Deep Learning Topology

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Topology

- Topology is about shapes
- Topology is also about functions between shapes
- Two shapes are topologically equivalent if they can be continuously deformed into one another, *e.g.*, by bending, twisting, stretching, crumpling, but not by tearing, cutting, gluing

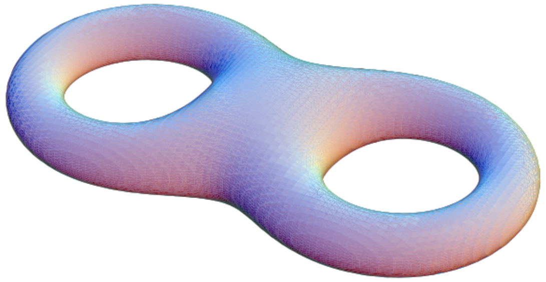
Formally: There exists a **homeomorphism** $f: X \to Y$

— f is one-to-one and onto

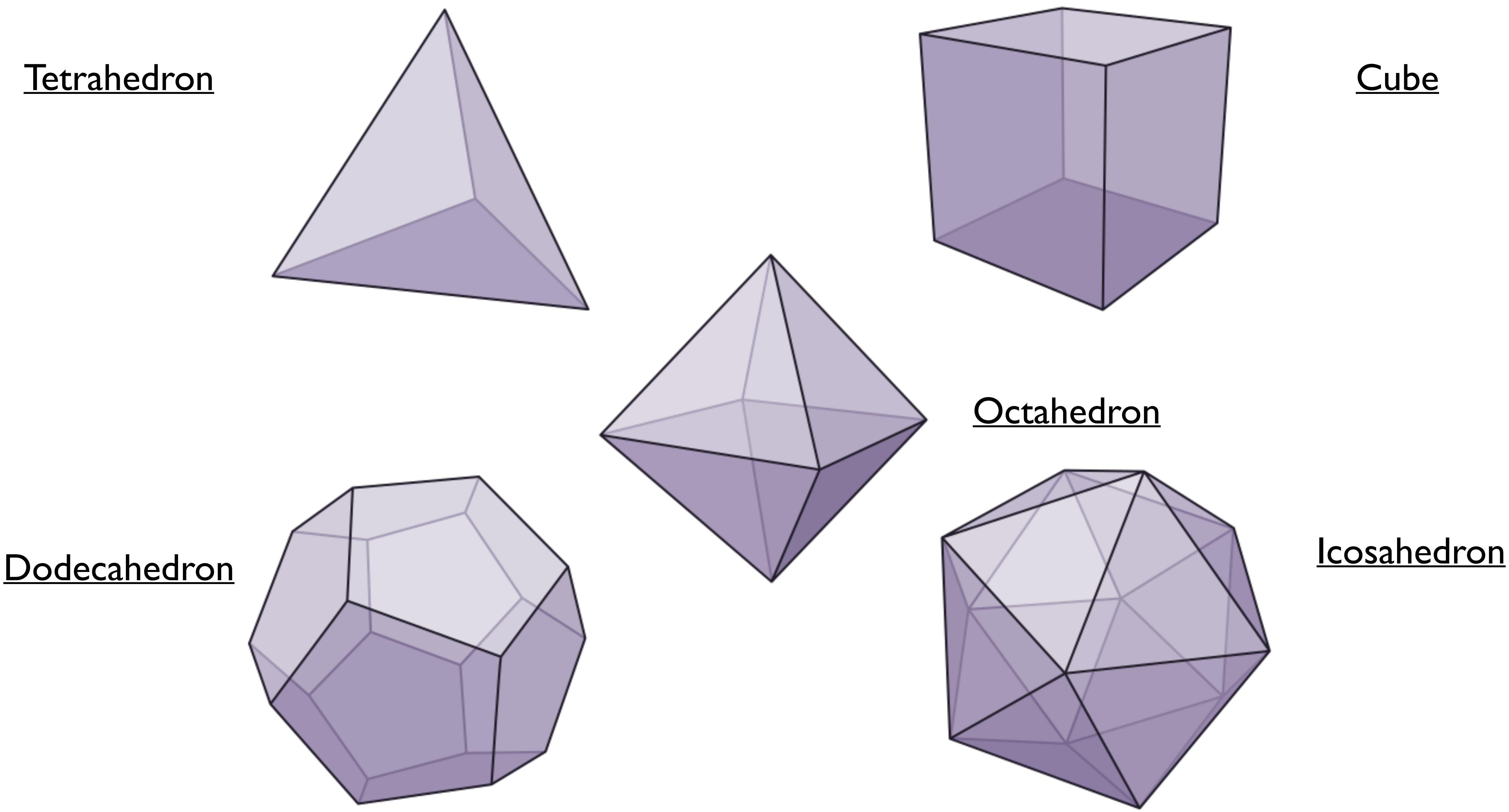
— f and f^{-1} are continuous

- You have an intuitive understanding of topology already





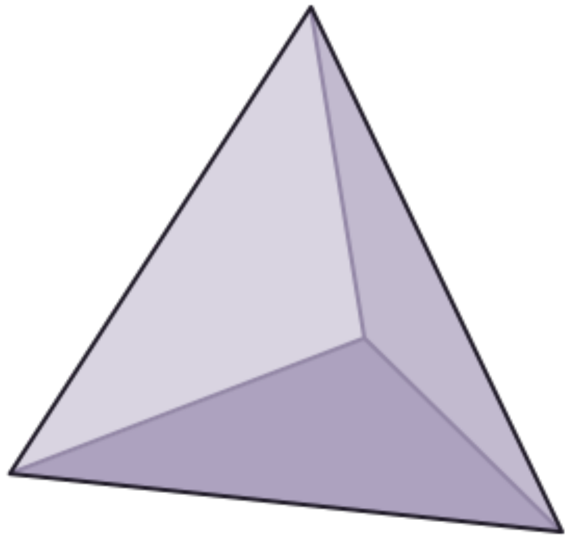

Euler Characteristic



- **Platonic solids:** convex, regular polyhedra in 3d

Theaetetus (c. 369 BCE)

Euler Characteristic

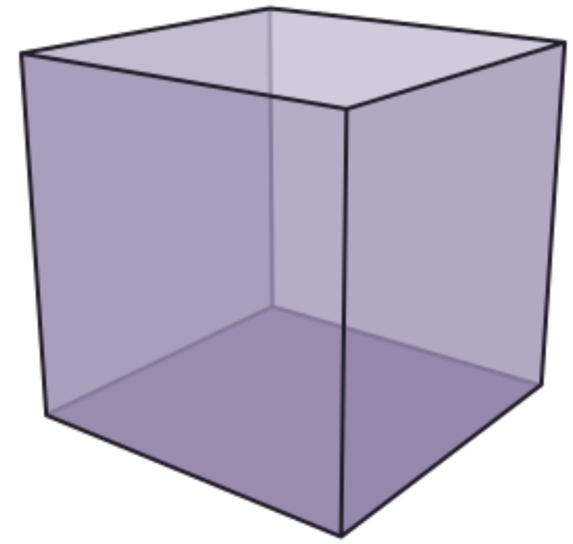


Tetrahedron

$V = 4$

$E = 6$

$F = 4$

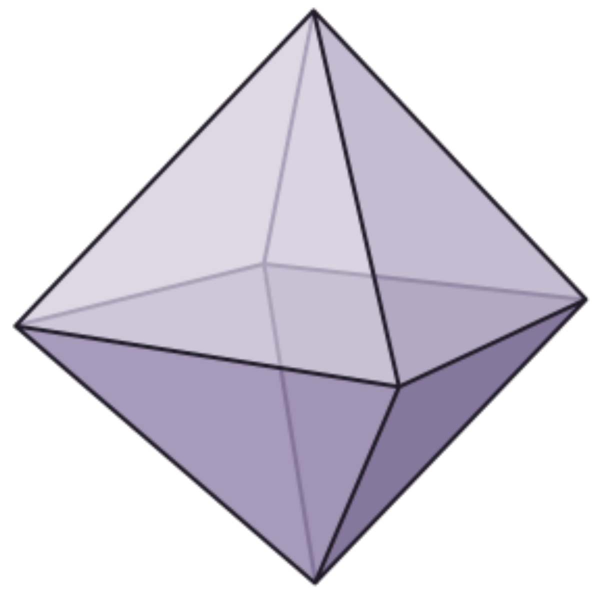


Cube

$V = 8$

$E = 12$

$F = 6$

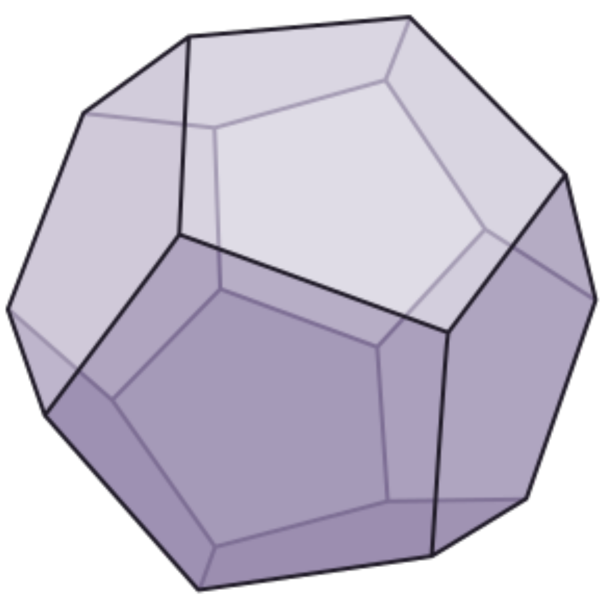


Octahedron

$V = 6$

$E = 12$

$F = 8$

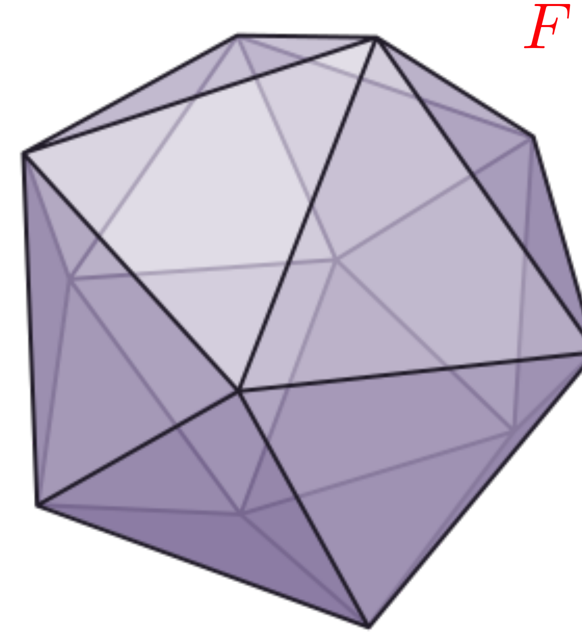


Dodecahedron

$V = 20$

$E = 30$

$F = 12$



Icosahedron

$V = 12$

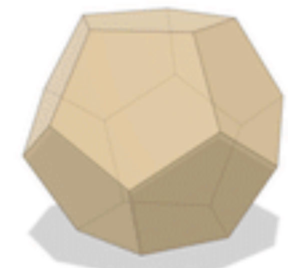
$E = 30$

$F = 20$

$$\boxed{\chi = V - E + F = 2}$$

Maurolico (1537, unpublished)
Euler (1752)

Euler Characteristic

	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Platonic Solid					
Spherical Polyhedron					

$$\chi = 2 - 2g$$

$$g = \text{genus}$$

- The Euler characteristic and genus are topological invariants

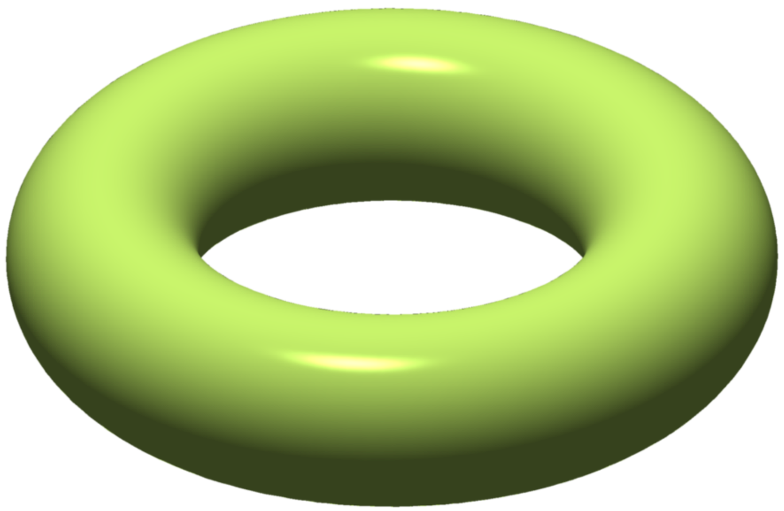
Euler Characteristic

genus 0

genus 1

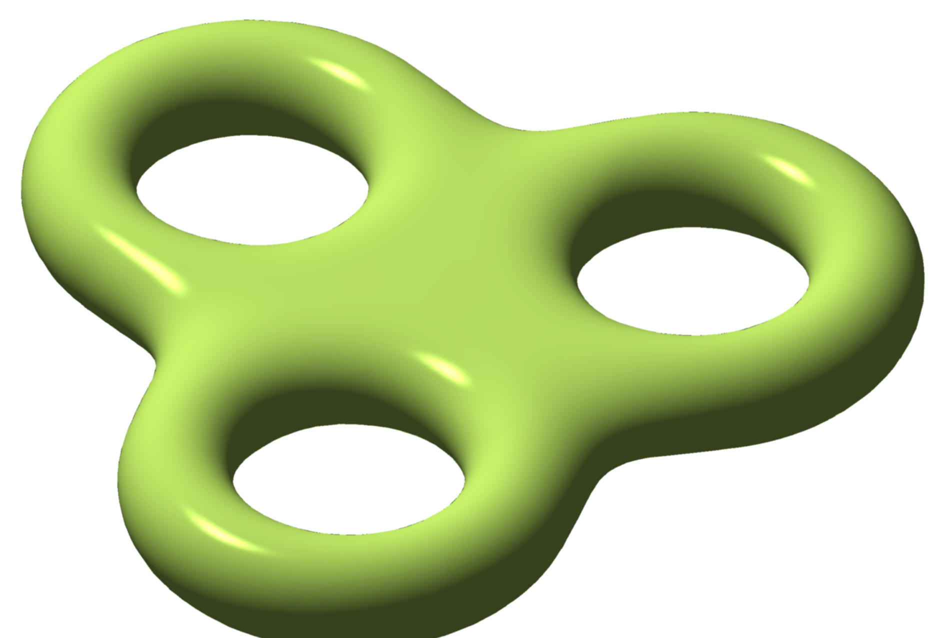
topological invariants

A Topologist's Morning



coffee

trousers

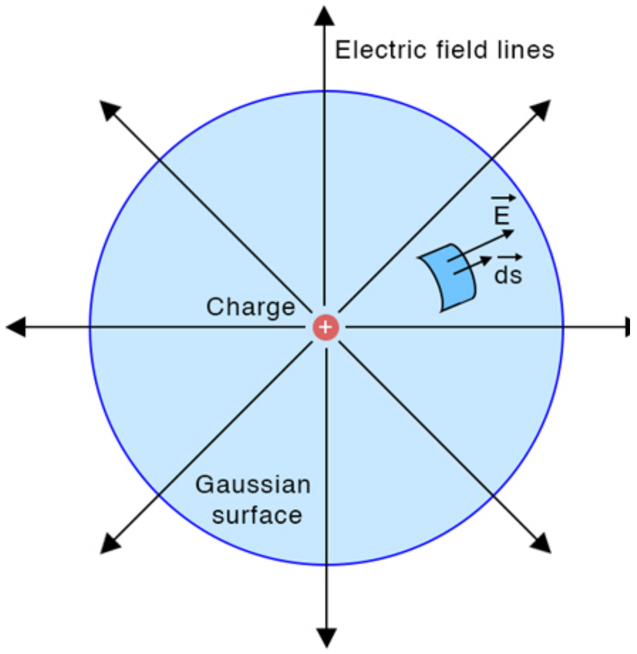


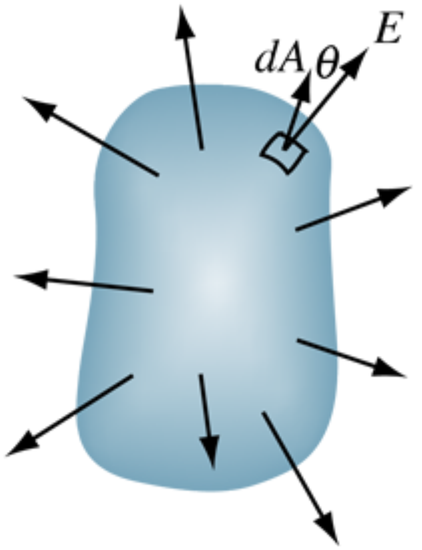
shirt

Topology in Physics

- Gauss's law

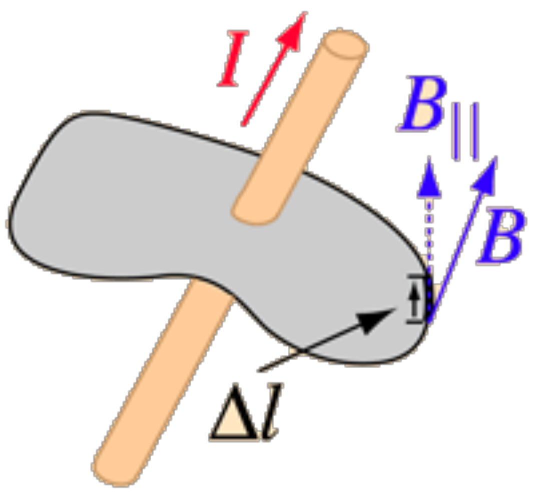
$$\Phi_E = \int_\Sigma \vec{\mathbf{E}} \cdot \mathrm{d}\vec{\mathbf{A}}$$





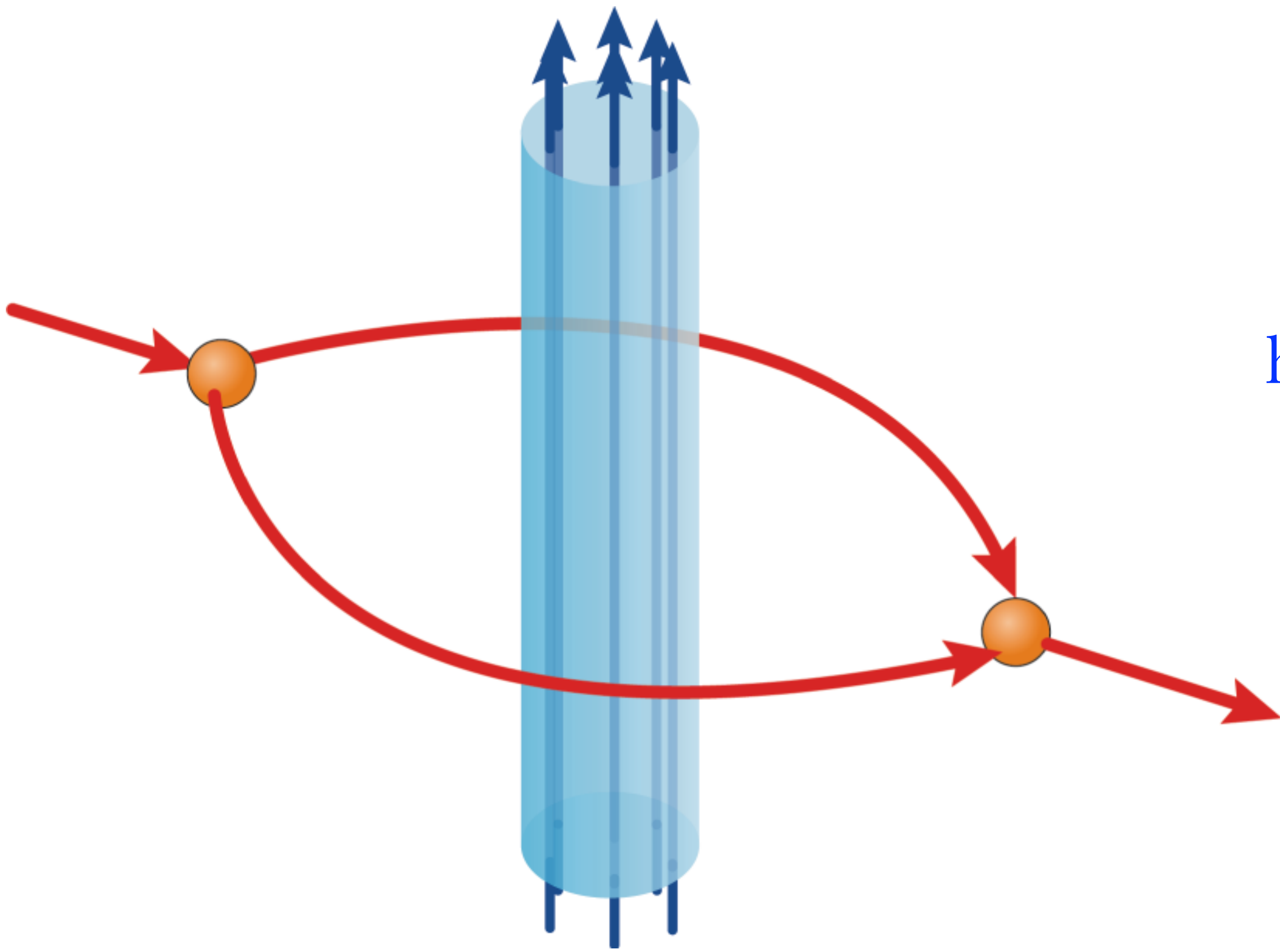
- Ampère's law

$$\oint_C \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\ell} = \int\int (\vec{\mathbf{J}} + \partial_t \vec{\mathbf{E}}) \cdot \mathrm{d}\vec{\mathbf{S}}$$



Topology in Physics

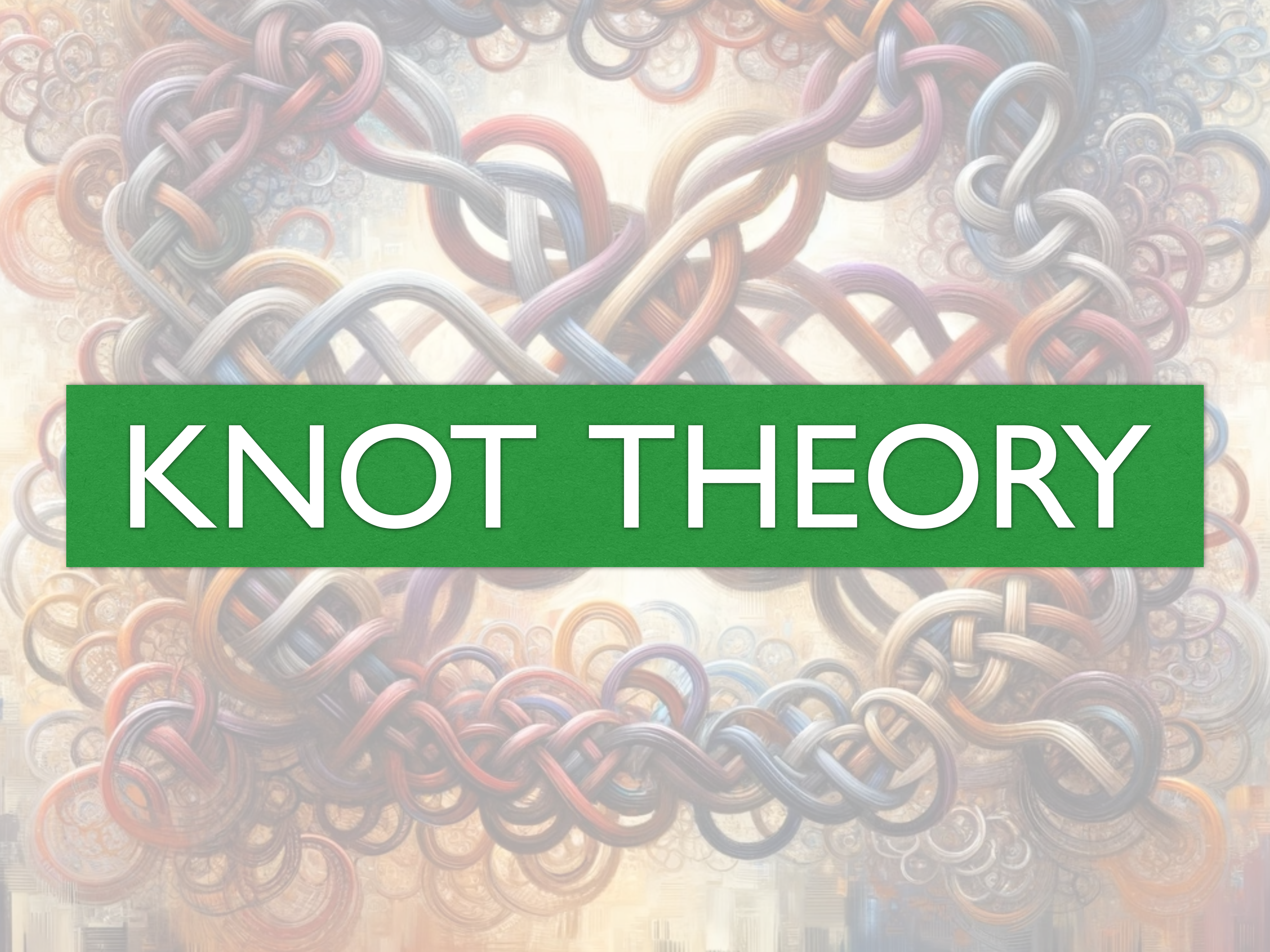
- Aharonov–Bohm effect



holonomy of gauge field
around closed loop

$$\Delta\varphi = \frac{e}{\hbar}\int_{\gamma} \vec{\mathbf{A}} \cdot \mathrm{d}\vec{\mathbf{x}}$$
$$= \frac{e\Phi_B}{\hbar}$$
$$= 2\pi n\ , \quad n \in \mathbb{Z}$$

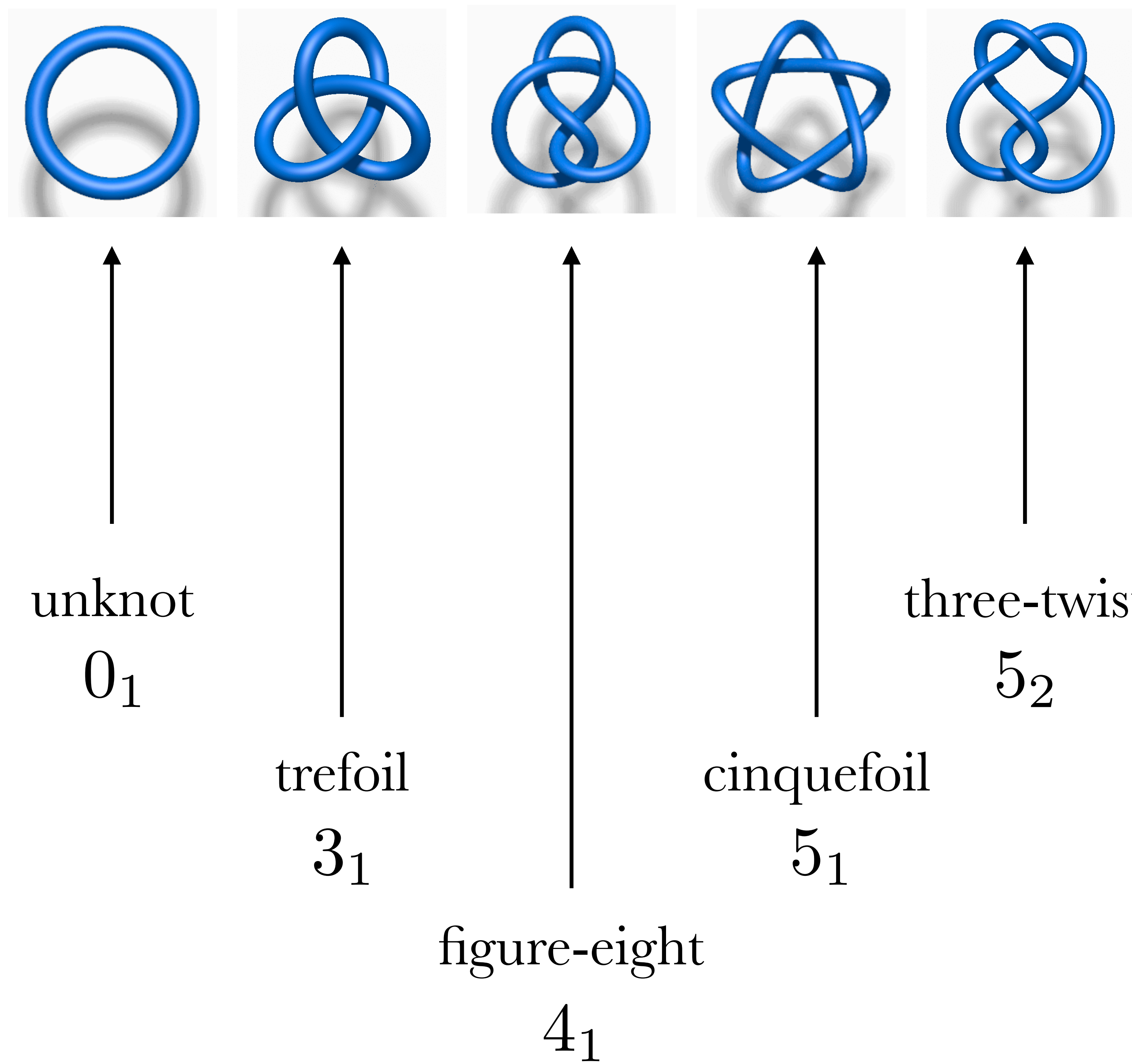
- Condensed matter systems, *e.g.*, quantum Hall effect



KNOT THEORY

Dramatis Personae

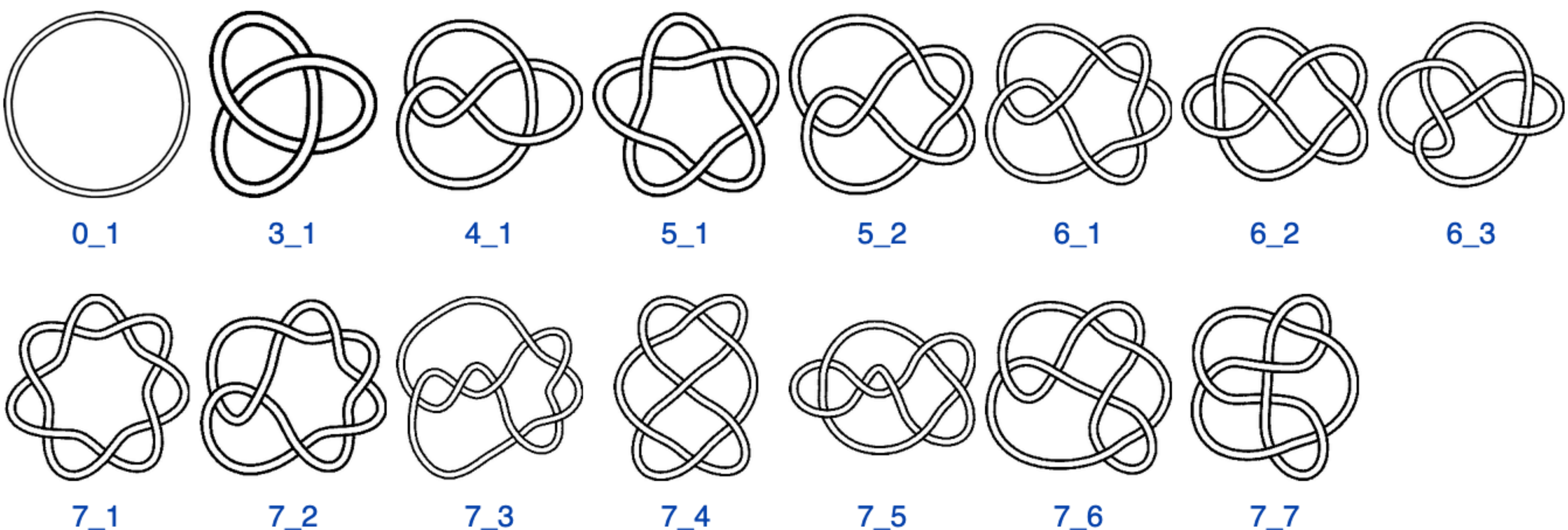
Knot: $S^1 \subset S^3$; *e.g.*,



Dramatis Personae

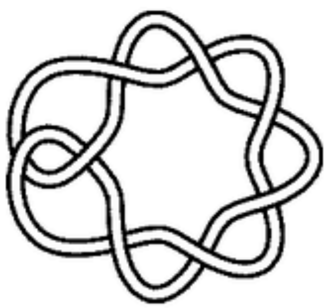
Knot: $S^1 \subset S^3$

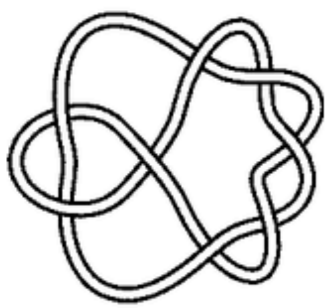
- Organized by crossing number, a proxy for complexity



Dramatis Personae

Knot: $S^1 \subset S^3$


8_1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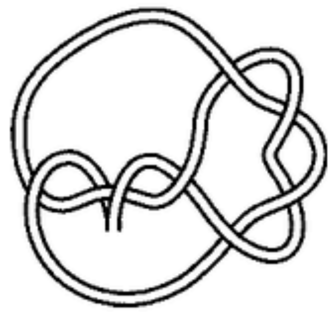
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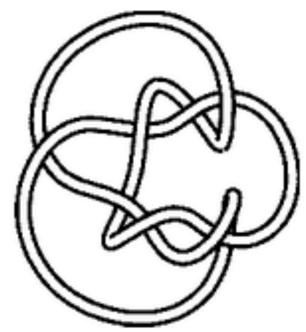
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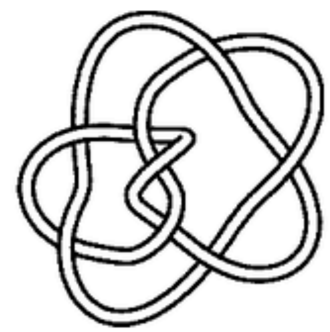
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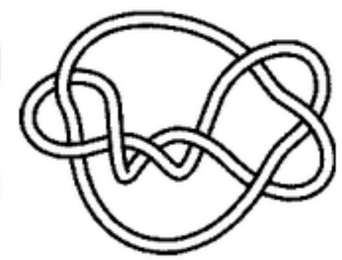
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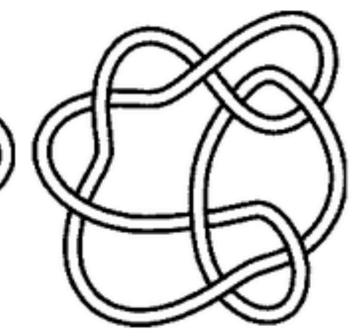
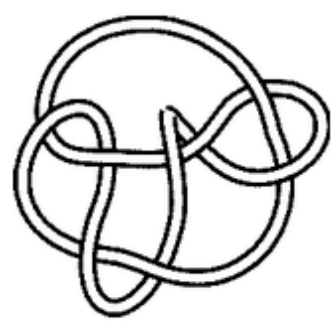
8_7


8_8


8_9


8_10


8_11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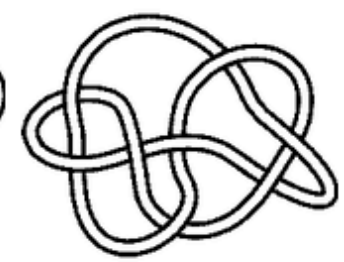
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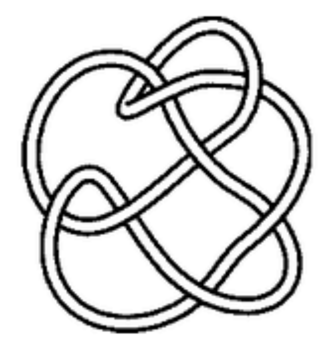
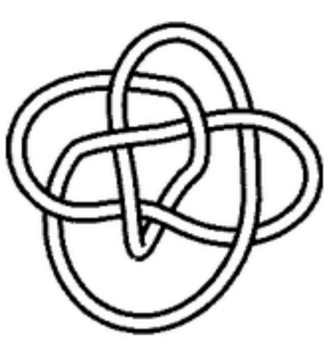

8_14


8_15


8_16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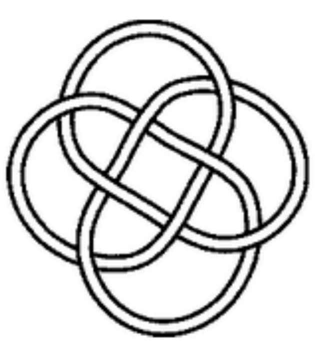
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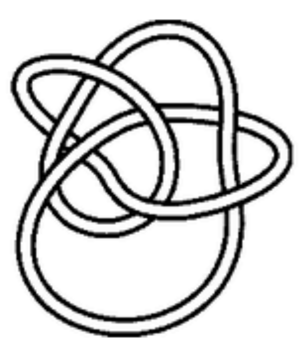
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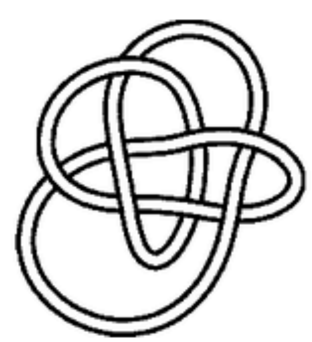
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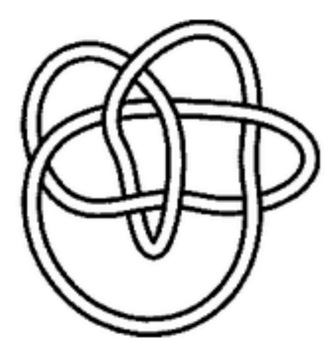
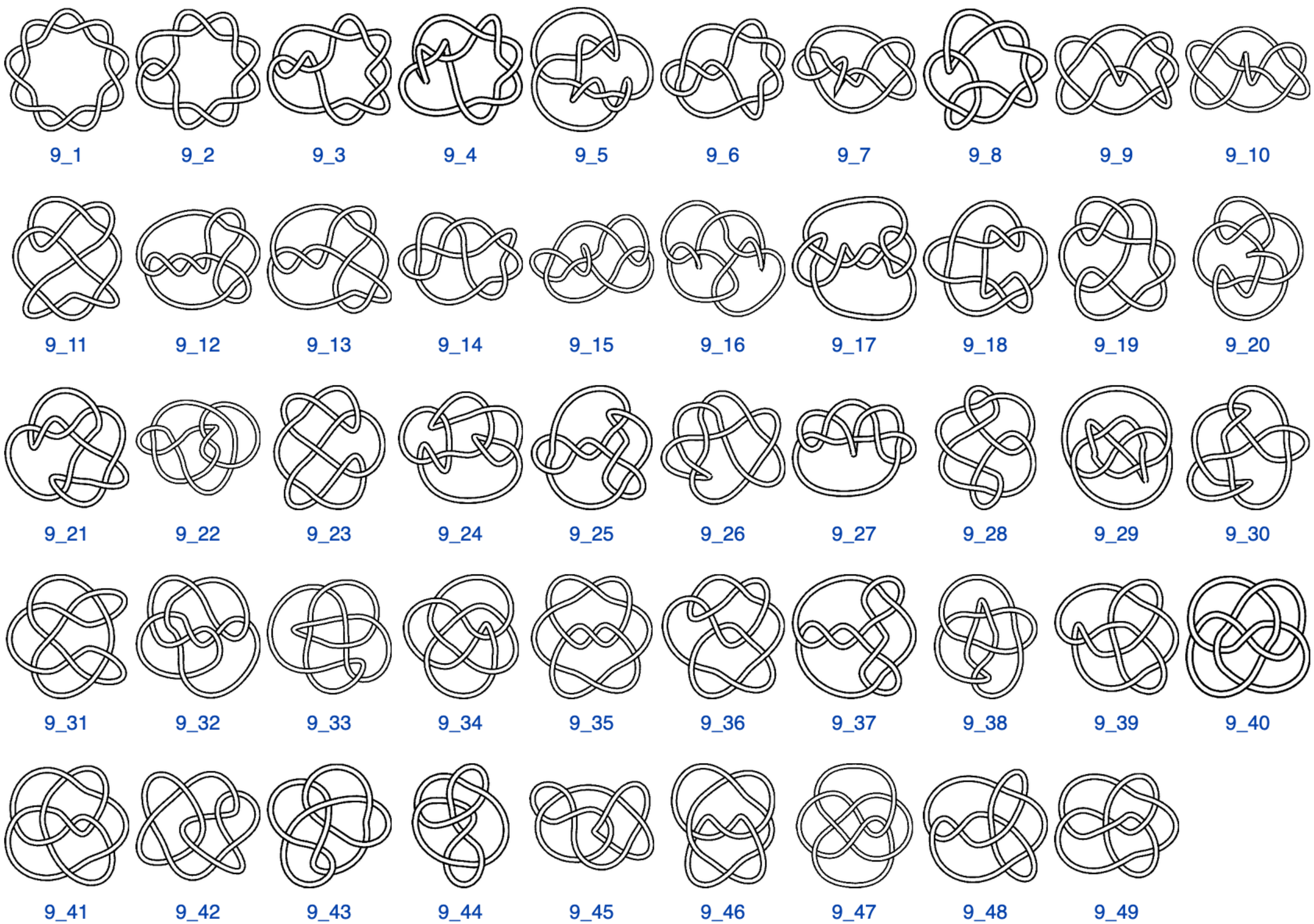

8_21

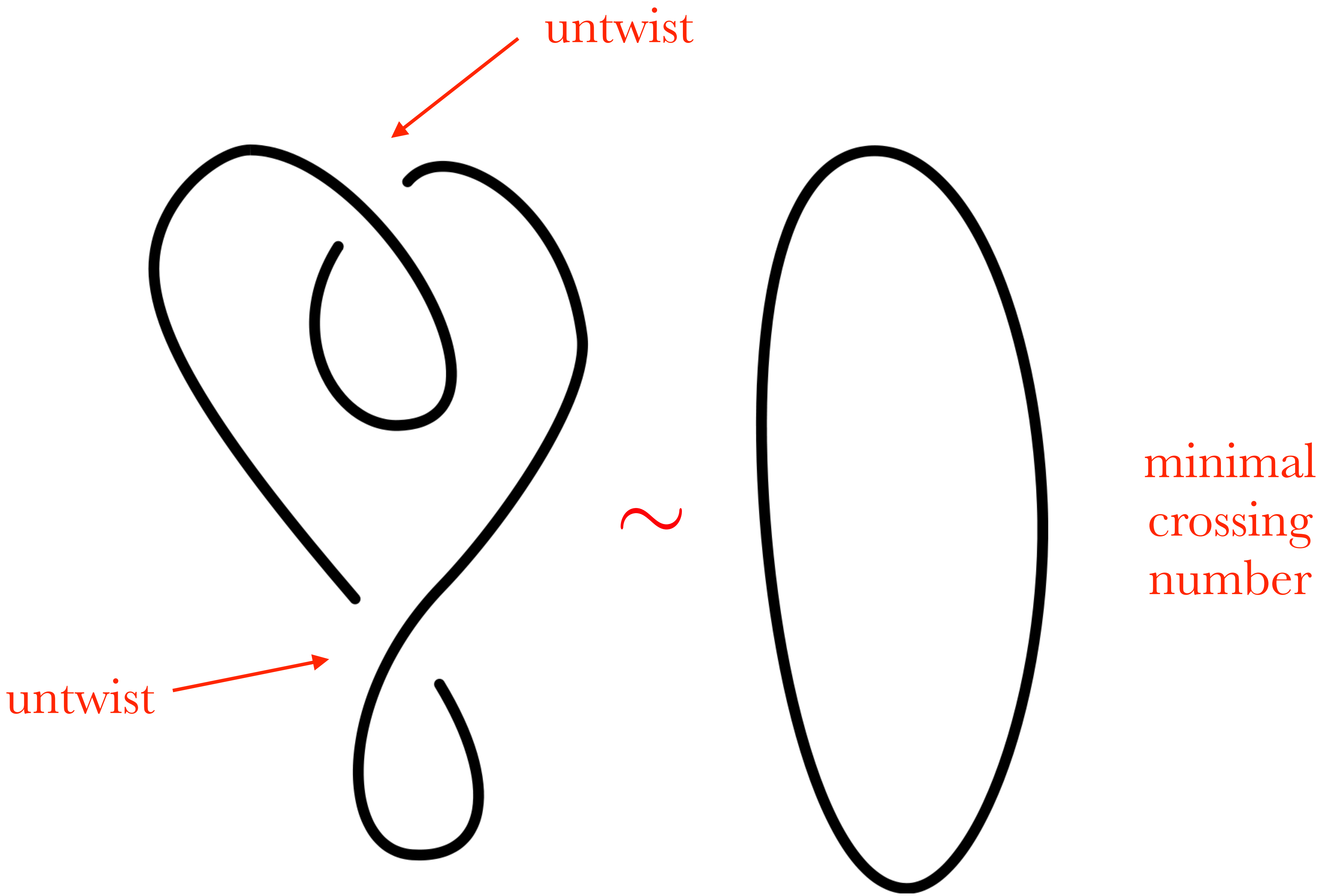
Dramatis Personae

Knot: $S^1 \subset S^3$



Dramatis Personae

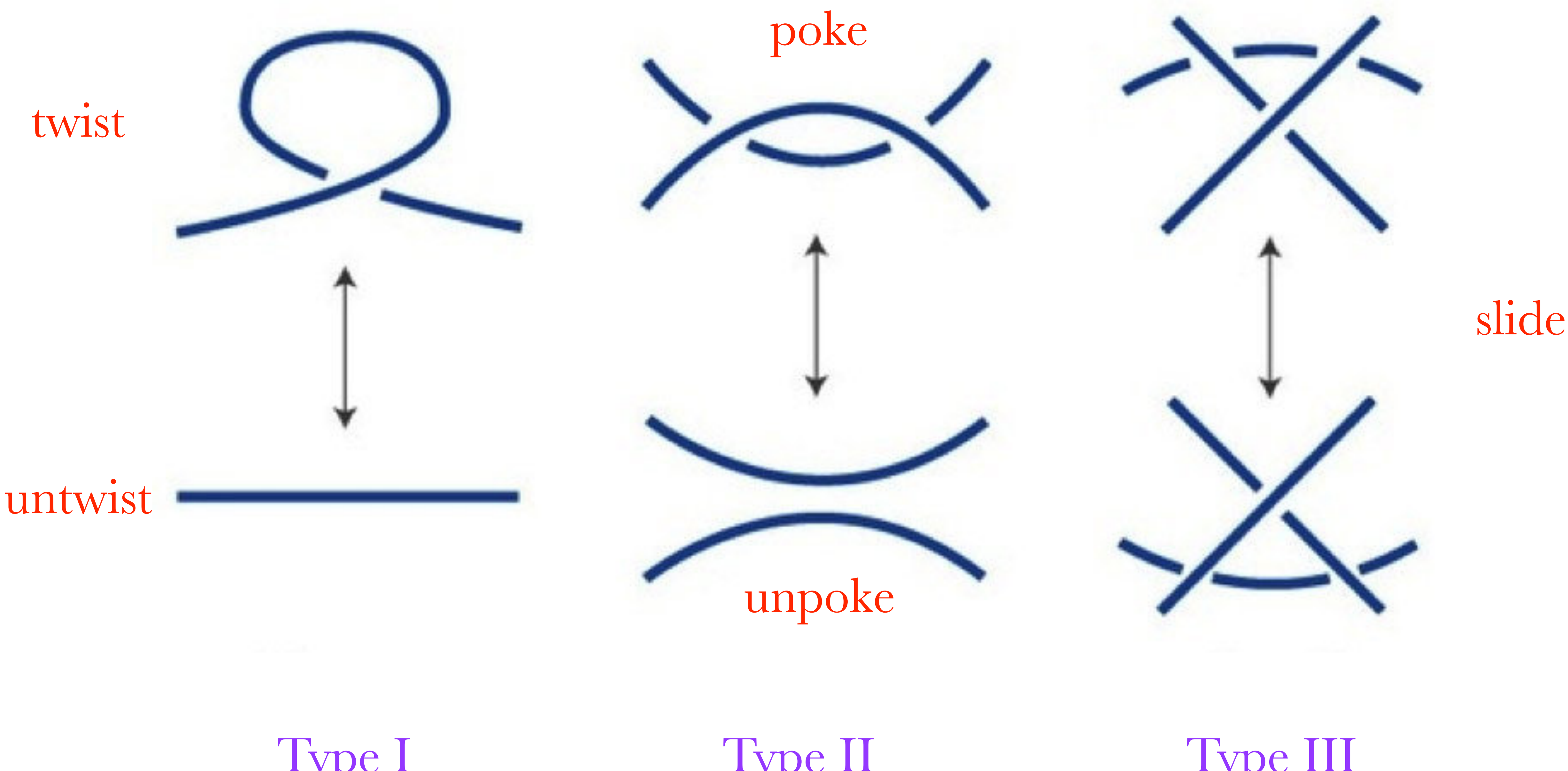
Knot: $S^1 \subset S^3$



Dramatis Personae

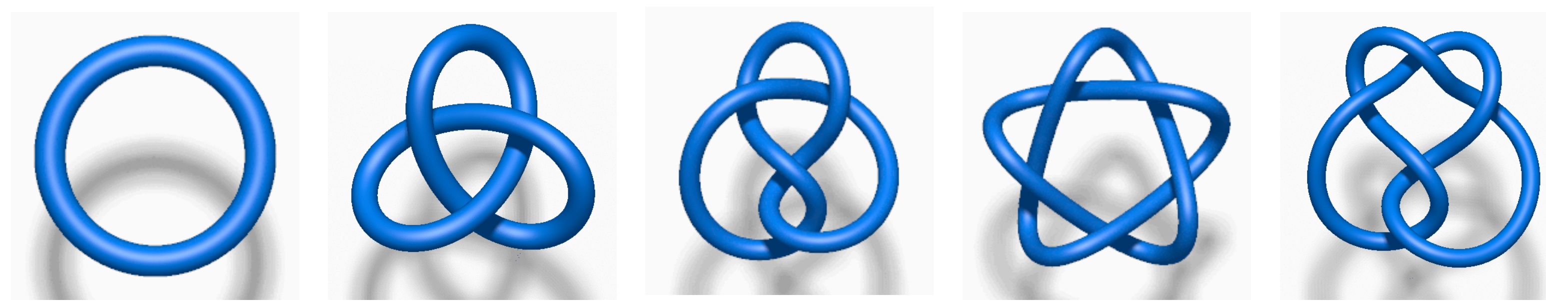
Knot: $S^1 \subset S^3$

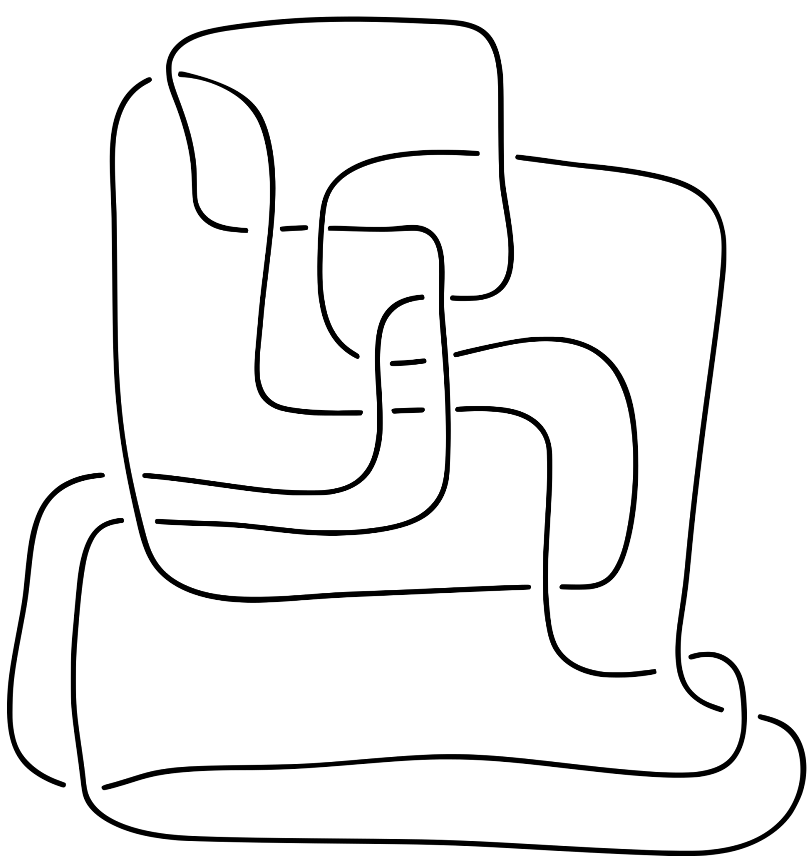
Reidemeister moves



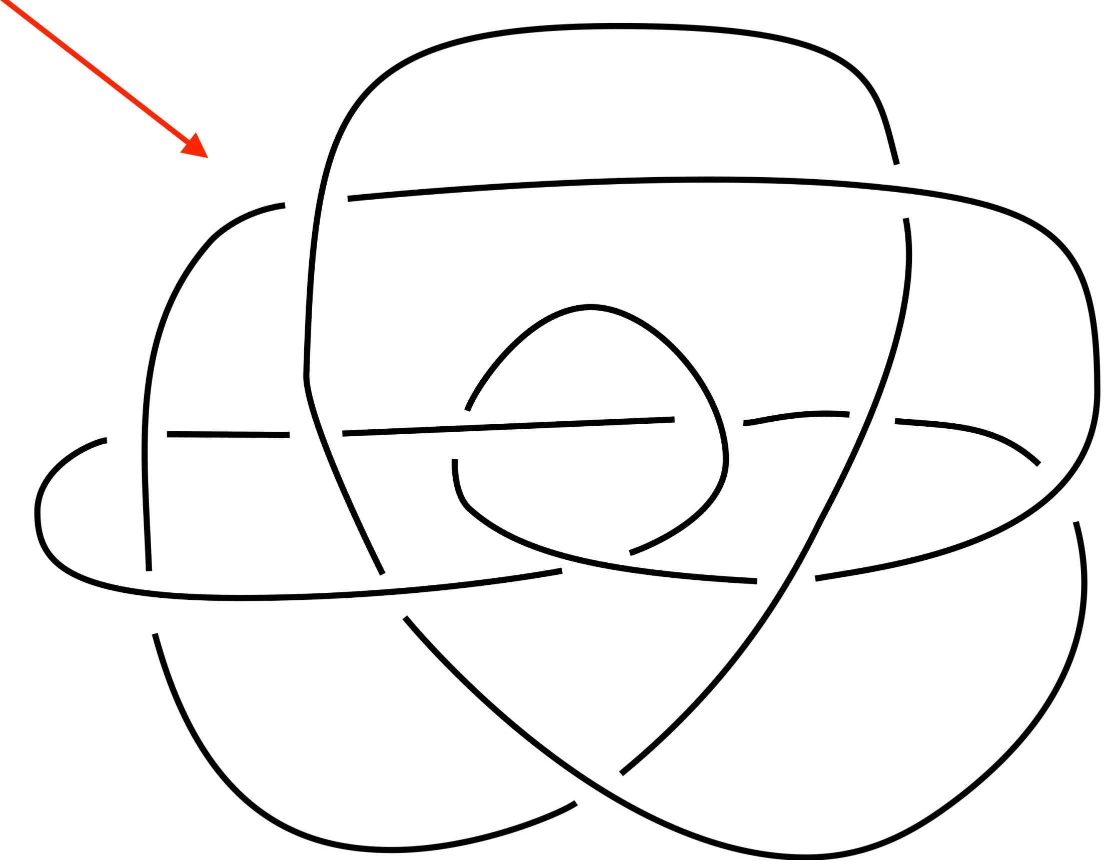
Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,





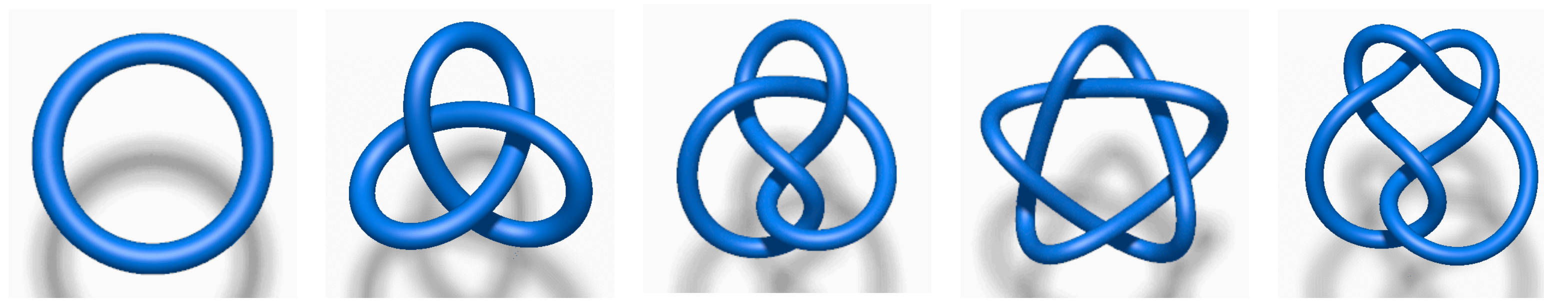
Thistlethwaite unknot

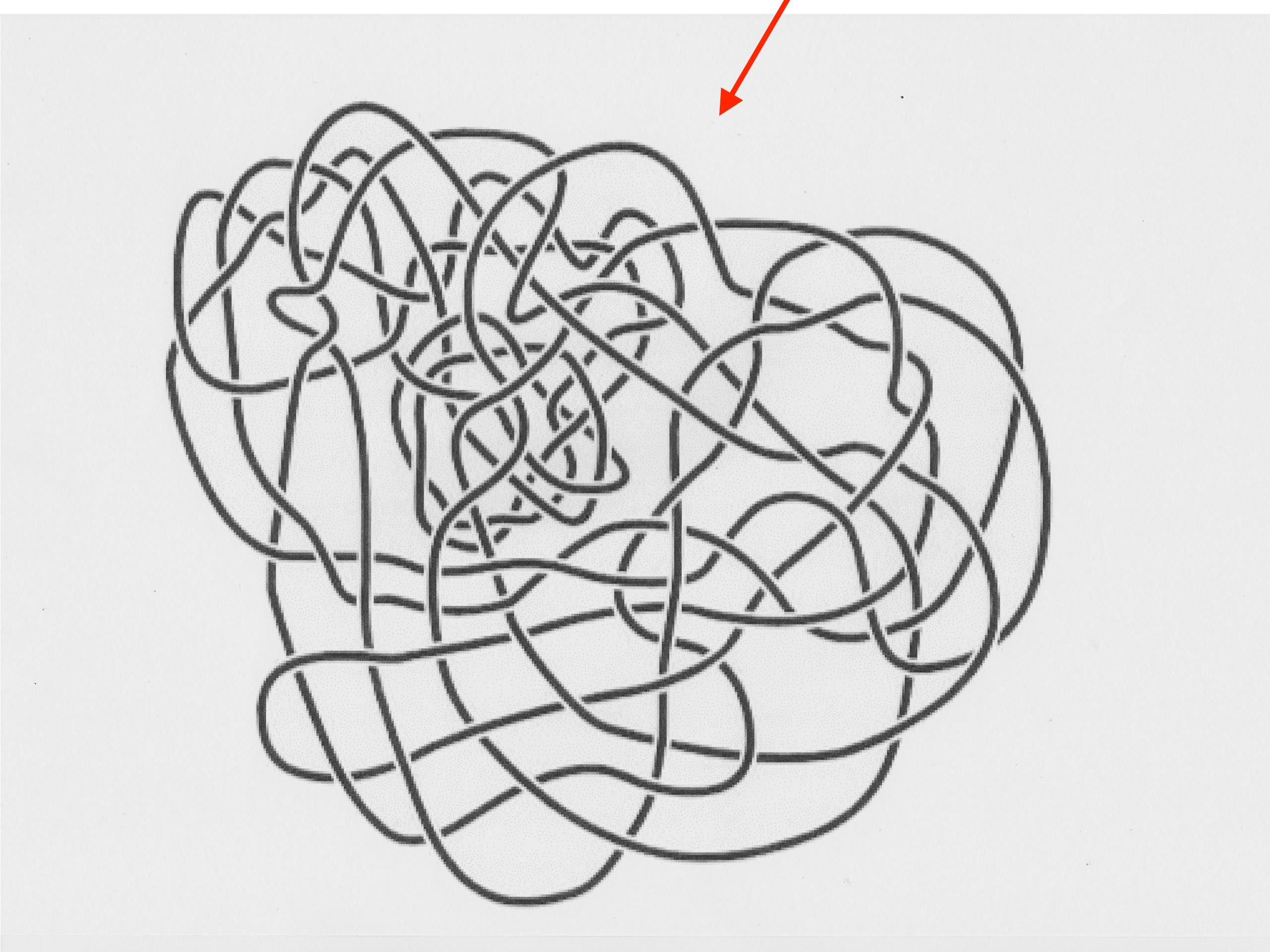


Ochiai unknot

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,





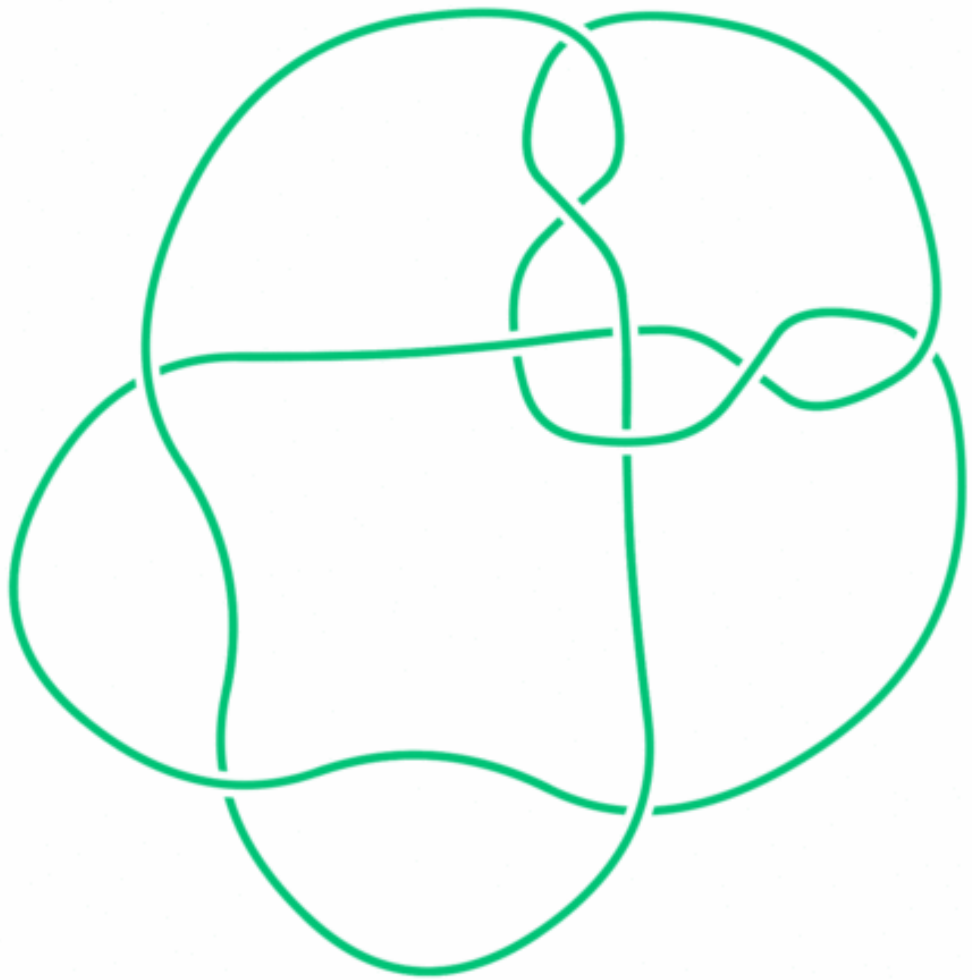
Burton (2015)

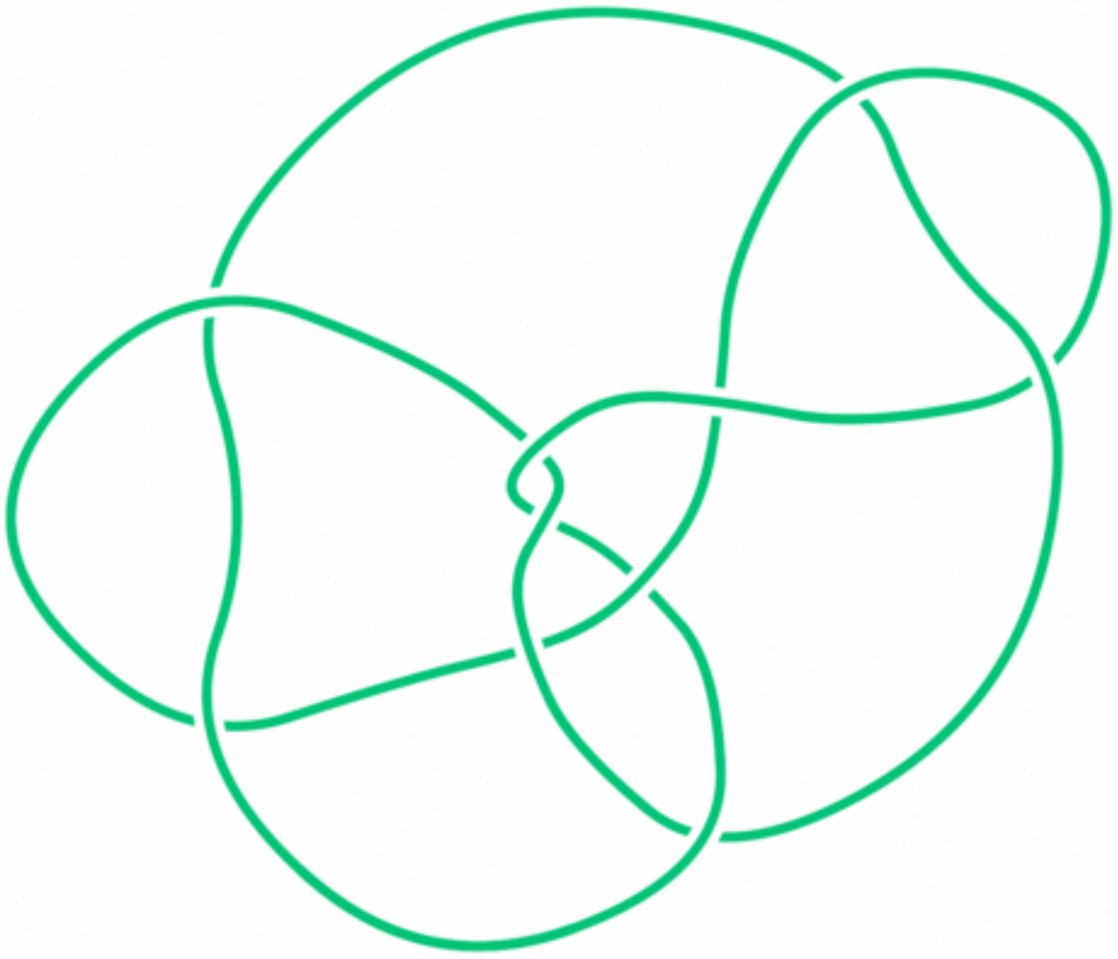
Haken or Gordian unknot

Dramatis Personae

Knot: $S^1 \subset S^3$

Rolfsen table

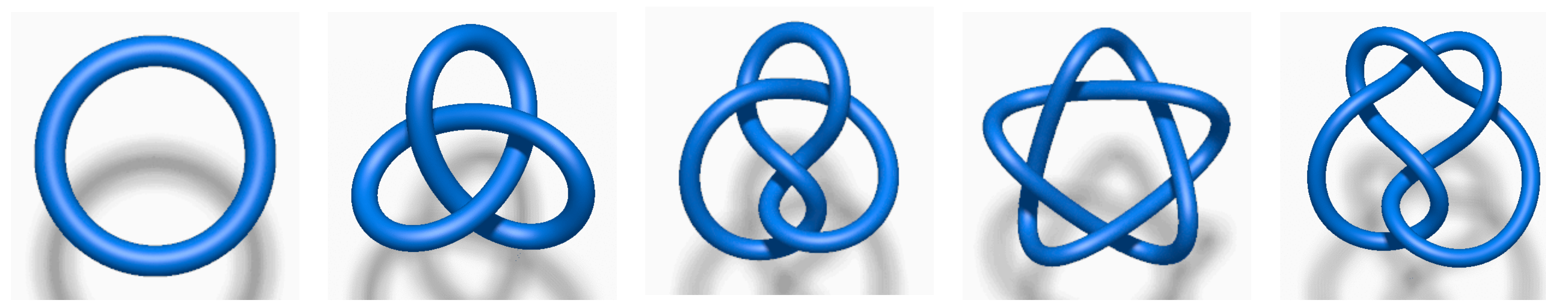




10_{161} ⟵ Perko pair (1973) ⟶ 10_{162}

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,



Jones polynomial: $J(K;q) = (-q^{\frac{3}{4}})^{w(K)} \dfrac{\langle K \rangle}{\langle \bigcirc \rangle}$

$$\langle \times \rangle = q^{\frac{1}{4}} \langle \asymp \rangle + \frac{1}{q^{\frac{1}{4}}} \langle)(\rangle$$

$w(K) =$ overhand $-$ underhand

$J(\bigcirc;q) = 1$

Jones (1985)

topological invariant: independent of how knot is drawn

Question: how to calculate these?

Answer: quantum field theory!

Topological Invariants

- On a manifold $\mathcal{M}$ with metric $g_{\mu\nu}$, a topological invariant enjoys:

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = 0$$

- In Chern–Simons theory, the operators are Wilson loops

$$U_R(\gamma) = \mathrm{tr}_R \, \mathcal{P} \exp \left(i \oint_\gamma A \right)$$

- The colored Jones polynomial is a knot invariant in 3d:

$$J_n(K; q = e^{2\pi i/(k+2)}) = \frac{\int_{\mathcal{U}} [DA] \; U_n(K) e^{iS_{\mathrm{CS}}(A)}}{\int_{\mathcal{U}} [DA] \; U_n(0_1) e^{iS_{\mathrm{CS}}(A)}} = \frac{\langle U_n(K) \rangle}{\langle U_n(0_1) \rangle}$$

$$S_{\mathrm{CS}}(A) = \frac{k}{4\pi} \int_{\mathcal{M}} \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) , \quad Z(\mathcal{M}) = \int_{\mathcal{U}} [DA] \; e^{iS_{\mathrm{CS}}(A)}$$

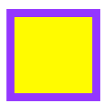
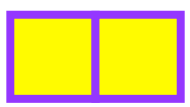
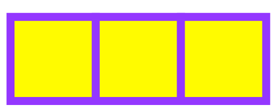
$$k \in \mathbb{Z}$$

simplest non-trivial quantum field theory

Witten (1989)

Group Theory

- Irreducible representations of $SU(2)$: $R(g_1 g_2) = R(g_1)R(g_2)$

	defining (fundamental)	2×2	matrices
	adjoint	3×3	matrices
		4×4	matrices
$\vdots$			

- Algebra: $[S_i, S_j] = i\hbar\epsilon_{ijk}S_k$

$$S_i = \frac{\hbar}{2}\sigma_i\ , \qquad \sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

spin-$\frac{1}{2}$

$$S_1 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{pmatrix}, \quad S_2 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} & -i & \\ i & & -i \\ & i & \end{pmatrix}, \quad S_3 = \hbar\begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

spin-1

- Colored Jones polynomials correspond to traces in different representations

Dramatis Personae

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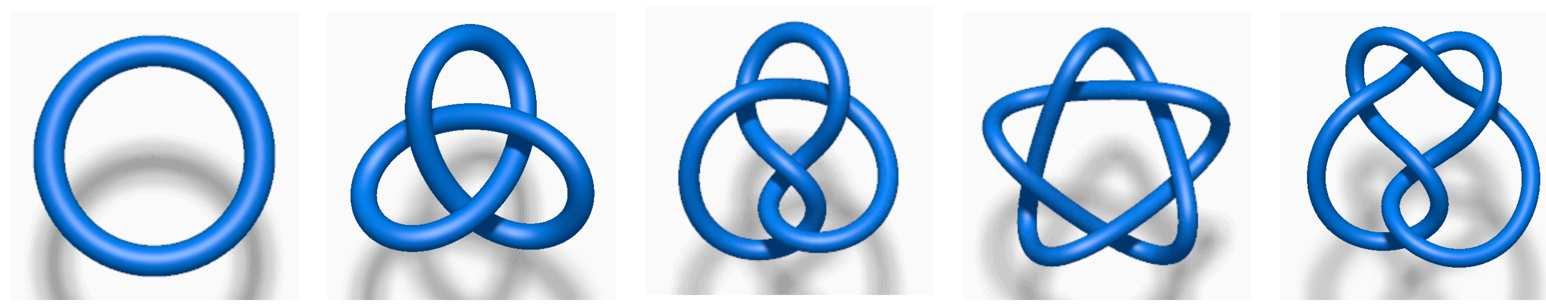

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$w(K) = \text{overhand} - \text{underhand}$

vev of Wilson loop operator along K in
$\square$ for $SU(2)$ Chern–Simons on S^3

Jones (1985)
Witten (1989)

$$J_2(4_1;q) = q^{-2} - q^{-1} + 1 - q + q^2 \ , \quad q = e^{\frac{2\pi i}{k+2}}$$

Open question: Is unknot unique knot for which $J_2(q) = 1$?

Open question: Unknot recognition problem — is this NP?

Dehn (1910)
Turing (1954)
Haken (1961)
Lackenby (2021)

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,

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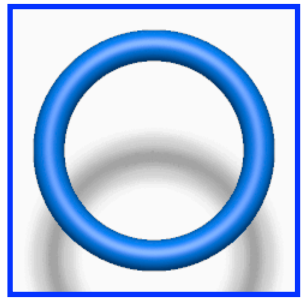
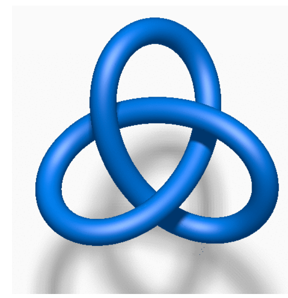
Hyperbolic volume: volume of $S^3 \backslash K$ is another knot invariant in 3d

computed from tetrahedral decomposition of knot complement

Thurston (1978)
Mostow (1968)

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,

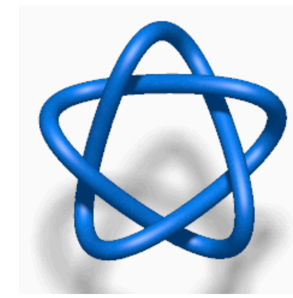
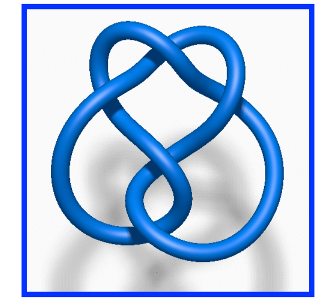



Table A1

# Crossings	# Knots	# Torus	# Sat.
0	1		
3a	1	1	
4a	1		
5a	2	1	
6a	3		
7a	7	1	
8a	18		
8n	3	1	
9a	41	1	
9n	8		
10a	123		
10n	42	1	
11a	367	1	
11n	185		
12a	1,288		
12n	888		
13a	4,878	1	
13n	5,110		2
14a	19,536		
14n	27,436	1	2
15a	85,263	1	
15n	168,030	1	6
16a	379,799		
16n	1,008,906	1	10

1701936 knots up to 16 crossings

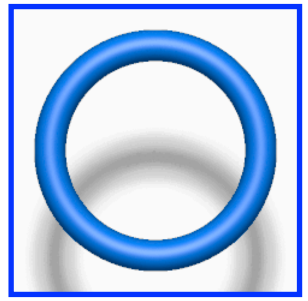
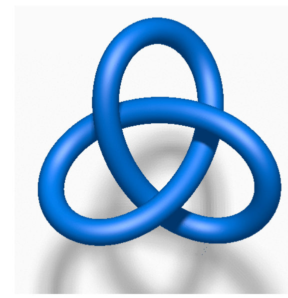
All but 32 are hyperbolic

Hoste, Thistlewaite, Weeks (1998)

Conjecture [Adams]: proportion of hyperbolic knots approaches 1 as crossing number goes to ∞

Dramatis Personae

Knot: $S^1 \subset S^3$; *e.g.*,

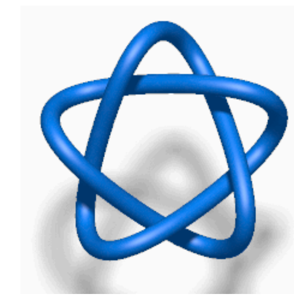
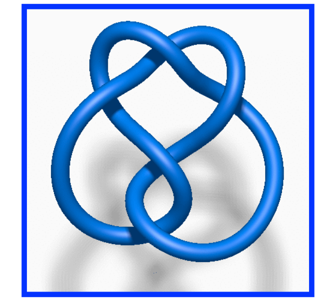



Table A1

# Crossings	# Knots	# Torus	# Sat.
0	1		
3a	1	1	
4a	1		
5a	2	1	
6a	3		
7a	7	1	
8a	18		
8n	3	1	
9a	41	1	
9n	8		
10a	123		
10n	42	1	
11a	367	1	
11n	185		
12a	1,288		
12n	888		
13a	4,878	1	
13n	5,110		2
14a	19,536		
14n	27,436	1	2
15a	85,263	1	
15n	168,030	1	6
16a	379,799		
16n	1,008,906	1	10

1701936 knots up to 16 crossings

All but 32 are hyperbolic

Hoste, Thistlewaite, Weeks (1998)

Conjecture [Adams]: proportion of hyperbolic knots approaches 1 as crossing number goes to ∞

Contradicts another **conjecture**: crossing number of composite knot not less than that of each of its factors

Malyutin (2016)

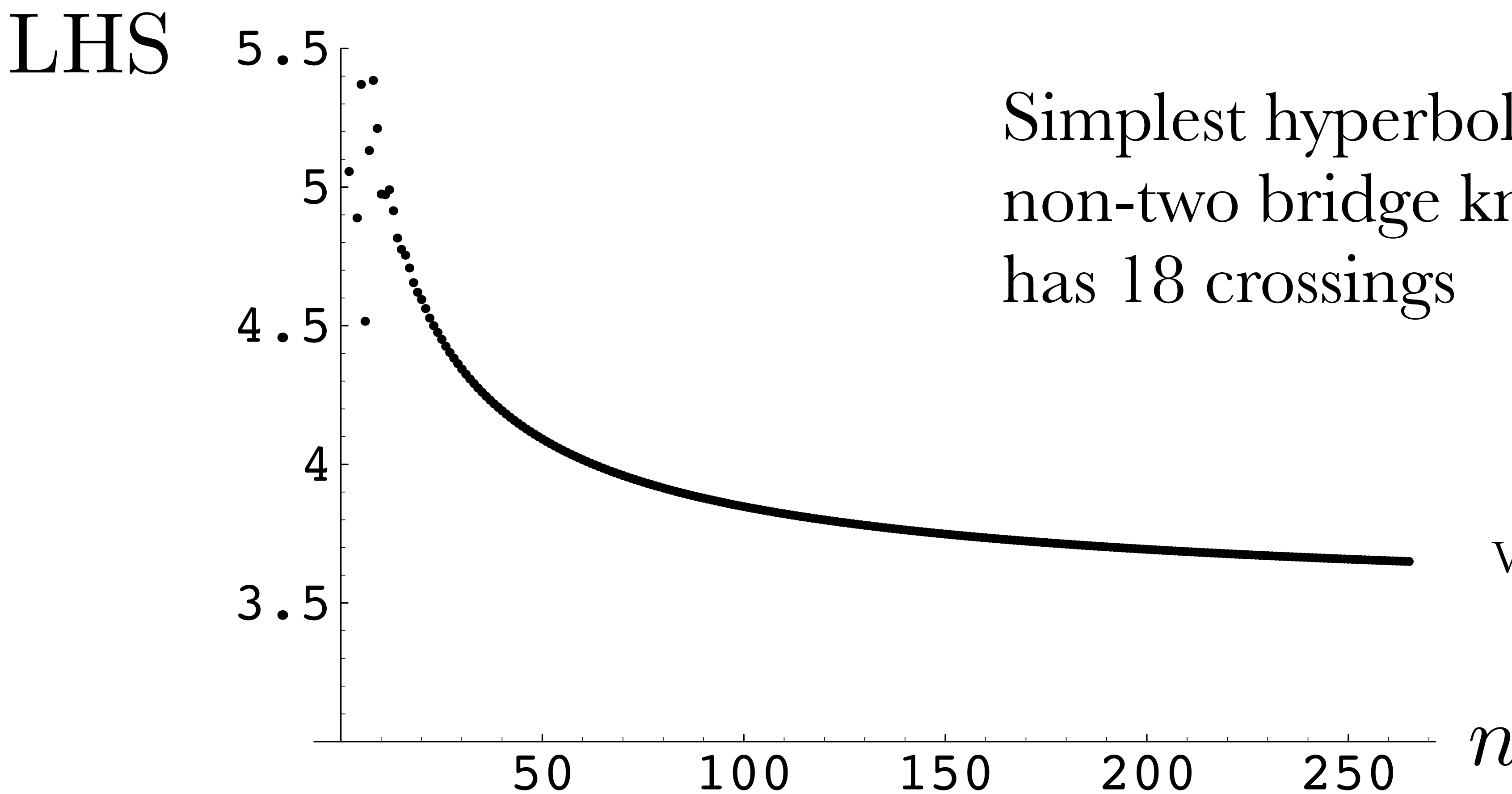
Dramatis Personae

Volume conjecture: $\lim_{n\to\infty} \frac{2\pi \log|J_n(K;\omega_n)|}{n} = \mathrm{Vol}(S^3 \setminus K)$

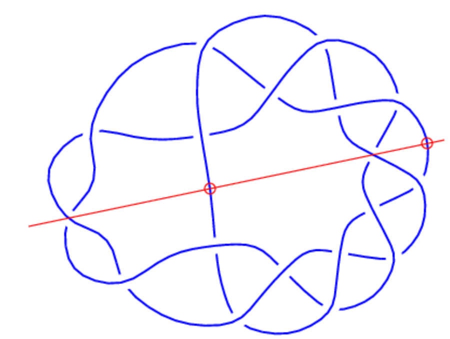
Kashaev (1997)
Murakami x 2 (2001)
Gukov (2005)

$\omega_n = e^{\frac{2\pi i}{n}}$

In fact, we take $n, k \to \infty$



Simplest hyperbolic non-two bridge knot, has 18 crossings



$\sigma_1^2(\sigma_1\sigma_2)^8$

Behavior is not monotonic!

Garoufalidis, Lan (2004)

Dramatis Personae

Volume conjecture: $$\lim_{n\to\infty} \frac{2\pi \log |J_n(K;\omega_n)|}{n} = \mathrm{Vol}(S^3 \setminus K)$$

$$\omega_n = e^{\frac{2\pi i}{n}}$$

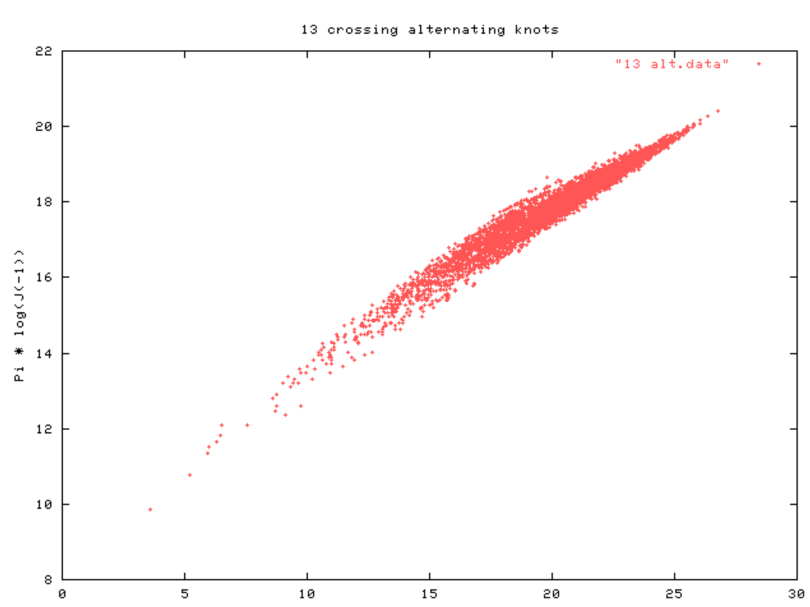
Kashaev (1997)
Murakami x 2 (2001)
Gukov (2005)

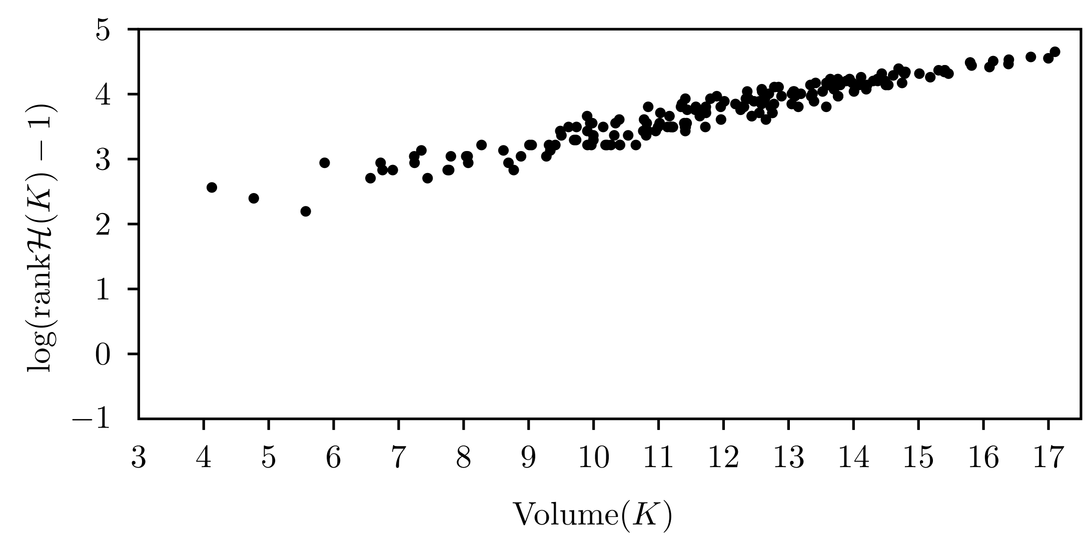
Khovanov homology: a homology theory $\mathcal{H}_K$ whose graded Euler characteristic is $J_2(K;q)$; explains why coefficients are integers

Khovanov (2000)
Bar-Natan (2002)

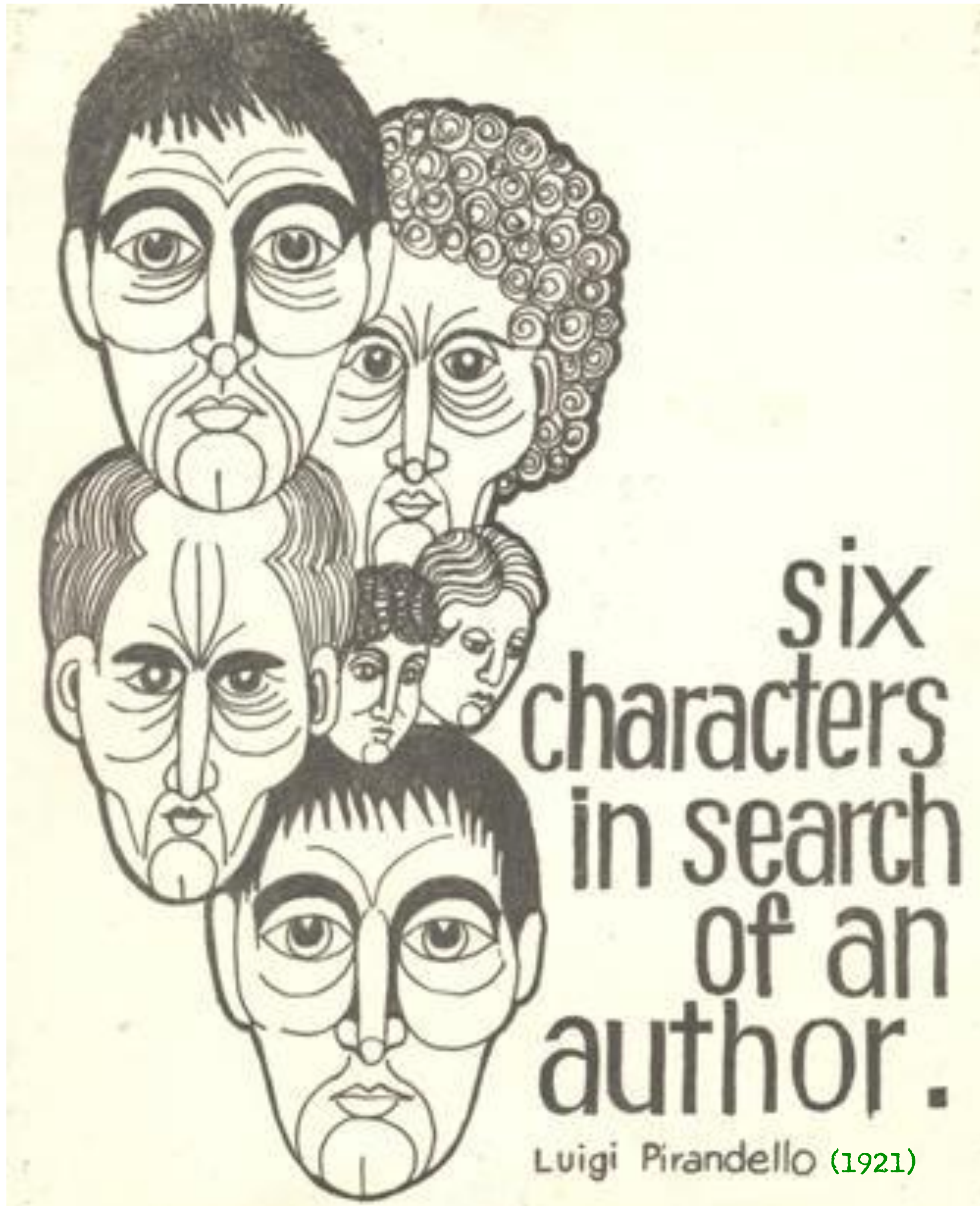
$$\log |J_2(K;-1)| \ , \ \ \log(\mathrm{rank}(\mathcal{H}_K) - 1) \propto \mathrm{Vol}(S^3 \setminus K)$$

Dunfield (2000)
Khovanov (2002)

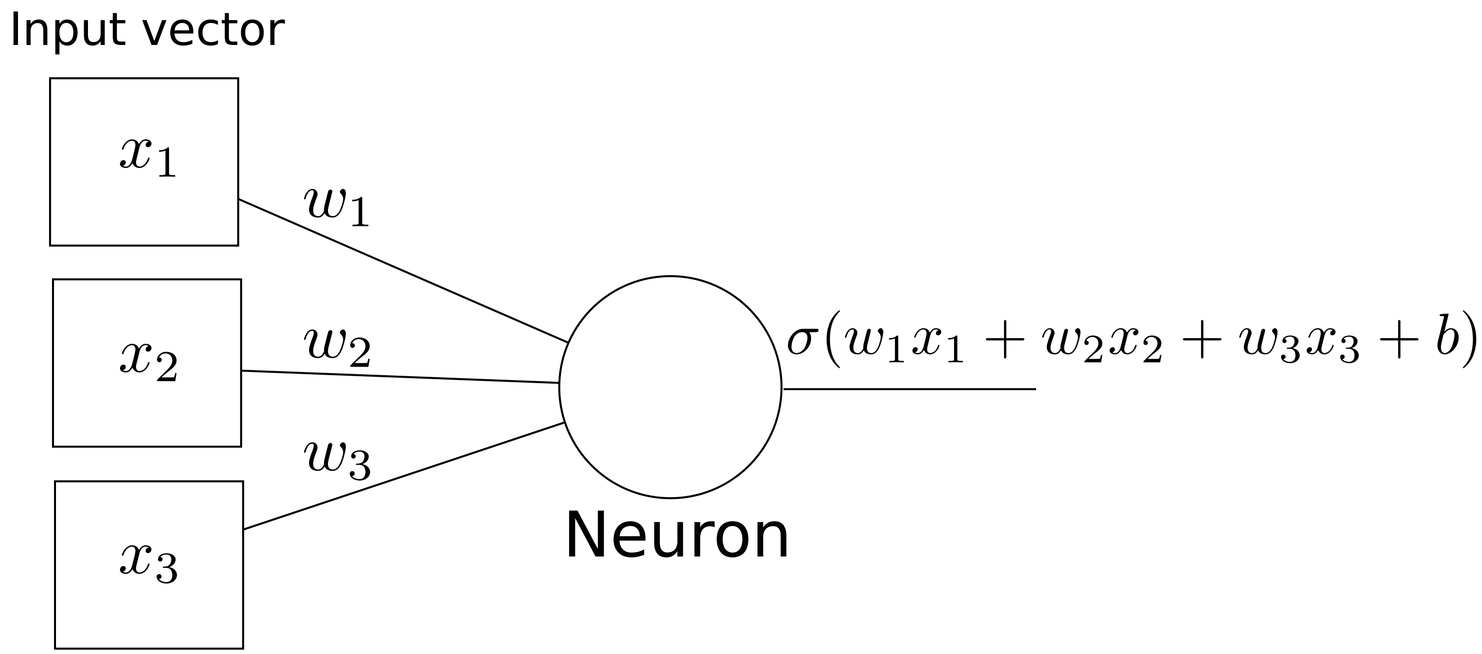




Dramatis Personae



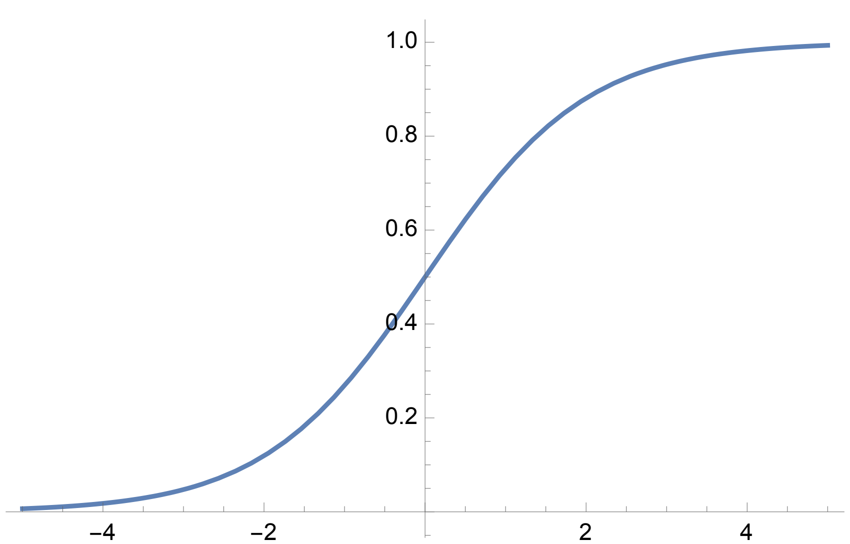
Feedforward Neural Networks



Rosenblatt (1957)

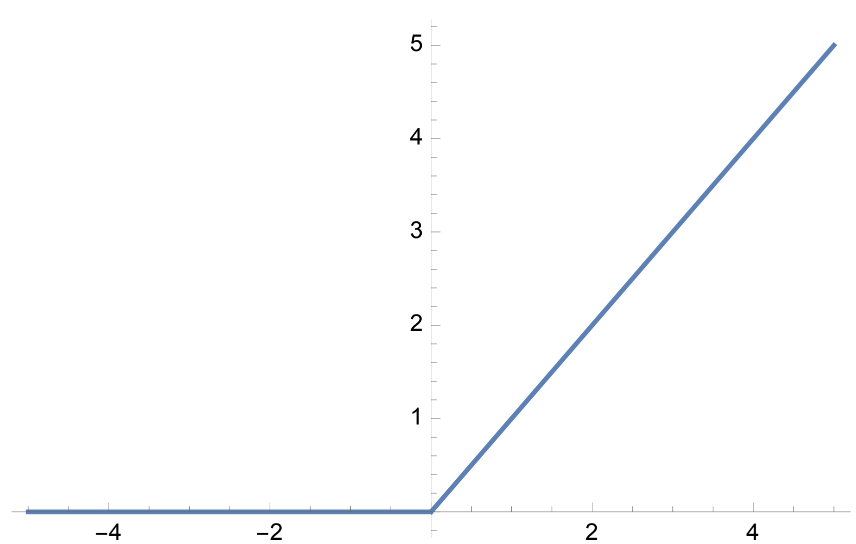
non-linearity

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



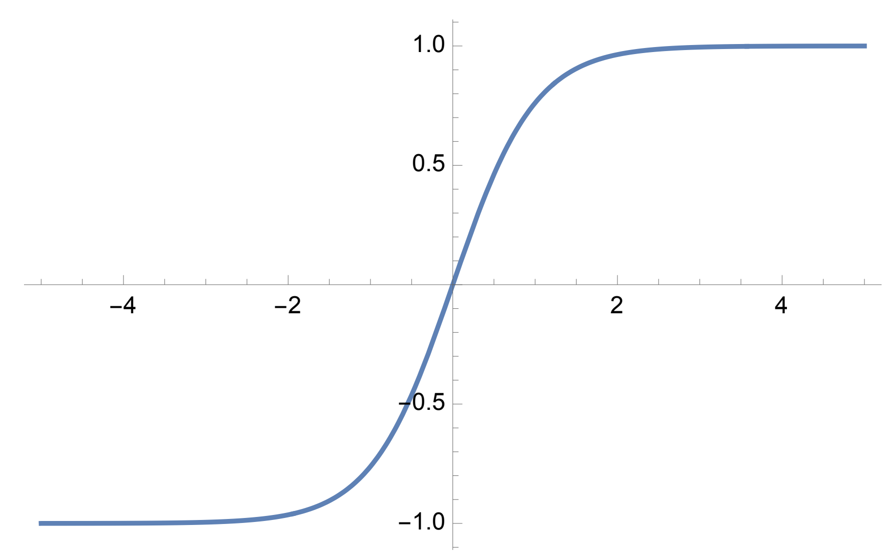
logistic sigmoid

$$\sigma(x) = \max(0, x)$$



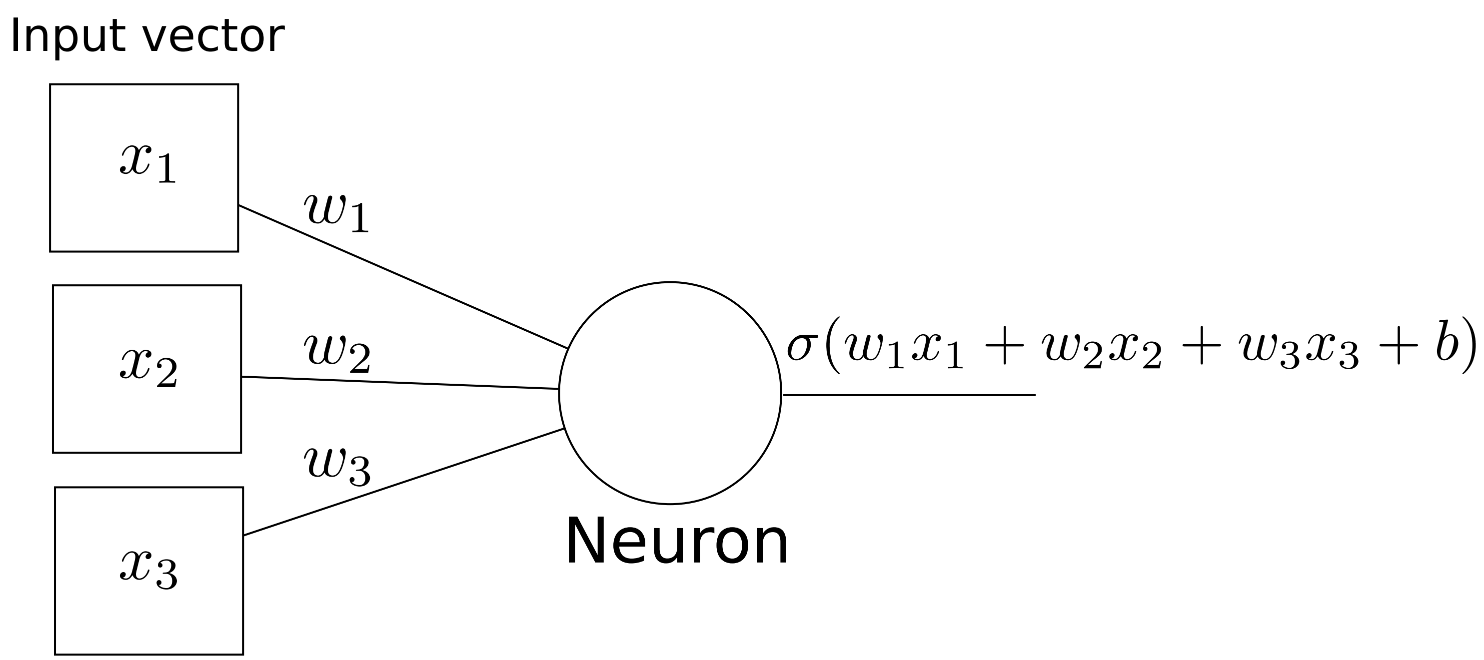
ReLU

$$\sigma(x) = \tanh x$$

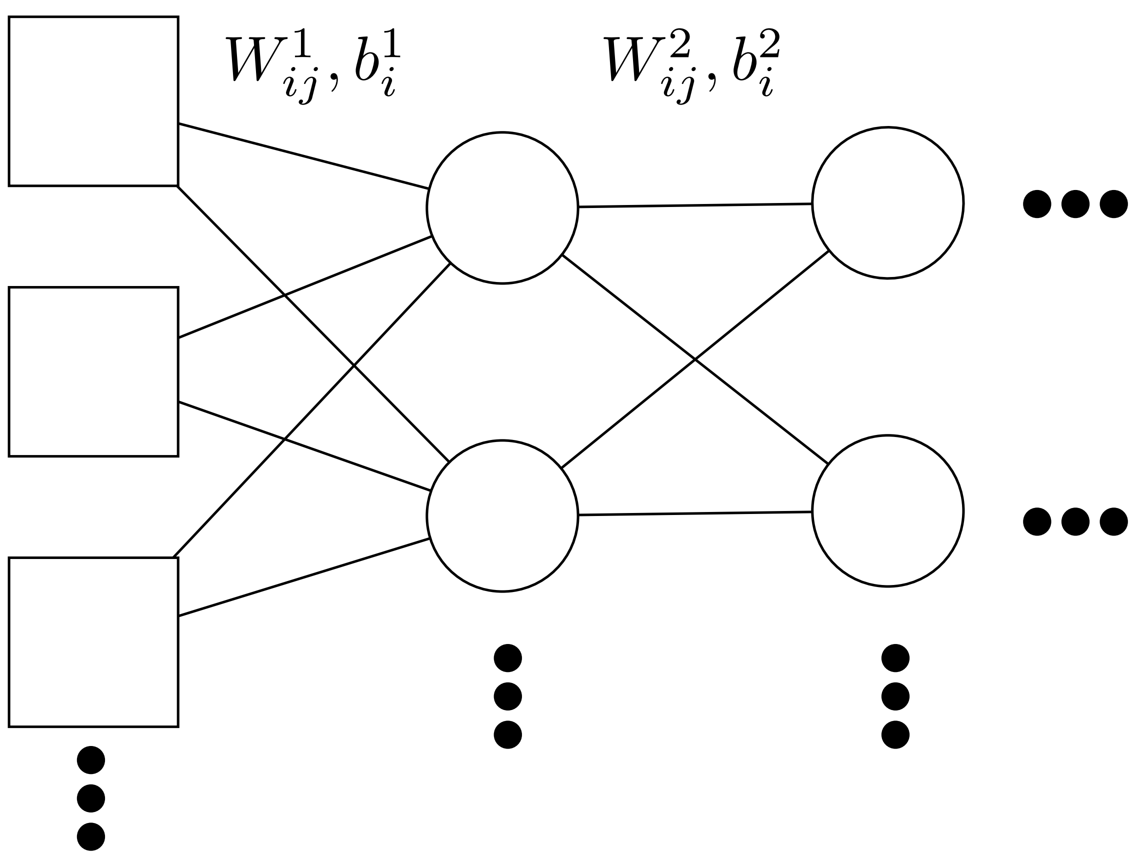


tanh

Feedforward Neural Networks



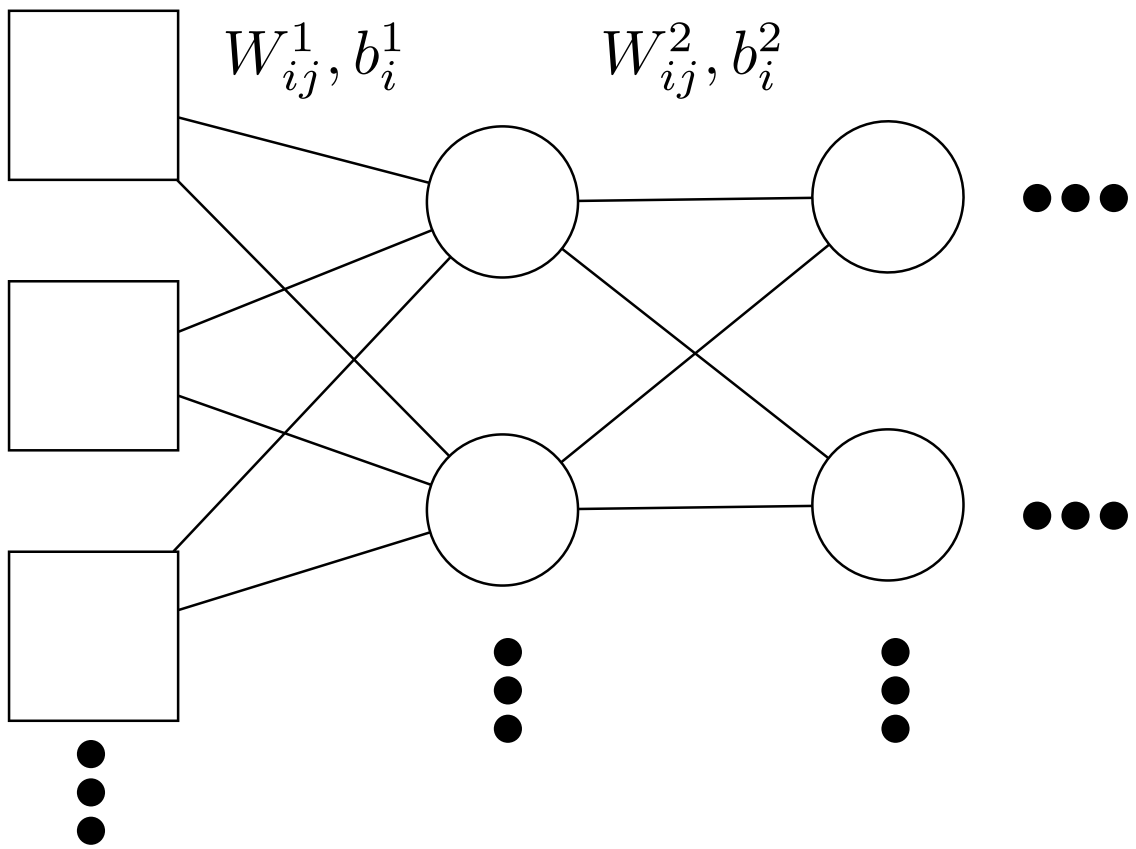
Rosenblatt (1957)



Feedforward Neural Networks

$$f_i : \quad \mathbb{R}^{n_{i-1}} \to \mathbb{R}^{n_i}$$
$$\mathbf{v}_{i-1} \mapsto \mathbf{v}_i = \sigma(\mathbf{v}'_i) \qquad \mathbf{v}'_i = W^i \cdot \mathbf{v}_{i-1} + \mathbf{b}_i$$

$$\mathbf{v}_\ell = f_\ell(f_{\ell-1}(\cdots(f_1(\mathbf{v}_0)\cdots))) \; , \qquad a_{\text{out}} := \sum_{m=1}^{n_\ell} \mathbf{v}_\ell^m$$



Feedforward Neural Networks

$$\begin{array}{llll} f_i : & \mathbb{R}^{n_{i-1}} & \to & \mathbb{R}^{n_i} \\ & \mathbf{v}_{i-1} & \mapsto & \mathbf{v}_i = \sigma(\mathbf{v}'_i) \end{array}$$

number of neurons in i-th layer

$$\mathbf{v}'_i = W^i \cdot \mathbf{v}_{i-1} + \mathbf{b}_i$$

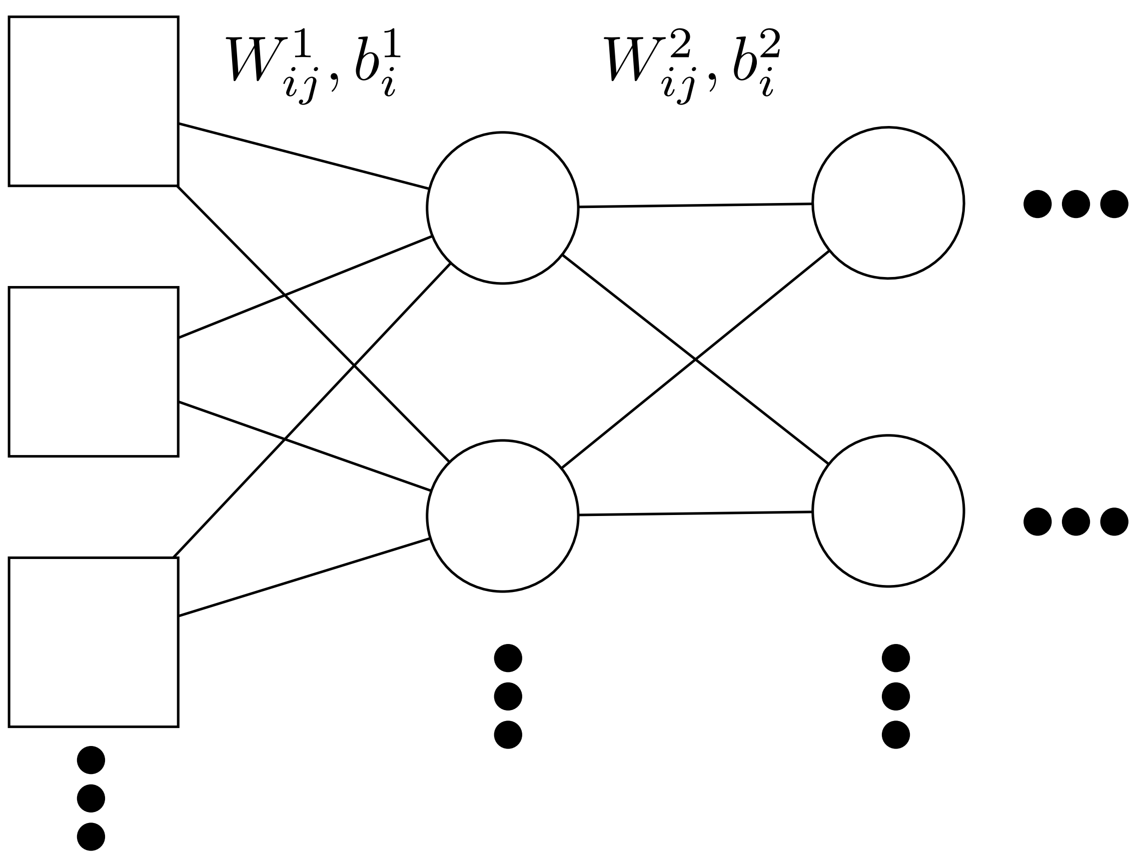
weight matrix

bias vector

$$\mathbf{v}_\ell = f_\ell(f_{\ell-1}(\cdots(f_1(\mathbf{v}_0)\cdots))) \;, \qquad a_{\text{out}} := \sum_{m=1}^{n_\ell} \mathbf{v}_\ell^m$$

number of layers

compare prediction
to ground truth



Feedforward Neural Networks

$$\begin{aligned} f_i : \quad \mathbb{R}^{n_{i-1}} &\to \mathbb{R}^{n_i} \\ \mathbf{v}_{i-1} &\mapsto \mathbf{v}_i = \sigma(\mathbf{v}'_i) \end{aligned}$$

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$$\mathbf{v}'_i = W^i \cdot \mathbf{v}_{i-1} + \mathbf{b}_i$$

weight matrix

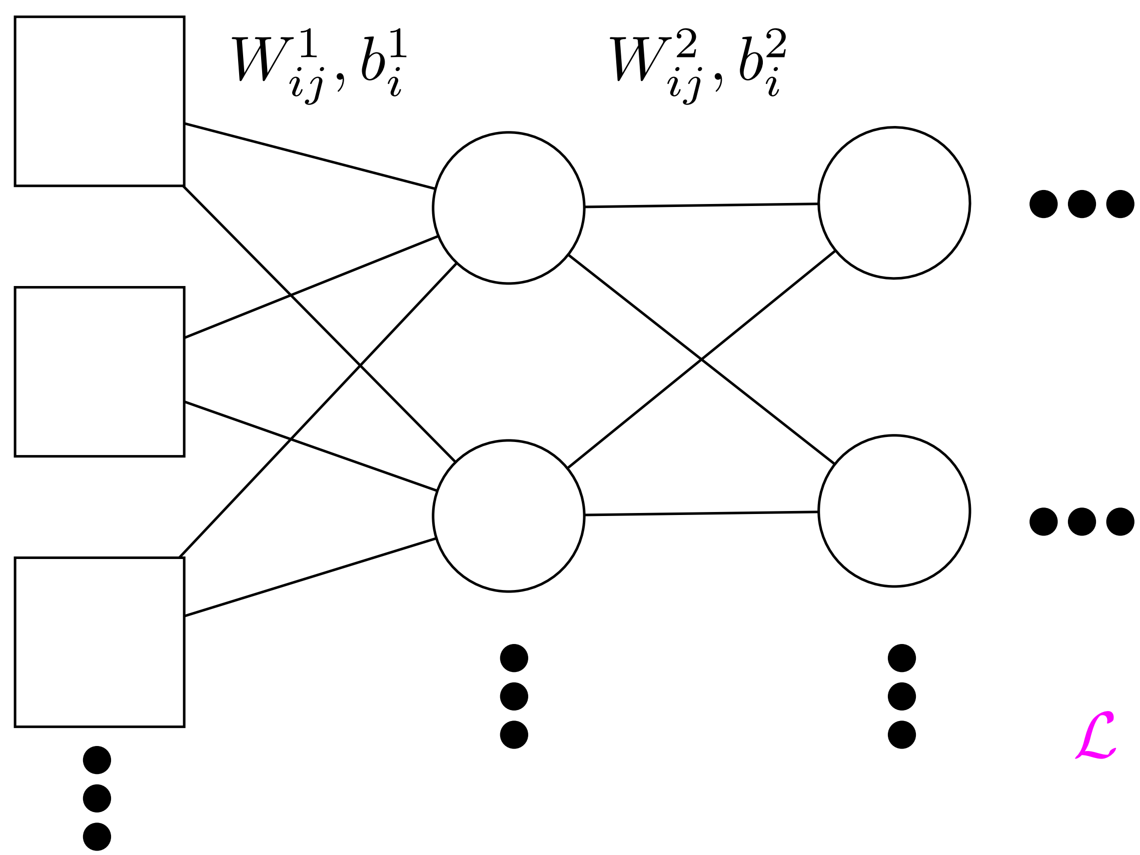
bias vector

$$\mathbf{v}_\ell = f_\ell(f_{\ell-1}(\cdots(f_1(\mathbf{v}_0)\cdots))) \ ,$$

number of layers

$$a_{\text{out}} := \sum_{m=1}^{n_\ell} \mathbf{v}_\ell^m$$

compare prediction
to ground truth

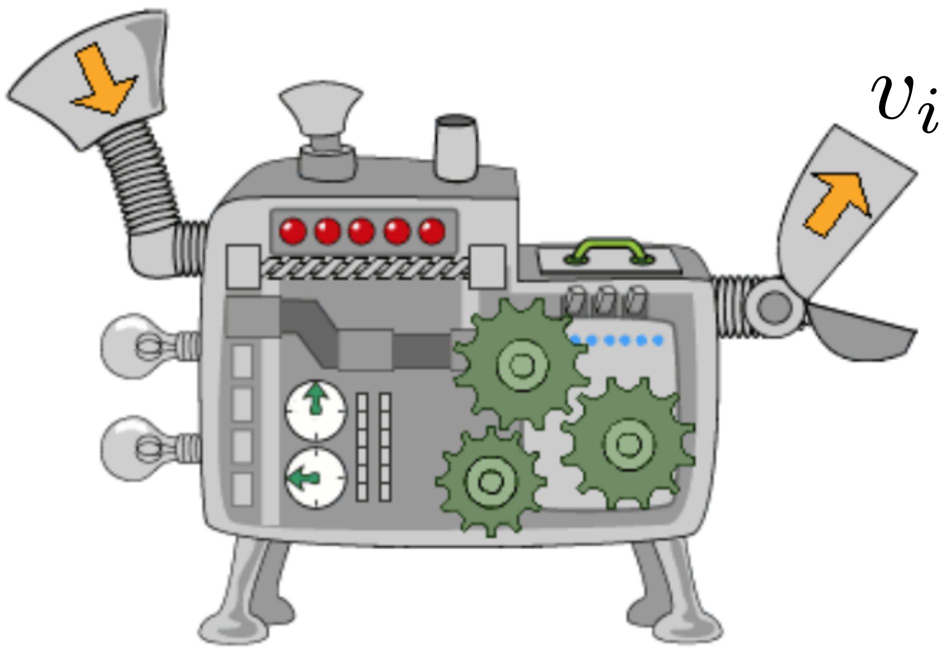


tune weights and biases
to minimize **loss function**
during **training**

$$\mathcal{L} = \sum_{\text{batch}} |\text{predicted} - \text{actual}|^2$$

Neural Network

J_i



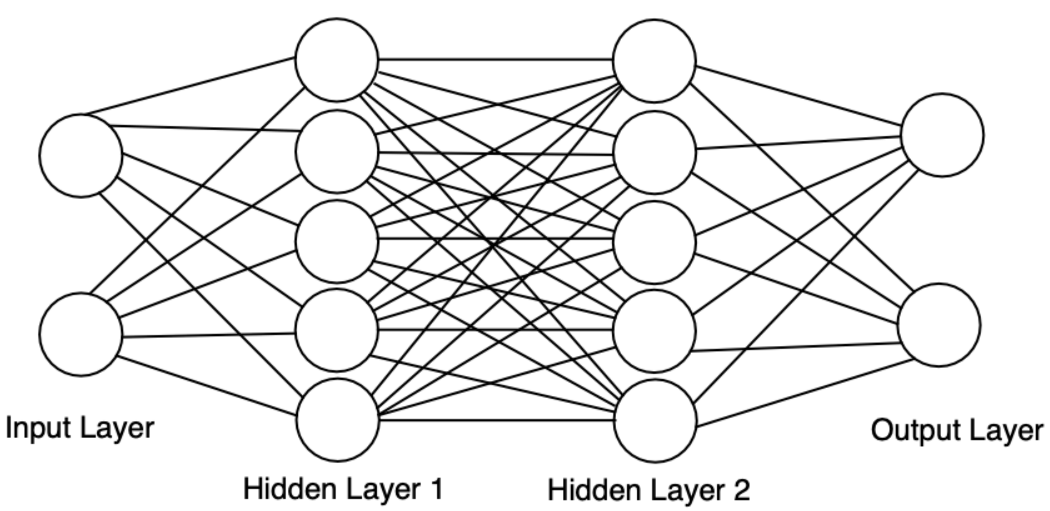
v_i

$$\{J_1, \ldots, J_n\} \longrightarrow \{v_1, \ldots, v_n\}$$

$$J_i \in T$$

$$\{J'_1, \ldots, J'_m\} \longrightarrow ???$$

$$J'_i \in T^c$$



$\vec{J}$ $\quad 100 \times 18 \quad 100 \times 100 \quad \sum_{a=1}^{100}$

12000 hyperparameters

Jones polynomials are represented as 18-vectors

$$\vec{J}_K = (\text{min}, \text{max}, \text{coeffs}, 0, \ldots, 0)$$

Two layer neural network in Mathematica

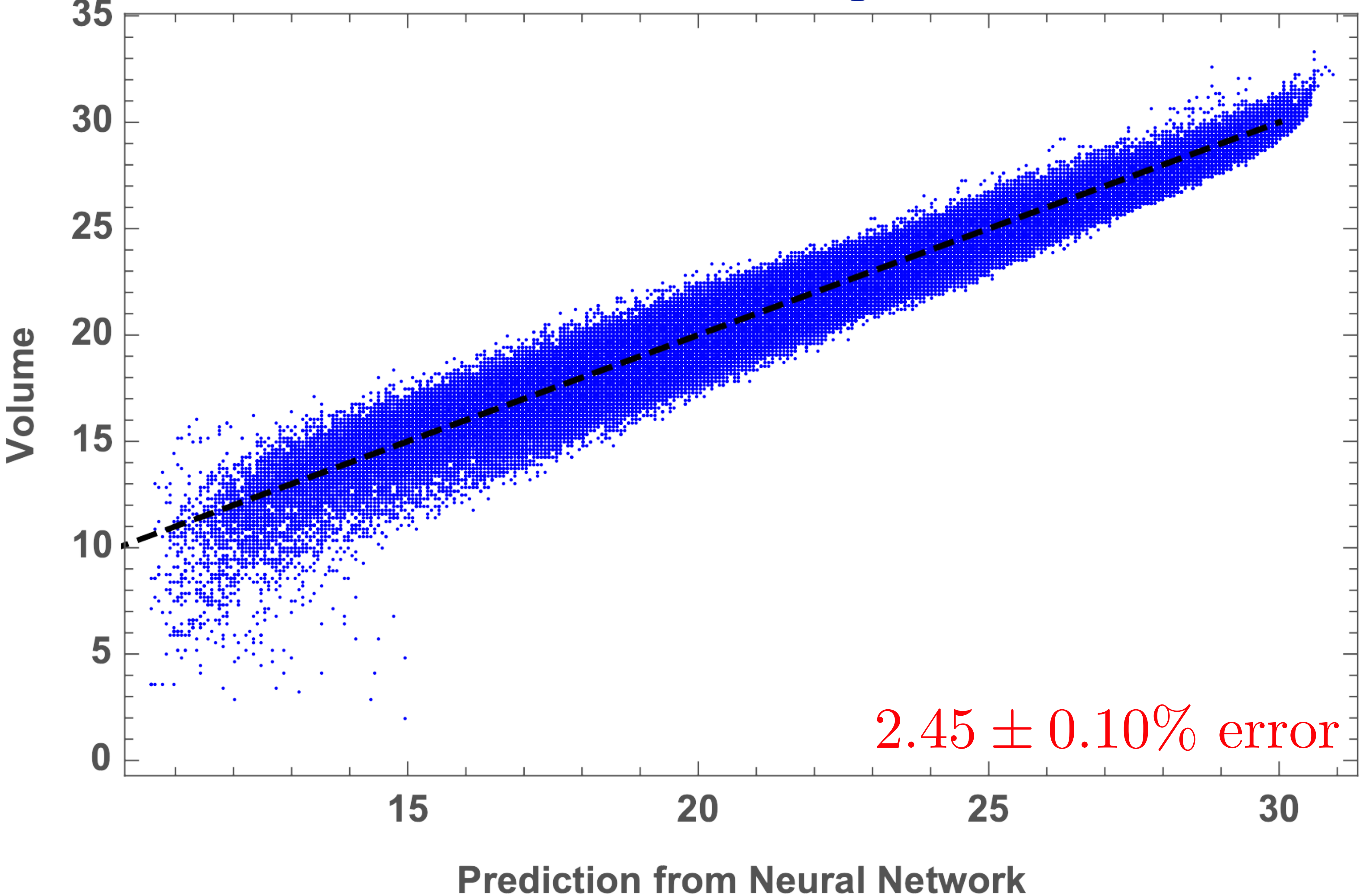
$$f_\theta(\vec{J}_K) = \sum_a \sigma\left(W^2_\theta \cdot \sigma(W^1_\theta \cdot \vec{J}_K + \vec{b}^1_\theta) + \vec{b}^2_\theta\right)^a$$

Logistic sigmoids for the hidden layers

Mean squared loss function

VJ, Kar, Parrikar (2019)

Volume from Jones



trained on 10% of the $313,209$ knots up to 15 crossings

VJ, Kar, Parrikar (2019)

Mathematica Code

knot run.nb

```
In[•]:= f[frac_] :=
  Module[{y, list, listcomp, joneslist, vollist, joneslistcomp, vollistcomp,
    test, aa, KnotNet0, TrainedKnotNet0},
   y = frac;
   list = Sort[RandomSample[Range[1, len], IntegerPart[len y]]];
   listcomp = Complement[Table[i, {i, 1, len}], list];
   joneslist = Table[joneses[[list[[i]]]], {i, 1, Length[list]}];
   vollist = Table[volumes[[list[[i]]]], {i, 1, Length[list]}];
   joneslistcomp = Table[joneses[[listcomp[[i]]]], {i, 1, Length[listcomp]}];
   vollistcomp = Table[volumes[[listcomp[[i]]]], {i, 1, Length[listcomp]}];
   KnotNet0 = NetChain[{LinearLayer[100], ElementwiseLayer[LogisticSigmoid],
      LinearLayer[100], ElementwiseLayer[LogisticSigmoid], SummationLayer[]},
     "Input" → {maxlen}];
   TrainedKnotNet0 = NetTrain[KnotNet0, joneslist → vollist] ;
   test = TrainedKnotNet0[joneses];
   Show[ListPlot[Table[{test[[i]], volumes[[i]]}, {i, 1, len}],
     AxesLabel → {"Predicted Volume", "Volume"}],
    Plot[x, {x, 0, 40}, PlotRange → Full, PlotStyle → {Black, Dashed}]]
  ]
f[.1]
```

Result

$$v_i = f(J_i) + \text{small corrections}$$

- J_i does not uniquely identify a knot

  e.g., 4_1 and K11n19 have same Jones polynomial, different volumes

- $174,619$ unique Jones polynomials

  2.83% average spread in volumes for a Jones polynomial

  intrinsic mitigation against overfitting

- Same applies to 1,701,903 hyperbolic knots up to 16 crossings

  841,139 unique polynomials

  (database compiled from **Knot Atlas** and `SnapPy`)

VJ, Kar, Parrikar (2019)

Result

$$v_i = f(J_i) + \text{small corrections}$$

- **Universal Approximation Theorem:** feedforward neural network, sigmoid activation function, single hidden layer with finite number of neurons can approximate continuous functions on compact subsets of $\mathbb{R}^n$

Cybenko (1989)
Hornik (1991)
cf. Kidger, Lyona (2019)

- Surprise here is simplicity of architecture that does the job

- We want a **not** machine learning knot result

Entr'acte

$$v_i = f(J_i) + \text{small corrections}$$

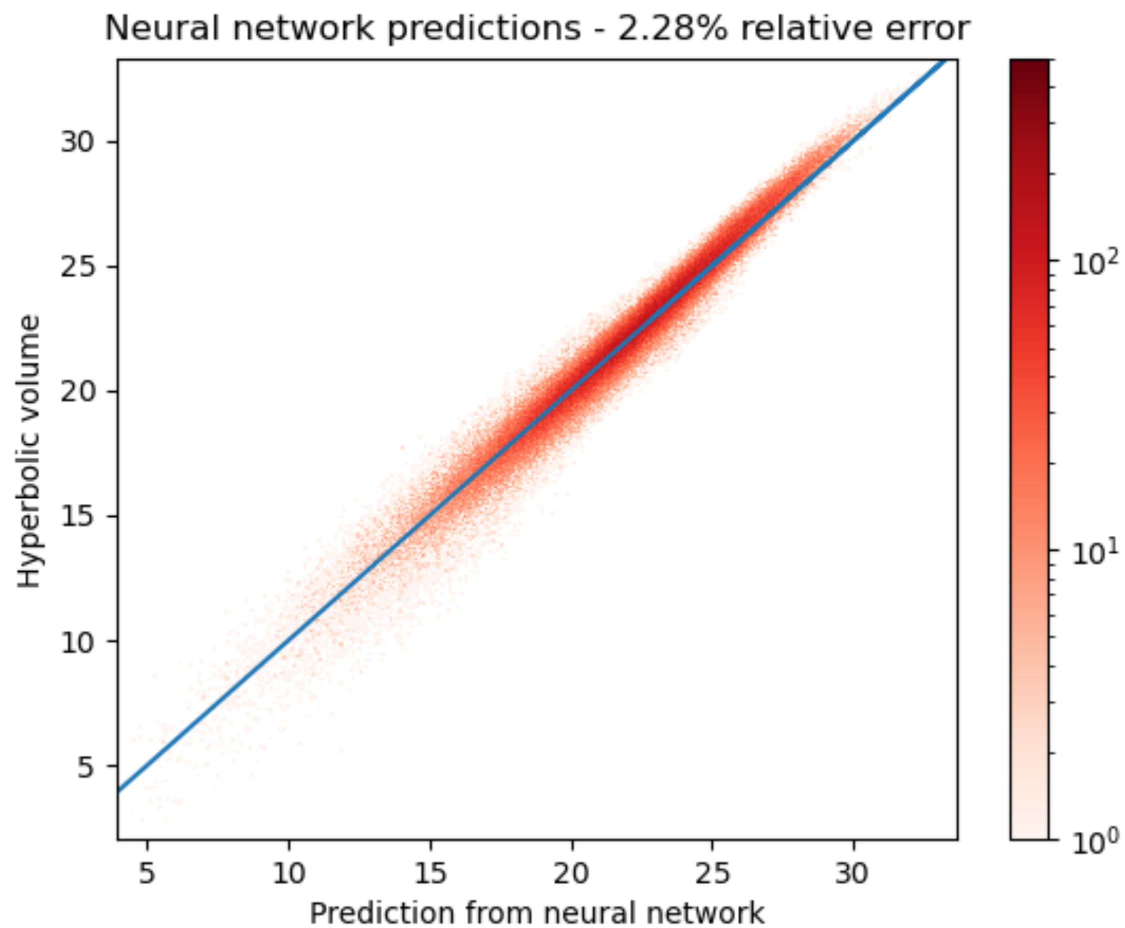
We seek to reverse engineer the neural network to obtain an analytic expression for the volume as a function of the Jones polynomial

To interpret the formula, we use machinery of analytically continued Chern–Simons theory

No Degrees Needed

- Suppose we drop the degrees and provide only the coefficients; Jones polynomial is no longer recoverable from the input vector

- Results are unchanged!

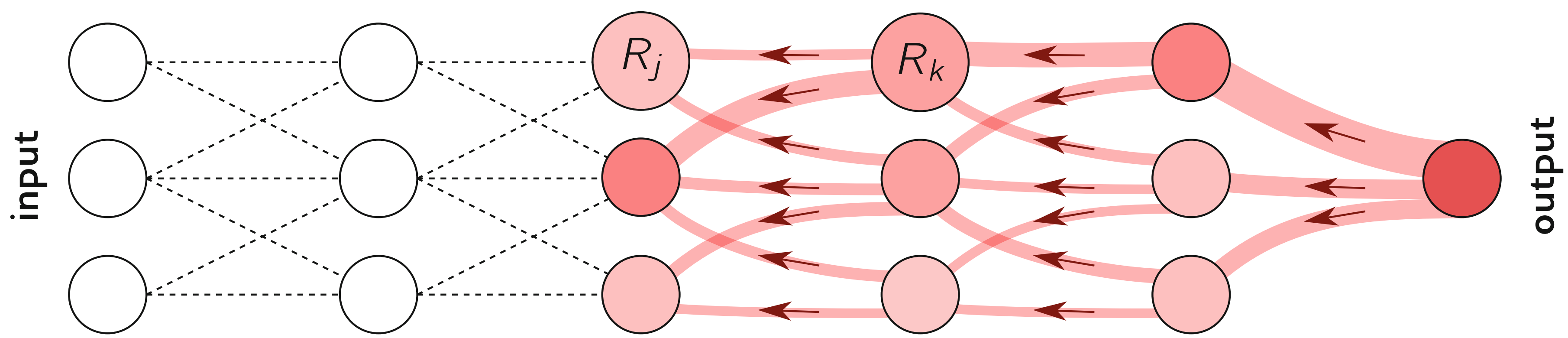


__N.B.__: we have switched to `Python 3` using `GPU-Tensorflow` with `Keras` wrapper
two hidden layers, 100 neurons/layer, ReLu activation, mean squared loss, `Adam` optimizer

Craven, VJ, Kar (2020)

Layer-wise Relevance Propagation

- To determine which inputs carry the most weight, propagate backward starting from output layer employing a conservation property



Montavon et al. (2019)

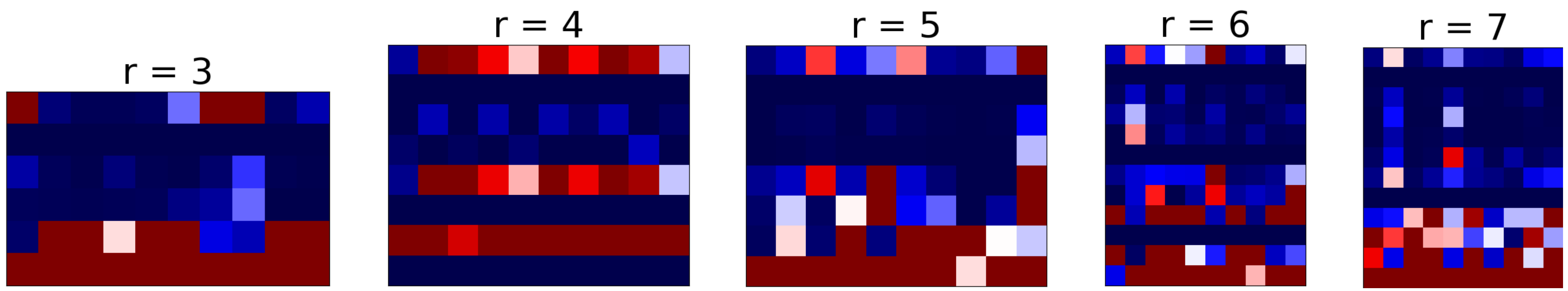
- Compute relevance score for a neuron using activations, weights, and biases

$$R_j^{m-1} = \sum_k \frac{a_j^{m-1} W_{jk}^m + N_{m-1}^{-1} b_k^m}{\sum_l a_l^{m-1} W_{lk}^m + b_k^m} R_k^m \quad , \quad \sum_k R_k^m = 1$$

j^{th} neuron in layer $m-1$

Craven, VJ, Kar (2020)

Layer-wise Relevance Propagation



- Each column is a single input corresponding to evaluations of the Jones polynomial at phases $e^{\frac{2\pi i p}{r+2}}$, $0 \leq 2p \leq r+2$, $p \in \mathbb{Z}$

- Ten different knots

- We show the relevances (red is most relevant) and notice that the same input features light up

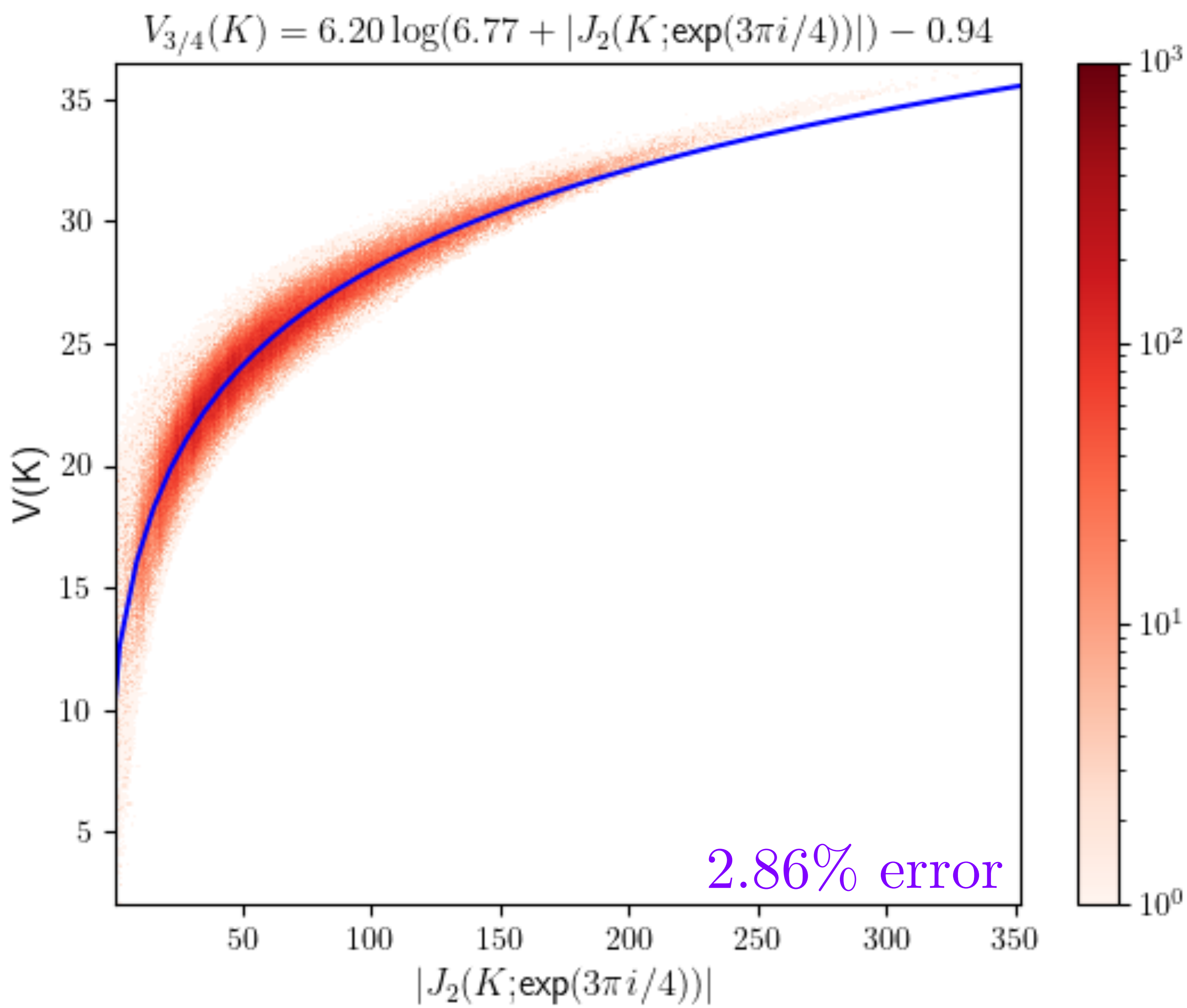
Craven, VJ, Kar (2020)

Phenomenological Function

$$\omega = e^{\frac{3\pi i}{4}}$$

$$\Updownarrow$$

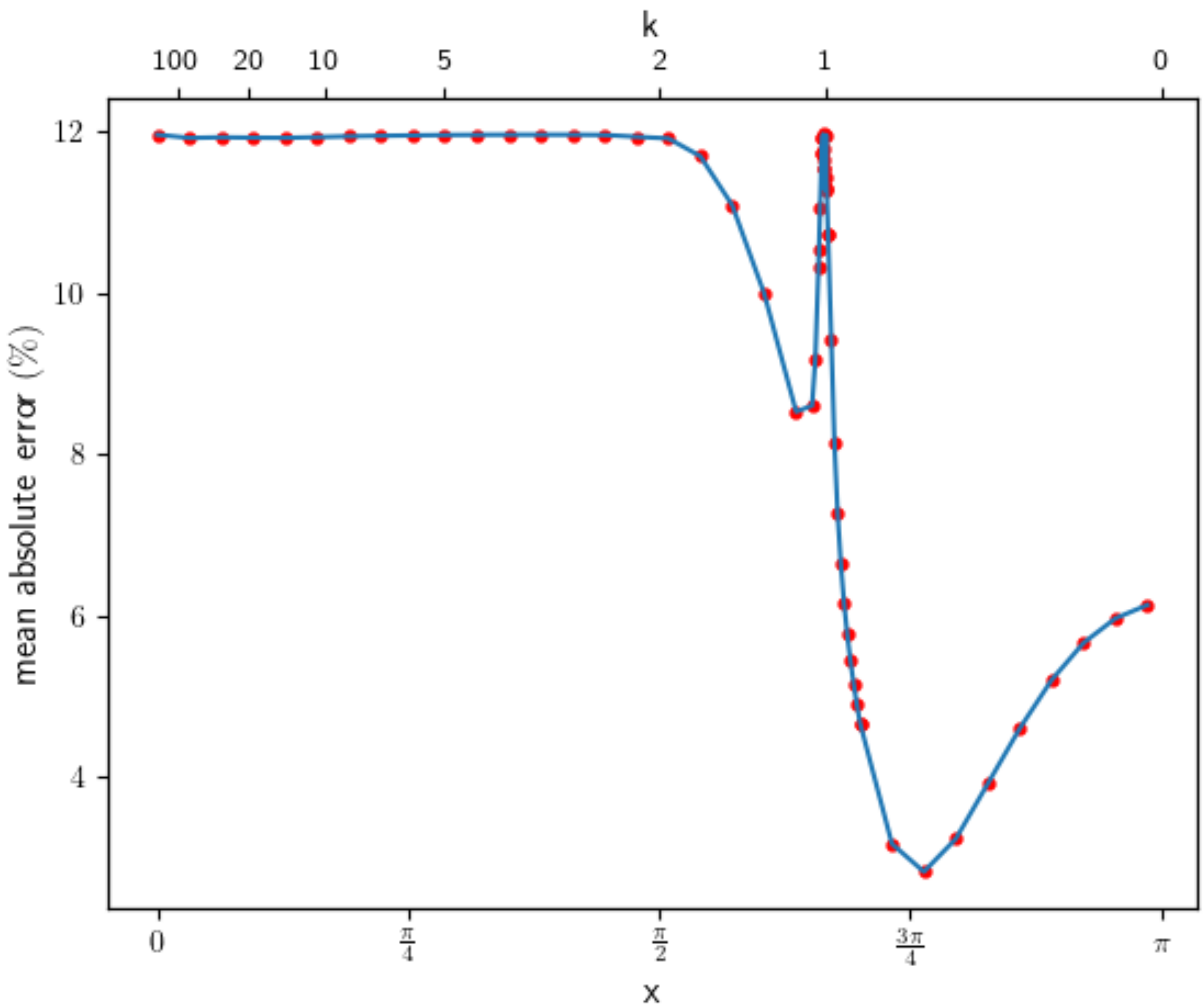
$$k = \frac{2}{3}$$



- 1,701,903 knots up to 16 crossings
- Behavior explainable in Witten's analytic continuation of Chern–Simons theory

Craven, VJ, Kar (2020)

The Shape of Things



Craven, VJ, Kar (2020)

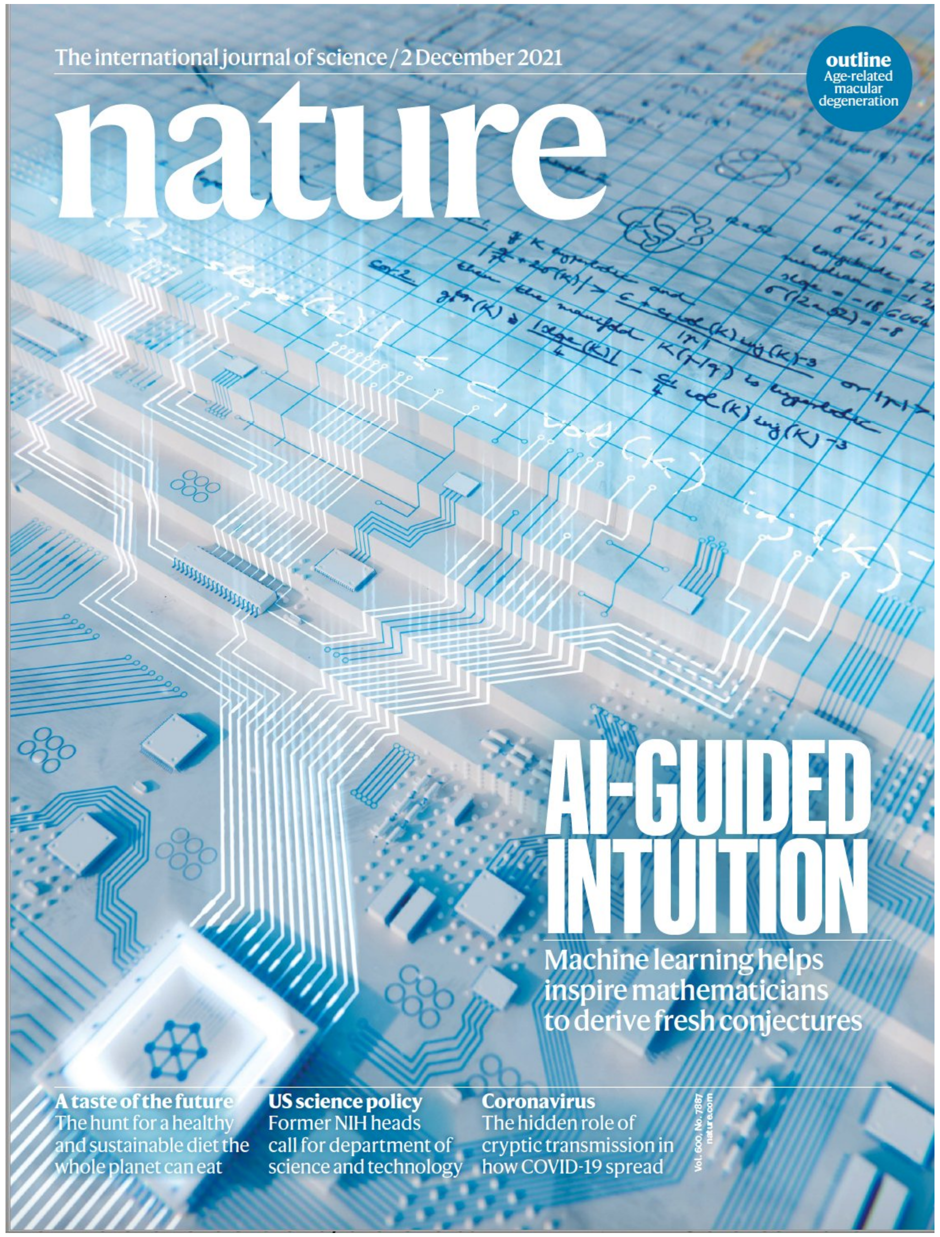
Explanation

The approximation formula works well for levels k for which $\mathcal{A}_+$ makes a contribution to the Chern–Simons path integral, and its accuracy increases with fraction of knots in dataset that receive such a contribution

$$Z \sim e^{iS(\mathcal{A}_+)}\left(1 - e^{2\pi i k}\right)$$

Craven, VJ, Kar (2020)

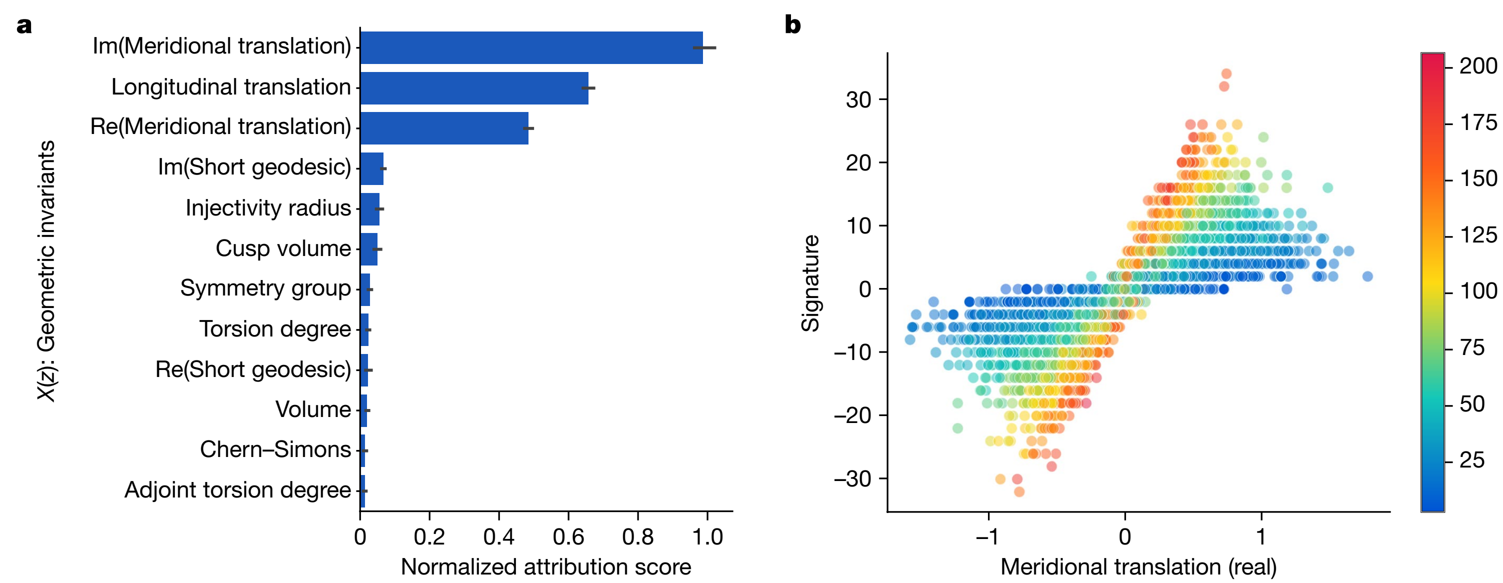
Other Experiments



- Obtain a theorem in knot theory from saliency analysis of ML experiments

Theorem: There exists a constant c such that for every hyperbolic knot,

$$|2\sigma(K) - \mathrm{slope}(K)| \leq c\,\mathrm{Vol}(K)\,\mathrm{inj}(K)^{-3}$$



Davies, et al. (2021)

- See also: Hughes (2016)

  Levit, Hajij, Sazdanovic (2019)

  Gukov, Halverson, Ruehle, Sulkowski (2020)

  Craven, Hughes, VJ, Kar (2021, 2022)

  Gukov, Halverson, Manolescu, Ruehle (2023)

Poincaré Conjecture

- A homotopy *n*-sphere has the same homotopy and homology groups as S^n

- **Conjecture:** Any homotopy *n*-sphere is homeomorphic, PL-isomorphic, diffeomorphic to S^n

  — homeomorphic is always true

  — piecewise linear isomorphic is true for $n \neq 4$

  — $n = 1$ is trivial

  — $n = 2$ proved by Poincaré (1907) and Koebe (1907)

  — $n \geq 5$ wins Fields Medal for Smale (1966); *cf.* Newman (1966)

  — $n = 4$ wins Fields Medal for Freedman (1986)

  — $n = 3$ wins Fields Medal for Perelman (2006, declined)

SPC4

- **Conjecture:** Any manifold homotopy equivalent to S^n is also diffeomorphic to S^n

isomorphism of smooth manifolds, *i.e.*, f, f^{-1} continuously differentiable

- False generally, but true if $n = 1, 2, 3, 5, 6, 12, 56, 61$

- $n = 7$: 28 different smooth structures on the sphere — Milnor (1956)

Remark: These are interpreted as gravitational instantons — Witten (1985)

dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
# structures	1	1	1	?	1	1	28	2	8	6	992	1	3	2	16256	2	16	16

Observation: spikes at $n = 3 \bmod 4$

Kervaire, Milnor (1963)

SPC4

- **Conjecture:** Any manifold homotopy equivalent to S^n is also diffeomorphic to S^n

  isomorphism of smooth manifolds, *i.e.*, f, f^{-1} continuously differentiable

- False generally, but true if $n = 1, 2, 3, 5, 6, 12, 56, 61$

- $n = 7$: 28 different smooth structures on the sphere Milnor (1956)

  Remark: These are interpreted as gravitational instantons Witten (1985)

- $n = 4$: answer unknown; equivalent to PL-isomorphic statement
  [also unknown for $n = 126$]

- If certain knots have certain topological invariants, this supplies counterexamples to SPC4

$$S^3 = \partial B_4 \ , \quad K = \partial \Sigma$$

$$K \subset S^3 \ , \quad \Sigma \subset B_4$$

SPC4: Any smooth manifold B with $\partial B = S^3$ homotopy equivalent to B_4 is also diffeomorphic to B_4

CALABI-YAU MANIFOLDS

The Forces of Nature

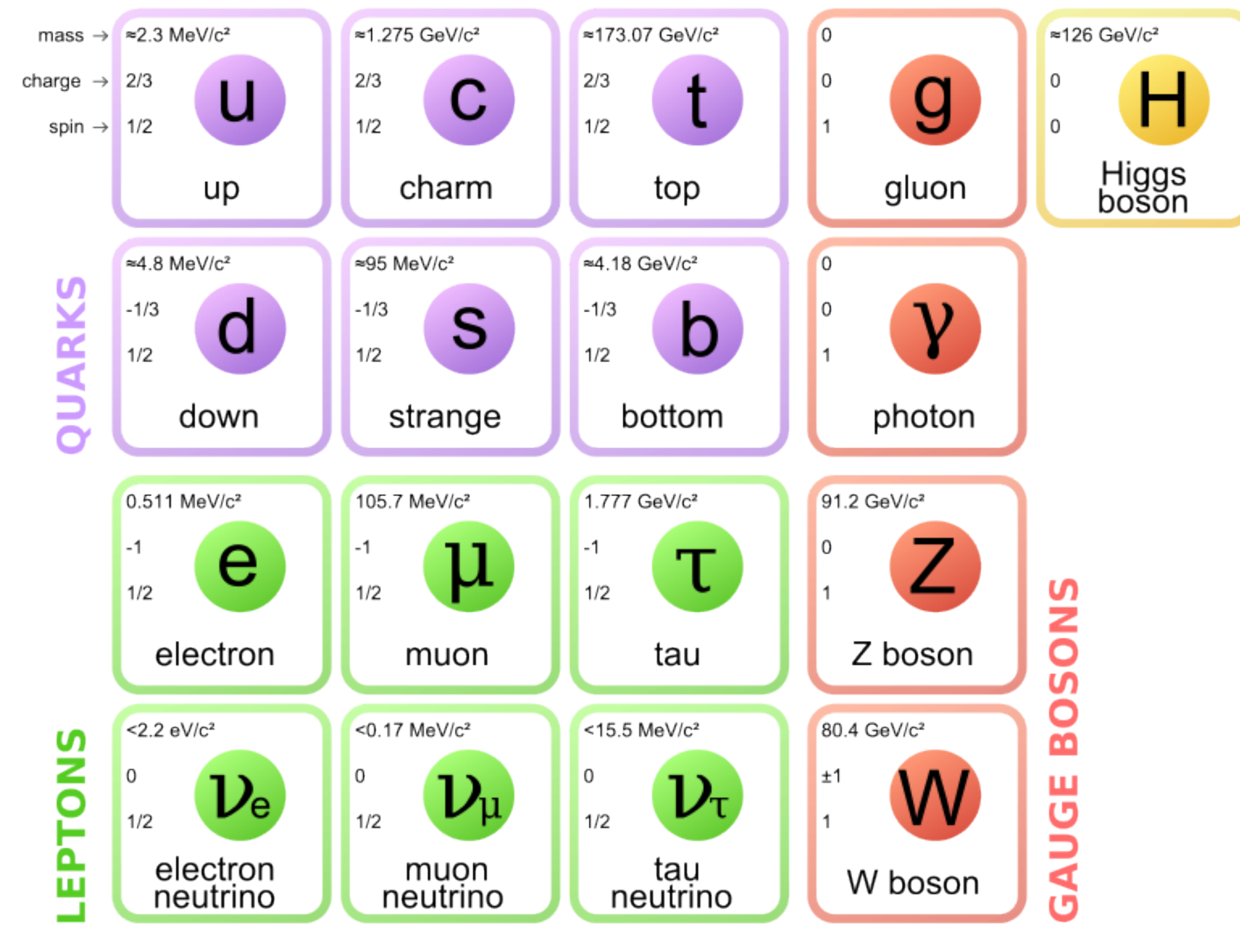
- General relativity (gravity) is geometry of spacetime in presence of energy

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu}$$

- Standard Model of particle physics

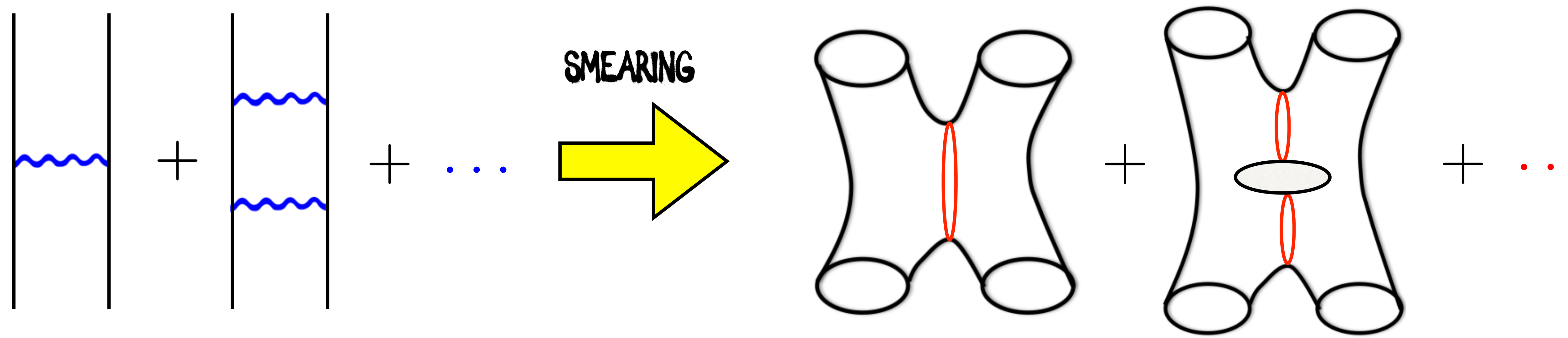
electromagnetism
weak force
strong force
Higgs effect

$$\alpha_{\rm exp}^{-1} = 137.035999139(31)$$
$$\alpha_{\rm th}^{-1} = 137.035999173(35)$$



String Theory

Gravity as a QFT | **Gravity from String Theory**



$d = 3 + 1: \ \infty$ | $d = 9 + 1: \ \text{finite}$

- Map from string worldsheet to target space (spacetime)
  $X^\mu: \ \Sigma \to \mathcal{M}$

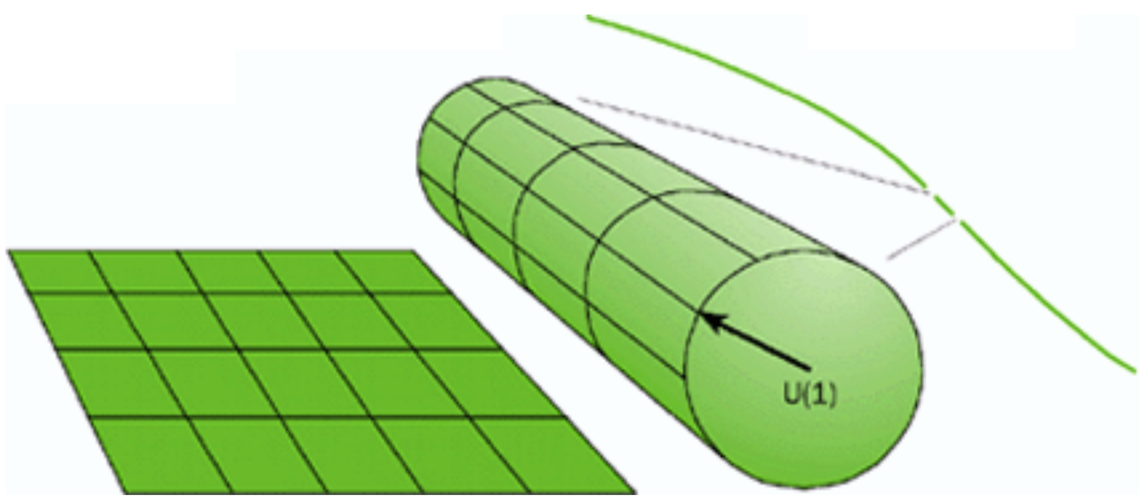
- X^μ are fields on string worldsheet; consistency of this QFT gives Einstein equations in ten dimensional spacetime

Compactification

- Non-gravitational interactions are not encoded as geometry

  Theorem [Coleman–Mandula]**:** symmetry group in 4 dimensions is Poincaré x internal

- **Clever loophole:** internal symmetries may arise from higher dimensional geometry



$\mathbb{R}^{1,3} \times S^1$

Kaluza–Klein: 5d Einstein equations give 4d Einstein + Maxwell equations

$S^1 \longleftrightarrow U(1)$

Geometric Engineering

12 dimensions F-theory

T^2

10 dimensions Heterotic $E_8 \times E_8$ Type IIB

$\vdots$

CY_3 CY_4

4 dimensions 4d $\mathcal{N} = 1$ effective field theory

e.g., minimal supersymmetric Standard Model

The Unreal World

- The objective is to obtain the real world from a string compactification

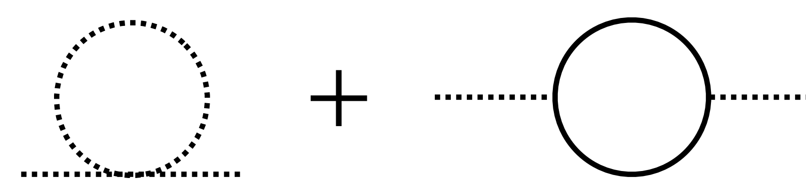
- We would happily settle for a modestly unreal world

$$\boxed{\mathcal{N} = 1 \text{ supersymmetry in 4 dimensions}}$$

No experimental evidence so far!

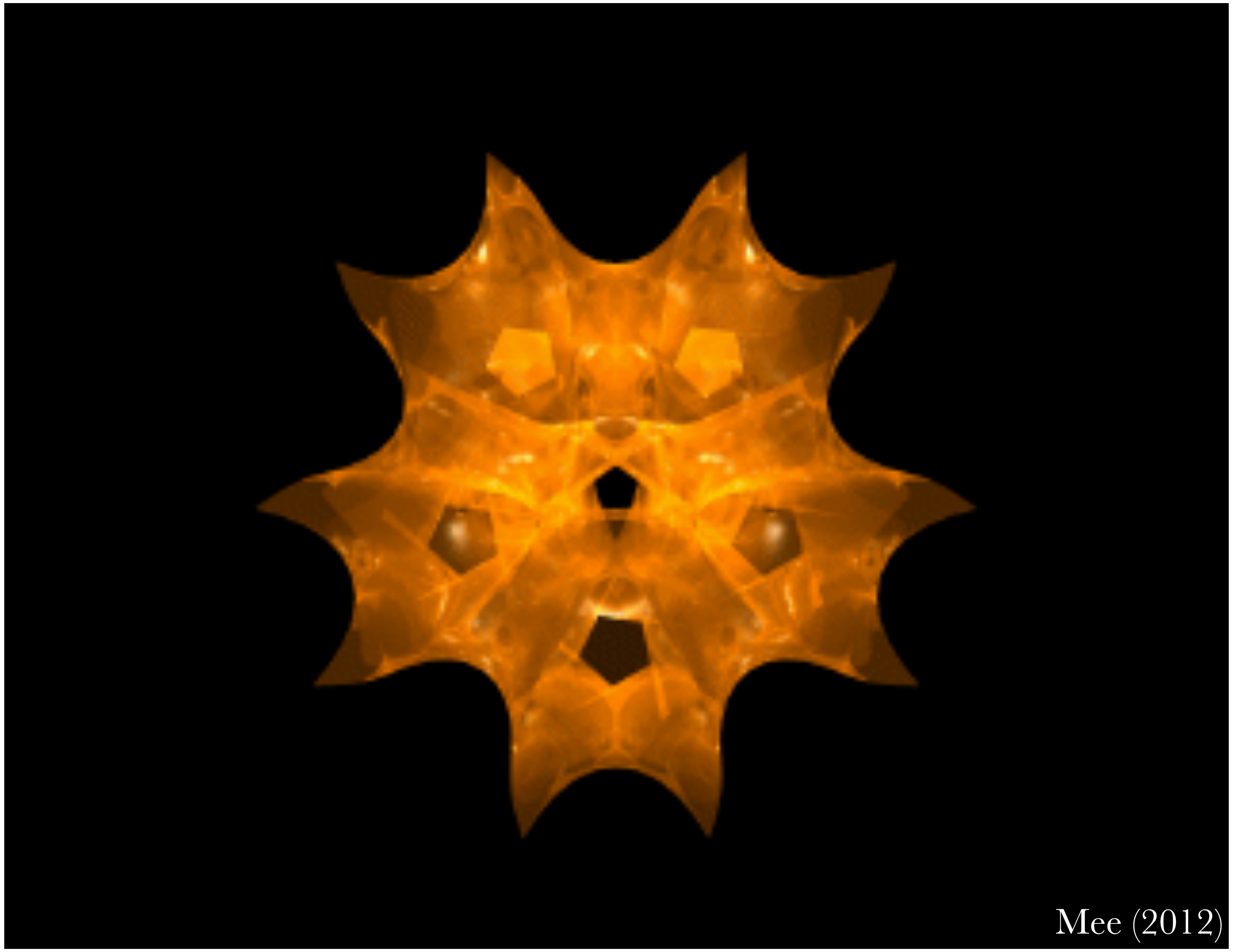
$$Q|\lambda\rangle \sim |\lambda \pm \frac{1}{2}\rangle$$

$$|\text{boson}\rangle \longleftrightarrow |\text{fermion}\rangle$$



$$m_H \ll m_P$$

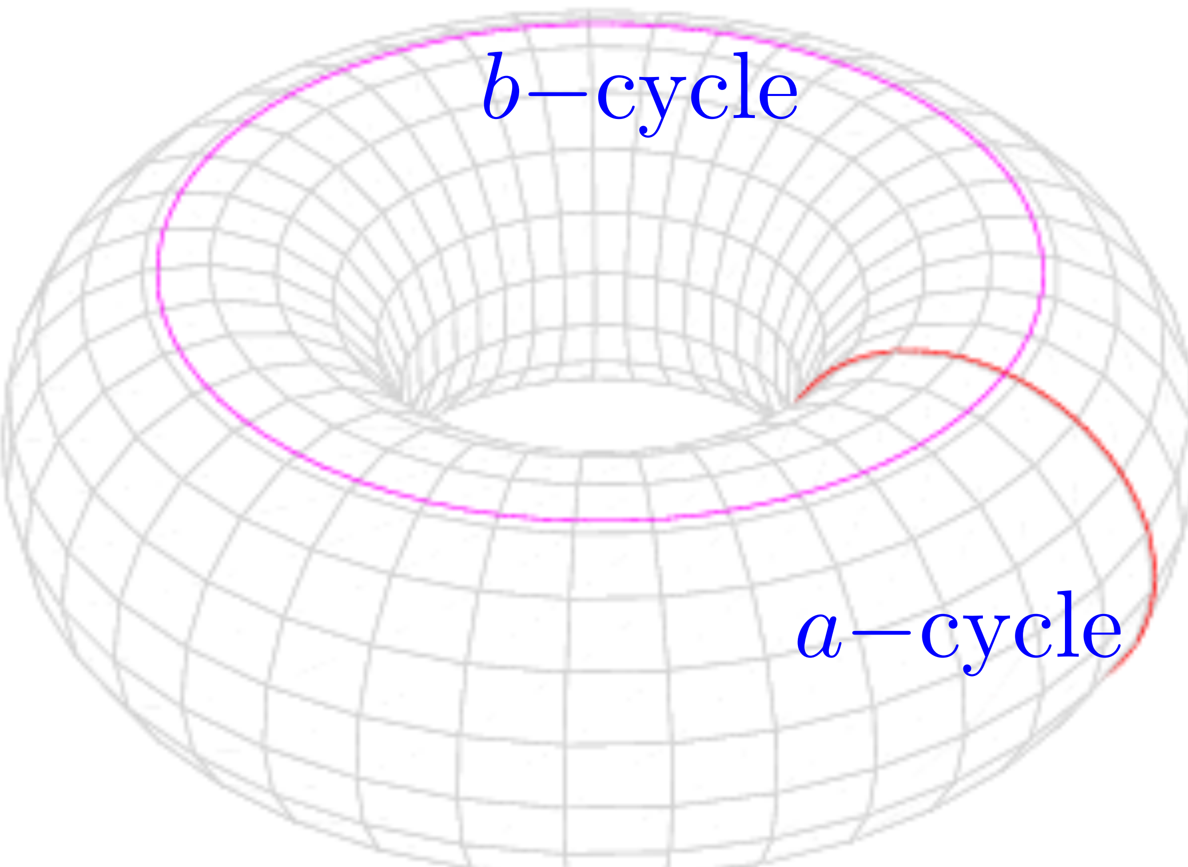
Calabi–Yau



$u^5 + v^5 + x^5 + y^5 + z^5 = 0 \subset \mathbb{P}^4$

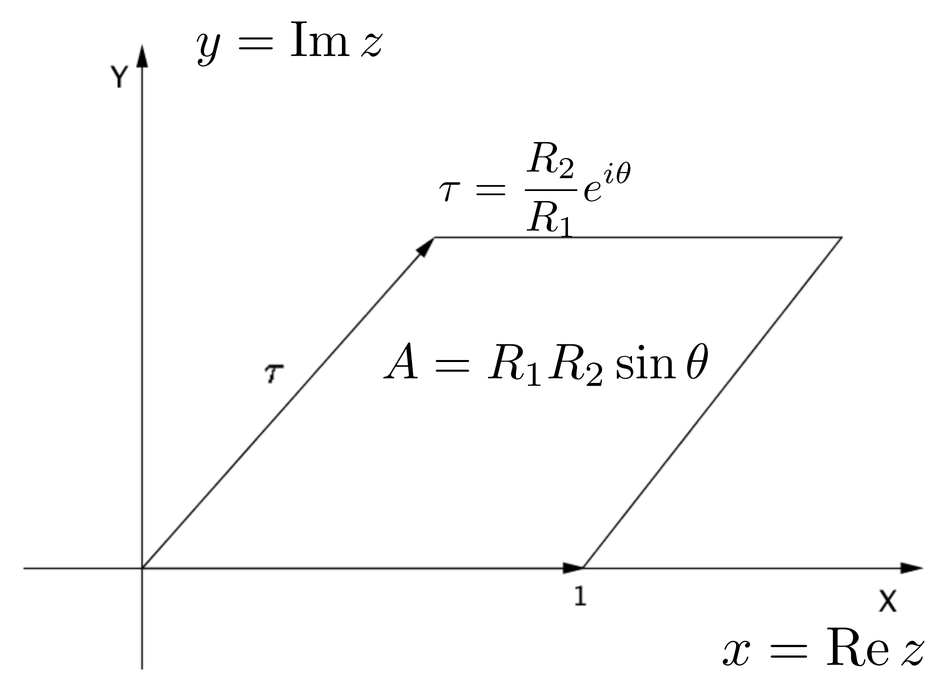
- Complex, Kähler manifold that admits Ricci-flat metric
- Numerical approximations of metric using neural networks

Torus



Flat, but has non-trivial homotopy

There are non-contractible cycles



Kähler parameter: area A **size**

complex structure parameter: τ **shape**

1-fold: $z_0^3 + z_1^3 + z_2^3 = 0 \subset \mathbb{P}^2$

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

$\chi = 2 - 2g = 0$

Calabi–Yau n-folds

2-folds (K3):

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\qquad \chi = 24
$$

3-folds:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad \chi = 2(h^{1,1} - h^{2,1})
$$

2-cycles, 4-cycles

3-cycles

- Learn topological invariants of 7890 Calabi–Yau threefolds realized as complete intersections of homogeneous polynomials in projective space

Bull, He, VJ, Mishra (2018, 2019)

Toric CYs

- Starting from a reflexive polytope, one can realize a Calabi–Yau geometry as a smooth hypersurface in a toric variety

Batyrev (1993)
Batyrev, Borisov (1994)

- 4319 3d polytopes give K3; 473,800,776 4d polytopes yield toric Calabi–Yau threefolds with 30,108 unique pairs of Hodge numbers; more than 185,269,499,015 5d maximal polytopes yield toric Calabi–Yau fourfolds

Kreuzer, Skarke (1997, 1998, 2003)
Schöller, Skarke (2019)

- New formula for Euler characteristic in 4d (*i.e.*, for threefolds)

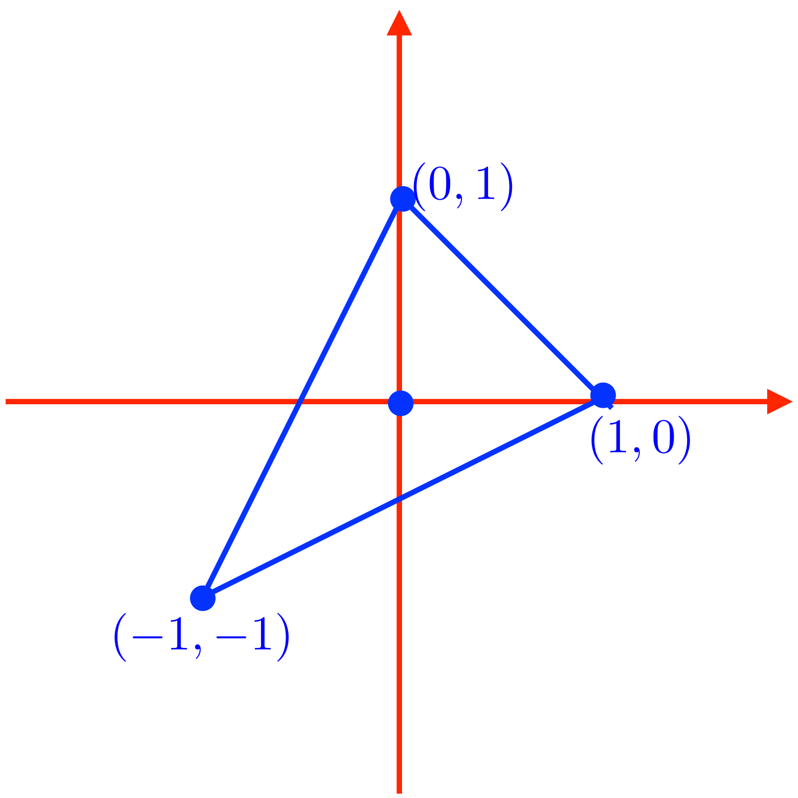
$$\chi(\mathcal{M}) = 2\big(\ell(2\Delta) - \ell(2\Delta^\circ)\big) + 18\big(\ell(\Delta^\circ) - \ell(\Delta)\big)$$

Berglund, Campbell, VJ (2021)

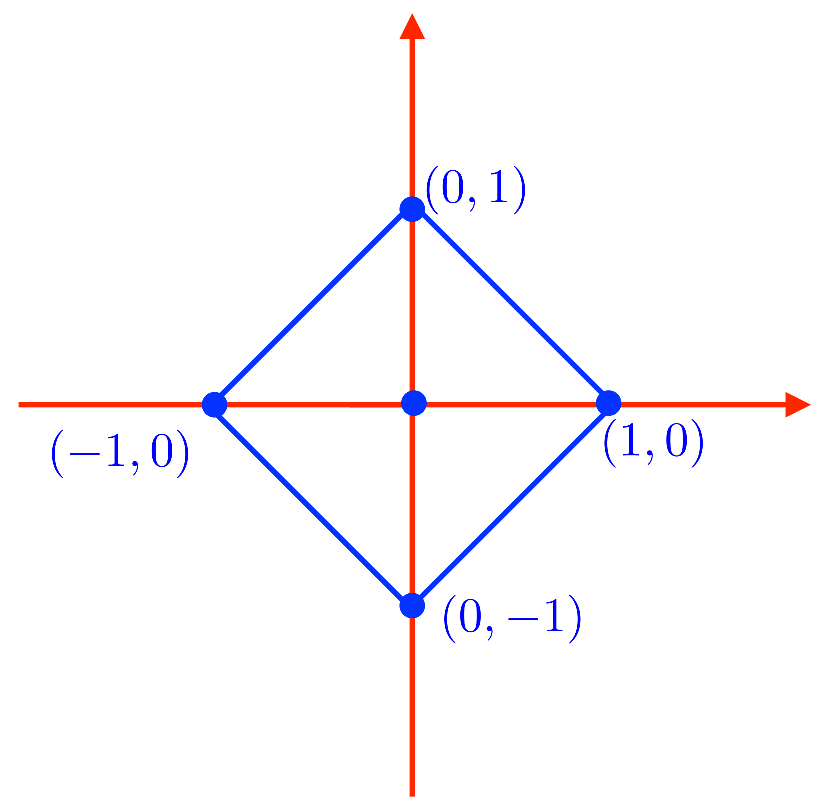
- New reflexive polytopes in 5d with new Hodge numbers, designer properties

Berglund, He, Heyes, Hirst, VJ, Lukas (2023)

2d Reflexive Polytopes



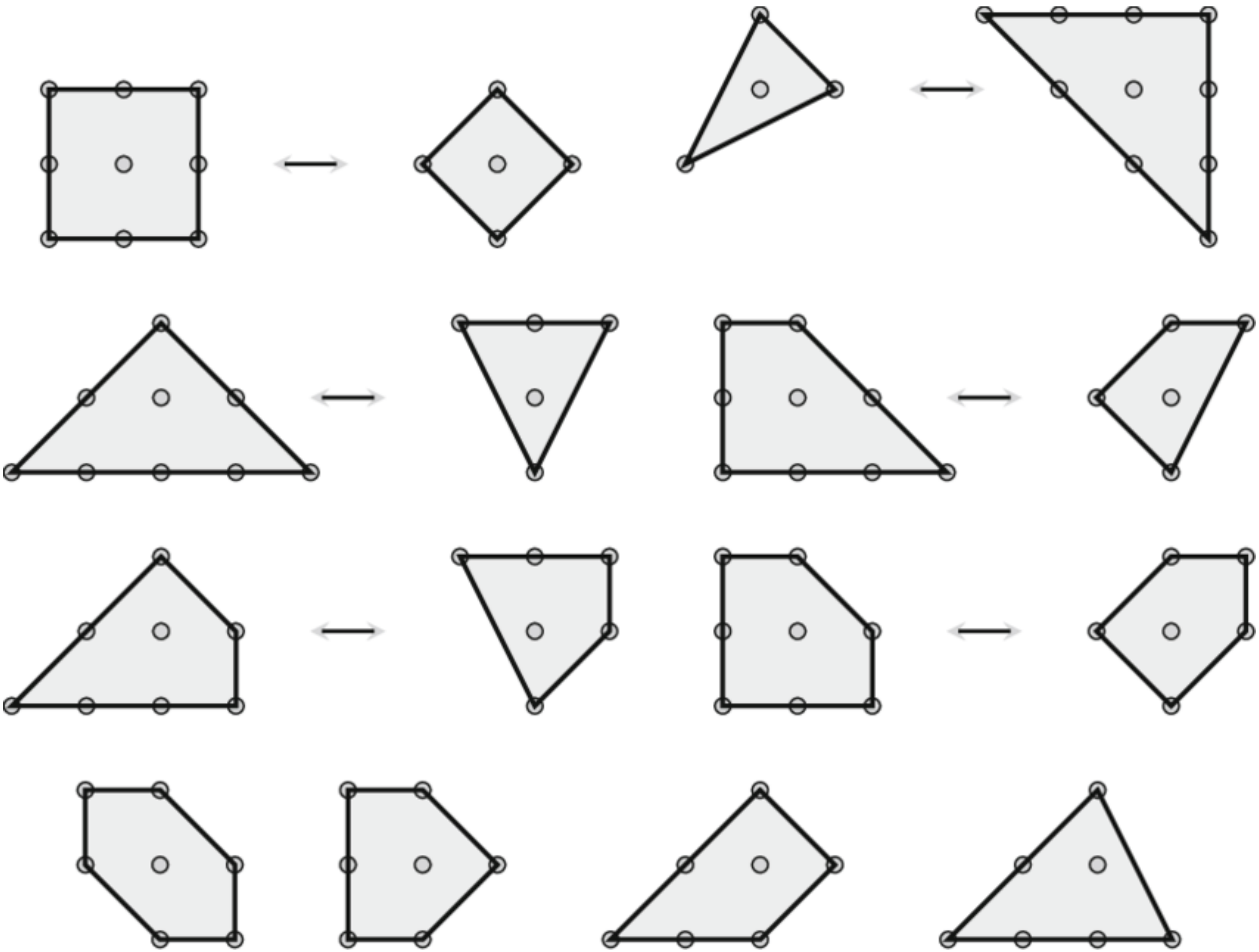
$$x+y=1$$
$$x-2y=1$$
$$-2x+y=1$$



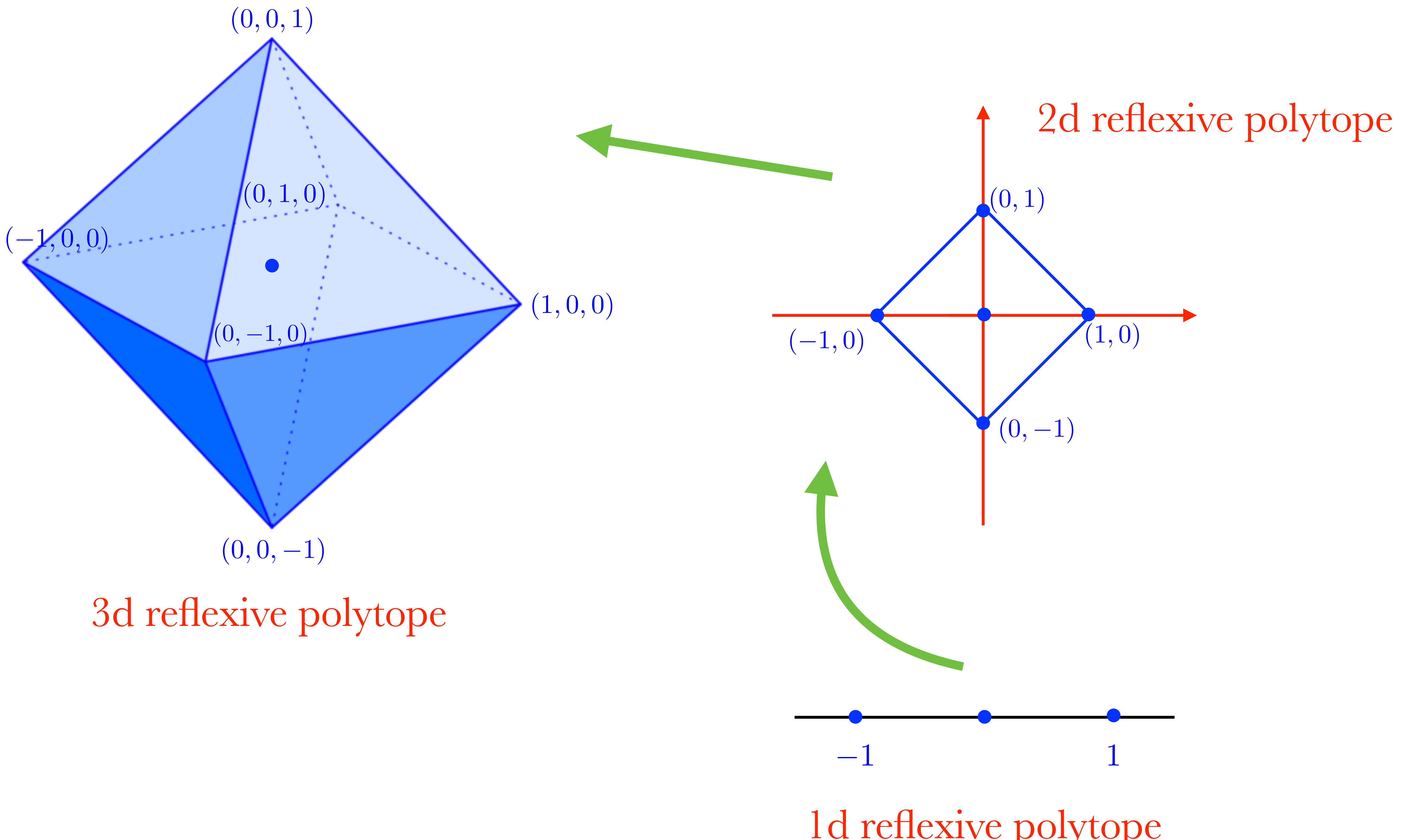
$$x+y=1$$
$$x-y=1$$
$$-x+y=1$$
$$-x-y=1$$

$$d_i(\Delta)=1$$

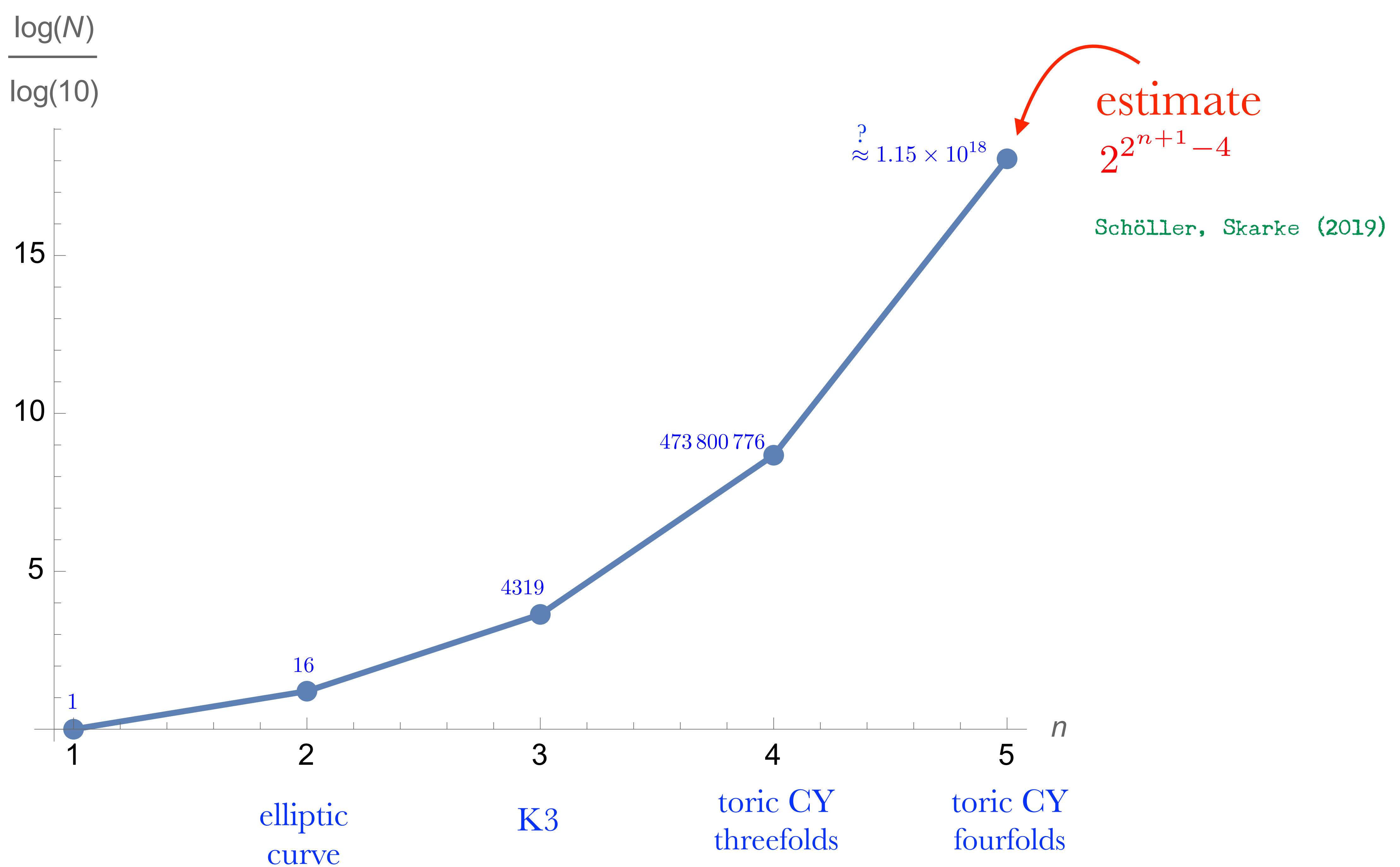
2d Reflexive Polytopes



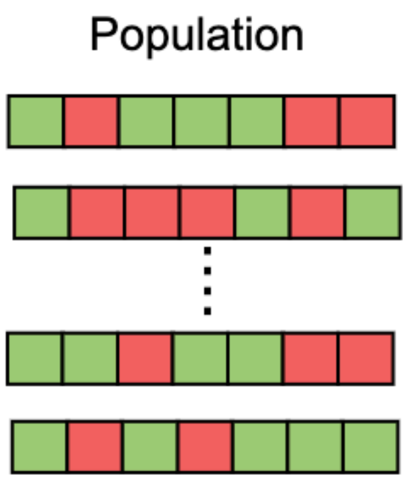
3d Reflexive Polytopes



n-d Reflexive Polytopes

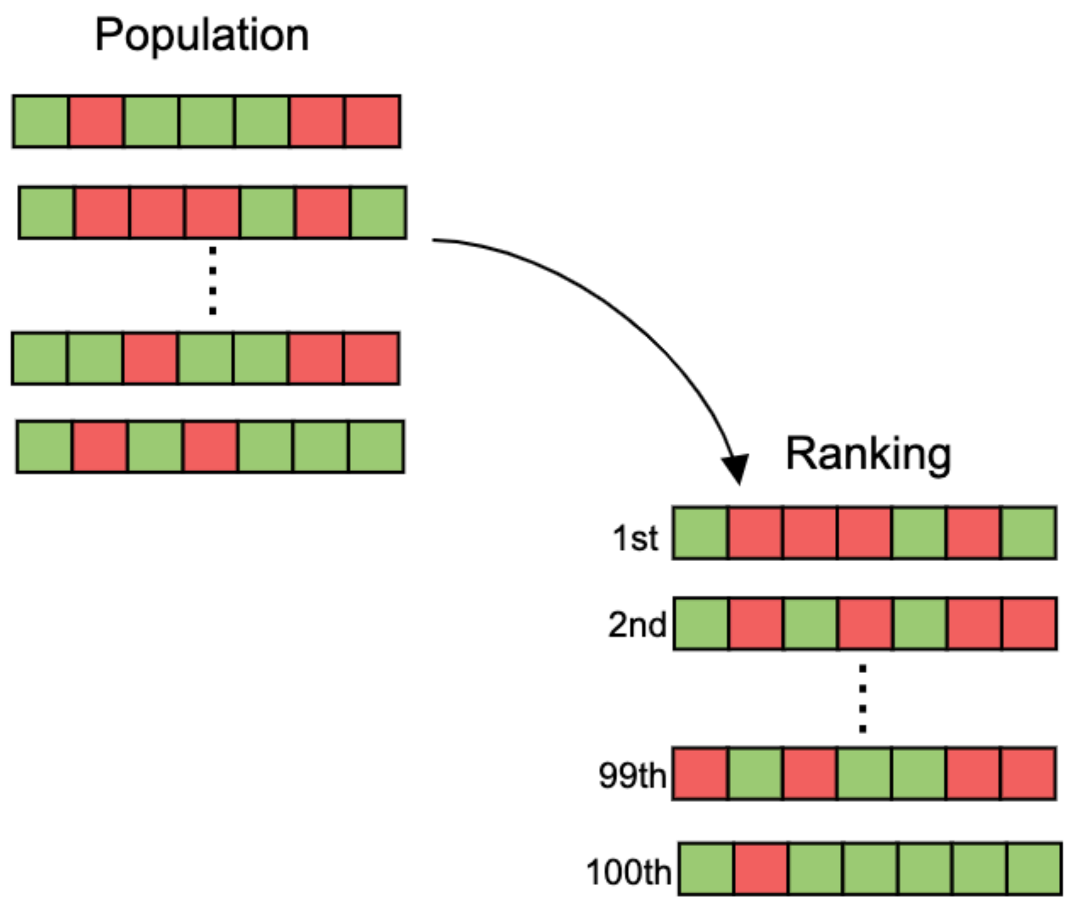


Genetic Algorithm



Darwin (1859)
Turing (1950)
Barricelli (1954)
Holland (1992)

Genetic Algorithm



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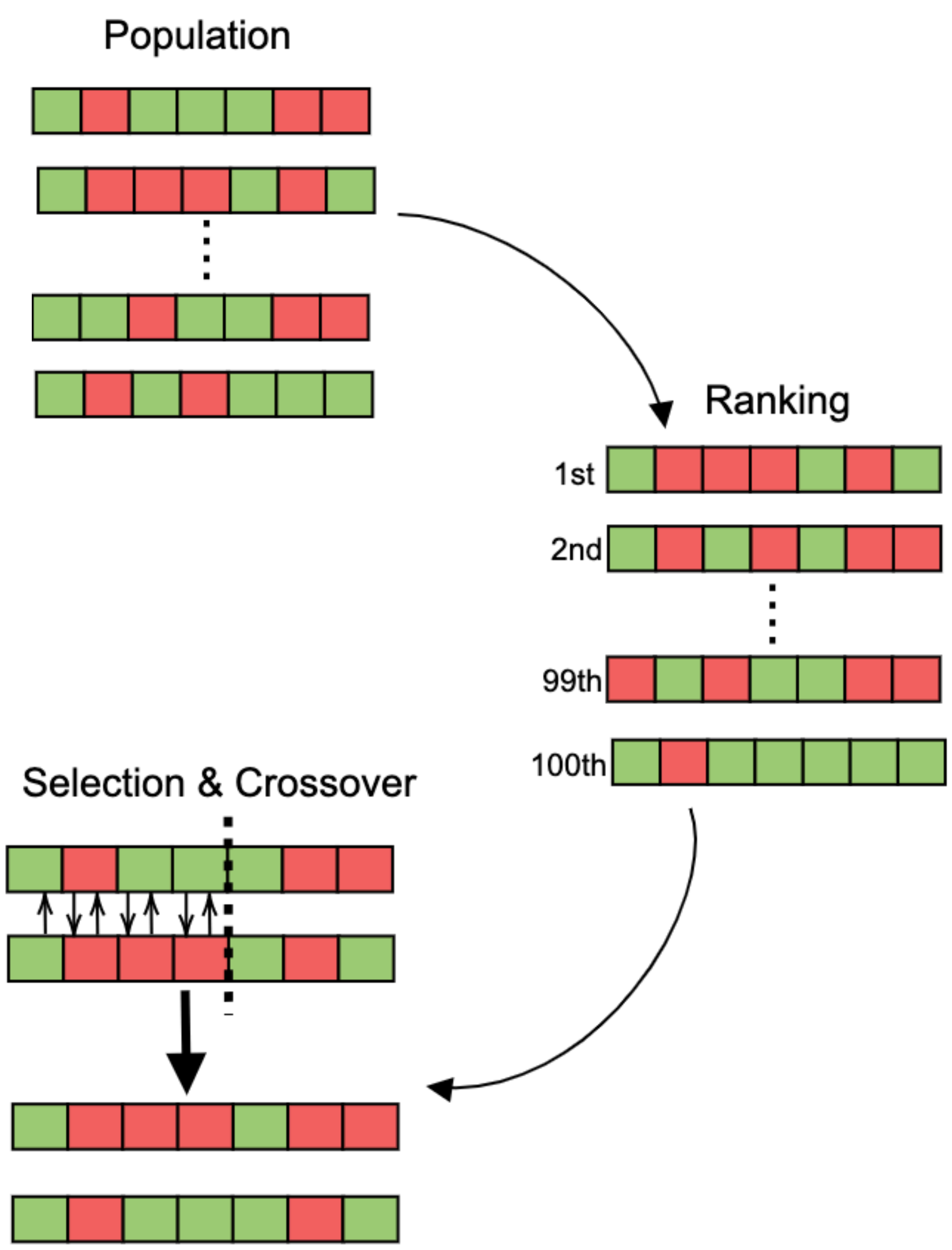

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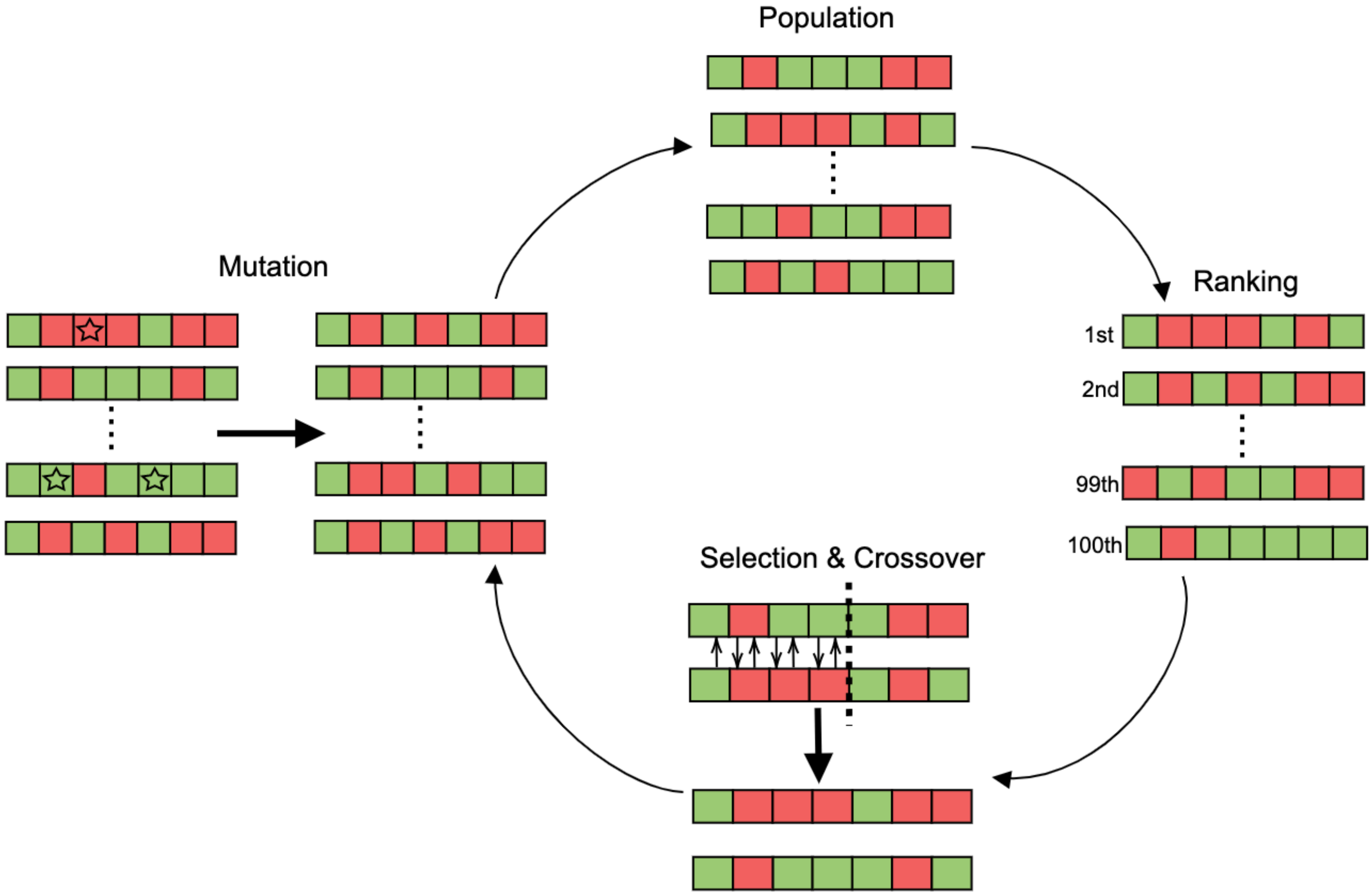

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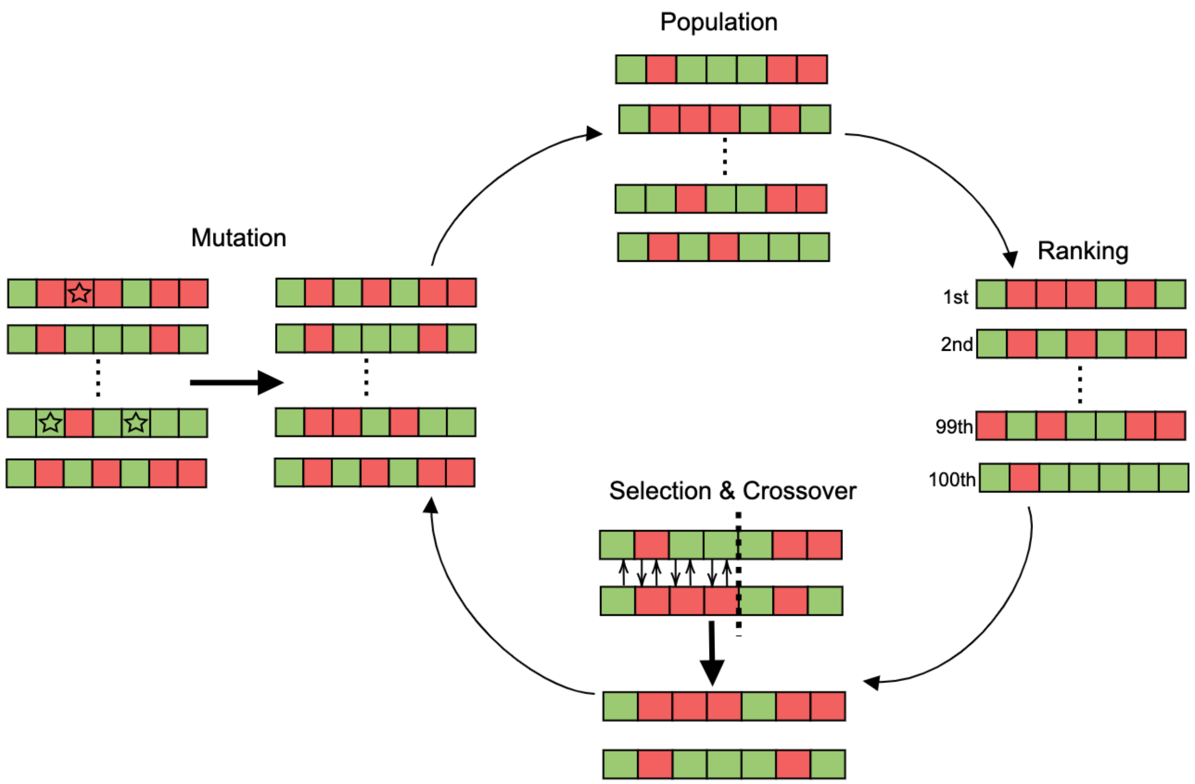


1. Create a random population

2. Evolve the population

3. Extract all generated reflexive polytopes ← these optimize cost function

4. Remove redundancy in this list

5. Repeat until all reflexive polytopes with desired properties realized

Berglund, He, Heyes, Hirst, VJ, Lukas (2023)

Fitness Function

- How to rank polytopes?

- Let $\Delta = \{m \in M_{\mathbb{R}} \mid \langle u_i, m \rangle \leq a_i \ , \ i = 1, \ldots, k\}$, where $u_i \in N_{\mathbb{R}} \ , \ a_i \in \mathbb{R}$

- Reflexive iff IP and $a_i = 1 \ \forall \ i$

- Fitness function: $f(\Delta) = w_1(\mathrm{IP}(\Delta) - 1) - \frac{w_2}{k} \sum_{i=1}^{k} |d_i(\Delta) - 1|$

  $w_1 \ , \ w_2 \in \mathbb{R}_{\geq 0}$ are weights

  $\mathrm{IP}(\Delta) = 1$ if Δ enjoys IP property and 0 otherwise

  fitter polytopes have larger $f(\Delta)$

Berglund, He, Heyes, Hirst, VJ, Lukas (2023)

5d Reflexive Polytopes

- $h^{1,1} = 1:$ $$V = \begin{pmatrix} 0 & 4 & -1 & 2 & -1 & -2 \\ 3 & -1 & -4 & 9 & 0 & -3 \\ 2 & 2 & -2 & 2 & -2 & 2 \\ 1 & -3 & 0 & 3 & 0 & -1 \\ 3 & -5 & 1 & -1 & 1 & -1 \end{pmatrix}$$ $h^{2,1} = 0$, $h^{3,1} = 111$, $h^{2,2} = 492$, $\chi = 720$

- **Conjecture:** There are 15 reflexive polytopes with $h^{1,1} = 1$

- $h^{1,1} = 2:$ $$V = \begin{pmatrix} 0 & -2 & -4 & -1 & 10 & 8 & -2 \\ 4 & 0 & 2 & -1 & -6 & -4 & 0 \\ -1 & 1 & -3 & 0 & 1 & 3 & 1 \\ 2 & -2 & 0 & 0 & -2 & 0 & 1 \\ 2 & 0 & -2 & 1 & -6 & -2 & 3 \end{pmatrix}$$ $h^{2,1} = 0$, $h^{3,1} = 111$, $h^{2,2} = 496$, $\chi = 726$

- *N.B.*: Calabi–Yau fourfolds inherit Hodge numbers from reflexive polytope. These are new geometries!

Berglund, He, Heyes, Hirst, VJ, Lukas (2023)

5d Reflexive Polytopes

- 11d supergravity compactified on a fourfold to break $\mathcal{N}=2$ to $\mathcal{N}=1$ in 3d needs χ divisible by 24, 224, 504

Berg, Haack, Samtleben (2003)

- $$\chi = 2016 : \quad V = \begin{pmatrix} -1 & -1 & 0 & -1 & 0 & 1 & -1 & 0 \\ 1 & 1 & 2 & 0 & -1 & 2 & 0 & 0 \\ -1 & 3 & 2 & 3 & -1 & -1 & 2 & 0 \\ 0 & 4 & 3 & 4 & -2 & 0 & 3 & 1 \\ -3 & 4 & 0 & 4 & 0 & 0 & 2 & -2 \end{pmatrix} \qquad \begin{aligned} h^{1,1} &= 331 \\ h^{2,1} &= 9 \\ h^{3,1} &= 6 \\ h^{2,2} &= 1374 \end{aligned}$$

- We can do targeted searches for desired phenomenology

Berglund, He, Heyes, Hirst, VJ, Lukas (2023)

- Use reinforcement learning to construct Calabi–Yau manifolds explicitly

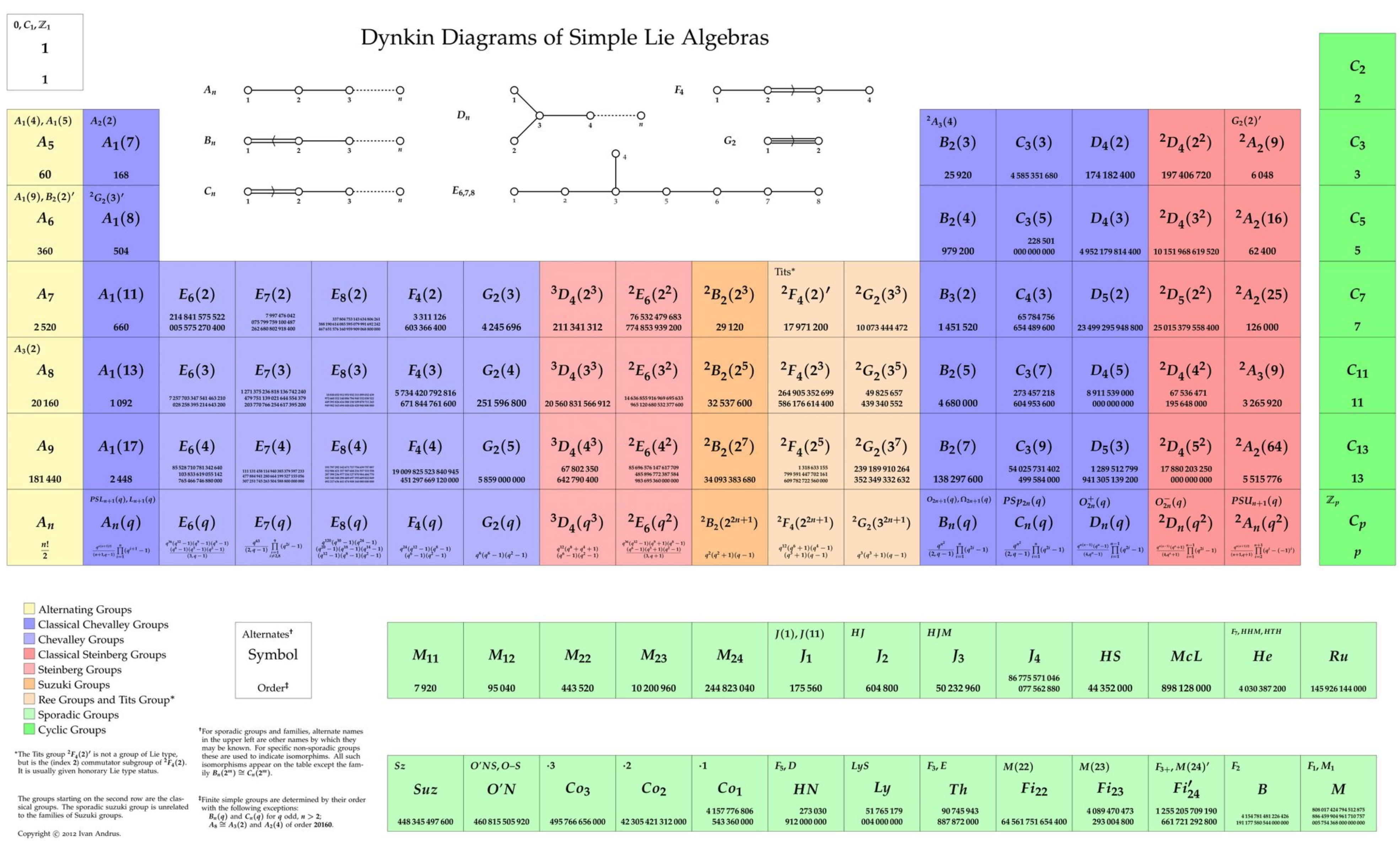
Berglund, Butbaia, He, Heyes, Hirst, VJ (2024)

PROSPECTUS

Challenge

- Utility of computers for mathematics

The Periodic Table Of Finite Simple Groups

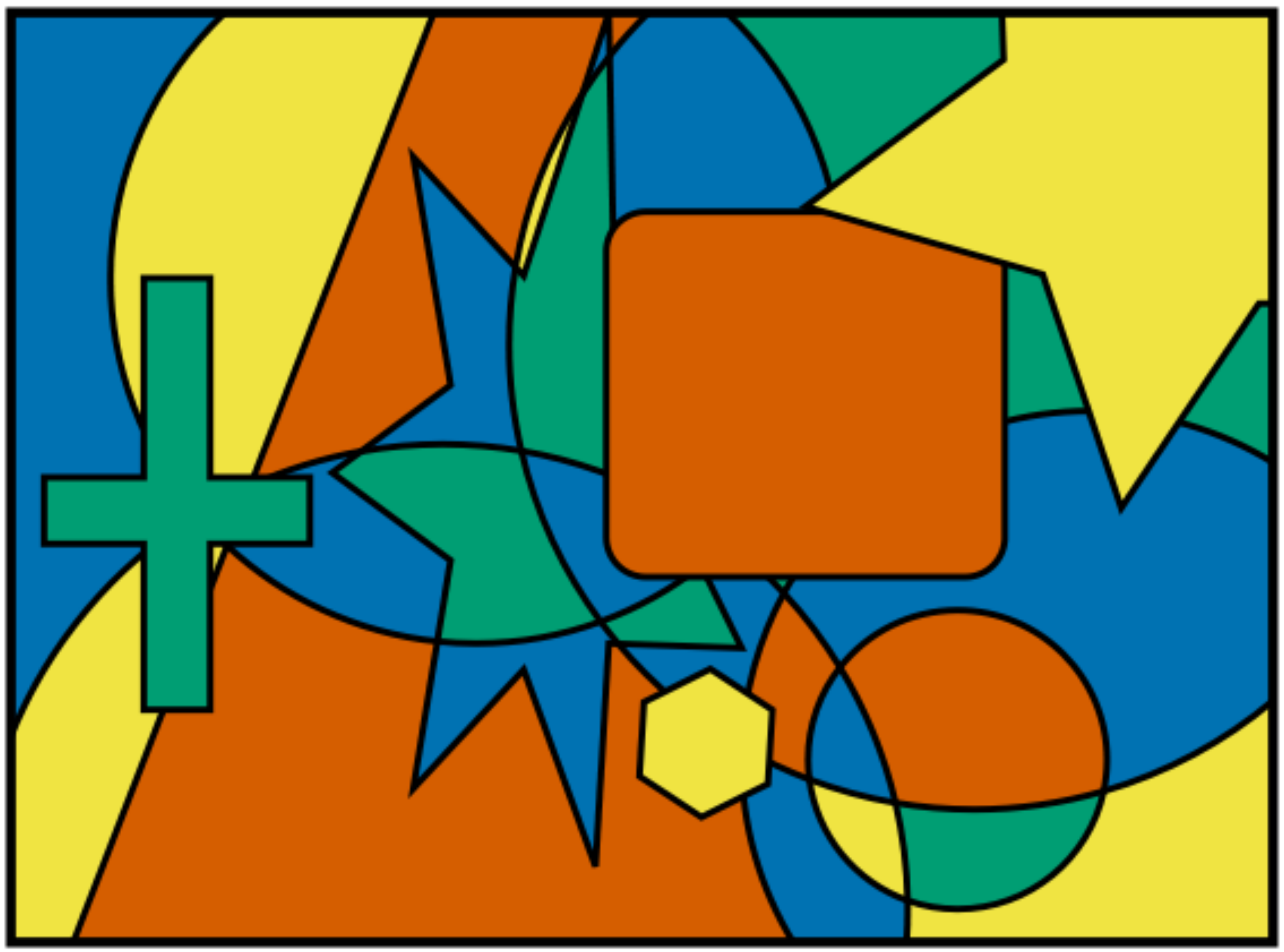


Gorenstein (1982, 1983)
Ashbacher, Lyons, Smith (2011)

Challenge

- Utility of computers for mathematics

Four color theorem



Appel, Haken (1976)

Challenge

Applications [edit]

Theorems proved with the help of computer programs [edit]

Inclusion in this list does not imply that a formal computer-checked proof exists, but rather, that a computer program has been involved in some way. See the main articles for details.

- Four color theorem, 1976
- Mitchell Feigenbaum's universality conjecture in non-linear dynamics. Proven by O. E. Lanford using rigorous computer arithmetic, 1982
- Connect Four, 1988 – a solved game
- Non-existence of a finite projective plane of order 10, 1989
- Double bubble conjecture, 1995[2]
- Robbins conjecture, 1996
- Kepler conjecture, 1998 – the problem of optimal sphere packing in a box
- Lorenz attractor, 2002 – 14th of Smale's problems proved by Warwick Tucker using interval arithmetic
- 17-point case of the Happy Ending problem, 2006
- NP-hardness of minimum-weight triangulation, 2008
- Optimal solutions for Rubik's Cube can be obtained in at most 20 face moves, 2010
- Minimum number of clues for a solvable Sudoku puzzle is 17, 2012
- In 2014 a special case of the Erdős discrepancy problem was solved using a SAT-solver. The full conjecture was later solved by Terence Tao without computer assistance.[3]
- Boolean Pythagorean triples problem solved using 200 terabytes of data in May 2016.[4]
- Applications to the Kolmogorov-Arnold-Moser theory[5][6]
- Kazhdan's property (T) for the automorphism group of a free group of rank at least five
- Schur number five, the proof that S(5) = 161 was announced in 2017 by Marijn Heule and took up 2 petabytes of space[7][8]
- Keller's conjecture in dimension 7 the only remaining case in 2020 with a 200 gigabyte proof[9][10][11]

Theorems for sale [edit]

In 2010, academics at The University of Edinburgh offered people the chance to "buy their own theorem" created through a computer-assisted proof. This new theorem would be named after the purchaser.[12][13] This service now appears to no longer be available.

Challenge

- How does a black box learn semantics without knowing syntax?

  — Knowing that there are approximate functions can we find analytic expressions by opening the black box?

- Computers are creative — *cf.*, new fuseki, jōseki in go `AlphaZero (2017)`

- Can artificial intelligence do interesting research?

  — Proofs in real analysis `Ganesalingam, Gowers (2013)`

  — Proof assistants `Voevodsky (2014)`

  — Lean, Xena Project, Liquid Tensor Experiment `Buzzard (2019)` `Clausen, Scholze (2020)` `Commelin (2021)` `Tao (2023)`

- Logical reasoning + natural language — how do we collaborate effectively?

  — does silicon replace carbon? when?

THANK YOU!

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Foundation