Module – 3 Lecture Notes – 6

Other Algorithms for Solving Linear Programming Problems

Introduction

So far, *Simplex algorithm*, *Revised Simplex algorithm*, *Dual Simplex method* are discussed. There are few other methods for solving LP problems which have an entirely different algorithmic philosophy. Among these, *Khatchian's ellipsoid method* and *Karmarkar's projective scaling method* are well known. In this lecture, a brief discussion about these new methods in contrast to Simplex method will be presented. However, *Karmarkar's projective scaling method* will be discussed in detail.

Comparative discussion between new methods and Simplex method

Khatchian's ellipsoid method and *Karmarkar's projective scaling method* seek the optimum solution to an LP problem by moving through the interior of the feasible region. A schematic diagram illustrating the algorithmic differences between the Simplex and the Karmarkar's algorithm is shown in figure 1. *Khatchian's ellipsoid method* approximates the optimum solution of an LP problem by creating a sequence of ellipsoids (an ellipsoid is the multidimensional analog of an ellipse) that approach the optimal solution.

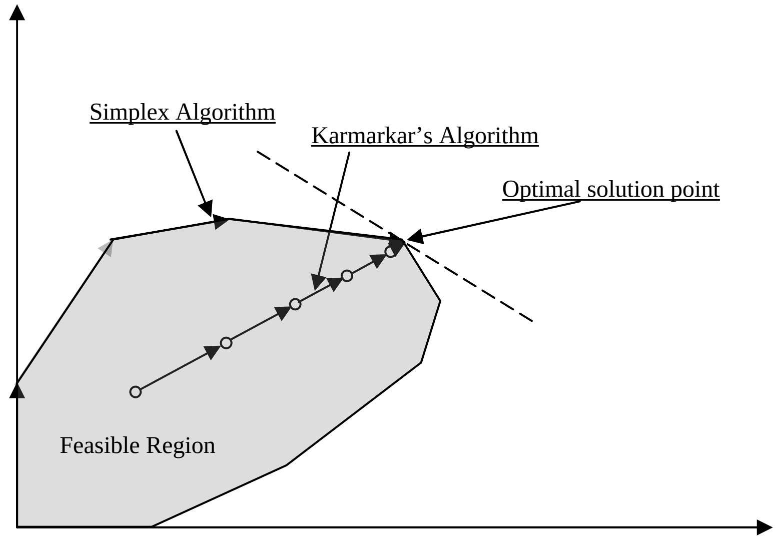

Figure 1 Difference in optimum search path between Simplex and Karmarkar's Algorithm

Both *Khatchian's ellipsoid method* and *Karmarkar's projective scaling method* have been shown to be polynomial time algorithms. This means that the time required to solve an LP problem of size n by the two new methods would take at most an^b where a and b are two positive numbers.

On the other hand, the Simplex algorithm is an exponential time algorithm in solving LP problems. This implies that, in solving an LP problem of size n by Simplex algorithm, there exists a positive number c such that for any n the Simplex algorithm would find its solution in a time of at most $c2^n$. For a large enough n (with positive a, b and c), $c2^n > an^b$. This means that, in theory, the polynomial time algorithms are computationally superior to exponential algorithms for large LP problems.

Karmarkar's projective scaling method

Karmarkar's projective scaling method, also known as *Karmarkar's interior point LP algorithm*, starts with a trial solution and shoots it towards the optimum solution.

To apply Karmarkar's projective scaling method, LP problem should be expressed in the following form

$$\begin{aligned} \text{Minimize} \quad & Z = \mathbf{C}^{\mathrm{T}}\mathbf{X} \\ \text{subject to :} \quad & \mathbf{AX} = \mathbf{0} \\ & \mathbf{1X} = 1 \\ \text{with :} \quad & \mathbf{X} \geq \mathbf{0} \end{aligned}$$

where $\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $\mathbf{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{(1\times n)}$, $\mathbf{A} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$ and $n \geq 2$. It is also assumed that $\mathbf{X}_0 = \begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}$ is a feasible solution and $Z_{\min} = 0$. The two other variables are defined as $r = \dfrac{1}{\sqrt{n(n-1)}}$, $\alpha = \dfrac{(n-1)}{3n}$.

Iterative steps are involved in Karmarkar's projective scaling method to find the optimal solution.

In general, k^{th} iteration involves following computations:

a) Compute $\mathbf{C}_p = \left[\mathbf{I} - \mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\mathbf{P}^{\mathrm{T}}\right)^{-1}\mathbf{P}\right]\overline{\mathbf{C}}^{\mathrm{T}}$

where $\mathbf{P} = \begin{pmatrix} \mathbf{A}\mathbf{D}_k \\ \mathbf{1} \end{pmatrix}$, $\overline{\mathbf{C}} = \mathbf{C}^{\mathrm{T}}\mathbf{D}_k$ and $\mathbf{D}_k = \begin{bmatrix} \mathbf{X}_k(1) & 0 & 0 & 0 \\ 0 & \mathbf{X}_k(2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \mathbf{X}_k(n) \end{bmatrix}$

If $\mathbf{C}_p = \mathbf{0}$, any feasible solution becomes an optimal solution. Further iteration is not required. Otherwise, compute the following

b) $\mathbf{Y}_{new} = \mathbf{X}_0 - \alpha\, r \dfrac{\mathbf{C}_p}{\left\|\mathbf{C}_p\right\|}$,

c) $\mathbf{X}_{k+1} = \dfrac{\mathbf{D}_k \mathbf{Y}_{new}}{\mathbf{1}\mathbf{D}_k \mathbf{Y}_{new}}$. However, it can be shown that for $k = 0$, $\dfrac{\mathbf{D}_k \mathbf{Y}_{new}}{\mathbf{1}\mathbf{D}_k \mathbf{Y}_{new}} = \mathbf{Y}_{new}$.

Thus, $\mathbf{X}_1 = \mathbf{Y}_{new}$.

d) $Z = \mathbf{C}^{\mathrm{T}}\mathbf{X}_{k+1}$

e) Repeat the steps (a) through (d) by changing k as $k+1$.

Consider the following problem:

$$\begin{array}{ll} \text{Minimize} & Z = 2x_2 - x_3 \\ \text{subject to:} & x_1 - 2x_2 + x_3 = 0 \\ & x_1 + x_2 + x_3 = 1 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Thus, $n = 3$, $\mathbf{C} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$, $\mathbf{X}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ \vdots \\ 1/3 \end{bmatrix}$, $r = \dfrac{1}{\sqrt{n(n-1)}} = \dfrac{1}{\sqrt{3(3-1)}} = \dfrac{1}{\sqrt{6}}$,

$\alpha = \dfrac{(n-1)}{3n} = \dfrac{(3-1)}{3 \times 3} = \dfrac{2}{9}$.

Iteration 0 (k=0):

$$\mathbf{D}_0 = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \mathbf{C}^{\mathrm{T}}\mathbf{D}_0 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 2/3 & -1/3 \end{bmatrix}$$

$$\mathbf{AD}_0 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \end{bmatrix}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{AD}_0 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{PP}^{\mathrm{T}} = \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1/3 & 1 \\ -2/3 & 1 \\ 1/3 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\left(\mathbf{PP}^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\mathbf{P}^{\mathrm{T}}\left(\mathbf{PP}^{\mathrm{T}}\right)^{-1}\mathbf{P} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

$$\mathbf{C}_p = \left[\mathbf{I} - \mathbf{P}^{\mathrm{T}}\left(\mathbf{PP}^{\mathrm{T}}\right)^{-1}\mathbf{P}\right]\overline{\mathbf{C}}^{\mathrm{T}} = \begin{bmatrix} 1/6 \\ 0 \\ -1/6 \end{bmatrix}$$

$$\left\|\mathbf{C}_p\right\| = \sqrt{(1/6)^2 + 0 + (1/6)^2} = \frac{\sqrt{2}}{6}$$

$$\mathbf{Y}_{new} = \mathbf{X}_0 - \alpha\, r \frac{\mathbf{C}_p}{\left\|\mathbf{C}_p\right\|} = \begin{bmatrix} 1/3 \\ 1/3 \\ \vdots \\ 1/3 \end{bmatrix} - \frac{\frac{2}{9} \times \frac{1}{\sqrt{6}}}{\frac{\sqrt{2}}{6}} \times \begin{bmatrix} 1/6 \\ 0 \\ -1/6 \end{bmatrix} = \begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix}$$

$$\mathbf{X}_1 = \mathbf{Y}_{new} = \begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix}$$

$$Z = \mathbf{C}^{\mathrm{T}}\mathbf{X}_1 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2692 \\ 0.3333 \\ 0.3974 \end{bmatrix} = 0.2692$$

<u>Iteration 1</u> <u>(k=1):</u>

$$\mathbf{D}_1 = \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix}$$

$$\overline{\mathbf{C}} = \mathbf{C}^{\mathrm{T}}\mathbf{D}_1 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} = \begin{bmatrix} 0 & 0.6667 & -0.3974 \end{bmatrix}$$

$$\mathbf{AD}_1 = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 0 & 0 \\ 0 & 0.3333 & 0 \\ 0 & 0 & 0.3974 \end{bmatrix} = \begin{bmatrix} 0.2692 & -0.6666 & 0.3974 \end{bmatrix}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{AD}_1 \\ \mathbf{1} \end{pmatrix} = \begin{bmatrix} 0.2692 & -0.6667 & 0.3974 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{PP}^{\mathrm{T}} = \begin{bmatrix} 0.2692 & -0.6667 & 0.3974 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 0.2692 & 1 \\ -0.6667 & 1 \\ 0.3974 & 1 \end{bmatrix} = \begin{bmatrix} 0.675 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\left(\mathbf{PP}^{\mathrm{T}}\right)^{-1} = \begin{bmatrix} 1.482 & 0 \\ 0 & 0.333 \end{bmatrix}$$

$$\mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\mathbf{P}^{\mathrm{T}}\right)^{-1}\mathbf{P}=\begin{bmatrix}0.441 & 0.067 & 0.492\\ 0.067 & 0.992 & -0.059\\ 0.492 & -0.059 & 0.567\end{bmatrix}$$

$$\mathbf{C}_p=\left[\mathbf{I}-\mathbf{P}^{\mathrm{T}}\left(\mathbf{P}\mathbf{P}^{\mathrm{T}}\right)^{-1}\mathbf{P}\right]\overline{\mathbf{C}}^{\mathrm{T}}=\begin{bmatrix}0.151\\ -0.018\\ -0.132\end{bmatrix}$$

$$\left\|\mathbf{C}_p\right\|=\sqrt{(0.151)^2+\left(-0.018\right)^2+(-0.132)^2}=0.2014$$

$$\mathbf{Y}_{new}=\mathbf{X}_0-\alpha\, r\frac{\mathbf{C}_p}{\left\|\mathbf{C}_p\right\|}=\begin{bmatrix}1/3\\ 1/3\\ \vdots\\ 1/3\end{bmatrix}-\frac{\frac{2}{9}\times\frac{1}{\sqrt{6}}}{0.2014}\times\begin{bmatrix}0.151\\ -0.018\\ -0.132\end{bmatrix}=\begin{bmatrix}0.2653\\ 0.3414\\ 0.3928\end{bmatrix}$$

$$\mathbf{D}_1\mathbf{Y}_{new}=\begin{bmatrix}0.2692 & 0 & 0\\ 0 & 0.3333 & 0\\ 0 & 0 & 0.3974\end{bmatrix}\times\begin{bmatrix}0.2653\\ 0.3414\\ 0.3928\end{bmatrix}=\begin{bmatrix}0.0714\\ 0.1138\\ 0.1561\end{bmatrix}$$

$$\mathbf{1D}_1\mathbf{Y}_{new}=\begin{bmatrix}1 & 1 & 1\end{bmatrix}\times\begin{bmatrix}0.0714\\ 0.1138\\ 0.1561\end{bmatrix}=0.3413$$

$$\mathbf{X}_2=\frac{\mathbf{D}_1\mathbf{Y}_{new}}{\mathbf{1D}_1\mathbf{Y}_{new}}=\frac{1}{0.3413}\times\begin{bmatrix}0.0714\\ 0.1138\\ 0.1561\end{bmatrix}=\begin{bmatrix}0.2092\\ 0.3334\\ 0.4574\end{bmatrix}$$

$$Z=\mathbf{C}^{\mathrm{T}}\mathbf{X}_2=\begin{bmatrix}0 & 2 & -1\end{bmatrix}\times\begin{bmatrix}0.2092\\ 0.3334\\ 0.4574\end{bmatrix}=0.2094$$

So far, two successive iterations are shown for the above problem. Similar iterations can be followed to get the final solution upto some predefined tolerance level.

It may be noted that, the efficacy of Karmarkar's projective scaling method is more convincing for 'large' LP problems. Rigorous computational effort is not economical for 'not-so-large' problems.

References / Further Reading:

1. Rao S.S., *Engineering Optimization – Theory and Practice*, Third Edition, New Age International Limited, New Delhi, 2000.
2. Ravindran A., D.T. Phillips and J.J. Solberg, *Operations Research – Principles and Practice*, John Wiley & Sons, New York, 2001.
3. Taha H.A., *Operations Research – An Introduction*, Prentice-Hall of India Pvt. Ltd., New Delhi, 2005.