CMPT 409: Theoretical Computer Science **Summer 2017**

Lecture 3: Propositional Logic (29 May - 2 June)

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3.1 Review

Recall consistency means that there exists a structure which satisfies all sentences in the set. Recall the compactness theorem for f.o. logic. It does not hold for finite models and second order logic. Then we talked about elementary classes (and elementary in the wider sense). We concluded with the Löwenheim-Skolem Theorem. Induction and reachability (in graphs) are *NOT* first order axiomatizable!

3.2 Definability in a Structure

Let $\mathcal{M}$ be a structure, ϕ a formula s.t. all free vars are among $v_1, ..., v_k$. Construct a relation:

$$\{\langle a_1, ..., a_k\rangle : \mathcal{M} \vDash \phi[a_1, ..., a_k]\}$$

where $\mathcal{M}$ satisfies ϕ with an assignment σ such that $\sigma(v) = a_i$, $1 \leq i \leq k$. Then ϕ **defines** this relation R. A k-ary relation on $|\mathcal{M}|$ is definable in $\mathcal{M}$ iff there is a formula which defines it there. So relations can be defined in-terms of formulas with free variables. Consider $\mathcal{L}_A$ with the $\mathbb{N}$ structure. If we want the $<$ relations then $\phi(x, y) := \exists k, x + s(k) = y$. Consider also language L with $=, f$ and structure $\mathcal{M} = \{\mathbb{Z}; \cdot\}$. Then what relation does the following relation define:$(\exists u, fxu = y) \wedge (\exists v, fxv = z)$?

With structure $\mathcal{L}_A$, you can define a formula for the primes! So we need some formula with one free variable $\phi(x)$ as follows:

Proposition 3.1 *There is no FO formula with free variables x, y which defines the relation:* x is reachable from y *in the class of a directed graphs.*

Proof: By contradiction. Suppose there is a formula $\Phi(x, y)$ which defines the relation above. ∎