Magnetic field dependence of critical current in a Josephson junction

Pradyot pritam sahoo*, 1811106, P452
National Institute of Science Education and Research

One of the most striking features of the behavior of Josephson structure is the occurrence of diffraction and interference phenomena of supercurrents when magnetic fields are applied. This is a consequence of the wave-like nature of Cooper pairs and the phase coherence through Josephson links. The extremely high sensitivity of the Josephson current to magnetic fields is the key to the most important applications of the Josephson effect. Furthermore, a careful study of the dependence of the maximum d.c. Josephson current on the applied magnetic field represents a very powerful method to investigate important aspects of the junction behavior.

OBJECTIVE

- Study of magnetic field dependence of critical current in a Josephson junction in nonuniform current density.

THEORY

Here we are considering SIS junctions in which the current crosses the insulating layer perpendicularly to the $x-y$ plane i.e. in the z direction, and the external field H_y is applied along its width, i.e. in the y direction. By extension, the parameters t, W and L of the insulating layers will be referred as the thickness, the width and the length of the Josephson junctions. In practice, the insulating layer can be inserted between two superconductors S_1 and S_2, with different geometry. Damped by the screening current, the field decreases exponentially from each side of the insulating layer towards the superconducting parts, with respective penetration depths λ_1 and λ_2. From basic Josephson junction relation in stationary case

$$J = J_1 \sin\phi \;;\; \frac{d\phi}{dt} = 0 \tag{1a}$$

$$\Delta_{x,y}\phi = \left(\frac{2ed}{\hbar c}\right) H \times n \tag{1b}$$

where $d = \lambda_1 + \lambda_2 + t$ and λ_1 and λ_2 are effective London penetration depths of the two superconductors and t is the physical barrier thickness. H represents the actual magnetic field in the plane of the junction including both externally applied magnetic field and the field generated by the currents flowing in the junction, n is the unit vector normal to the plane of the junction. We now consider junctions with dimensions that are small compared to the Josephson penetration depth. We refer to a coordinate system with the z axis normal to the plane of the junction. Let us consider an external magnetic field H applied in the y direction; inside the junction the magnetic field is constant and equal to the external value (absence of self-fields). Thus by integration of eq(1b)

$$\phi(x) = \left(\frac{2\pi d}{\Phi_0} H_y\right) x + \phi_0 = kx + \phi_0$$

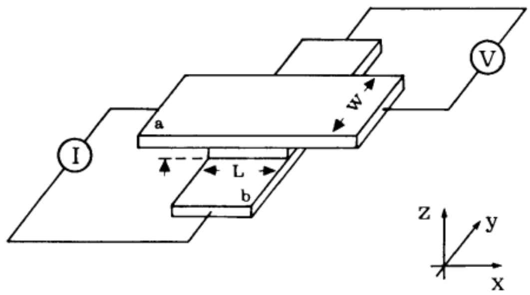

FIG. 1: Tunneling junction: The dimensions are L and W; a and b are the two superconducting films.

Where $\Phi_0 = hc/2e$ is the flux quantum and ϕ_0 is an integration constant. The phase shows a linear spatial variation along the x direction, the rate of variation being proportional to the externally applied magnetic field H_y. Furthermore from eq(1a) it follows that

$$J(x) = J_1 \sin\left(\frac{2\pi d}{\Phi_0} H_y x + \phi_0\right)$$

Therefore the Josephson current density exhibits a periodic distribution inside the junction. We can see that the periodic behavior of $J(x)$ leads, for particular values of the external magnetic field, to situations of zero net current through the junction. So the total current in the junction

$$\begin{aligned} I(k,\phi) &= \int\int dx \;\; dy J_1(x,y) sin\,(kx+\phi_0) \\ &= \int_{-L/2}^{+L/2} dx \mathscr{I}(x) sin(kx+\phi_0) \\ &= Im\left\{e^{j\phi_0}\int_{-L/2}^{+L/2} dx \mathscr{I}(x) e^{jkx}\right\} \end{aligned}$$

Where the integral is calculated over the junction area and $\mathscr{I}(x) = \int J_1(x,y)dy$. This expression of $I(k,\phi)$, maximized with respect to ϕ_0 gives the maximum Josephson current $I_1(k)$ as

$$I_1(k) = \left|\int_{-L/2}^{+L/2} dx \mathscr{I}(x) e^{jkx}\right|$$

For convenience we can calculate this integral between $-\infty$ and $+\infty$ assuming $\mathscr{I}(x) = 0$ for $|x| > L/2$, so that

$$I_1(k) = \left| \int_{-\infty}^{+\infty} dx \mathscr{I}(x) e^{jkx} \right| \tag{2}$$

Thus the maximum Josephson current, at a given applied magnetic field, is represented by the modulus of the Fourier transform of $\mathscr{I}(x)$.

VARIOUS CURRENT DENSITY

In actual junctions it is important to take into account effects of the Josephson current density($J_1 \neq$ constant) distribution due to nonuniform tunneling barriers. Study of the dependence of the maximum Josephson current, on the external magnetic field, provide a useful means to make evident spatial variations of $J_1 = J_1(x,y)$, since these are reflected in peculiar features of the I_1 vs. H patterns. Here we consider various current density profiles which account for nonuniformities localized in the barrier layer and calculate the corresponding $I_1 - H$ patterns.

Steplike current density

Let us assume in the rectangular junction geometry the magnetic field is applied in the y direction. The total Josephson current is given by

$$\mathscr{I}(x) = \begin{cases} 0 & \text{for } |x| > L/2 \\ \int_{-W/2}^{+W/2} J_1(x,y)dy = J_1(x)W & \text{for } |x| \leq L/2 \end{cases}$$

Now assume that the Josephson tunneling current density J_1 is spatially constant(step-like). Also $J_1(x,y)$ the profile only varies along x direction only. The corresponding analytical expression of $J_1(x)$ is

$$J_1(x) = J_0 \left[\xi p_{l/2}(x) + p_{s/2}\left(x - \frac{l+s}{2}\right) + p_{s/2}\left(x + \frac{l+s}{2}\right)\right] \tag{3}$$

Where ξ is a parameter ($\xi << 1$) and $l = L - 2s$. The meaning of J_o is the maximum current density. Where the p_τ function is defined as

$$p_\tau(x) = \begin{cases} 1 & \text{for} |x| \leq \tau \\ 0 & \text{for} |x| > \tau \end{cases}$$

So from $eq(2)$ the maximum Josephson current is $I_1(k)$ will be

$$J_0 W \int_{-\infty}^{+\infty} \xi p_{l/2}(x) + p_{s/2}(x - \frac{l+s}{2}) + p_{s/2}(x + \frac{l+s}{2}) e^{jkx} dx \tag{4}$$

After integrating the above equation analytically we got the expression

$$\begin{aligned} I\left(\frac{\Phi}{\Phi_0}\right) = I_1(0)\left(\frac{2s}{\xi l + 2s}\right) | \frac{\xi l}{2s} \frac{sin(\pi \frac{l}{L}\frac{\Phi}{\Phi_0})}{\pi \frac{l}{L}\frac{\Phi}{\Phi_0}} \\ + \frac{sin(\pi \frac{s}{L}\frac{\Phi}{\Phi_0})}{\pi \frac{s}{L}\frac{\Phi}{\Phi_0}} cos[\pi (\frac{l+s}{L})\frac{\Phi}{\Phi_0}]| \end{aligned} \tag{5}$$

Where $\Phi = H_y L d$. For $k = 0$ we have the maximum Josephson current

$$I_1(0) = J_0 W L \left(\xi \frac{l}{L} + \frac{2s}{L}\right)$$

Thus it is evident from above equation that the higher tunneling current value assumed at the edges of a rectangular junction leads to I_1 vs. H_e dependence which is different from what obtained assuming uniform current density distribution.

One-parameter current density

Let us consider now another current distribution peaked at the edges of the junction given by

$$J_1(x) = J_1 \frac{cosh(ax)}{cosh(aL/2)} \tag{6}$$

Where the parameter a has the dimension of the inverse of a length and gives a measure of the peak to valley ratio. In this case the maximum joshephson current will be

$$\begin{aligned} I_1(k) &= \left| \frac{J_1 W}{cosh(\frac{aL}{2})} \int_{-L/2}^{L/2} cosh(ax) e^{jkx} dx \right| \\ &= \left| \frac{J_1 W}{cosh(\frac{aL}{2})} \int_{-L/2}^{L/2} cosh(ax) cos(kx) dx \right| \end{aligned} \tag{7}$$

By integrating above equation analytically we can show that

$$I_1(k) = I_1(0) \frac{a^2}{a^2 + k^2} \left| \frac{k \sin(kL/2)}{a \tanh(aL/2)} + cos\left(\frac{kL}{2}\right) \right| \tag{8}$$

Where the maximum joshephson current $I_1(0)$ is given by

$$I_1(0) = \frac{2J_1}{a} W tanh\left(\frac{aL}{2}\right)$$

By defining $\chi = aL/2$ the above equation turned into

$$I\left(\frac{\Phi}{\Phi_0}, \chi\right) = I_1(0) \frac{\chi^2}{\chi^2 + (\pi\frac{\Phi}{\Phi_0})^2} \left| \frac{\pi \frac{\Phi}{\Phi_0} sin(\frac{\pi\Phi}{\Phi_0})}{\chi tanh(\chi)} + cos(\pi \frac{\Phi}{\Phi_0}) \right| \tag{9}$$

This dependence, computed for different values of the parameter χ. By equating to zero we get for the minima

$$\pi\frac{\Phi}{\Phi_0}tan\left(\pi\frac{\Phi}{\Phi_0}\right) = -\chi tanh(\chi)$$

It is possible to solve this equation by a simple graphical construction. The minima are given by the intersection of a horizontal line corresponding to a given value of $\chi \geq 0$.

Triangular current density

Lets consider now the opposite situation in which the current density distribution is maximum at the center of the junction. So assume a simple triangular profile, such that

$$J_1(x) = J_1 a_{L/2}(x) \tag{10}$$

Where,

$$a_{L/2}(x) = \begin{cases} 1 - \frac{|x|}{L/2} & \text{for} \quad |x| \leq L/2 \\ 0 & \text{for } |x| > L/2 \end{cases}$$

In this case maximum joshephson current will be

$$I_1(k) = \left|\int_{-\infty}^{+\infty} J_1(x)e^{jkx}dx\right| = \left|\int_{-\infty}^{+\infty} J_1 a_{L/2}(x)e^{jkx}dx\right| \tag{11}$$

By solving above $eq(11)$ analytically we got

$$I_1(k) = \left|J_1 \frac{WL}{2}\frac{sin^2(kL/4)}{kL/4}\right| \tag{12}$$

We recall incidentally, that the lowering of the secondary maxima occurs also in circular junctions with a uniform current density distribution.

RESULT

For numerical integration of $eq(4)$ I have used the *scipy.integrate.quad* module in python library, where we can integrate from a to b (possibly infinite interval) using a technique from the *Fortran* library *QUADPACK*. Also for $eq(7), eq(11)$ I have used Gaussian quadrature method with tolerance($\varepsilon = 10^-5$).

Steplike current density

By integrating the $eq(4)$ numerically, we have showed the I_1 vs H_e dependence. It is interesting to observe the very good agreement between experiments and calculations obtained assuming a steplike current density distribution as reported in the inset. Here we have reported the dependence given for several values of the parameters ξ and $s' = s/L$. It is clear that s' influences the period of the superimposed modulation.

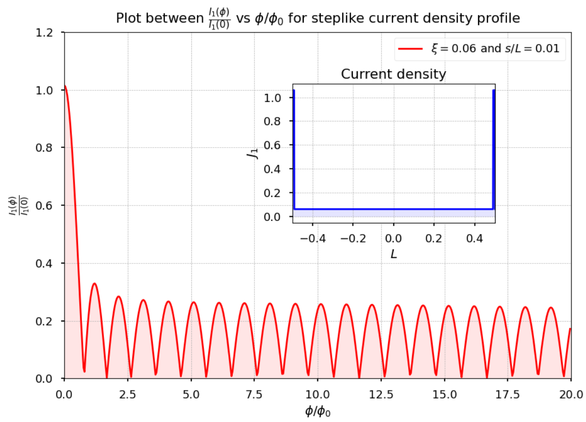

FIG. 2: $I_1(x)$ vs H_e with steplike current density($\xi = 0.06$ and $s/L = 0.01$)

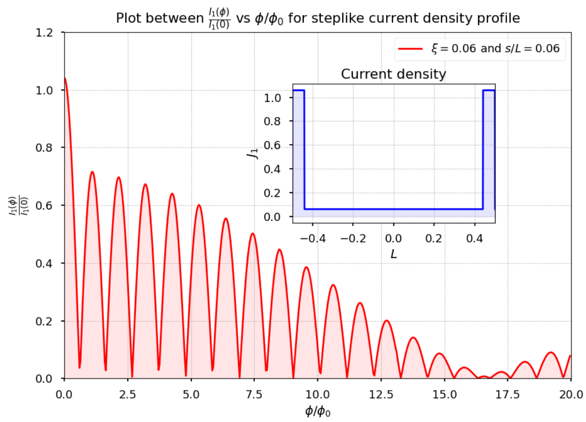

FIG. 3: $I_1(x)$ vs H_e with steplike current density($\xi = 0.06$ and $s/L = 0.06$)

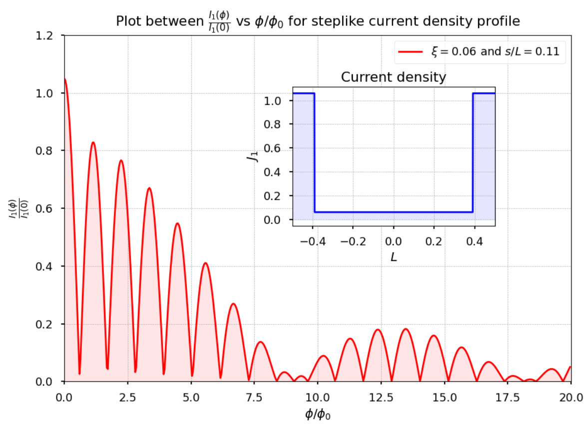

FIG. 4: $I_1(x)$ vs H_e with steplike current density($\xi = 0.06$ and $s/L = 0.11$)

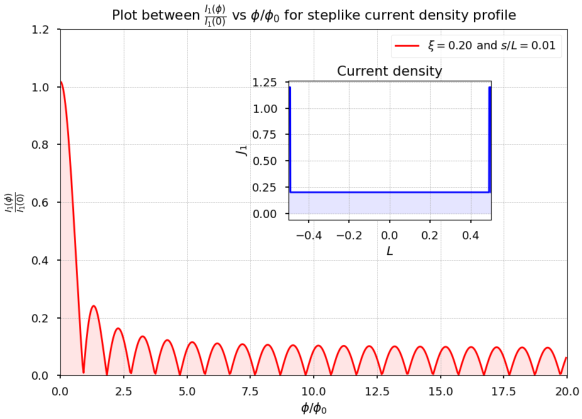

FIG. 5: $I_1(x)$ vs H_e with steplike current density($\xi = 0.20$ and $s/L = 0.01$)

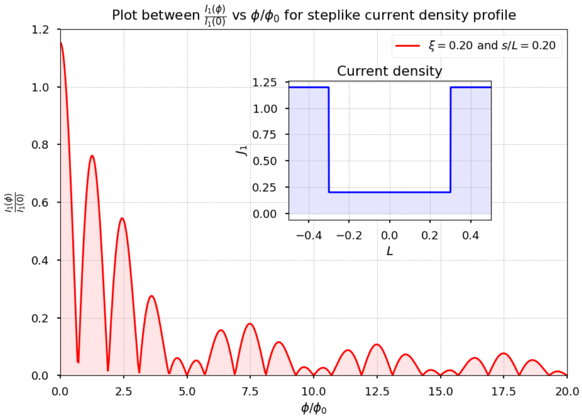

FIG. 6: $I_1(x)$ vs H_e with steplike current density ($\xi = 0.20$ and $s/L = 0.20$)

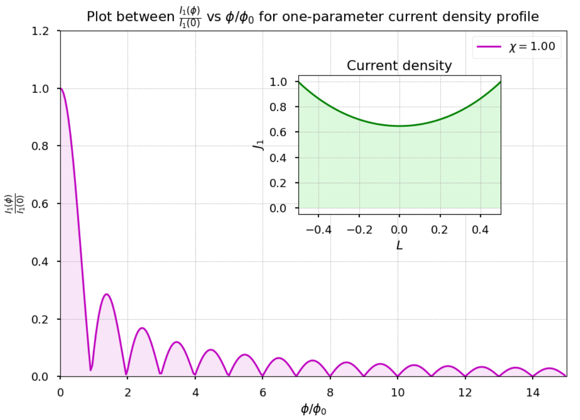

FIG. 7: $I_1(x)$ vs H_e with one-parameter current density($\chi = 1.0$)

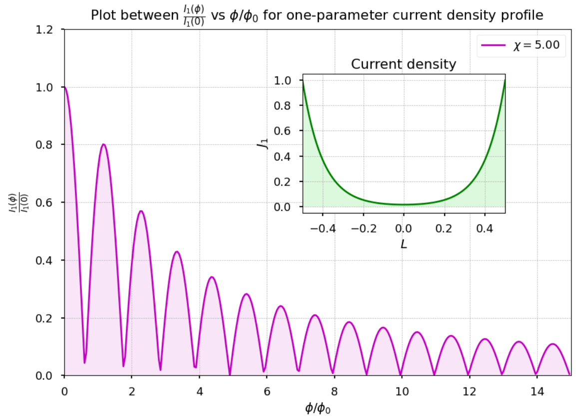

FIG. 8: $I_1(x)$ vs H_e with one-parameter current density ($\chi = 5.0$)

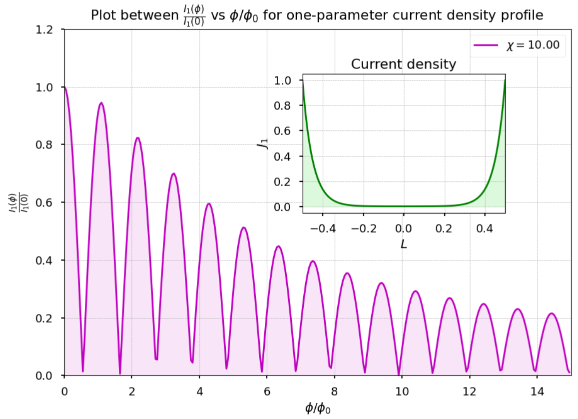

FIG. 9: $I_1(x)$ vs H_e with one-parameter current density ($\chi = 10.0$)

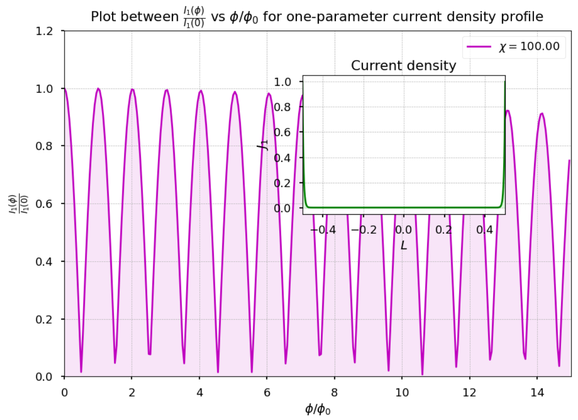

FIG. 10: $I_1(x)$ vs H_e with one-parameter current density ($\chi = 100.0$)

One parameter current density

For the dependence of I_1 vs H_e, we have integrate $eq(7)$ numerically. Here we got that for $\chi \to 0$ which is a uniform current density distribution, we find the expected positions for the minima in the Fraunhofer pattern, that is, the distance between the first two minima (symmetric with respect to zero) is twice the distance between consecutive minima. In the limit of large χ, the distance between all the consecutive minima becomes the same as shown in $fig(10)$.

Triangular current density

In triangular current density profile we have integrated $eq(11)$ numerically. When current density distributions decreasing from the center to the edges of the junction are considered, a lowering of the secondary maxima occurs in the I_1 vs. H_e patterns as shown in $fig(11)$. This agrees with the experimental observation by Dynes and Fulton who have calculated also the corresponding current density profile.

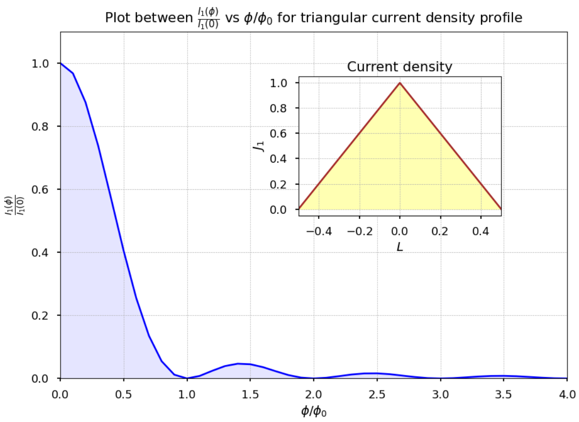

FIG. 11: Magnetic field dependence of $I_1(x)$ for a junction with a triangular current density

CONCLUSION

It is important to take into account of the Josephson current density distribution due to nonuniform tunneling barriers like actual junction. One of the important features of Josephson structures is the occurrence of diffraction and interference phenomena of supercurrents when magnetic fields are applied. Here we have considered various current density profiles which account for nonuniformities localized in the barrier layer and calculated the corresponding $I_1 - H_e$ patterns. The extremely high sensitivity of the Josephson current to magnetic fields is the key to the most important applications of the Josephson effect. Furthermore, a careful study of the dependence of the maximum $d.c.$ Josephson current on the applied magnetic field represents a very powerful method to investigate important aspects of the junction behavior.

ACKNOWLEDGMENT

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[1] A. Barone and G. Paterno, Physics and Application of Josephson effect *A wiley-interscience publication*
[2] P. Mangin and R. Kahn, Superconductivity An introduction *springer*

Appendixes:Python Code

Github file link : `https://github.com/pradyotpsahoo/P452-Computational-physics`