CMPT 409: Theoretical Computer Science — Summer 2017

Lecture 6: Incompleteness (13 July - 4 Aug)

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6.1 Gödel Incompleteness Theorem

Definition 6.1 ***True Arithmetic*** *(TA) is the set of all sentence in*

$$\delta_A = [0, s, +, \cdot, =]$$

that are true in the standard model $\underline{\mathbb{N}}$.

Notation: $s_0 = 0$ *and* $s_{k+1} = ss_k$ *for all* $k = 0, 1...$ *so* $s_3 = 3$. s_k *is a syntactic object which represent semantic object (the number* k*). Further* $A(s_{\vec{a}})$ *means* $A(s_{a_1}, ..., s_{a_n})$.

Definition 6.2 *If* R *is an* n*-ary relation* $A(\vec{x})$ *is a formula such that all free variables among* $x_1, ..., x_n$. *Then* $A(\vec{x})$ ***represents*** R *if and only if* $\forall \vec{a} \in \mathbb{N}^n$

$$R(\vec{a}) \iff \underline{\mathbb{N}} \vdash A(s_{\vec{a}})$$

Definition 6.3 *A relation* R *is* ***arithmetical*** *if and only if* R *is representable by some formula (in the vocabulary of* δ_A*).*

Consider some arithmetical relations

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Definition 6.4 *Let* Δ_0 *be the set of all bounded formulas.*

$R(\vec{x})$ *is a* Δ_0 ***relation*** *if and only if some* Δ_0 *formula* A *represents* R. *This implies what about* Δ_0 *relations and arithmetical?*

Lemma 6.5 *The* Δ_0 *relations are closed under* $\wedge, \vee, \neg$ *and bounded* $\forall \leq$, $\exists \leq$.

Proof:

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Theorem 6.6 *TA is not a recursive set (not even RE) and does not have a recursive set of axioms.*

Proof:

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Since TA is so unwieldy, we will consider instead the subset of Peano Arithmetics (PA).

Theorem 6.7 *(Gödel's Second Incompleteness Theorem) the consistency of PA cannot be proved in PA.*

Proof:

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Even though Peano arithmetics is incomplete, a simpler arithmetics (Presburger Arithmetics, containing only the + operation) is decidable.