CSC2429: Proof Complexity	**Winter 2018**

Lecture 4: SOS

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4.1 SOS

First lets consider some simple polynomials such as $f : \{0,1\}^n \to \mathbb{R}$ over the boolean hyper-cube for $x_1, ..., x_n$. It turns out that: f is non-negative if and only if f has a SOS certificate (what you want to consider here is the function $\sqrt{f}$ and find the unique multinomial g such that $g = \sqrt{f}$ over the boolean hyper-cube).

4.1.1 SOS Certificates for PSD Matrices

Lemma 4.1 *Let $f : \mathbb{R}[x_1, ..., x_n] \to \mathbb{R}$ be a degree $2d$ polynomial. Let $\mathbf{z}$ be a vector of degree $\leq d$ monomials i.e. $\mathbf{z} = [1, x_1, ..., x_n, x_1^2, x_2^2, ...]$. Then f is a sum-of-squares if and only if there exists $Q \succeq 0$ such that $f(\mathbf{x}) = \mathbf{z}^T Q \mathbf{z}$.*

If we were to take $f : \{0,1\}^n \to \mathbb{R}$, then the degree needs to go up to n. To see why this is we will show that SAT is solvable in this paradigm. Let $F = C_1 \wedge \cdots \wedge C_m$ be an unsatisfiable k-CNF formula over $x_1, ..., x_n$. Where we are choose F to be a random collection of $\sim 10n$ equations modulo 2 or a random 3-CNF with $\sim 10n$ clauses or F could be the particular family of modulo 2 equations coming from a 3-regular expander with an odd number of vertices (for some reason you want the expansion factor to be large).

Theorem 4.2 *For n sufficiently large, $\{g_n\}$ a degree 3 expander graphs and let F_n be the Tseitin 3-CNF on g_n (which is unsatisfiable) then any SOS certificate for P_f has degree $\Omega(n)$, where $P_f : \{0,1\}^n \to \mathbb{R}$.*

4.1.2 Positivestellensatz

Recall from the results of Hilbert's 17$^{\text{th}}$ problem