Problem Set 5

6

Let A, B be a compact and closed set, respectively. Then, $A \cap B$ is bounded, since A is compact so it is closed and bounded. Also, the intersection of closed sets is closed, so $A \cap B$ is closed. Thus, by Heine-Borel, $A \cap B$ is compact.

7

TODO.

8

a

Consider the complement of the set X. Let $x \in \mathbb{R}$ such that $f(x) \neq 0$. By continuity of $f, \exists \delta > 0$ such that $x_0 \in (x - \delta, x + \delta) \implies |f(x_0) - f(x)| < f(x)$. That is, $0 < f(x_0)$. So, $f(x_0) \neq 0$, so $(x - \delta, x + \delta \subseteq X)$.

b

This is equivalent to $\{x \in \mathbb{R} : h(x) = 0\}$, where $h(x) = f(x) - g(x)$. Since f, g continuous, h is continuous. This was solved in a.

9

$\{x \in \mathbb{R} : f(x) = g(x)\} \iff \{x \in \mathbb{R} : (f - g)(x) = 0\}$

Claim: $h : \mathbb{R} \to \mathbb{R}$ continuous, $D \subseteq \mathbb{R}$ dense, $h\ |_D = 0 \implies h = 0$ is a reduction.

Suppose $a \in \mathbb{R}$ is such that $h(a) \neq 0$. Choose $\epsilon = |h(a)|/2$. By continuity of h at a, can choose δ such that $|x - a| < \delta \implies |h(x) - h(a)| < \epsilon$, so $h(a) - |h(a)|/2 < h(x) < h(a) + |h(a)|/2$. So there is a neighborhood around a such that values near $h(a)$ are non zero. These inputs giving non-zero outputs form an open set, which is disjoint from D, which is impossible, if D is dense.

10

Let $\epsilon > 0$. Let $\delta = \min\{1, \frac{\epsilon}{7}\}$. Suppose $|x - 1| < \delta$. Then $|f(x) - 0| = |x^3 - 1| = |(x - 1)(x^2 + x + 1)| < \frac{\delta}{7} \cdot 7 = \epsilon$.

11

TODO.

12

$\forall\epsilon > 0, \exists a < 0, b > 0$ such that $\forall x \in (-\infty, a) \cup (b, \infty), |f(x)| < \epsilon/2$. Then $f |_[a, b]$ is continuouns on a closed and bounded set, so it is uniformly continuous.

Let $\epsilon > 0$, Choose $\delta_1 > 0$ such that $\forall x_1, x_2 \in [a, b], |x_1 - x_2| < \delta_1 \implies |f(x_1) - f(x_2)| < \epsilon$. Choose $\delta_a > 0$ such that $\forall x \in \mathbb{R}$, $0 < |x - a| < \delta_0 \implies |f(x) - f(a)| < \epsilon/2$. (similarily for δ_b). Then $\delta = \min\{\delta_1, \delta_a, \delta_b\}$ works. Then, consider the 3 cases where both x are less than a, both are within δ of a, or within both distance δ from b, or both greater than b. TODO.

13

Let $\epsilon > 0$. Let $a \in A$. Choose δ_f such that $\forall x \in A, |x - a| < \delta \implies |f(x) - f(a)| < \epsilon/2$, by uniform continuity of f. Similarily choose δ_g for g. Choose $\delta = \min\{\delta_f, \delta_g\}$. Then $|f(x) + g(x) - f(a) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| \leq \epsilon/2 + \epsilon/2 = \epsilon$. $\blacksquare$

14

a

Let $a, x, \in \mathbb{R}$. Choose $\delta = \epsilon$. Let $x' \in \mathbb{R}$. Suppose $|x - x'| < \delta$. Then $||x - a| - |x' - a|| \leq |x - a - (x' - a)| = |x - x'| < \delta = \epsilon$. $\blacksquare$

b

Using f defined in a, this is equivalent to finding the global minimum of f. Since K is compact, f admits its global minimum by Extreme Value Theorem. $\blacksquare$