Unit 2: Logic

Logic is a science of the necessary laws of thought, without which no employment of the understanding and the reason takes place - Immanuel Kant, 1785

Mathematician deal with **statement**
Usually the statements are about **numbers**
The statements may be **true** or **false**
To decide whether a statement is true or false requires a **proof**

1 Propositions and logical operations

Definition 1 (Proposition/Statement)
A statement or proposition is a declarative sentence that is either true or false but not both.

A proposition does not contain a variable(s).

Example 1

(a) $2 + 4 = 5$ [False or 0]

(b) Kathmandu is the capital of Nepal. [True or 1]

(c) Nepal is a South-Asian country. [True]

(d) Kathmandu university is one of the university of Nepal. [True]

(e) For all $n \in \mathbb{N}, n^2$ is a prime. [False]

The above statements are called **primitive** or **simple** statements, for there is really no way to break down into anything simpler.

Example 2

The followings are not a statement.

(a) Do you speak English?

(b) $x = 5$

(c) Hurray!

(d) May you live long.

(e) $x + 1 = y$

So the following types of sentences do not include logical statements.

(a) Interrogative - Question, ask, enquire

(b) Imperative

(c) Exclamatory

(d) Wish/Prayer

(e) Conceptual

1.1 Logical connectives

Definition 2 (Logical connectives)
Those words or symbols which are used to construct a statement by combining two statements are called **sentential connectives** *or* **logical connectives** *or simply* **connectives***.*

The following connective words and their symbols are in use:

Connective word	**Connective symbol**
negation, not	$\neg$ or $\sim$
and	$\wedge$
or	$\vee$
if $\cdots$ then $\cdots$	$\Longrightarrow$
if and only if or iff	$\Longleftrightarrow$

Definition 3 (primitive or simple statement)
A statement that does not involve any connectives is called a simple statment.

Example 3

(a) Today is Monday.

(b) The sun is hot.

(c) $3 + 5 = 10$.

(d) 27 is a multiple of 3.

Definition 4 (Compound statement)
A statement involving the use of one or more connectives is called a compound statement.

Example 4

(a) It is dark **and** the weather is humid.

(b) KU is in Kavre district **or** 5 is an even number.

(c) If 256 is a perfect square, then 1024 is a perfect square.

(d) $n \in \mathbb{N}$ is a multiple of 12 **if and only if** it is a multiple of 3 **and** a multiple of 4.

Definition 5 (Truth table)
True or False value of a statement are called **truth values***, and the table which shows the truth values of a compound statement in terms of its component parts, is called a* **truth table***.*

Definition 6 (Negation)
If p is a statement, the **negation** *of p is the statement* **not** *p, denoted by $\sim p$ or $\neg p$. Thus $\sim p$ is the statement* **it is not the case that** *p.*

From this definition, it follows that if p is true, then $\sim p$ is false and if p is false, then $\sim p$ is true.

Truth Table

p	$\sim p$
T	F
F	T

Example 5 *Consider the statement*

$$\begin{aligned} p &: \textit{The sun is hot.} \\ \sim p &: \textit{The sun is not hot.} \end{aligned}$$

Definition 7 (Conjunction)
If p and q are statements. The conjunction of p and q is a compound statement

$$p \textit{ and } q$$

denoted by $p \wedge q$. The compound statement $p \wedge q$ is true when both p and q are true; otherwise, it is false.

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 6

1.

$$\begin{aligned} p &: \text{It is snowing} \\ q &: 3 > 5 \\ p \wedge q &: \text{It is snowing } \textbf{and } 3 > 5 \end{aligned}$$

2.

$$\begin{aligned} p &: 4 - 2 = 10 \\ q &: 2 < 5 \\ p \wedge q &: 4 - 2 = 10 \textbf{ and } 2 < 5 \end{aligned}$$

Definition 8 (Disjunction)
If p and q are statements. The disjunction of p and q is a compound statement

$$p \textit{ or } q$$

denoted by $p \vee q$. The compound statement $p \vee q$ is true if at least one of p or q is true; it is false when both p and q are false.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 7

1.

$$\begin{aligned} p &: \text{ 2 is a positive integer} \\ q &: 2+3 \neq 5 \\ p \vee q &: \text{ 2 is a positive integer } \textbf{or } 2+3 \neq 5 \end{aligned}$$

2.

$$\begin{aligned} p &: \sqrt{2} \text{ is irrational number} \\ q &: \text{The sun is cold} \\ p \vee q &: \sqrt{2} \text{ is irrational number } \textbf{or} \text{ the sun is cold} \end{aligned}$$

1.2 Quantifiers

Two types:

1. Universal quantifier : $\forall$ for all
2. Existential quantifier : $\exists$ there exists

Definition 9 (Predicate)
The set builder notation

$$\{x \mid p(x)\}$$

is just the collection of all objects x for which p is sensible and true. Such a sentence $p(x)$ is called **predicate**.

$p(x)$ *is also called a* **propositional function** *because each choice of x produces a proposition $p(x)$ that is either true or false.*

Use of predicate

1. In universal quantification:
   The universal quantification of a predicate $p(x)$ is the statement

   $$\text{for all values of } x,\ p(x) \text{ is true}$$

   The universal quantification of $p(x)$ is denoted by

   $$\forall x \quad p(x)$$

Example 8 *The sentence* $p(x) : -(-x) = x$ *is a predicate that makes sense for real numbers* x*. The universal quantification of* $p(x)$*,* $\forall x \quad p(x)$ *is a true statement, because for all real numbers*

$$-(-x) = x$$

Example 9 *Let* $q(x) : x + 1 < 4$*. Then* $\forall x \geq 0 \quad q(x)$ *is a false statement, because* $q(5)$ *is not true.*

2. In existential quantification:
   The existential quantification of a predicate $p(x)$ is the statement

   there exists a value of x for which $p(x)$ is true

   The existential quantification of $p(x)$ is denoted by

   $$\exists x \quad p(x)$$

   Example 10 $q(x) : x+1 < 4$*. The existential quantification of* $q(x)$*,* $\exists x\ q(x)$ *is a true statement because* $q(2)$ *is a true statement.*

   Example 11 *The statement* $\exists y\ y + 2 = y$ *is false. There is no value of* y *for which the propositional function* $y + 2 = y$ *produces a true statement.*

   Exercise 1 *Find the truth value of the following statements.*

   (a) $(\forall x)(\exists y)(x + y < 3)$
   (b) $(\exists y)(\forall x)(x + y < 3)$
   (c) $(\exists y)(\forall x)(x + y = 3)$

3. In programming:
   Two common constructs are

   - if $p(x)$, then execute certain steps, and
   - while $q(x)$, do specified actions.

   In this case the predicate $p(x)$ and $q(x)$ are called **guards** for the block of programming code.

1.3 Conditional statement or implication

If p and q are statements, then the compound statement

if p then q

is called a **conditional statement** or **implication**. It is denoted by

$$p \implies q$$

The statement p is called the **hypothesis** or **antecedant**, and the statement q is called the **conclusion** or **consequent**.

The rule for $p \implies q$ is false only when the hypothesis p is true and the conclusion q is false. In all other situations, it is true. The rule can be seen in the following truth table of $p \implies q$.

Truth Table

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:

(i) If a compound statement s contains n component statements, there will need to be 2^n rows in the truth table of s.

(ii) The symbol $p \implies q$ is read as:

$$\begin{aligned} p &\text{ implies } q \\ \text{if } p &\text{ then } q \\ p &\text{ is sufficient for } q \\ q &\text{ is necessary for } p \\ p &\text{ only if } q \\ q &\text{ if } p \\ q &\text{ follows from } p \end{aligned}$$

(iii) A conditional statement with False hypothesis
Consider the statement:

$$\text{If } 0 = 1, \text{ then } 1 = 2$$

As strange as it may seem, since the hypothesis of this statement is False, the statement as a whole is True.

1.3.1 Converse/Inverse/Contrapositive

Definition 10
If $p \implies q$ is an implication, then

- *the* **converse** *of $p \implies q$ is the implication $q \implies p$.*
- *the* **inverse** *of $p \implies q$ is the implication $\sim p \implies \sim q$.*
- *the* **contrapositive** *of $p \implies q$ is the implication $\sim q \implies \sim p$.*

Truth Table

p	q	$p \implies q$	$q \implies p$	$\sim p$	$\sim q$	$\sim p \implies \sim q$	$\sim q \implies \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Note that the truth value of the compound statements $p \implies q$ and $\sim q \implies \sim p$ are logically equivalent.

Example 12

Write down the converse of the following statement

$$\text{if } n \text{ is a multiple of 12 then } n \text{ is a multiple of 4}$$

Is the original statement True or False? Is the converse statement True or False? Justify your answer by giving a proof or a counter example.
Solution The converse is:

$$\text{if } n \text{ is a multiple of 4 then } n \text{ is a multiple of 12}$$

The original statement is True, because if $n = 12m, m \in \mathbb{N}$ then $n = 4 \times (3m)$. The converse statement is False: 8 is a counter example.

1.3.2 Biconditional statement

Definition 11
If p and q are statements, then the compound statement

$$p \textit{ if and only if } q$$

is called a biconditional or equivalence statement and is denoted by

$$p \iff q$$

The rule for $p \iff q$ is true if p and q have the same truth values; it is false if p and q have different truth values. The rule can obviously be seen in the following truth table of $p \iff q$.

Truth Table

p	q	$p \iff q$
T	T	T
T	F	F
F	T	F
F	F	T

Note that **if and only if** is also written as **iff**.

Example 13

1.

$$\begin{aligned} p &: \text{ 2 is an even number} \\ q &: \text{ 4 is an even number} \\ p \iff q &: \text{ 2 is an even number iff 4 is an even number. This is a \textbf{True} statement.} \end{aligned}$$

2.

$$\begin{aligned} p &: \text{2 is an even number} \\ q &: \text{5 is an even number} \\ p \iff q &: \text{2 is an even number iff 5 is an even number. This is a \textbf{False} statement.} \end{aligned}$$

Exercise 2 *Is the biconditional statement*

$$3 > 2 \textit{ if and only if } 0 < 3 - 2$$

true?

1.3.3 Tautology/contradiction/contingency

Definition 12

- *A statement that is* **true** *for all possible values of its propositional variables is called a* **tautology**.
- *A statement that is* **false** *for all possible values of its propositional variables is called a* **contradiction** *or* **absurdity**.
- *A statement that can be either* **true** *or* **false**, *depending on the truth values of its propositional variables is called a* **contingency**.

Example 14

(*a*) The statement $p \wedge \sim p$ is a contradiction. It can obviously be seen from the truth table of $p \wedge \sim p$.

Truth Table

p	$\sim p$	$p \iff q$
T	F	F
F	T	F

(*b*) The statement $(p \implies q) \iff (\sim q \implies \sim p)$ is a tautology. It can obviously be seen from the truth table of $(p \implies q) \iff (\sim q \implies \sim p)$.

Truth Table

p	q	$p \implies q$	$\sim q$	$\sim p$	$\sim q \implies \sim p$	$(p \implies q) \iff (\sim q \implies \sim p)$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(*c*) The statement $(p \implies q) \wedge (p \vee q)$ is a contingency. It can obviously be seen from the truth table of $(p \implies q) \wedge (p \vee q)$.

Truth Table

p	q	$p \implies q$	$p \vee q$	$(p \implies q) \wedge (p \vee q)$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	F

1.3.4 Logically equivalent

Definition 13
Two statements p and q are logically equivalent or simply equivalent if $p \iff q$ is a tautology, and we denote it by $p \equiv q$.

In other words, two statements are logically equivalent if they have the same truth values.

Example 15

(a) $p \implies q$ is logically equivalent to $\sim p \vee q$. It can be seen from the following truth table.

Truth Table

p	q	$\sim p$	$p \implies q$	$\sim p \vee q$	$(p \implies q) \iff (\sim p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(b) $p \implies q$ is logically equivalent to $\sim q \implies \sim p$.

(c) $\sim (p \implies q)$ is logically equivalent to $p \wedge \sim q$.

(d) $p \implies q \equiv (p \implies q) \wedge (q \implies p)$

Example 16

(a) A conditional statement and its converse are *not* logically equivalent.

(b) A conditional statement and its inverse are *not* logically equivalent.

(c) The converse and inverse of a conditional statement are logically equivalent.

Example 17 *Show that* $\sim (\forall x\ p(x)) \equiv \exists x\ \sim p(x)$.

Solution $\forall x\ p(x)$ is False if there is at least one x for which $p(x)$ is False: in other words, if $\exists x\ \sim p(x)$ is True. Thus we can express this relationship in symbolic form as

$$\sim (\forall x\ p(x)) \equiv \exists x\ \sim p(x)$$

Example 18 *Show that* $\exists x\ (p(x) \vee q(x) \equiv \exists x\ p(x) \vee \exists x\ q(x)$.

Solution The statement $\exists x\ (p(x) \vee q(x)$ is True if and only if $\exists x$ such that either $p(x)$ or $q(x)$ is true, but this means either $\exists x\, p(x)$ or $\exists x\, q(x)$ is True. This holds if and only if $\exists x\, p(x) \vee \exists x\, q(x)$ is True.

Theorem 1 (Logical equivalence)

For any three statements p, q and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalence hold.

(a) Commutative property

$$p \vee q \equiv q \vee p$$
$$p \wedge q \equiv q \wedge p$$

(b) Associative property

$$p \vee (q \vee r) \equiv (p \vee q) \vee r$$
$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$$

(c) Distributive property

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

(d) Identity laws

$$p \wedge \mathbf{t} \equiv p$$
$$p \vee \mathbf{c} \equiv p$$

(e) Negation laws

$$p \wedge \sim p \equiv \mathbf{c}$$
$$p \vee \sim p \equiv \mathbf{t}$$

(f) Double negation law

$$\sim (\sim p) \equiv p$$

(g) Idempotent property

$$p \vee p \equiv p$$
$$p \wedge p \equiv p$$

(h) Universal bound laws

$$p \vee \mathbf{t} \equiv \mathbf{t}$$
$$p \wedge \mathbf{c} \equiv \mathbf{c}$$

(i) De Morgon's laws

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$
$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

(j) Absorption laws

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

(k) Negations of **t** and **c**

$$\sim \mathbf{t} \equiv \mathbf{c}$$
$$\sim \mathbf{c} \equiv \mathbf{t}$$

Example 19 *Verify the logical equivalence*

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$$

with reasons.

Solutions

		Reasons
	$\sim(\sim p \wedge q) \wedge (p \vee q)$	
$\equiv$	$(\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	De Morgon's law
$\equiv$	$(p \vee \sim q) \wedge (p \vee q)$	Double negation
$\equiv$	$p \vee (\sim q \wedge q)$	Distributive law
$\equiv$	$p \vee \mathbf{c}$	Negation law
$\equiv$	p	Identity law

This we can prove constructing the truth table as well.

EXERCISES

1. Prove that each of the following is a tautology.
   (a) $(p \wedge q) \implies p$
   (b) $p \implies (p \vee q)$
   (c) $\sim p \implies (p \implies q)$
   (d) $(p \wedge (p \implies q)) \implies q$
   (e) $(\sim q \wedge (p \implies q)) \implies \sim p$
   (f) $((p \implies q) \wedge (q \implies r)) \implies (p \implies r)$

2. Let p : I will study discrete math, q : I will go to a movie and r : I am in a good mood. Write the following statements in terms of p, q, r and logical connectives.
   (a) If I am not ina good mood, then I will go to a movie.
   (b) I will not go to a movie and I will study discrete math.
   (c) I will go to a movie only if I will not study discrete math.
   (d) If I will not study discrete math, then I am not in a good mood.

3. Let p : I will study discrete math, q : I will go to a movie and r : I am in a good mood. Write the English sentences corresponding to the following statements.
   (a) $((\sim p) \wedge q) \implies r$
   (b) $r \implies (p \vee q)$
   (c) $(\sim r) \implies ((\sim q) \vee p)$
   (d) $(q \wedge (\sim p)) \iff r$

4. Write down the converse of the following statement

$$\text{if } n \text{ is a multiple of 3 then } n \text{ is not a multiple of 7}$$

Is the original statement True or False? Is the converse statement True or False? Justify your answer by giving a proof or a counter example.

5. Which of the statements

$$\text{if } a \text{ and } b \text{ are even numbers then } a+b \text{ is an even number}$$

$$\text{if } a+b \text{ is an even number then } a \text{ and } b \text{ are even numbers}$$

is True and which is False? Justify your answers by giving a proof or a counter example.

6. Show that

(a) $\sim (\exists x\ p(x)) \equiv \forall x\ (\sim p(x))$

(b) $\exists x\ (p(x) \implies q(x)) \equiv \forall x\ p(x) \implies \exists x\ q(x)$

(c) $\exists x\ p(x) \implies \forall x\ q(x) \equiv \forall x\ (p(x) \implies q(x))$

2 Method of proof

To begin the idea, let us construct the truth table of the compound statement $(p \wedge (p \implies q)) \implies q$.

Truth Table

p	q	$p \implies q$	$p \wedge (p \implies q)$	$(p \wedge (p \implies q)) \implies q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

The statement $(p \wedge (p \implies q)) \implies q$ is a **tautology**. Therefore this implication is true regardless of the truth values of its components p and $p \implies q$. In this case, we say that q **logically follows** from p and $p \implies q$. This we can write as

$$\begin{array}{l} p \\ p \implies q \\ \text{— — — — — — — — — — — —} \\ \therefore \quad q \end{array}$$

In general if a statement q follows from the statements $p_1, p_2, \cdots, p_n$, that is, the implication $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \implies q$ is a tautology, we write

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ -\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,- \\ \therefore \quad q \end{array}$$

This means that if we know that p_1 is true, p_2 is true, $\cdots$, p_n is true, then we know q is true. The p_i's are called the **hypotheses** or **premises**, and q is called the **conclusion**.

The above assertion is called an **argument**, that is, by an argument, we mean a sequence of statements that end with a conclusion.

Definition 14 (Argument or inference)
An **argument or inference** *is an assertion (logic of reasoning) that a given set of propositions $p_1, p_2, \cdots, p_n$ called hypotheses or premises yield another proposition q called the conclusion.*

Definition 15 (Valid argument)
An argument $p_1 \wedge p_2 \wedge \cdots \wedge p_n \implies q$ is said to be **valid** *if the statement*

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \implies q$$

is a tautology.

Thus we can say that the argument

$$p_1 \wedge p_2 \wedge \cdots \wedge p_n \implies q$$

is valid if q is true whenever all the premises $p_1, P_2, \cdots, p_n$ are true. [Note that if any one of $p_1, P_2, \cdots, p_n$ is false, then the hypothesis $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ is false and the implication $p_1 \wedge p_2 \wedge \cdots \wedge p_n \implies q$ is automatically true, regardless of the truth value of q]

An argument which is not valid is called a **fallacy**.

It is to be noted that the validity of an argument depends only on the form of the statements involved and not on the truth values of the variable that they contain. Such arguments are called the **rule of inference**.

A very important rule of inference is

$$\begin{array}{l} p \\ p \implies q \\ -\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,- \\ \therefore \quad q \end{array}$$

This rule of inference is well known as **Rule of detachment** or **modus pones** (Modus pones comes from Latin and may be translated as *the method of affirming*), and is universally valid. Another important rule of inference

$$\begin{array}{l} p \implies q \\ q \implies r \\ \hline \therefore \quad p \implies r \end{array}$$

is universally valid (Verify it showing the conjunction $((p \implies q) \wedge (q \implies r) \implies (p \implies r))$ is a tautology). This rule of inference is known as **law of syllogism**.

Other rule of inference called **Modus Tollens** (Modus Tolles is a Latin word meaning *method of denying*) is given by

$$\begin{array}{l} p \implies q \\ \sim q \\ \hline \therefore \quad \sim p \end{array}$$

Example 20

Is the following argument valid?

Smoking is healthy.
If smoking is healthy, then cigarettes are prescribed by physicians.
– –
∴ Cigarettes are prescribed by physicians.

Solution The argument is valid since it is of the form **modus pones**. However, the conclusion is **false**. Observe that the first premise

$$p : \text{ smoking is healthy}$$

is **false**. The second premises

$$p \implies q$$

is then **true** and $(p \wedge (p \implies q))$, the conjunction of the two premises, is **false**.

Example 21

Is the following argument valid?

If you invest in the stock market, then you will get rich.
If you get rich, then you will be happy.
– –
∴ If you invest in the stock market, then you will be happy.

Solution The argument is valid since it is of the form **law of syllogism**. Although the conclusion may be false.

Example 22

Determine the validity of the following argument

If I drive to work, then I will arive tired.
I do not drive to work.
– –
$\therefore$ I will not arrive tired.

Solution Translate the argument in symbolic form. Let

$$p: \text{ I drive to work}$$

and

$$q\text{: I will arrive tired}$$

Then the argument is of the form

$$p \implies q$$
$$\sim p$$
– – – – – – – –
$$\therefore \quad \sim q$$

Now to test the validity of the argument, construct the truth table of the statement $(p \implies q) \wedge (\sim p) \implies (\sim q)$.

Truth Table

p	q	$\sim p$	$\sim q$	$p \implies q$	$(p \implies q) \wedge (\sim p)$	$(p \implies q) \wedge (\sim p) \implies (\sim q)$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Hence the statement $(p \implies q) \wedge (\sim p) \implies (\sim q)$ is not a tautology, and so the given argument is not valid.

EXERCISES

1. Determine the validity of the following argument

   I will become famous or I will be a writer.
   I will not be a writer.
   –
   $\therefore$ I will become famous.

2. Determine the validity of the following argument

   If I graduate this semester, then I will have passed the math course.
   If I do not study math for 10 hours a week, then I will not pass math.
   If I study math for 10 hours a week, then I cannot play football.
   –
   $\therefore$ If I play football, I will not graduate this semester.

2.1 Theorem

Definition 16 (Theorem)
A true statement is known as a **Theorem***, and the reason why it is true is its Proof.*

Theorem 2 15 *is a multiple of* 3.

Proof $15 = 5 \times 3$. □

But the statement

20 is a multiple of 3

is false. So the statement is not a Theorem.

Theorem 3 20 *is not a multiple of* 3.

Proof If 20 were a multiple of 3, there would be a $n \in \mathbb{N}$ such that $n \times 3 = 20$. Since $7 \times 3 = 21, n$ must be less than 7. But none of the values $1, 2, 3, 4, 5, 6$ works. □

Example 23 *Prove or disprove the statement that if $x, y \in \mathbb{R}$, $(x^2 = y^2) \iff (x = y)$.*

Solution The statement can be restated as

$$(\forall x)(\forall y)(\mathbb{R}(x, y)), \quad (x^2 = y^2) \iff (x = y)$$

Thus, to prove this result, we would need to provide steps, each of which would be true for all $x, y \in \mathbb{R}$. To disprove this result, we need only find one example for which this implication is false.

Since $(-3)^2 = 3^2$, but $-3 \neq 3$, the result is false. Our example is called a

counter example

and any other counter example would do just as well. □

From the above example, we observe that

- A disprove of a statement is a proof that the statement is false.
- A counter example to a statement of the form $(\forall x \in D)p(x)$ is an element $y \in D$ for which $p(y)$ is false. Thus the method of disprove by counter example is based on the following fact:

$$\sim [(\forall x \in D)p(x)] \iff (\exists x \in D)[\sim p(x)]$$

2.1.1 Meaning to prove the theorem

To **prove the theorem** means to show that the **implication**

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \implies q$$

is a tautology.

Note that we are not trying to show that q (the conclusion) is true, but only that q will be true if all the p_i's are true. For this reason, mathematical proofs often begin with the statement **suppose that** $p_1, p_2, \cdots, p_n$ **are true** and conclude with the statement **therefore** q **is true**. The proof does not show that q is true, but simply shows that q has to be true if the p_i's are all true.

2.1.2 Proving invalidity of an argument

To prove that an argument or an inference pattern is invalid, produce a counter example to the inference pattern.
An example for which an inference/argument is false, is called a counter example.

Example 24 $n^2 - n + 41$ *is a prime for all* $n \in \mathbb{N}$.

Solution Counter example $n = 41$ results a composite number.

2.1.3 Proving the validity of an argument

To prove that an inference pattern is valid, prove that the conditional

$$\text{Conjuction of premises} \implies \text{Conclusion}$$

is a tautology.

Generally two methods in proving validity:

(*a*) Direct proof

(*b*) Indirect proof

(*c*) Mathematical induction

(a) Direct Proof
Starting from the true premises (in mathematical proof), the process of reaching the true conclusion is called direct proof. Generally two rule of inference are in use

(*i*) First rule of inference or modus pones

$$(p \wedge (p \implies q)) \implies q$$

(*ii*) Second rule of inference or law of syllogism

$$((p \implies q) \wedge (q \implies r)) \implies (p \implies r)$$

(b) Indirect Proof

(i) Method of contrapositive
To prove the implication $p \implies q$, prove the implication $\sim q \implies \sim p$. That is, assume q is false, and show that p is then false. It is based on the tautology

$$(p \implies q) \iff ((\sim q) \implies (\sim p))$$

(ii) Method of contradiction
This method is based on the tautology

$$((p \implies q) \wedge (\sim q)) \implies (\sim p)$$

2.1.4 Proof techniques

- In order to show that an existential statement is true: give an example.
- In order to show that a universal statement is false: give a counter example.
- In order to show that a statement is true, assume the opposite and show that it leads to a false conclusion.

Example 25 *Let n be an integer. Prove that if n^2 is odd, then n is odd.*

Proof

(a) **Contrapositive method**
Let

$$p : n^2 \text{ is odd}$$

and

$$q : n \text{ is odd}$$

Then we have to prove

$$p \implies q$$

is true. We prove it by method of contrapositive $\sim q \implies \sim p$. Here

$$\sim q : n \text{ is not odd}$$

so that n is even. Then $n = 2k$, where k is an integer. We have

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

so n^2 is even. We thus have shown that if n is even, then n^2 is even, that is, we have shown $\sim q \implies \sim p$. Hence proved.

(b) **Direct method**
Write

$$n = p_1 p_2 \cdots p_s$$

where $p_1, p_2, \cdots, p_s$ are the (not necessarily distinct) prime factors of n. Then

$$n^2 = p_1^2 p_2^2 \cdots p_s^2$$

Assume that n^2 is odd, then none of the p_i's equals 2. Hence 2 is not a prime factor of n, and therefore n is also odd.

(c) **Contradiction method**
Assume that n^2 is odd and n is even. Then

$$n^2 + n$$

is the sum of an odd and an even number. Hence $n^2 + n$ is odd. But

$$n^2 + n = n(n + 1)$$

is a product of consecutive positive integers, one of which must be even. Hence $n^2 + n$ is also an even number, because odd $\times$ even $=$ even. So, it brings the contradiction due to assumption that n is even, hence n is odd.

Example 26 *Prove or disprove: The sum of any five consecutive integers is divisible by* 5.

Solution Let n be an integer. Then the sum of five consecutive integers is

$$n + (n + 1) + (n + 2) + (n + 3) + (n + 4) = 5(n + 2)$$

Therefore

$$5|5(n + 2)$$

So the sum of any five consecutive integers is divisible by 5.

EXERCISES

1. Prove or disprove the statement $(\forall x, y \in \mathbb{R})(x^2 < y^2 \implies x < y)$.
2. Prove or disprove the statement: *every prime number is odd.*
3. Prove or disprove: $n^2 + 41n + 41$ is a prime number for every integer n.
4. Prove that the sum of two even numbers is even.
5. Prove that the sum of two odd numbers is odd.
6. Prove that n^2 is even if and only if n is even.
7. Prove that $(\forall n \in \mathbb{N})(n^2 + n)$ is even.

2.1.5 Necessary and sufficient conditions

The phrases *necessary and sufficient condition*, as used in formal English, correspond exactly to their definition in logic.

Definition 17 *If p and q are statements*

- *p is sufficient condition for q means* **if** p **then** q, *that is,* $(p \implies q)$
- *p is necessary condition for q means* **if not** p **then not** q, *that is,* $(\sim p \implies \sim q)$

In other words, to say

$$p \text{ is sufficient condition for } q$$

means that the occurrence of p is *sufficient* to guarantee the occurrence of q. On the other hand, to say

$$p \text{ is necessary condition for } q$$

means that if p does not occur, then q can not occur either. The occurrence of p is *necessary* to obtain the occurrence of q. Note that because of the equivalence between a statement and its contrapositive,

$$p \text{ is necessary condition for } q \text{ means } \textbf{if } p \textbf{ then } q$$

Consequently,

$$p \text{ is necessary and sufficient condition for } q \text{ means } p \textbf{ if and only if } q$$

Example 27 (Interpreting of necessary and sufficient conditions)

Consider the statement

If Suvesh is eligible to vote, then he is at least 18 years old

The truth of the condition

Suvesh is eligible to vote

is *sufficient* to ensure the truth of the condition

Suvesh is at least 18 years old

In addition, the truth of the condition

Suvesh is at least 18 years old

is **necessary** for the condition

Suvesh is eligible to vote

to be true. If Suvesh were younger than 18. then he would not be eligible to vote.

Example 28 (Converting a sufficient condition to if $\cdots$ then form)

Rewrite the following statement in the form **if** p **then** q

Robin's birth on Nepal soil is a sufficient condition for him to be a Nepali citizen

Solution

If Robin were born on Nepal soil, then he is a Nepali citizen

Example 29 (Converting a necessary condition to if $\cdots$ then form)

Use the contrapositive to rewrite the following statement in the ways

Susan's attaining age 35 is a necessary condition for his being president of Nepal

Solution
Version 1

If Susan has not attained the age of 35, then he can not be a president of Nepal

Version 2

If Susan can be a president of Nepal, then he has attained the age of 35

3 Mathematical induction

A substantial part of mathematics is the notion of proof of a mathematical statement, which constitutes a theorem, and in general we use the following three logically equivalent methods to prove such a mathematical statement.

(i) Direct method

(ii) Contradiction method (reductio-ad-absurdum)

(ii) Indirect method (Contrapostive method)

Besides the above three methods, there is also an important approach to prove a theorem about positive integers, and this is the **Mathematical Induction**. In natural sciences, Induction infers something general from a few cases where the statement is known to hold true. If you have only known liars throughout your life, you may infer that the next person you meet is a liar. This is an induction, as you would be making a conjecture based on a finite number of cases known to you. But, in mathematics, a statement involving a natural number might turn out to be erroneous even if it happens to be true for the first ten, or thousand, or even million natural numbers.

For instance, $n^2 - n + 41$ is prime for $n = 1, 2, 3, \cdots, 40$, but for $n = 41$, it is 41×41, and so it is not a prime for $n = 41$. An even more convincing example is the following. The number $2^{2^0} + 1 = 3, 2^{2^1} + 1 = 5, 2^{2^2} + 1 = 17, 2^{2^3} + 1 = 257, 2^{2^4} + 1 = 65537$ are all prime numbers and the 17th century mathematician Pierre de Fermat suggested that $2^{2^n} + 1$ must be prime for every positive integer n. However, a century later, another great mathematician Leonard Euler showed that $2^{2^5} + 1 = 641 \times 6700417$. This tells us that in mathematics; a lot of care is needed to establish an induction procedure, which proves a mathematical theorem for each of an infinite sequence of cases, without exception. So the method of mathematical induction is such a procedure that involves extremely **believable** logic that we accept it as valid reasoning.

Then the axiom we require is called the **principle of mathematical induction** or **just the principle of induction**.

3.1 Principle of induction

The induction technique is used to prove a statement which can be put in the form

$$\forall n \geq n_0, \quad p(n)$$

where $n \in \mathbb{Z}^+$ and n_0 is a fixed integer. That is, suppose we wish to show that $p(n)$ is true for all integers $n \geq n_0$.

The principle of induction is the rule of inference for proving that all items in the list $n_0, n_1, \cdots$ has the property $p(n)$. The principle has the two forms:

Weak form:
Weak form has the rule of inference as

(i) $p(n_0)$ is true **Basic step**
(ii) $(\forall k \geq n_0)$ $[p(k)$ true $\implies p(k+1)$ true$]$ **Induction step**
– –
$\therefore (\forall n \geq n_0)$ $[p(n)$ true$]$ **Conclusion**

The above forms of mathematical induction using quantifiers is

$$p(n_0) \wedge [\forall k \geq n_0, \quad p(k) \implies p(k+1)] \implies \forall n \geq n_0, \quad p(n)$$

is a tautology.

Strong form:
Strong form has the rule of inference as

(i) $p(n_0)$ is true **Basic step**
(ii) $(\forall k \geq n_0)$ $[p(n_0+1), p(n_0+2), \cdots, p(k)$ true$]$ **Induction step**
– –
$\therefore (\forall n \geq n_0)$ $[p(n)$ true$]$ **Conclusion**

The above forms of mathematical induction using quantifiers is

$$p(n_0) \wedge p(n_0+1) \wedge p(n_0+2) \wedge p(k) \implies p(k+1)$$

is a tautology.

Example 30 *Use mathematical induction to prove that the following statement holds for all $n \in \mathbb{Z}^+$*

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(n+2)}{6}$$

Solution Let $p(n)$ be the statement

$$p(n) : 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(n+2)}{6}$$

Basic step We start with the statement $p(1)$:

$$p(1) : 1^2 = \frac{1(1+1)(1+2)}{6} = 1$$

which is true. So $p(1)$ is true.

Induction step We assume that the statement it true for some positive integer k, and we then show that it holds for the next positive integer $k+1$. Assume

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(k+2)}{6}$$

holds and we then prove that

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(k+3)}{6}$$

Now LHS of $p(k+1)$:

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(k+2)}{6} + (k+1)^2 \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right] \\
&= \frac{(k+1)(k+2)(k+3)}{6}
\end{aligned}$$

Thus $p(k+1)$ is true. Hence by mathematical induction $p(n)$ holds for all $n \in \mathbb{Z}^+$.

Example 31 *Prove by mathematical induction that* $3|(n^3 - n)$.

Solution Let $p(n)$ be the statement

$$p(n) : 3|(n^3 - n)$$

Basic step We start with $n_0 = 1$ and show that $p(1)$ is true. For $n_0 = 1$

$$p(1) = 3|(0^3 - 0) = 3|0$$

So $p(1)$ is true.

Induction step Suppose that the statement $p(n)$ is true for some integer $k \in \mathbb{Z}^+$. That is

$$p(k) : 3|(k^3 - k)$$

is true. We shall now prove that

$$p(k+1) : 3|\left((k+1)^3 - (k+1)\right)$$

is true. Now

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k - k$$

Since

$$3|(k^3 - k) \implies \exists m \in \mathbb{Z}^+ \text{ such that } k^3 - k = 3m$$

Using this relation

$$(k+1)^3 - (k+1) = 3m + 3(k^2 + k) = 3(m + k^2 + k)$$

Since $m + k^2 + k \in \mathbb{Z}^+$, so $3|\left((k+1)^3 - (k+1)\right)$. Therefore $p(k+1)$ is true, and hence by mathematical induction $3|(n^3 - n)$ for all $n \in \mathbb{Z}^+$.

EXERCISES

1. Use the method of induction to show that, for all $n \in \mathbb{N}, n^2 + n$ is an even number.

2. Prove that, for all $n \in \mathbb{N}, n^3 + 5n$ is a multiple of 6.

3. Use the method of induction to show that the following statement is true for all natural number n:

   (a) $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

   (b) $\sum_{r=1}^{n} r(r+2)(r+4) = \frac{n(n+1)(n+4)(n+5)}{4}$

4. Use the principle of induction to prove that

   (a) $1 + 2^n < 3^n$ for $n \geq 2$.

   (b) $n < 2^n$ for $n \geq 1$.

   (c) $1 + 2 + 3 + \cdots + n < \frac{(2n+1)^2}{8}$

5. Prove by mathematical induction that if $A_1, A_2, \cdots, A_n$ are n sets, then

$$\overline{\left(\bigcap_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \overline{A_i}$$

6. Use induction to show that if p is a prime and $p \mid a^n$ for some $n > 1$, then $p \mid a$.

7. Let $p(n) : 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2 + 4}{4}$

   (a) Use $p(k)$ to show $p(k+1)$.

   (b) Is $p(n)$ true for all $n \geq 1$?

4 Pigeonhole principle

Many results in combinatorial theory (theory of counting problems) come from the almost obvious statement, known as **Pigeonhole principle**.

Theorem 4 (Pigeonhole principle)
Suppose n be the numbers of pigeons and m be the number of pigeonholes such that $m < n$. Then at least one pigeonhole contains two or more pigeons.

This principle can be applied to many problems where we want to show that a given situation can occur.

Example 32

Suppose a department contains 13 students. Then two of the students (pigeons) were born in the same month (pigeonhole).

Example 33

Find the minimum number of elements that one needs to take from the set $S = \{1, 2, \cdots, 9\}$ to be sure that two of the numbers add up to 10.
Solution Here the pigeonholes are the five sets

$$\{1, 9\},\ \{2, 8\},\ \{3, 7\},\ \{4, 6\},\ \{5, 5\}$$

Thus any choice of 6 elements (pigeons) from S will guarantee that two of the numbers add up to 10. Hence the minimum number of elements from S to be chosen such that two of the numbers add up to 10 is six.

Theorem 5 (Extended pigeonhole principle)
If n pigeons are assigned to m pigeonholes, then one of the pigeonholes must contain at least

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

pigeons.

Here $\lfloor x \rfloor$ is the floor value of x, and is the greatest integer less than equal to x. For example, $\lfloor 3.6 \rfloor = 3$.
Proof Suppose no pigeonhole contains more than $\left\lfloor \frac{n-1}{m} \right\rfloor$ pigeons. Then

$$\begin{aligned} \text{maximum number of pigeons} &= m \cdot \left\lfloor \frac{n-1}{m} \right\rfloor \\ &\leq m \cdot \left\lfloor \frac{n-1}{m} \right\rfloor \\ &= n - 1 \end{aligned}$$

This contradicts our assumption that there are n pigeons. Thus, one pigeonhole must contain at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons. $\square$

Example 34

Six friends discover that they have a total of Rs.2161 with them on trip to the movies. Show that one or more of them must have at least Rs. 36.
Solution Let Rs. be the pigeons and the friends be the pigeonholes. Then by the extended pigeonhole principle, one student must have at least Rs.

$$\begin{aligned} &\left\lfloor \frac{n-1}{m} \right\rfloor + 1 \\ &= \left\lfloor \frac{2161-1}{6} \right\rfloor + 1 \\ &= \left\lfloor \frac{2160}{6} \right\rfloor + 1 \\ &= 360 + 1 \\ &= 361 \end{aligned}$$

Example 35

Show that if 7 colors are used to paint 50 bicycles, at least 8 bicycles will be the same color.
Solution Let $n = 50$ bicycles be the pigeons and $m = 7$ be the pigeonholes. Then by extended pigeonhole principle, at least

$$\begin{aligned} & \left\lfloor \frac{n-1}{m} \right\rfloor + 1 \\ &= \left\lfloor \frac{50-1}{7} \right\rfloor + 1 \\ &= \lfloor 7 \rfloor + 1 \\ &= 8 \end{aligned}$$

bicycles will be the same color.

EXERCISES

1. Show that if seven numbers from 1 to 12 are chosen, then two of them will add up to 13.
2. Let T be an equilateral triangle whose sides are of length 1 unit. Show that if any five points are chosen lying on or inside the triangle, then two of them must be no more than $\frac{1}{2}$ unit apart.
3. Show that if any eight positive integers are chosen, two of them will have the same remainder when divided by 7.
4. Show that if five points are selected in a square whose sides have length 1 inch, at least two of the points must be no more than $\sqrt{2}$ inches apart.
5. Show that in any 11-digit integer, at least two digits are the same.
6. If we select any group of 1000 students on campus, show that at least three of them must have the same birthday.