Lecture 4: Stochastic Thinking and Random Walks

Relevant Reading

- Pages 235-238
- Chapter 14

The World is Hard to Understand

- Uncertainty is uncomfortable
- But certainty is usually unjustified

Newtonian Mechanics

- Every effect has a cause
- The world can be understood causally

Copenhagen Doctrine

- Copenhagen Doctrine (Bohr and Heisenberg) of causal nondeterminism
  - At its most fundamental level, the behavior of the physical world cannot be predicted.
  - Fine to make statements of the form “x is highly likely to
  - occur,” but not of the form “x is certain to occur.”
- Einstein and Schrödinger objected
  - “God does not play dice.” -- Albert Einstein

Does It Really Matter

Did the flips yield

2 heads

2 tails

1 head and 1 tail?

The Moral

- The world may or may not be inherently unpredictable
- But our lack of knowledge does not allow us to make accurate predictions
- Therefore we might as well treat the world as inherently unpredictable
- Predictive nondeterminism

Stochastic Processes

- An ongoing process where the next state might depend on both the previous states and some random element

```
def rollDie():
    """returns an int between 1 and 6"""

def rollDie():
    """returns a randomly chosen int
       between 1 and 6"""
```

Implementing a Random Process

```
import random

def rollDie():
    """returns a random int between 1 and 6"""
    return random.choice([1,2,3,4,5,6])

def testRoll(n = 10):
    result = ''
    for i in range(n):
        result = result + str(rollDie())
    print(result)
```

Probability of Various Results

- Consider `testRoll(5)`
- How probable is the output 11111?

Probability Is About Counting

- Count the number of possible events
- Count the number of events that have the property of interest
- Divide one by the other
- Probability of 11111?
  - 11111, 11112, 11113, …, 11121, 11122, …, 66666
  - 1/(6**5)
  - ~0.0001286

Three Basic Facts About Probability

- Probabilities are always in the range 0 to 1. 0 if impossible, and 1 if guaranteed.
- If the probability of an event occurring is p, the probability of it not occurring must be
- When events are independent of each other, the probability of all of the events occurring is equal to a product of the probabilities of each of the events occurring.

Independence

- Two events are independent if the outcome of one event has no influence on the outcome of the other
- Independence should not be taken for granted

Will One of the Patriots and Broncos Lose?

- Patriots have winning percentage of 7/8, Broncos of 6/8
- Probability of both winning next Sunday is 7/8 * 6/8 = 42/64
- Probability of at least one losing is 1 – 42/64 = 22/64
- What about Sunday, December 18
  - Outcomes are not independent
  - Probability of one of them losing is much closer to 1 than to 22/64!

A Simulation of Die Rolling

```
def runSim(goal, numTrials, txt):
    total = 0
    for i in range(numTrials):
        result = ''
        for j in range(len(goal)):
            result += str(rollDie())
        if result == goal:
            total += 1
    print('Actual probability of', txt, '=',
          round(1/(6**len(goal)), 8))
    estProbability = round(total/numTrials, 8)
    print('Estimated Probability of', txt, '=',
          round(estProbability, 8))

runSim('11111', 1000, '11111')
```

Output of Simulation

- Actual probability = 0.0001286
- Estimated Probability = 0.0
- Actual probability = 0.0001286
- Estimated Probability = 0.0

- How did I know that this is what would get printed?
- Why did simulation give me the wrong answer?

Let's try 1,000,000 trials

Morals

- Moral 1: It takes a lot of trials to get a good estimate of the frequency of occurrence of a rare event. We'll talk lots more in later lectures about how to know when we have enough trials.
- Moral 2: One should not confuse the sample probability with the actual probability
- Moral 3: There was really no need to do this by simulation, since there is a perfectly good closed form answer. We will see many examples where this is not true.
- But simulations are often useful.

The Birthday Problem

- What's the probability of at least two people in a group having the same birthday
- If there are 367 people in the group?
- What about smaller numbers?
- If we assume that each birthdate is equally likely
  - $1-\frac{366!}{366^N * (366-N)!}$
- Without this assumption, VERY complicated

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Approximating Using a Simulation

```
def sameDate(numPeople, numSame):
    possibleDates = range(366)
    birthdays = [0]*366
    for p in range(numPeople):
        birthDate = random.choice(possibleDates)
        birthdays[birthDate] += 1
    return max(birthdays) >= numSame
```

Approximating Using a Simulation

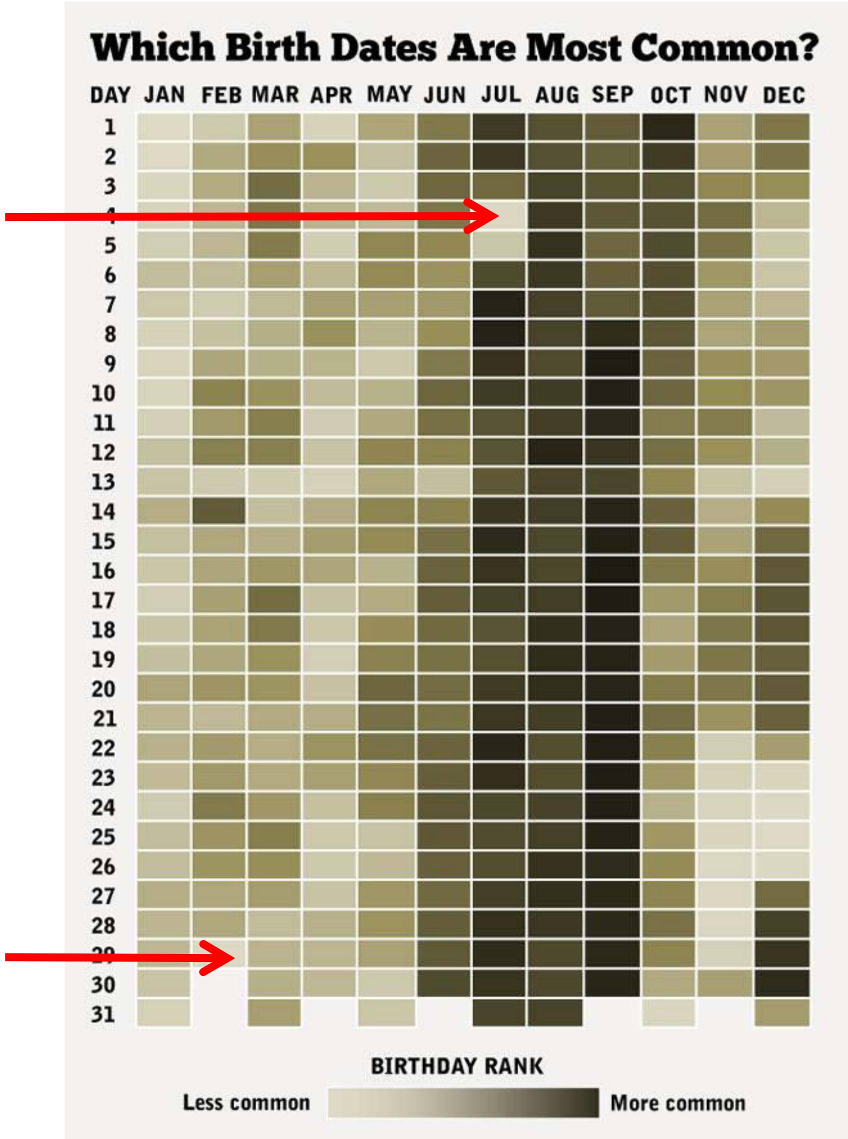
```
def birthdayProb(numPeople, numSame, numTrials):
    numHits = 0
    for t in range(numTrials):
        if sameDate(numPeople, numSame):
            numHits += 1
    return numHits/numTrials

for numPeople in [10, 20, 40, 100]:
    print('For', numPeople,
          'est. prob. of a shared birthday is',
          birthdayProb(numPeople, 2, 10000))
    numerator = math.factorial(366)
    denom = (366**numPeople)*math.factorial(366-numPeople)
    print('Actual prob. for N = 100 =',
          1 - numerator/denom)
```

Suppose we want the probability of 3 people sharing

Why 3 Is Much Harder Mathematically

- For 2 the complementary problem is “all birthdays distinct”
- For 3 people, the complementary problem is a complicated disjunct
  - All birthdays distinct or
  - One pair and rest distinct or
  - Two pairs and rest distinct or
  - …
- But changing the simulation is dead easy

But All Dates Are Not Equally Likely



Are you exceptional?

Another Win for Simulation

- Adjusting analytic model a pain
- Adjusting simulation model easy

```
def sameDate(numPeople, numSame):
    possibleDates = 4*list(range(0, 57)) + [58]\
                    + 4*list(range(59, 366))\
                    + 4*list(range(180, 270))
    birthdays = [0]*366
    for p in range(numPeople):
        birthDate = random.choice(possibleDates)
        birthdays[birthDate] += 1
    return max(birthdays) >= numSame
```

Simulation Models

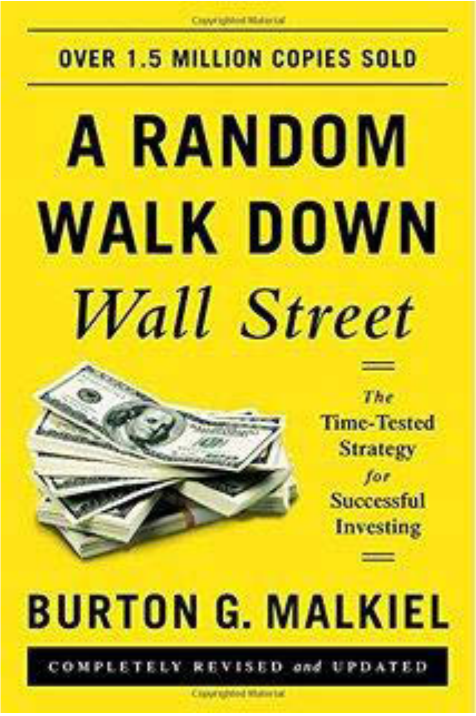
- A description of computations that provide useful information about the possible behaviors of the system being modeled
- Descriptive, not prescriptive
- Only an approximation to reality
- “All models are wrong, but some are useful.” – George Box

Simulations Are Used a Lot

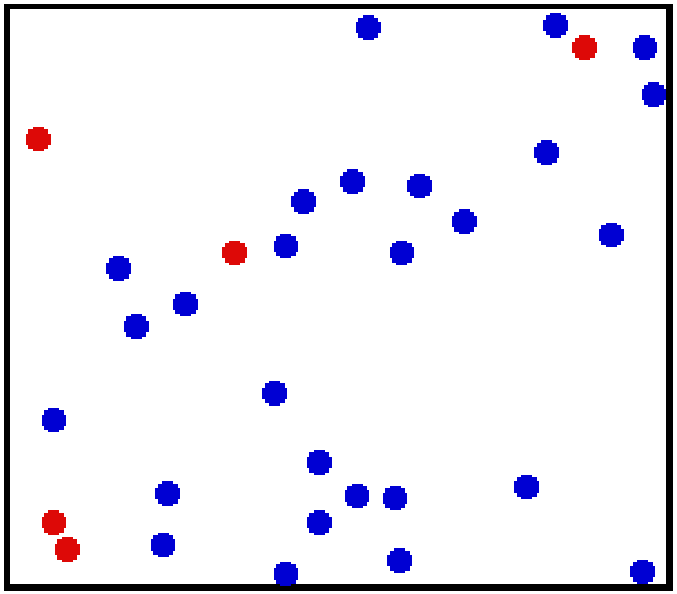
- To model systems that are mathematically intractable
- To extract useful intermediate results
- Lend themselves to development by successive refinement and "what if" questions
- Start by simulating random walks

Why Random Walks?

- Random walks are important in many domains
  - Understanding the stock market (maybe)
  - Modeling diffusion processes
  - Etc.
- Good illustration of how to use simulations to understand things
- Excuse to cover some important programming topics
  - Practice with classes
  - More about plotting

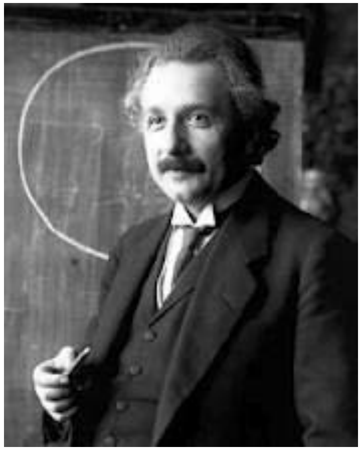


Brownian Motion Is a Random Walk



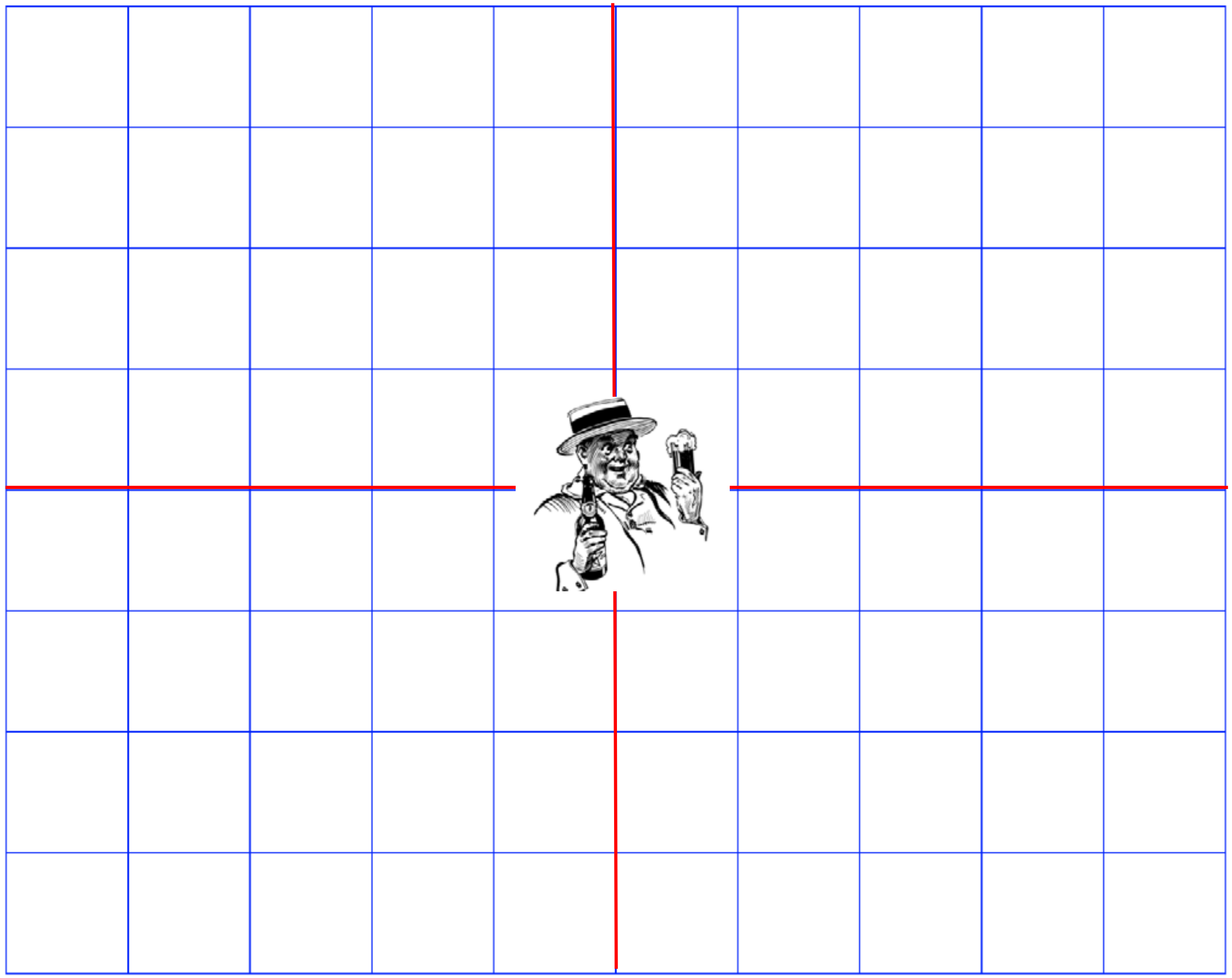
Robert Brown 1827

Louis Bachelier 1900

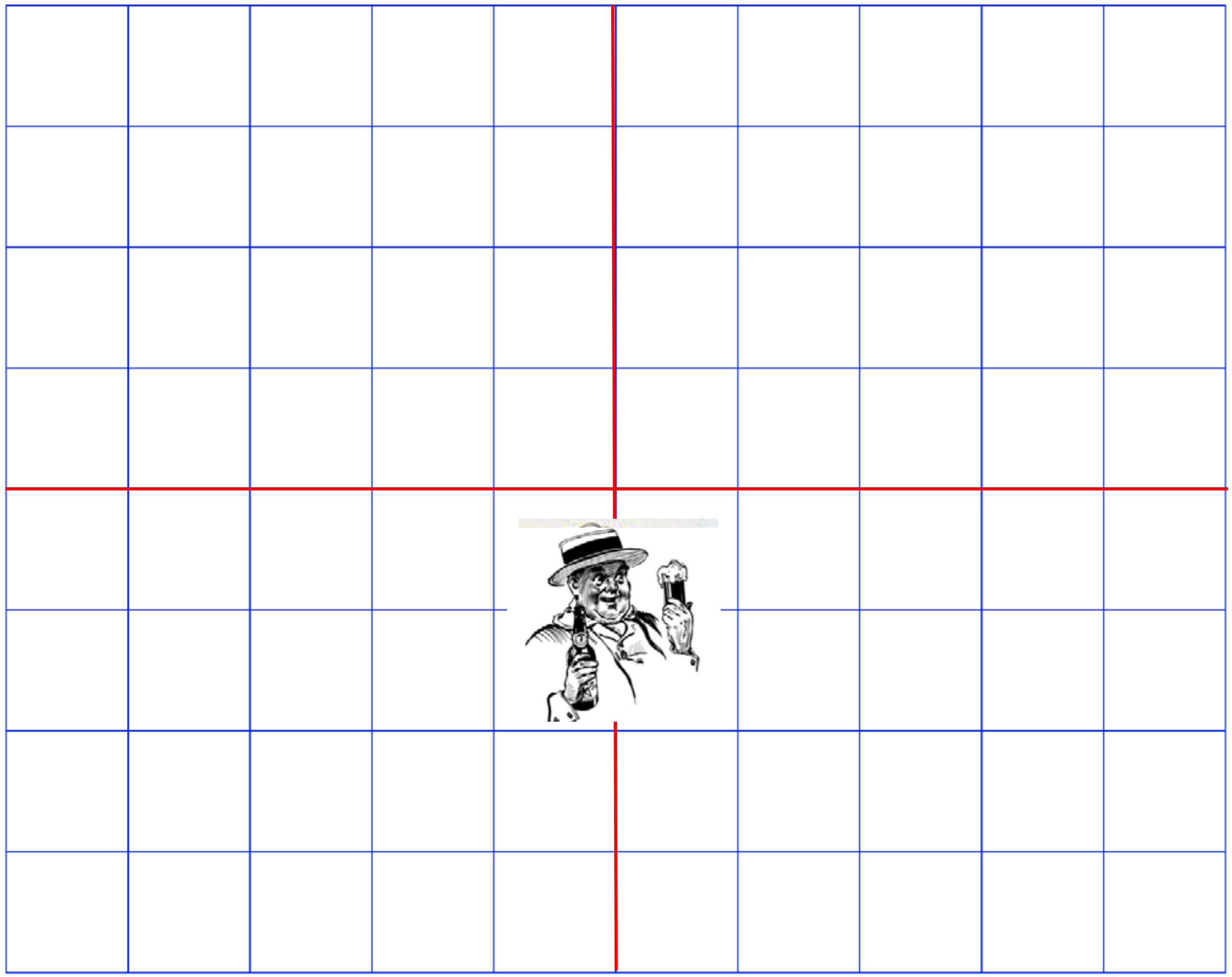


Albert Einstein 1905

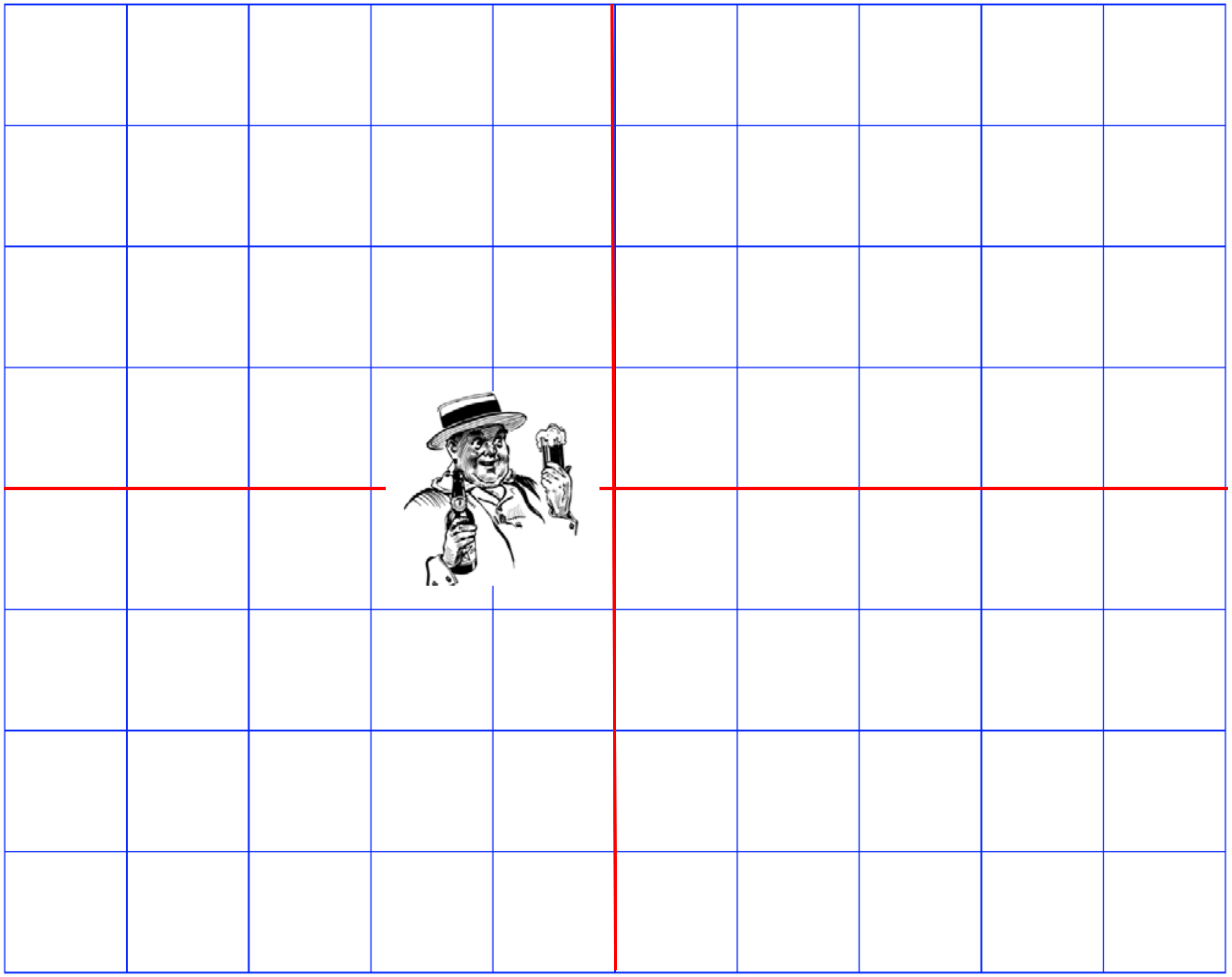
Drunkard's Walk



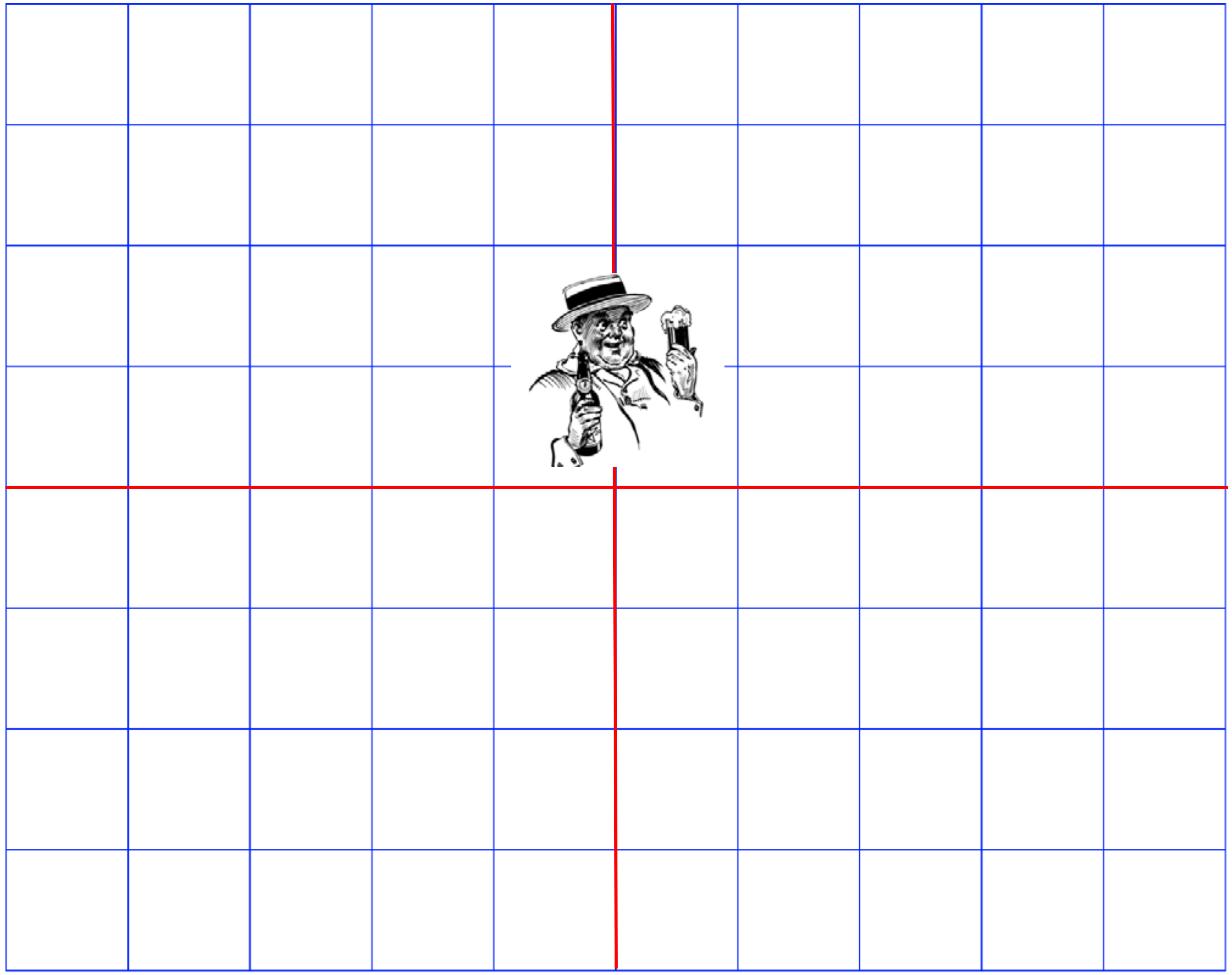
One Possible First Step



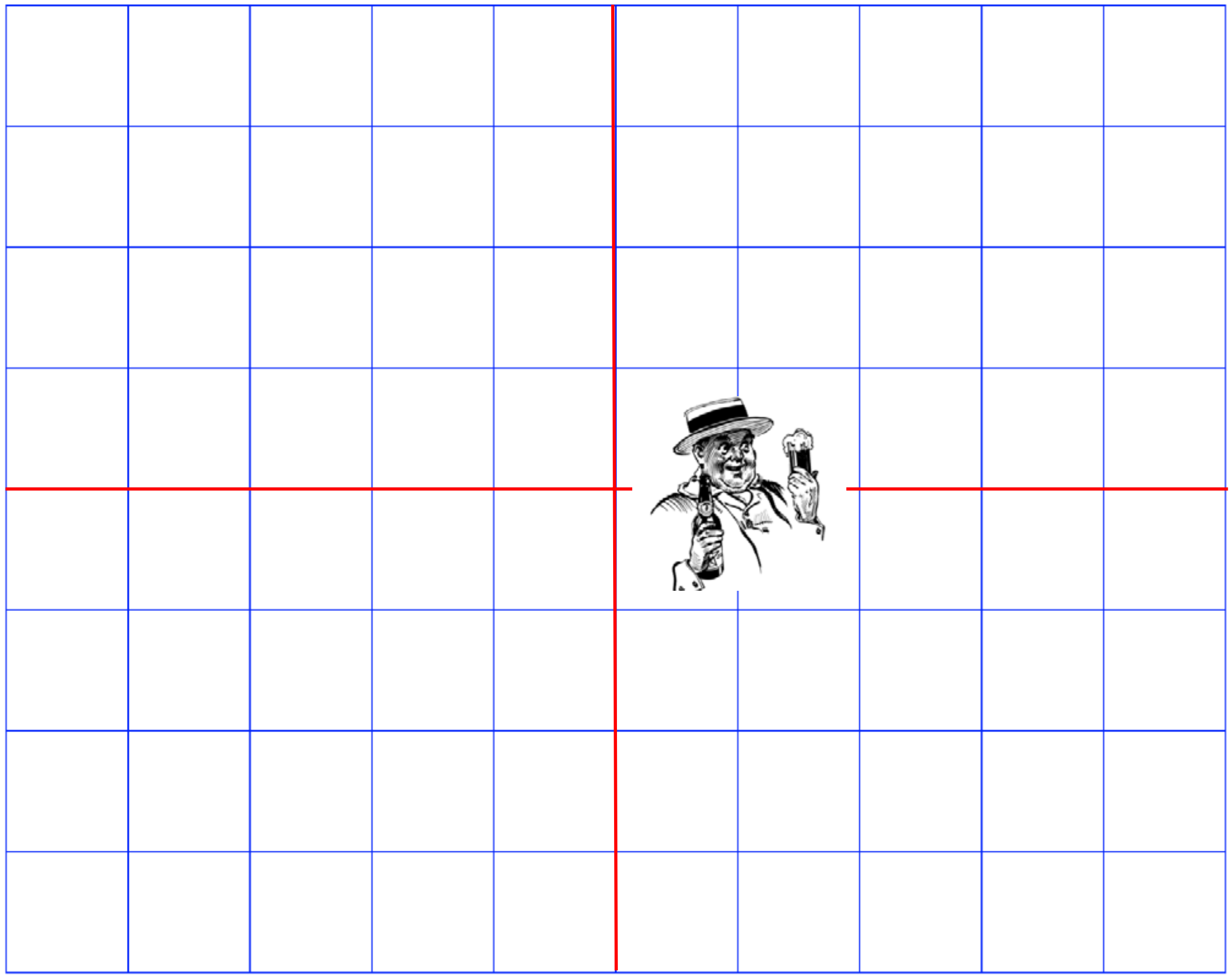
Another Possible First Step



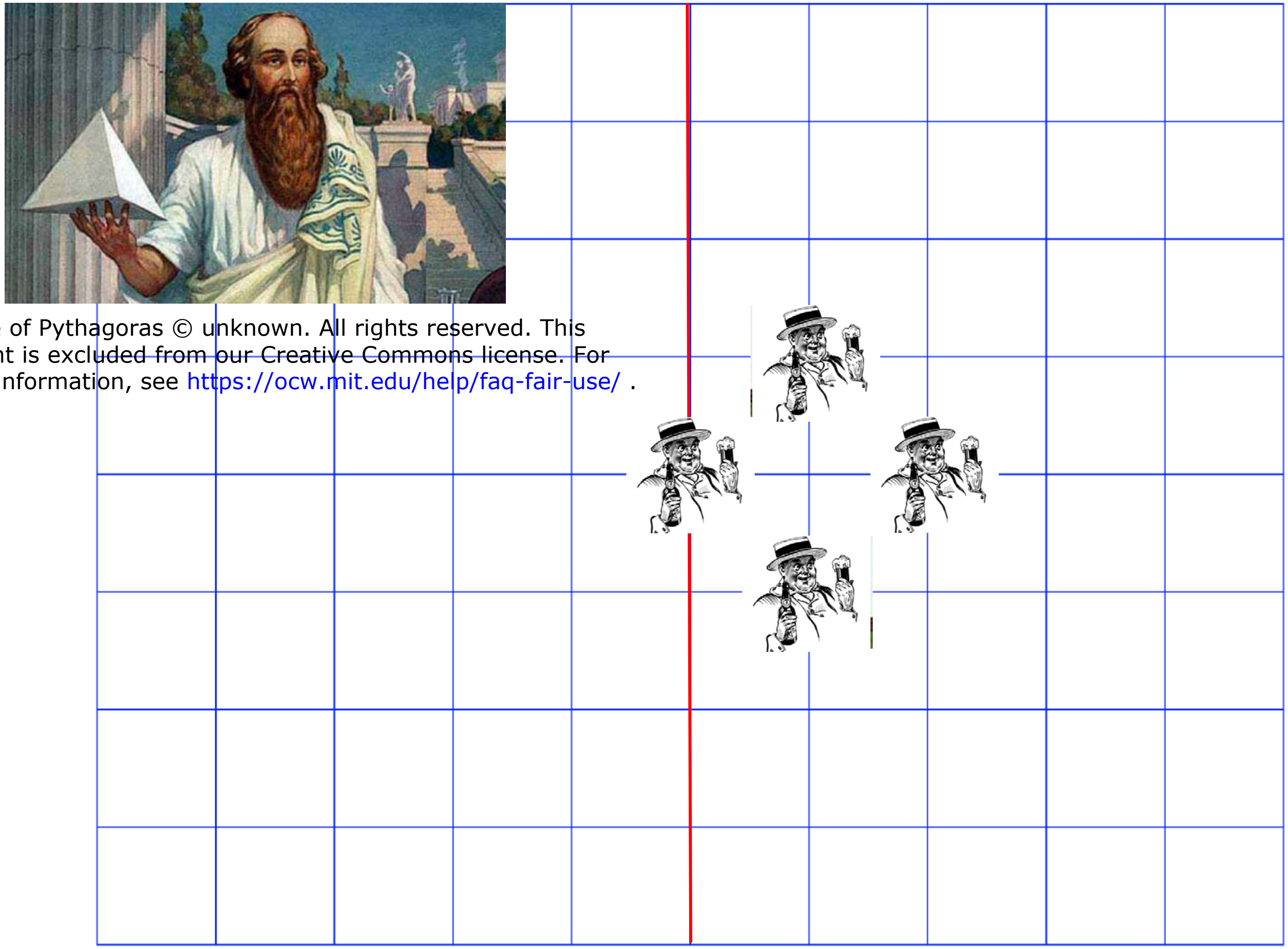
Yet Another Possible First Step



Last Possible First Step



Possible Distances After Two Steps



Expected Distance After 100,000 Steps?

- Need a different approach to problem
- Will use simulation
- But not until the next lecture