Question Set

Problem 1. *4.1.2 (Discrepancy of the product of set systems) Let $\mathcal{S}$ and $\mathcal{T}$ be set systems on finite sets. Let $\mathcal{S} \times \mathcal{T} = \{S \times T : S \in \mathcal{S}, T \in T\}$.*

1. Show that $\mathsf{disc}(\mathcal{S} \times \mathcal{T}) \leq \mathsf{disc}(\mathcal{S})\mathsf{disc}(\mathcal{T})$.

   Proof. Let $\mathsf{disc}(\mathcal{S}) = d_{\mathcal{S}}$ and $\mathsf{disc}(\mathcal{T}) = d_{\mathcal{T}}$. Then there exists colorings $\chi_{\mathcal{S}}$ of $X_{\mathcal{S}}$ and $\chi_{\mathcal{T}}$ of $X_{\mathcal{T}}$ such that $\mathsf{disc}(\chi_{\mathcal{S}}, \mathcal{S}) = d_{\mathcal{S}}$ and $\mathsf{disc}(\chi_{\mathcal{T}}, \mathcal{T}) = d_{\mathcal{T}}$. We will construct a coloring $\chi : X_{\mathcal{S}} \times X_{\mathcal{T}} \to \{-1, 1\}$ such that for all $S \times T \subseteq \mathcal{S} \times \mathcal{T}$, $|\chi(S \times T)| \leq d_{\mathcal{S}} \cdot d_{\mathcal{T}}$. For $s \in X_{\mathcal{S}}$ and $t \in X_{\mathcal{T}}$ define: $\chi(s,t) = \chi_{\mathcal{S}}(s) \cdot \chi_{\mathcal{T}}(t)$. Observe that

   $$\begin{aligned} |\chi(S \times T)| &= \left| \sum_{(s,t) \in \mathcal{S} \times \mathcal{T}} \chi(s,t) \right| = \left| \sum_{(s,t) \in \mathcal{S} \times \mathcal{T}} \chi_{\mathcal{S}}(s) \cdot \chi_{\mathcal{T}}(t) \right| \\ &\leq \left| \sum_{s \in \mathcal{S}} \chi_{\mathcal{S}}(s) \right| \cdot \left| \sum_{t \in \mathcal{T}} \chi_{\mathcal{T}}(t) \right| \\ &= |\chi_{\mathcal{S}}(S)| \cdot |\chi_{\mathcal{T}}(T)| = d_{\mathcal{S}} \cdot d_{\mathcal{T}} \end{aligned}$$

   where the inequality on the second line follows from Cauchy-Schwarz. Thus $\mathsf{disc}(\mathcal{S} \times \mathcal{T}) \leq \mathsf{disc}(\mathcal{S})\mathsf{disc}(\mathcal{T})$ as required. □

2. Find an example with $\mathsf{disc}(\mathcal{S}) > 0$ and $\mathsf{disc}(\mathcal{S} \times \mathcal{S}) = 0$.

   Solution. Well, its not fair now since I looked at the answers but something like

   $$\{\{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{1,2,3,4,5,6\}\}$$

   works since you have all the properties that you need.

Problem 2. *4.3.3 Show that the set system $\mathcal{S} = \{\{1\}, \{2\}, ..., \{n\}, \{1,2,...,n\}\}$ has hereditary discrepancy 1 and linear discrepancy at least $2 - \frac{2}{n+1}$.*

Solution. To show that the hereditary discrepancy is 1 it suffices to show that for set system $\mathcal{S}_1 = \{\{1\}\}$ and set system $\mathcal{S}_2 = \{\{1, ..., n\}\}$ it is the case that $\mathsf{disc}(\mathcal{S}_1) = 1$ and that $\mathsf{disc}(\mathcal{S}_2) = 1$ (all other subsets of $\mathcal{S}$ can be reduced to these two). $\mathsf{disc}(\mathcal{S}_1) = 1$ is trivial. In the case that $\mathsf{disc}(\mathcal{S}_2)$ simply color half of the -1 and the other half 1. With this coloring $|\chi(\{1, ..., n\})| \leq 1$ with equality when n is odd.

To show that the linear discrepancy is at least $2 - \frac{2}{n+1}$ we will consider the adjacency matrix of $\mathcal{S}$. This is just the $n \times n$ identity matrix with an extra row of all ones. Consider the weight vector $\mathbf{w} = \langle 1 - \frac{2}{n+1}, ..., 1 - \frac{2}{n+1} \rangle$. Observe that:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \cdot (\mathbf{x} - \mathbf{w}) = \begin{bmatrix} x_1 - 1 + \frac{2}{n+1} \\ x_2 - 1 + \frac{2}{n+1} \\ x_3 - 1 + \frac{2}{n+1} \\ \vdots \\ x_n - 1 + \frac{2}{n+1} \\ \sum_{i=1}^{n} \left(x_i - 1 + \frac{2}{n+1} \right) \end{bmatrix}$$

If $x_i = -1$ then the i^{th} element would be $\geq 2 - \frac{2}{n+1}$. However, ever if $x_i = 1$ for every i then the last entry would be $\geq 2 - \frac{2}{n+1}$. Thus the linear discrepancy is $\geq 2 - \frac{2}{n+1}$.