Linear Discrepancy is Π_2-Hard

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Abstract

Maybe the following shows that Linear Discrepancy is Hard?

1 Introduction

Král and Nejedlý showed that Group Coloring is Π_2^P-Complete [2]. We will show that linear discrepancy is Π_2-Hard by reducing $\mathsf{GroupColor}_{\mathbb{Z}_3}$ to linear discrepancy (hence forth denoted LD). First we define $\mathsf{GroupColor}_{\mathbb{Z}_3}$ and LD and their associated decision problems.

Definition 1 *The input to* $\mathsf{GroupColor}_{\mathbb{Z}_3}$ *is a directed graph* $G = (V, E)$*. An* edge labeling *of* G *is a function* $\phi : E \to \mathbb{Z}_3$*. Let* $\chi : V \to \mathbb{Z}_3$ *be a coloring of the vertices using elements of* $\mathbb{Z}_3$*.* χ satisfies *edge label* ϕ *if for every directed edge* $e = uv$*,* $\chi(u) - \chi(v) \neq \phi(e)$*.* G *is* $\mathbb{Z}_3$-colorable *if every edge label is satisfied by some coloring of the vertex set.*

The associated decision problem asks if G *is* $\mathbb{Z}_3$*-colorable.*

Definition 2 *Given an* $m \times n$ *matrix* A *with entries in* $\mathbb{R}$*, the linear discrepancy of* A *is defined as:*

$$\mathsf{lindisc}(A) = \max_{w \in [0,1]^n} \min_{x \in \{0,1\}^n} \|A(x - w)\|_\infty.$$

The decision problem associated with linear discrepancy asks if $\mathsf{lindisc}(A) \leq d$ *for some constant* $d \geq 0$*.*

The structure of the proof will be similar to Haviv and Regev's hardness result for the Covering Radius Problem [1]. In particular we will use the basis of their lattice $\mathcal{L}_G$ — with some slight modifications — as our matrix A.

2 Completeness

Consider a Yes-instance of $\mathsf{GroupColor}_{\mathbb{Z}_3}$. That is, consider some digraph $G = (V, E)$, where $|V| = n$ and $|E| = m$, which is $\mathbb{Z}_3$-colorable. Consider the $m \times n$ matrix C where

$$C_{i,j} = \begin{cases} 1, & \text{if } e_i = v_j u \text{ for some vertex } u \\ -1, & \text{if } e_i = u v_j \text{ for some vertex } u \\ 0 & \text{otherwise.} \end{cases}$$

The lattice $\mathcal{L}_G = \{y \in \mathbb{Z}^m : \exists x \in \mathbb{Z}^n \text{ such that } y = Cx \mod 3\}$ is the set of all edge labels induced by colorings of V. Let $\mathcal{B}_G$ be the basis of $\mathcal{L}_G$ consisting of vectors $y \in \{0, 1, 2\}^m$. Observe that each column of $\mathcal{B}_G$ corresponds to some edge labeling modulo 3, thus all the entries of $\mathcal{B}_G$ are positive. Let $A = \mathcal{B}_G$ be an $m \times m$ matrix — since $\mathcal{B}_G$ spans $\mathbb{R}^m$ — and let $d = 1$. We will show that $\mathsf{lindisc}(A) \leq d$.

Consider any vector $w \in [0, 1]^m$. Define $\mathcal{U}_G \subset \mathcal{L}_G$ to be the *unit polytope* of $\mathcal{L}_G$ such that $y^* \in U_G$ if and only if $y^* = \mathcal{B}_G x$ for some $x \in \{0, 1\}^m$. The *Hardness of the Covering Radius Problem on Lattices* [1] showed that for every $z \in \mathbb{R}^m$ there exists a lattice point y such that $\|y - z\|_\infty \leq \frac{3-1}{2} = 1$ (we can reproduce the proof here if necessary). Thus for $y \in \mathcal{L}_G$ such that

$$y = \arg \min_{y \in \mathcal{L}_G} \|y - Aw\|_\infty,$$

y satisfies $\|y - Aw\|_\infty \leq 1$. By Lemma 3, we know that y must be some point in $\mathcal{U}_G$. Thus there exists a vector $x \in \{0, 1\}^m$ such that $y = Ax$. Then $\mathsf{lindisc}(A) = \max_{w \in [0,1]^m} \min_{x \in \{0,1\}^m} \|A(x - w)\|_\infty \leq 1$ as required.

Lemma 3 *Let $\mathcal{B}_G$ be the basis of lattice $\mathcal{L}_G$ with entries in $\{0, 1, 2\}$. For each $w \in [0, 1]^w$ there exists a point y in the unit polytope (i.e. $y = \mathcal{B}_G x$ for $x \in \{0, 1\}^w$) such that*

$$y = \arg \min_{y \in \mathcal{L}_G} \|y - \mathcal{B}_G w\|_\infty.$$

Proof. Consider any $z \in \mathcal{L}_G$ such that

$$z = \arg \min_{z \in \mathcal{L}_G} \|z - \mathcal{B}_G w\|_\infty$$

(that is: z is the closest lattice point to $\mathcal{B}_G w$ using the infinity norms as the metric). We will find a point y in the unity polytope such that

$$\|y - \mathcal{B}_G w\|_\infty \leq \|z - \mathcal{B}_G w\|_\infty$$

Since $z \in \mathcal{L}_G$, $z = \mathcal{B}_G c$ for some $c \in \mathbb{Z}^m$. Let $d \in \{0,1\}^m$ be defined as follows:

$$d_i = \begin{cases} c_i & \text{if } c_i \in \{0,1\} \\ 1 & \text{if } c_i > 1 \\ 0 & \text{if } c_i < 0 \end{cases}$$

Compare each element of $c - w$ with the associated element of $d - w$ and observe that $|d_i - w_i| \leq |c_i - w_i|$ (recall: $w_i \in [0,1]$). Since $\mathcal{B}_G$ is a matrix with positive entries, it follow that $\|\mathcal{B}_G(d-w)\|_\infty \leq \|\mathcal{B}_G(c-w)\|_\infty$ as required. □

3 Soundness

Consider a No-instance of $\mathsf{GroupColor}_{\mathbb{Z}_3}$. That is, consider a graph $G = (V, E)$ which is not $\mathbb{Z}_3$-colorable. Again, construct the lattice $\mathcal{L}_G$ from the induced edge-labels and a basis $\mathcal{B}_G$ for $\mathcal{L}_G$ with entries in $\{0,1,2\}$.

The *Hardness of the Covering Radius Problem on Lattices* [1] showed that there exists a $z \in \mathbb{Z}^m$ such that $\|y - z\|_\infty \geq \frac{3}{2}$ for every $y \in \mathcal{L}_G$. We can write $z = \mathcal{B}_G w$ for some $w \in \mathbb{R}^m$ since $\mathcal{B}_G$ is a spanning set of $\mathbb{R}^m$. Let $w' \in [0,1]^m$ be the vector such that $0 \leq w'_i \leq 1$ and $w_i = w'_i + c_i$ for some integer c_i. Thus

$$\mathsf{lindisc}(A) \geq \min_{x \in \{0,1\}^m} \|\mathcal{B}_G(x - w')\|_\infty \geq \frac{3}{2} > 1.$$

References

[1] Haviv, I., and Regev, O. Hardness of the covering radius problem on lattices. In *Computational Complexity, 2006. CCC 2006. Twenty-First Annual IEEE Conference on* (2006), IEEE, pp. 14–pp.

[2] Nejedlỳ, P., et al. Group coloring and list group coloring are π2p-complete. In *International Symposium on Mathematical Foundations of Computer Science* (2004), Springer, pp. 274–286.