CSC2429: Proof Complexity **Winter 2018**

Lecture 3: SA and SOS

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3.1 SA Review

Recall: Each feasible solution α to the degree-d SA LP corresponds to a linear functional $E_\alpha : [x_1, ..., x_n]_d \to \mathbb{R}$. These functionals are the *pseudo-distributions*. The set of all such functions is $\mathcal{E}_d(\mathcal{H})$ where $\mathcal{H} = \{A\mathbf{x} - \mathbf{b} \geq 0, 1 \geq \mathbf{x} \geq 0, 1 \geq 0\}$ are our original inequalities (which are linear).

Lemma 3.1 *$\mathcal{H} = \{A\mathbf{x} - \mathbf{b} \geq 0, 1 \geq \mathbf{x} \geq 0, 1 \geq 0\}$ and $\mathcal{F}$ as before. Then*

$$\min\{E(P) : E \in \mathcal{E}_d(\mathcal{H})\}$$

equals $\max\{c_0$ *: there is a degree-d SA derivation of $P \geq c_0$ from $\mathcal{H}, \mathcal{F}$*$\}$.

3.1.1 How hard is it to find SA proofs?

If the LP is infeasible then we can find a SA refutation of degree-d

3.1.2 Proof Systems/ LP tightening related to SA

1. Dynamic SA (LS_d): normal SA is a one shot system. The dynamic variant takes the original LP, lifts to degree-d, then project down to your original dimension. If you did not get the solution that you want yet, add some of the higher degree inequalities to your original set and repeat.

2. Nullsatz (polynomial calculus, PC): over any field, not just $\mathbb{R}$, were we only allow equalities. Consider the k-CNF $F = C_1 \wedge \cdots \wedge C_m$ over $x_1, ..., x_n$. We will convert each clause C_i to an equality as follow (an example will suffice):
$$x_1 \vee \bar{x_2} \vee x_3 \implies (1 - x_1)x_2(1 - x_3) = 0$$
and including $x_i^2 - x_i = 0$. Observe that the only way the polynomial would *not* be satisfied is if the clause is false.

   There is a dynamic variant of Nullsatz.

3. Cutting Planes: this is *not* a lift and project system (unlike SA and Nullsatz). The rules are

   (a) Add non-negative linear combinations of previously derived inequalities.

   (b) Perform division with rounding. That is, for
   $$\sum_{i=1}^{n} a_i x_i \geq a_0$$
   where $a_1, ..., a_n$ *is* divisible by k and a_0 is not, we can derive the inequality:
   $$\sum_{i=1}^{n} \frac{a_i}{k} x_i \geq \lceil a_0/k \rceil.$$

It is unknown if this system is automatizable, but this is widely used in optimization applications for solving LP.