MATH 3122A FALL 2023 HOMEWORK ASSIGNMENT 2

Please hand in your solutions to T. Barron on or before Wednesday October 4 (10:30 am)

Problem 1. Consider the sequence of functions

$$f_n : [0,1] \to \mathbb{R}$$

defined by

$$f_n(x) = \begin{cases} 0, \text{ if } 0 \le x \le 1/2 \\ nx - \frac{n}{2}, \text{ if } 1/2 < x \le \frac{1}{2} + \frac{1}{n} \\ 1, \text{ if } \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

$n = 1, 2, 3, ...$

Prove that the sequence (f_n) is Cauchy in $C[0,1]$ equipped with the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)| dx.$$

Explain why it diverges.

Show that the sequence (f_n) is *not* Cauchy in ($C[0,1]$, sup metric).

Problem 2. Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that each of the following is a metric on $X \times Y$:

$$d_1((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$$
$$d_\infty((x,y),(x',y')) = \max\{d_X(x,x'), d_Y(y,y')\}$$

Show that $S \subset (X \times Y)$ is open with respect to d_1 if and only if it is open with respect to d_∞.