**CSC2420: Algorithm Design, Analysis and Theory** **Fall 2017**

# Lecture 6: Randomized Algorithms

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## 6.1 From Last Time...

Recall the $\{0,1\}$-metric labeling problem is $\mathsf{NP}-\mathsf{Hard}$ for three or more labels.

Next consider an IP/LP relation solution for the makespan problem in the unrestricted machine model. We can use the indicator variables $x_{ij}$ to say if job $b_i$ has been put on machine $h_j$ or not. Further you have $1 \geq x_{ij} \geq 0$ and $\sum x_{ij} \leq T$ for the maximum makespan $T$ on any machine. The exact solution is integral. The relation allows any real value between 0 and 1. However you should be able to see that the integrality gap could be as bad as possible.

Concluding remarks for LP rounding. Note that the integrality gap is in relation to the LP formulation so it may help to add more constraints. This shrinks the poly-tope. One useful technique is the LS lift and project method.

## 6.2 Duality

(In terms of LPs) The dual of a primal maximization instance is the dual minimization instance and vice versa. Consider the following example:

**Example 6.1** *Recall the set cover problem in its IP/LP formulation:*

$$\begin{aligned}&\textit{minmize } \sum_j w_j x_j \\ &\textit{subject to } \sum_{j:e_i \in S_j} x_j \geq 1 \textit{ for all } i; \quad e_i \in U \\ &x_j \in \{0,1\} \quad (x_j \geq 0)\end{aligned}$$

*Now consider its dual:*

$$\begin{aligned}&\textit{maximize } \sum_i y_i \\ &\textit{subject to } \sum_{i:e_i \in S_j} y_i \leq w_j \textit{ for all } j \\ &y_i \geq 0\end{aligned}$$

*It is a fact that the optimal value for the primal is the same as the dual if both exist and are finite.*

The **Weak Duality** for a minimization problem is: that if $x$ and $y$ are primal and dual solutions respectively then $\mathcal{D}(y) \leq \mathcal{P}(x)$.

## 6.3 Randomized Algorithms

Problems in randomized poly-time not known to be in poly-time:

1. Symbolic determinant.
2. Given $n$ find prime in $[2^n, 2^{n+1}]$.
3. Estimating volume of a convex body given by a set of linear inequalities.
4. Solving a quadratic equation in $\mathbb{Z}_p[x]$.
5. Polynomial Identity Testing (via Schwartz Zipple)

### 6.3.1 Naive Application

Consider the example of $\mathsf{Max}-\mathsf{k}-\mathsf{SAT}$. When we are talking about approximations here we are going to use fractional ratios $c < 1$ (what $c \cdot P_{OPT}$ of the clauses can you cover, where $P_{OPT}$ is the maximum number of clauses coverable given any assignment).

Naive randomized approximation algorithm? Randomly assign value the variables! Suppose we are just consider exact $\mathsf{k}-\mathsf{SAT}$. Each clause has probability $1 - \frac{1}{2^k}$ of being satisfied.