CSC236H Exercise 1

Sample Solutions

Winter 2016

1. Use induction to prove that $3^{2n} - 1$ is divisible by 8, for all $n \in \mathbb{N}$.

   Solution: Let $P(n)$ denote the assertion that $3^{2n} - 1$ is divisible by 8.

   Base Case: Let $k = 0$.
   Then $3^{2k} - 1 = 3^0 - 1 = 0$. Since 0 is divisible by any natural number, including 8, we can conclude that $P(0)$ is true.

   Induction Step: Let $k \in \mathbb{N}$. Suppose $P(k)$ is true, i.e., $3^{2k} - 1$ is divisible by 8. **[IH]**
   WTP: $P(k+1)$ holds, i.e., $3^{2(k+1)} - 1$ is divisible by 8.

$$
\begin{aligned}
3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\
&= 3^{2k}3^2 - 1 \\
&= (3^{2k} - 1 + 1)3^2 - 1 && \text{\# added and subtracted 1} \\
&= (3^{2k} - 1)3^2 + 3^2 - 1 && \text{\# distributed } 3^2 \text{ over } (3^{2k} - 1 + 1)
\end{aligned}
$$

   By IH, $3^{2k} - 1$ is divisible by 8, so there must exists $j \in \mathbb{N}$ such that $3^{2k} - 1 = 8j$. Then we have

$$
\begin{aligned}
3^{2(k+1)} - 1 &= 8j \times 3^2 + 9 - 1 \\
&= 8(9j + 1)
\end{aligned}
$$

   Therefore, $3^{2(k+1)} - 1$ is divisible by 8, and so $P(k+1)$ holds.

2. Assume $x \in \mathbb{R}$ and $(x + \frac{1}{x}) \in \mathbb{Z}$. Use induction to prove that for all $n \in \mathbb{N}$

$$(x^n + \frac{1}{x^n}) \in \mathbb{Z}.$$

   Solution: Let $P(n)$ denote the assertion that $x^n + \frac{1}{x^n}$ is an integer.

   Base Case: Let $k = 0$.
   Then $x^n + \frac{1}{x^n} = x^0 + \frac{1}{x^0} = 1 + 1 = 2$. Therefore, $P(0)$ holds.

   Notice that in the problem statement, $P(1)$ is given as an assumption.

   Induction Step: Let $k \in \mathbb{N}$ and $k \geq 1$. Suppose for all $j \in \mathbb{N}$, $0 \leq j \leq k$, $P(j)$ is true, i.e., $x^j + \frac{1}{x^j}$ is an integer. **[IH]**

(Note that we need to include the condition $k \geq 1$ in the IH because in the inductive step we go to $k-1$.)
WTP: $P(k+1)$ holds, i.e., $x^{k+1} + \frac{1}{x^{k+1}}$ is an integer.

By IH, we know that $x + \frac{1}{x}$ and $x^k + \frac{1}{x^k}$ are both integers. Since integer numbers are closed under multiplication, $(x + \frac{1}{x})(x^k + \frac{1}{x^k})$ is also an integer.

$$\begin{aligned}(x + \frac{1}{x})(x^k + \frac{1}{x^k}) &= x^{k+1} + \frac{1}{x^{k-1}} + x^{k-1} + \frac{1}{x^{k+1}} \\ &= (x^{k-1} + \frac{1}{x^{k-1}}) + (x^{k+1} + \frac{1}{x^{k+1}})\end{aligned}$$

Since $(x + \frac{1}{x})(x^k + \frac{1}{x^k})$ is an integer, we have $\left((x^{k-1} + \frac{1}{x^{k-1}}) + (x^{k+1} + \frac{1}{x^{k+1}})\right) \in \mathbb{Z}$.

On the other hand, by IH , we know that $x^{k-1} + \frac{1}{x^{k-1}} \in \mathbb{Z}$ (We can apply the IH for $k-1$ since $k \geq 1$, and so $0 \leq k-1 \leq k$). Therefore, $(x^{k+1} + \frac{1}{x^{k+1}})$ must also be an integer.