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HW 4

Math 3124
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1. We have $\dot{w}(t) = \cos t - i \sin t$, so

$$\begin{aligned}\int_0^{2\pi} \frac{1}{w(t)} \dot{w}(t) dt &= \int_0^{2\pi} \frac{\cos t - i \sin t}{\sin t + i \cos t} dt \\ &= \int_0^{2\pi} \frac{(\cos t - i \sin t)(\sin t - i \cos t)}{\sin^2 t + \cos^2 t} dt \\ &= \int_0^{2\pi} \frac{-i(\sin^2 t + \cos^2 t)}{\sin^2 t + \cos^2 t} dt \\ &= \int_0^{2\pi} -i dt \\ &= -2\pi i\end{aligned}$$

This is the negation of the integral parameterized by the previous curve, since $w(t)$ traces out the unit circle in the opposite direction of $z(t)$.

2. *Proof.* Since $\int_C f(z)dz = re^{i\theta}$ for some $r, \theta \in \mathbb{R}$, $r = \int_C e^{-i\theta} f(z)dz$. Also, $|re^{i\theta}| = r = |\int_C f(z)dz|$, so, using a reparameterization of $z = e^{it}$ for $0 \le t \le 2\pi$,

$$\begin{aligned}\left|\int_C f(z)dz\right| &= \int_C e^{-i\theta} f(z)dz \\ &= \int_0^{2\pi} e^{-i\theta} f(e^{it}) ie^{it} dt \\ &= \int_0^{2\pi} ie^{i(t-\theta)} f(e^{it}) dt \\ &= \int_0^{2\pi} (i \cos(t-\theta) + \sin(\theta - t)) f(e^{it}) dt \\ &\le \text{Re}\left(\int_0^{2\pi} (i \cos(t-\theta) + \sin(\theta - t)) f(e^{it}) dt\right) \\ &= \int_0^{2\pi} \sin(\theta - t) f(e^{it}) dt \\ &\le \left|\int_0^{2\pi} \sin(\theta - t) f(e^{it}) dt\right| \\ &\le \int_0^{2\pi} |\sin(\theta - t) f(e^{it})| dt \\ &\le \int_0^{2\pi} |\sin(\theta - t)| dt \\ &= 4\end{aligned}$$

where we use the fact that $|f(e^{it})| \le 1$ in the last inequality. $\square$

3. (a) *Proof.* Let $f(z) = \frac{z^{k+1}}{k+1}$. Then f is a function analytic throughout C, and $z^k = f'(z)$. So by the Closed Curve Theorem and its corollary, $\int_C z^k dz = 0$. $\square$

   (b) *Proof.* Let $z(\theta) = Re^{i\theta}$. Then $\dot{z}(\theta) = iRe^{i\theta}$, so

$$
\begin{aligned}
\int_C z^k dz &= \int_0^{2\pi} i(Re^{i\theta})^{k+1} d\theta \\
&= \left. \frac{(Re^{i\theta})^{k+1}}{k+1} \right|_{\theta=0}^{2\pi} \\
&= \frac{R^{k+1}}{k+1} - \frac{R^{k+1}}{k+1} \\
&= 0
\end{aligned}
$$

$\square$

4. *Proof.* Let $a, b \in D$. Define $C : z(t) = (1-t)a + bt, 0 \leq t \leq 1$. Note then that the start and endpoints of C are a, b, respectively. Then by Proposition 4.12, since f is analytic in a convex region,

$$
\begin{aligned}
|f(b) - f(a)| &= \left| \int_C f'(z) dz \right| \\
&= \left| \int_0^1 f'(z(t)) \dot{z}(t) dt \right| \\
&= \int_0^1 \left| f'((1-t)a + bt)(b-a) \right| dt \\
&\leq \int_0^1 |b-a| dt \\
&= |b-a|
\end{aligned}
$$

where we use the fact that $|f'| \leq 1$ in the inequality. $\square$