Discrete Mathematics

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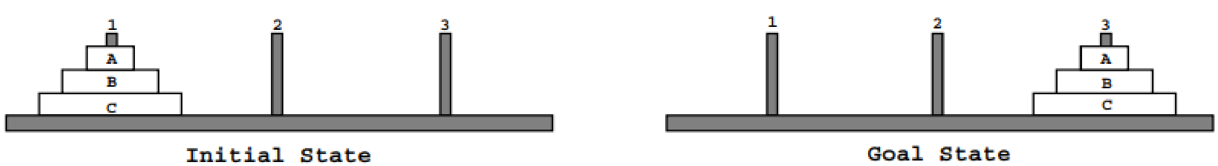
"Tower of Hanoi: Exploring Recursive Algorithms and Optimal Solutions"

Abstract

This report delves into the Tower of Hanoi problem, exploring its origin, problem statement, and significance as a beginner's introduction to recursive algorithms. It discusses various approaches to finding optimal solutions, highlights the Hanoi constant, and reveals its intriguing connection with Gray codes through bit representation. Additionally, the report examines the practical application of the Tower of Hanoi in backup techniques within computer science. It introduces variations like the Reves puzzle and touches upon the Frame-Stewart algorithm, an optimal solution that remains unproven. This comprehensive overview aims to provide a clear understanding of the Tower of Hanoi, showcasing its relevance in theoretical algorithms and real-world scenarios.

Introduction

The Tower of Hanoi is a classic math puzzle first introduced by French mathematician Edouard Lucas in 1883. There are old stories suggesting that this puzzle is inspired by ancient history. According to these tales, a successful transfer of the 64 golden disks from one post to another would bring about the end of the world—intriguing, right? But don't worry— even if we take just a second to move one disk, it would actually take about 585 billion years to complete, which is 42 times longer than the estimated age of the universe. We'll find out why that number is significant shortly.



Let's now understand what the puzzle is?

The puzzle involves three pegs and a set of disks stacked in ascending order of size. The challenge is simple to move the entire stack from one peg to another while following three simple rules:

- ➢ Only one disc may be moved at a time,
- ➢ It is the top disc of a tower that may be moved from one peg to the other, and
- ➢ No disc may rest upon a smaller disc at any time.

Mathematically,

The Tower of Hanoi problem involves three pegs (P1,, P2 and P3) and n discs (D1, D2, D3 . . ., Dn) such that D1 < D2 < D3 < • • • < Dn, where D1 is the smallest disc. Initially, all the n discs are placed on a peg Pi as a pyramid with D1 on the top. The task is to move these n discs from Pi to Pj such that i != j , subject to the constraints as listed above.

Over the years, mathematicians have devised various methods to determine the optimal solution to the Tower of Hanoi problem. It has been proven that the minimum number of moves required to solve a Tower of Hanoi problem with n disks is 2^n - 1. This is how the staggering number 585 billion (2^64 - 1) comes, showcasing the exponential nature of the solution space. Notably, this minimum number of moves is unique, and any deviation from the algorithm would result in a suboptimal solution.
The recursive algorithm remains the preferred approach for solving the Tower of Hanoi problem. It serves as an excellent illustration for recursion, demonstrating how a seemingly intricate problem can be systematically broken down into simpler subproblems.

While the recursive approach is powerful, an alternative lies in the iterative method. This straightforward approach follows a set of rules, enabling the problem to be solved with a large number of disks even by human intervention. Unlike the recursive method, the iterative approach offers a practical advantage in competitions, where you can neither accurately recall all unique steps from memory nor you can hold the stack frame in your mind if tried following recursive method.

Reve's problem, the Tower of Hanoi problem with 4 pegs and the unproven optimal solution of it, Frame Stewart Conjecture is also discussed later in this report, touching on interesting facts like its relation with Gray Code, Sierpinski triangle and the Hanoi Constant.

Existing Literature

Numerous studies have explored the topics mentioned in this report. One such paper is "A Representation Approach to the Tower of Hanoi Problem" by M. C. Er (Department of Computing Science, The University of Wollongong, 1981). It looks into the iterative algorithm proposed by Hayes, Buneman, and Levy. The paper uses tree and bits representation to make it easier for us to understand why the assumptions in the iterative method for the classical puzzle are the way they are. Another paper, "An efficient implementation of Tower of Hanoi using Gray Codes" by Hari Krishnan.V, Sandhya M.K, and Monica Jeniffer.B, helped me understand the connection between the Tower of Hanoi puzzle and Gray codes. There's also "Iterative Algorithm for the Reve's Puzzle" by A van de Liefvoort, which expands the iterative approach to more than three pegs and introduces the concept of the super disc. These and many other papers collectively contribute to a better understanding of the Tower of Hanoi problem.

Topic of Study

This section includes everything in detail which I could came across and learn in the logical algorithmic field via this simple Tower of Hanoi puzzle. It includes lthe different approaches to the puzzle, its variations and interesting facts related to the mathematical and computing world.

A. Recursive Algorithm:

For this puzzle, the basic idea is to understand the fact that if there are n discs on peg P1, then what we do is temporarily shift the top n-1 discs on peg P3, slide the largest one to peg P2 and then bring the n-1 discs on top of the largest disc on P2. This might intuitively seem contrary to the condition C1 but try to digress it.

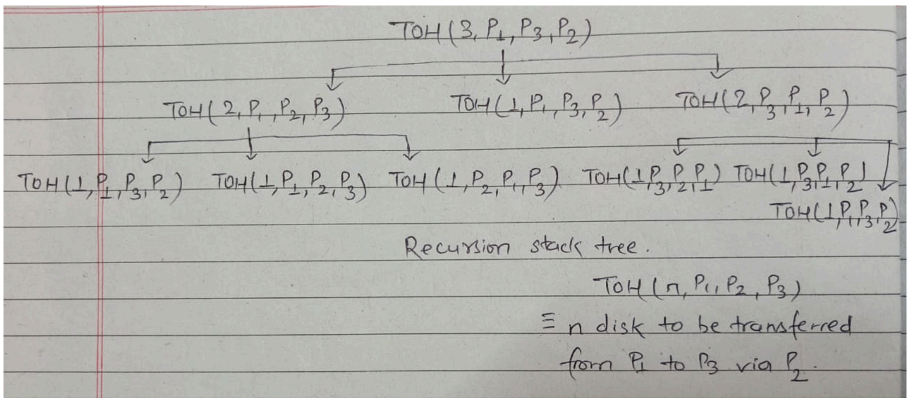
This is a recursive algorithm which typically consists of a general recursive step and then a base condition. The recursive steps bring down the complex task down to the base condition for which we know the solution.

for n=1: we can simply shift the only one D1 from P1 to P2,
for n=2: we shift D1 from P1 to P3, D2 from P1 to P2, bring back D1 from P3 to P2.
for n=3: simply do what you did for n=2, but this time shift them to P3, next shift D3 to P2, and again as done for n=2, shift the two disks D1 and D2 from P3 to P2.

This way for any number of disks, the puzzle can be solved, with the simple recursion logic.

The complexity remains O(2^n) due to the recursion stack. For n discs, there will be n recursive calls terminated when we reach n=1. The code for this algorithm is also provided with the report.

Recursion tree:



The minimal steps is 2^n - 1 is found by solving the recurrence relation:

$$
\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\
&= 2^n - 1.
\end{aligned}
$$

B. Iterative Algorithm:

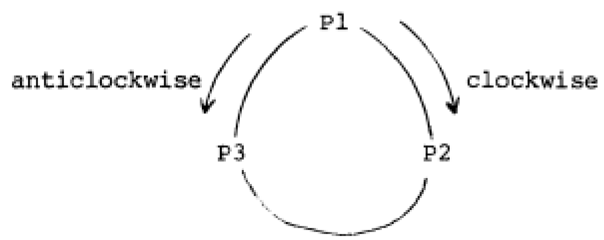


Figure 1 Arrangement of Pegs

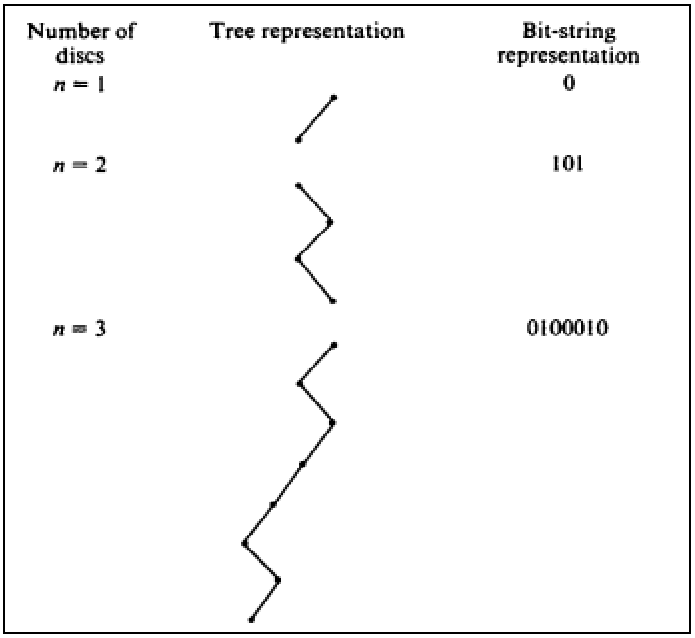
Hayes (1977) and Buneman and Levy (1980) proposed an iterative algorithm for the Tower of Hanoi, introducing a triangular arrangement of the three pegs, unlike the typical linear configuration, as shown in the above figure.
The algorithm works with n discs (D1, D2, ..., Dn), where D1 is the smallest. P1 is the initial peg, P2 is the target and P3 the auxiliary peg.

Algorithm:

- ➢ If the total count of disks is even, move D1 clockwise(else anticlockwise) as the first move and then every other turn.
- ➢ This leaves a single possible way to move the other disks without violating the TOH constraints, iterate over the two steps and after exactly 2^n -1 moves, the puzzle gets solved.

Adhering to these principles and Tower of Hanoi constraints yields a uniquely determined, minimal solution, as it leaves out a single valid move each time.

Visualization using a binary tree structure: left children represent clockwise moves, right children represent anticlockwise moves.
For bits representation, Huffman's coding is used which employs 0 for left children and 1 for right children.



Observation from above table:

The recursive nature of the problem can be seen here as well, for eg, if we need to make the bits representation for n discs, we find it as the 1's complement of bits representation for n-1 disks, then the bit 0 and then again 1's complement of bits representation for n-1 discs.

By this the work becomes so easy,

for n=4, 0

1_0_1

010_0_010

→ 1011101_0_1011101, interpret 1 as anticlock, 0 as clockwise direction,

Relation with Gray Codes:

The Tower of Hanoi and Gray codes share a significant connection when the number of bits in the Gray code corresponds to the number of disks in the Tower of Hanoi. In this setup, counting in Gray code from zero upwards, each code represents a unique state, and the bit that changes in consecutive codes indicates the disk to be moved. The least significant bit represents the smallest disk, while the most significant bit represents the largest. By counting moves from 1 and identifying disks in order of increasing size, the position of the disk to move during a given move is determined by the number of times the move can be divided by 2.

3. Reve’s puzzle

The Canteburry Puzzles from Henry Ernest Dudeney in 1896 was where this variation to the classic Tower of Hanoi first introduced. Its a simple variation, one with 4 pegs and n disks, however Dudeney discusses solution for 3, 4 and 5 pegs, forming up an interesting table.

Iterative Algorithm:

The same three phases of the standard solution to the 3-peg problem with n discs, (moving all but the largest disk to the auxiliary peg, then placing the largest in the target peg and finally, bringing back the other disks from auxiliary to the target peg) is generalized here for more than 3 pegs.

Whats done here is the disks are grouped as super disks and follow the same algo in an abstract manner.

Lets say for n = 6 (1+2+3) and 4 pegs, the three superdisks are:

S1-> D1, S2-> D2, D3 and S3 -> D4, D5, D6

Its simple, think of them as the three disk - three pegs problem, and then place those seven steps, but since you cant be moving the S2 and S3 at once, those are abstractly moved by using the 4th peg.

4. Frame- Stewart Conjecture

Dudeney asserted the minimal steps for the Tower of Hanoi problem, but without proof. In 2014, Thiery Bousch validated the optimality of the 4-peg solution, but for pegs more than that, it still remains unproved.

Frame Stewart proposed an algorithm for the optimality, though still a conjecture. The approach involves dividing the n disks into two sets and employing a trial-and-error method to find the split that minimizes moves. For instance, with 7 disks and 4 pegs, considering splits like 5 and 2, the algorithm calculates moves for each split. While a split of 5 and 2 may take 29 moves, a split of 4 and 3 achieves a faster 25 moves. Despite this empirical evidence, a formal proof of optimality remains elusive. The algorithm leverages known optimal solutions for up to 3 pegs, yet its conjectured optimality for more than 4 pegs lacks conclusive proof.

The Frame- Stewart's Algorithm:

Let n be the number of disks and r be the number of pegs and we define T(n,r) to be the minimum number of moves required to transfer n disks using r pegs.

The algorithm can be described recursively:

1. For some k, $1 \le k <$, transfer the top k disks to a single peg other than the start or destination pegs, taking T(k,r) moves.
2. Without disturbing the peg that now contains the top k disks, transfer the remaining n-k disks to the destination peg, using only the remaining r-1 pegs, taking T(n-k,r-1) moves.
3. Finally, transfer the top k disks to the destination peg, taking T(k,r) moves. The entire process takes 2T(k,r)+T(n-k,r-1) moves. Therefore, the count k should be picked for which this quantity is minimum

5. Sierpinski triangle and Hanoi constant

The state graph of the Tower of Hanoi puzzle comprises vertices representing all conceivable transition states, with connecting lines indicating the feasibility of moving between these states in a single move. What this diagram closely resembles is with the Sierpinski triangle.

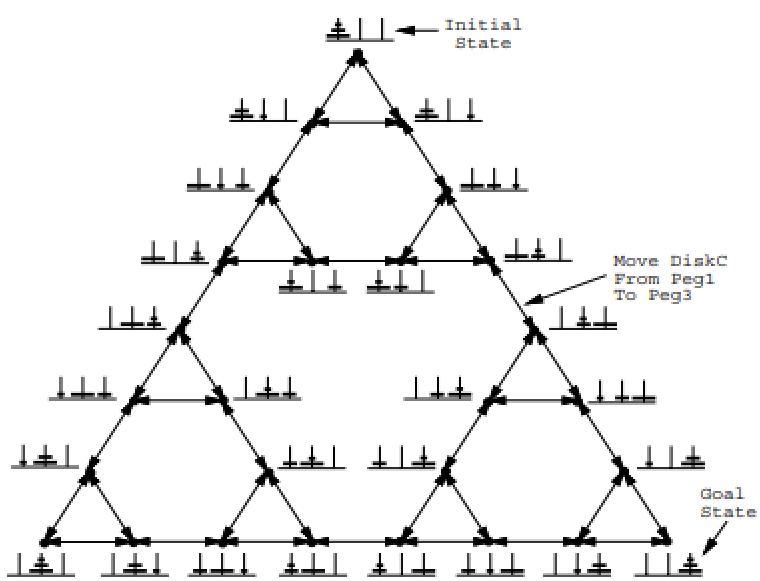


Figure 2: State Space for the Three-Disk Tower of Hanoi Puzzle

The top and bottom right joining straight line is the minimal solution route.
The number 466/885 is termed as the Hanoi constant which comes as the approximate multiple of the average minimal steps between any two states to the maximum minimal length.

Conclusion

In wrapping up my exploration of the Tower of Hanoi, I've learned a lot about recursive algorithms, the intuitive iterative approach using bits representation, Gray code, and cool math concepts like the Sierpinski triangle and Hanoi constant. The Frame Stewart conjecture, proposing an optimal algorithm, caught my interest, suggesting a cool area for more research. Understanding its links with Gray codes and its practical use in backup strategies adds to the puzzle's significance. Looking ahead, proving the Frame Stewart conjecture and exploring connections like those with Gray codes could open up new possibilities in math and computing.

References:

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3. H Mayer and D Perkins, “Tower of Hanoi Revisited, A Non Recursive Surprise”, pp 80-84
4. An efficient implementation of Tower of Hanoi using Gray Codes, Hari Krishnan.V Sandhya M.K Monica Jeniffer.B