CHAPTER

Mathematical Morphology 7

Historically, the study of shape took place over a long period of time and resulted in a highly variegated set of algorithms and methods. Over the past 30 years the formalism of mathematical morphology was set up, and provided a background theory into which many of the individual advances could be slotted. This chapter takes a journey through this interesting subject, but aims to steer an intuitive path between the many mathematical theorems, concentrating on finding practically useful results.

Look out for:

- how the concepts of expanding and shrinking are transformed into the more general concepts of dilation and erosion.
- how dilation and erosion operations may be combined to form more complex operations whose properties may be predicted mathematically.
- how the concepts of closing and opening are defined, and how they are used to find defects in binary object shapes, via residue (or "top hat") operations.
- how mathematical morphology is generalized to cover grayscale processing.
- how noise affects morphological grouping operations.

This theory in the present chapter is especially valuable because of the way in which it integrates a range of topics. Once the methodology has been learnt, morphology should be of distinct value in taking the earlier ideas forward and optimizing algorithms that use them.

7.1 INTRODUCTION

In Chapter 2, we have discussed the operations of erosion and dilation: in Chapter 9 we will apply them to the filtering of binary images, and will show that with suitable combinations of these operators it is possible to eliminate certain types of object from images, and also to locate other objects. These possibilities are not fortuitous, but on the contrary reflect fundamental properties of shape, which are dealt with in the subject known as mathematical morphology. This subject has grown up over the past two or three decades, and over the past few years knowledge in this area has become consolidated and is now understood in considerable depth. It is the purpose of this chapter to give some insight into this vital area of study. Note that mathematical morphology is especially important because it provides a backbone for the whole study of shape and thus is able to unify techniques as disparate as noise suppression, shape analysis, feature recognition, skeletonization, convex hull formation, and a host of other topics.

Section 7.2 starts the discussion by extending the concepts of expanding and shrinking first encountered in Section 2.2. Section 7.3 then develops the theory of mathematical morphology, arriving at many important results—with emphasis deliberately being placed on understanding of concepts rather than mathematical rigor. Section 7.4 goes on to show how morphology can be generalized to cope with grayscale images. The chapter also includes a discussion (Section 7.5) on the noise behavior of morphological grouping operations and arrives at a formula explaining the shifts introduced by noise.

7.2 DILATION AND EROSION IN BINARY IMAGES

7.2.1 Dilation and Erosion

As we have seen in Chapter 2, dilation expands objects into the background and is able to eliminate "salt" noise within an object. It can also be used to remove cracks in objects that are less than 3 pixels in width.

In contrast, erosion shrinks binary picture objects, and has the effect of removing "pepper" noise. It also removes thin object "hairs" whose widths are less than 3 pixels.

As we shall see in more detail below, erosion is strongly related to dilation, in that a dilation acting on the inverted input image acts as an erosion, and vice versa.

7.2.2 Cancellation Effects

An obvious question is whether erosions cancel out dilations, or vice versa. We can easily answer this question: for if a dilation has been carried out, salt noise and cracks will have been removed, and once they are gone, erosion cannot bring them back; hence, exact cancellation will not occur in general. Thus, for the set S of object pixels in a general image I, we may write:

$$\text{erode}\,(\text{dilate}(S)) \neq S \tag{7.1}$$

equality only occurring for certain specific types of image (these will lack salt noise, cracks, and fine boundary detail). Similarly, pepper noise or hairs that are eliminated by erosion will not in general be restored by dilation:

$$\text{dilate}\,(\text{erode}(S)) \neq S \tag{7.2}$$

Overall, the most general statements that can be made are:

$$\text{erode (dilate}(S)) \supseteq S \tag{7.3}$$

$$\text{dilate (erode}(S)) \subseteq S \tag{7.4}$$

We may note, however, that large objects will be made 1 pixel larger all round by dilation, and will be reduced by 1 pixel all round by erosion, so a considerable amount of cancellation will normally take place when the two operations are applied in sequence. This means that sequences of erosions and dilations provide a good basis for filtering noise and unwanted detail from images.

7.2.3 Modified Dilation and Erosion Operators

It sometimes happens that images contain structures that are aligned more or less along the image axes' directions, and in such cases it is useful to be able to process these structures differently. For example, it might be useful to eliminate fine vertical lines, without altering broad horizontal strips. In that case the following "vertical erosion" operator will be useful:

```
for all pixels in image do{
  sigma = A1 + A5;
  if (sigma < 2) B0 = 0; else B0 = A0;
}
```
(7.5)

although it will be necessary to follow it with a compensating dilation operator[1] so that horizontal strips are not shortened:

```
for all pixels in image do {
  sigma = A1 + A5;
  if (sigma > 0) B0 = 1; else B0 = A0;
}
```
(7.6)

This example demonstrates some of the potential for constructing more powerful types of image filter. To realize these possibilities, we next develop a more general mathematical morphology formalism.

7.3 MATHEMATICAL MORPHOLOGY

7.3.1 Generalized Morphological Dilation

The basis of mathematical morphology is the application of set operations to images and their operators. We start by defining a generalized dilation mask as a set of locations within a 3×3 neighborhood. When referred to the center of the neighborhood as origin, each of these locations causes a shift of the image in the direction defined by the vector from the origin to the location. When several

[1] Here and elsewhere in this chapter, any operations required to restore the image to the original image space are not considered or included.

shifts are prescribed by a mask, the 1 locations in the various shifted images are combined by a set union operation.

The simplest example of this type is the identity operation *I*, which leaves the image unchanged:

	1	

(Note that we leave the 0s out of this mask, as we are now focussing on the set of elements at the various locations, and set elements are either present or absent.)

The next operation to consider is:

1		

which is a left shift, equivalent to the one discussed in Section 2.2. Combining the above two operations into a single mask:

1	1	

leads to a horizontal thickening of all objects in the image, by combining it with its left-shifted version. An isotropic thickening of all objects is achieved by the operator:

1	1	1
1	1	1
1	1	1

(clearly, this is equivalent to the dilation operator discussed in Sections 2.2 and 7.2), whereas a symmetrical horizontal thickening operation (see Section 7.2.3) is achieved by the mask:

1	1	1

A rule of such operations is that if we want to guarantee that all the original object pixels are included in the output image, then we must include a 1 at the center (origin) of the mask.

Finally, there is no compulsion for all masks to be 3×3. Indeed, all but one of those listed above are effectively smaller than 3×3, and in more complex cases larger masks could be used. To emphasize this point, and to allow for asymmetrical masks in which the full 3×3 neighborhood is not given, we shall shade the origin—as shown in the above cases.

7.3.2 Generalized Morphological Erosion

We now move on to describe erosion in terms of set operations. The definition is somewhat peculiar in that it involves reverse shifts, but the reason for this will become clear as we proceed. Here the masks define directions as before, but in this case we shift the image in the reverse of each of these directions and perform intersection operations to combine the resulting images. For masks with a single element (as for the identity and shift left operators in Section 7.3.1), the intersection operation is improper and the final result is as for the corresponding dilation operator, but with a reverse shift. For more complex cases, the intersection operation results in objects being reduced in size. Thus, the mask:

1	1	

has the effect of stripping away the left sides of objects (the object is moved right and *and*ed with itself). Similarly, the mask:

1	1	1
1	1	1
1	1	1

results in an isotropic stripping operation, and is hence identical to the erosion operation described in Section 7.2.1.

7.3.3 Duality Between Dilation and Erosion

We shall write the dilation and erosion operations formally as $A \oplus B$ and $A \ominus B$, respectively, where A is an image and B is the mask of the relevant operation:

$$A \oplus B = \cup_{b \in B} A_b \tag{7.7}$$

$$A \ominus B = \cap_{b \in B} A_{-b} \tag{7.8}$$

In these equations, A_b indicates a basic shift operation in the direction of element b of B and A_{-b} indicates the reverse shift operation.

We next prove an important theorem relating the dilation and erosion operations:

$$(A \ominus B)^{c} = A^{c} \oplus B^{r} \tag{7.9}$$

where A^c represents the complement of A, and B^r represents the reflection of B in its origin. We first note that:[2]

$$x \in A^{c} \Leftrightarrow x \notin A \tag{7.10}$$

and

$$b \in B^{r} \Leftrightarrow -b \in B \tag{7.11}$$

[2]The sign "⇔" means "if and only if," i.e., the statements so connected are equivalent.

We now have:[3]

$$\begin{aligned} x \in (A \ominus B)^{c} &\Leftrightarrow x \notin A \ominus B \\ &\Leftrightarrow \exists \;\; b \;\; \text{such that} \;\; x \notin A_{-b} \\ &\Leftrightarrow \exists \;\; b \;\; \text{such that} \;\; x + b \notin A \\ &\Leftrightarrow \exists \;\; b \;\; \text{such that} \;\; x + b \in A^{c} \\ &\Leftrightarrow \exists \;\; b \;\; \text{such that} \;\; x \in (A^{c})_{-b} \\ &\Leftrightarrow x \in \cup_{b \in B} \;\; (A^{c})_{-b} \\ &\Leftrightarrow x \in \cup_{b \in B^{r}} \;\; (A^{c})_{b} \\ &\Leftrightarrow x \in A^{c} \oplus B^{r} \end{aligned} \tag{7.12}$$

This completes the proof. The related theorem:

$$(A \oplus B)^{c} = A^{c} \ominus B^{r} \tag{7.13}$$

is proved similarly.

The fact that there are two such closely related theorems, following the related union and intersection definitions of dilation and erosion given above, indicates an important duality between the two operations. Indeed, as stated earlier, erosion of the objects in an image corresponds to dilation of the background, and vice versa. However, the two theorems indicate that this relation is not absolutely trivial, on account of the reflections of the masks required in the two cases. It is perhaps curious that in contrast with the case of the de Morgan rule for complementation of an intersection:

$$(P \cap Q)^{c} = P^{c} \cup Q^{c} \tag{7.14}$$

the effective complementation of the dilating or eroding mask is its reflection rather than its complement *per se*, while that for the operator is the alternate operator.

7.3.4 Properties of Dilation and Erosion Operators

Dilation and erosion operators have some very important and useful properties. First, note that successive dilations are associative:

$$(A \oplus B) \oplus C = A \oplus (B \oplus C) \tag{7.15}$$

whereas successive erosions are not. In fact, the corresponding relation for erosions is:

$$(A \ominus B) \ominus C = A \ominus (B \oplus C) \tag{7.16}$$

Clearly, the apparent symmetry between the two operators is more subtle than their simple origins in expanding and shrinking might indicate.

Next, the property:

$$X \oplus Y = Y \oplus X \tag{7.17}$$

[3]This proof is based on that of Haralick et al. (1987). The sign "∃" means "there exists," and in this context should be interpreted as "there is a value of." The symbol " ∈ " means "is a member of the following set": the symbol "∉" means "is *not* a member of the following set."

means that the order in which dilations of an image are carried out does not matter, and the same applies to the order in which erosions are carried out:

$$(A \oplus B) \oplus C = (A \oplus C) \oplus B \tag{7.18}$$

$$(A \ominus B) \ominus C = (A \ominus C) \ominus B \tag{7.19}$$

In addition to the above relations, which use only the morphological operators $\oplus$ and $\ominus$, there are many more relations that involve set operations. In the examples that follow, great care must be exercised to note which particular distributive operations are actually valid:

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C) \tag{7.20}$$

$$A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C) \tag{7.21}$$

$$(A \cap B) \ominus C = (A \ominus C) \cap (B \ominus C) \tag{7.22}$$

In certain other cases, where equality might *a priori* have been expected, the strongest statements that can be made are typified by the following:

$$A \ominus (B \cap C) \supseteq (A \ominus B) \cup (A \ominus C) \tag{7.23}$$

Note that the associative relations are of value in showing how large dilations and erosions might be factorized so that they can be implemented more efficiently as two smaller dilations and erosions applied in sequence. Similarly, the distributive relations show that a large mask may be split into two separate masks, which may then be applied separately and the resulting images *or*ed together to create the same final image. These approaches can be useful for providing efficient implementations, especially in cases where very large masks are involved. For example, we could dilate an image horizontally and vertically by two separate operations, which would then be merged together—as in the following instance:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline 1 & 1 & 1 \\ \hline & & \\ \hline\end{array} \quad \text{followed by} \quad \begin{array}{|c|c|c|}\hline & 1 & \\ \hline & 1 & \\ \hline & 1 & \\ \hline\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline\end{array}$$

Next, let us consider the importance of the identity operation I, which corresponds to a mask with a single 1 at the central ($A0$) position:

$$\begin{array}{|c|c|c|}\hline & & \\ \hline & 1 & \\ \hline & & \\ \hline\end{array}$$

By way of example, we take Eqs. (7.20) and (7.21) and replace C by I in each of them. If we write the union of B and I as D, so that mask D is bound to contain a central 1 (i.e., $D \supseteq I$), we have:

$$A \oplus D = A \oplus (B \cup I) = (A \oplus B) \cup (A \oplus I) = (A \oplus B) \cup A \tag{7.24}$$

which always contains A:

$$A \oplus D \supseteq A \tag{7.25}$$

Similarly:

$$A \ominus D = A \ominus (B \cup I) = (A \ominus B) \cap (A \ominus I) = (A \ominus B) \cap A \tag{7.26}$$

which is always contained within A:

$$A \ominus D \subseteq A \tag{7.27}$$

Operations (such as dilation by a mask containing a central 1) which give outputs that are guaranteed to contain the inputs are termed *extensive*, whereas those (such as erosion by a mask containing a central 1) for which the outputs are guaranteed to be contained by the inputs are termed *antiextensive*. Clearly, extensive operations extend objects and antiextensive operations contract them, or in either case, leave them unchanged.

Another important type of operation is the *increasing* type of operation. An increasing operation is one, such as union, which preserves order in the size of the objects on which it operates. If object F is small enough to be contained within object G, then applying erosions or dilations will not affect the situation, even though the objects change their sizes and shapes considerably. We can write these conditions in the form:

$$\text{if} \quad F \subseteq G \tag{7.28}$$

$$\text{then} \quad F \oplus B \subseteq G \oplus B \tag{7.29}$$

$$\text{and} \quad F \ominus B \subseteq G \ominus B \tag{7.30}$$

Next, we note that erosion can be used for locating the boundaries of objects in binary images:[4]

$$P = A - (A \ominus B) \tag{7.31}$$

There are many practical applications of dilation and erosion, which follow particularly from using them together, as we shall see below.

Finally, we explore why the morphological definition of erosion involves a reflection. The idea is so that dilation and erosion are able, under the right circumstances, to cancel each other out. Take the left-shift dilation operation and the right-shift erosion operation. These are both achieved via the mask:

1	1	

but in the erosion operation it is applied in its reflected form, thereby producing the right shift required to erode the left edge of any object. This makes it clear why an operation of the type $(A \oplus B) \ominus B$ has a chance of canceling to give A.

[4]Technically, we are here dealing with sets, and the appropriate set operation is the *andnot* function / rather than minus. However, the latter admirably conveys the required meaning without ambiguity.

More specifically, there must be shifts in opposite directions as well as appropriate subtractions produced by *and*ing instead or *or*ing in order for cancellation to be possible. Of course, in many cases the dilation mask will have 180° rotation symmetry, and then the distinction between B^{r} and B will be purely academic.

7.3.5 Closing and Opening

Dilation and erosion are basic operators from which many others can be derived. Earlier, we were interested in the possibility of an erosion canceling a dilation and vice versa. Hence, it is an obvious step to define two new operators that express the degree of cancellation: the first is called *closing* since it often has the effect of closing gaps between objects and the other is called *opening* because it often has the effect of opening gaps (Fig. 7.1). Closing (•) and opening (◦) are formally defined by the formulae:

$$A \bullet B = (A \oplus B) \ominus B \tag{7.32}$$

$$A \circ B = (A \ominus B) \oplus B \tag{7.33}$$

Closing is able to eliminate salt noise, narrow cracks or channels, and small holes or concavities.[5] Opening is able to eliminate pepper noise, fine hairs, and small protrusions. Thus, these operators are extremely important for practical applications. Furthermore, by subtracting the derived image from the original image, it is possible to locate many sorts of defect, including those cited above as being eliminated by opening and closing: this possibility makes the two operations even more important. For example, we might use the following operation to locate all the fine hairs in an image:

$$Q = A - A \circ B \tag{7.34}$$

This operator and its dual using opening:

$$R = A \bullet B - A \tag{7.35}$$

are extremely important for defect detection tasks. They are often, respectively, called the white and black top-hat operators.[6] (Practical applications of these two operators include location of solder bridges and cracks in printed circuit board tracks.)

Closing and opening have the interesting property that they are idempotent: this means that repeated application of either operation has no further effect.

[5] Here we continue to take the convention that dark objects have become 1s in binary images, and light background or other features have become 0s.

[6] It is dubious whether "top hat" is a very appropriate name for this type of operator: *a priori*, the term "residue function" (or simply "residue") would appear to be better, as it conjures up the right functional connotations.

```
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . 1 1 . . 1 . . . .
. . . 1 1 1 1 1 . . . .
. . . 1 1 1 1 1 . . . .
. . 1 1 1 1 1 1 1 . . .
. . . 1 1 1 1 1 1 . . .
. . . 1 1 1 1 1 . . . .
. . . . 1 1 1 1 . . . .
. . . 1 . . . . . 1 . .
. . . . . . . 1 1 1 . .
. . . . . . . . . . . .
. . . . . . . . . . . .
```
(a)

```
. . . . . . . . . . . .
. . 1 1 1 1 1 1 1 . . .
. . 1 1 1 1 1 1 1 . . .
. . 1 1 1 1 1 1 1 . . .
. 1 1 1 1 1 1 1 1 1 . .
. 1 1 1 1 1 1 1 1 1 . .
. 1 1 1 1 1 1 1 1 1 . .
. . 1 1 1 1 1 1 1 1 . .
. . 1 1 1 1 1 1 1 1 1 .
. . 1 1 1 1 1 1 1 1 1 .
. . 1 1 1 . 1 1 1 1 1 .
. . . . . . 1 1 1 1 1 .
. . . . . . . . . . . .
```
(b)

```
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . 1 1 1 . . . . .
. . . . 1 1 1 . . . . .
. . . . 1 1 1 . . . . .
. . . . . 1 1 . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
```
(c)

```
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . 1 1 1 1 1 . . . .
. . . 1 1 1 1 1 . . . .
. . . 1 1 1 1 1 . . . .
. . 1 1 1 1 1 1 1 . . .
. . . 1 1 1 1 1 1 . . .
. . . 1 1 1 1 1 1 . . .
. . . 1 1 1 1 1 1 . . .
. . . 1 . . . 1 1 1 . .
. . . . . . . 1 1 1 . .
. . . . . . . . . . . .
. . . . . . . . . . . .
```
(d)

```
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . 1 1 1 1 1 . . . .
. . . 1 1 1 1 1 . . . .
. . . 1 1 1 1 1 . . . .
. . . 1 1 1 1 1 . . . .
. . . 1 1 1 1 1 . . . .
. . . . 1 1 1 1 . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
. . . . . . . . . . . .
```
(e)

FIGURE 7.1

Results of morphological operations. (a) The original image, (b) the dilated image, (c) the eroded image, (d) the closed image, and (e) the opened image.

Source: © *World Scientific 2000*

(This property contrasts strongly with what happens when dilation and erosion are applied a number of times.) We can write these results formally as follows:

$$(A \bullet B) \bullet B = A \bullet B \tag{7.36}$$

$$(A \circ B) \circ B = A \circ B \tag{7.37}$$

From a practical point of view these properties are to be expected, since any hole or crack that has been filled in remains filled in, and there is no point in repeating the operation. Similarly, once a hair or protrusion has been removed, it cannot again be removed without first recreating it. Not quite so obvious is the fact that the combined closing and opening operation is idempotent:

$$\{[(A \bullet B) \circ C] \bullet B\} \circ C = (A \bullet B) \circ C \tag{7.38}$$

The same applies to the combined opening and closing operation. A simpler result is the following:

$$(A \oplus B) \circ B = (A \oplus B) \tag{7.39}$$

which shows that there is no point in opening with the same mask that has already been used for dilation: essentially, the first dilation produces some effects that are not reversed by the erosion (in the opening operation), and the second dilation then merely reverses the effects of the erosion. The dual of this result is also valid:

$$(A \ominus B) \bullet B = (A \ominus B) \tag{7.40}$$

There are a number of other properties of closing and opening; among the most important ones are the following set containment properties, which apply when $D \supseteq I$:

$$A \oplus D \supseteq A \bullet D \supseteq A \tag{7.41}$$

$$A \ominus D \subseteq A \circ D \subseteq A \tag{7.42}$$

Thus, closing an image will, if anything, increase the sizes of objects, while opening an image will, if anything, make objects smaller, although there are clear limits on how much change closing and opening operations can induce.

Finally, note that closing and opening are subject to the same duality as for dilation and erosion:

$$(A \bullet B)^{\mathrm{c}} = A^{\mathrm{c}} \circ B^{\mathrm{r}} \tag{7.43}$$

$$(A \circ B)^{\mathrm{c}} = A^{\mathrm{c}} \bullet B^{\mathrm{r}} \tag{7.44}$$

7.3.6 Summary of Basic Morphological Operations

The past few sections have by no means exhausted the properties of the morphological operations, dilate, erode, close, and open. However, these sections have outlined some of their properties and have demonstrated some of the practical results obtained using them. Perhaps the main aim of including the mathematical analysis has been to show that these operations are not *ad hoc* and that their properties are mathematically provable. Furthermore, the analysis has also indicated (a) how sequences of operations can be devised for a number of eventualities and (b) how sequences of operations can be analyzed to save computation (for instance) by taking care not to use idempotent operations repeatedly and by breaking masks down into smaller more efficient ones.

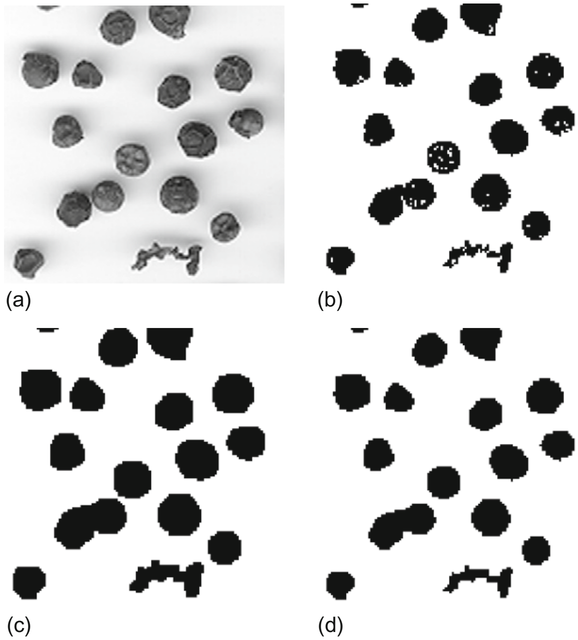


FIGURE 7.2

Use of the closing operation. (a) A peppercorn image, (b) the result of thresholding, (c) the result of applying a 3×3 dilation operation to the object shapes, and (d) the effect of subsequently applying a 3×3 erosion operation. The overall effect of the two operations is a "closing" operation. In this case closing is useful for eliminating the small holes in the objects: this would, e.g., be useful for helping to prevent misleading loops from appearing in skeletons. For this picture, extremely large window operations would be required to group peppercorns into regions.

Source: © *World Scientific 2000*

Overall, the operations devised here can help to eliminate noise and irrelevant artifacts from images so as to obtain more accurate recognition of shapes; they can also help to identify defects on objects by locating specific features of interest. In addition, they can perform grouping functions such as locating regions of images where small objects such as seeds may reside (Section 7.5). In general, elimination of artifacts is carried out by operations such as closing and opening, while location of such features is carried out by finding how the results of these operations differ from the original image (cf. Eqs. (7.20) and (7.21)); and locating regions where clusters of small objects occur may be achieved by larger scale closing operations. Clearly, care in the choice of scales and mask sizes is of vital importance in the design of complete algorithms for all these tasks. Figures 7.2 and 7.3 illustrate some of these possibilities in the case of a peppercorn image: some of the interest in this image relates to the presence of a twiglet and how it is eliminated from consideration and/or identified.

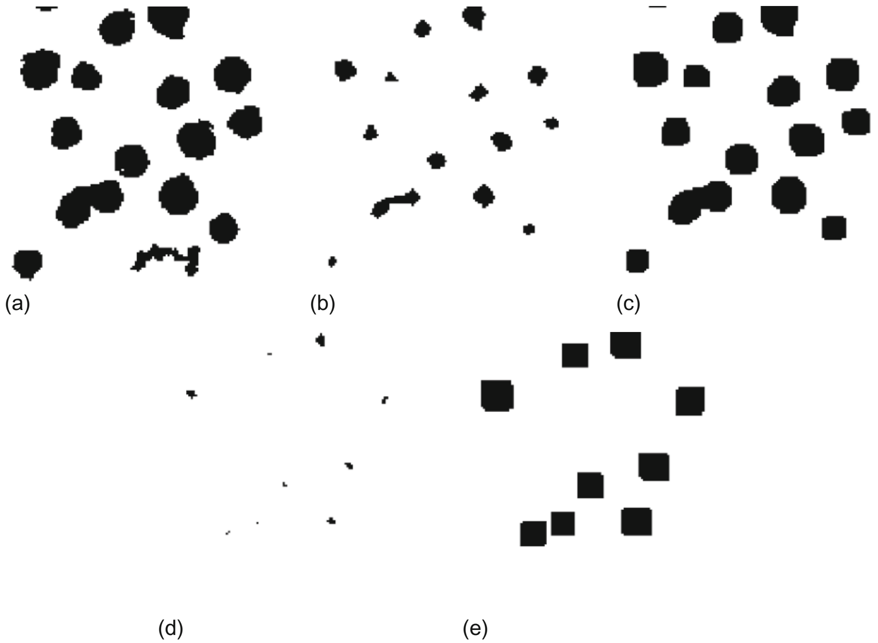


FIGURE 7.3

Use of the opening operation. (a) A thresholded peppercorn image. (b) The result of applying a 7×7 erosion operation to the object shapes. (c) The effect of subsequently applying a 7×7 dilation operation. The overall effect of the two operations is an "opening" operation. In this case, opening is useful for eliminating the twiglet. (d) and (e) The same respective operations when applied within an 11×11 window. Here some size filtering of the peppercorns has been achieved and all the peppercorns have been separated—thereby helping with subsequent counting and labeling operations.

Source: © *World Scientific 2000*

7.4 GRAYSCALE PROCESSING

The generalization of morphology to grayscale images can be achieved in a number of ways. A particularly simple approach is to employ "flat" structuring elements. These perform morphological processing in the same way for each of the gray levels, acting as if the shapes at each level were separate, independent binary images. If dilation is carried out in this way, the result turns out to be identical to the effect of applying a maximum intensity operation of the same shape: i.e., we replace set inclusion by a magnitude comparison; needless to say, this is mathematically identical in action for a normal binary image, but when applied to a grayscale image it neatly generalizes the dilation concept. Similarly, erosion can be carried out by applying a minimum intensity structuring element of the same shape as the original binary structuring element. This discussion assumes that we focus on light objects against dark backgrounds[7]: these will be dilated when the maximum intensity operation is applied, and eroded when the minimum intensity operation is applied; we could of course reverse the convention, depending on what type of objects we are concentrating on at any moment, or in any application. We can summarize the situation as follows:

$$A \oplus B = \max_{b \in B} A_b \tag{7.45}$$

$$A \ominus B = \min_{b \in B} A_{-b} \tag{7.46}$$

There are other more complex grayscale analogs of dilation and erosion: these take the form of 3-D structuring elements whose output at any gray level depends not only on the shape of the image at that gray level, but also on the shapes at a number of nearly gray levels. Although such "nonflat" structuring elements are useful, for a good many applications they are not necessary, as flat structuring elements already embody a very considerable amount of generalization relative to the binary case.

Next we consider how edge detection is carried out using grayscale morphology.

7.4.1 Morphological Edge Enhancement

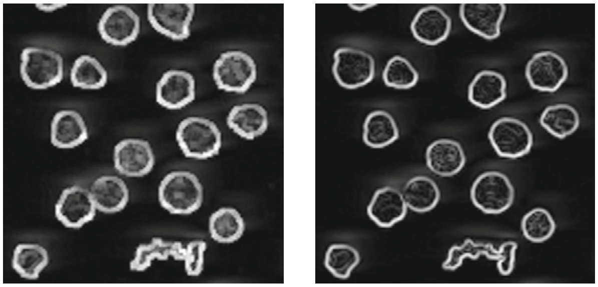
In Section 2.2.2, we have shown how edge detection can be carried out in binary images. We have defined edge detection asymmetrically, in the sense that the edge is the part of the object that is next to the background. This is useful because including the part of the background next to the object would merely have served to make the boundary wider and less precise. However, edge detection in grayscale images does not need to embody such an asymmetry,[8] because it starts by performing edge enhancement and then carrying out a thresholding type of operation—the width being controlled largely by the manner of thresholding and whether nonmaximum suppression or other factors are brought to bear. Here we start by formulating the original binary edge detector in morphological form. Then we make it more symmetric. Finally we generalize it to grayscale operation.

The original binary edge detector may be written in the form (cf. Eq. (7.31)):

$$E = A - (A \ominus B) \tag{7.47}$$

[7]In fact, this is the opposite convention to that employed in Chapter 2, but, as we shall see below, in grayscale processing it is probably more general to focus on intensities rather than on specific objects.

[8]Indeed, any asymmetry would lead to an unnecessary bias and hence inaccuracy in the location of edges.



(a) (b)

FIGURE 7.4

Determination of the morphological gradient of an image. The original image is that of Fig. 7.2(a). (a) The morphological gradient, obtained using 3×3 window operations. (b) The result for a Sobel operator: note that the latter gives less diffuse responses.

Source: © *World Scientific 2000*

Making it symmetrical merely involves adding the background edge $((A \oplus B) - A)$:

$$G = (A \oplus B) - (A \ominus B) \tag{7.48}$$

To convert to grayscale operation involves employing maximum and minimum operations in place of dilation and erosion. In this case we are concentrating on intensity *per se*, and so these respective assignments of dilation and erosion are used (the alternate arrangement would result in negative edge contrast). Thus, we here use Eqs. (7.45) and (7.46) to *define* dilation and erosion for grayscale processing, so Eq. (7.48) already represents morphological edge enhancement for grayscale images. The argument G is often called the morphological gradient of an image (Fig. 7.4). Note that it is not accompanied by an accurate edge orientation value, although approximate orientations can, of course, be computed by determining which part of the structuring element gives rise to the maximum signal.

7.4.2 Further Remarks on the Generalization to Grayscale Processing

In the previous subsection we found that reinterpreting Eq. (7.48) permitted edge detection to be generalized immediately from binary to gray scale. This is a consequence of the extremely powerful *umbra homomorphism theorem*. This starts with the knowledge that intensity I is a single-valued function of position within the image. This means that it represents a surface in a 3-D (grayscale) space. However, as we have seen, it is useful to take into account the individual gray levels. In particular, we note that the set of pixels of gray level g_i is a subset of the set of pixels of gray level g_{i-1}, where $g_i \geq g_{i-1}$. The important step forward is to interpret the 3-D volume containing all these gray levels under the intensity surface as constituting an umbra—a 3-D shadow region for the relevant part of the surface. In fact, we write the umbra volume of I as $U(I)$, and clearly we also have $I = T(U(I))$, where the operator $T(\cdot)$ recalculates the top surface.

The umbra homomorphism theorem then states that a dilation has to be defined and interpreted as an operation on the umbras:

$$U(I \oplus K) = U(I) \oplus U(K) \tag{7.49}$$

To find the relevant intensity function we merely need to apply the top-surface operator to the umbra:

$$I \oplus K = T(U(I \oplus K)) = T(U(I) \oplus U(K)) \tag{7.50}$$

A similar statement applies for erosion.

The next step is to note that the generalization from binary to grayscale dilation using flat structuring elements involved applying a maximum operation in place of a set union operation. The operation can, for the simple case of a 1-D image, be rewritten in the form:

$$(I \oplus K)(x) = \max(I(x - z) + K(z)) \tag{7.51}$$

where $K(z)$ takes value 0 or 1 only, and $x - z$, z must lie within the domains of I and K. However, another vital step is to notice that this form generalizes to nonbinary $K(z)$, whose values can, e.g., be the integer gray-level values. This makes the dilation operation considerably more powerful, yielding the nonflat structuring element concept—which will clearly also work for 2-D images with full gray scale.

It can be tedious working out the responses of this sort of operation, but it is susceptible to a neat geometric interpretation. If the function $K(z)$ is inverted and turned into a template, this may be run over the image $I(x)$ in such a way as to remain just in contact with it. Thus, the origin of the inverted template will trace out the top surface of the dilated image. The process is depicted in Fig. 7.5 for the case of a triangular structuring element in a 1-D image.

Similar relations apply for erosion, closing, opening, and a variety of set functions. This means that the standard binary morphological relations, Eqs. (7.15)–(7.23), apply for grayscale images as well as for binary images. Furthermore, the dilation–erosion and closing–opening dualities (Eqs. (7.9), (7.13), (7.43), and (7.44)) also apply for grayscale images. These are extremely powerful results, and allow one to apply morphological concepts in an intuitive manner. In that case the practically important factor devolves into choosing the right grayscale structuring element for the application.

Another interesting factor is the possibility of using morphological operations instead of convolutions in the many places where the latter are employed throughout image analysis. We have already seen how edge enhancement and detection can be performed using morphology in place of convolution. In addition, noise suppression by Gaussian smoothing can be replaced by opening and closing operations. However, it must always be borne in mind that convolutions are linear operations, and are thereby grossly restricted, whereas morphological operations

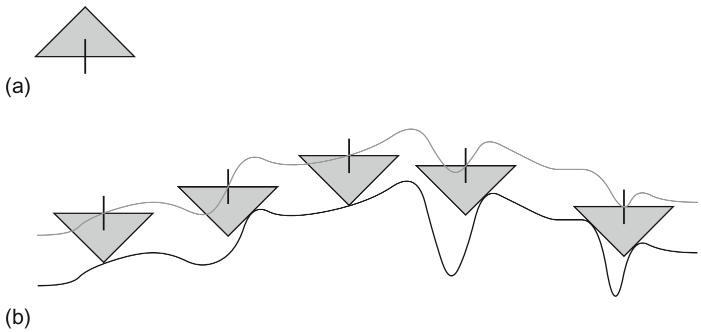


FIGURE 7.5

Dilation of 1-D grayscale image by triangular structuring element. (a) The structuring element, with the vertical line at the bottom indicating the origin of coordinates. (b) The original image (continuous black line), several instances of the inverted structuring element being applied, and the output image (continuous gray line). This geometric construction automatically takes account of the maximum operation in Eq. (7.51). Note that as no part of the structuring element is below the origin in (a), the output intensity is increased at every point in the image.

are highly nonlinear, their very structure embodying multiple "if" statements, so outward appearances of similarity are bound to hide deep differences of operation, effectiveness, and applicability. Consider, e.g., the optimality of the mean filter for suppressing Gaussian noise and the optimality of the median filter (a morphological operator) for eliminating impulse noise.

One example of this is worth including here: when performing edge enhancement prior to edge detection, it is possible to show that if both a differential gradient (e.g., Sobel) operator and a morphological gradient operator are applied to a noise-free image with a steady intensity gradient, the results will be identical, within a constant factor. However, if one impulse noise pixel arises, the maximum or minimum operations of the morphological gradient operator will select this value in calculating the gradient, whereas the differential gradient operator will average its effect over the window, giving significantly less error.

Space prevents grayscale morphological processing from being considered in more detail here (see, e.g., Haralick and Shapiro (1992) and Soille (2003)).

7.5 EFFECT OF NOISE ON MORPHOLOGICAL GROUPING OPERATIONS

Texture analysis is an important area of machine vision, and is relevant not only for segmenting one region of an image from another (as in many remote sensing applications), but also for characterizing regions absolutely—as is necessary when performing surface inspection (e.g., when assessing the paint finish on automobiles). Many methods have been employed for texture analysis. These range from the widely used gray-level co-occurrence matrix approach to Law's texture energy approach and from the use of Markov random fields to fractal modeling (Chapter 8).

In fact, there are approaches that involve even less computation and which are applicable when the textures are particularly simple and the shapes of the basic texture elements are not especially critical. For example, if it is required to locate regions containing small objects, simple morphological operations applied to thresholded versions of the image are often appropriate (Fig. 7.6) (Haralick and Shapiro, 1992; Bangham and Marshall, 1998). Such approaches can be used for locating regions containing seeds, grains, nails, sand, or other materials, either for assessing the overall quantity or spread or for determining whether there are regions that have not yet been covered. The basic operation to be applied is the dilation operation, which combines the individual particles into fully connected regions. This method is suitable not only for connecting individual particles but also for separating regions containing high and low densities of such particles. The expansion characteristic of the dilation operation can be largely canceled by a subsequent erosion operation, using the same morphological kernel. Indeed, if the particles are always convex and well separated, the erosion should exactly cancel the dilation, although in general the combined closing operation is not a null operation, and this is relied upon in the above connecting operation.

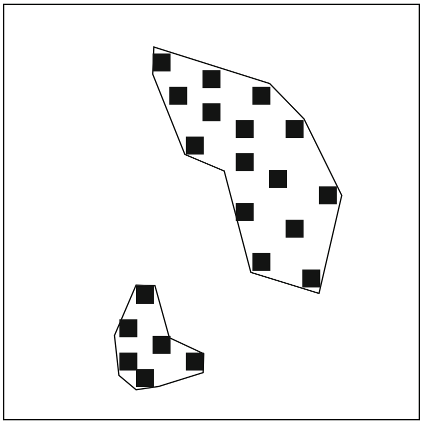


FIGURE 7.6

Idealized grouping of small objects into regions such as might be attempted using closing operations.

Source: © IEE 2000

Closing operations have been applied to images of cereal grains containing dark rodent droppings in order to consolidate the droppings (which contain significant speckle—and therefore holes when the images are thresholded) and thus to make them more readily recognizable from their shapes (Davies et al., 1998a). However, the initial result was rather unsatisfactory as dark patches on the grains tend to combine with the dark droppings: this has the effect of distorting the shapes and also makes the objects larger. This problem was partially overcome by a subsequent erosion operation so that the overall procedure is dilate + erode + erode (for further details, see Chapter 21). Originally, adding this final operation seemed to be an *ad hoc* procedure, but on analysis it was found that the size increase actually applies quite generally when segmentation of textures containing different densities of particles is carried out. It is this general effect that we now consider.

7.5.1 Detailed Analysis

Let us take two regions containing small particles with occurrence densities ρ_1 and ρ_2, where $\rho_1 > \rho_2$. In region 1, the mean distance between particles will be d_1 and in region 2, the mean distance will be d_2, where $d_1 < d_2$. If we dilate using a kernel of radius a, where $d_1 < 2a < d_2$, this will tend to connect the particles in region 1 but should leave the particles in region 2 separate. To *ensure* connecting the particles in region 1, we can make $2a$ larger than $\frac{1}{2}(d_1 + d_2)$, but this may risk connecting the particles in region 2 (the risk will be reduced when the subsequent erosion operation is taken into account). Selecting an optimum value of a clearly depends not only on the mean distances d_1 and d_2 but also on their distributions. This is not discussed in detail due to space limitation: here we assume that a suitable selection of a is made, and that it is effective. The problem that is tackled next is whether the size of the final regions matches the *a priori* desired segmentation, i.e., whether any size distortion takes place. We start by taking this to be an essentially 1-D problem, which can be modeled as in Fig. 7.7 (in what follows, the 1-D particle densities are given an x suffix).

Suppose first that $\rho_{2x} = 0$. Then in region 2, the initial dilation will be counteracted exactly (in 1-D) by the subsequent erosion. Next take $\rho_{2x} > 0$: when dilation occurs, a number of particles in region 2 will be enveloped, and the erosion process will not exactly reverse the dilation. If a particle in region 2 is within $2a$ of an outermost particle in region 1, it will merge with region 1, and will remain merged when erosion occurs. The probability P that this will happen is the integral over a distance $2a$ of the probability density for particles in region 2. In addition, when the particles are well separated we can take the probability density as being equal to the mean particle density ρ_{2x}. Hence:

$$P = \int_0^{2a} \rho_{2x} \mathrm{d}x = 2a\rho_{2x} \tag{7.52}$$

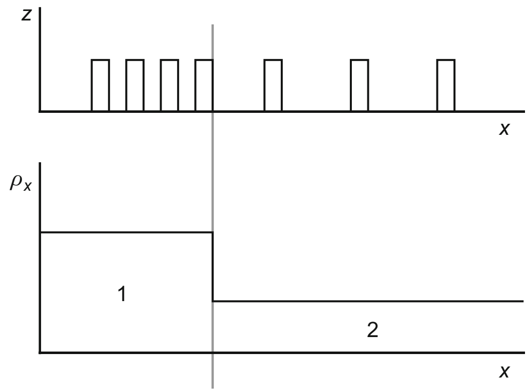


FIGURE 7.7

1-D particle distribution. z indicates the presence of a particle, and ρ_x shows the densities in the two regions.

Source: © *IEE 2000*

If such an event occurs, then region 1 will be expanded by amounts ranging from a to $3a$, or 0 to $2a$ after erosion, although these figures must be increased by b for particles of width b. Thus, the *mean* increase in size of region 1 after dilation + erosion is $2a\rho_{2x} \times (a + b)$, where we have assumed that the particle density in region 2 remains uniform right up to region 1.

Next we consider what additional erosion operation will be necessary to cancel this increase in size. In fact, we just make the radius $\tilde{a}_{1\text{-D}}$ of the erosion kernel equal to the increase in size:

$$\tilde{a}_{1\text{-D}} = 2a\rho_{2x}(a + b) \tag{7.53}$$

Finally, we must recognize that the required process is 2-D rather than 1-D, and take y to be the lateral axis, normal to the original (1-D) x-axis. For simplicity, we assume that the dilated particles in region 2 are separated laterally, and are not touching or overlapping (Fig. 7.8). As a result, the change of size of region 1 given in Eq. (7.53) will be diluted relative to the 1-D case by the reduced density along the direction (y) of the border between the two regions: i.e., we must multiply the right-hand side of Eq. (7.53) by $b\rho_{2y}$. We now obtain the relevant 2-D equation:

$$\tilde{a}_{2\text{-D}} = 2ab\rho_{2x}\rho_{2y}(a + b) = 2ab\rho_2(a + b) \tag{7.54}$$

where we have finally reverted to the appropriate 2-D *area* particle density ρ_2.

Clearly, for low values of ρ_2 an additional erosion will not be required, whereas for high values of ρ_2 substantial erosion will be necessary, particularly if b is comparable to or larger than a. If $\tilde{a}_{2\text{-D}} < 1$, it will be difficult to provide an accurate correction by applying an erosion operation, and all that can be done is

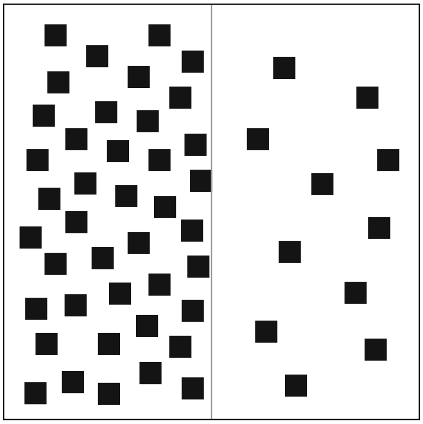


FIGURE 7.8

Model of the incidence of particles in two regions. Region 2 (on the right) has sufficiently low density that the dilated particles will not touch or overlap.

Source: © *IEE 2000*

to bear in mind that any measurements made from the image will require correction. (Note that if, as often happens, $a \gg 1$, $\tilde{a}_{2\text{-D}}$ could well be at least 1.)

7.5.2 Discussion

The work of the author described above (Davies, 2000c) was motivated by analysis of cereal grain images containing rodent droppings, which had to be consolidated by dilation operations to eliminate speckle, followed by erosion operations to restore size.[9] It has been found that if the background contains a low density of small particles that tend, upon dilation, to increase the sizes of the foreground objects, additional erosion operations will in general be required to accurately represent the sizes of the regions. The effect would be similar if impulse noise were present, although theory shows what is observed in practice—that the effect is enhanced if the particles in the background are not negligible in size. The increases in size are proportional to the occurrence density of the particles in the background, and the kernel for the final erosion operation is calculable, the overall process being a necessary measure rather than an *ad hoc* technique.

7.6 CONCLUDING REMARKS

Binary images contain all the data needed to analyze the shapes, sizes, positions, and orientations of objects in two dimensions, and thereby to recognize them

[9]For further background on this application, see Chapter 21.

and even to inspect them for defects. As we shall see in Chapters 9 and 10, many simple small neighborhood operations exist for processing binary images and moving toward the goals stated earlier. At first sight these may appear a somewhat random set, reflecting historical development rather than systematic analytic tools. However, in the past few years, mathematical morphology has emerged as a unifying theory of shape analysis: we have aimed to give the flavor of the subject in this chapter. In fact, mathematical morphology, as its name suggests, is mathematical in nature, and this can be a source of difficulty,[10] but there are a number of key theorems and results, which are worth remembering: a few of these have been considered here and placed in context. For example, generalized dilation and erosion have acquired a central importance, since further vital concepts and constructs are based on them—closing, opening, template matching, and even connectedness properties (although space has prevented a detailed discussion of its application to the last two topics). For further information on grayscale morphological processing, see Haralick and Shapiro (1992) and Soille (2003).

> Mathematical morphology is one of the standard methodologies that has evolved over the past few decades. This chapter has demonstrated how the mathematical aspects make the subject of shape analysis rigorous and less ad hoc. Its extensions to grayscale image processing are interesting and useful, although at this stage they have not ousted more traditional approaches.

7.7 BIBLIOGRAPHICAL AND HISTORICAL NOTES

The book by Serra (1982) was an important early landmark in the development of morphology. Many subsequent papers helped to lay the mathematical foundations, perhaps the most important and influential being that by Haralick et al. (1987); see also Zhuang and Haralick (1986) for methods for decomposing morphological operators, and Crimmins and Brown (1985) for more practical aspects of shape recognition. The papers by Dougherty and Giardina (1988), Heijmans (1991), and Dougherty and Sinha (1995a, 1995b) were important in the development of methods for grayscale morphological processing, while the work of Huang and Mitchell (1994) on grayscale morphology decomposition and that of Jackway and Deriche (1996) on multiscale morphological operators gave further impetus to the subject.

[10]The rigor of mathematics is a cause for celebration, but at the same time it can make the arguments and the results obtained from them less intuitive. On the other hand, the real benefit of mathematics is to leapfrog what is possible by intuition alone and to arrive at results that are new and unexpected.

One problem is that it is by no means obvious how to decide on the sequence of morphological operations that is required in any application. This is an area where genetic algorithms have contributed to the systematic generation of complete systems (see, e.g., Harvey and Marshall, 1994).

Work up to 1998 is reviewed in a useful tutorial paper by Bangham and Marshall (1998): more recently Soille (2003) produced a thoroughgoing volume on the subject. Gil and Kimmel (2002) address the problem of rapid implementation of dilation, erosion, opening, and closing algorithms, and arrive at a new approach based on deterministic calculations: these give low computational complexity for calculating max and min functions, and similar complexities for the four cited filters and other derived filters. The paper goes on to state some open problems, and to suggest how they might be tackled: it is clear that some apparently simple tasks that ought perhaps to have been dealt with before the 2000s are still unsolved—such as how to compute the median in better than $O(\log^2 p)$ time per filtered point (in a $p \times p$ window) or how to *optimally* extend 1-D morphological operations to circular rather than square window (2-D) operations. Note that the new implementation is immediately applicable to determining the morphological gradient of an image.

7.7.1 More Recent Developments

More recently, Bai and Zhou (2010) have designed a top-hat selection transformation for locating and enhancing small dim infrared targets typified by aircraft in the sky. The selection transformation is based on the classical top-hat (residue) operator. A necessary parameter in the analysis is the value of n, the minimum difference in intensity between the target and the background, and methods are given for estimating it. Jiang et al. (2007) also use a residue operator to find thin low-contrast edges. The method uses five basic 5×5 masks to detect edges of the right widths. Very high resistance to noise is demonstrated by the particular combination of techniques applied in this approach. Soille and Vogt (2009) show how binary images may be segmented to identify a range of different types of pattern. These include the following mutually exclusive foreground categories: core, islet, connector (loop and bridge), boundary (perforation and edge), branch, and segmented binary pattern. Lézoray and Charrier (2009) describe a new approach to color image segmentation, by analysis of color projections in 2-D histograms to find the dominant colors: the important factor is that clustering in 2-D histograms can proceed very effectively using standard image processing techniques, including morphological processing. Valero et al. (2010) use directional mathematical morphology to detect roads in remote sensing images. The paper starts by taking roads to be linear connected paths; however, curved road segments and other network details can be dealt with using "path openings" and "path closings" in order to obtain the required structural information.

7.8 PROBLEM

1. A morphological gradient binary edge enhancement operator is defined by the formula:

$$G = (A \oplus B) - (A \ominus B)$$

Using a 1-D model of an edge, or otherwise, shows that this will give wide edges in binary images. If grayscale dilation ($\oplus$) is equated to taking a local *maximum* of the intensity function within a 3×3 window, and grayscale erosion ($\ominus$) is equated to taking a local *minimum* within a 3×3 window, sketch the result of applying the operator G. Show that it is similar in effect to a Sobel edge enhancement operator, if edge orientation effects are ignored by taking the Sobel magnitude:

$$g = (g_x^2 + g_y^2)^{1/2}$$