Matrix Decomposition (factorization) :-

* data is represented as matrix

Q.) How to summarize matrix ?
Q.) How matrix can be decomposed ?
Q.) How these decomposition can be used for matrix approximation ?

I :- Summarizing Matrix :-

Determinants & Trace

Eigenvalues & Eigenvectors

(1) Determinant :-

→ Only defined for square matrix, $A \in \mathbb{R}^{n \times n}$
→ det (A)
→ Determinant of a square matrix $A \in \mathbb{R}^n$ is a funcn that maps A onto real number

* → For any square matrix $A \in \mathbb{R}^{n \times n}$ it holds that A is invertible if and only if

$$\det(A) \neq 0.$$

* → Determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is the product of its eigenvalues i.e. $\det(A) = \prod_{i=1}^{n} \lambda_i$

for $n=1$,

$$\det(A) = \det(a_{ij}) = a_{11}$$

for $n=2$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

For $n=3$,

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33}$$

* we compute det(A) by using Laplace expansion.

Trace :-

Note :- Trace of matrix A is the sum of its eigenvalue, $tr(A) = \sum_{i=1}^{n} \lambda_i$

→ Trace of sq matrix $A \in \mathbb{R}^{n \times n}$ is defined as:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

→ ∴ Trace is sum of all diagonal elts of A.

Characteristic Polynomial :- For $\lambda \in \mathbb{R}$ and a sq matrix $A \in \mathbb{R}^{n \times n}$.

$$p_A(\lambda) = \det(A - \lambda I)$$
$$= c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1} + (-1)^n \lambda^n$$

where, $c_0, \ldots c_{n-1} \in \mathbb{R}$ is the characteristic polynomial of A.

$$c_0 = \det(A) \rightarrow \text{determin of } A$$
$$c_{n-1} = (-1)^{n-1} \, tr(A) \rightarrow \text{trace of } A$$

$\boxed{\text{Eigen Values \& Eigen Vectors}}$

Def :- Let $A \in \mathbb{R}^{n \times n}$ be a sq matrix. Then $\lambda \in \mathbb{R}$ is an eigenvalue of A & $x \in \mathbb{R}^n$ is the corresponding eigenvector of A.

$$Ax = \lambda x \longrightarrow \text{Eigenvalue eq}^n$$

* Collinearity :- Two vectors are collinear if they point in the same or opposite direction.

Codirection :- Two vector that point in same the same direction are called codirected.

* Remark :- If x is eigenvector of A associated with eigenvalue λ, then for any $c \in \mathbb{R} \setminus \{0\}$ it holds that cx is an eigenvector of A with same eigenvalue

$$A(cx) = cAx = c\lambda x = \lambda(cx)$$

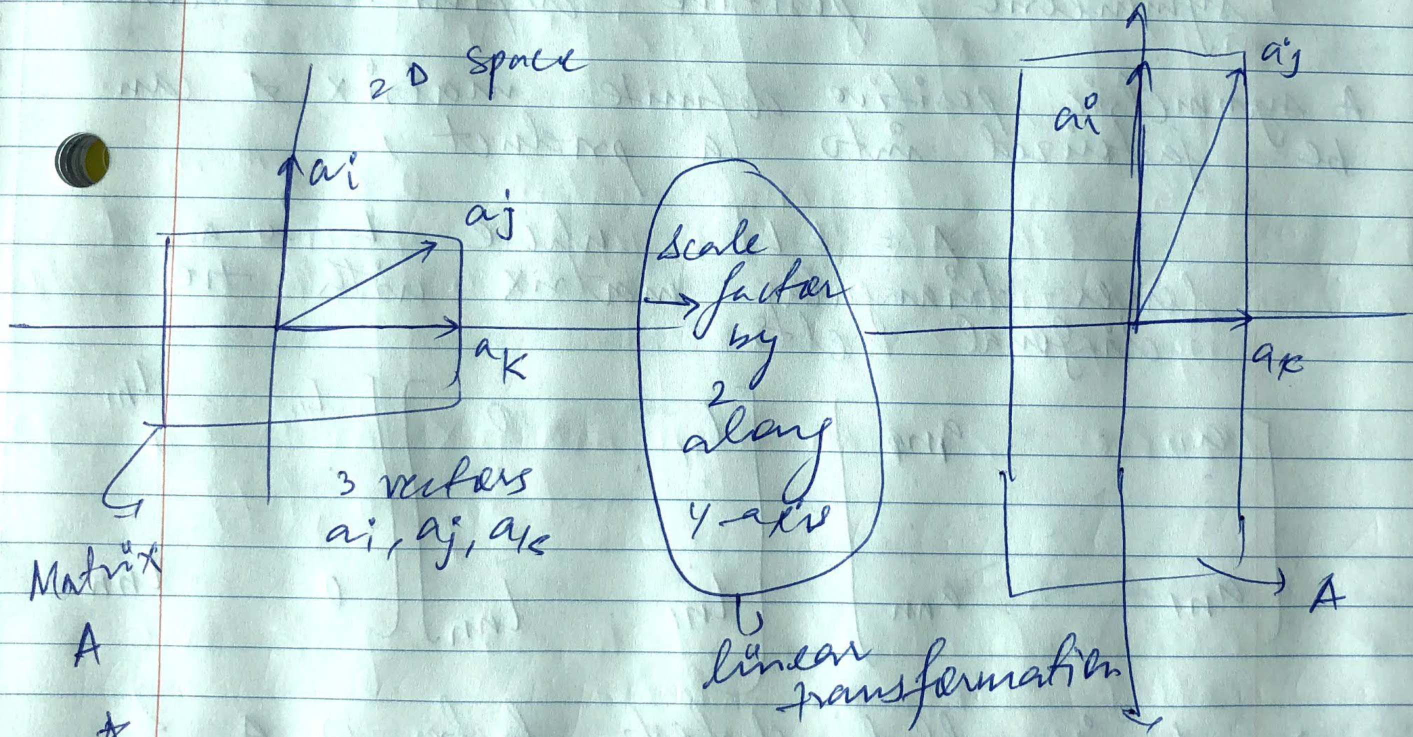
Thus, all vectors that are collinear to x are also eigenvector of A.

(E_λ)

Eigenspace :- For $A \in \mathbb{R}^{n \times n}$, the set of all eigenvectors of A associated with an eigenvalue λ spans a subspace of $\mathbb{R}^n$, which is called the eigenspace of A w.r.t of λ and is denoted by E_λ.

Eigenspectrum :- The set of all eigenvalues of A is called the eigenspectrum, or just spectrum of A.

$\boxed{\text{Geometric Intuition}}$



* $a_K \rightarrow$ has same scale & dir^n after transformat^n.
$a_i \rightarrow$ scale changed, dir^n is same
$a_j \rightarrow$ scale & dir^n both changed.

$\therefore$ a_K & a_i are eigenvectors and their scale change is eigenvalue.

Cholesky Decomposition

→ Matrix factoru Method

* For +ve real numbers, we have the sq-root operations that gives us a decomposition of nos into identical components.

$$9 = 9 = 3 \cdot 3$$

* The cholesky decomposition / factorization provides square root equivalent operation on symmetric, positive definite matrices.

A symmetric, positive definite matrix A can be factorized into a product,

$A = LL^T$, where L is a lower-triangular matrix with +ve diagonal elts.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & \cdots & l_{n1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_{nn} \end{bmatrix}$$

→ L is called cholesky factor of A & L is unique.

→ $A = LL^T$

$$\left.\begin{aligned} \det(A) &= \det(L) \cdot \det(L^T) \\ &= \det(L)^2 \end{aligned}\right\} \text{L is triangular matrix, det is product of diagonal elt}$$

Eigendecomposition & Diagonalization

* A diagonal matrix is a matrix that has value zero on all off-diagonal elt.

$$D = \begin{bmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_n \end{bmatrix}$$

* $\det(D)$ = product of all diagonal elts.
* Matrix power (D^k) = each digonal elt raised to power k.
* Inverse (D^{-1}) = Reciprocal of its diagonal elt if all of them are non-zero.

Def:- A matrix $A \in \mathbb{R}$ is diagonalizable if it is similar to a diagonal matrix, i.e. if there exist an invertible matrix.

$$P \in \mathbb{R}^{n \times n} \ \& \ D = P^{-1} A P$$

$\boxed{\text{Eigendecompos}}$:

A sq matrix $A \in \mathbb{R}^{n \times n}$ can be factored into,

$$A = P D P^{-1}$$

where, $P \in \mathbb{R}^{n \times n}$ & D is a diagonal matrix whose diagonal entries are eigenvalues of A.

* A Symmetric matrix $S \in \mathbb{R}^n$ can always be diagonalized.
* Eigendecomposition requires square matrices.

SVD

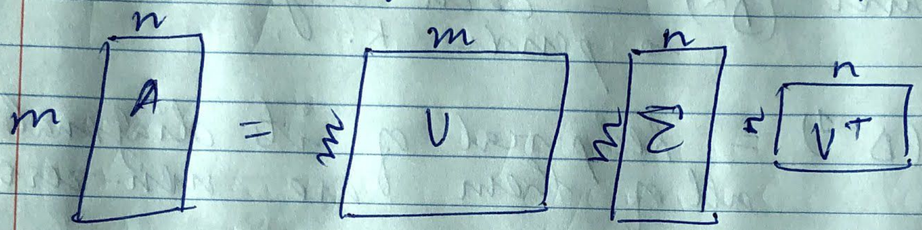
(Singular Value Decomposition)

* Can be applied to all matrices
* always exist.

SVD Theorem

Let $A^{m\times n}$ be a rectangular matrix of rank $r \in [0, \min(m,n)]$.

The SVD of A, is a decomposition of the form,



with an orthogonal matrix $U \in \mathbb{R}^{m\times m}$ with cols vectors u_i, $i = 1, \cdots m$ and an orthogonal matrix $V \in \mathbb{R}^{n\times n}$ with col vectors v_j, $j = 1, \cdots n$.

Moreover, Σ is an $m\times n$ matrix with $\Sigma_{ii} = \sigma_i \geq 0$ and $\Sigma_{ij} = 0$, $i \neq j$

* The diagonal entries σ_i, $i = 1, \cdots r$ of Σ are called singular values.

* u_i are called the left-singular vectors.
* v_j are called the right-singular vectors.

By convention, the singular values are ordered i.e. $\sigma_1 \geq \sigma_2 \geq \sigma_r \geq 0$

* Singular value matrix Σ is unique.

* The singular value matrix Σ is unique, but it requires some attention. Σ is of the same size as A.

↳ This means that Σ has a diagonal submatrix that contains the singular values & need additional zero padding.

If $m > n$,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \cdots & \sigma_n \\ 0 & \cdots & 0 \\ 0 & & 0 \\ 0 & 0 \cdots & 0 \end{bmatrix}$$

If $m < n$,

$$\Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & \cdots & \sigma_m & & 0 \end{bmatrix}$$

<u>Remark</u>:- The SVD exist for any matrix $A \in \mathbb{R}^{m \times n}$.

Q.) <u>Intution</u>:- How SVD works?

→ SVD perform a basis change via V^T followed by scalaing and augmentation in dimensionality via the singular matrix Σ.

→ Finally, it perform a second basis change via U.

Computing SVD :-

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

We need to compute,
right singular vectors v_j
right singular values σ_k
left singular vectors u_i

① Right-singular vectors as the eigenbasis of A^TA.

$$A^TA = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We compute the singular values & right-singular vectors $\{v_j\}$ through eigenvalue decomposition of A^TA,

$$A^TA = PDP^T = \begin{bmatrix} \frac{5}{\sqrt{30}} & 0 & -1/\sqrt{6} \\ \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{5}} & -2/\sqrt{6} \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{30}} & \frac{-2}{\sqrt{30}} & \frac{1}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and we obtain right-singular vectors as the columns of P

$$V = P = \begin{bmatrix} \frac{5}{\sqrt{30}} & 0 & \frac{-1}{\sqrt{6}} \\ \frac{-2}{\sqrt{30}} & 1/\sqrt{5} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

(2.) Singular-value Matrix

As the singular values σ_i are the sq root of the eigenvalues of A^TA we obtain them straight from D.

Since rank $rk(A) = 2$, there are only two non-zero singular values. The size of A & Σ must be same.

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(3) Left-singular vectors as the normalized image of the right singular vectors.

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5/\sqrt{30} \\ -2/\sqrt{30} \\ 1/\sqrt{30} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$U = [u_1\ u_2] = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

let A be a matrix $\in \mathbb{R}^{n \times n}$

Eigenvalue Decomposition	Singular Value Decomposition
$A = PDP^{-1}$	$SVD = U \Sigma V^T$
* Only defined for sq matrix. and only exist if we can find a basis of eigenvector of $\mathbb{R}^n$	* SVD always exist for any matrix.
* Vectors in eigendecomp are not orthogonal. i.e change of basis is not a simple rotation & scaling	* Vectors in U & V in SVD are orthogonal so they represents rotations

* Both eigendecomposition & SVD are composition of 3 linear mapping:

→ change of basis in the domain
→ Independent scaling of each new basis vector & mapping from domain to co-domain.
→ change of basis in the codomain.

* The basis change matrices P and P^{-1} are inverse to each other.	* In SVD, U & V are generally not inverse to each other. They perform basis change in different vector spaces.
* Not necessary that diagonal elt in D are non-negative & real	* Σ are all real & non-negative.

* The SVD and the eigendecomposition are closely related through their projections.

→ The left-singular vectors of A are eigenvectors of AA^T

→ The right-singular vector of A are eigenvectors of A^TA.

→ The non-zero singular values of A are the sq root of the non-zero eigenvalues of AA^T and are equal to the nonzero eigenvalues of A^TA.

* For symmetric matrix $A \in \mathbb{R}^{n \times n}$, the eigenvalue decomposition & SVD are the one & same.