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# HW 1

Math 3122
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1. (a) $\frac{3}{20} - \frac{1}{20}i$ (multiplying the numerator and the denominator by the conjugate)
   (b) $\frac{4+7i}{1-i} = \frac{(4+7i)(1+i)}{2} = -\frac{3}{2} + \frac{11}{2}i$
   (c) Note that $\theta = \operatorname{Arg} z = \frac{\pi}{6}$ and $r = |z| = 1$, so $z^4 = r^n \operatorname{cis}(4\theta) = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3}) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
   (d) $-1 + 0i, 0 - 1i, 1 + 0i, 0 + i, -1 + 0i, \ldots,$

2. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, $z = a + bi$.
   (a) $\overline{z_1 + z_2} = \overline{(a_1 + a_2) + (b_1 + b_2)i} = (a_1 + a_2) - (b_1 + b_2)i = a_1 - b_1 i + a_2 - b_2 i = \overline{z}_1 + \overline{z}_2$.
   (b) $\overline{z_1 z_2} = \overline{a_1a_2 - b_1b_2 + (a_1b_2 + a_2b_1)i} = a_1a_2 - b_1b_2 - (a_1b_2 + a_2b_1)i = (a_1 - b_1 i)(a_2 - b_2 i) = \overline{z}_1\overline{z}_2$
   (c) $\overline{P(z)} = \overline{\sum_{k=0}^n a_k z^k} = \sum_{k=0}^n \overline{a_k z^k} = \sum_{k=0}^n a_k \overline{z^k} = \sum_{k=0}^n a_k \overline{z}^k = P(\overline{z})$ (by (a) and (b))
   (d) $\overline{\overline{z}} = \overline{a - bi} = a + bi = z$

3. *Proof.* Suppose $P(z) = 0$. Then by (2c), $P(\overline{z}) = \overline{P(z)} = \overline{0} = 0$. Now suppose $P(\overline{z}) = 0$. Then by (2c), $\overline{P(z)} = 0$, so $\overline{\overline{P(z)}} = P(z) = 0$ by (2d). □

4. *Proof.* Let $\epsilon > 0$. Suppose the series converges to $L$, then there is some $N \in \mathbb{N}$ such that $|s_{n-1} - L| < \epsilon/2$ and $|s_n - L| < \epsilon/2$. Then, $|z_n| = |s_n - s_{n-1}| = |(s_n - L) + (L - s_{n-1})| \leq |s_n - L| + |s_{n-1} - L| < \epsilon$. So $\lim_{n\to\infty} z_n = 0$. □

5. *Proof.* ($\rightarrow$) Suppose $S$ is closed. Let $z_0 \in \mathbb{C}$ and $(z_n) \subseteq S$ such that $\lim_{n\to\infty} z_n = z_0$. Suppose for the sake of contradiction that $z_0 \notin S$. Then $z_0 \in \mathbb{C} \setminus S$, which is open. Choose $\epsilon > 0$ such that $D(z_0; \epsilon) \subseteq C \setminus S$. So, $\forall n \in \mathbb{N}$, $|z_n - z_0| \geq \epsilon$, contradicting the assumption that $\lim_{n\to\infty} z_n = z_0$.
   ($\leftarrow$) Assume the right side is true and suppose for the sake of contradiction that $S$ is not closed. Then $\mathbb{C} \setminus S$ is not open. So, choose an $z_0 \in \mathbb{C} \setminus S$ such that $D(z_0; 1/n) \not\subseteq \mathbb{C} \setminus S$ for all $n \in \mathbb{N}$. Then, the sequence $(z_n)$ such that $z_n \in D(z_0; 1/n) \cap S$ is in $S$ and converges to $z_0$ but $z_0 \notin S$; a contradiction. □

6. *Proof.* Suppose $S$ is polygonally connected. Assume for the sake of contradiction that $S$ is disconnected. Then, there exists open sets $A, B$ such that $S \subseteq A \cup B$, $A \cap B = A \cap S = B \cap S = \emptyset$. Let $a \in A \cap S$ and $b \in B \cap S$. Since $S$ is polygonally connected, then $a$ and $b$ can be connected with a finite union of line segments $[z_0, z_1], \ldots, [z_{n-1}, z_n] \subseteq S$ with $a = z_0$ and $b = z_n$. Then, there must be some point $z$ along the path such that for every disk $D(z; \epsilon)$, $\epsilon > 0$, $D \not\subseteq A$ and $D \not\subseteq B$, since otherwise the line path could not cross between the disjoint sets $A$ to $B$ (i.e., $z_0 \in A$ and $z_n \in B$). But then, by the openness of $A, B$, $z \notin A$ and $z \notin B$, contradicting that $S \subseteq A \cup B$. □

7. *Proof.* We have $|P(z)| = \left|\sum_{k=1}^n a_k z^k\right| \geq \sum_{k=1}^n |a_k||z^k| = |z^n|(|a_n||z^0| + |a_{n-1}||z^{-1}| + \cdots + |a_0||z^{-n}|) \to |a_n||z^n|$ as $z \to \infty$, since each of terms with $z$ to a negative power vanish, and $|a_n||z^0| = |a_n|$ for all $z \in \mathbb{C}$. Since $n \geq 1$, then $|z^n| \to \infty$ as $z \to \infty$, so $P(z) \to \infty$. □