CS663, Assignment 5 **Yash Sanghvi, Pradyot Prakash**
Instructor: Prof. Ajit Rajwade 13D070042, 130050008
Question 5

Problem Statement

Consider a set of N vectors $X = x_1, x_2, ..., x_N$ each in R^d, with average vector $\overline{x}$. We have seen in class that the direction e such that $||x_i - \overline{x} - (e.(x_i - \overline{x})e)||^2$ is minimized, is obtained by maximizing $e^T Ce$, where C is the covariance matrix of the vectors in X . This vector e is the eigenvector of matrix C with the highest eigenvalue. Prove that the direction f perpendicular to e for which $f^T Cf$ is maximized, is the eigenvector of C with the second highest eigenvalue. For simplicity, assume that all non-zero eigenvalues of C are distinct and that $rank(C) > 2$ [**10 points**]

Consider the direction f s.t. it is perpendicular to e & has unit norm

$$\Rightarrow \quad f^T e = e^T f = 0$$

$$||f||^2 = f^T f = 1$$

To max. $f^T C f$

or min $-f^T C f$, we'll use Lagrange Multiplier method.

$$J(f) = -f^T C f + \lambda (f^T f - 1) + \mu (f^T e)$$

$$\frac{\partial J(f)}{\partial f} = -2Cf + 2\lambda f + \mu e = 0$$

On Premultiplying by f^T:

$$-2f^T Cf + 2\lambda f^T f + \mu f^T e = 0$$

$$\lambda = f^T C f \quad \left(\text{as} \begin{array}{l} f^T f = 1 \\ f^T e = 0 \end{array} \right)$$

Premult by e^T

$$-2e^T C f + 2\lambda (e^T f) + \mu e^T e = 0$$

$$\mu = 2e^T C f$$

$$\Rightarrow \quad 2Cf = 2f^TCf^2 + 2e^TCfe$$

$$Cf = (f^TCf)f + (\cancel{e}\,C^Te)^T fe$$

$$Cf = (f^TCf)f + (Ce)^T fe$$

$$Cf = (f^TCf)f + \lambda e^T fe$$

where λ = largest eigv.
Corresponding to e

$$Cf = (f^TCf)f + \lambda(0)e$$

[Since e and f are orthogonal to each other]

$$\Rightarrow \quad Cf = (f^TCf)f$$

$\Rightarrow$ f is eigenvector of C. (Let the eigenvalue be λ_0)

$$\Rightarrow \quad Cf = \lambda_0 f$$

$$\Rightarrow \quad f^TCf = \lambda_0$$

Thus to max. f^TCf, λ_0 (Eigenvalue) needs to be as high as possible

$\Rightarrow \lambda_0$ is second highest eigenvector.