CMPT 407: Computational Complexity	**Summer 2017**

Lecture 4: Randomized Computation (6 - 9 June)

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4.1 Review

Theorem 4.1 $\mathsf{EXP} \subset \mathsf{PolySize} \implies \mathsf{EXP} = \Sigma_2^p$.

Proof: For all L in EXP there exists a TM M, which runs in time $2^{n^c} = t$ for an input x of size n. Imagine a txt grid which describes the operation of M. Where each row is a configuration of M. This transcript is valid if and only if all windows (three consecutive cells in row i and the associated cell in row $i+1$) are consistent. Consider the function $T : [t] \times [t] \to \Sigma^*$ where $T(i,j) =$ cell j at time i. Since we assumed $\mathsf{EXP} \subset \mathsf{PolySize}$ all we need to do is show that $T \in \mathsf{EXP}$. Well, that's pretty obvious, simply execute the TM M. Now show that $T \in \Sigma_2$ as follows: $\exists C \forall i, j$: window (i, j) (in the tableau) is consistent and the tableau ends in an accepting state. ∎

Note: (by IKW) it is possible to generalize this implication for NEXP, namely $\mathsf{NEXP} \subset \mathsf{PolySize} \implies \mathsf{NEXP} = \Sigma_2$. Proving this is quite a bit more difficult and requires more tool.

4.2 Circuits

Let us consider the set of inclusion of circuit complexity: $\mathsf{AC}_0 \subset \mathsf{TC}_0 \subset \mathsf{NC}_1 \subset \mathsf{PolySize}$

Claim 4.2 $\mathsf{NC}_1 = \mathsf{PolySize}$ *formula.*

Proof: $\mathsf{NC}_1 \subseteq \mathsf{PolySize}$ formula is the easy direction. Now lets attempt to show th other direction, $\mathsf{PolySize}$ formula $\subseteq \mathsf{NEXP}$. Now a normal expansion of a formula F in $x_1, ..., x_n$ might be of depth $O(n)$. But if you think about it you realize that there are not a lot of "stuff" so a long path can be restructured to be made shorter and wider. In particular cut F into two pieces F_1 and F_2 each of depth approximately half. Let $f(F)$ be the formula associated the the circuit of F. Then ∎

4.2.1 Valiant's Challenge

Find an explicit function $f : \{0,1\}^n \to \{0,1\}$ that cannot be computed by a circuit of size $O(n)$ and depth $O(\log n)$.

Definition 4.3 *A majority gate is defined as follows: $maj_n : \{\}$*

Example 4.4 *Let $a_1, a_2, ..., a_n$ be n, n digit numbers. We want to show that this problem in the domain of Valiant's Challenge. So we need to demonstrate a $O(\log n)$ circuit of $O(n)$ size to solve this problem. The algorithm here requires a trick as follows:*

Theorem 4.5 *Finding the parity of n numbers is in* TC_0.

Proof: First we need to construct a threshold function. ■

4.3 AC_0

Theorem 4.6 *The addition of two n bit numbers is in* AC_0.