Orthogonal Matrices

Definition: A set of vectors is called **orthonormal** if they are mutually orthogonal and all of length 1.

Definition: An **orthogonal matrix** is a square matrix whose columns form an orthonormal set.

Proposition: Q is an orthogonal matrix if and only if $Q^T Q = I$.

Proof: Since the columns form an orthonormal basis, the dot product of any two non-equal vectors is 0, where the dot product of each vector with itself (along the diagonal of $Q^T Q$ is 1). ■

Remark: The above proposition holds even for non-square matrices.

The following properties follow from the above:

- $Q^T = Q^{-1}$
- Q^T is orthogonal
- The rows of Q form an orthonormal set.

Proposition: In $\mathbb{R}^2$, every orthogonal matrix is either a reflection or a rotation.

Gram-Schmidt Decomposition

Gram-Schmidt Decomposition: Given $\{x_1, x_2 \ldots, x_n\}$ as a basis for a subspace, to get an othogonal basis, let

$$\begin{aligned}
v_1 &= x_1 \\
v_2 &= x_2 - \operatorname{proj}_{v_1} x_2 \\
v_3 &= x_3 - \operatorname{proj}_{v_1} x_3 - \operatorname{proj}_{v_2} x_3
\end{aligned}$$

In general, the idea is to subtract from each vector x_1 the components parallel to the basis vectors already defined.

QR Factorization

Given an $m \times n$ matrix A with linearly independent columns, then we can factorize A as

$$A = QR$$

where Q is a matrix formed by the orthonormal basis of the columns of A and R is upper triangular.

Note that since the columns of Q^T are orthogonal, $Q^{-1} = Q^T$, so $R = Q^T A$.

Orthogonal Diagonalization

Remark: The columns of Q are the normalized eigenvectors of A.

Remark: The diagonal elements of D are the eigenvalues of A.

Example: *Find the orthogonal diagonalization of*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 4$ *and* $\lambda_2 = 1$ *(multiplicity 2).*

First we find the eigenvectors corresponding to the eigenvalues by solving $(A - \lambda I)v = v$, which gives $v_1 = [1 \;\; 1 \;\; 1]^T$. The eigenvectors corresponding to λ_1 form an eigenspace with span $\{[-1 \;\; 1 \;\; 0]^T, [-1 \;\; 0 \;\; 1]^T\}$. For the columns of Q, we need unit vectors that are mutually orthogonal, so perform Gram-Schmidt to obtain the equivalent span $\{[-1 \;\; 0 \;\; 1]^T, [-1/2 \;\; 1 \;\; -1/2]^T\}$.

Theorem: Orthogonally diagonalizable matrices are symmetric.

Theorem: The eigenvectors corresponding to distinct eigenvalues are orthogonal.

Theorem (Spectral): A real matrix is symmetric if and only if it is orthogonally diagonalizable.

Theorem: A real symmetric matrix has real eigenvalues and orthogonal eigenvectors (for eigenvectors with distinct eigenvalues).

Theorem (Spectral Decomopsition): If A is a real symmetric matrix with eigenvalues λ_i and eigenvectors q_i each with unit length, then

$$A = \sum_i \lambda_i q_i q_i^T$$

Example: *Find a matrix with eigenvalues* 2 *and* 1 *with corresponding eigenvectors* $[1 \;\; -2]^T$ *and* $[2 \;\; 1]^T$.

First normalize the eigenvectors then use Spectral Decomposition to form A.

Quadratic Forms

Definition: A **quadratic form** is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form $f(u) = u^T A u$, where A is symmetric.

Sketching Quadratic Forms

Let $Q^T D Q = A$ be the orthogonal diagonalization of A. The eigenvectors of Q are the principal axis of the quadratic curve, and the eigenvalues determine the curve shape.

Principle Axes Theorem Given a quadratic form $u^T Au$, if $Q^T AQ = D$ is the orthogonal diagonlization, then the change of variables $v = Q^T u$ gives $u^T Au = v^T Dv$.

Theorem: Let A be a symmetric real matrix and f its quadratic form.

- if A is positive definite (all eigenvalues are positive), then the quadratic form is positive everywhere except the origin
- if A is positive semi-definite (all eigenvalues are non-negative), then the quadratic form is non-negative everywhere
- if A is indefinite (eigenvalues are opsitive and negative)\$, f is positive or negative or 0.