Correlated Quantum Systems
Project Report

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Summary

We study the system of interacting bosons, first we apply methods of quantum field theory to the weak interactions and find the spectrum from first order perturbation calculations[3][1]. Then we try to calculate higher order perturbations, but as it turns out the series diverges and we are required to introduce an effective potential[2]. We find out that the energy and momentum have a linear relationship for small momentum, we then find the spectrum near the threshold point, where the linear relationship ceases to hold[4]. Next we move to strongly correlation in bosons.

Chapter 1

System of Interacting Bosons

In an ideal Bose gas at low temperatures, an arbitrarily large number of particles can be concentrated in the state with zero momentum. As the we introduce interaction, the density of bosons outside the condensate starts to increase.

We take T=0 and seperate the operators for bosons in the condensate b_0 and out of condensate $b(x)$

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{k=0}^{N} a_k e^{ikx} \qquad \psi^\dagger(x) = \frac{1}{\sqrt{V}} \sum_{k=0}^{N} a_k^\dagger e^{-ikx} \qquad (1.1)$$

$$\psi(x) = b_0 + b(x) \qquad \psi^\dagger(x) = b_0^\dagger + b^\dagger(x) \qquad (1.2)$$

where $b_0 = \frac{a_0}{\sqrt{V}}$, from boson commutation relationship we know

$$a_0 a_0^\dagger - a_0^\dagger a_0 = 1 \qquad (1.3)$$

Therefore,

$$b_0 b_0^\dagger - b_0^\dagger b_0 = \frac{1}{V} \qquad (1.4)$$

as $V \to \infty$ the RHS tends to zero hence these operators commute and can be regarded as numbers to a first approximation. Hence $b_0 = b_0^\dagger$. As we know $a_0^\dagger a_0 = N_0$ (where N_0 is the number of particles in the condensate) therefore

$$b_0^\dagger b_0 = \frac{N_0}{V} = n_0 \qquad (1.5)$$

$$b_0^\dagger = b_0 = \sqrt{n_0} \qquad (1.6)$$

where n_0 is the number density of the particles in the condensate.

Thermodynamic variables : μ, n_0

In an bose ideal gas, the chemical potential is zero below the condensation temperature. However, in the interacting problem, it becomes a non-zero thermodynamic variable. Instead of keeping the total number of particles constant, we treat the chemical potential as an independent variable. It can be calculated based on the condition that the average number of particles in the system is equal to the given actual number of particles. Therefore, in our problem, we consider the chemical potential μ and number density of the particles in the condensate n_0 as the independent variables. The total number of particles is not constant.

1.1 Green's Function

Hamiltonian of the system is given by

$$H = H_0 + H_{int} \tag{1.7}$$

$$H_{int} = \frac{1}{2} \int \psi^\dagger(x)\psi^\dagger(x')U(\mathbf{r}-\mathbf{r}')\psi(x')\psi(x)d\mathbf{r}d\mathbf{r}' \tag{1.8}$$

One particle Green's function $G(x, x')$ is defined by

$$G(x - x') = -\iota\langle T(\psi(x)\psi^\dagger(x'))\rangle \tag{1.9}$$

where S-Matrix is given by $S = Texp\{-\iota \int H_{int}\, d^4x\}$. Now, when we substitute 1.2 in 1.8 we get sixteen terms and each contributes to different Feynman diagrams in the perturbation series. We introduce three types of Green's function which can be used to write the original Green's function $(G(x - x'))$. We write these Green's Functions in momentum basis

$$G'(k, t' - t) = -\iota\langle T(b_k(t')b_k^\dagger(t))\rangle \tag{1.10}$$

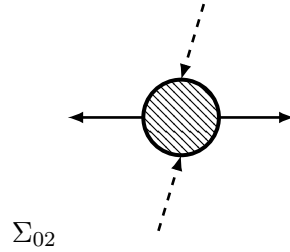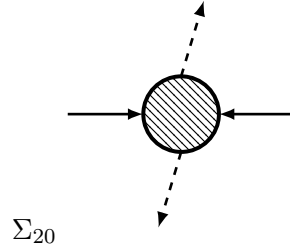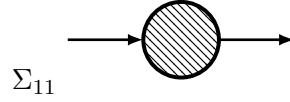$$F(k, t' - t) = -\iota\langle T(b_k^\dagger(t')b_k^\dagger(t))\rangle \tag{1.11}$$

$$\tilde{F}(k, t' - t) = -\iota\langle T(b_k(t')b_k(t))\rangle \tag{1.12}$$

Green's Function for ideal bose gas is given by

$$G^{(0)}(k) = \frac{1}{\omega - (k^2/2m) + \mu + i\delta} \tag{1.13}$$

1.1.1 Self Energy

The Feynman diagrams for the perturbation series of the green function will have three types of self energy diagrams Σ_{11}, Σ_{02} and Σ_{20}, the subscripts indicates incoming and outgoing out-of-condensate particles
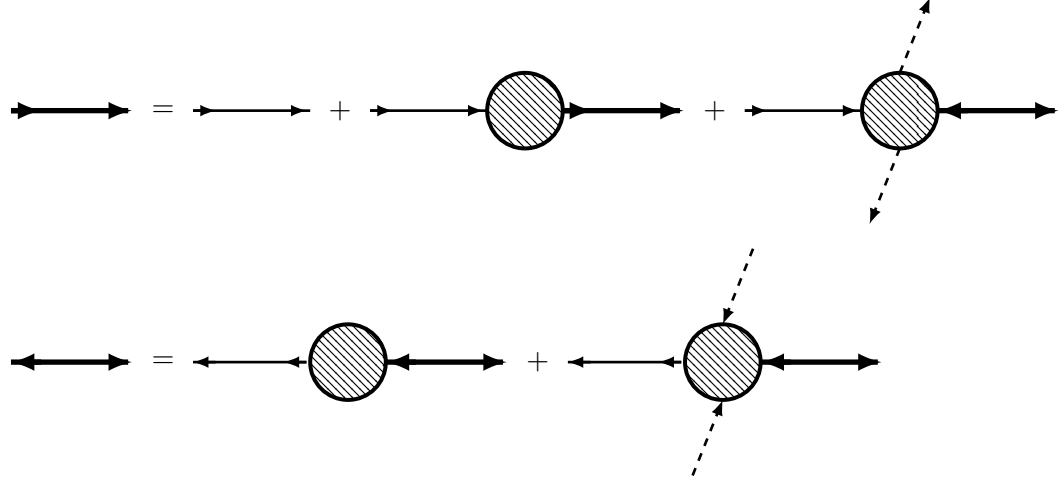

Σ_{11}

Σ_{20}

Σ_{02}

1.1.2 Dyson Equation

Diagrammatically the Dyson equation for the out-of-condensate particles can be written in the following way, first we define the lines

out-of-condensate particle	⟶
condensate particle	⇢
$G^{(0)}(k)$	⟶
$G'(k)$	⟶
$F(k)$	⟷

Dyson Equation diagrammatically is represented as

Analytically we write these equations as

$$G'(k) = G^{(0)}(k) + G^{(0)}(k)\Sigma_{11}(k)G'(k) + G^{(0)}\Sigma_{20}F(k) \qquad (1.14)$$

$$F(k) = G^{(0)}(-k)\Sigma_{11}(-k)G'(k) + G^{(0)}(-k)\Sigma_{02}G'(k) \qquad (1.15)$$

using 1.13

$$[\omega - \epsilon_0(\vec{k}) + \mu - \Sigma_{11}(k)]G'(k) - \Sigma_{20}(k)F(k) = 1 \qquad (1.16)$$

$$[-\omega - \epsilon_0(\vec{k}) + \mu - \Sigma_{11}(-k)]F(k) - \Sigma_{02}(k)G'(k) = 0 \qquad (1.17)$$

where $\epsilon_0(\vec{k}) = \frac{\mathbf{k}^2}{2m}$, Solving for G'(k) and F(k)

$$G'(k) = \frac{\omega + \epsilon_0(\vec{k}) + S(k) + A(k) - \mu}{[\omega - A(k)]^2 - [\epsilon_0(\vec{k}) + S(k) - \mu]^2 + \Sigma_{20}(k)\Sigma_{02}(k)} \qquad (1.18)$$

$$F(k) = -\frac{\Sigma_{02}(k)}{[\omega - A(k)]^2 - [\epsilon_0(\vec{k}) + S(k) - \mu]^2 + \Sigma_{20}(k)\Sigma_{02}(k)} \qquad (1.19)$$

where

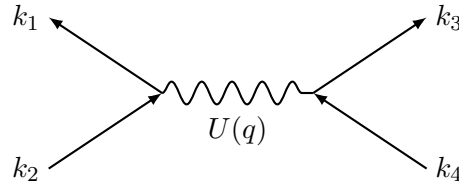$$S(k) = \frac{\Sigma_{11}(k) + \Sigma_{11}(-k)}{2} \qquad (1.20)$$

$$A(k) = \frac{\Sigma_{11}(k) - \Sigma_{11}(-k)}{2} \qquad (1.21)$$

First order perturbation

Using 1.8

$$H_{int} = \frac{1}{2} \int \psi^\dagger(x)\psi^\dagger(x')U(\mathbf{r} - \mathbf{r}')\psi(x')\psi(x)d\mathbf{r}d\mathbf{r}'$$

The H_{int} will break up into 16 terms as we substitute 1.2. But they will have a similar structure of two particles coming in and two particles going out, but we will be considering the condensate particles .

$$U(q)$$

Self energies in first order

$$\Sigma_{11}^{(1)}(k) = n_0[U(0) + U(\vec{k})] \tag{1.22}$$

$$\Sigma_{20}^{(1)}(k) = \Sigma_{02}^{(1)}(k) = n_0U(\vec{k}) \tag{1.23}$$

and chemical potential is given by

$$\mu = \Sigma_{11}(0) - \Sigma_{20}(0) \tag{1.24}$$

Substituting these values in 1.19 we get

$$G'(k) = \frac{\omega + \epsilon_0(\vec{k}) + n_0U_k}{\omega^2 - \epsilon_0^2(\vec{k}) - 2n_0U_k\epsilon_0(\vec{k}) + \iota\delta} \tag{1.25}$$

Hence the spectrum can be obtained from the poles of the Green's function

$$\omega = \sqrt{\epsilon_0^2(\vec{k}) + 2n_0U_k\epsilon_0(\vec{k})} \tag{1.26}$$

For small values of $\left\|\vec{k}\right\| = k$

$$\omega = \sqrt{\frac{n_0U_k}{m}} \ k \tag{1.27}$$

1.2   Dilute Non-ideal Bose Gas

First order perturbation is only valid for small values of k, when we include higher order of perturbations in the calculation of self energy we find that the series does not converge.

### 1.2.1   Low density approximation

We assume the interaction potential to be

$$U(\vec{k}) = \begin{cases} U_0 & \text{for } \|\vec{k}\| \ll \frac{1}{a} \\ 0 & \text{for } \|\vec{k}\| \gg \frac{1}{a} \end{cases}$$

where $a \sim f_0$ is the radius of the particles, in order of magnitude. For this problem we form two dimensionless parameters

$$\alpha \sim \frac{mU_0}{a}, \quad \beta = \sqrt{n_0 a^3} \tag{1.28}$$

We can now solve the Green's Function by calculating the self energies using perturbation theory. The series does not converge in α, as a result we introduce the effective interaction potential, represented by (for example Σ_{02})
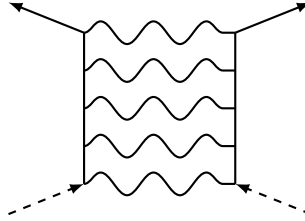
Theoretically it is not possible to calculate the effective interaction potential in a general form without knowing the specific properties of the interaction potential. But it solves the issue of diverging perturbation series, as this is bounded and the first order approximation spectrum obtained using the effective potential matches that of first order perturbation. But in the next section we calculate the spectrum near the cutoff frequency, i.e. where the energy stop being proportional to k.

1.3   Spectrum near cut-off point

Unlike fermi liquid undamped Bose excitation can exist. But when the momentum is increased we reach a threshold, above which the excitation is unstable. Pitaevskii in his paper[4] calculates the spectrum near the cutoff region where the interacting bose gas particles' energy stop being proportion to its momentum. It is assumed that for some value of p (momentum) this relationship breaks an the threshold momentum is represented as p_c

1.3.1   Green's Function near the cut-off point

We assume that a particle near the threshold is unstable and decay into two excitations. Momentum and energy are conserved

$$\epsilon(\vec{p}) = \epsilon(\vec{q}) + \epsilon(\vec{p} - \vec{q}) \tag{1.29}$$

Analytically we can write

$$G^{-1}(p) = G_0^{-1}(p) + \frac{1}{(2\pi)^4} \iota \int \Gamma(p; q, p - q) G(q) G(p - q) \Gamma_0(p; q, p - q) d^4 q \tag{1.30}$$

Where Γ is the triple vertex which is supposed to be proportional to k for small k. As we are probing near the cutoff point of linear relationship. The energy relation would have minima near the cut off point $p = p_c$

$$\epsilon(\vec{p}) = \epsilon(\vec{p} - \vec{k}) + \omega(k) \tag{1.31}$$

Also for small k, ωk becomes $\omega(k) = ck - \alpha k^2$

Solving the above equations we get the following spectrum

$$\epsilon(p) = \epsilon_c + c(p - p_c) + \beta(p - p_c)^2 + a(p - p_c)^3 ln(p_c - p) \tag{1.32}$$

If $p \geq p_c$ then we has an additional imaginary term $-a\pi(p - p_c)^3$. Hence the excitation above p_c will be damped and their lifetime inversely proportional to $(p - p_c)^3$.

Chapter 2

Strong correlation in bosons

2.1 Mean field approach

In the previous chapter we discussed the weak limit of interaction between bosons, which can be solved using perturbation theory. But when the interaction between the bosons is strong, the perturbation theory does not work. Here we first attack the problem using a mean field approach on the kinetic energy term for the bosons on the lattice. Hamiltonian of the Hubbard model for bosons in a lattice is given by

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} b_i^\dagger b_j + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i$$

Approximation of the hopping term

$$b_i^\dagger b_j = \langle b_i^\dagger \rangle b_j + b_i^\dagger \langle b_j \rangle - \langle b_i^\dagger \rangle \langle b_j \rangle$$

Therefore,

$$\mathcal{H}^{MF} = \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i \ - t(\sum_j \langle b_i^\dagger \rangle b_j + \sum_i b_i^\dagger \langle b_j \rangle - \sum_i \langle b_i^\dagger \rangle \langle b_j \rangle)$$

we substitute $\psi = \langle b_i^\dagger \rangle = \langle b_i \rangle$

$$\mathcal{H}^{MF} = \sum_i \frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - \mu \hat{n}_i \ - t\psi(b_i + b_i^\dagger) + t\psi^2$$

Where ψ is the superfluid order parameter, we can now write Hamiltonian for a site

$$\mathcal{H}_i^{MF} = \frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - \mu \hat{n}_i \ - t\psi(b_i + b_i^\dagger) + t\psi^2$$
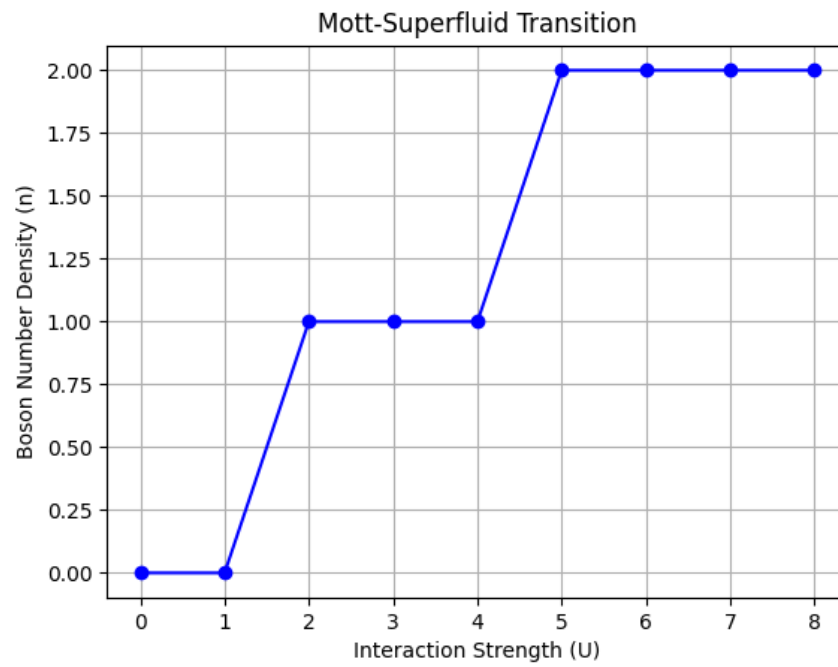
Figure 2.1: Representative diagram of variation of n versus U

References

Literature

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[4] L.P. Pitaevskii. "Properties of the spectrum of elementary excitations near the disintegration threshold of the excitations". In: *Soviet Physics JETP* 36(9), Number 4 (October 1959), pp. 830–837 (cit. on pp. 2, 9).