Unit-1

Electromagnetic Theory

- Differential form of Gauss's Law of Electrostatics
- Poisson's and Laplace's Equations
- Continuity Equation
- Maxwell's Equations and Physical Significance

Differential form of Gauss's Law

Let the charges are distributed over volume, ρ be the volume charge density

Then $$\boldsymbol{q} = \int_v \rho dv \qquad \text{-------(1)}$$

We have Gauss's Law $$\varepsilon_0 \oint \vec{E}.\overrightarrow{dS} = q$$

Therefore (1) becomes

$$\varepsilon_0 \oint \vec{E}.\overrightarrow{dS} = \int_v \rho dv \qquad ---- -(2)$$

Using Gauss's divergence theorem

$$\oint \vec{E}.\overrightarrow{dS} = \int_v \nabla.E\,dv$$

From (2)

$$\varepsilon_0 \int_v (\nabla.E)\,dv = \int_v \rho dv$$

Since the equality holds good for every volume, the integrands of L.H.S. and R.H.S. should be equal

$$\varepsilon_0(\nabla . E) = \rho \qquad ----(3)$$

$$(\nabla . \varepsilon_0 E) = \rho \qquad (\varepsilon_0\text{=constant P.Q})$$

Here, $\varepsilon_0 E = D$ is the electric flux density.

$$(\nabla . D) = \rho$$

This is differential form of Gauss's Law of electrostatics.

Another Form

From (3), differential form of Gauss's Law can also be written as

$$(\nabla . E) = \frac{\rho}{\varepsilon_0}$$

Poisson's and Laplace's Equations

We have $\vec{\nabla}.\vec{D} = \rho$ Gauss's Law in a medium

$\vec{D} = \varepsilon\vec{E}$ Which is electric flux density

Because $\vec{E} = -\vec{\nabla}V$

$$\nabla^2\text{V} = -\rho\ /\varepsilon$$ This is the Poisson's Eq.

The solution to Poisson's equation is the potential field caused by a given electric charge distribution; with the potential field known, one can then calculate electrostatic field.

For charge free medium that is ρ=0, then

$$\nabla^2\text{V} = 0$$ This is the Laplace's Eq.

It is a useful approach to the determination of the electric potentials in free space or region.

Continuity Equation

It states that the total current flowing out through some volume must be equal to the rate of decrease of charge within that volume, if the charge is neither being created nor lost within that volume.

Since the charge is flowing, let the volume charge density ρ is a function of time. Let "V" be the volume which is enclosed by a surface S.

Therefore

$$I = \frac{dq}{dt} = \frac{d}{dt}\int_v \rho dv \qquad --(1)$$

$$I = \oint \vec{J}.\overrightarrow{dS} \qquad --(2)$$

As the charge decreases, means ρ is decreasing. Therefore from (1) and (2)

$$\oint \vec{J}.\overrightarrow{dS} = -\frac{dq}{dt} = -\frac{d}{dt}\int_v \rho dv \qquad --(3)$$

Therefore $$\oint \vec{J}.\overrightarrow{dS} = -\int_v \frac{\partial \rho}{\partial t} dv ------(4)$$

By using Gauss's divergence theorem $\oint \vec{J}.\overrightarrow{dS} = \int_v (\nabla.J)dV$

From (4) $$\int_v (\nabla.J)dV = -\int_v \frac{\partial \rho}{\partial t} dv -----(5)$$

Since (5) holds good for all arbitrary volume

Therefore $$(\nabla.J) + \frac{\partial \rho}{\partial t} = 0$$ This is continuity equation

In case of stationary currents, that is when the charge density at any point within the region remains constant, but the charges are moving.

$$\frac{\partial \rho}{\partial t} = 0$$

So that $\nabla.J = 0$ which expresses the fact that there is no net outward flux of current density J.

Maxwell's Equations (Differential Form)

1. $\vec{\nabla}.\vec{D} = \rho = \overrightarrow{\nabla}.\varepsilon_0\vec{E}$ *Gauss's Law of electrostatics*

2. $\vec{\nabla}.\vec{B} = 0$ *Gauss's Law of Magnetostatics*

3. $\nabla \times E = -\frac{\partial B}{\partial t}$ *Faraday's law*

4. $\nabla \times B = \mu_0 \left(J + \frac{\partial D}{\partial t} \right)$ *Ampere − Maxwell Law*

$$= \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \mu_0 J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$ (Here $\mu_0\varepsilon_0 = \frac{1}{c^2}$)

The equations are named after the physicist and mathematician **James Clerk Maxwell**.

Maxwell's 1st Equation

According to the Gauss law, **the total flux linked with a closed surface is 1/ε_0 times the charge enclosed by the closed surface.**

$$\Phi = \oiint E \, . \, ds = \frac{Q}{\varepsilon_0}$$

Consider a surface S bounding a volume V in a dielectric medium, which is kept under E, it polarizes the dielectric medium and bound charges are induced.

Then total charge density at a point in a small volume element dV would be $(\rho + \rho_P)$. Where ρ is free charge density and ρ_P polarization charge density, and $\rho_P = -\nabla . P$

$$\vec{\nabla} . \overrightarrow{P} = amount\ of\ polarised\ charge$$

Maxwell's 1st Equation

The net volume charge density = $\rho - \vec{\nabla}.\vec{P}$ ……(1)

Thus Gauss's theorem Can be written as

$$\oiint E\,.\,ds = \iiint (\nabla.E)dV = \frac{1}{\varepsilon_0}\iiint (\rho - \vec{\nabla}.\vec{P})dV \;\ldots\ldots(2)$$

$$\varepsilon_0 \iiint (\nabla.E)dV = \iiint (\rho - \vec{\nabla}.\vec{P})d\mathrm{V}$$

$$\iiint (\nabla.\varepsilon_0 E)dV = \iiint (\rho - \vec{\nabla}.\vec{P})d\mathrm{V}$$

$$\iiint (\nabla.\,\varepsilon_0 E + \nabla.\vec{P})dV = \iiint (\rho)d\mathrm{V}$$

Maxwell's 1st Equation

Since the equation is true for all the arbitrary volumes, the integrands in this equation must be equal.

$$\left(Div\ \varepsilon_0\vec{E} + Div\ \vec{P}\right) = \rho$$

$$Div\ \left(\varepsilon_0\vec{E} +\ \vec{P}\right) = \rho$$

$\left(\varepsilon_0\vec{E} +\ \vec{P}\right) = \left(\vec{D}\right) =$ Electric displacement

$$Div\ (D) = \rho$$

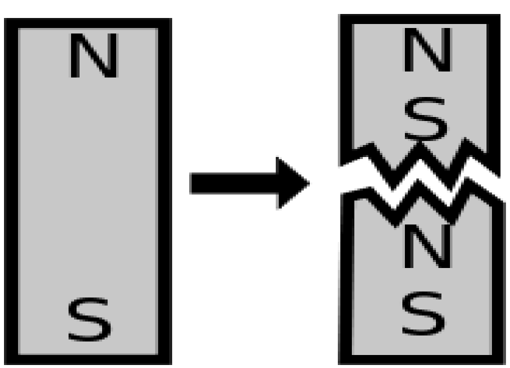
Or $\vec{\nabla}.\vec{D} = \rho$ Is the Maxwell's First Equation

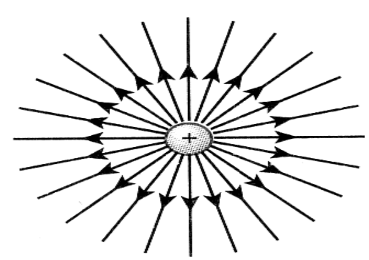
Maxwell's 2nd Equation

It is also called as Gauss's Law in magnetostatics. Since magnetic lines of force entering any arbitrary surface is exactly the same as leaving it. It means

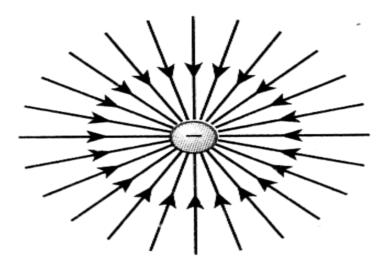
$$\vec{\nabla}.\vec{B} = 0$$

This result in that no monopole exists in nature but electric monopole exists. Since isolated magnetic poles and magnetic currents due to them have no significance.





(a) Positive charge



(b) Negative charge

Maxwell's 2nd Equation

Mathematically the Gauss's Law can be written as:

$$\oiint B\,.\,ds = 0 \qquad \ldots (3)$$

$$by\ divergence\ theorem$$

$$\oiint B\,.\,ds = \iiint div\ B dv = 0$$

The integrand should vanish for the surface boundary as the volume is arbitrary.

Div B =0

This is Maxwell's second equation.

Maxwell’s 3rd Equation

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

“It is nothing but the differential form of Faraday’s law and Lenz’s law of electromagnetic induction.”

According to Faraday’s law of em induction, it is known that e.m.f. (Electromotive force) induced in a closed loop is defined as negative rate of change of magnetic flux i.e.

$$e = -\frac{d\phi}{dt} = -\iint \frac{\partial B}{\partial t} . ds \quad \ldots\ldots\ldots . (4)$$

Maxwell's 3rd Equation

Since e.m.f. is the line integral of the electric field over the circuit

Therefore $$e = \oint \vec{E}.\overrightarrow{dl}$$

Equation (4) becomes

$$\oint \vec{E}.\overrightarrow{dl} \quad = -\iint \frac{\partial B}{\partial t}.ds \quad (5)$$

Using Stoke's theorem

$$\oint \vec{E}.\overrightarrow{dl} \qquad = \iint (\nabla \times E).ds$$

Maxwell's 3rd Equation

Equation (5) becomes

$$\iint (\nabla \times E).ds = -\iint \frac{\partial B}{\partial t}.ds$$

This equation is true for any surface whether small or large in the field. Therefore the two vectors in the integrands must be equal at every point.

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

This is the 3rd Maxwell's equation.