Erogodicity

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Outline

Why Ergodicity?

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} f(T^k x) = \frac{1}{\mu(X)} \int f \, d\mu$$

Boltzmann Hypothesis

Boltzmann

1. Hypothesis : Phase space trajectories for an isolated physical system having constant energy pass through every point on the energy surface.
2. It is valid for one dimensional phase space

Ehrenfests

1. They proposed quasi-ergodic hypothesis. It states that some orbit of the flow will pass arbitrarily close to every point of phase space.

Birkoff and von Neumann

1. Formally : A system is said to be ergodic if and only if it is "metrically transitive", that is, provided that the trajectories on the energy surface cannot be decomposed into two invariant sets having positive measure
2. Less Formally : Ergodicity requires that most phase space trajectories for an isolated physical system densely cover the energy surface
3. Not Enough : Ergodicity does not ensure that a system "forgets" its initial state

Mixing and Equilibrium

1. Mixing : After a sufficiently long time interval, the fractional amount of B_t in every arbitrary small, but nonzero, set A is the same as the original fractional amount of B in E

$$\lim_{t\to\infty} \frac{\mu(A \cap B_t)}{\mu(A)} = \frac{\mu(B)}{\mu(E)}$$

2. Ergodicity is at the bottom of the hierarchy of properties which relates to the time evolution of ensembles.
3. Bernoulli System $\rightarrow$ K-System $\rightarrow$ Mixing $\rightarrow$ Ergodicity

Recurrence and Reversibility

1. Poincaré recurrence theorem : Dynamical system will after a long but finite time, return to their initial state.
2. Loschmidt's paradox is the objection that it should not be possible to deduce an irreversible process from time-symmetric dynamics

KAM Theorem

Consider N harmonic oscillators with non-linear coupling

$$H = \frac{1}{2}\sum_{k=1}^{N}\omega_k(P_k^2 + Q_k^2) + \gamma[V_3 + V_4 + ...]$$

Where $V_3, V_4, \ldots$ are cubic, quartic, . . ., polynomials in Q_k and P_k

Intuitively one might think that perturbation will permit the energy from one mode to another and thus lead to ergodicity.

KAM Theorem

KAM rigorously proved that, excepting a set of small measure, most trajectories of Hamiltonians are quasi-periodic orbits lying on smooth $N-$dimensional integral surfaces embedded in $2N-$dimensional $\Gamma-$space provided,

- γ or, equivalently, the total energy is sufficiently small, and
- the harmonic frequencies ω_k do not satisfy low-order resonance conditions of the form $\Sigma_k n_k \omega_k \cong 0$ for integers n_k

1954: Kolmogorov A.N. Preservation of Conditionally Periodic Movements with Small Change in the Hamiltonian Function, Doklady Academii Nauk (Report of the Academy of Sciences) SSSR vol. 98 : 527-530.

Significance of KAM Theorem

1. If we chose γ and ω_k suitably we will have an integrable dynamic system.
2. If violate the conditions mentioned above, we will have non-integrable - chaotic - dynamic system.
3. Thus we get a boundary between non-statistical behaviour and stochastic behaviour

Phase Diagram of an anharmonic oscillator

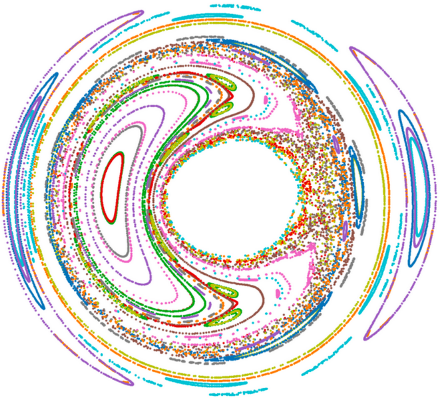

Figure: Non-ergodicity of an anharmonic oscillator system

Definition

1. Sinai Billiards is a dynamical system of a point mass m within a region Ω that has a smooth boundary with elastic reflection.
2. Boltzmann gas of elastically colliding hard balls in a box can be easily reduced to a billiard.

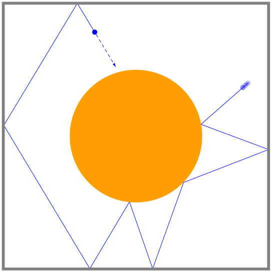

Figure: Sinai Billiard

Billiards in a Polygon

1. Billiards in rational polygons are nonergodic because of a finite number of possible directions of their orbits.
2. However, a billiard in a typical polygon is ergodic (Kerckhoff et al., 1986).

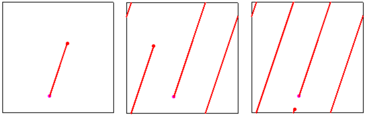

Figure: Trajectory with a rational slope

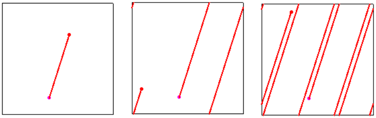

Figure: Trajectory with a irrational slope

Sinai Billiards

1. In 1963, Sinai proved that the system shown below is ergodic and it was the first time anyone proved a dynamic system was ergodic.
2. In general billiards demonstrate chaotic behaviour, because a typical billiard table has at least one non-flat component of the boundary

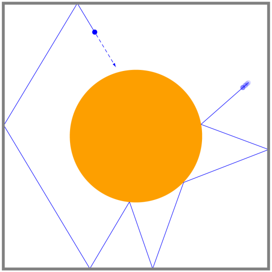

Figure: Sinai Billiards

Integrable Billiards

1. Examples of integrable billiards are billiards in circles and ellipses
2. Configuration spaces of these billiards are foliated by caustics, which are smooth curves (or surfaces) such that if one segment of the billiard orbit is tangent to it, then every other segment of this orbit is also tangent to it

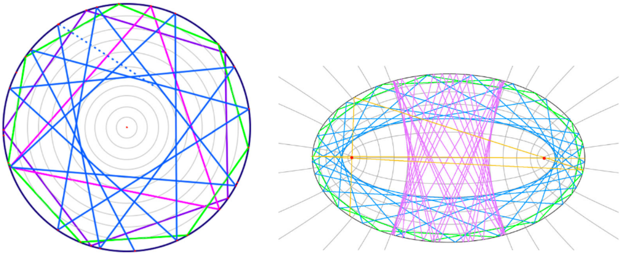

Figure: Billiards in circles and ellipses

KAM Island and Chaotic sea

1. Chaos does not imply Ergodicity.
2. Phase spaces of typical Hamiltonian systems are divided into KAM-islands and chaotic sea(s). Therefore they are neither integrable nor chaotic ones.
3. Each individual chaotic region is ergodic in itself, but since trajectories cannot cross the regular, invariant barriers between those regions, the systems as a whole is not ergodic.

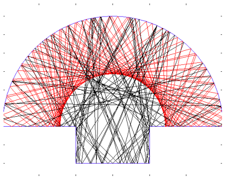

Figure: Mushroom

Conclusion

1. The Boltzmann-Sinai Hypothesis dates back to 1963 as Sinai's modern formulation of Ludwig Boltzmann's statistical hypothesis in physics, actually as a conjecture: Every hardball system on a flat torus is (completely hyperbolic and) ergodic after fixing the values of the obviously invariant kinetic quantities.
2. Sinai himself proved the hyperbolicity and ergodicity for the case $N = 2$ and $v = 2$ hard ball gas in 1970
3. N Simanyi proved the Boltzmann-Sinai Ergodic Hypothesis in 2005.

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Thank You!