CS663, Assignment 5 **Yash Sanghvi, Pradyot Prakash**
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Question 4

We have,

$$g_1 = f_1 + h_2 * f_2$$
$$g_2 = h_1 * f_1 + f_2$$

On taking the Fourier transform, we get,

$$G_1 = F_1 + H_2F_2$$
$$G_2 = H_1F_1 + F_2$$

Solving these gives us,

$$F_1 = \frac{G_1 - H_2G_2}{1 - H_1H_2}$$

$$F_2 = \frac{G_2 - H_1G_1}{1 - H_1H_2}$$

We can recover the original images back by taking the inverse Fourier transform, which gives us,

$$\widehat{f_1} = F^{-1}\left(\frac{G_1 - H_2G_2}{1 - H_1H_2}\right)$$

$$\widehat{f_2} = F^{-1}\left(\frac{G_2 - H_1G_1}{1 - H_1H_2}\right)$$

Since h_1 and h_2 are defocussing kernels, let's assume that their frequency responses are Gaussian in nature.

$$H_1(u,v) = k_1 e^{-(u^2+v^2)/\sigma_1^2}$$

$$H_1(u,v) = k_2 e^{-(u^2+v^2)/\sigma_2^2}$$

$$H(u,v) = H_1(u,v)H_2(u,v) = k_1k_2 e^{-(u^2+v^2)/\sigma^2}$$

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

Both these terms have the term $(1 - H_1(u,v)H_2(u,v))$ in the denominator. If $H(u,v) = H_1(u,v)H_2(u,v)$ is close to 1, then the inverse transform blows up. This leads to erroneous results and the inverse transform is meaningless. Since $H(u,v)$ is a complex number, so in the case that it's real part is 1 and the imaginary part is close to 0 then such a situation is presented.

For our specific kernel, assume that $H(u,v) = 1 \pm \epsilon$

$$
\begin{aligned}
k_1 k_2 e^{-(u^2+v^2)/\sigma^2} &= 1 \pm \epsilon \\
e^{(u^2+v^2)/\sigma^2} &= \frac{k_1 k_2}{1 \pm \epsilon} \\
u^2 + v^2 &= \sigma^2 \ln\left(\frac{k_1 k_2}{1 \pm \epsilon}\right) \\
\sigma^2 \ln\left(\frac{k_1 k_2}{1 + \epsilon}\right) \le u^2 + v^2 &\le \sigma^2 \ln\left(\frac{k_1 k_2}{1 - \epsilon}\right)
\end{aligned}
$$

Hence, we get a circular ring where the value of $H(u, v)$ will be close to 1. The parameter ϵ is a small positive number close to 0.

If some noise were also present in the image, then also we would be facing a tough situation.

Suppose η_1 and η_2 are noise in the two images.

$$
\begin{aligned}
g_1 = f_1 + h_2 * f_2 + \eta_1 \\
g_2 = h_1 * f_1 + f_2 + \eta_2
\end{aligned}
$$

On taking the Fourier transform, we get,

$$
\begin{aligned}
G_1 = F_1 + N_1 + H_2 F_2 \\
G_2 = H_1 F_1 + F_2 + N_2
\end{aligned}
$$

Solving these gives us,

$$
\begin{aligned}
F_1 &= \frac{G_1 - H_2 G_2}{1 - H_1 H_2} - \frac{N_1 - N_2 H_2}{1 - H_1 H_2} \\
F_2 &= \frac{G_2 - H_1 G_1}{1 - H_1 H_2} - \frac{N_2 - N_1 H_1}{1 - H_1 H_2}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_1 &= \widehat{f_1} - F^{-1}\left(\frac{N_1 - N_2 H_2}{1 - H_1 H_2}\right) \\
f_2 &= \widehat{f_2} - F^{-1}\left(\frac{N_2 - N_1 H_1}{1 - H_1 H_2}\right)
\end{aligned}
$$

Even if we make a conservative assumption that $\widehat{f_1}$ is not affected much by the small denominator, the noise can severely affect the result. Since, we are assuming h_1 and h_2 to be blurring kernels, so N_1 would dominate the $N_2 H_2$ term. Hence even a small amount of noise can lead to a drastic change in the image f_1 if $H_1(u, v) H_2(u, v)$ is close to 1. Similar analysis holds for f_2 as well.