

GATE 2023-ME-50

Pragnidhved Reddy
EE23BTECH11050

Question GATE 23 ME 50:

The initial value problem $\frac{dy}{dt} + 2y = 0, y(0) = 1$ is solved numerically using the forward Euler's method with a constant and positive time step of δ .

Let y_n represent the numerical solution obtained after n steps. The condition $|y_{n+1}| \leq |y_n|$ is satisfied if and only if δ does not exceed

Solution:

Numerical solution: -

By forward Euler's method formula

$$y(n+1) = y(n) + \delta f(x, y) \quad (1)$$

From question we get

$$\frac{dy}{dx} = -2y = f(x, y) \quad (2)$$

From (2) in (1)

$$y(n+1) - y(n) = -2\delta y(n) \quad (3)$$

$$y(n+1) = (1 - 2\delta)y(n) \quad (4)$$

$$y(n) \xleftrightarrow{Z} Y(z) \quad (5)$$

$$y(n+1) \xleftrightarrow{Z} zY(z) - y(0) \quad (6)$$

$$\Rightarrow zY(z) - y(0) = (1 - 2\delta)Y(z) \quad (7)$$

$$Y(z) = \frac{1}{z - 1 + 2\delta} \quad (8)$$

$$\frac{1}{z - (1 - 2\delta)} \xleftrightarrow{Z} (1 - 2\delta)^n u(n) \quad (9)$$

Now using the condition given in question

$$|y(n+1)| \leq |y(n)| \quad (10)$$

$$|(1 - 2\delta)^2| \leq |1 - 2\delta| \quad (11)$$

$$|1 - 2\delta| \leq 1 \quad (12)$$

$$0 \leq \delta \leq 1 \quad (13)$$

From this we can say that the maximum value of δ is 1

Theoretical solution: -

By properties of Laplace transform: -

$$Y(s) = \mathcal{L}y(s) \quad (14)$$

$$\mathcal{L}y' = sY(s) - y(0) \quad (15)$$

Given equation: -

$$y' + 2y = 0 \quad (16)$$

$$\mathcal{L}(y' + 2y) = 0 \quad (17)$$

From (14) and (15)

$$sY(s) - 1 + 2Y(s) = 0 \quad (18)$$

$$\frac{1}{s+2} = Y(s) \quad (19)$$

$$y(t) = \mathcal{L}^{-1}Y(s) \quad (20)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) \quad (21)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+k}\right) = e^{-kt}u(t) \quad (22)$$

$$\Rightarrow y(t) = e^{-2t}u(t) \quad (23)$$

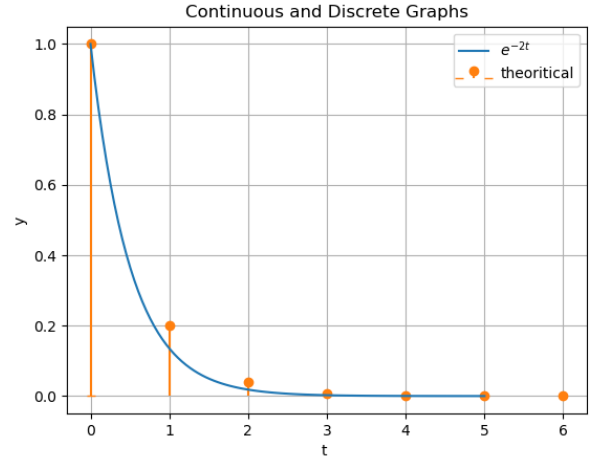


Fig. 1. simulation vs analysis