

# CSC 411: Lecture 06 - Decision Trees

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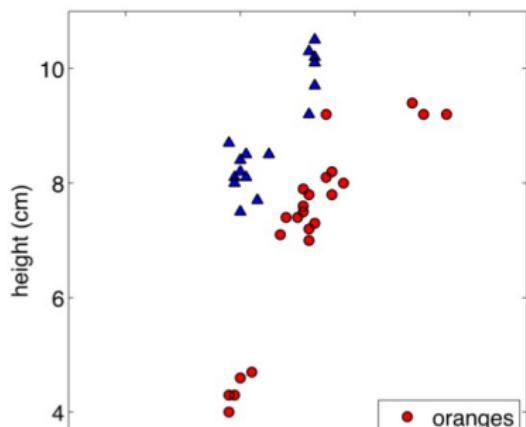
# Today

- Decision Trees

- ▶ entropy
- ▶ information gain

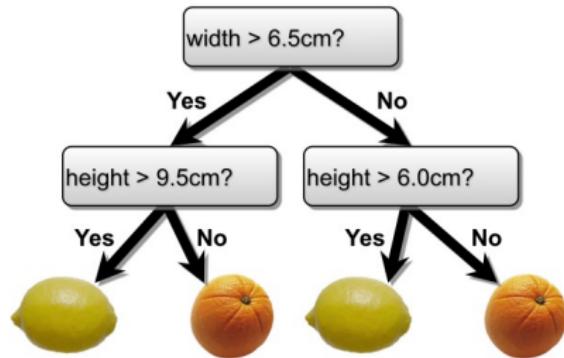
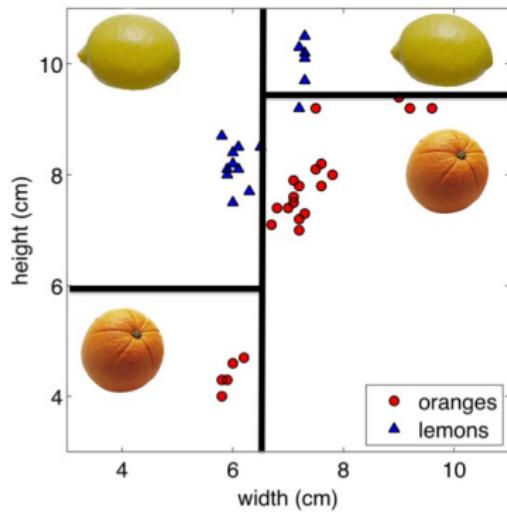
# Another Classification Idea

- We learned about linear classification (e.g., logistic regression), and nearest neighbors. Any other idea?
- Pick an attribute, do a simple test
- Conditioned on a choice, pick another attribute, do another test
- In the leaves, assign a class with majority vote
- Do other branches as well

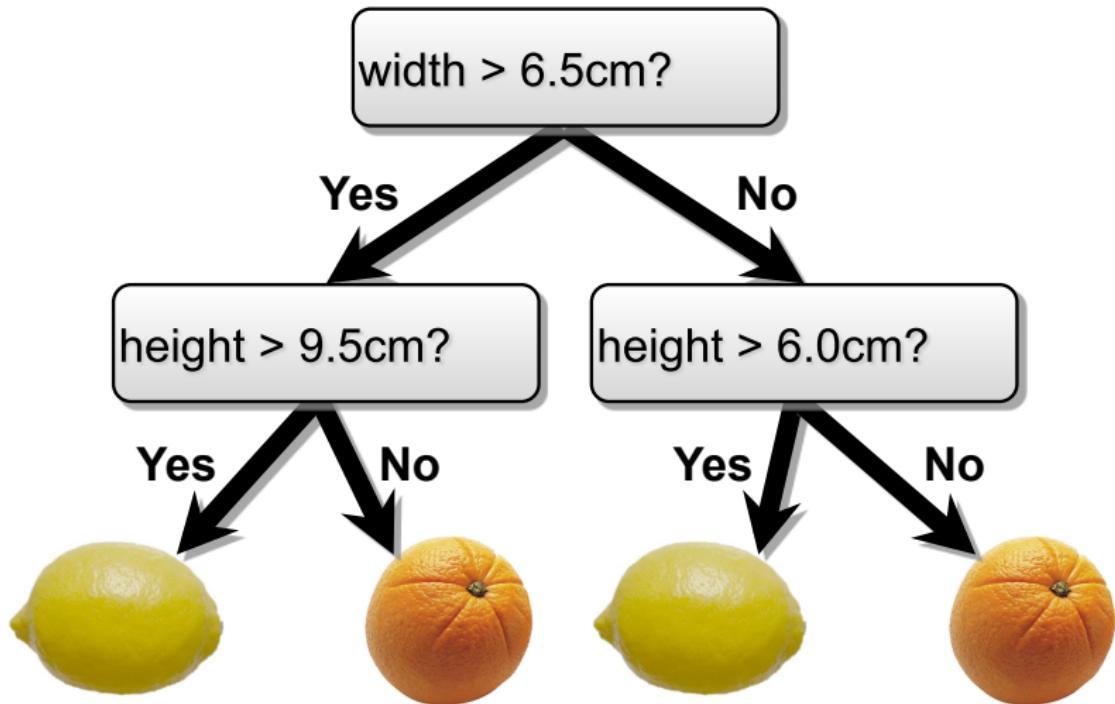


# Another Classification Idea

- Gives axes aligned decision boundaries

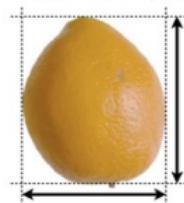


## Decision Tree: Example



# Decision Tree: Classification

Test example



width > 6.5cm?

Yes

No

height > 9.5cm?

Yes

No

height > 6.0cm?

Yes

No



# Example with Discrete Inputs

- What if the attributes are discrete?

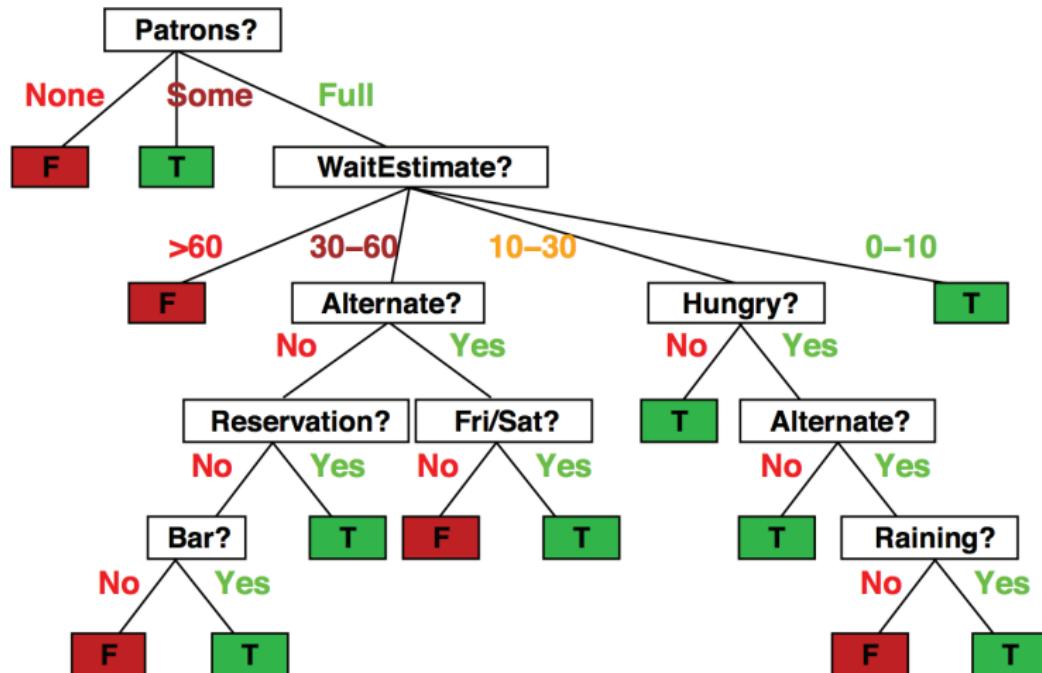
Example	Input Attributes										Goal <i>WillWait</i>
	Alt	Bar	Fri	Hun	Pat	Price	Rain	Res	Type	Est	
$x_1$	Yes	No	No	Yes	Some	\$\$\$	No	Yes	French	0-10	$y_1 = \text{Yes}$
$x_2$	Yes	No	No	Yes	Full	\$	No	No	Thai	30-60	$y_2 = \text{No}$
$x_3$	No	Yes	No	No	Some	\$	No	No	Burger	0-10	$y_3 = \text{Yes}$
$x_4$	Yes	No	Yes	Yes	Full	\$	Yes	No	Thai	10-30	$y_4 = \text{Yes}$
$x_5$	Yes	No	Yes	No	Full	\$\$\$	No	Yes	French	>60	$y_5 = \text{No}$
$x_6$	No	Yes	No	Yes	Some	\$\$	Yes	Yes	Italian	0-10	$y_6 = \text{Yes}$
$x_7$	No	Yes	No	No	None	\$	Yes	No	Burger	0-10	$y_7 = \text{No}$
$x_8$	No	No	No	Yes	Some	\$\$	Yes	Yes	Thai	0-10	$y_8 = \text{Yes}$
$x_9$	No	Yes	Yes	No	Full	\$	Yes	No	Burger	>60	$y_9 = \text{No}$
$x_{10}$	Yes	Yes	Yes	Yes	Full	\$\$\$	No	Yes	Italian	10-30	$y_{10} = \text{No}$
$x_{11}$	No	No	No	No	None	\$	No	No	Thai	0-10	$y_{11} = \text{No}$
$x_{12}$	Yes	Yes	Yes	Yes	Full	\$	No	No	Burger	30-60	$y_{12} = \text{Yes}$

1.	Alternate: whether there is a suitable alternative restaurant nearby.
2.	Bar: whether the restaurant has a comfortable bar area to wait in.
3.	Fri/Sat: true on Fridays and Saturdays.
4.	Hungry: whether we are hungry.
5.	Patrons: how many people are in the restaurant (values are None, Some, and Full).
6.	Price: the restaurant's price range (\$, \$\$, \$\$\$).
7.	Raining: whether it is raining outside.
8.	Reservation: whether we made a reservation.
9.	Type: the kind of restaurant (French, Italian, Thai or Burger).
10.	WaitEstimate: the wait estimated by the host (0-10 minutes, 10-30, 30-60, >60).

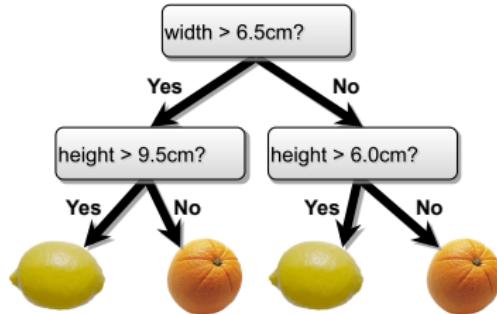
Attributes:

# Decision Tree: Example with Discrete Inputs

- The tree to decide whether to wait (T) or not (F)



# Decision Trees



- Internal nodes **test attributes**
- Branching is determined by **attribute value**
- Leaf nodes are **outputs** (class assignments)

# Decision Tree: Algorithm

- Choose an attribute on which to descend at each level
- Condition on earlier (higher) choices
- Generally, restrict only one dimension at a time
- Declare an output value when you get to the bottom
- In the orange/lemon example, we only split each dimension once, but that is not required

# Decision Tree: Classification and Regression

- Each path from root to a leaf defines a region  $R_m$  of input space
- Let  $\{(x^{(m_1)}, t^{(m_1)}), \dots, (x^{(m_k)}, t^{(m_k)})\}$  be the training examples that fall into  $R_m$
- **Classification tree:**
  - ▶ discrete output
  - ▶ leaf value  $y^m$  typically set to the most common value in  $\{t^{(m_1)}, \dots, t^{(m_k)}\}$
- **Regression tree:**
  - ▶ continuous output
  - ▶ leaf value  $y^m$  typically set to the mean value in  $\{t^{(m_1)}, \dots, t^{(m_k)}\}$

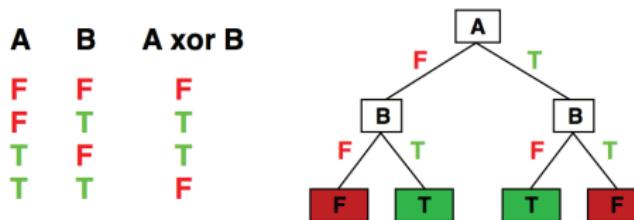
Note: We will only talk about classification

[Slide credit: S. Russell]

# Expressiveness

- **Discrete-input, discrete-output case:**

- ▶ Decision trees can express any function of the input attributes
- ▶ E.g., for Boolean functions, truth table row → path to leaf:



- **Continuous-input, continuous-output case:**

- ▶ Can approximate any function arbitrarily closely
- Trivially, there is a consistent decision tree for any training set w/ one path to leaf for each example (unless  $f$  nondeterministic in  $x$ ) but it probably won't generalize to new examples

Need some kind of regularization to ensure more **compact** decision trees

[Slide credit: S. Russell]

# How do we Learn a Decision Tree?

- How do we construct a useful decision tree?

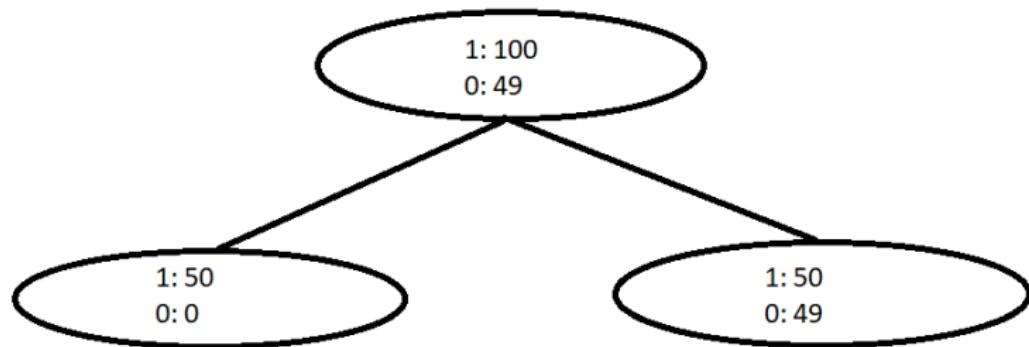
# Learning Decision Trees

Learning the simplest (smallest) decision tree is an NP complete problem [if you are interested, check: Hyafil & Rivest'76]

- Resort to a **greedy heuristic**:
  - ▶ Start from an empty decision tree
  - ▶ Split on next best attribute
  - ▶ Recurse
- What is **best** attribute?
- How about accuracy?

# Choosing a Good Split

- Why isn't accuracy a good measure?



- Is this split good? Zero accuracy gain.
- We will use intuition from [information theory](#)

**Idea:** Use counts at leaves to define probability distributions, so we can measure uncertainty

# Choosing a Good Split

- Which attribute is better to split on,  $X_1$  or  $X_2$ ?
  - ▶ Deterministic: good (all are true or false; just one class in the leaf)
  - ▶ Uniform distribution: bad (all classes in leaf equally probable)
  - ▶ What about distributions in between?

Note: Let's take a slight detour and remember concepts from information theory

[Slide credit: D. Sontag]

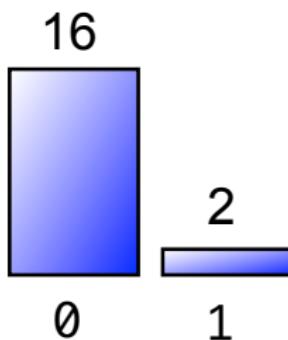
# We Flip Two Different Coins

Sequence 1:

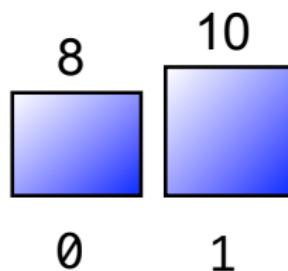
0 0 0 1 0 0 0 0 0 0 0 0 0 0 1 0 0 ... ?

Sequence 2:

0 1 0 1 0 1 1 1 0 1 0 0 1 1 1 0 1 0 1 ... ?



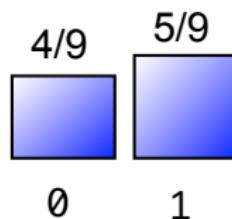
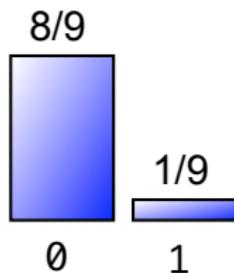
versus



# Quantifying Uncertainty

Entropy  $H$ :

$$H(X) = - \sum_{x \in X} p(x) \log_2 p(x)$$



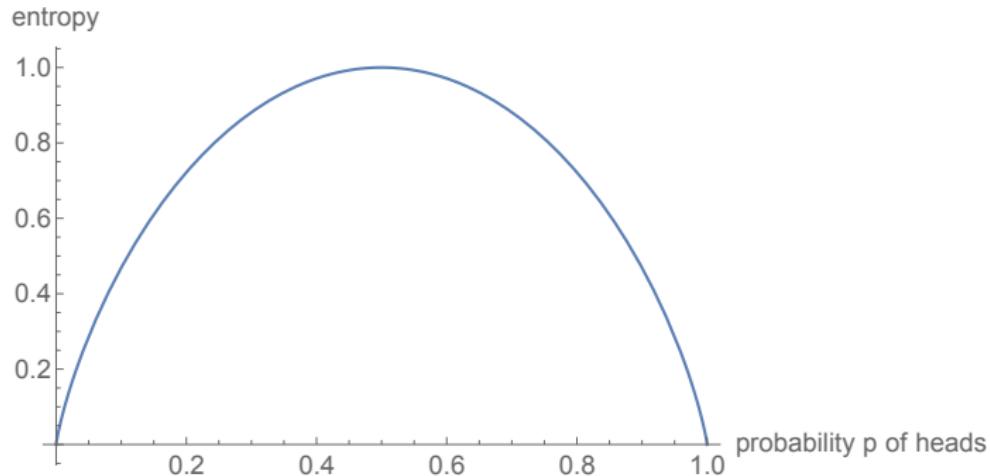
$$-\frac{8}{9} \log_2 \frac{8}{9} - \frac{1}{9} \log_2 \frac{1}{9} \approx \frac{1}{2}$$

$$-\frac{4}{9} \log_2 \frac{4}{9} - \frac{5}{9} \log_2 \frac{5}{9} \approx 0.99$$

- How surprised are we by a new value in the sequence?
- How much information does it convey?

# Quantifying Uncertainty

$$H(X) = - \sum_{x \in X} p(x) \log_2 p(x)$$



# Entropy

- “High Entropy”:
  - ▶ Variable has a uniform like distribution
  - ▶ Flat histogram
  - ▶ Values sampled from it are less predictable
- “Low Entropy”
  - ▶ Distribution of variable has many peaks and valleys
  - ▶ Histogram has many lows and highs
  - ▶ Values sampled from it are more predictable

[Slide credit: Vibhav Gogate]

# Entropy of a Joint Distribution

- Example:  $X = \{\text{Raining, Not raining}\}$ ,  $Y = \{\text{Cloudy, Not cloudy}\}$

		Cloudy	Not Cloudy
Raining	24/100	1/100	
Not Raining	25/100	50/100	

$$\begin{aligned} H(X, Y) &= - \sum_{x \in X} \sum_{y \in Y} p(x, y) \log_2 p(x, y) \\ &= -\frac{24}{100} \log_2 \frac{24}{100} - \frac{1}{100} \log_2 \frac{1}{100} - \frac{25}{100} \log_2 \frac{25}{100} - \frac{50}{100} \log_2 \frac{50}{100} \\ &\approx 1.56 \text{ bits} \end{aligned}$$

# Specific Conditional Entropy

- Example:  $X = \{\text{Raining, Not raining}\}$ ,  $Y = \{\text{Cloudy, Not cloudy}\}$

		Cloudy	Not Cloudy
Raining	24/100	1/100	
Not Raining	25/100	50/100	

- What is the entropy of cloudiness  $Y$ , given that it is raining?

$$\begin{aligned} H(Y|X=x) &= -\sum_{y \in Y} p(y|x) \log_2 p(y|x) \\ &= -\frac{24}{25} \log_2 \frac{24}{25} - \frac{1}{25} \log_2 \frac{1}{25} \\ &\approx 0.24 \text{ bits} \end{aligned}$$

- We used:  $p(y|x) = \frac{p(x,y)}{p(x)}$ , and  $p(x) = \sum_y p(x,y)$  (sum in a row)

# Conditional Entropy

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

- The expected conditional entropy:

$$\begin{aligned} H(Y|X) &= \sum_{x \in X} p(x)H(Y|X=x) \\ &= -\sum_{x \in X} \sum_{y \in Y} p(x,y) \log_2 p(y|x) \end{aligned}$$

# Conditional Entropy

- Example:  $X = \{\text{Raining, Not raining}\}$ ,  $Y = \{\text{Cloudy, Not cloudy}\}$

		Cloudy	Not Cloudy
Raining	24/100	1/100	
Not Raining	25/100	50/100	

- What is the entropy of cloudiness, given the knowledge of whether or not it is raining?

$$\begin{aligned}H(Y|X) &= \sum_{x \in X} p(x)H(Y|X=x) \\&= \frac{1}{4}H(\text{cloudy}| \text{is raining}) + \frac{3}{4}H(\text{cloudy}| \text{not raining}) \\&\approx 0.75 \text{ bits}\end{aligned}$$

# Conditional Entropy

- Some useful properties:
  - ▶  $H$  is always non-negative
  - ▶ Chain rule:  $H(X, Y) = H(X|Y) + H(Y) = H(Y|X) + H(X)$
  - ▶ If  $X$  and  $Y$  independent, then  $X$  doesn't tell us anything about  $Y$ :  
 $H(Y|X) = H(Y)$
  - ▶ But  $Y$  tells us everything about  $Y$ :  $H(Y|Y) = 0$
  - ▶ By knowing  $X$ , we can only decrease uncertainty about  $Y$ :  
 $H(Y|X) \leq H(Y)$

# Information Gain

	Cloudy	Not Cloudy
Raining	24/100	1/100
Not Raining	25/100	50/100

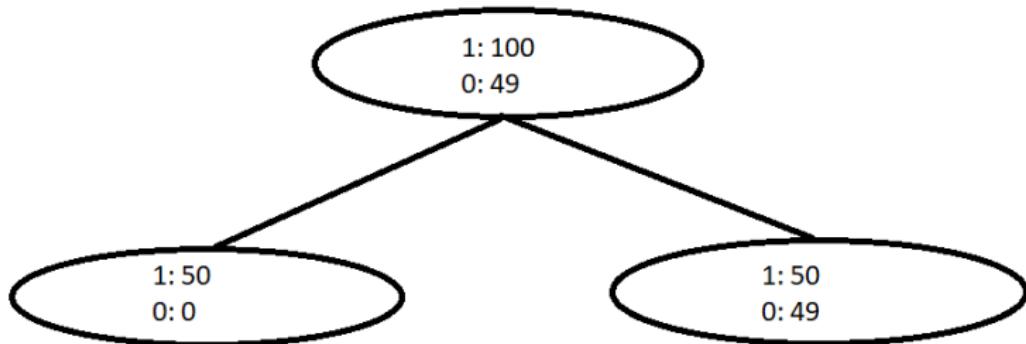
- How much information about cloudiness do we get by discovering whether it is raining?

$$\begin{aligned}IG(Y|X) &= H(Y) - H(Y|X) \\&\approx 0.25 \text{ bits}\end{aligned}$$

- Also called **information gain** in  $Y$  due to  $X$
- If  $X$  is completely uninformative about  $Y$ :  $IG(Y|X) = 0$
- If  $X$  is completely informative about  $Y$ :  $IG(Y|X) = H(Y)$
- How can we use this to construct our decision tree?

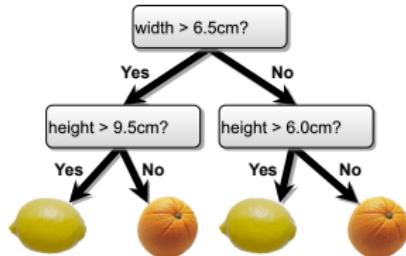
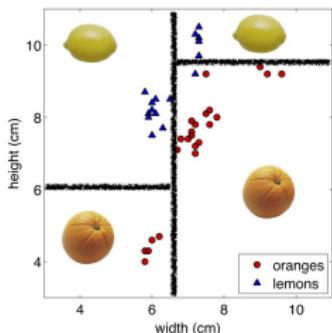
## Revisiting Our Original Example

- What is the information gain of this split?



- Root entropy:  $H(Y) = -\frac{49}{149} \log_2(\frac{49}{149}) - \frac{100}{149} \log_2(\frac{100}{149}) \approx 0.91$
- Leaf's entropy:  $H(Y|left) = 0, H(Y|right) \approx 1.$
- $IG(split) \approx 0.91 - (\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 1) \approx 0.24 > 0$

# Constructing Decision Trees



- I made the fruit data partitioning just by eyeballing it. 
- We can use the [information gain](#) to automate the process.
- At each level, one must choose:
  - Which variable to split.
  - Possibly where to split it.
- Choose them based on how much information we would gain from the decision! (choose attribute that gives the highest gain)

# Decision Tree Construction Algorithm

- Simple, greedy, recursive approach, builds up tree node-by-node
1. pick an attribute to split at a non-terminal node
  2. split examples into groups based on attribute value
  3. for each group:
    - ▶ if no examples – return majority from parent
    - ▶ else if all examples in same class – return class
    - ▶ else loop to step 1



# Back to Our Example

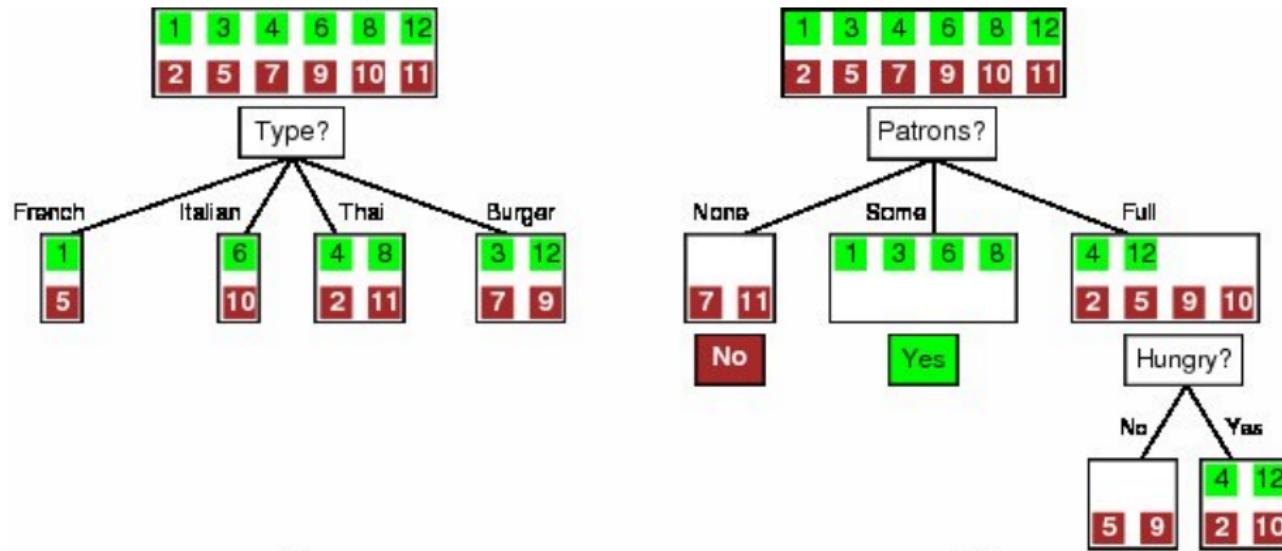
Example	Input Attributes										Goal WillWait
	Alt	Bar	Fri	Hun	Pat	Price	Rain	Res	Type	Est	
x <sub>1</sub>	Yes	No	No	Yes	Some	\$\$\$	No	Yes	French	0-10	y <sub>1</sub> = Yes
x <sub>2</sub>	Yes	No	No	Yes	Full	\$	No	No	Thai	30-60	y <sub>2</sub> = No
x <sub>3</sub>	No	Yes	No	No	Some	\$	No	No	Burger	0-10	y <sub>3</sub> = Yes
x <sub>4</sub>	Yes	No	Yes	Yes	Full	\$	Yes	No	Thai	10-30	y <sub>4</sub> = Yes
x <sub>5</sub>	Yes	No	Yes	No	Full	\$\$\$	No	Yes	French	>60	y <sub>5</sub> = No
x <sub>6</sub>	No	Yes	No	Yes	Some	\$\$	Yes	Yes	Italian	0-10	y <sub>6</sub> = Yes
x <sub>7</sub>	No	Yes	No	No	None	\$	Yes	No	Burger	0-10	y <sub>7</sub> = No
x <sub>8</sub>	No	No	No	Yes	Some	\$\$	Yes	Yes	Thai	0-10	y <sub>8</sub> = Yes
x <sub>9</sub>	No	Yes	Yes	No	Full	\$	Yes	No	Burger	>60	y <sub>9</sub> = No
x <sub>10</sub>	Yes	Yes	Yes	Yes	Full	\$\$\$	No	Yes	Italian	10-30	y <sub>10</sub> = No
x <sub>11</sub>	No	No	No	No	None	\$	No	No	Thai	0-10	y <sub>11</sub> = No
x <sub>12</sub>	Yes	Yes	Yes	Yes	Full	\$	No	No	Burger	30-60	y <sub>12</sub> = Yes

1. Alternate: whether there is a suitable alternative restaurant nearby.
2. Bar: whether the restaurant has a comfortable bar area to wait in.
3. Fri/Sat: true on Fridays and Saturdays.
4. Hungry: whether we are hungry.
5. Patrons: how many people are in the restaurant (values are None, Some, and Full).
6. Price: the restaurant's price range (\$, \$\$, \$\$\$).
7. Raining: whether it is raining outside.
8. Reservation: whether we made a reservation.
9. Type: the kind of restaurant (French, Italian, Thai or Burger).
10. WaitEstimate: the wait estimated by the host (0-10 minutes, 10-30, 30-60, >60).

Attributes:

[from: Russell & Norvig]

# Attribute Selection

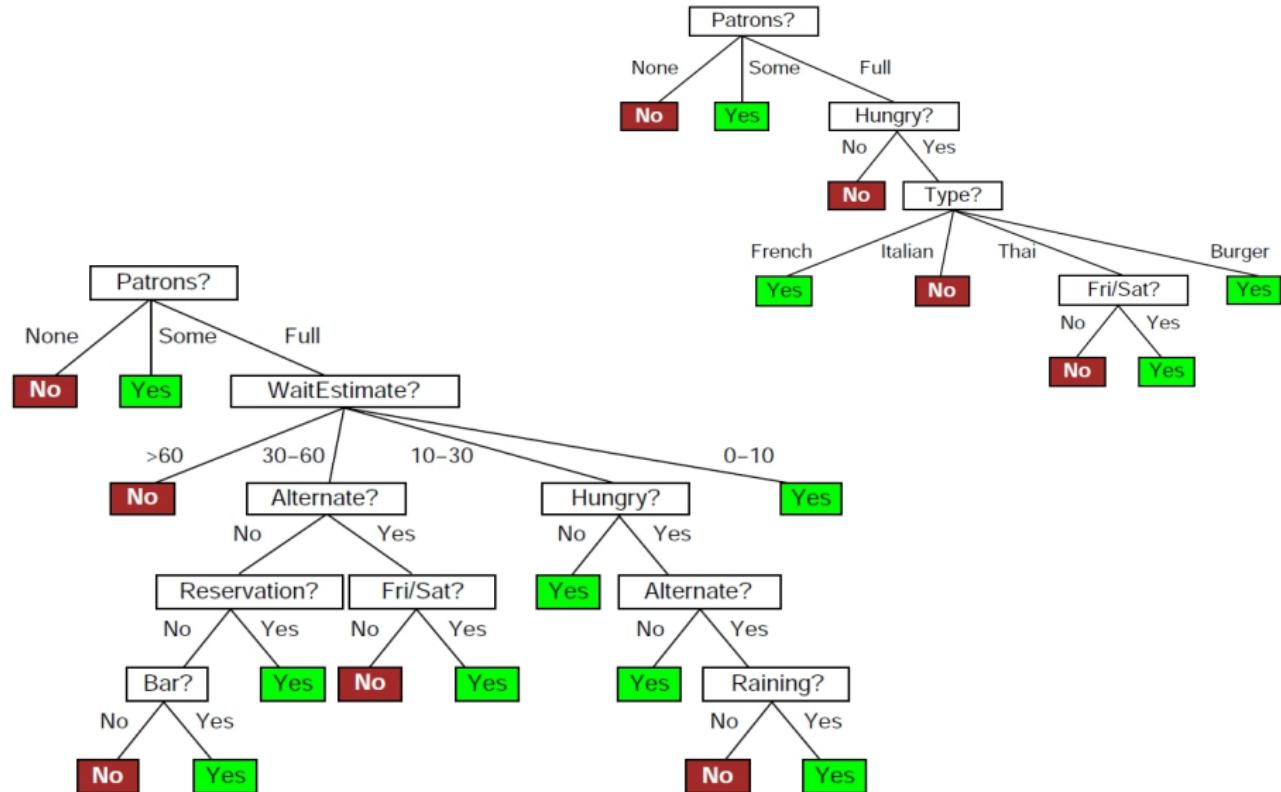


$$IG(Y) = H(Y) - H(Y|X)$$

$$IG(type) = 1 - \left[ \frac{2}{12} H(Y|Fr.) + \frac{2}{12} H(Y|It.) + \frac{4}{12} H(Y|Thai) + \frac{4}{12} H(Y|Bur.) \right] = 0$$

$$IG(Patrons) = 1 - \left[ \frac{2}{12} H(0, 1) + \frac{4}{12} H(1, 0) + \frac{6}{12} H\left(\frac{2}{6}, \frac{4}{6}\right) \right] \approx 0.541$$

# Which Tree is Better?



# What Makes a Good Tree?

- Not too small: need to handle important but possibly subtle distinctions in data
- Not too big:
  - ▶ Computational efficiency (avoid redundant, spurious attributes)
  - ▶ Avoid over-fitting training examples
- **Occam's Razor:** find the simplest hypothesis (smallest tree) that fits the observations
- **Inductive bias:** small trees with informative nodes near the root

# Decision Tree Miscellany

- Problems:
  - ▶ You have exponentially less data at lower levels
  - ▶ Too big of a tree can **overfit** the data
  - ▶ Greedy algorithms don't necessarily yield the global optimum
- In practice, one often **regularizes** the construction process to try to get small but highly-informative trees
  - ▶ Pruning.
  - ▶ Minimum gain for split.
  - ▶ Minimum node size.
- Decision trees can also be used for regression on real-valued outputs, but it requires a different formalism

# Comparison to k-NN

## K-Nearest Neighbors

- Decision boundaries: piece-wise linear
- Test complexity: non-parametric, few parameters besides (all?) training examples

## Decision Trees

- Decision boundaries: axis-aligned, tree structured
- Test complexity: attributes and splits

# Applications of Decision Trees: XBox!

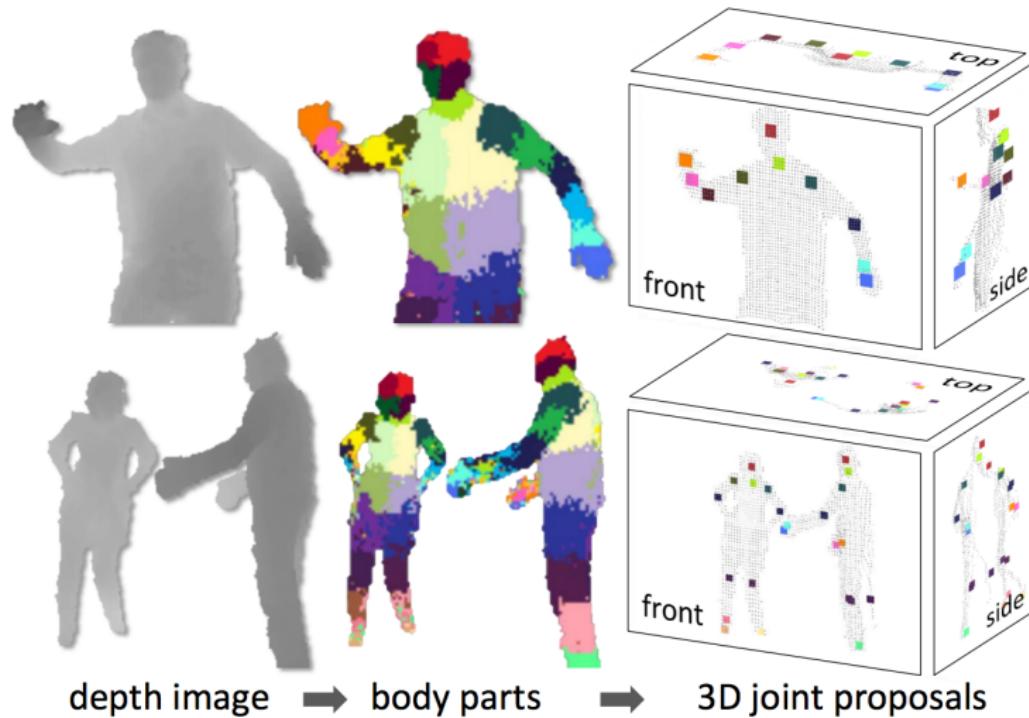
- Decision trees are in XBox



[J. Shotton, A. Fitzgibbon, M. Cook, T. Sharp, M. Finocchio, R. Moore, A. Kipman, A. Blake. Real-Time Human Pose Recognition in Parts from a Single Depth Image. CVPR'11]

# Applications of Decision Trees: XBox!

- Decision trees are in XBox: Classifying body parts



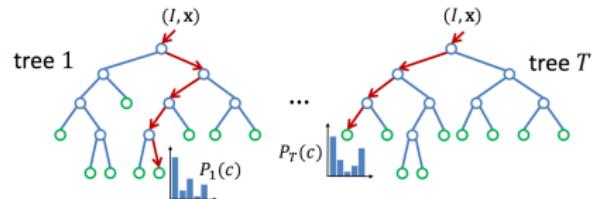
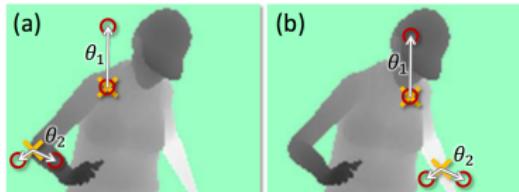
# Applications of Decision Trees: XBox!

- Trained on million(s) of examples

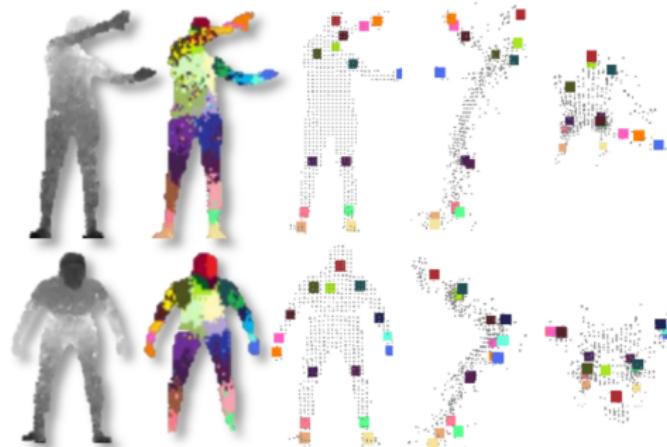


# Applications of Decision Trees: XBox!

- Trained on million(s) of examples



- Results:



# Applications of Decision Trees

- Can express any Boolean function, but most useful when function depends critically on few attributes
- Bad on: parity, majority functions; also not well-suited to continuous attributes
- Practical Applications:
  - ▶ Tubular data
  - ▶ Flight simulator: 20 state variables; 90K examples based on expert pilot's actions; auto-pilot tree
  - ▶ Yahoo Ranking Challenge
  - ▶ Random Forests: Microsoft Kinect Pose Estimation