Energy-based Switched Control Law for a Swirling Pendulum

G. S. Prajwal[1], Sujay Kadam[2], Nidhish Raj[3], Ravi Banavar[4], P. M. Harish[5]

***Abstract*— The article proposes a state-based switching control law for a mechanism we call the swirling pendulum. The mechanism falls in the earlier studied genre of underactuated systems, but differs significantly in the orientation of the joint axes combined with the manner in which gravity affects the two links to warrant a new control strategy. The control strategy involves both energy notions and a state-based switching mechanism. The motivation for a switched control law, detailed mathematical proofs and representative simulations are presented.**

I. Introduction

Underactuated systems have fascinated control engineers for many years, both from the theoretical and practical perspective. While on the theoretical front they open up a gamut of problems in mechanics and control, on the practical front they represent situations that one is often confronted with, either under loss of actuation in many fully actuated systems or, under purposeful underactuation by design. One of the early problems to captivate the community in this field was the acrobot [1], a two-link, planar, two revolute joints, underactuated system. Since then, there have been other representative systems - the pendubot [2], the pendulum on a cart [3], the Furuta pendulum [4] - all of which have been studied extensively in the control community. For the pendulum on a cart system, Chung and Hauser [3] propose a nonlinear controller to regulate the swinging energy of the pendulum resulting in the closed-loop system possessing an asymptotically stable periodic orbit.

In Spong [5], two different control strategies based on partial feedback linearisation are proposed for the swing-up of the acrobot, one of which employs an energy-based algorithm. Spong and Block [2] use feedback linearisation techniques for the swing-up control of the pendubot and employ a linear quadratic regulator on the linearised model about the desired unstable equilibrium for balancing control.

[1]G. S. Prajwal (gsprajwal96@gmail.com) is a Project Research Assistant at SysCon, IIT Bombay, India. [2]Sujay D. Kadam (kadam.sujay@iitgn.ac.in) is a doctoral student at SysIDEA Lab, IIT Gandhinagar, India. [3]Nidhish Raj (nraj@iitk.ac.in) is a doctoral student at the Department of Aerospace, IIT Kanpur. [4]Ravi Banavar (banavar@iitb.ac.in) is a Professor, SysCon, IIT Bombay, India. [5]P. M. Harish (harish@iitgn.ac.in) is an Associate Professor, SysIDEA Lab, Mechanical Engineering, IIT Gandhinagar, India.

Fantoni *et al.* [6] present a control strategy based on energy approach and passivity properties of the pendubot with a complete stability analysis based on LaSalle's invariance theorem. In [7], Shiriaev *et al.* present a passivity based control for partial stabilisation of underactuated systems with the example of the swing-up of the pendubot. [8] proposes a control for an almost global energy level stabilisation and upright equilibrium stabilisation of a pendulum. Xin *et al.* [9] present a discussion on the choice of parameters in the control law proposed by Fantoni *et al.* [6] to ensure convergence of the total energy of the pendubot to its potential energy at the upright equilibrium. Olfati-Saber and Megretski [10] give a global change of coordinates that transforms the dynamics of an underactuated system into a lower-order nonlinear subsystem plus a linear subsystem. A control law is formulated for the lower-order subsystem and backstepping procedure is used to reconstruct the control for the original system.

Mahindrakar and Banavar [11] propose a control law extracted from an energy-based Lyapunov function, which reduces the effective DOF of the acrobot to one by aligning the links and simultaneously pumps in the desired energy to drive the system to a neighbourhood of the upward equilibrium, about which a linear feedback controller is switched on for stabilisation. In [12], Shiriaev *et al.* propose a state-feedback regulator using the speed-gradient method for the swing-up of the spherical pendulum to its upright equilibrium. It shows the inability of a simple energy-based passive controller to achieve this objective and incorporates a state-based switching in the proposed control law.

In this article, we present the modelling and an energy-based swing-up control of a two-link, two-joint underactuated mechanism - the swirling pendulum (SP) (see Figure 1), which is constructed by simply rotating the direction of the gravity vector in a Furuta pendulum by 90° with the second joint being actuated. This has interesting consequences, especially with regard to swing-up of the mechanism to an upward target equilibrium. An experimental set up of this mechanism is found in IIT-Gandhinagar (for more details, please see http://research.iitgn.ac.in/sysidea/). The following are the points of contrast of the SP with other existing benchmark problems in the literature:

- ***Furuta pendulum and pendulum on a cart:*** Like the SP, both these have mutually perpendicular joint axes. But, in the case of the Furuta pendulum, gravity has a direct influence on the dynamics of the second link alone. The acceleration of the first link, lying in the horizontal plane, is not directly affected by the gravitational force. So is the case with the pendulum on a cart system, where gravity does not directly contribute to the acceleration of the cart. Further, for these mechanisms, the objective of swing-up entails getting the second link alone to an upright position. However, the accelerations of both the links of the SP have direct gravity contributions and the swing-up involves elevating the centers of gravity of both the links.
- ***Acrobot and pendubot:*** These are similar to the SP in terms of gravitational influence on the links and the objective of swing-up. However, the joint axes of these sytems are parallel to each other. As a consequence, the control torque at the actuated joint has a direct influence on the accelerations of both the links throughout the configuration space. The SP, with its perpendicular joint axes, doesn't enjoy this privilege as it loses controllability at certain configurations.

Given the features of the SP that have been enumerated above, we propose a swing-up control law based on two notions - one being an energy principle and the other being an appropriate state-based switching law that ensures convergence to the desired equilibrium in reasonable time.

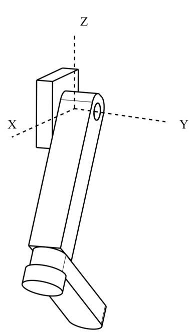


Fig. 1. Swirling Pendulum

II. Modelling and Preliminary Analysis

In the schematic, Figure 2 shown above, there are 3 coordinate systems - one fixed at O with the y-axis along $A_1 O A_1'$, the second attached to the first link at Q, termed $x'y'z'$, and the third attached to the second link $x''y''z''$.

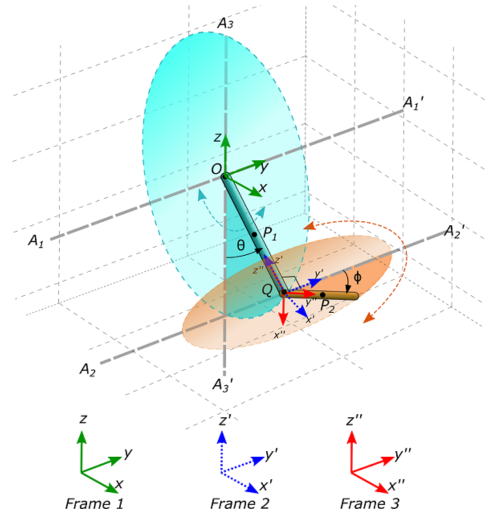


Fig. 2. Frames of reference and schematic of the SP

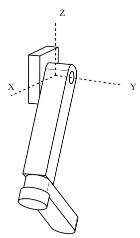
The half-length and the mass of link-i are l_i and m_i respectively. The configuration manifold of the system is $\mathbb{S}^1 \times \mathbb{S}^1$ and the state $x = [x_1, x_2, x_3, x_4]' \triangleq [\theta, \phi, \dot{\theta}, \dot{\phi}]'$ evolves on the tangent bundle TM. The control torque $u(t)$ is applied at the second joint to the second link. Using a Lagrangian approach, the state-space can be represented as: *(refer Appendix VII-A for the functions $f_3(x)$, $f_4(x)$, $g_3(x)$ and $g_4(x)$)*

$$\dot{x} = f(x) + g(x)u$$

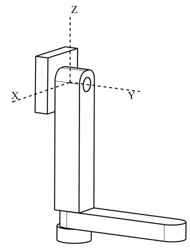
$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ f_3(x) \\ f_4(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_3(x) \\ g_4(x) \end{bmatrix} u \tag{1}$$

The equilibrium solutions of (1) for $u = 0$ are enumerated in Table I, where $\alpha = \tan^{-1}\left(\frac{m_2 l_2}{(2m_2+m_1)l_1}\right)$. Figure 3 shows four qualitatively different equilibrium configurations (the remaining four are mirrored images of these). The stability of these equilibria can be determined by evaluating the Hessian of the SP's potential energy. The Hessian is positive-definite only at the downward equilibria, $(\pm\alpha, \mp\frac{\pi}{2}, 0, 0)$, making these the only stable equilibria.

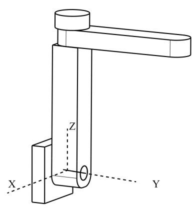
The main control objective is to swing up the SP to an upward unstable equilibrium. We intend to employ a stabilizing linear controller that kicks in when the nonlinear swing-up control brings the system into a sufficiently small neighbourhood of the desired equilibrium. On carrying out a linear controllability analysis, it is seen that the linearisations at those equilibria with $x_2 = \pm\frac{\pi}{2}$ are uncontrollable. This is essentially due to a complete loss of inertial coupling when link-2 is in the plane of motion of link-1. From (1), consider $|g_3(x)| = \left|\frac{-9l_1 \cos x_2}{\gamma(x_2)}\right|$ which quantifies the control torque's influence on $\dot{x}_3$ i.e.



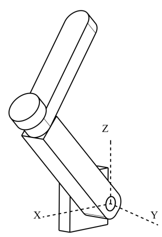
(a) $(\alpha, -\frac{\pi}{2}, 0, 0)$



(b) $(0, 0, 0, 0)$



(c) $(\pi, 0, 0, 0)$



(d) $\left(\alpha - \pi, -\frac{\pi}{2}, 0, 0\right)$

Fig. 3. Representative equilibrium configurations

$\ddot{\theta}$. $|g_3(x)| = 0$ at $x_2 = \pm\frac{\pi}{2}$. In view of this, we pick one of the controllable upward equilibria, $(\pi, 0, 0, 0)$, as the target equilibrium for swing-up. The rest of this article is devoted to swing-up control of the SP to this equilibrium.

TABLE I
EQUILIBRIA

Sl No	Equilibrium	Name	Stability
1	$(\alpha, -\frac{\pi}{2}, 0, 0)$	Downward	Stable
2	$(-\alpha, \frac{\pi}{2}, 0, 0)$		
3	$(0, 0, 0, 0)$	Downward Hanging	Unstable
4	$(0, \pi, 0, 0)$		
5	$(\pi, 0, 0, 0)$	Upright Hanging	Unstable
6	$(\pi, \pi, 0, 0)$		
7	$(\alpha - \pi, -\frac{\pi}{2}, 0, 0)$	Upright	Unstable
8	$(-(\alpha - \pi), \frac{\pi}{2}, 0, 0)$		

TABLE II
SP PARAMETERS

m_1	m_2	l_1	l_2
$0.2\ kg$	$0.4\ kg$	$0.2\ m$	$0.4\ m$

III. ENERGY-BASED SWING-UP PHILOSOPHY

In this section we present an energy-based swing-up of the SP and towards the end discuss why this control law is inadequate for our objective.

The total energy of the swirling pendulum is given by:

$$\begin{aligned} E(x) = {} & \frac{x_3^2}{3}[2(m_1 + 3m_2)l_1^2 + 2m_2 l_2^2 \sin^2 x_2] \\ & + \frac{x_4^2}{3}(2m_2 l_2^2) + x_3 x_4 (2 l_1 l_2 m_2 \cos x_2) \\ & - (2m_2 + m_1) g l_1 \cos x_1 + m_2 g l_2 \sin x_1 \sin x_2 \end{aligned} \tag{2}$$

The energy at the desired upward equilibrium $(\pi, 0, 0, 0)$ is $E_d = g(m_1 l_1 + 2m_2 l_1)$. Define $\widehat{E}(x) = E(x) - E_d$. Imposing a restriction of $x_2 = x_4 = 0$ reduces the effective degrees of freedom of the system to one. $\widehat{E}(x) = 0$, under this restriction, therefore implies the attainment of the desired equilibrium. Consider the Lyapunov function $V : M \to \mathbb{R}$
(where the state-space $M = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}^2$)

$$V(x) = \frac{1}{2}\left(k_p x_2^2 + k_d x_4^2 + k_e \widehat{E}^2(x)\right) \tag{3}$$

where k_p, k_d and k_e are strictly positive constants, thereby making V positive semi-definite. Differentiating the Lyapunov function (3) along the trajectories of the system (1) gives

$$\dot{V} = x_4\left(k_\mathrm{p} x_2 + k_\mathrm{d}\dot{x}_4 + k_\mathrm{e}\widehat{E}(x)u\right)$$

$\dot{V}$ is made negative semi-definite by setting

$$k_\mathrm{p} x_2 + k_\mathrm{d}\dot{x}_4 + k_\mathrm{e}\widehat{E}(x)u = -x_4 \tag{4}$$

giving us

$$\dot{V} = -x_4^2 \tag{5}$$

Substituting the expression for $\dot{x}_4$ from (1) in (4) we get $k_\mathrm{p} x_2 + k_\mathrm{d}(f_4(x) + g_4(x)u) + k_\mathrm{e}\widehat{E}(x)u = -x_4$ which gives the control law as

$$u(x) = \frac{-\{x_4 + k_\mathrm{p} x_2 + k_\mathrm{d} f_4(x)\}}{k_\mathrm{d} g_4(x) + k_\mathrm{e}\widehat{E}(x)} \tag{6}$$

For the control law to avoid singularities we need to have,

$$k_\mathrm{d} g_4(x) + k_\mathrm{e}\widehat{E}(x) \neq 0 \tag{7}$$

Noting that $g_4(x) > 0$ throughout the state-space, choosing k_e and k_d such that, for $g_{\min} := \min_{x \in M}\{g_4(x)\}$ and $E_{\min} := \min_{x \in M}\{\widehat{E}(x)\}$ and some $\epsilon > 0$:

$$\frac{k_\mathrm{e}}{k_\mathrm{d}} + \epsilon \leqslant \frac{g_{\min}}{|E_{\min}|} \tag{8}$$

ensures that (7) is satisfied throughout the state space. The control law ensures negative semi-definiteness of $\dot{V}$.

Theorem 1. *The state of the system (1), starting from any initial condition, with control $u = u(x)$ as given by (6) for a choice of k_e and k_d satisfying (8), converges to the set $C_{\mathrm{eq}} := \{x \in M \mid F(x) = 0\}$ where, $F(x) := f(x) + g(x)u(x)$.*

(a) (1, 1, 0.95) (b) (1, 1, 2.61)

Fig. 4. Controlled equilibria for different $(k_\mathrm{p}, k_\mathrm{d}, k_\mathrm{e})$

Proof: The given choice of k_d and k_e ensures that the control is free of singularities throughout the state-space. The Lyapunov function $V : M \to \mathbb{R}$ is a continuously differentiable, radially unbounded, positive definite function on M such that $\dot{V}(x) \leqslant 0$ for all $x \in M$. Define $S = \{x \in M | \dot{V}(x) = 0\}$. For a solution to stay identically in S , $\dot{V}(x) \equiv 0$. From (5), $\dot{V}(x) \equiv 0 \Rightarrow x_4 \equiv 0$. From (1), $\dot{x_2} = x_4 = 0 \Rightarrow x_2 =$ constant. From (4), $\widehat{E}(x)u =$ constant. From (3), $\widehat{E}(x) =$ constant which together imply that $u =$ constant. Since $x_4 \equiv 0 \Rightarrow \dot{x}_4 \equiv 0$, from (1) we have: $f_4(x) + ug_4(x) = 0 \Rightarrow f_4(x) = -ug_4(x) =$ constant (say C) ($\because$ g_4 is a function of x_2 alone). Substituting $x_4 = 0$ in (2) and solving for x_3 we get, $x_3 = \sqrt{c_1 + c_2 \cos x_1 + c_3 \sin x_1}$ where c_1, c_2 and c_3 are constants (refer Appendix VII-B). Substituting $x_4 = 0$ and $x_3 = \sqrt{c_1 + c_2 \cos x_1 + c_3 \sin x_1}$ in $f_4(x) = C$, we get

$$A \sin x_1 + B \cos x_1 = C' \tag{9}$$

where A, B and C' are constants (refer Appendix VII-B). From (9), $x_1 =$ constant $\Rightarrow x_3 \equiv 0$. In summary, for a solution to stay identically in S, $\{x_3, x_4\} \equiv 0$ (i.e x_1, x_2 and u must remain constant with time) for the dynamics given by (1) with the control input as given by (6)

It follows that the system (1) with control (6) is bound to converge to one of these constant solutions (we term them controlled equilibria). Now, $C_\mathrm{eq} = \{x \in M \mid F(x) = 0\}$ is the set of all possible controlled equilibrium states. Hence, the state of the system (1) converges to C_eq. $\square$

Figure 4 shows the plots of controlled equilibrium configurations, $(x_1, x_2) \in \mathbb{S}^1 \times \mathbb{S}^1$, for the parameters in Table II and for different choices of Lyapunov gains satisfying (8). For $(k_\mathrm{p}, k_\mathrm{d}, k_\mathrm{e}) = (1, 1, 0.95)$, $C_\mathrm{eq} = \{(0, 0, 0, 0), (\pi, 0, 0, 0)\}$ and for $(k_\mathrm{p}, k_\mathrm{d}, k_\mathrm{e}) = (1, 1, 2.61)$:

$$C_\mathrm{eq} = \left\{ \begin{matrix} (0, 0, 0, 0), (\pm 0.1853, \mp 2.9050, 0, 0), \\ (\pm 0.6691, \mp 1.7233, 0, 0), (\pi, 0, 0, 0) \end{matrix} \right\}$$

Clearly, C_eq consists of states other than the target equilibrium, $(\pi, 0, 0, 0)$. So, the system could get stuck at any of these undesired controlled equilibria.

Consider a favourable case where the system does not get stuck at these points. To eliminate control singularities throughout the state space, the Lyapunov gains k_d and k_e were chosen to satisfy (8). Under the assumption of the aforementioned favourable case, such a choice should, in theory, make the system converge to $(\pi, 0, 0, 0)$. However, the analysis does not speak of the time taken for the convergence. For the given system parameters, if $\frac{k_\mathrm{e}}{k_\mathrm{d}}$ is specified to be very small by the condition (8), then the time taken for convergence becomes unreasonably high owing to the small weighting(k_e) for the term containing $\widehat{E}(x)$ in the Lyapunov function (3). In other words, the controller underplays the need to pump in the desired energy. Hence, even under the assumption of the undesired controlled equilibria being avoided during the swing-up, it takes an unreasonable amount of time for the SP to be energized enough to reach a sufficiently small neighbourhood of the target equilibrium for the stabilizing linear controller to take over.

To circumvent these issues, we propose to employ a switched-controller obtained by using multiple values of k_e in (6) and an algorithm which orchestrates the switching between these values.

IV. Switched Control

Define an index set, $I := \{i \in Z \mid 0 \leqslant i \leqslant m\}$ for some positive integer m and $\kappa := \left\{k_{\mathrm{e}_i} \in \mathbb{R}^+ \mid k_{e_{i-1}} \leqslant k_{\mathrm{e}_i} \ \forall \ 1 \leqslant i \leqslant m\right\}_{i \in I}$. For fixed values of k_p and k_d, and $k_\mathrm{e} \in \kappa$, (3) through (6) result in a family of Lyapunov functions:

$$V_\mathrm{fam} := \left\{ V_i\left(x\right) = \frac{1}{2}\left(k_p x_2^2 + k_d x_4^2 + k_{\mathrm{e}_i} \widehat{E}^2\left(x\right)\right) \right\}_{i \in I}$$

and a family of control laws:

$$u_\mathrm{fam} := \left\{ u_i(x) = \frac{-\{x_4 + k_\mathrm{p} x_2 + k_\mathrm{d} f_4(x)\}}{k_\mathrm{d} g_4(x) + k_{\mathrm{e}_i} \widehat{E}(x)} \right\}_{i \in I}$$

each of which render

$$\dot{V}_i = -x_4^2 \qquad \forall \ i \in I \tag{10}$$

We obtain a family of autonomous closed-loop subsystems by substituting $u = u_i(x)$, $i \in I$ in the system (1) given by $S_\mathrm{fam} := \{\dot{x} = F_i(x)\}_{i \in I}$. The set of all controlled equilibria and the set of undesired controlled equilibria of the i^{th} subsystem are given by $C_{\mathrm{eq}_i} := \{x \in M \mid F_i(x) = 0\} \ \forall i \in I$ and $C_{\mathrm{ueq}_i} := C_{\mathrm{eq}_i} - \{(\pi, 0, 0, 0)\} \ \forall \ i \in I$. Also define $v_{\mathrm{ueq}_i} := \left\{V_i(x) \mid x \in C_{\mathrm{ueq}_i}\right\} \ \forall \ i \in I$ and $V_{\mathrm{min\_ueq}_i} := \min(v_{\mathrm{ueq}_i}) \ \forall \ i \in I$. Noting that the behavior of each subsystem, $\dot{x} = F_i(x)$, and therefore the elements of the set C_{eq_i}, depend on the choice of k_{e_i}, it is observed that a low enough value of k_{e_i} renders $C_{\mathrm{eq}_i} = \{(0, 0, 0, 0), (\pi, 0, 0, 0)\}$ (as can be seen for the lower value of k_e from Figure 4). We pick a low enough value for k_{e_0} which, in addition to rendering

$$\begin{aligned} C_{\mathrm{eq}_0} &= \{(0,0,0,0),(\pi,0,0,0)\} \\ \Rightarrow V_{\mathrm{min\_ueq}_0} &= V_0\big((0,0,0,0)\big) \end{aligned} \tag{11}$$

, satisfies (8) i.e. for some $\epsilon > 0$

$$\frac{k_{\mathrm{e}_0}}{k_\mathrm{d}} + \epsilon \leqslant \frac{g_{\min}}{|E_{\min}|} \tag{12}$$

For $1 \leqslant i \leqslant m$, define the following constants: $a_i := \frac{(k_\mathrm{d} g_{\min} - \epsilon)^2}{2k_{\mathrm{e}_i}}$ for some $0 < \epsilon < k_\mathrm{d} g_{\min}$, $b_i := V_{\mathrm{min\_ueq}_i} - \epsilon$ for some $0 < \epsilon < V_{\mathrm{min\_ueq}_i}$ and $c_i := \min\{a_i, b_i\}$. With $M \ni x \longmapsto i(x) := \max\big(\{0\} \cup \{j \mid V_j(x) \leqslant c_j, 1 \leqslant j \leqslant m\}\big) \in I$, we define a state-dependent switched-control law given by:

$$u_{\mathrm{sw}}(x) := u_{i(x)}(x) = \frac{-\{x_4 + k_\mathrm{p} x_2 + k_\mathrm{d} f_4(x)\}}{k_\mathrm{d} g_4(x) + k_{\mathrm{e}_{i(x)}} \widehat{E}(x)} \tag{13}$$

Lemma 2. *The control law $u = u_{\mathrm{sw}}(x)$ given by (13) does not encounter singularities for any $x \in M$*

Proof: From (13) $u_{i(x)}(x)$ is well-defined if

$$k_\mathrm{d} g_4(x) + k_{\mathrm{e}_{i(x)}} \widehat{E}(x) \neq 0 \tag{14}$$

Case 1: $x \in M \mid \widehat{E}(x) \geqslant 0$.
Since $k_\mathrm{d} > 0$, $k_{\mathrm{e}_{i(x)}} > 0$ and $g_4(x) \geqslant g_{\min} > 0 \;\forall\; x \in M$,

$$\widehat{E}(x) \geqslant 0 \Rightarrow k_\mathrm{d} g_4(x) + k_{\mathrm{e}_{i(x)}} \widehat{E}(x) \geqslant k_\mathrm{d} g_{\min} > 0$$

. It follows that (14) is satisfied $\forall\; x \in M \mid \widehat{E}(x) \geqslant 0$
Case 2: $x \in M \mid \widehat{E}(x) < 0$.
(12) ensures that (14) is satisfied $\forall\; x \in M \mid \widehat{E}(x) < 0, i(x) = 0$. If $i(x) = l$ where $0 < l \leqslant m$, from the definition of $i(x)$, we have $V_l(x) \leqslant c_l \leqslant a_l = \frac{(k_\mathrm{d} g_{\min} - \epsilon)^2}{2k_{\mathrm{e}_l}}$ for some $0 < \epsilon < k_\mathrm{d} g_{\min}$. Noting that $k_{\mathrm{e}_l} \widehat{E}(x)^2 \leqslant 2V_l(x) \;\forall\; x \in M$ (by definition of $V_l(x)$) $k_{\mathrm{e}_l}^2 \widehat{E}(x)^2 \leqslant 2V_l(x) k_{\mathrm{e}_l} \leqslant (k_\mathrm{d} g_{\min} - \epsilon)^2 \Rightarrow k_{\mathrm{e}_l} |\widehat{E}(x)| \leqslant (k_\mathrm{d} g_{\min} - \epsilon) \leqslant (k_\mathrm{d} g_4(x) - \epsilon)$. Since $\widehat{E}(x) < 0$ under the current case, $k_\mathrm{d} g_4(x) + k_{\mathrm{e}_l} \widehat{E}(x) \geqslant \epsilon$. So, (14) is satisfied $\forall\; x \in M \mid \widehat{E}(x) < 0$. It follows that the control law $u = u_{\mathrm{sw}}(x)$ is well-defined $\forall\; x \in M$. □

Lemma 3. *For the system (1) with control $u = u_i(x)$ and an initial state*

$$x(t_0) \in \begin{cases} M & \text{for } i = 0 \\ \Omega_i := \{x \in M \mid V_i(x) \leqslant a_i\} & \text{for } 0 < i \leqslant m \end{cases},$$

$\dot{V}_i(x) \equiv 0 \Rightarrow x \in C_{\mathrm{eq}_i}$

Proof: From the proof of Lemma 2 $u_0(x)$ is well-defined $\forall\, x \in M$ and $u_i(x)$, $0 < i \leqslant m$, is well-defined $\forall\, x \in \Omega_i$. Since $\dot{V}_i(x) \leqslant 0$ (from (10)), the set Ω_i is positively invariant under the flow of the closed-loop system. So, for $0 < i \leqslant m$, given that $x(t_0) \in \Omega_i$, the control does not encounter singularities for any $t \geqslant t_0$. It then follows from Theorem 1 that $\dot{V}_i(x) \equiv 0 \Rightarrow x \in C_{\mathrm{eq}_i}$. □

Lemma 4. *For the system (1) with control $u = u_{\mathrm{sw}}(x)$ given by (13), if at some time instant $t = \tau$, $i(x(\tau)) = l$, where $l \in I$, then $i(x(t)) \geqslant l \quad \forall\; t \geqslant \tau$.*

Proof: The statement holds trivially for $l = 0$ since $i(x) \geqslant 0$ by definition. For $l > 0$: Let us assume that $i(x(t))$ switches from l to some $l' \in I$, $0 \leqslant l' < l$, at some $t = \tau' > \tau$. Since $i(x(\tau')) = l' < l$,

$$V_l((x(\tau')) > c_l \tag{15}$$

But, since $i(x(\tau)) = l$, $V_l((x(\tau)) \leqslant c_l$. Given that $i(x(t)) = l \;\forall\; \tau \leqslant t < \tau'$, from (10) we have, $\dot{V}_l(x(t)) \leqslant 0 \;\forall\; \tau \leqslant t < \tau' \Rightarrow V_l(x(t)) \leqslant V_l(x(\tau)) \;\forall\; \tau \leqslant t \leqslant \tau'$

$$\Rightarrow V_l(x(\tau')) \leqslant V_l(x(\tau)) \leqslant c_l \tag{16}$$

Clearly, (16) contradicts (15). The lemma indicates that once a subsystem is active, a switch to another subsystem of a lower index cannot occur. □

Lemma 5. *For the system (1) with an initial state $x(t_0) \in \big\{x \in M \mid V_0(x) < V_0\big((0,0,0,0)\big)\big\}$ and control $u = u_{\mathrm{sw}}(x)$ given by (13), if at some time instant $t = \tau \geqslant t_0$, $i(x(\tau)) = l$, where $0 \leqslant l < m$, then there exists a time $t = \tau' > \tau$ such that $i(x(\tau')) > l$.*

Proof: If $i(x(\tau)) = l = 0$, then $i(x(t)) \leqslant 0 \;\forall t_0 \leqslant t \leqslant \tau$. The reason being that, if for some earlier time $\tau_{\mathrm{earlier}} < \tau$, $i(x(\tau_{\mathrm{earlier}})) > 0$, then Lemma 4 would be violated. Also, $i(x) \geqslant 0$ by definition. So, $i(x(\tau)) = l = 0 \Rightarrow i(x(t)) = 0 \;\forall\; t_0 \leqslant t \leqslant \tau$. $V_0(x(t_0)) < V_0\big((0,0,0,0)\big)$ (given). Given that $i(x(t)) = 0 \;\forall\; t_0 \leqslant t < \tau$, from (10) we have, $\dot{V}_0(x(t)) \leqslant 0 \;\forall\; t_0 \leqslant t < \tau \Rightarrow V_0(x(t)) \leqslant V_0(x(t_0)) \;\forall\; t_0 \leqslant t \leqslant \tau \Rightarrow V_0(x(\tau)) \leqslant V_0(x(t_0)) < V_0\big((0,0,0,0)\big) = V_{\mathrm{min\_ueq}_0}$ *(From (11))*. If $0 < i(x(\tau)) = l \leqslant m$, then from the definition of $i(x)$, $V_l(x(\tau)) \leqslant c_l \leqslant b_l = (V_{\mathrm{min\_ueq}_l} - \epsilon)$ *for some* $0 < \epsilon < V_{\mathrm{min\_ueq}_l} \Rightarrow V_l(x(\tau)) < V_{\mathrm{min\_ueq}_l}$

$$\Rightarrow V_l(x(\tau)) < V_{\mathrm{min\_ueq}_l} \;\forall\; l \in I \tag{17}$$

Let us assume that $i(x(t)) \leqslant l \;\forall\; t \geqslant \tau$. But, according to Lemma 4, $i(x(t)) \geqslant l \;\forall\; t \geqslant \tau$

$$\Rightarrow i(x(t)) = l \quad \forall\; t \geqslant \tau \tag{18}$$

Given (18), from (10), we have: $\dot{V}_l(x(t)) \leqslant 0 \;\forall\; t \geqslant \tau$

$$\Rightarrow V_l(x(t)) \leqslant V_l(x(\tau)) < V_{\mathrm{min\_ueq}_l} \quad \forall\; t \geqslant \tau \text{ (Using (17))}$$

$$\Rightarrow x(t) \notin C_{\mathrm{ueq}_l} \quad \forall\; t \geqslant \tau \tag{19}$$

Given (18), from Lemma 3, we have:

$$\dot{V}_l(x) \equiv 0 \Rightarrow x \in C_{\mathrm{eq}_l} \tag{20}$$

From (19) and (20), $\dot{V}_l(x(t)) \equiv 0 \Rightarrow x(t) \in C_{\text{eq}_l} - C_{\text{ueq}_l} = \{\pi, 0, 0, 0\} \Rightarrow x(t) = (\pi, 0, 0, 0) \Rightarrow V_l(x(t)) = 0$. But, $V_l(x(t)) = 0 \Rightarrow \dot{V}_l(x(t)) \equiv 0$ (can be easily verified).

$$\therefore \dot{V}_l(x(t)) \equiv 0 \Leftrightarrow V_l(x(t)) = 0 \tag{21}$$

From (10) and (21), $\forall\ \delta > 0\ \exists\ t \geqslant \tau \mid V_l(x(t)) < \delta$. For some $h \in I \mid h > l$, we have $V_h(x) \leqslant \cdot \frac{k_{e_h}}{k_{e_l}} V_l(x) \quad \forall\, x \in M$ (Refer Appendix VII-C).

$$\Rightarrow \exists\ t \geqslant \tau \mid V_h(x(t)) \leqslant \frac{k_{e_h}}{k_{e_l}} V_l(x(t)) \leqslant c_h$$
$$\Rightarrow \exists\ t \geqslant \tau \mid i(x(t)) > l \tag{22}$$

(22) contradicts (18). So, there exists some $t = \tau' > \tau \mid i(x(\tau')) > l$.
Note that the lemma excludes the $l = m$ case since $m = \max(I)$ i.e. m is the maximum value $i(x)$ can take in the index set, I. □

Theorem 6. *The system (1) with an initial state $x(t_0) \in \{x \in M \mid V_0(x) < V_0((0, 0, 0, 0))\}$ and control $u = u_{\text{sw}}(x)$ given by (13) converges to the target equilibrium* $(\pi, 0, 0, 0)$.

Proof: Lemma 4 precludes $i(x(t))$ from switching to lower values in the index set, I. From Lemma 5 for every $t = \tau \mid 0 \leqslant i(x(\tau)) < m\ \exists\ t = \tau' > \tau \mid i(x(\tau')) > i(x(\tau))$. So, as time proceeds, $i(x(t))$ progressively keeps switching to higher values in the index set (I) until $i(x(t)) = \max(I) = m$. Let $t = \tau_m$ be the first instant of time at which $i(x(t)) = m$. From Lemma 4 and the fact that m is the maximum value that $i(x)$ can take, we have $i(x(t)) = m\ \ \forall\ \ t \geqslant \tau_m$. Noting that the switched control law is free from singularities (Lemma 2), on following the steps of Theorem 1 for $u = u_m(x)$, it follows that the m^{th} (final) subsystem converges to the set C_{eq_m}. But, $i(x(t)) = m\ \ \forall\ \ t \geqslant \tau \Rightarrow V_m(x(t)) \leqslant c_m \leqslant b_m < V_{min\_ueq_m}\ \ \forall\ \ t \geqslant \tau \Rightarrow x(t) \notin C_{\text{ueq}_m}\ \ \forall\ \ t \geqslant \tau$. So, the final subsystem (and therefore the switched-system) converges to $C_{\text{eq}_m} - C_{\text{ueq}_m} = \{(\pi, 0, 0, 0)\}$. □

V. Simulation Results

The strategy proposed in the article is demonstrated through a numerical experiment. The simulations in Figures 5 and 6 correspond to $k_\text{d} = k_\text{p} = 1$, $\kappa = \{0.95,\ 3.32,\ 72.15,\ 2430.75\}$ and an initial state $x(t_0) = (\frac{\pi}{4}, 0, 0, 0)$ with the switched control in action. Once the system is in a sufficiently small neighbourhood of the target equilibrium, an LQR is switched on. The weighting and gain matrices for the LQR are $Q = I_{4\times4}$, $R = 1$ and $K = [-326.6\ \ -264.3\ \ -63.7\ \ -60]$. Figure 5 shows the evolution of the two links (plotted on a cylinder) with time.

Figure 7 shows the energy profile for a simple energy-based control law stated in Section III for $k_\text{p} = k_\text{d} = 1$, $k_\text{e} = 0.95$[1] with the same initial conditions. Clearly, the energy is pumped in at a very slow rate.

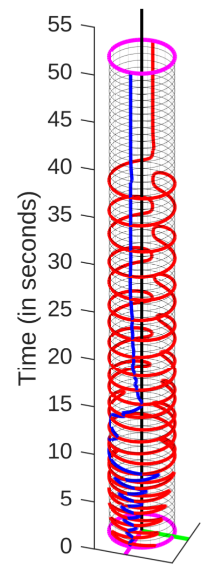


Fig. 5. θ (Red) & ϕ (Blue) Vs Time

VI. Conclusions

The article shows the inadequacy of a simple energy-based strategy for the swing-up of the SP and proposes a state-based switched control law which, besides guaranteeing convergence to the target equilibrium, accelerates the rate at which the system is energized so as to achieve the control objective in reasonable time. The numerical experiments are in support of the theoretical results.

VII. Appendix

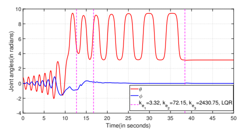
A. Dynamic Model

$g = 9.81 m/s^2$ is the acceleration due to gravity.

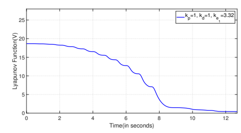
$$\gamma(x_2) := 8l_1^2 l_2 m_1 + 24 l_1^2 l_2 m_2 + 8 l_2^3 m_2 \sin^2 x_2 - 18 l_1^2 l_2 m_2 \cos^2 x_2$$

$$f_3(x) = \frac{-1}{\gamma(x_2)} \Big(6 g l_1 l_2 m_1 \sin x_1 + 12 g l_1 l_2 m_2 \sin x_1$$

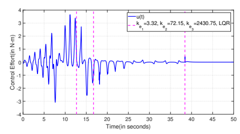
[1]The choice of $\frac{k_\text{e}}{k_\text{d}} = 0.95$ is made to render $C_\text{eq} = \{(0, 0, 0, 0), (\pi, 0, 0, 0)\}$. For the simple energy-based control, a higher value of $\frac{k_\text{e}}{k_\text{d}}$ introduces more controlled equilibria which the system could get stuck at.



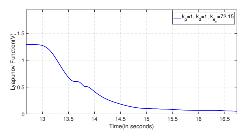
(a) Joint Angles



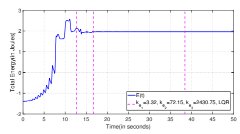
(b) $V_1(t)$



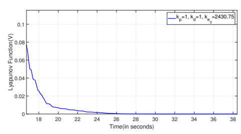
(c) Control Effort



(d) $V_2(t)$



(e) Energy Profile



(f) $V_3(t)$

Fig. 6. Swing-up Plots

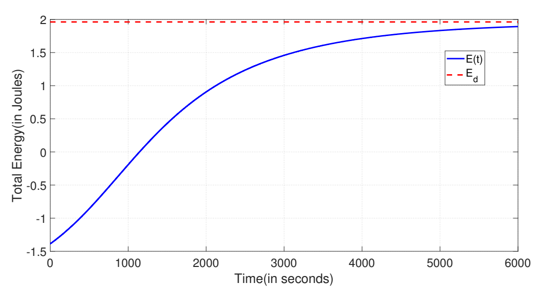


Fig. 7. Energy Profile for $k_p = k_d = 1, k_e = 0.95$

$$
\begin{aligned}
&+ 16m_2 \sin x_2 l_2^3 x_4 x_3 \cos x_2 + 12 l_1 m_2 \sin x_2 l_2^2 x_3^2 \cos^2 x_2 \\
&- 12 l_1 m_2 \sin x_2 l_2^2 x_4^2 + 6gm_2 \cos x_1 \sin x_2 l_2^2 \\
&- 9gl_1 l_2 m_2 \sin x_1 \cos^2 x_2 \Big)
\end{aligned}
$$

$$
\begin{aligned}
f_4(x) = &\frac{1}{\gamma(x_2)} \Big(- 6gl_2^2 m_2 \cos x_2 \sin^2 x_2 \sin x_1 \\
&+ 3gl_1^2 m_1 \cos x_2 \sin x_1 + 8l_2^3 m_2 x_3^2 \cos x_2 \sin^3 x_2 \\
&- 18 l_1^2 l_2 m_2 x_4^2 \cos x_2 \sin x_2 + 8 l_1^2 l_2 m_1 x_3^2 \cos x_2 \sin x_2 \\
&+ 24 l_1^2 l_2 m_2 x_3^2 \cos x_2 \sin x_2 + 9gl_1 l_2 m_2 \cos x_2 \cos x_1 \sin x_2 \\
&+ 24 l_1 l_2^2 m_2 x_4 x_3 \cos^2 x_2 \sin x_2 \Big)
\end{aligned}
$$

$$
g_3(x) = \frac{-9l_1 \cos x_2}{\gamma(x_2)} \; ; \; g_4(x) = \frac{6l_1^2 m_1 + 18 l_1^2 m_2 + 6 l_2^2 m_2}{l_2 m_2 \gamma(x_2)}
$$

B. Constants related to Theorem 1

$$
c_1 = \frac{3\left(\widehat{E}(x) + g(m_1 l_1 + 2m_2 l_1)\right)}{(2(m_1 + 3m_2) l_1^2 + 2m_2 l_2^2 \sin^2 x_2)}
$$

$$
c_2 = \frac{(6m_2 + 3m_1) g l_1}{(2(m_1 + 3m_2) l_1^2 + 2m_2 l_2^2 \sin^2 x_2)}
$$

$$
c_3 = \frac{-3m_2 g l_2 \sin x_2}{(2(m_1 + 3m_2) l_1^2 + 2m_2 l_2^2 \sin^2 x_2)}
$$

$$
\begin{aligned}
A = &- 6gl_2^2 m_2 \cos x_2 \sin^2 x_2 + 8 l_2^3 m_2 c_3 \sin^3 x_2 \cos x_2 \\
&+ 8 l_1^2 l_2 m_1 c_3 \cos x_2 \sin x_2 + 24 l_1^2 l_2 m_2 c_3 \cos x_2 \sin x_2 \\
&+ 3gl_1^2 m_1 \cos x_2
\end{aligned}
$$

$$
\begin{aligned}
B = &8 l_2^3 m_2 c_2 \sin^3 x_2 \cos x_2 + 8 l_1^2 l_2 m_1 c_2 \cos x_2 \sin x_2 \\
&+ 24 l_1^2 l_2 m_2 c_2 \cos x_2 \sin x_2 + 9gl_1 l_2 m_2 \cos x_2 \sin x_2
\end{aligned}
$$

$$
\begin{aligned}
C' = &C - \left(8 l_2^3 m_2 c_1 \sin^3 x_2 \cos x_2 + 8 l_1^2 l_2 m_1 c_1 \cos x_2 \sin x_2 \right. \\
&\left. + 24 l_1^2 l_2 m_2 c_1 \cos x_2 \sin x_2 \right)
\end{aligned}
$$

C. Proof for $V_h(x) \leqslant \frac{k_{e_h}}{k_{e_l}} V_l(x) \quad \forall x \in M$ for some $\{h, l\} \in I \;\; | \; h > l$

$$
V_h(x) = \frac{1}{2}\left(k_p x_2^2 + k_d x_4^2 + k_{e_h} \widehat{E}^2(x)\right) \tag{23}
$$

$$
\left(\frac{k_{e_h}}{k_{e_l}}\right) V_l(x) = \frac{1}{2}\left(\frac{k_{e_h}}{k_{e_l}}\right)\left(k_p x_2^2 + k_d x_4^2\right) + \frac{1}{2}\left(k_{e_h} \widehat{E}^2\right) \tag{24}
$$

Noting that $k_{e_h} > k_{e_l}$, (24) - (23) gives,

$$
\left(\frac{k_{e_h}}{k_{e_l}}\right) V_l(x) - V_h(x) = \frac{1}{2}\left(\frac{k_{e_h}}{k_{e_l}} - 1\right)\left(k_p x_2^2 + k_d x_4^2\right) \geqslant 0
$$

$$
\Rightarrow V_h(x) \leqslant \frac{k_{e_h}}{k_{e_l}} V_l(x) \quad \forall x \in M
$$

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