Fun Number Theory Meeting
(Computation, Factors, Exponent Patterns, Factorials, Perfect Squares and Cubes)

Topic
This meeting's topics fall in the category of number theory. Students will apply their knowledge of factoring to answer a variety of problems. Students also will investigate number patterns and ways of minimizing or maximizing differences and products.

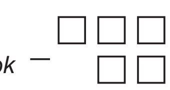
Materials Needed
- ♦ Copies of the Number Theory problem set (Problems and answers can be viewed below. Complete solutions and a more student-friendly version of the problems—with pictures and larger text—are available for download from www.mathcounts.org on the MCP Members Only page of the Club Program section.)
- ♦ No calculators. This will encourage students to use more of their number theory knowledge!

Meeting Plan
A wide range of topics is included in this problem set. There are a number of terms and concepts you may wish to review with students before they get started: *factor*, *factorial* (!), *exponent*, *perfect square*, *perfect cube* and *cube root*.

The problems are arranged in five categories and increase in difficulty throughout the set. Therefore, depending on your students' ability levels, you may wish to work on the later problems as a group to encourage discussion and further learning.

Computation

1. If the digits 4, 5, 6, 7 and 9 are placed in the boxes shown such that there is one digit in each box and each of the five digits is used, what is the smallest possible result of the subtraction problem? *2006-2007 School Handbook Warm-Up 4-2*



2. In the following addition procedure, each letter represents a distinct digit: MQM + NQ = QHHH. What is the value of N? *2002-2003 School Handbook Workout 8-1*

3. If x and y are positive integers with $x + y < 40$, what is the largest possible product xy? *2008-2009 School Handbook Warm-Up 3-7*

Factors and Divisibility

4. What is the least positive integer divisible by the four smallest odd, positive integers? *2007 School Competition Sprint Round #3*

5. A positive two-digit integer is divisible by n and its units digit is n. What is the greatest possible value of n? *2006 School Competition Countdown Round #8*

6. A day can be evenly divided into 86,400 periods of 1 second; 43,200 periods of 2 seconds; or in many other ways. In total, how many ways are there to divide a day into n periods of m seconds, where n and m are positive integers? *2006-2007 School Handbook Workout 1-10*

7. What is the smallest positive perfect square that is divisible by both 2 and 3? *2007 School Competition Countdown Round #22*

8. Johnny had a full bag of apple seeds. He found that if he repeatedly removed the apple seeds 2 at a time, 1 seed remained in the bag at the end. Similarly, if he repeatedly removed the seeds 3, 4, 5 or 6 at a time from the full bag, 1 seed remained in the bag at the end. However, if he removed the seeds 7 at a time from the full bag, no seeds remained at the end. What is the least number of seeds he could have in the full bag? *2006-2007 School Handbook Workout 2-10*

Exponents and Patterns

9. What is the ones digit of the integer form of $(2 + 3)^{23}$? *2007 School Competition Countdown Round #21 (modified)*

10. When the expression $3^{444} + 4^{333}$ is written as an integer, what is the units digit? *2007-2008 School Handbook Warm-Up 3-2*

Factorials

11. What is the value of n for which $(3!)(5!)(7!) = n!$? *2007-2008 School Handbook Warm-Up 3-7*

12. What is the greatest perfect square that is a factor of 7!? *2008-2009 School Handbook Warm-Up 2-3*

Perfect Squares and Cubes

13. How many factors of 8000 are perfect squares? *2006-2007 School Handbook Workout 4-6*

14. The product of integers 240 and k is a perfect cube. What is the smallest possible positive value of k? *2006-2007 School Handbook Workout 3-8*

15. If k is an integer and $k > 100$, what is the smallest possible integer value of the cube root of k^2? *2007-2008 School Handbook Workout 2-6*

Answers: 1) 359; 2) 8; 3) 380; 4) 105; 5) 9; 6) 96 ways; 7) 36; 8) 301 seeds; 9) 5; 10) 5; 11) 10; 12) 144; 13) 8 factors; 14) 900; 15) 25

Possible Next Steps

If there is still time, you can have students explore some of the number theory categories in a bit more detail.

Can you


the patterns?

Related to Exponents and Factors... ask students to consider each digit 1 through 9 and determine the pattern created by the units digits associated with its positive, integral powers. For example, when 7 is raised to consecutive, integral powers, starting with 7^1 (so 7^1, 7^2, 7^3, ...), the units digits of the results are 7, 9, 3, 1, 7, 9, 3, 1, ..., which has a repeating cycle of four values.

1 raised to consecutive powers always results in a units digit of 1.
2 raised to consecutive powers produces the four-value cycle 2, 4, 8, 6, 2, 4, 8, 6, ...
3 raised to consecutive powers produces the four-value cycle 3, 9, 7, 1, 3, 9, 7, 1, ...
4 raised to consecutive powers produces the two-value cycle 4, 6, 4, 6, ...
5 raised to consecutive powers always results in a units digit of 5.
6 raised to consecutive powers always results in a units digit of 6.
7 raised to consecutive powers produces the four-value cycle 7, 9, 3, 1, 7, 9, 3, 1, ...
8 raised to consecutive powers produces the four-value cycle 8, 4, 2, 6, 8, 4, 2, 6, ...
9 raised to consecutive powers produces the two-value cycle 9,1, 9, 1, ...
*This list assumes the *consecutive powers* are the set of positive integers.

Students now can find the units digit of any of the following expressions: $8^3 + 7^3 – 9^3$; $2^{34} + 3^{42} + 4^{23}$; $(6^7)(2^3)(5^9)$. Ask students to create their own problems to challenge each other!

Fun Number Theory Student Sheet

Computation

1. ____________ If the digits 4, 5, 6, 7 and 9 are placed in the boxes shown such that there is one digit in each box and each of the five digits is used, what is the smallest possible result of the subtraction problem?

$$\begin{array}{r} \square\square\square \\ -\quad \square\square \\ \hline \end{array}$$

2. ____________ In the following addition procedure, each letter represents a distinct digit. What is the value of N?

$$\begin{array}{r} MQM \\ +\quad NQ \\ \hline QHHH \end{array}$$

3. ____________ If x and y are positive integers with $x + y < 40$, what is the largest possible product xy?

Factors and Divisibility

4. ____________ What is the least positive integer divisible by the four smallest odd, positive integers?

5. ____________ A positive two-digit integer is divisible by n and its units digit is n. What is the greatest possible value of n?

6. __________ways A day can be evenly divided into 86,400 periods of 1 second; 43,200 periods of 2 seconds; or in many other ways. In total, how many ways are there to divide a day into n periods of m seconds, where n and m are positive integers?

7. ____________ What is the smallest positive perfect square that is divisible by both 2 and 3?

8. __________seeds Johnny had a full bag of apple seeds. He found that if he repeatedly removed the apple seeds 2 at a time, 1 seed remained in the bag at the end. Similarly, if he repeatedly removed the seeds 3, 4, 5 or 6 at a time from the full bag, 1 seed remained in the bag at the end. However, if he removed the seeds 7 at a time from the full bag, no seeds remained at the end. What is the least number of seeds he could have in the full bag?

Exponents and Patterns

9. ____________ What is the ones digit of the integer form of $(2 + 3)^{23}$?

10. ____________ When the expression $3^{444} + 4^{333}$ is written as an integer, what is the units digit?

Factorials

11. ____________ What is the value of n for which $(3!)(5!)(7!) = n!$?

12. ____________ What is the greatest perfect square that is a factor of 7!?

Perfect Squares and Cubes

13. ________ factors How many factors of 8000 are perfect squares?

14. ____________ The product of integers 240 and k is a perfect cube. What is the smallest positive value of k?

15. ____________ If k is an integer and $k > 100$, what is the smallest possible integer value of the cube root of k^2?

Fun Number Theory Solutions

Computation

1. To get the smallest possible difference, we want to subtract the largest possible two-digit number we can make (97) from the smallest possible positive three-digit number we can make (456). The result is 456 – 97 = **359**.

2. The letter M must represent the digit 9, since no two-digit addend would be enough to add to a three-digit number below 901 and make is a four-digit number. The letter Q must represent the digit 1, for similar reasons. We now have 919 + _1 = 1 _ _ _. The letter H must represent the digit 0, since 9 + 1 in the units place equals 10, which leaves a 0 in the units place. The only possible sum is 919 + 81 = 1000, so N is **8**.

3. We should make $x + y = 39$, and x and y should be as close in value to each other as possible to get the greatest product. Let's try $x = 20$ and $y = 19$. Then $xy = 20 \times 19 =$ **380**.

Factors and Divisibility

4. The four smallest odd, positive integers are 1, 3, 5 and 7. Since 3, 5 and 7 are each prime and don't share any common factors, their least common multiple is 3 × 5 × 7 = **105**.

5. Since we are looking for the greatest possible one-digit number, let's try 9. Are there any two-digit integers with a units digit of 9 that are divisible by 9? Yes... 99 works. So n = **9**.

6. We just need to know how many possible factors of 86,400 there are. Rather than testing the integers 1 through 86,400 (or the positive integers less than $\sqrt{86{,}400}$) to see if they are factors of 86,400, we can quickly determine the number of factors of 86,400. First we need to determine the prime factorization of 86,400. This is $2^7 \times 3^3 \times 5^2$. From here we can calculate the total number of factors. First, take the exponent of each of the prime factors and increase each of them by 1. This gives us 8, 4 and 3. Now find the product of these new numbers: 8 × 4 × 3 = 96. The number 86,400 has **96** factors, and each of these factors could be the number of periods into which our day could be divided.

7. The smallest positive perfect square that is divisible by both 2 and 3 must have 2 and 3 as factors. In fact, it must have the square of every one of its prime factors as a factor. Since 2 and 3 are relatively prime, there is no overlap, and our number is $2^2 \times 3^2 =$ **36**.

8. We are looking for a multiple of 7 that is one more than a multiple of 2, 3, 4, 5 and 6. To create a number that is one more than a multiple of 2, 3, 4, 5 and 6, we find the least common multiple (LCM) of these number, which is 60, and add 1. But 61 is not a multiple of 7. Let's try 2 × 60 + 1 = 121. This is not a multiple of 7 either. We keep creating numbers that are 1 more than a multiple of 60 until we find a multiple of 7. The least such number is 5 × 60 + 1 = 301, which is also 43 × 7. The least number of apple seeds Johnny could have had in his full bag is **301** seeds.

Exponents and Patterns

9. Simplifying the expression, we have 5^{23}. It is true that 5 raised to any positive integer results in an integer with a units digit of **5**.

10. First we observe a pattern in the units digit of the powers of 3: $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, and so on. The units digits cycle through the four digits 3, 9, 7, 1. Since 444 is a multiple of 4, the expression 3^{444} must end in a 1. The units digits of the powers of 4 simply alternate between the digits 4 and 6. Since 333 is odd, the expression 4^{333} must end in a 4. Thus, the units digit of the value of $3^{444} + 4^{333}$ must be 1 + 4 = **5**.

Factorials

11. The value of (3!)(5!)(7!) must be greater than 7! alone. Using factors from 3! and 5!, we can create the factors of 8, 9 and 10 as follows: (3!)(5!) = 3 × 2 × 1 × 5 × 4 × 3 × 2 × 1 = (2 × 4) × (3 × 3) × (2 × 5) = 8 × 9 × 10. Therefore, (3!)(5!)(7!) = (10!), and *n* must be **10**.

12. We don't need to know that 7! = 5040 to find the greatest perfect square factors of 7!. We should look at the prime factorization of 7!, which is 7 × 6 × 5 × 4 × 3 × 2 × 1 = 7 × (2 × 3) × 5 × (2 × 2) × 3 × 2 × 1. A perfect square will have a prime factorization where each factor is raised to a power that is an even number. We have four factors of 2 and two factors of 3, so the greatest perfect square factor of 7! is $2^4 \times 3^2 = 16 \times 9 =$ **144**.

Perfect Square and Cubes

13. Let's first determine the factors of 8000. The prime factorization is $2^6 \times 5^3$. If we increase each of the powers of the prime factors by 1 and then find the product of these new values, we see that 8000 has (6 + 1)(3 + 1) = 28 factors. However, many of these are not perfect square factors. We can use a similar process though. Rewrite the factorization as $(2^2)^3 \times (5^2)^1 \times 5$. Now we see there are three of the 2^2 perfect factors and one of the 5^2 perfect square factors. Increasing the exponent of the 2^2 factor and the 5^2 factor by one each and multiplying, we see there are (3 + 1)(1 + 1) = **8** perfect square factors of 8000.

14. The prime factorization of 240 is $2^4 \times 3 \times 5 = 2^3 \times 2 \times 3 \times 5$. If we wish to make the smallest positive perfect cube, we will need two more factors of 2, two more factors of 3 and two more factors of 5 to get $2^3 \times 2^3 \times 3^3 \times 5^3$. The value of *k* must be $2^2 \times 3^2 \times 5^2 = 4 \times 9 \times 25 =$ **900**.

15. Squaring *k* will double the number of each of its prime factors. If we want the value of k^2 to be a perfect cube, then *k* must start out as a perfect cube, which is to say that there must be a multiple of 3 of each of its prime factors. The smallest perfect cube greater than 100 is 125, which is 5^3. If $k = 5^3$, then $k^2 = (5^3)^2 = 5^6$. The cube root of 5^6 is equal to 5^2, which is **25**.