Problem 1:

a) Suppose for the sake of contradiction that $\mathbb{Q}$ is finitely generated. Then, $\exists A = \{a_1, ..., a_n\} \subseteq \mathbb{Q}$ s.t. $\mathbb{Q} = \langle A \rangle$.

Recall that $\langle A \rangle = \{a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n} | a_1, ..., a_n \in \mathbb{Z}\}$. So, assuming that $\langle A \rangle = \mathbb{Q}$ tells us that $\forall q \in \mathbb{Q}, q = a_1^{\alpha_1} a_2^{\alpha_2} ... a_n^{\alpha_n}$.

It must be true then, that $\frac{1}{p} \in \langle A \rangle, \forall p \in \mathbb{Z}$ s.t. p is prime. So, since there are infinitely many prime numbers, we can take some prime p s.t. $\frac{1}{p} \notin A$.

Then it must be that $\sum\limits_{i \in I} a_i^{\alpha_i} = \frac{1}{p}$ for some index set I. But this is impossible, as p is prime, and $a_i^{\alpha_i} \neq \frac{1}{p}, \forall a_i \in A, \alpha_i \in \mathbb{Z}$. Hence, by contradiction, we find that $\mathbb{Q}$ is not finitely generated.

$\square$

b) An example of a subgroup of $\mathbb{Q}$ that is proper but not cyclic is the subgroup generated by $\{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$. We will show that it is a subgroup.

The identity element is present: $\frac{0}{2^n} = 0 \in \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$. Also $\left(\frac{m}{2^n}\right)^{-1} = \frac{-m}{2^n} \in \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$. And finally, for some $\frac{h}{2^l}, \frac{d}{2^k} \in \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}, \frac{h}{2^l} + \frac{d}{2^k} = \frac{h2^k + d2^l}{2^{k+l}} = \frac{h2^{\max(k-l,0)} d2^{\max(l-k),0}}{2^{\max(k,l)}} \in \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$

But clearly, $\frac{1}{3} \notin \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$, so it is proper.

Now, suppose this subgroup were cyclic, then it would be generated by some element q. Then it would have to be that $q \in \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$, so that $q = \frac{k}{2^l}$ for some $k, l \in \mathbb{Z}$. But notice that $\frac{q}{2} \notin \langle q \rangle$, but that $\frac{q}{2} = \frac{k}{2^{l+1}} \in \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$. So, it cannot be that $\langle q \rangle = \{\frac{m}{2^n} | m, n \in \mathbb{Z}\}$.

Hence, we have found a proper subgroup that cannot be generated by any single element in the subgroup, hence it is not cyclic.

$\square$

c) We wish to show that $\mathbb{Q}^+$ is generated by the set $\{\frac{1}{p} | p \text{ is prime}\}$. This quickly follows from the fact that $\mathbb{Z}$ is a UFD (Unique Factorization Domain).

Take some $q \in \mathbb{Q}^+$, then $q = \frac{\alpha}{\beta}$, where $\beta \in \mathbb{Z}$, hence $\beta = p_1^{\lambda_1} p_2^{\lambda_2} p_3^{\lambda_3} ....$ The exact same hold for α, as it is an integer as well, so $\alpha = p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} ....$

So, overall we have that for any $q = \frac{\alpha}{\beta} \in \mathbb{Q}^+$, we find $\frac{\alpha}{\beta} = \frac{1}{p_1^{\lambda_1} p_2^{\lambda_2} p_3^{\lambda_3} ...} \left(\frac{1}{p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} ...}\right)^{-1}$, which is clearly in the group generated by $\{\frac{1}{p} | p \text{ is prime}\}$

$\square$

Problem 2:

a) We wish to use the lattice diagrams of Q_8 to find the centralizer and normalizer of each subgroup of Q_8. For $\langle i \rangle$, we note that the subgroup itself is in its centralizer (since it's cyclic, hence abelian). We then note that Q_8 is not abelian, and that $ik \neq ki$. Combining these two facts with the fact that the centralizer itself is a subgroup, we see, from the lattice diagram, that the centralizer $C_{Q_8}(\langle i \rangle) = \langle i \rangle$.

Considering that the argument for both $\langle j\rangle$ and $\langle k\rangle$ is identical to the one above, we find that $C_{Q_8}(\langle j\rangle) = \langle j\rangle$ and $C_{Q_8}(\langle k\rangle) = \langle k\rangle$.

The centralizer of $\langle -1\rangle$ can be found by once again noting that $\langle -1\rangle \leq C_{Q_8}$. If we also take into account that both elements $\{1, -1\} = \langle -1\rangle$ commute with all i, j, k, we find that the centralizer $C_{Q_8}(\langle -1\rangle) = Q_8$.

As for the normalizers of the group, we begin by noting from a fact given in class, that $C_G(A) \leq N_G(A)$. So, it follows immediately that $N_{Q_8}(\langle -1\rangle) = Q_8$.

As for the other three, wlog, notice $j\langle i\rangle j^{-1} = \{-1, 1, i, -i\} = \langle i\rangle$. And so, we found that $j \in N_{Q_8}(\langle i\rangle) \leq Q_8 \Rightarrow N_{Q_8}(\langle i\rangle) = Q_8$, by looking at the lattice structure.

The argument for the other two remaining subgroups is identical to the one above, so that $N_{Q_8}(\langle j\rangle) = N_{Q_8}(\langle k\rangle) = Q_8$.

$\square$

b) We wish now to find the normal subgroups of Q_8 and the isomorphism type of their quotient. Recall that $N \trianglelefteq G \Leftrightarrow N_G(N) = G$. So, we have already shown that the normalizer of every subgroup in Q_8 is just Q_8, hence every subgroup of Q_8 is normal.

As for the quotient type of their isomorphism, notice that for $\{1, -1\} = \langle -1\rangle$ we get $|Q_8 : \langle -1\rangle| = 4$. So, $Q_8/\langle -1\rangle \cong V_4$ or $\mathbb{Z}_4$. Notice that $\forall g \in Q_8/\langle -1\rangle$, $g^2\langle -1\rangle = \pm 1\langle -1\rangle = 1\langle -1\rangle$. So, all non-identity elements $(i\langle -1\rangle, j\langle -1\rangle, k\langle -1\rangle)$ have order 2, hence it cannot be that $Q_8/\langle -1\rangle \cong \mathbb{Z}_4$. So, we find that $Q_8/\langle -1\rangle = V_4$.

Now, notice that $|Q_8 : \langle i\rangle| = 2$. Since there is only one group of order 2, we find immediately that $Q/\langle i\rangle \cong \mathbb{Z}_2$.

As for the other two, we use an identical argument to the one above to find that $Q/\langle j\rangle \cong \mathbb{Z}_2, Q/\langle k\rangle \cong \mathbb{Z}_2$.

$\square$

Problem 6:

We know that $xH = Hy$, which tells us that $\forall xh_1 \in xH, \exists h_2 \in H$ s.t. $xh_1 = h_2y \in Hy$. In particular, let $h_1 = 1 \in H$. Then $xh_1 = h_2y \Rightarrow x = h_2y \Rightarrow xy^{-1} = h_2 \in H$. And we know that $xy^{-1} \in H \Rightarrow Hx = Hy$. Hence, $xH = Hy = Hx$.

And if $xH = Hx \Rightarrow xHx^{-1} = H$, thus $x \in N_GH$.

$\square$

Problem 7:

We have that G is a finite group and $H \leq G$, and the order of H is relatively prime to the index $|N : G|$.

Notice that since G is a finite group, we have that $|G : N| = \frac{|G|}{|N|} \Rightarrow |N| = \frac{|G|}{|G:N|}$.

We know, from corollary 15 in chapter 3, that $HN \leq G$, since N is normal. So, we know then that $|HN| = \frac{|H||N|}{|H\cap N|}$.

So, $|G| = \frac{|H||N|}{|H\cap N|}|G : HN| = \frac{|H||G|}{|H\cap N||G:N|}|G : HN|$. Then if we cancel the $|G|$ on either

side we get $|H||G : HN| = |H \cap N||G : N|$. From here we use unique prime factorization of integers (orders and indexes are integer values) and remember that $(|H|, |G : N|) = 1$, telling us that $|H| = |H \cap N|$, and $|G : HN| = |G : N|$. Hence, $H \leq N$.
$\square$

Problem 8:
If M is a subgroup of G s.t. G $|G : M| = p$, then either $i)(|H|, |G : M|) = 1$ or $ii)(|H|, |G : M|) = p$.

In the first case, it follows immediately from the result proved in problem 7 that $H \leq M$.

For the second case, we recall corollary 15 that tells us that HM is a subgroup, and that we derived in question 7: $|H||G : HM| = |H \cap M||G : M|$. We know that $|HM| = \frac{|H|}{|H \cap K}|K| = |H : H \cap K||K|$, so $|HM| = a|K|, a \in \mathbb{Z}_+$, so it follows that either $|G : HM| = p$ or else $|G : HM| = 1$.

If $|G : HM| = p \Rightarrow$ the equation from 7 tells us $|H| = |H \cap M| \Rightarrow H \leq M$.

If instead $|G : HM| = 1 \Rightarrow |HM| = |G| \Rightarrow HM = G$ and we can also see that $|HM| = p|M|$. But we know from earlier that $|HM| = |H : H \cap M||K| \Rightarrow |H : H \cap M| = p$.
$\square$

Problem 9: If $M, N \trianglelefteq G$ s.t. $G = MN$. We define the homomorphism $\phi : G \to G/M \times G/N$ as $\phi(a) = (aN, aM)$.

To show ϕ is well-defined suppose $g_1 = g_2 \in G/(M \cap N)$, then $g_1 = g_2 m$ for some $m \in M \cap N$. Then $g_1 M = g_2 M$, and $g_1 N = g_2 N$ i.e. $(g_1 M, g_1 N) = (g_2 M, g_2 N)$ and thus $\phi(g_1) = \phi(g_2)$, hence ϕ is well-defined.

To show that ϕ is a homomorphism, take $a, b \in G/(M \cap N)$, then

$$\phi(ab) = (abN, abM) = (aM, aN)(bM, bN) = \phi(a)\phi(b)$$

Hence ϕ is a homomorphism.

To show that ϕ is surjective, take some $(aN, bM) \in (G/N) \times (G/M)$. Since $G = MN$, we have that $\forall g \in G, g = mn$ for some $m \in M, n \in N$. So, we can say $a = m_1 n_1$ and $b = m_2 n_2$, and thus

$$aM = m_1 n_1 M = m_1 M n_1 = n_1 M$$

Where we used the fact that M is normal, hence $gM = Mg, \forall g \in G$.

Also,

$$bN = m_2 n_2 N = m_2 N$$

Thus $\phi(m_2 n_1) = (m_2 n_1 M, m_2 n_1 N) = (m_2 M n_1, m_2 N) = (n_1 M, m_2 N) = (aM, bN)$.

Hence, ϕ is surjective.

Since we have a surjective homomorphism $\phi : G \to G/M \times G/N$, by The First Isomorphism Theorem, to show now that $G/(M \cap N) \cong G/M \times G/N$ it suffices to show that $\ker(\phi) = M \cap N$.

We can easily see that $\phi(a) = (1M, 1N) \Rightarrow a \in M, a \in N \Rightarrow a \in M \cap N$. Hence $\ker(\phi) = M \cap N$.

$\square$