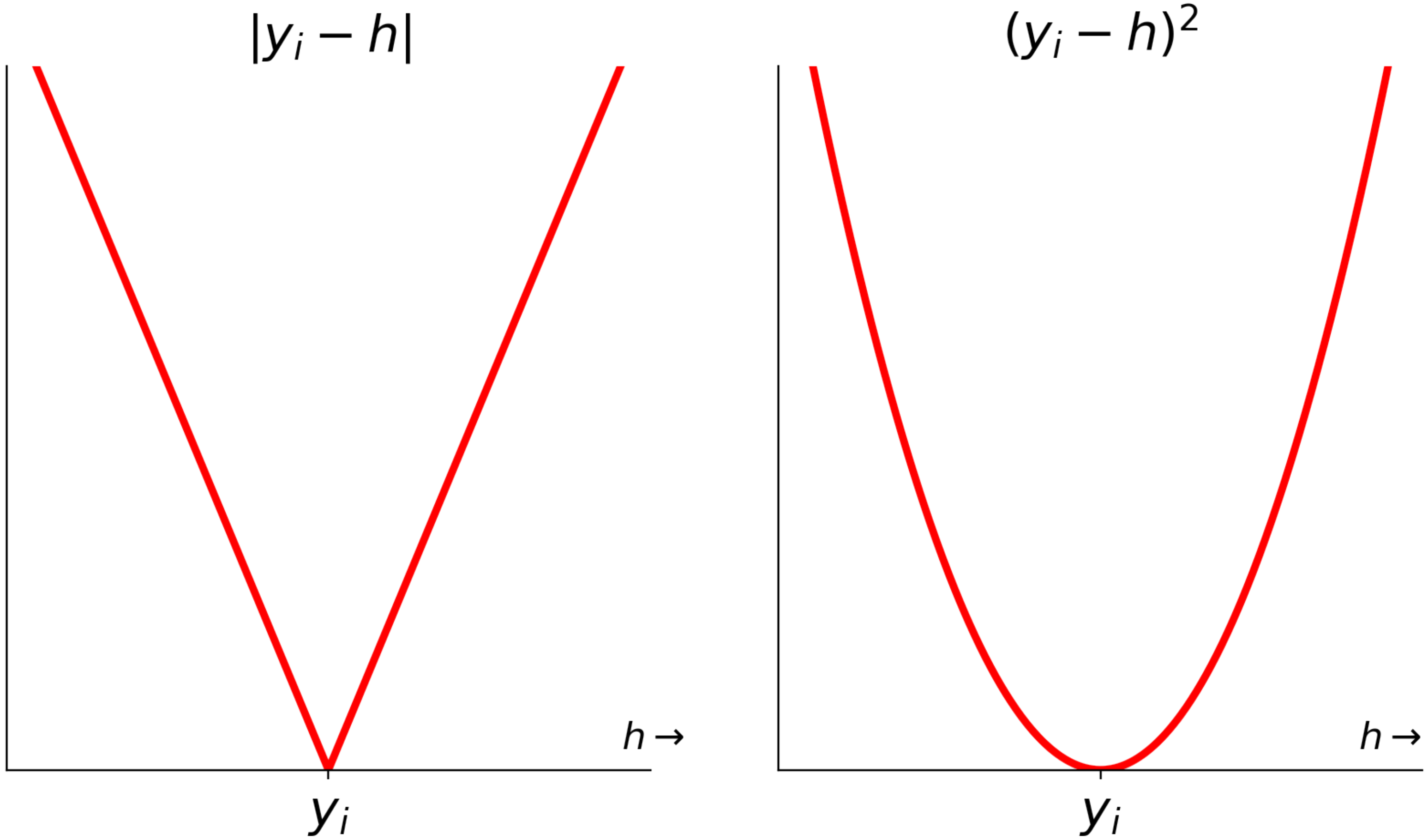
- The solution was to **square** the errors, so that all are non-negative. The resulting loss function is called **squared loss**.

$$L_{\text{sq}}(y_i, H(x_i)) = (y_i - H(x_i))^2$$

- Another loss function, which also measures how far $H(x_i)$ is from y_i, is **absolute loss**.

$$L_{\text{abs}}(y_i, H(x_i)) = |y_i - H(x_i)|$$

Absolute loss vs. squared loss



Mean absolute error

- Suppose we collect n commute times, $y_1, y_2, \ldots, y_n$.
- The **average absolute loss**, or **mean absolute error (MAE)**, of the prediction h is:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

- We'd like to find the best constant prediction, h^*, by finding the h that minimizes **mean absolute error** (a new objective function).
- Any guesses?

The median minimizes mean absolute error!

- It turns out that the constant prediction h^* that minimizes mean absolute error,

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

  is:

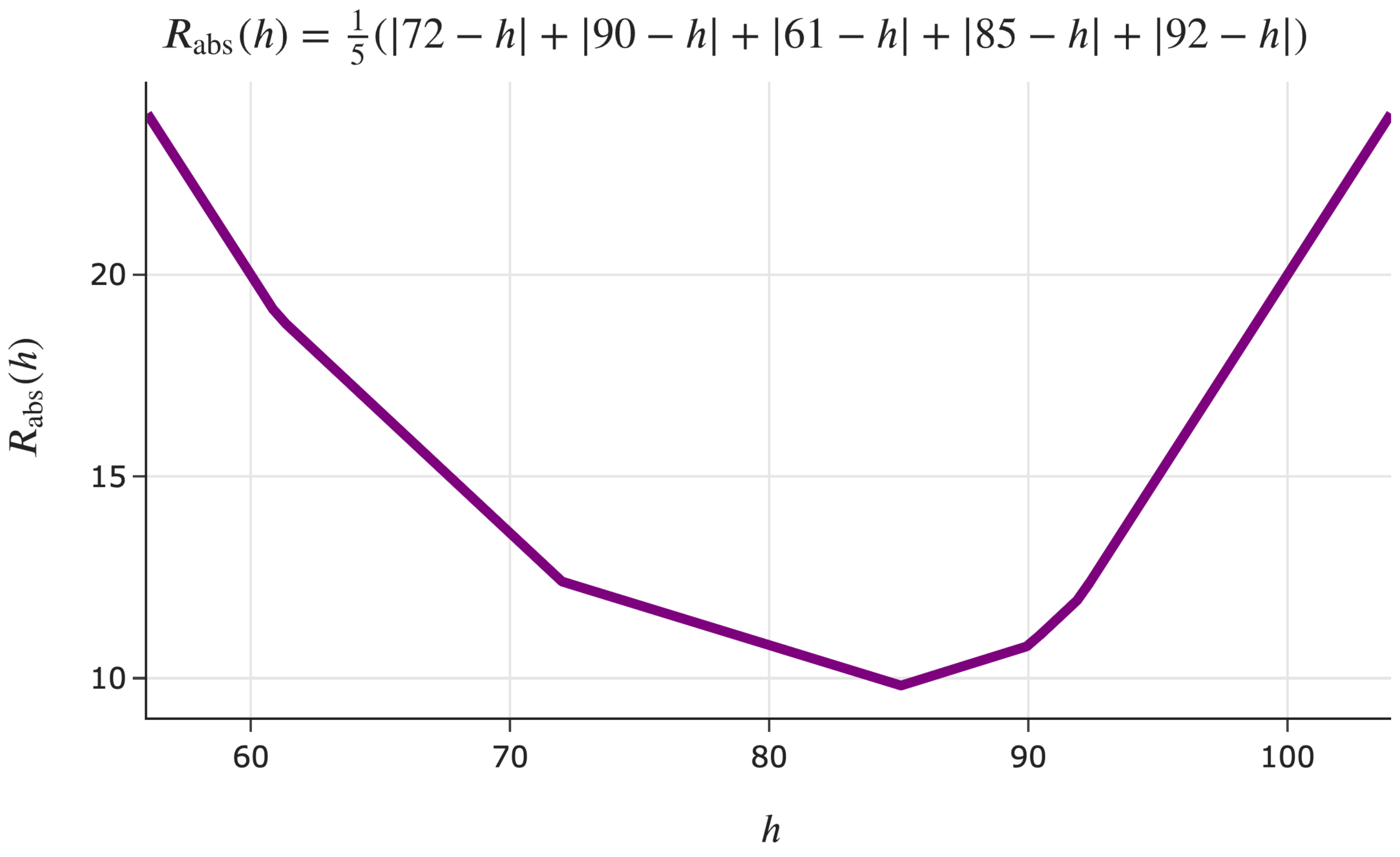
$$h^* = \text{Median}(y_1, y_2, \ldots, y_n)$$

- We won't prove this in lecture, but this extra video walks through it.

  Watch it!

- To make a bit more sense of this result, let's graph $R_{\text{abs}}(h)$.

Visualizing mean absolute error

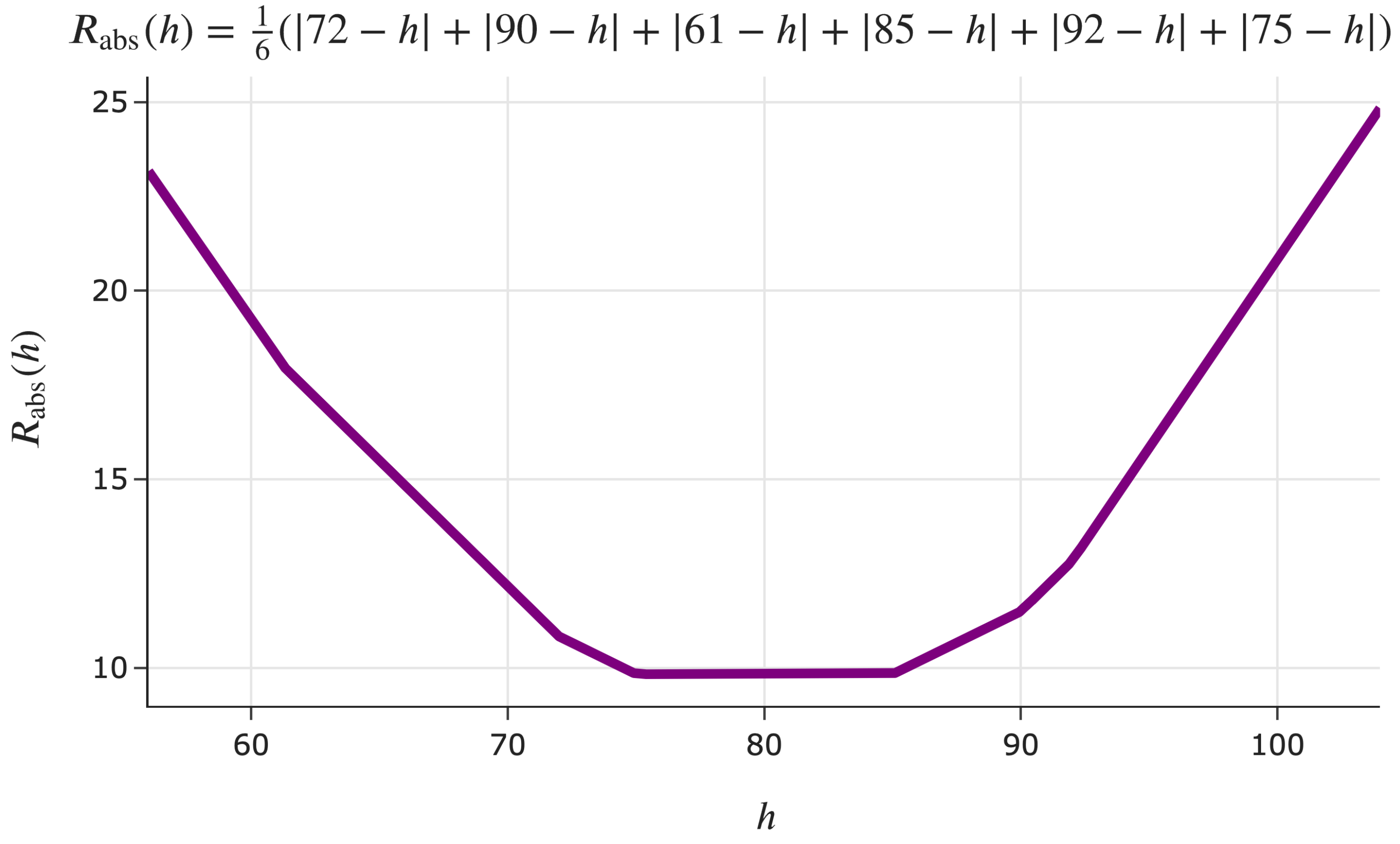


- Consider, again, our example dataset of five commute times.

$$72, 90, 61, 85, 92$$

- Where are the "bends" in the graph of $R_{\text{abs}}(h)$ – that is, where does its slope change?

Visualizing mean absolute error, with an even number of points



- What if we add a sixth data point?

$$72, 90, 61, 85, 92, 75$$

- Is there a unique h^*?

The median minimizes mean absolute error!

- The new problem we set out to solve was, find the h^* that minimizes:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h|$$

- The answer is:

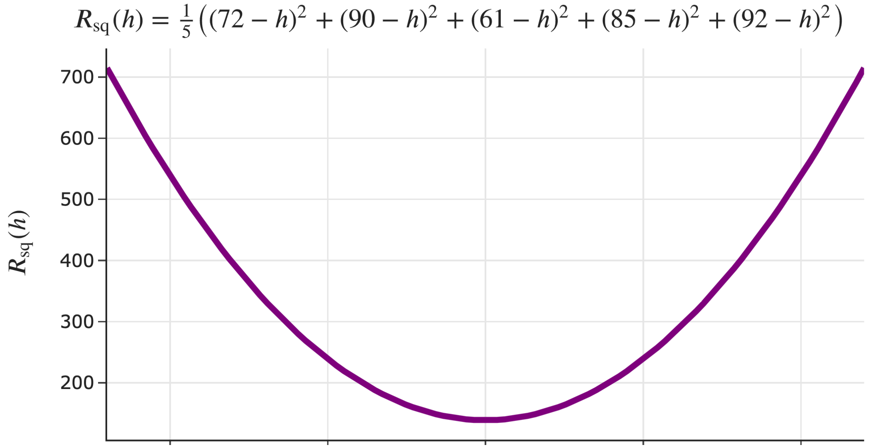
$$h^* = \text{Median}(y_1, y_2, \ldots, y_n)$$

- The **best constant prediction**, in terms of mean absolute error, is always the **median**.
  - When n is odd, this answer is unique.
  - When n is even, any number between the middle two data points (when sorted) also minimizes mean absolute error.
  - When n is even, define the median to be the mean of the middle two data points.

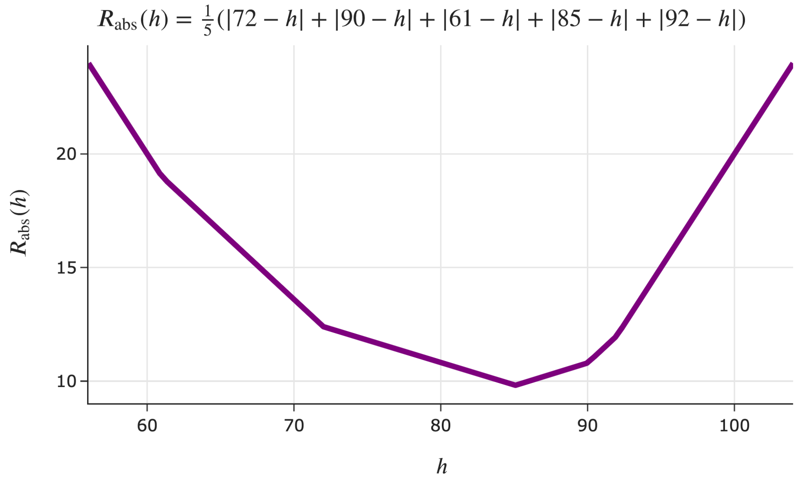
Choosing a loss function

- For the constant model $H(x_i) = h$, the **mean** minimizes mean **squared** error.
- For the constant model $H(x_i) = h$, the **median** minimizes mean **absolute** error.
- In practice, squared loss is the more common choice, as the resulting objective function is more easily **differentiable**.

Mean squared error



Mean absolute error



- But how does our choice of loss function impact the resulting optimal prediction?

Comparing the mean and median

- Consider our example dataset of 5 commute times.

$$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 92$$

- As of now, the **median is 85** and the **mean is 80**.
- What if we add 200 to the largest commute time, 92?

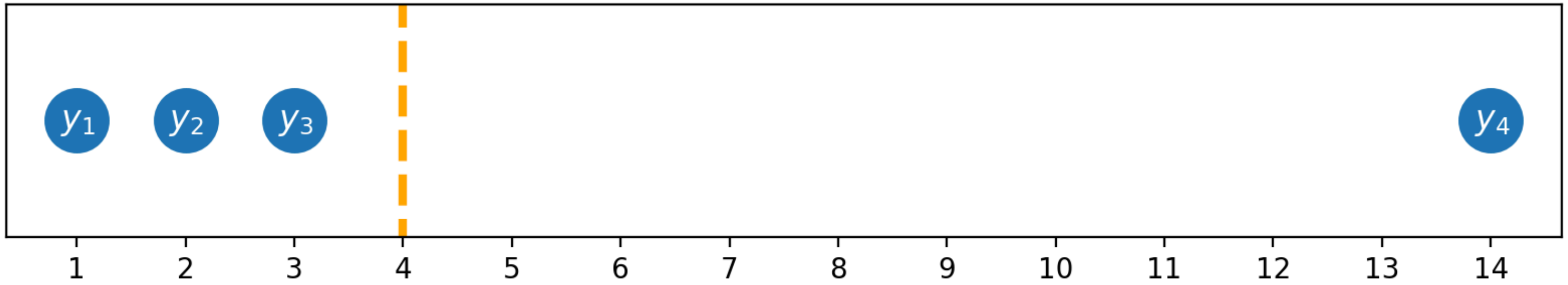
$$y_1 = 72 \qquad y_2 = 90 \qquad y_3 = 61 \qquad y_4 = 85 \qquad y_5 = 292$$

- Now, the median is but the mean is !
- **Key idea**: The mean is quite **sensitive** to outliers.

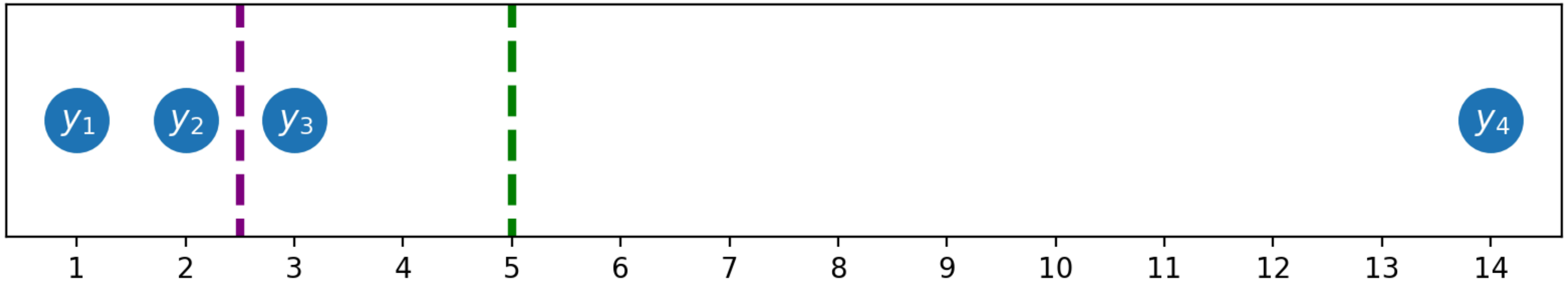
  But why?

Outliers

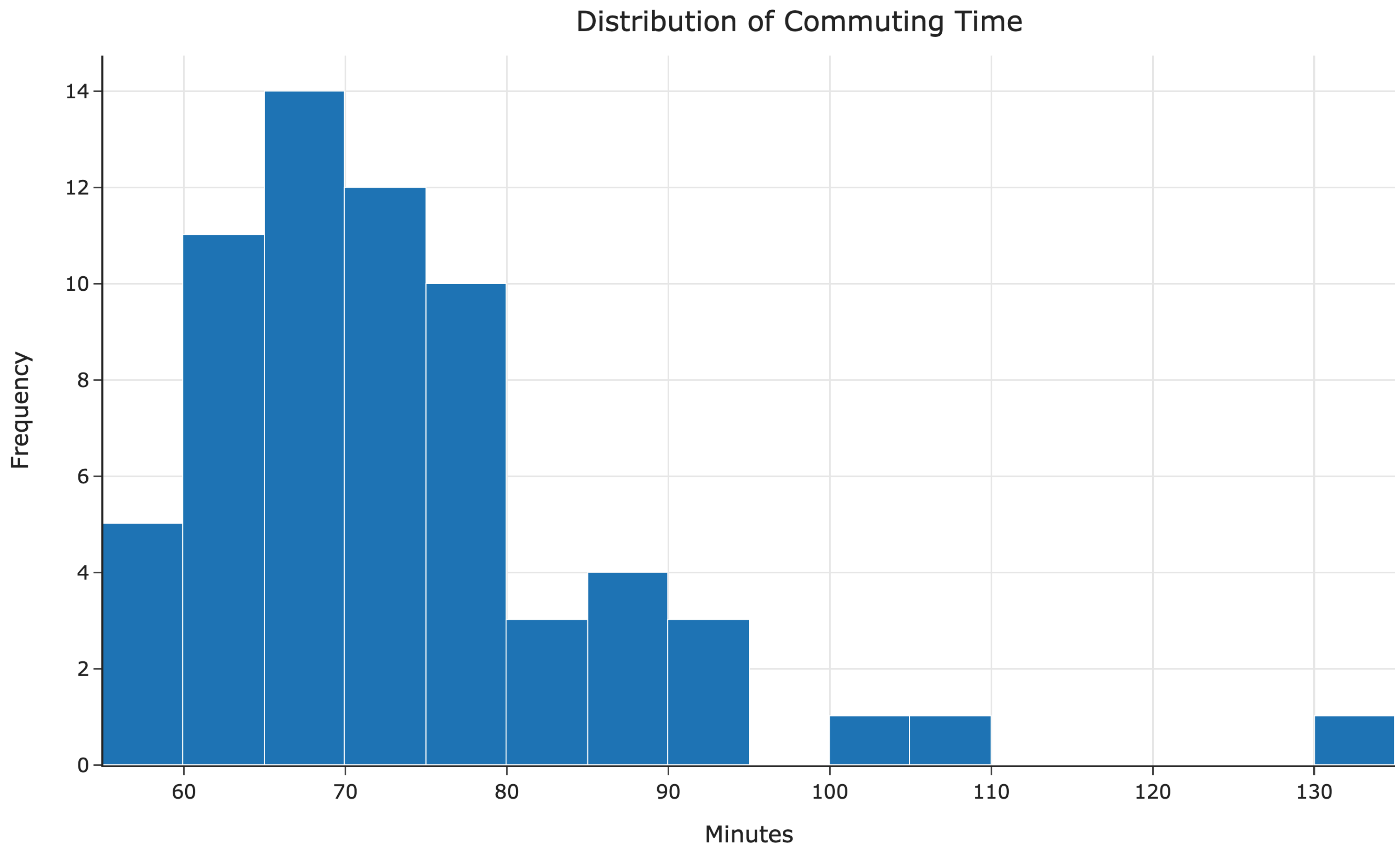
- Below, $|y_4 - h|$ is 10 times as big as $|y_3 - h|$, but $(y_4 - h)^2$ is 100 times $(y_3 - h)^2$.



- The result is that the **mean** is "pulled" in the direction of outliers, relative to the **median**.

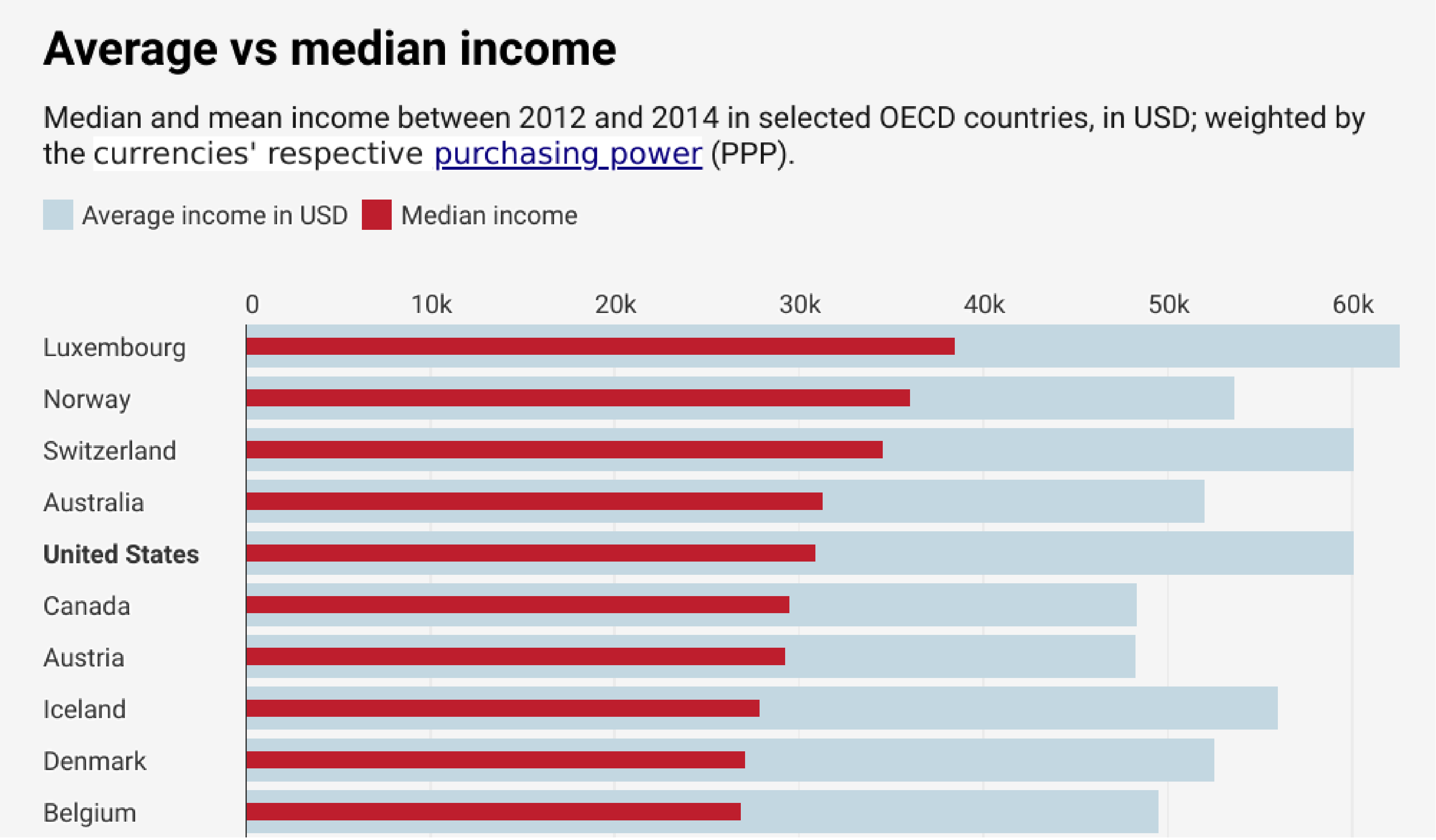


- As a result, we say the **median** – and absolute loss more generally – is **robust**.



Distribution of Commuting Time
Frequency
Minutes
0
2
4
6
8
10
12
14
60
70
80
90
100
110
120
130

Example: Income inequality



Summary: Choosing a loss function

- **Key idea**: Different loss functions lead to different best predictions, h^*!

Loss	Minimizer	Always Unique?	Robust to Outliers?	Differentiable?
$L_{\text{sq}}(y_i, h) = (y_i - h)^2$	mean	yes ✅	no ❌	yes ✅
$L_{\text{abs}}(y_i, h) = \lvert y_i - h \rvert$	median	no ❌	yes ✅	no ❌
$L_{0,1}(y_i, h) = \begin{cases} 0 & y_i = h \\ 1 & y_i \neq h \end{cases}$	mode	no ❌	yes ✅	no ❌
$L_\infty(y_i, h)$ See HW 6.	???	yes ✅	no ❌	no ❌

- The optimal predictions, h^*, are all **summary statistics** that measure the **center** of the dataset in different ways.

Question 🤔

Answer at practicaldsc.org/q

What questions do you have?

The modeling recipe

- We've now made two full passes through our modeling recipe.

    1. Choose a model.

    2. Choose a loss function.

    3. Minimize average loss to find optimal model parameters.

Empirical risk minimization

- The formal name for the process of minimizing average loss is **empirical risk minimization**; another name for "average loss" is **empirical risk**.
- When we use the squared loss function, $L_{\text{sq}}(y_i, h) = (y_i - h)^2$, the corresponding empirical risk is mean squared error:

$$R_{\text{sq}}(h) = \frac{1}{n} \sum_{i=1}^{n} (y_i - h)^2 \implies h^* = \text{Mean}(y_1, y_2, \dots, y_n)$$

- When we use the absolute loss function, $L_{\text{abs}}(y_i, h) = |y_i - h|$, the corresponding empirical risk is mean absolute error:

$$R_{\text{abs}}(h) = \frac{1}{n} \sum_{i=1}^{n} |y_i - h| \implies h^* = \text{Median}(y_1, y_2, \dots, y_n)$$

Empirical risk minimization, in general

- **Key idea**: If L is **any** loss function, and H is any hypothesis function, the corresponding empirical risk is:

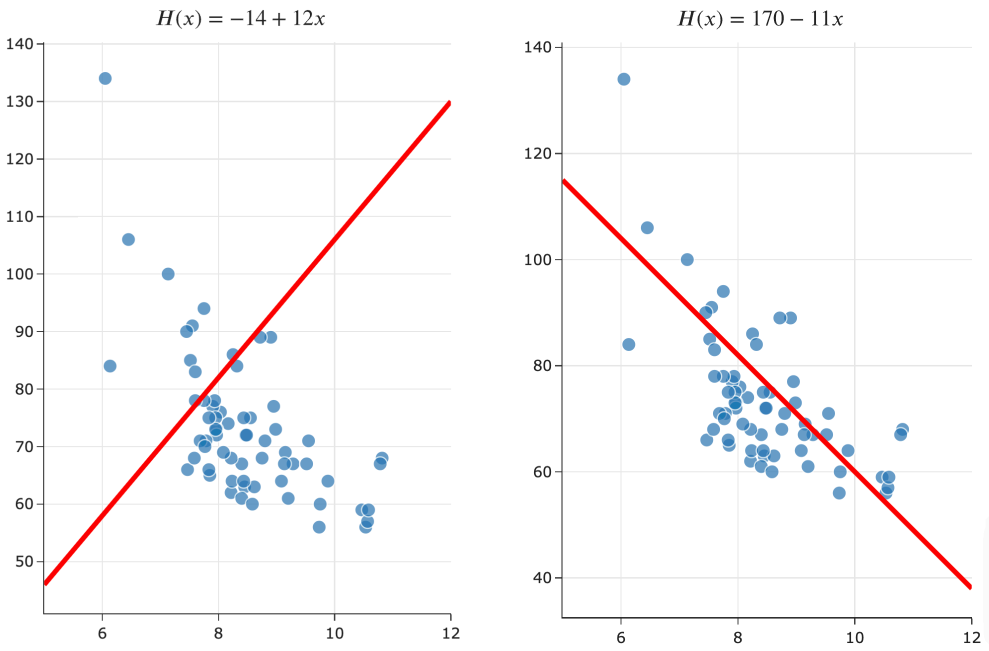
$$R(H) = \frac{1}{n} \sum_{i=1}^{n} L(y_i, H(x_i))$$

- In Homework 6 and tomorrow's discussion, there are several questions where:
  - You are given a new loss function L.
  - You have to find the optimal parameter h^* for the constant model $H(x_i) = h$.

Towards simple linear regression

Recap: Hypothesis functions and parameters

- A hypothesis function, H, takes in an x_i as input and returns a predicted y_i.
- **Parameters** define the relationship between the input and output of a hypothesis function.
- **Example**: The simple linear regression model, $H(x_i) = w_0 + w_1 x$, has two parameters: w_0 and w_1.



The modeling recipe

1. Choose a model.

2. Choose a loss function.

3. Minimize average loss to find optimal model parameters.

Minimizing mean squared error for the simple linear model

- We'll choose squared loss, since it's the easiest to minimize.
- Our goal, then, is to find the linear hypothesis function $H^*(x)$ that minimizes empirical risk:

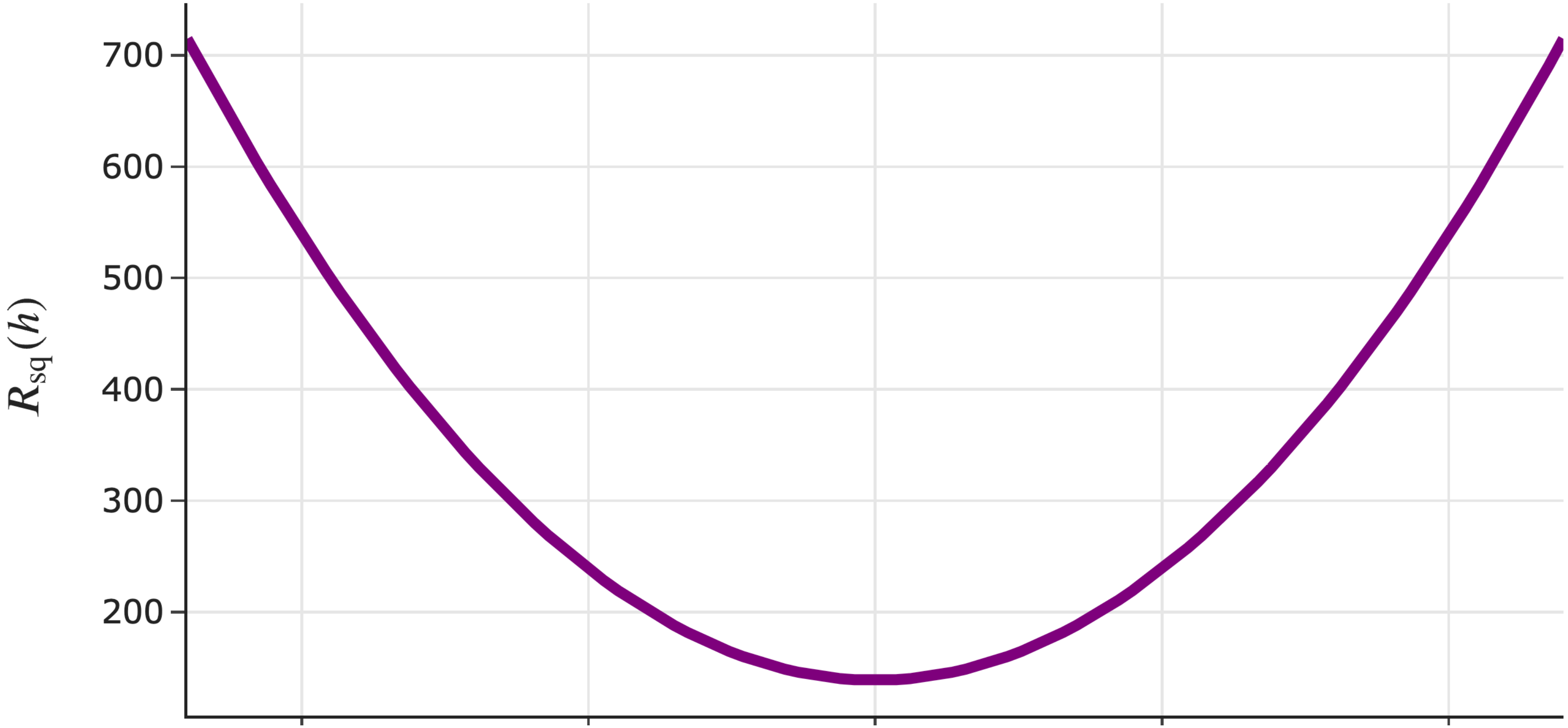
$$R_{\text{sq}}(H) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - H(x_i)\right)^2$$

- Since linear hypothesis functions are of the form $H(x_i) = w_0 + w_1 x_i$, we can re-write R_{sq} as a function of w_0 and w_1:

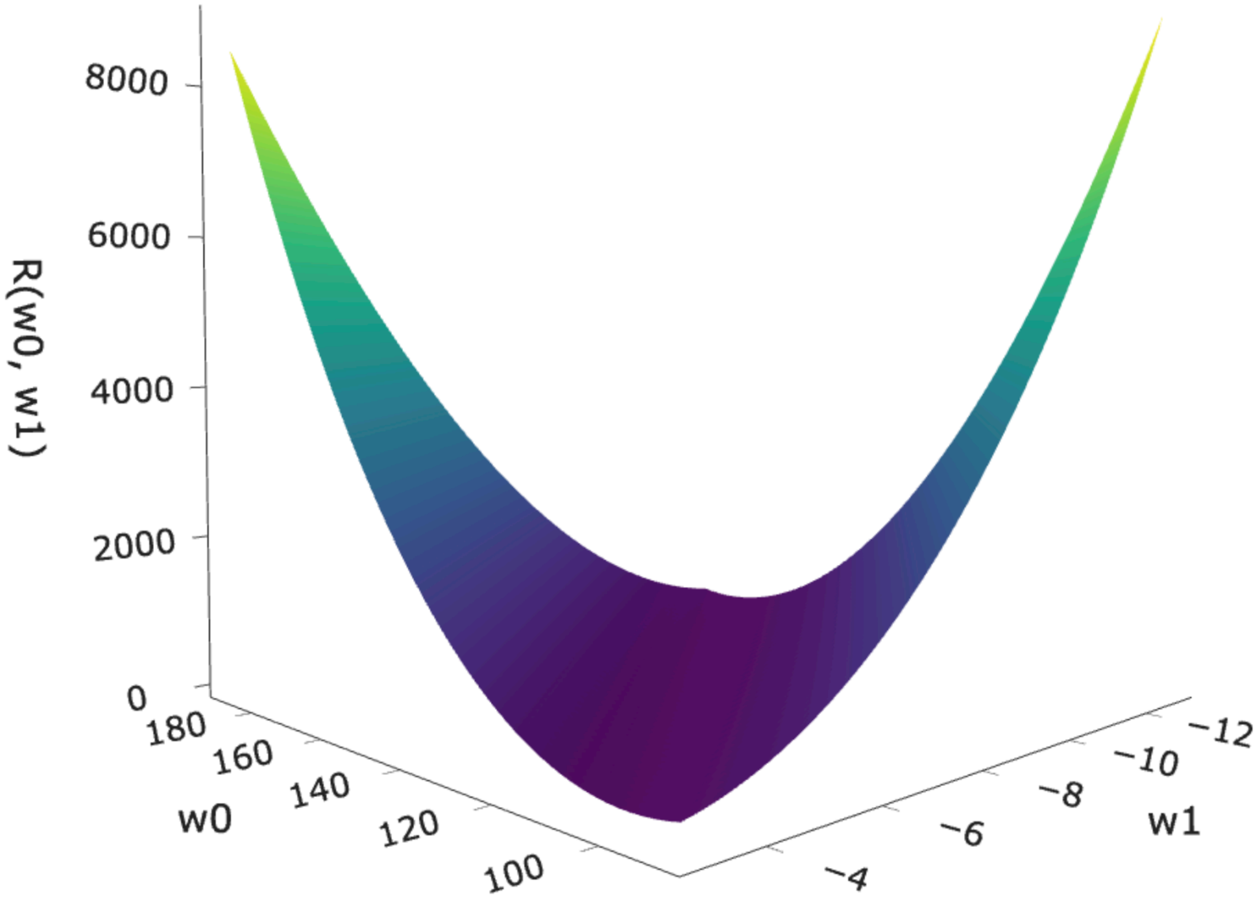
$$\boxed{R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - (w_0 + w_1 x_i)\right)^2}$$

- **How do we find the parameters w_0^* and w_1^* that minimize $R_{\text{sq}}(w_0, w_1)$?**

$$R_{\text{sq}}(h) = \frac{1}{5}\left((72-h)^2 + (90-h)^2 + (61-h)^2 + (85-h)^2 + (92-h)^2\right)$$



For the constant model, the graph of $R_{\text{sq}}(h)$ looked like a parabola.



The graph of $R_{\text{sq}}(w_0, w_1)$ for the simple linear regression model is 3 dimensional **bowl**, and is called a **loss surface**.

Minimizing mean squared error for the simple linear model

Minimizing multivariate functions

- Our goal is to find the parameters w_0^* and w_1^* that minimize mean squared error:

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (w_0 + w_1 x_i))^2$$

- R_{sq} is a function of two variables: w_0 and w_1, and is a bowl-like shape in 3D.
- To minimize a function of multiple variables:
  - Take partial derivatives with respect to each variable.
  - Set all partial derivatives to 0 and solve the resulting system of equations.
  - Ensure that you've found a minimum, rather than a maximum or saddle point (using the second derivative test for multivariate functions).
- To save time, we won't do the derivation live in class, but you are responsible for it! Here's a video of me walking through it, and the slides will be annotated with it.

Example

Find the point (x, y, z) at which the following function is minimized.

$$f(x, y) = x^2 - 8x + y^2 + 6y - 7$$

Minimizing mean squared error

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - (w_0 + w_1 x_i)\right)^2$$

To find the w_0^* and w_1^* that minimize $R_{\text{sq}}(w_0, w_1)$, we'll:

1. Find $\frac{\partial R_{\text{sq}}}{\partial w_0}$ and set it equal to 0.
2. Find $\frac{\partial R_{\text{sq}}}{\partial w_1}$ and set it equal to 0.
3. Solve the resulting system of equations.

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (w_0 + w_1 x_i))^2$$

$$\frac{\partial R_{\text{sq}}}{\partial w_0} =$$

$$R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} \left(y_i - (w_0 + w_1 x_i)\right)^2$$

$$\frac{\partial R_{\text{sq}}}{\partial w_1} =$$

Strategy

- We have a system of two equations and two unknowns (w_0 and w_1):

$$-\frac{2}{n}\sum_{i=1}^{n}(y_i - (w_0 + w_1 x_i)) = 0 \qquad -\frac{2}{n}\sum_{i=1}^{n}(y_i - (w_0 + w_1 x_i))x_i = 0$$

- To proceed, we'll:

1. Solve for w_0 in the first equation.

   The result becomes w_0^*, because it's the "best intercept."

2. Plug w_0^* into the second equation and solve for w_1.

   The result becomes w_1^*, because it's the "best slope."

Solving for w_0^*

$$-\frac{2}{n}\sum_{i=1}^{n}(y_i - (w_0 + w_1 x_i)) = 0$$

Solving for w_1^*

$$-\frac{2}{n}\sum_{i=1}^{n}(y_i - (w_0 + w_1 x_i))x_i = 0$$

Least squares solutions

- We've found that the values w_0^* and w_1^* that minimize R_{sq} are:

$$w_1^* = \frac{\displaystyle\sum_{i=1}^{n}(y_i - \bar{y})x_i}{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})x_i} \qquad\qquad w_0^* = \bar{y} - w_1^*\bar{x}$$

where:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad\qquad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

- **These formulas work, but let's re-write w_1^* to be a little more symmetric.**

An equivalent formula for w_1^*

- Claim:

$$w_1^* = \frac{\displaystyle\sum_{i=1}^{n}(y_i - \bar{y})x_i}{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})x_i} = \frac{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- Proof:

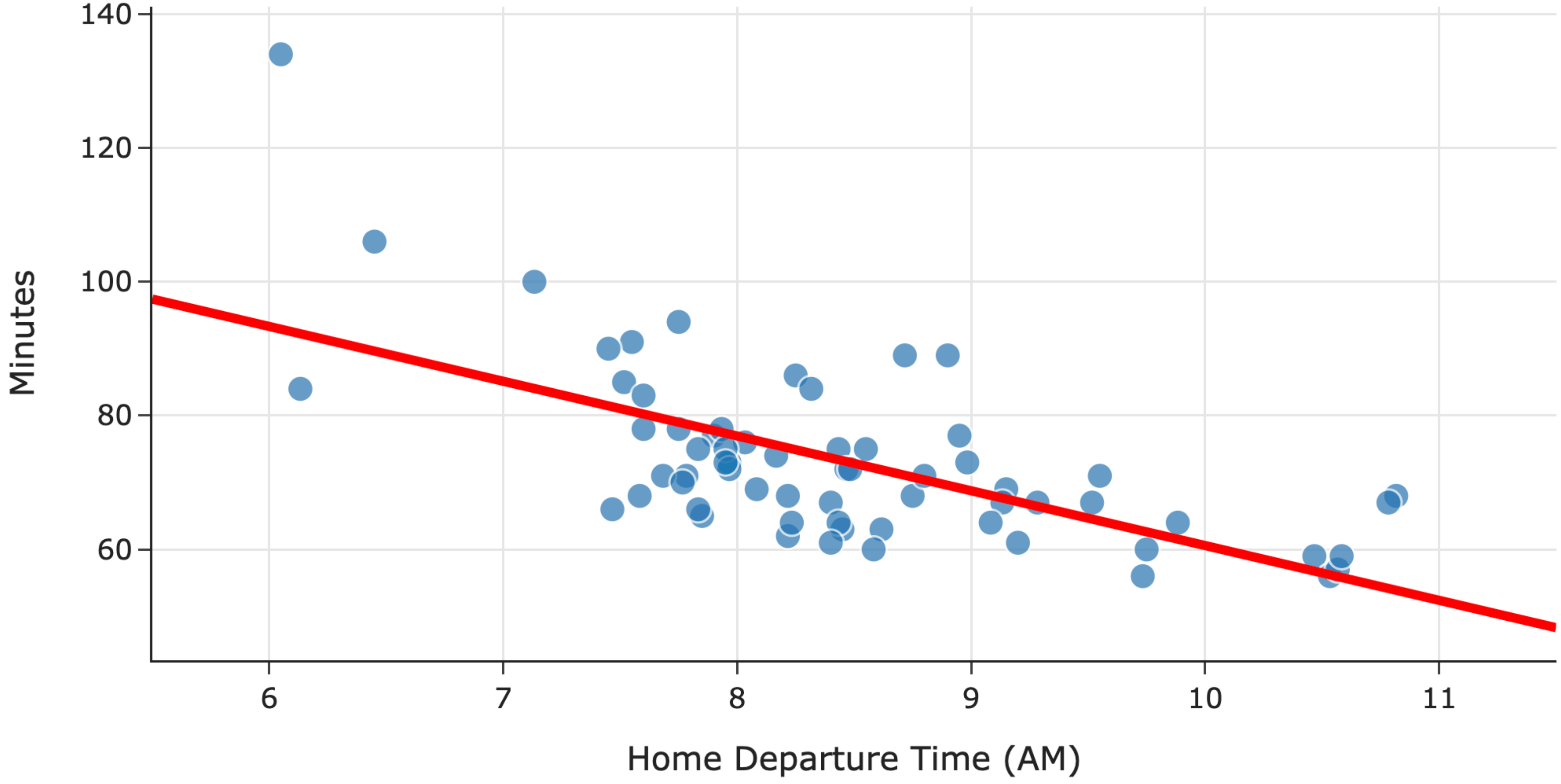
Least squares solutions

- The **least squares solutions** for the intercept w_0 and slope w_1 are:

$$\boxed{w_1^* = \frac{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad\qquad w_0^* = \bar{y} - w_1^* \bar{x}}$$

- We say w_0^* and w_1^* are **optimal parameters**, and the resulting line is called the **regression line**.
- The process of minimizing empirical risk to find optimal parameters is also called "**fitting to the data**."
- To make predictions about the future, we use $\boxed{H^*(x) = w_0^* + w_1^* x}$.

Predicted Commute Time = 142.25 - 8.19 * Departure Hour



Question 🤔

Answer at practicaldsc.org/q

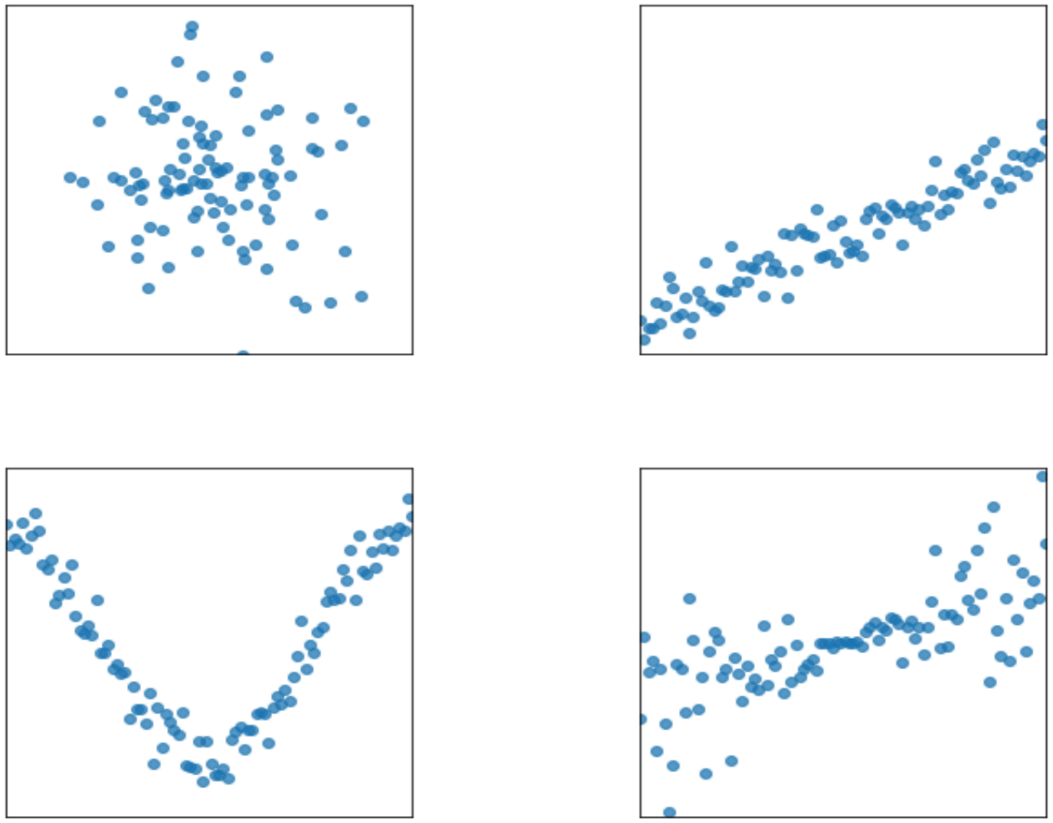
Consider a dataset with just two points, $(2, 5)$ and $(4, 15)$. Suppose we want to fit a linear hypothesis function to this dataset using squared loss. What are the values of w_0^* and w_1^* that minimize empirical risk?

- A. $w_0^* = 2$, $w_1^* = 5$
- B. $w_0^* = 3$, $w_1^* = 10$
- C. $w_0^* = -2$, $w_1^* = 5$
- D. $w_0^* = -5$, $w_1^* = 5$

Correlation

Quantifying patterns in scatter plots

- The **correlation coefficient**, r, is a measure of the strength of the **linear association** of two variables, x and y.
- Intuitively, it measures how tightly clustered a scatter plot is around a straight line.
- It ranges between -1 and 1.

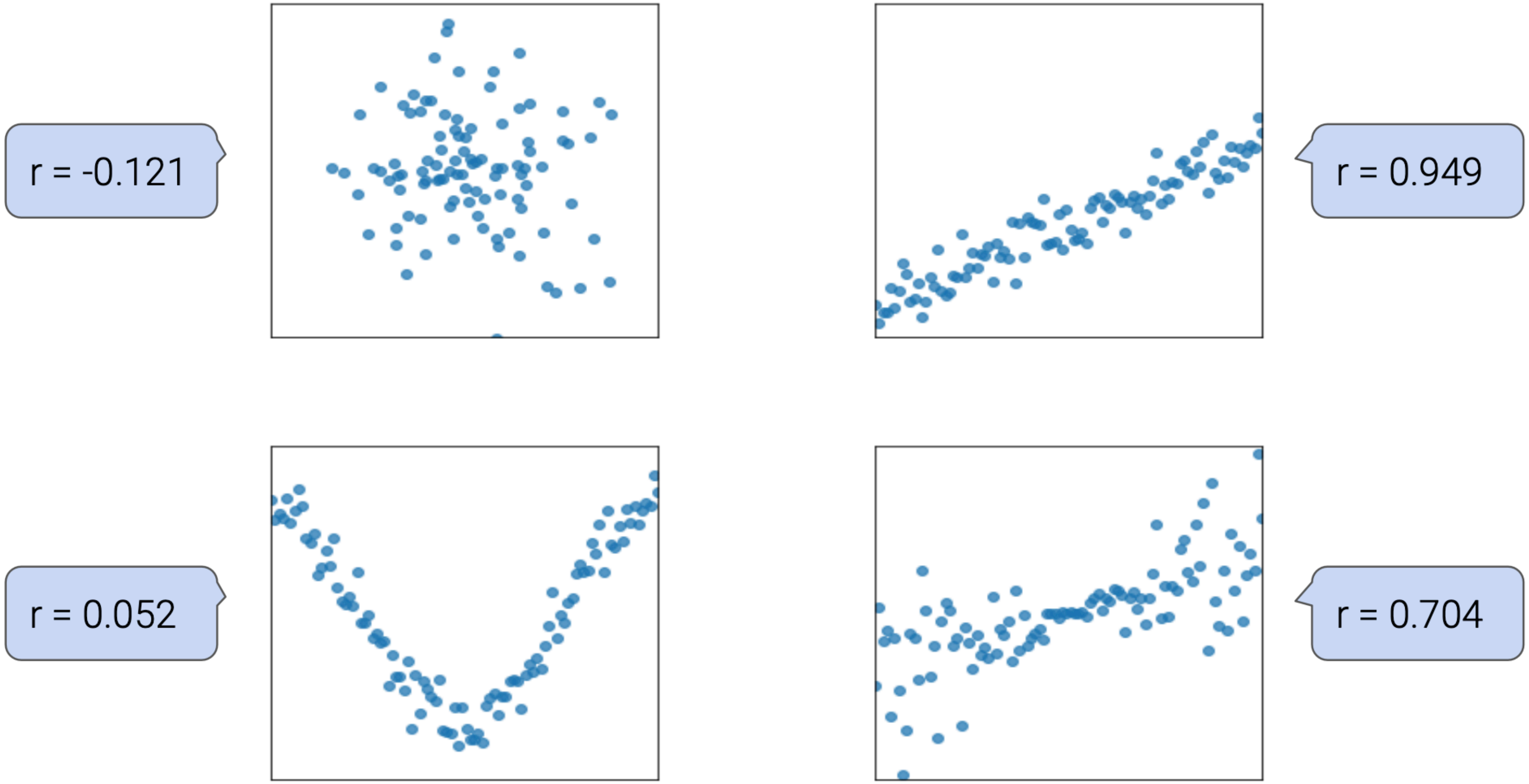


The correlation coefficient

- The correlation coefficient, r, is defined as the **average of the product of x and y, when both are *standardized***.
- Let σ_x be the standard deviation of the x_is, and $\bar{x}$ be the mean of the x_is.
- x_i standardized is $\frac{x_i - \bar{x}}{\sigma_x}$.
- The correlation coefficient, then, is:

$$r = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma_x} \right) \left(\frac{y_i - \bar{y}}{\sigma_y} \right)$$

The correlation coefficient, visualized



Another way to express w_1^*

- It turns out that w_1^*, the optimal slope for the linear hypothesis function when using squared loss (i.e. the regression line), can be written in terms of r!

$$w_1^* = \frac{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\displaystyle\sum_{i=1}^{n}(x_i - \bar{x})^2} = r\frac{\sigma_y}{\sigma_x}$$

- It's not surprising that r is related to w_1^*, since r is a measure of linear association.
- Concise way of writing w_0^* and w_1^*:

$$w_1^* = r\frac{\sigma_y}{\sigma_x} \qquad w_0^* = \bar{y} - w_1^*\bar{x}$$

Proof that $w_1^* = r\frac{\sigma_y}{\sigma_x}$

Recap: Simple linear regression

- **Goal**: Use the modeling recipe to find the "best" simple linear hypothesis function.

  1. **Model**: $H(x_i) = w_0 + w_1 x_i$.
  2. **Loss function**: $L_{\text{sq}}(y_i, H(x_i)) = (y_i - H(x_i))^2$.
  3. **Minimize empirical risk**: $R_{\text{sq}}(w_0, w_1) = \frac{1}{n} \sum_{i=1}^{n} (y_i - (w_0 + w_1 x_i))^2$.

$$\implies w_1^* = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = r \frac{\sigma_y}{\sigma_x} \qquad w_0^* = \bar{y} - w_1^* \bar{x}$$

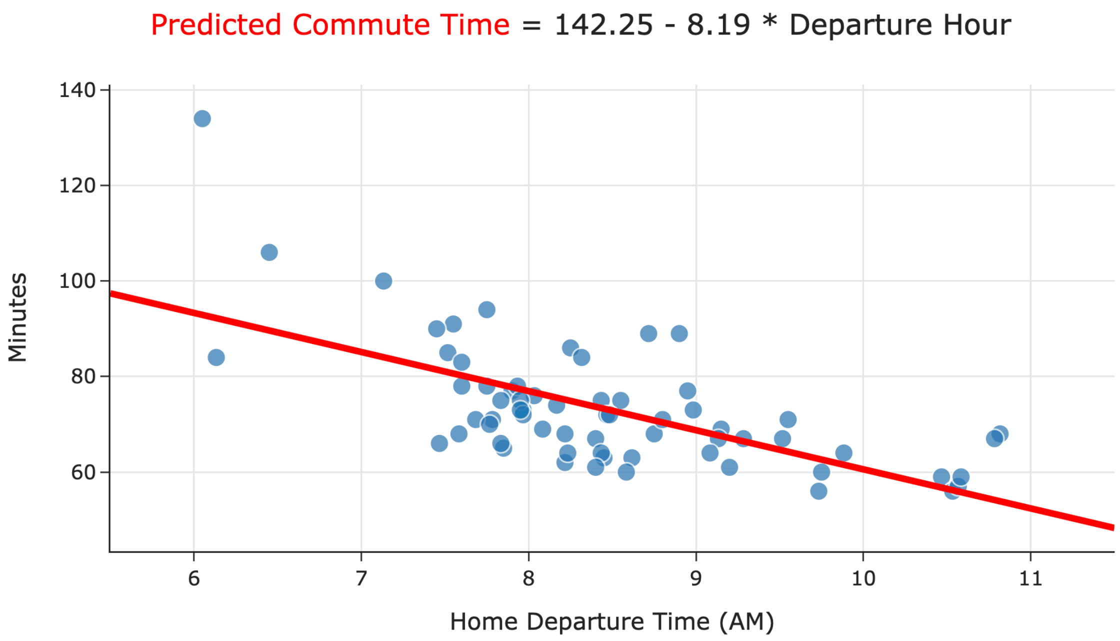
- The resulting line, $H^*(x) = w_0^* + w_1^* x$, is the line that minimizes mean squared error.

  It's often called the **(least squares) regression line**, and the **optimal linear predictor**.

Interpreting the formulas

Causality

- Can we conclude that leaving later **causes** you to get to school earlier?



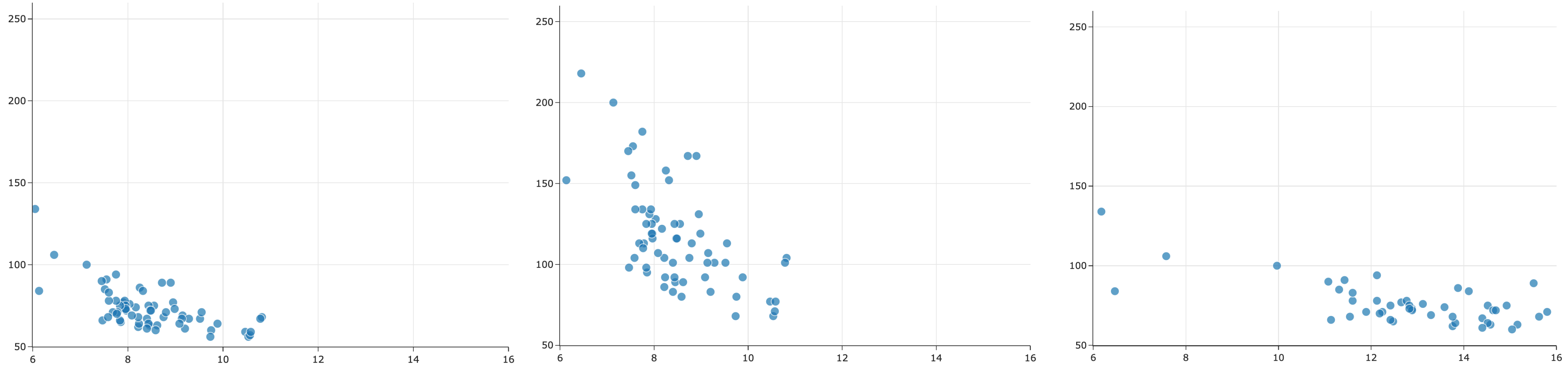
Interpreting the slope

$$w_1^* = r \frac{\sigma_y}{\sigma_x}$$

- The units of the slope are **units of y per units of x**.
- In our commute times example, in $H^*(x) = 142.25 - 8.19x$, our predicted commute time **decreases by 8.19 minutes per hour**.

Interpreting the slope

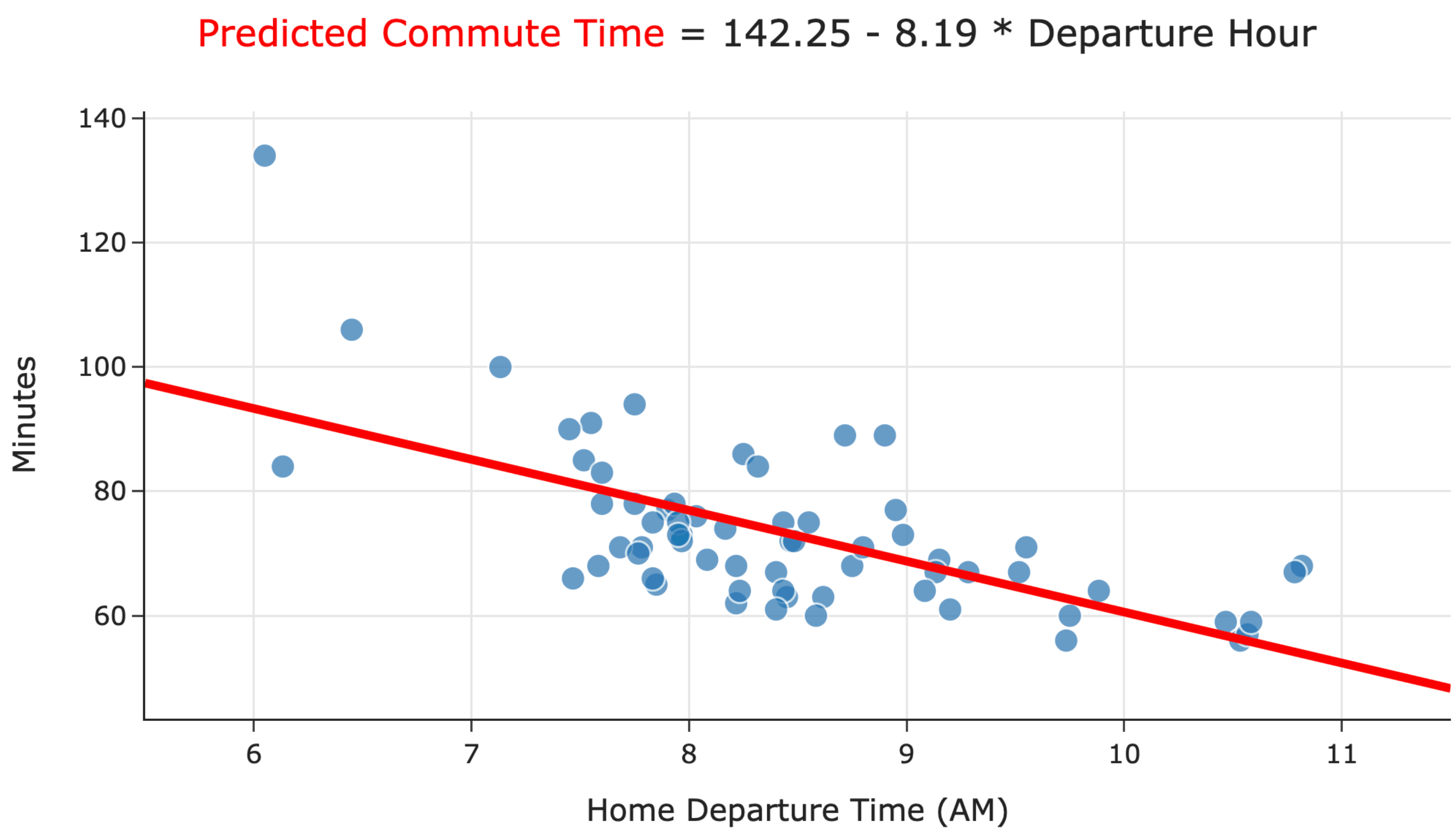
$$w_1^* = r\frac{\sigma_y}{\sigma_x}$$



- Since $\sigma_x \geq 0$ and $\sigma_y \geq 0$, the slope's sign is r's sign.
- As the y values get more spread out, σ_y increases, so the slope gets steeper.
- As the x values get more spread out, σ_x increases, so the slope gets shallower.

Interpreting the intercept

$$w_0^* = \bar{y} - w_1^* \bar{x}$$



- What are the units of the intercept?
- What is the value of $H^*(\bar{x})$?

Question 🤔

Answer at practicaldsc.org/q

We fit a regression line to predict commute times given departure hour. Then, we add 75 minutes to all commute times in our dataset. What happens to the resulting regression line?

- A. Slope increases, intercept increases.
- B. Slope decreases, intercept increases.
- C. Slope stays the same, intercept increases.
- D. Slope stays the same, intercept stays the same.