Assignment 1

Problem 1. *In the following we will prove that any one-tape TM deciding the language* PAL *where* $\mathsf{PAL} = \{w \in \{0,1\}^* : w \text{ is a palindrome }\}$ *requires time* $\Omega(n^2)$.

The proof will take place over several steps. First we define the notion of a **crossing sequence** $S_w(i)$ for input w denoting the behavior of the tape head as it crosses between tape cells i and $i+1$. Such a sequence has the form $(q_{k_1}, D_1), (q_{k_2}, D_2), \ldots$ where q_{k_i} is a state and $D_i \in \{R, L\}$.

1. Show that for any two strings $x = x_1x_2$ and $y = y_1y_2$ such that $|x_1| = |y_1| = i$ and $S_x(i-1) = S_y(i-1)$, if a TM M accepts x and y, the it also accepts the strings x_1y_2 and x_2y_1.

   Proof. Consider the execution of the TM M on the input x_1y_2. It runs just as it would on input x until the first element of $S_x(i-1)$. Since $S_x(i-1) = S_y(i-1)$, if M crosses over to the cell of y_2, M runs just as it would on input y. Every time the tape head crosses between cells i and $i+1$, since the crossing sequences are identical, it is as if M was running on input x or y. Eventually M will ending in the terminating state of x or y so M will accept. The same is true for input y_1x_2. □

2. Let M be any one-tape TM deciding PAL and x, y be two distinct strings of length n each. If $X = x0^nx^R$ and $Y = y0^ny^R$ where x^R is the reverse of x. Show that for every $n < i < 2n$, it must be the case that $S_X(i) \neq S_Y(i)$.

   Proof. Suppose for a contradiction that for some $i \in (n, 2n)$, $S_X(i) \neq S_Y(i)$. Since X is a palindrome, X is accepted by M. Let $X = x_1x_2$ and $Y = y_1y_2$ where $|x_1| = |y_1| = i$. By the previous part we know that x_1y_2 is also accepted by M. However, since x and y is distinct, x_1y_2 is not a palindrome. This contradicts the fact that M decides PAL. □

3. Show that there exists a constant $\epsilon > 0$ such that for every $n < i < 2n$, the number of strings of the form $x0^nx^R$ with $|x| = n$ that have $|S_X(i)| < \epsilon n$ is less than $2^{n/2}$.

   Proof. To choose ϵ we first need a handle on the moving parts of $S_X(i)$. Let there be k states in M and let $p = \log k$. q represents the number of bits used to represent each state. The maximum number of bits needed to express a crossing sequences of length less than ϵn is $(p+1)$. Thus the total number of such crossing sequences is $2^{\epsilon(p+1)n}$ (each bit can be 0 or 1). By choosing $\epsilon = \frac{1}{3(p+1)}$, for sufficiently large n, $2^{\epsilon(p+1)n} < 2^{n/2}$. □

4. Argue that there will always exists a string $X = x0^nx^R$ for large enough n such that $|S_X(i)| \geq \epsilon n$ for every $n < i < 2n$.

   Proof. From the previous part we know that the number of palindromes X which have $|S_X(i)| < \epsilon n$ is less than $2^{n/2}$. □

Problem 2. *Show that the language $L = \{0\}$ is complete for* P*, under polytime reduction.*

Proof. L is certainly in P. Consider any language $L' \in \mathsf{P}$. Then there exists a TM M which solves L' in polynomial time. For your reduction simply simulate the input x to L' on M. If M accepts, output 0; otherwise if M rejects, output 1. Thus $w \in L'$ if and only if the the reduction is in L. □

Problem 3. *Show that if $P = \mathsf{NP}$, then* $\mathsf{EXP} = \mathsf{NEXP}$.

Proof. The trick here is to pad liberally. □

Problem 4.

Proof. □

Problem 5.

Proof. □

Problem 6.

1. *Proof.* □
2. *Proof.* □

Problem 7.

Proof. □