CSCI 301 Homework 3

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1 Problem 1

Consider the relation $|$ (divides) on the set $\mathbb{Z}$.

1.1 Part a

We will prove that $|$ is reflexive. Take any integer, $x \in \mathbb{Z}$. Obviously, $x \mid x$ for any value of x, so the relation $|$ is true on the set $\mathbb{Z}$. $\blacksquare$

1.2 Part b

We will now submit disproof that $|$ is symmetric by counterexample. The relation $|$ is symmetric by definition if for all $\forall x, y \in \mathbb{Z}, x \mid y \implies y \mid x$. Observe that for $x = 4$ and $y = 8, 4 \mid 8$ yet $8 \nmid 4$. $\blacksquare$

1.3 Part c

Finally, we will submit contrapositive proof that that $|$ is transitive. Consider that if $|$ is transitive, then $\forall x, y, z \in \mathbb{Z}, ((x \mid y)(y \mid z)) \implies x \mid z$. Then let us suppose that $x \mid y$ and $y \mid z$. Then $y = za$, where a is some integer, by definition of divisibility. Also, $z = yc$, where c is some integer. Adding these statements, we find that

$$\begin{aligned}
y + z &= xa + yc \\
z &= xa + xa * c + xa \\
z &= 2xa + xac \\
z &= x(2a + ac) \\
z &= x * d.
\end{aligned}$$

where d is some integer. Thus, $x \mid z$. $\blacksquare$

2 Problem 2

Assume R and S are two equivalence relations on a set A.

2.1 Part a

We will prove that $R \cup S$ is reflexive. Consider an integer $x \in A$. Since R and S are reflexive, as they are both equivalence relations, then $(x, x) \in R$ and $(x, x) \in S$. Therefore, $(x, x) \in R \cup S$ and $R \cup S$ is reflexive by definition. $\blacksquare$

2.2 Part b

We will now prove that $R \cup S$ is symmetric. Consider two integers, $x, y, \in A$. Suppose $(x, y) \in R \cup S$. Then, since R and S are symmetric, at least one of them must contain (y, x). So $(x, y) \in R \cup S \implies (y, x) \in R \cup S$. Therefore, $R \cup S$ is symmetric by definition. $\blacksquare$

2.3 Part c

We will now offer disproof by counterexample that $R \cup S$ is transitive. That is, $\forall x, y, z \in A, ((xR \cup Sy)(yR \cup Sz)) \implies xR \cup Sz$. Now for our counterexample: Suppose $A = \{a, b, c\}, R = \{(a, b)\}, andS = \{(b, c)\}$. Then $R \cup S = \{(a, b), (b, c)\}$. Then $R \cup S$ is not transitive because although (a, b) and $(b, c) \in R \cup S, (a, c)$ is not. $\blacksquare$

3 Problem 3

Consider the function $\theta : \{0, 1\} * \mathbb{N} \implies \mathbb{Z} definedas \theta(a, b) = a - 2ab + b.$

3.1 Part a

We will now prove that $\theta(a, b)$ is injective. That is to say, $\forall(a, b), (c, d) \in \{0, 1\} * \mathbb{N}, (a, b) \neq (c, d) \implies \theta(a, b) \neq \theta(c, d)$. We will do so via contrapositive proof.

Suppose $\theta(a, b) = \theta(c, d)$, where a and c are either 1 or 0, and b and d are elements of the set of Natural Numbers, and therefore integers more than 0. Then

$$a - 2ab + b = c - 2cd + d \tag{1}$$

Note that our assumption does not hold if, without loss of generality, $a = 0$ and $c = 1$. This is easily proven as follows:

$$\begin{aligned} 0 - 0 + b &\neq 1 - 2d + d \\ b &\neq 1 - 2d + d \\ b &\neq 1 - d \end{aligned}$$

Notice that this is the case since, as previously defined, $b > 1$ and $d > 1$. Therefore, it must be the case that $a = c$. So it follows that whether $a = c = 0$ or $a = c = 1, b = d$. Observe that when $a = c = 0$...

$$\begin{aligned} 0 - 0 + b &= 0 - 0 + d \\ b &= d \end{aligned}$$

And that when $a = c = 1$...

$$\begin{aligned} 1 - 2b + b &= 1 - 2d + d \\ 1 - b &= 1 - d \\ b &= d \end{aligned}$$

Therefore, $\theta(a, b) = \theta(c, d) \implies (a, b) = (c, d)$. Hence, $\theta(a, b)$ is injective. ■

3.2 Part b

Finally, we shall provide disproof that $\theta(a, b)$ is surjective. Specifically, $\theta(a, b)$ is not surjective since there exists $c = -1 \in \mathbb{Z}$ for which $a - 2ab + b \neq -1 \forall (a, b) \in \{0, 1\} * \mathbb{N}$. Notice that if $a = 0$, then...

$$\begin{aligned} -1 &\neq 0 - 0 + b \\ -1 &\neq b \end{aligned}$$

Since $b \in \mathbb{N}$ and $b > 0$ as a result. And if $a = 1$, then...

$$\begin{aligned} -1 &\neq 1 - 2b + b \\ -1 &\neq 1 - b \end{aligned}$$

Again, since b is necessarily greater than 0. ■