Question Set

Problem 1. *1.3.2 Let $\mathcal{K}_2$ denote the collection of all closed convex sets in the plane. Show that $D(n, \mathcal{K}_2) = o(n)$ and $\mathsf{disc}(n, \mathcal{K}_2) \geq \frac{n}{2}$.*

Solution. We choose a triangular lattice of points. This obtains a Lebesgue-measure discrepancy of about $\sqrt{n} \in o(n)$ according to the book (page 3). To show that $\mathsf{disc}(n, \mathcal{K}_2) \geq \frac{n}{2}$ consider the following set of points P in the plane where $|P| = n$. Let P be n points evenly spaced about the circle of radius $\frac{1}{2}$ centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ within the unit cube. Denote this circle by C. By the pigeon hole principle every coloring χ of C will color one at least $\frac{n}{2}$ of the points of P the same color. Let the closed convex polygon formed by these points be denoted by G. Then $|\chi(P \cap G)| \geq \frac{n}{2}$.

Problem 2. *4.2.4 Let $A = \frac{1}{2}(H + J)$ be the incidence matrix of set system $\mathcal{S}$. Show that the eigenvalue bound is quite weak for A, namely that the smallest eigenvalue of $A^T A$ is $O(1)$*

Proof. Similar to the proof of Proposition 4.4 (Hadamard set system) in the book we first calculate $A^T A$:

$$
\begin{aligned}
A^T A &= \frac{1}{4}\left(H^T + J^T\right)(H + J) \\
&= \frac{1}{4}\left(H^T H + J^T J + H^T J + J^T H\right) \\
&= \frac{n}{4}\left(I + J + R + R^T\right)
\end{aligned}
$$

where R is the $n \times n$ matrix whose first row is all ones and all the remaining entries are zeros. The eigenvalue bound says that $\mathsf{disc}(A) \geq \mathsf{disc}_2(A) \geq \sqrt{\lambda_n}$ where λ_n is the smallest eigenvalue. Since $\lambda_n = \min_{\|x\|=1} x^T A^T A x$ we will find an x such that $\|x\| = 1$ and $x^T A^T A x \in O(1)$. Consider

$$
x = \left\langle -\sqrt{\frac{n-1}{n+3}}, \frac{2}{\sqrt{(n-1)(n+3)}}, \frac{2}{\sqrt{(n-1)(n+3)}}, ..., \frac{2}{\sqrt{(n-1)(n+3)}} \right\rangle.
$$

Observe that:

$$
\begin{aligned}
x^T A^T A x &= \frac{n}{4} x^T \left(I + J + R + R^T\right) x \\
&= \frac{n}{4}\left(\sum_{i=1}^{n} (x_i)^2 + \left(\sum_{i=1}^{n} x_i\right)^2 + 2x_1\left(\sum_{i=1}^{n} x_i\right) + \right) \\
&= \frac{n}{4}\left(1 + \frac{n-1}{n+3} - \frac{2(n-1)}{n+3}\right) \\
&= \frac{n}{4}\left(1 - \frac{(n-1)}{n+3}\right) \\
&= \frac{n}{n+3} \leq 1
\end{aligned}
$$

Thus the eigenvalue bound is not tight. $\square$

Problem 3. *4.3.2 Find a set system $(X, \mathcal{S})$ and a set $A \subset X$ such that $\mathsf{disc}(\mathcal{S}) = 0$ but $\mathsf{disc}(\mathcal{S} \cup \{A\})$ is arbitrarily large.*

Solution. Let $X = Y \cup Z$ where Y and Z are two disjoint sets of size n. Let $\mathcal{S}$ be the set of all subsets of X of the form $Y' \cup Z'$ where $Y' \subset Y$, $Z' \subset Z$ and $|Y'| = |Z'|$. As we have discussed, $\mathsf{disc}(\mathcal{S}) = 0$ by coloring elements of Y color 1 and the elements of Z color -1. Next let $A = Y$. Consider $\mathsf{disc}(\mathcal{S} \cup \{A\})$. Consider any coloring χ of X. Suppose without loss of generality that $Y' \subseteq Y$ such that for all $y \in Y'$, $\chi(y) = 1$ and $|Y'| \geq \frac{n}{2}$. Either $|Y'| - (|Y| - |Y'|) \geq \frac{n}{4}$ or $|Y'| - (|Y| - |Y'|) < \frac{n}{4}$. In the former case $\mathsf{disc}(\mathcal{S} \cup \{A\}) \geq \frac{n}{4} \in O(n)$. In the latter case $|Y'|$ and $|Y| - |Y'|$ (i.e. the number of elements colored 1 and -1 respectively by χ) differ by at most $\frac{n}{4}$. Consider a subset $Z' \subset Z$ such that $|Z'| = |Y'|$. And let $Z' = Z_1 \cup Z_{-1}$ such that for all $z \in Z_1$, $\chi(z) = 1$ and for all $z \in Z_{-1}$, $\chi(z) = -1$ Either $|Z_1| \geq \frac{n}{4}$ or $|Z_{-1}| \geq \frac{n}{4}$. In the former case $\chi(Y' \cup Z') \geq \frac{n}{4}$. In the latter case $\chi((Y - Y') \cup Z'') \geq \frac{n}{4}$ where $Z'' \subset Z'$, $Z_{-1} \subset Z''$ and $|Z''| = |Y - Y'|$. Since $Y' \cup Z'$ and $(Y - Y') \cup Z'' \in \mathcal{S}$, in both case we have $\mathsf{disc}(\mathcal{S} \cup \{A\}) \geq \frac{n}{4} \in O(n)$.

Problem 4. *4.3.5 Let A be an $m \times n$ real matrix and set*

$$\Delta = \max_{w \in \{-1,0,1\}^n} \min_{x \in \{-1,1\}^n} \|A(x - w)\|_\infty$$

*(*linear discrepancy with weights -1, 0, 1*). Prove that* $\mathsf{lindisc}(A) \leq 2\Delta$.

Proof. From our discussion two weeks ago, we observe that the proof of $\mathsf{lindisc} \leq 2 \cdot \mathsf{herdisc}$ is essentially the same as that for $\mathsf{lindisc}(A) \leq 2 \cdot \Delta$. We will reproduce the proof here in our own words to make sure it sticks. Δ can be interpreted as the moment when $B = \cup_{a \in \{-1,1\}^n} U + a$ covers the points $w \in \{-1, 0, 1\}^n$. We claim that at the same moment $C = \cup_{a \in \{-1,1\}^n} 2U + a$ covers all points $x \in [-1, 1]^n$. Since C is closed (all boundary points of C are in C), we can restrict our attention to the dense set of dyadic rationals of the form $v = \frac{z}{2^k} \in [-1, 1]^n$ where $z \in \mathbb{Z}^n$. The base case holds since B covers the points w described above. Consider any point $v = \frac{z}{2^k} \in [-1, 1]^n$. Then $2v = \frac{z}{2^{k-1}} \in [-2, 2]^n$. There exists $b \in \{-1, 1\}^n$ such that $2v - b \in [-1, 1]^n$. Since $2v - b = \frac{z + 2^{k-1} b}{2^{k-1}}$, by the induction hypothesis there exists some $a \in \{-1, 1\}^n$ such that $2v - b \in U + a$ and $v \in U + \frac{a+b}{2}$ (1). $\frac{a+b}{2} \in \{-1, 0, 1\}^n$. Since B covers all such points, there exists some $c \in \{-1, 1\}^n$ such that $\frac{a+b}{2} \in U + c$. By substituting this into (1) we have

$$v \in U + (U + c) = 2U + c.$$

It is important to note that U is convex here. Otherwise it might not be the case that elements of U are closed under addition. □

Problem 5. *4.3.4 Show that the $1 \times n$ matrix $[1, 2, 2^2, ..., 2^{n-1}]$ has hereditary discrepancy 2^{n-1} and linear discrepancy at most 2.*

Proof. (I know you talked about solution to this problem last time, but I just want to write it down to make sure.) Consider the subset of the universe consisting of the element 2^{n-1}. Coloring this element ± 1 will yield hereditary discrepancy of 2^{n-1}. Consider any $w \in [0, 1]^n$ and let $[1, 2, 2^2, ..., 2^{n-1}] \cdot w = k \in [0, 2^n - 1]$. Observe that each of the 2^n elements of $x \in -1, 1^n$ yields a distinct integer in the range $\{0, 1, ..., 2^n - 1\}$. Thus $[1, 2, 2^2, ..., 2^{n-1}] \cdot x$ is within distance 1 to any value k. Now if we consider the definition of linear discrepancy where $w \in [-1, 1]^n$ and $x \in \{-1, 1\}^n$, we see that the linear discrepancy is at most 2. □