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Lecture 1: Introductions

Instructor: Dmitriy Panchenko *Scribe: Lily Li*

Definition 1. *A* probability space *is a triple* $(\Omega, \mathcal{A}, \mathbb{P})$ *where* Ω *is the* sample space*,* $\mathcal{A}$ *is an algebra, defined below, and* $\mathbb{P} : \mathcal{A} \to [0, 1]$ *is a* probability measure*, or* measure*. We call a subset of* Ω*, an event.*

A collection of events of Ω*,* $\mathcal{A}$*, is an* algebra *if*

1. $\Omega \in \mathcal{A}$*: we can talk about the probability of the entire sample space.*
2. *if* $C, B \in \mathcal{A}$*, then* $C \cup B, C \cap B \in \mathcal{A}$*: we can talk about the union and intersection of events.*
3. *if* $C \in \mathcal{A}$*, then* $\Omega \backslash C = C^c \in \mathcal{A}$*: we can talk about the complement of an event.*

A σ*-algebra is an algebra which further satisfies:* $C_i \in \mathcal{A}$ *for countably many* C_i*, then* $\bigcup_{i=1}^{\infty} C_i \in \mathcal{A}$*. This means that we are allowed to take the union of countably many events.*

By themselves, the pair $(\Omega, \mathcal{A})$ *is a* measurable space *if* $\mathcal{A}$ *is a* σ*-algebra of subsets of* Ω*.*

Measure *means* something*... he went too fast and I couldn't follow.*

The probability measure $\mathbb{P}$ *satisfies:*

1. $\mathbb{P}(\Omega) = 1$*,*
2. $\mathbb{P}(A) \geq 0$*,*
3. $\mathbb{P}$ *is* countably additive *i.e.* $A_i \in \mathcal{A}$*,* $A_i \cap A_j \neq \emptyset$ *if* $i \neq j$*, then* $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$*; or alternatively,*
4. $\mathbb{P}$ *is* finitely additive *and* continuous *i.e. for any decreasing sequence* $B_n \supseteq B_{n+1} \in \mathcal{A}$*, if* $B = \bigcap_{i=1}^{\infty} B_n$ *then* $\mathbb{P}(B) = \lim_{n\to\infty} \mathbb{P}(B_n)$*.*

Claim 2. *A measure* $\mathbb{P}$ *is* countably additive *if and only if it is* finitely additive *and* continuous*.*

Proof. Suppose that $\mathbb{P}$ satisfies (3) above, then we show that it satisfies (4). Let C_i be the following sequence of disjoint sets: $C_i = B_i \backslash B_{i+1}$. Then $\mathbb{P}(B_n) = \mathbb{P}(B \cup (\cup_{i \geq n} C_i))$. Since the C_is and B are clearly disjoint, we have $\mathbb{P}(B_n) = \mathbb{P}(B) + \sum_{i \geq n} \mathbb{P}(C_i)$. Taking the limit of both sides,

$$\lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \left(\mathbb{P}(B) + \sum_{i \geq n} \mathbb{P}(C_i) \right) = \lim_{n\to\infty} \mathbb{P}(B) = \mathbb{P}(B)$$

since the tail of a convergent series, $\{C_i\}$, approaches zero in the limit.

Conversely if $\mathbb{P}$ satisfies (4) above, then it must satisfy (3). Let $B = \cap_{i=1}^{\infty} A_i$ and $B_n = \cup_{i \geq n+1} A_i$. Then

$$\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i) + \mathbb{P}(B_n).$$

Taking the limit of both sides

$$\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\mathbb{P}(A_i) + \mathbb{P}(B_n)\right) = \sum_{i=1}^{\infty}\mathbb{P}(A_i)$$

since $\lim_{n\to\infty}\mathbb{P}(B_n) = 0$. □

Example 3. *Consider the following probability space:* $([0,1], \mathbb{B}[0,1], \lambda)$ *where* $\mathbb{B}[0,1]$ *denotes the Borel* σ*-algebra over* $[0,1]$ *and* λ *is the Lebesgue measure. Here the Borel* σ*-algebra, also Borel Set, is the smallest* σ*-algebra containing all open sets.*

Think about why the power set of $[0,1]$ *is not an algebra.*