DGP : $y_i = \alpha + \beta x_i^* + \varepsilon_i^*$

Observed x : $x_i = x_i^* + \upsilon_i$

$x_i^*, \varepsilon_i^*, \upsilon_i$ uncorrelated

$y_i = \alpha + \beta x_i + \varepsilon_i$ regression model

a) $$b = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

b) $$y_i - \bar{y} = (\alpha + \beta x_i + \varepsilon_i) - (\alpha + \beta \bar{x} + \bar{\varepsilon}) = \beta(x_i - \bar{x}) + (\varepsilon_i - \bar{\varepsilon})$$

$$b = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(\beta(x_i - \bar{x}) + (\varepsilon_i - \bar{\varepsilon}))}{\sum_{i=1}^{n} (x_i - \bar{x})^2} = \beta + \frac{\sum_{i=1}^{n} (x_i - \bar{x})(\varepsilon_i - \bar{\varepsilon})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

c) $$x_i^* = x_i - \upsilon_i$$

$$y_i = \alpha + \beta x_i^* + \varepsilon_i^* = \alpha + \beta (x_i - \upsilon_i) + \varepsilon_i^*$$

$$= \alpha + \beta x_i + \underbrace{\varepsilon_i^* - \beta \upsilon_i}_{\varepsilon_i}$$

$$\Rightarrow \varepsilon_i = \varepsilon_i^* - \beta \upsilon_i$$

d) $$\operatorname{cov}(x_i, \varepsilon_i) = \operatorname{cov}(x_i^* + \upsilon_i, \varepsilon_i^* - \beta \upsilon_i)$$

$$= -\beta \operatorname{cov}(\upsilon_i, \upsilon_i) = -\beta \sigma_\upsilon^2$$

e) $$b = \beta + \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(\varepsilon_i - \bar{\varepsilon})}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \approx \mathrm{var}(x_i) = \mathrm{var}(x_i^* + u_i) = \sigma_*^2 + \sigma_u^2$$

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(\varepsilon_i - \bar{\varepsilon}) \approx \mathrm{cov}(x_i, \varepsilon_i) = -\beta\sigma_u^2$$

$$b - \beta \approx \frac{-\beta\sigma_u^2}{\sigma_*^2 + \sigma_u^2}$$

f) $$SN = \frac{\sigma_*^2}{\sigma_u^2}$$

$$b - \beta \approx \frac{-\beta\sigma_u^2}{\sigma_*^2 + \sigma_u^2} = \frac{-\beta}{\frac{\sigma_*^2}{\sigma_u^2} + 1} = \frac{-\beta}{SN + 1}$$

SN	Bias
1	-0.5
3	-0.25
10	-0.09