Pingala Series

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 ${\it Abstract} {\it \bf - This \ manual \ provides \ a \ simple \ introduction}$ to Transforms

1 JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \ge 1$$
 (1.1)

$$b_n = a_{n-1} + a_{n+1}, \quad n \ge 2, \quad b_1 = 1$$
 (1.2)

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \ge 1$$
 (1.3)

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \ge 1 \tag{1.5}$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: Download the python code for the above problem using.

\$ wget https://raw.githubusercontent.com/ prajwal-3-14159/ EE3900_ma20btech11013/main/pingala/ codes/problem 2-1.py

2 Pingala Series

2.1 The *one sided Z*-transform of x(n) is defined as

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C}$$
 (2.1)

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n), \quad x(0) = x(1) = 1, n \ge 0$$
(2.2)

Generate a stem plot for x(n).

Solution: The python code for stem plot:

\$ wget https://raw.githubusercontent.com/ prajwal-3-14159/ EE3900_ma20btech11013/main/pingala/ codes/problem_2-2.py

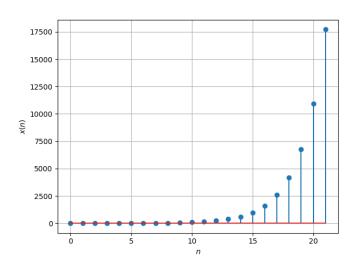


Fig. 2.2: The stem plot of x(n)

2.3 Find $X^{+}(z)$.

Solution: Take the one-sided Z-transform on both sides of the equation (2.2),

$$Z^{+}[x(n+2)] = Z^{+}[x(n+1)] + Z^{+}[x(n)]$$

$$(2.3)$$

$$z^{2}X^{+}(z) - z^{2}x(0) - zx(1) = zX^{+}(z) - zx(0) + zX^{+}(z)$$

$$(2.4)$$

Now, as
$$x(o) = x(1) = 1$$

$$(z^2 - z - 1)X^+(z) = z^2 (2.5)$$

$$X^{+}(z) = \frac{1}{1 - z^{-1} - z^{-2}}$$
 (2.6)

Let, α and β be solutions to eqn $x^2 - x - 1 = 0$, then we get,

$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}, \quad |z| > \alpha \qquad (2.7)$$

2.4 Find x(n).

Solution: On expanding the $X^+(z)$ as given in eqn (2.7) using partial fractions, we get

$$X^{+}(z) = \frac{1}{(\alpha - \beta)z^{-1}} \left[\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}} \right]$$
(2.8)

$$= \frac{1}{(\alpha - \beta)} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) z^{-n+1}$$
 (2.9)

$$=\sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^{-n+1}$$
 (2.10)

$$= \sum_{n=0}^{\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} z^{-n}$$
 (2.11)

As k := n + 1. hence,

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} u(n) = a_{n+1} u(n)$$
 (2.12)

therefore, $x(n) = a_{n+1}u(n)$.

2.5 Sketch

$$y(n) = x(n-1) + x(n+1), \quad n \ge 0$$
 (2.13)

Solution: Download the python code for plot of y(n) using:

\$ wget https://raw.githubusercontent.com/ prajwal-3-14159/ EE3900_ma20btech11013/main/pingala/ codes/problem 2-2.py

2.6 Find $Y^{+}(z)$.

Solution: Take the one-sided Z-transform on

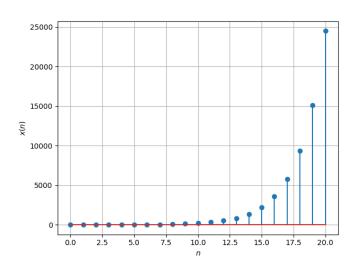


Fig. 2.5: The stem plot of y(n)

the both sides of the eqn (2.13),

$$Z^{+}[y(n)] = Z^{+}[x(n+1)] + Z^{+}[x(n-1)]$$
(2.14)

$$Y^{+}(z) = zX^{+}(z) - zx(0) + z^{-1}X^{+}(z) + zx(-1)$$
(2.15)

On substituting the values of $X^+(z)$, x(0), x(1) and x(-1). [here, x(-1) = 0, as $x(n) = 0 \ \forall n < 0$]

$$Y^{+}(z) = \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.16}$$

$$Y^{+}(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}, \quad |z| > \alpha$$
 (2.17)

2.7 Find y(n).

Solution: As $Y^+(z) = (1 + 2z^{-1})X^+(z)$, hence using the (2.1),

$$Y^{+}(z) = (1 + 2z^{-1}) \sum_{n=0}^{\infty} x(n)z^{-n}$$
 (2.18)

$$= \sum_{n=0}^{\infty} x(n)z^{-n} + \sum_{n=1}^{\infty} 2x(n-1)z^{-n} \quad (2.19)$$

$$= x(0) + \sum_{n=1}^{\infty} (x(n) + 2x(n-1)) z^{-n}$$
(2.20)

Now, as y(0) = x(0) = 1 and using the fact that

 α and β are the roots of the eqn $z^2 - z - 1 = 0$,

$$y(n) = \frac{\left(\alpha^{n+1} - \beta^{n+1}\right) + (2\alpha^n + 2\beta^n)}{\alpha - \beta}$$
 (2.21)

$$=\frac{\left(\alpha^{n+2}-\beta^{n+2}\right)+\left(\alpha^n+\beta^n\right)}{\alpha-\beta}\tag{2.22}$$

$$=\frac{\left(\alpha^{n+2}-\beta^{n+2}\right)-\alpha\beta\left(\alpha^{n}+\beta^{n}\right)}{\alpha-\beta} \quad (2.23)$$

$$\begin{aligned} \alpha - \beta \\ &= \frac{(\alpha - \beta) \left(\alpha^{n+1} + \beta^{n+1}\right)}{\alpha - \beta} \\ &= \alpha^{n+1} + \beta^{n+1} \end{aligned}$$
 (2.24)

$$= \alpha^{n+1} + \beta^{n+1} \tag{2.25}$$

Hence, $y(n) = \alpha^{n+1} + \beta^{n+1}$; $\forall n \ge 0$. as $\alpha + \beta = 1$. Comparing (2.22) with the definition of b_n , we get $y(n) = b_{n+1} = \alpha^{n+1} + \beta^{n+1}$ (: $b_n = \alpha^n + \beta^n$).

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1)$$
 (3.1)

Solution: From the eqn (2.12), we can see that $x(n) = 0 \ \forall \ n < 0,$

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k)$$
 (3.2)

$$=\sum_{k=-\infty}^{n-1} x(k) \tag{3.3}$$

$$= \sum_{k=-\infty}^{\infty} x(k)u(n-1-k)$$
 (3.4)

$$= x(n) * u(n-1)$$
 (3.5)

3.2 Show that

$$a_{n+2} - 1, \quad n \ge 1$$
 (3.6)

can be expressed as

$$[x(n+1)-1]u(n)$$
 (3.7)

Solution: From the eqn (2.12),

$$a_{n+2} - 1 = [x(n+1) - 1], \quad n \ge 0$$
 (3.8)

Now, using the definition of u(n),

$$a_{n+2} - 1 = [x(n+1) - 1]u(n)$$
 (3.9)

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10} X^+ (10) \quad (3.10)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k}$$
 (3.11)

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k}$$
 (3.12)

$$=\frac{1}{10}X^{+}(z)\tag{3.13}$$

$$= \frac{1}{10} \times \frac{100}{89} = \frac{10}{89} \tag{3.14}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \ge 1 \tag{3.15}$$

can be expressed as

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n)$$
 (3.16)

and find W(z).

Solution: Let, n = k + 1 in (3.15) and using the definition of u(n), we get:

$$\alpha^{n} + \beta^{n} = (\alpha^{k+1} + \beta^{k+1})u(k)$$
 (3.17)

Now, eqn (3.15) can be expressed as,

$$w(n) = (\alpha^{n+1} + \beta^{n+1})u(n) = y(n)$$
 (3.18)

Therefore,

$$W(z) = Y(z) = \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}}$$
 (3.19)

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+ (10) \quad (3.20)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k}$$
 (3.21)

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k}$$
 (3.22)

$$=\frac{1}{10}Y^{+}(z)\tag{3.23}$$

$$=\frac{1}{10}\times\frac{120}{89}=\frac{12}{89}\tag{3.24}$$

3.6 Solve the JEE 2019 problem.

Solution: As we know that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1)$$
 (3.25)

also as

$$x(n) * u(n-1) \stackrel{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z)$$
 (3.26)

$$x(n) * u(n-1) \stackrel{\mathcal{Z}}{\rightleftharpoons} X(z)z^{-1}U(z)$$
 (3.26)
=
$$\frac{z^{-1}}{(1-z^{-1}-z^{-2})(1-z^{-1})}$$
 (3.27)

$$= z \left[\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right]$$
 (3.28)

$$\stackrel{\mathcal{Z}}{\rightleftharpoons} z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n}$$
 (3.29)

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1}$$
 (3.30)

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n}$$
 (3.31)

And from (2.12), we get

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1 \tag{3.32}$$

We have already established the remaining options in the problems (3.3), (2.7), (3.5). \therefore the correct options are: a, b, and c

and, incorrect option is: d.