Transform the Non-Planar Graph to Planar Graph

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Abstract

We want to convert the given non-planar graph G(V,E) to planar graph G'(V',E') by removing minimum number of vertices or edges or both. To our knowledge, there are only few easily accessible algorithms to do so. We will make use of Maximum Planar Induced Subgraph algorithm to solve this problem. The non-planar vertex deletion or vertex deletion vd(G) of a graph G is the smallest non-negative integer k, such that the removal of k vertices from G produces a planar graph G'. In this case G' is said to be a maximum planar induced subgraph of G.

Keywords :- *Planar graph, Non-planar graph, Planar embedding, Complete graph, Nonplanar vertex deletion, Nonplanar edge deletion, Cubic graph*

1 Introduction

Graphs are a collection of nodes/ vertices and edges that represent relationships. There are two types of graphs- directed and undirected graphs. While directed graphs contain an ordered pair of vertices, undirected graphs contain an unordered pair of vertices. In this paper we will come across two types of graphs known as planar and non-plananr graphs. A graph is said to be planar if it can be drawn in a plane so that no edge cross whereas, a graph is said to be non planar if it cannot be drawn in a plane so that no edge cross. Following are the important notations and definitions that we will use frequently in this paper.

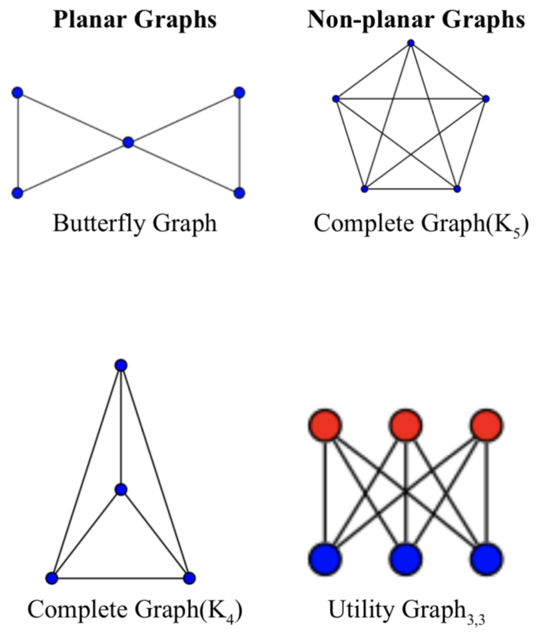
1.1 Basic Definitions

1. *Graph:* Graph is a pair G=(V,E) where,
   - V is a set of vertices (or nodes), and
   - E $\subseteq$ (VxV) is a set of edges.

   A graph may be directed or undirected, if graph is undirected, then adjacency relation defined by edges is symmetric.

2. *Directed and Undirected Graph:* A directed graph is graph, i.e., a set of objects (called vertices or nodes) that are connected together, where all the edges are directed from one vertex to another. A directed graph is sometimes called a digraph or a directed network. In contrast, a graph where the edges are bidirectional is called an undirected graph.

3. *Planar non-planar graph:* a planar graph is a graph that can be embedded in the plane i.e.,it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a plane graph or planar embedding of the graph. Some examples of planar graph and non-planar graphs are:



4. *Cubic graph:* All those graphs whose vertices have degree 3 are known as cubic graph.

5. *Maximum induced graph:* Let G=(V,E) be a graph, v $\in$ V and S $\subset$ V. The subgraph of

G induced by S is the maximal subgraph of G with vertex set S. The graph G- v is the subgraph of G induced by V{v}. The graph G-S is the subgraph of G induced by V\S.

6. *Subdivision of a graph:* A subdivision of an edge $e = uv$ replaces e by a path of length 2 connecting u and v, where the internal vertex of the path is a new vertex. A graph H is a subdivision of a graph G, if H is obtained from G by a sequence of edge subdivisions. We observe that a subdivision of a planar graph is also a planar graph.

7. *Contraction of an edge:* Acontraction of an edge e=uv replaces its end vertices u, v by a new vertex w whose neighbourhood N(w)= (N(u) ∪ N(v)) \ {u,v} (i.e., w is adjacent to every other vertex that was adjacent to u or v). The contraction of an edge in a planar graph produces another planar graph. We say that a graph G is contractible to a graph H, if H is obtained from G by a sequence of edge contractions. We say that a graph G has a graph H as a minor, if G has as a subgraph a graph contractible to H.

1.2 Motivation

Measures for nonplanarity and planarity have an important place in the study of planar graphs due to many industrial and combinatorial applications which involve planarity concepts. Testing a graph for planarity, non-planarity and embedding the planar graph in the plane have several applications. For example, the design of integrated circuits and the layout of printed circuit boards require testing whether a circuit can be embedded in the plane without edge crossings. Several cases of the routing problem have been shown to be equivalent to constructing planar embedding of special classes of graphs. Determining isomorphism of chemical structures is simplified if the structures are known to be planar. Because of the great practical interest, these two problems- planarity, non-planarity testing and planar embedding have been extensively studied in the literature.

There are several important measures for the nonplanarity of a graph, for instance, the minimum number of crossings in an embedding in the plane, the genus, the minimum number of edges whose removal defines a planar graph, the minimum number of edge-disjoint planar subgraphs whose sets of edges partition the set of edges of the graph.

1.3 Literatutre review

The nonplanar vertex deletion or vertex deletion vd(G) of a graph G is the smallest nonnegative integer such that the removal of vd(G) vertices from G produces a planar graph G′. The graph G′ is a maximum planar induced subgraph of G. The vertex deletion decision problem (VD) consists in deciding, given a graph G and a nonnegative integer k, whether vd(G)≤ k. The minimization problem of finding vd(G) of a given graph G = (V,E) is denoted by MINVD. The maximization problem of finding the number of vertices of a maximum planar induced subgraph of a given graph G = (V,E) is denoted by MAXPIS.

Yannakakis [1,2] proved that Vertex Deletion is NP-complete. Unless P = NP, there is no polynomial-time approximation algorithm with a fixed ratio for MAXPIS for graphs in general.So finding the maximum induced subgraph of a graph is NP-Hard problem and only approximation algorithms are known upto a max degree ≤ d. So in this article we will learn about a algorithm that is a polynomial-time 3/4 approximation algorithm for finding a maximum planar induced subgraph of a maximum degree 3 graph.

For a simple, connected, planar graph with v vertices and e edges and f faces, the following simple conditions hold for $v \geq 3$:

- $e \leq 3v - 6$
- If there are no cycles of length 3, then $e \leq 2v - 4$
- $f \leq 2v - 4$

2 Algorithm Design

We propose a simple greedy 3/4 -approximation algorithm for MAXPIS in a maximum degree 3 graph. Let x be a vertex of degree 2 adjacent to vertices a and b in a graph G. We smoothen x, when we remove x from G and we add edge ab. Note that the obtained graph may have multiple edges. When we smoothen a degree 2 vertex, we preserve the property of being planar.

2.1 Pseudo Code

Algorithm 1 Testing for planarity for graph G

Input: G(V,E), v, e
Output: True if graph is planar, false otherwise

if e $\nleq 3v - 6$ **then**
    return false
end if
if cycles of length 3 **OR** e $\nleq 2v - 4$ **then**
    return false
end if
$f = e - v + 2$
if f $\nleq 2v - 4$ **then**
    return false
end if
return false

Algorithm 2 planar induced subgraph

Input: Connected graph G = (V , E) with maximum degree 3
Output: Subset X of V , such that G[X] is a planar graph

X ← V
i ← 0
G_i ← G
while G_i is non-planar **do**
    while G_i is not a cubic simple graph **do**
        Removing vertices of degree 1 from G_i
        Smoothening vertices of degree 2 of G_i
        Replacing multiple edges by ordinary edges in G_i
        Removing the loops from G_i
    end while
    Select a noncut vertex u_i of G_i ; $N_{G_i}(u_i) = \{a_i , b_i , c_i\}$
    X ← X $\setminus\{u_i\}$
    $G_{i+1} \leftarrow G_i - u_i$
    i ← $i + 1$
end while
return X

3 Analysis

3.1 Time Complexity Analysis

- The runtime of algorithm 1 depends on [V] and on the topology of the graph.
- In the worst-case, the time complexity would be $O(|V|)^3$ or more. (a polynomial time complexity).
- The worst case of the proposed algorithm is reached when the selected vertex set X has size |X| = 3n/4
- Since it is a NP-Hard problem. The time complexity of the given algorithm is polynomial for all the cases. And this solution is 3/4- approximation solution feasible for graph with maximum degree 3 only.

<u>All Cases:</u>

$time_{best} = time_{average} = time_{worst} = O(V^3)$

where V is the total no. of vertices in the graph.

4 Experimental Study

4.1 Profiling

The best way to study an algorithm is by graphs and profiling.

4.1.1 Time Complexity

We have seen that

$time_{best} = time_{average} = time_{worst} = O(V^3)$

So,the graphs for all the cases remains the same.

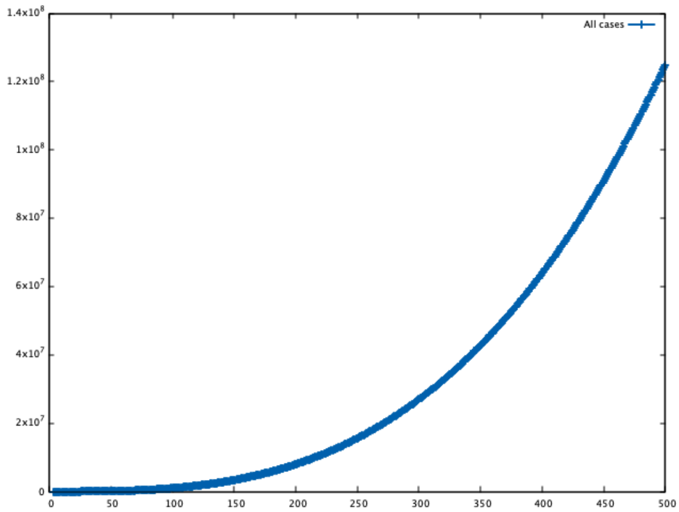


Figure 1: V vs T graph (All Cases)

5 Discussions

5.1 Applications

The concepts of graph theory are used widely to study and model various real time applications in different fields. The planar graphs has applications in several other areas of mathematical and computational study. In geographic information systems, flow networks and to cellular networks describing drainage divides. In checmistry Determining isomorphism of chemical structures is simplified if the structures are known to be planar.

6 Conclusion

Here we propose a polynomial time 3/4 -approximation algorithm for MAXPIS in a maximum degree 3 graph. The worst case of the proposed algorithm is reached when the selected vertex set X has size $|X| = 3n/4$. Note that the vertices discarded by the algorithm from V define an independent set. It is a property of the Max SNP-hard class that there are no polynomial time approximation schemes (PTASs) for problems in this class unless P = NP.

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