Linear multistep methods

Initial value problem (IVP)

Given an ODE,

$$\dot{u} = f(t, u)\,, \qquad u(0) = u_0$$

Define a mesh

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_N = T$$

Define a *mesh function* y_n

$$y_n \approx u(t_n)\,, \qquad n = 0, 1, \cdots, N$$

y_n satisfies a discretized form of the ODE.

Linear multistep methods

A *linear s-step method* is

$$\frac{a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s}}{\tau} = b_0 f_n + b_1 f_{n-1} + \cdots + b_s f_{n-s}$$

for $n = s, s+1, \ldots$.

Here

$$f_n \equiv f(t_n, y_n)\,,$$

τ is the step size,

a_k and b_k, $k = 0, \cdots s$ are n-independent.

Linear multistep methods

$$\frac{a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s}}{\tau} = b_0 f_n + b_1 f_{n-1} + \cdots + b_s f_{n-s}$$

for $n = s, s+1, \ldots$.

Start from $n = s$, then solve for y_n given $y_{n-1}, y_{n-2}, \cdots, y_{n-s}$.

Need s initial conditions. Take

$$y_0 = u_0$$

find $y_1, \ldots y_{s-1}$ via e.g. a Runge-Kutta method.

Linear multistep methods

$$\frac{a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s}}{\tau} = b_0 f_n + b_1 f_{n-1} + \cdots + b_s f_{n-s}$$

for $n = s, s+1, \ldots$.

- Unlike RK methods, only evaluate the r.h.s. at t_n.
- If $b_0 = 0$, the method is explicit. Otherwise, it is implicit.
- Without loss of generality, assume

$$\sum_{k=0}^{s} b_k = 1$$

Families of linear multistep methods

Adams methods

Take $a_0 = -a_1 = 1$, $a_k = 0$ for $k > 1$.

$$\frac{y_n - y_{n-1}}{\tau} = \sum_{k=0}^{s} b_k y_{n-k}$$

Select b_k coefficients to maximize the approximation order.

Adams-Bashforth schemes Take $b_0 = 0$.
Max order is s.
$s = 1$ is the Euler scheme.

Adams-Moulton schemes For $b_0 \neq 0$, methods are implicit.
Max order is $s + 1$.
$s = 1$ is the implicit Euler scheme.

Families of linear multistep methods : BDF

Backwards differentiation formulas (BDF)

- Take $b_0 = 1$.
- Approximate the derivative by an s-point finite difference formula.

$$
\begin{aligned}
s = 1 \qquad & y_n - y_{n-1} = \tau f_n \\
s = 2 \qquad & \frac{3}{2} y_n - 2 y_{n-1} + \frac{1}{2} y_{n-2} = \tau f_n \\
& \dots
\end{aligned}
$$

Zero-stability of linear multistep methods

Stability of ODEs and numerical schemes

Suppose that the IVP is stable.

$$\dot{u} = f(t, y)\,, \qquad u(0) = u_0$$

What are the conditions for the numerical scheme to be stable?

$$\frac{a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s}}{\tau} = b_0 f_n + b_1 f_{n-1} + \cdots + b_s f_{n-s}$$

Note that it needs s initial conditions. It might admit extra, spurious solutions.

Zero-stability of LMM

Only consider a homogenous equation,

$$\dot{u} = 0$$

Its discretized version reads

$$a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s} = 0$$

If a method is not zero-stable, it is not usable.

Zero-stability of LMM

For a homogenous recurrence relation of order s

$$a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s} = 0$$

Look for solutions in the form

$$y_n = q^n$$

Characteristic polynomial

$$a_0 q^s + a_1 q^{s-1} + \cdots + a_s = 0$$

A root of the CP defines a particular solution.

Zero-stability of LMM

For a homogenous recurrence relation of order s

$$a_0 y_n + a_1 y_{n-1} + \cdots + a_s y_{n-s} = 0$$

- A simple root, q, of the C.P. defines a particular solution

$$y_n = q^n$$

- A root of multiplicity r defines particular solutions

$$y_n = q^n,\ nq^n,\ n^2 q^n,\ \cdots,\ n^{r-1} q^n$$

Zero-stability of LMM

A method is zero-stable if and only if the *root condition* is satisfied:

- All roots of the characteristic polynomial have $|q| \leqslant 1$.
- All roots with $|q| = 1$ are simple.

Convergence of LMM

Let

- $|\partial f(t,u)/\partial u| < L$ for $0 \leqslant t \leqslant T$
- the root condition is satisfied.

Then, for all $n \geqslant s$,

$$|y_n - u(t_n)| \leqslant M \left(\max_{k<s} |y_k - u(t_k)| + \max_k |\psi_k| \right)$$

where $M = M(L,T)$ does not depend on n.