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22MAT230 MATHEMATICS FOR COMPUTING 4

LLM-Guided Regularization of Pseudoinverse for Ill-Posed Signal Reconstruction

TEAM 7 BATCH C

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Table of Content

Problem Statement : Why This Matters?

Real-World Inverse Problems
- True signals cannot be observed directly
- Measurements are distorted and noisy
- Reconstruction requires solving inverse problems

Practical Applications
- Medical imaging (CT, MRI, Ultrasound)
- Image deblurring & restoration
- Signal denoising & communications
- Remote sensing & scientific measurements

Core Challenge
- These problems are ill-posed
- Small noise causes large reconstruction errors
- Direct inversion is unstable

Impact
- Better image quality
- Reliable signal reconstruction
- Efficient algorithms for real systems

Stable and efficient solutions to inverse problems are essential for real-world signal and image recovery.

What Are ill-Posed Inverse Problems?

Forward Problem (Stable)

$$y = Ax$$

- System and signal are known
- Measurements are computed easily
- Small input changes makes small output changes

Inverse Problem (Unstable)

$$x = A^{-1}y^{\delta}$$

- Only measurements are known
- Original signal must be recovered
- Small noise makes large reconstruction error

Why Inverse Problems Are Hard

- Measurements always contain noise
- Matrix A is often ill-conditioned
- Inversion amplifies noise
- Errors grow rapidly in the solution

Mathematical Definition (Hadamard)

A problem is **ill-posed** if it violates any of:

- Solution does not exist
- Solution is not unique
- Solution must depend on the initial conditions.

Understanding Ill-Conditioning

$\kappa(A) = \frac{\sigma_{\max}}{\sigma_{\min}}$ (ratio of largest to smallest singular values)

Measures how sensitive the solution is to noise

Large $\kappa(A)$ leads to highly unstable inversion

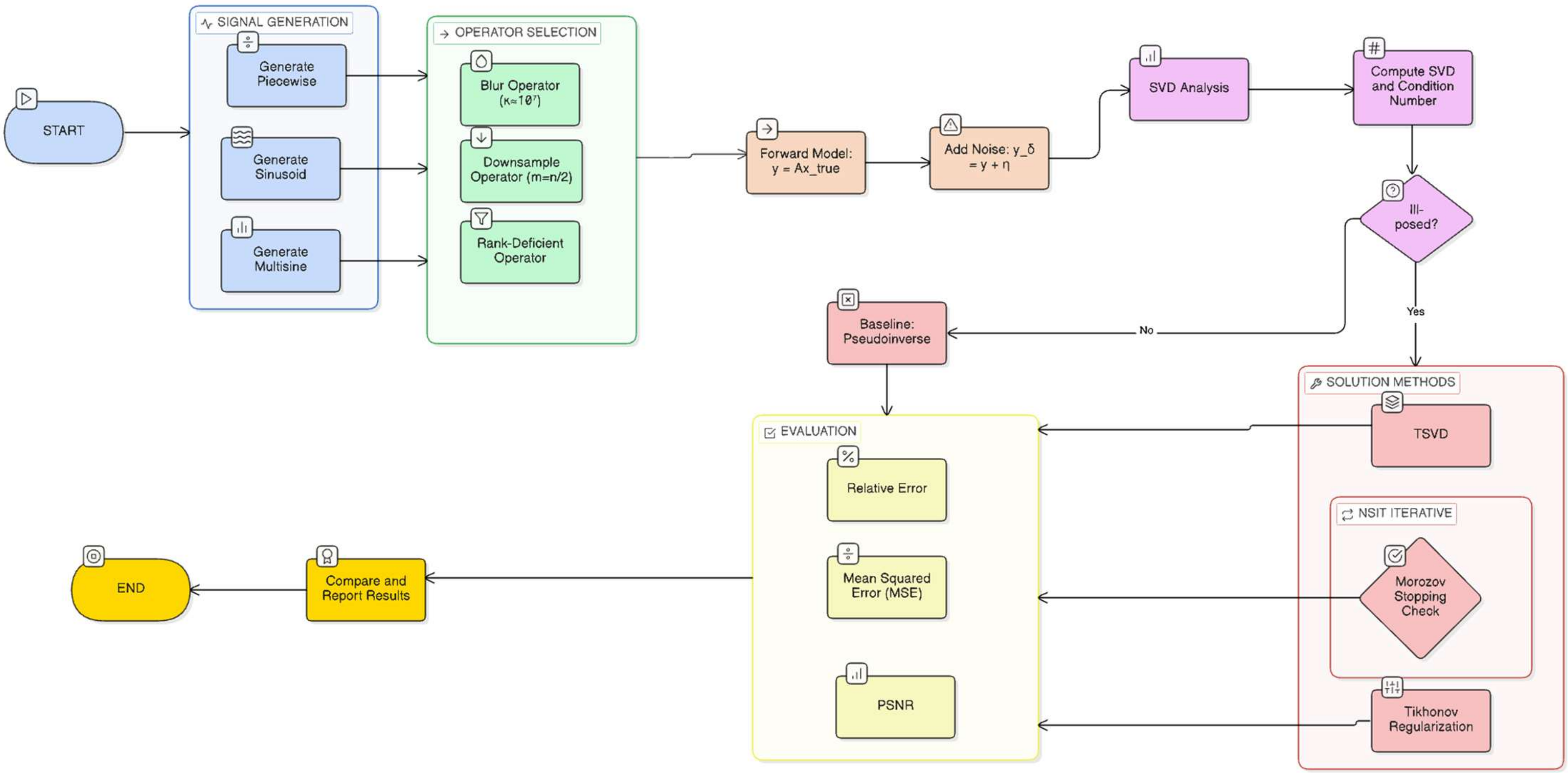
Blur operators have very small

In our project: $\kappa(A) \approx 10^7$

1% noise can become extremely large error in solution

This Explains why direct inversion fails for ill-posed problems

METHODOLOGY DIAGRAM



Mathematical Tool for Analysis

- Any matrix $\boldsymbol{A} \,\boldsymbol{\epsilon}\, \boldsymbol{R}^{(mxn)}$ can be decomposed as $\boldsymbol{A} = \boldsymbol{U\Sigma V^T}$
- Here $\boldsymbol{U} = \boldsymbol{R}^{(mxm)}$
- **Here** $\boldsymbol{\Sigma} = \boldsymbol{R}^{(mxn)}$
- Here $\mathrm{V} = \boldsymbol{R}^{(\mathrm{nxn})}$
- **Σ = diag**$(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \dots\dots, \boldsymbol{\sigma_r}, \mathbf{0}, \dots, \mathbf{0})$ **where** $\boldsymbol{\sigma_1} \geq \boldsymbol{\sigma_2} \geq \cdots .. \boldsymbol{\sigma_r} > \mathbf{0}$
- Right Singular Vectors $A^T u_i = \sigma_i v_i$, Columns $v_1, v_2, \dots\dots v_n$ forms an orthonormal basis for R^n
- Left Singular Vectors $Av_i = \sigma_i u_i$, Columns $u_1, u_2, \dots\dots u_m$ forms an orthonormal basis for R^m
- **Condition number**: $\kappa(A) = \sigma_1/\sigma_n$ (problem severity measure)

- For Real world problems $y = A\,x + \varepsilon$,Using SVD $y = U\Sigma V^T x + \varepsilon$,
- Rewrite in spectral form we get $U^T y = \Sigma\, V^T x + U^T \varepsilon$,

Method	Uses SVD for
Pseudoinverse	Direct inversion through $\Sigma^{(-1)}$
Tikhonov	Spectral Filtering $\sigma_i \to \sigma_i/\left(\sigma_i^2 + \lambda\right)$
TSVD	Hard Truncation, keeps k largest σ_i
NSIT	Iterative decay of regularization parameter

Pseudo inverse using SVD

- Let $\tilde{y} = U^T y$ and $\tilde{x} = V^T x$
- Then $\widetilde{y_i} = \sigma_i \widetilde{x_i} + \widetilde{\varepsilon_i}$

- Large σ_i, Strong components: Signal recoverable $\widetilde{x_i} \approx \widetilde{y_i}/\widetilde{\varepsilon_i}$ (stable) , Noise effect : $\widetilde{\varepsilon_i}$ / $\widetilde{y_i}$
- Small σ_i , Weak components: Signal unrecoverable as Noise dominates , Noise effect : $\widetilde{\varepsilon_i}/\widetilde{y_i} \rightarrow \infty$ as $\sigma_i \rightarrow 0$

- Pseudoinverse using SVD, $A^+ = V\Sigma^+U^T$
- We get, $x_{\text{pinv}} = A^+ y = V\Sigma^+U^T y$, $\Sigma^+_{ii} = 1/\sigma_i$
- $x_{pinv} = V\Sigma^+U^T(Ax_{true} + \epsilon)$ = $x_{true} + V\Sigma^+U^T\varepsilon$
- $x_{pinv} = \tilde{y}/\sigma_i$, if σ_i $is\ large$
- Error = $V\Sigma^+U^T\varepsilon$ = $\Sigma(\widetilde{\varepsilon_i}/\sigma_i)\ v_i$

The Picard Condition

When Does the Inverse Solution Make Sense?

- In ill-posed problems, not all signal components can be recovered
- Solution exists only if:
  - Signal information decays faster than noise amplification
- Problem in practice:
  - Small singular values boost noise heavily
  - High-frequency components become unstable
- Key Insight:
  - Noise dominates when singular values are too small
- Result:
  - Direct inversion fails
  - Regularization is required to suppress noisy components

- $u_i^T y \rightarrow$ how much data aligns with the i-th singular vector
- $\sigma_i \rightarrow$ strength of that component

Mathematical Form: $\sum_{i=1}^{n} \frac{(u_i^T y)^2}{\sigma_i^2} < \infty$

Radio analogy
Strong signal station → loud and clear
Weak signal station → full of static

- If you:
  Turn the volume knob very high to hear the weak station
  You amplify static more than the voice, That's noise amplification.

- Small singular value = weak station
- Inversion = turning volume to max
- Result = noise dominates

Picard condition tells us why we must ignore small singular values to get stable solutions.

Mathematical Setup

Discrete Ill-Posed Inverse Problem

$$y^{\delta} = Ax_{\text{true}} + \eta$$

$A \in R^{m \times n}$: Forward operator (system matrix)

$x_{\text{true}} \in R^{n}$: True signal

$y^{\delta} \in R^{m}$: Noisy measurements

$\eta \sim \mathcal{N}(0, \sigma^2 I)$: Additive Gaussian noise

Our **goal** is to estimate $\hat{x} \approx x_{\text{true}}$ from noisy data

Challenge

- Matrix A is ill-conditioned
- Noise dominates small singular values
- Direct inversion $A^{-1}y^{\delta}$ is unstable and unreliable

Pseudoinverse – The Baseline Failure

Why Naive Inversion Fails in Ill-Posed Problems

For the noisy inverse problem:

$$y^\delta = Ax + \eta$$

Where

$$U = [u_1, \dots, u_m] \quad \text{(left singular vectors)}$$
$$V = [v_1, \dots, v_n] \quad \text{(right singular vectors)}$$
$$\Sigma = diag(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n)$$

Moore Penrose Pesudo-Inverse

$$A^+ = V\Sigma^{-1}U^T$$

Reconstruction: (Substituting into $\hat{x} = A^+ y^\delta$)

$$\hat{x} = \sum_{i=1}^{n} \frac{u_i^T y^\delta}{\sigma_i} v_i$$

Amplification Factor

$$\text{Amplification factor} = \frac{1}{\sigma_i}$$

Since

$$y^\delta = y + \eta$$

We get:

$$\hat{x} = \sum_{i=1}^{n} \frac{u_i^T y}{\sigma_i} v_i + \sum_{i=1}^{n} \frac{u_i^T \eta}{\sigma_i} v_i$$

The second term represents amplified noise .
In ill-posed problems :

$$\sigma_i \to 0 \quad \Rightarrow \quad \frac{1}{\sigma_i} \to \infty$$

Even tiny noise components dominate the solution.
The reconstruction error satisfies:

$$\frac{|\Delta x|}{|x|} \leq \kappa(A) \frac{|\Delta y|}{|y|}$$

Where

$$\kappa(A) = \frac{\sigma_{max}}{\sigma_{min}} \gg 1$$

For our blur operator:

$$\kappa(A) \approx 10^7$$

Direct pseudoinverse inversion is unstable for noisy ill-posed problems

Regularization Philosophy

In ill posed problems, we face conflict between Accuracy and Stability
Fitting data perfectly $(\lambda \to 0)$: Reconstructs Noise as Signal and the solution diverges (unstable)
Ignoring data $(\lambda \to \infty)$: Smooth Stable solution but misses true signal features
Balancing $(\lambda \simeq \lambda^*)$: Suppress noise amplification and Recovers true signal structure

Ill-posed Problem $y_\delta = Ax + \varepsilon$ (unstable)
Well-posed Problem $\min_x ||Ax - y_\delta||^2 + \lambda R(x)$ (stable)
The regularized problem is **inherently different** from the original, but designed to recover the true signal when noise level and λ are balanced correctly.

Consistency	Solution approaches x_{true} as noise $\to 0$
Stability	Small noise changes → small solution changes
Convergence	With optimal $\lambda(\delta)$, error → 0 as noise → 0

Regularization Philosophy

Problem: Recover image/signal x from blurred noisy image/signal y
Forward: y = Blur(x) + ε, where Blur = A

Unregularized:
$x_{\mathrm{pinv}} = A^{+}y \rightarrow$ Noisy, oscillatory (FAILS)

Regularized:
$x_{\lambda} = \underset{x}{\operatorname{argmin}} \left|\left|Ax - y_{\delta}\right|\right|^{2} + \lambda\left|\left|x\right|\right|^{2}$; $x_{\lambda} = (A^{T}A + \lambda I)^{-1}A^{T}y$

STEP 1: Choose regularization method
STEP 2: Select regularization parameter λ
STEP 3: Solve optimization problem

Method 1 – Tikhonov Regularization

Smooth Regularization via Damping

- Problem with direct inversion - Small singular values amplify noise badly
- Tikhonov Regularization idea - Add a penalty term to stabilize the solution
- Optimization Problem: $\widehat{x_\lambda} = \arg\min_x\{|Ax - y|^2 + \lambda|x|^2\}$
- Closed-Form Solution: $\widehat{x_\lambda} = (A^T A + \lambda I)^{-1} A^T y$
- SVD-Based Form: $$\widehat{x_\lambda} = sum_{i=1}^{n}\left(f_i^{(Tikh)(\sigma_i)} * \frac{u_i^T y}{\sigma_i} * v_i\right)$$
- Filter Function: $f_i^{\mathrm{Tikh}} = \dfrac{\sigma_i^2}{\sigma_i^2 + \lambda^2}$
- Key Effect:
  - Large sigma → signal preserved
  - Small → noise suppressed

Tikhonov regularization smoothly damps small singular values to prevent noise explosion.

Tikhonov Filter Analysis $\boldsymbol{f_i(\lambda)} = \sigma_i^2 / \left(\sigma_i^2 + \lambda\right)$

The Tikhonov regularized solution in spectral form: $\widehat{x_\lambda} = f_i(\lambda).\left(u_i^T y_\delta\right)/\sigma_i$

Max amplification: $A_{max} = 1/\left(2\sqrt{\lambda}\right)$, at $\sigma_i = \sqrt{\lambda}$

- Small σ_i : suppressed (σ_i/λ)
- Large σ_i preserved $(1/\sigma_i)$
- Smooth transition, no sharp cutoff

- Large Singular Values $\sigma_i \gg \sqrt{\lambda}$
- $\boldsymbol{f_i(\lambda)} = \sigma_i^2 / \left(\sigma_i^2\right) = 1$ (filter ≈ 1)
- Amplification: $\sigma_i^2 / \left(\sigma_i^2 + \lambda\right) \cdot (1/\sigma_i) \approx 1/\sigma_i$

- Small Singular Values $\sigma_i \ll \sqrt{\lambda}$
- $\boldsymbol{f_i(\lambda)} \approx \sigma_i^2 / (\lambda) \approx 0$ (filter ≈ 0)
- Amplification: $\sigma_i^2 / \left(\sigma_i^2 + \lambda\right) \cdot (1/\sigma_i) \approx \sigma_i/\lambda$

Method 2 : Truncated SVD (TSVD)

•Pseudoinverse fails because **small singular values amplify noise**
•Idea: **Do not invert small singular values**
•TSVD achieves this by **explicit truncation**

Instead of using **all** singular values, we:
Keep only the **largest** k singular values
Discard components dominated by noise

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \gg \sigma_{k+1}, \ldots$$

TSVD Reconstruction Formula
Given SVD:

$$A = U\Sigma V^T$$

The **TSVD solution** is:

$$\boxed{\hat{x}_k = \sum_{i=1}^{k} \frac{u_i^T y^\delta}{\sigma_i} v_i}$$

- Only First K spectral components are used
- Noise-dominated components are ignored

Filter Function Interpretation
TSVD can be written using a **filter function**:

$$\hat{x}_k = \sum_{i=1}^{n} f_i^{TSVD} \frac{u_i^T y^\delta}{\sigma_i} v_i$$

where:

$$\boxed{f_i^{TSVD} = \begin{cases} 1, & i \leq k \\ 0, & i > k \end{cases}}$$

This is a **hard cutoff** in the spectral domain.
Why TSVD Works
Removes inversion of **small** σ_i
Prevents noise amplification
Retains dominant signal components
Mathematically:

$$\frac{1}{\sigma_i} \textit{ used only when } \sigma_i \textit{ is large}$$

TSVD STABILIZES INVERSION BY DISCARDING NOISE-DOMINATED SINGULAR COMPONENTS

Tikhonov vs TSVD Comparison

Filter Function Comparison

- **Key Differences:**
    - Tikhonov**:** Gradual transition, produces smooth solutions
    - TSVD**:** Abrupt transition, preserves strong features
- **When to Use:**
    - Tikhonov**:** General inverse problems, smooth signals
    - TSVD**:** Known dimensionality, feature-rich signals
- **Performance**
    - Both methods achieve ~96% error reduction

Tikhonov smooths the spectrum, TSVD cuts it off.

Forward Operators – Mathematical Details

1. Gaussian Blur Operator
Models smoothing effects in real systems
Causes loss of high-frequency information
Leads to ill-conditioning

Kernel: $k(i) = \left(\frac{1}{z}\right) exp\left(-\frac{i^2}{2\sigma^2}\right)$

Matrix: $A_{ij} = k\big((i - j)mod\ n\big)$

2. Downsampling Operator
Reduces number of measurements
Creates an underdetermined system
Common in signal compression

Aij = 1, j = 2i
0, otherwise

3. Rank-Deficient Operator
Artificially limits information content
Used to study severely ill-posed problems

$A = U \Sigma V^T, \quad \sigma_i = 0\ for\ i > r$

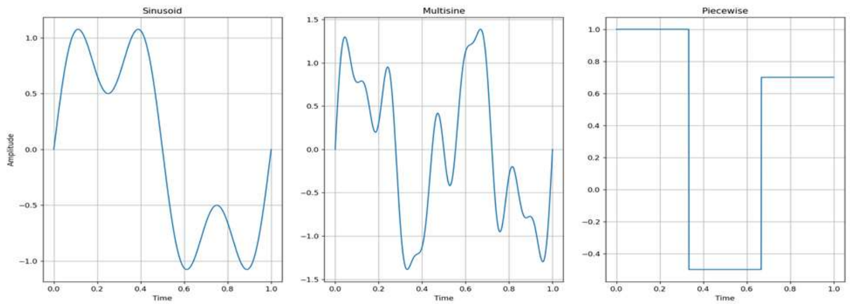
Different operators create different types of ill-posedness

Comprehensive Testing Framework

We use **three representative 1D signals**:

- **Sinusoidal signal**
- **Multi-sine signal**
  Superposition of multiple frequencies
- **Piecewise signal**
  Step function with sharp discontinuities

Covers **smooth, oscillatory, and discontinuous**



Each signal is distorted using **three operators**:

- **Gaussian Blur Operator**
  Convolution-based smoothing
- **Downsampling Operator**
  Dimension reduction
  Underdetermined system
- **Rank-Deficient Operator**
  Controlled loss of information
  Exact zero singular values

Reconstruction Methods Compared

- **Pseudoinverse (baseline – expected to fail)**
- **Tikhonov regularization**
- **Truncated SVD (TSVD)**

Error Metrics

Metric 1: Relative Error (Primary Metric)

$$RelErr = \frac{|x_{true} - \hat{x}|_2}{|x_{true}|_2}$$

Metric 2: Mean Squared Error (MSE)

$$\boldsymbol{MSE = \frac{1}{n}\sum_{i=1}^{n}\left(x_{true,i} - \hat{x}_i\right)^2}$$

Metric 3: Peak Signal-to-Noise Ratio (PSNR)

$$PSNR = 20\log_{10}\left(\frac{\max|x_{true}|}{\sqrt{MSE}}\right)$$

All metrics use the **Euclidean (ℓ_2) norm**:

$$\| x \|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Results - Overall Performance

Method	Parameter	Rel Error	MSE	Info
Pseudoinverse	N/A	1937.857605	2.323587e+06	Baseline (fails)
Tikhonov	λ=1.00e-01	0.021444	2.845296e-04	Optimized regularization
TSVD	k=17	0.017348	1.862208e-04	Optimized truncation
NSIT	n=6, τ=1.0	0.016913	1.770008e-04	Automatic stopping

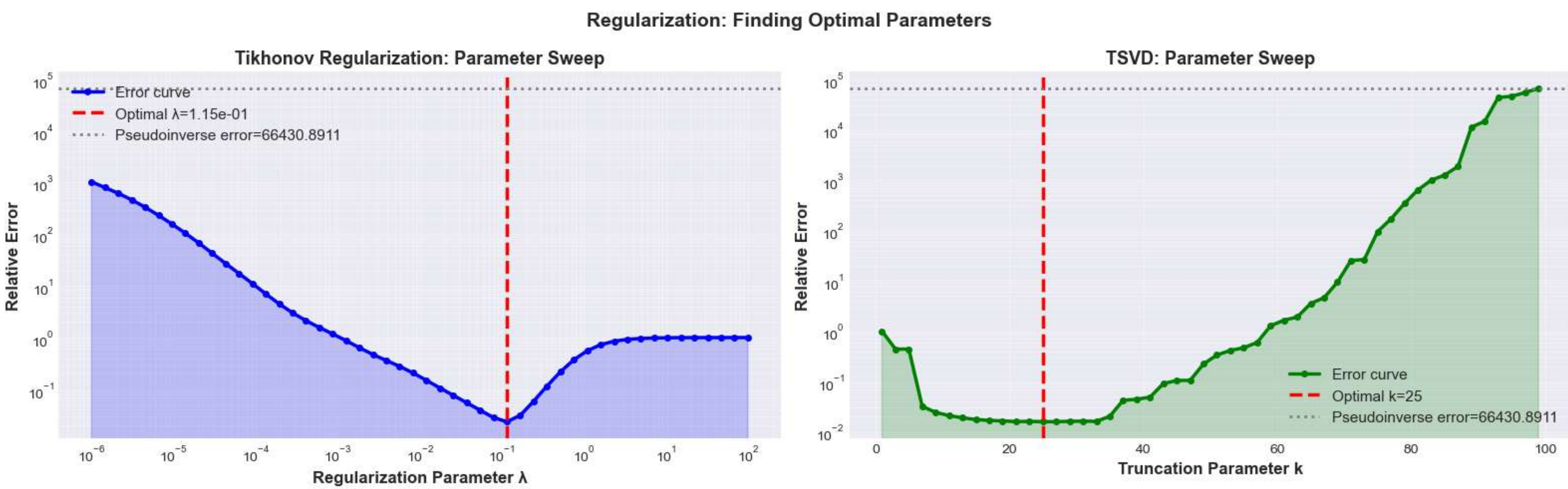
Parameter Optimization

A **parameter sweep** is a systematic process of testing multiple values of a parameter to find which value produces the best results.

Tikhonov Parameter Sweep Results:
Lambda range: 1.00e-06 to 1.00e+02
Optimal λ: 1.151395e-01
Optimal MSE: 0.000342
Optimal Relative Error: 0.023517

TSVD Parameter Sweep Results:
k range: 1 to 99
Optimal k: 25
Optimal MSE: 0.000157

Parameter Optimization



Results - Cross-Problem Robustness

- Tested across multiple problem settings
  - Different signals
  - Different forward operators
- Consistent performance observed
  - Similar reconstruction quality in all cases
  - No major failure cases
- Average Results:
  - Mean relative error ≈ 0.024
  - Very small variation across experiments
- Performance by Operator Type:
  - Gaussian blur → best performance
  - Downsampling → slightly higher error
  - Rank-deficient → still stable
- Key Observation:
  - Regularization methods are robust to problem structure
  - Visual Suggestion:

Regularization works reliably across different inverse problems.

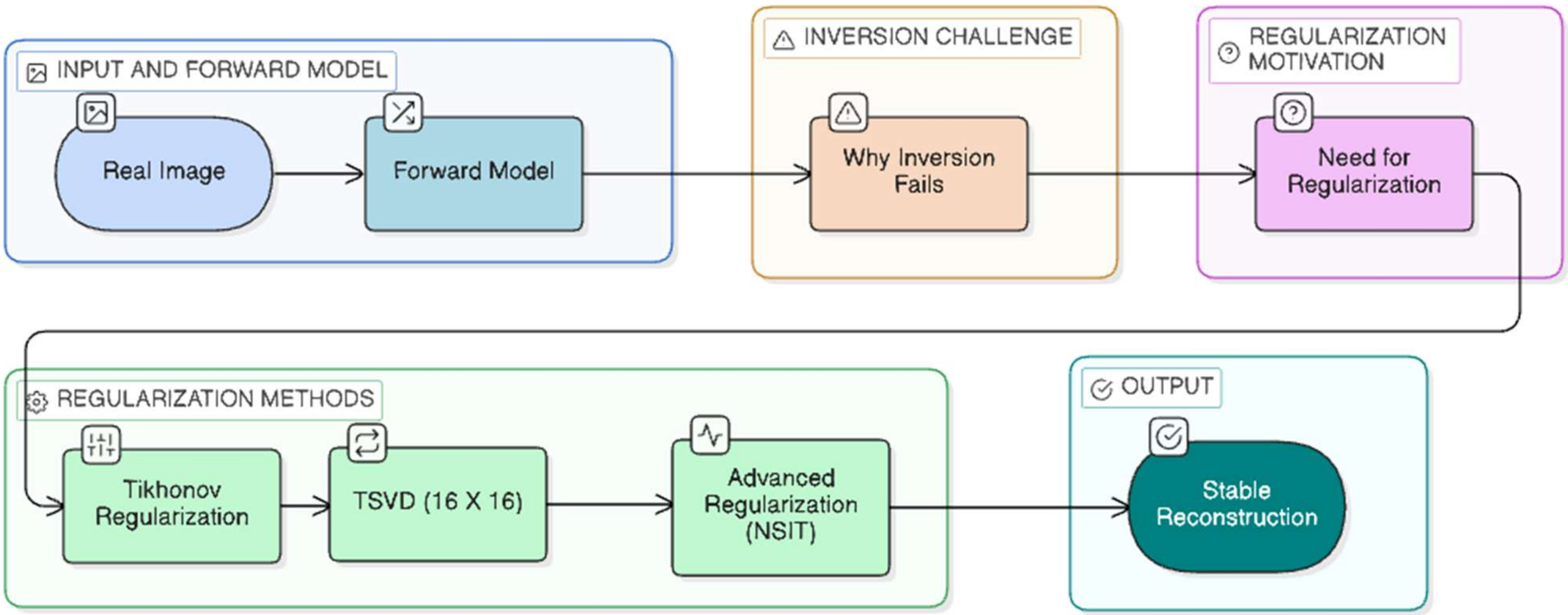
Image Reconstruction

In **Real-world situation ,** We start with a **clean image** (unknown): $x \in \mathbb{R}^n$
Also The imaging system (camera, optics, sensor): blurs the image , introduces noise This gives the **observed image**: $y \in \mathbb{R}^n$

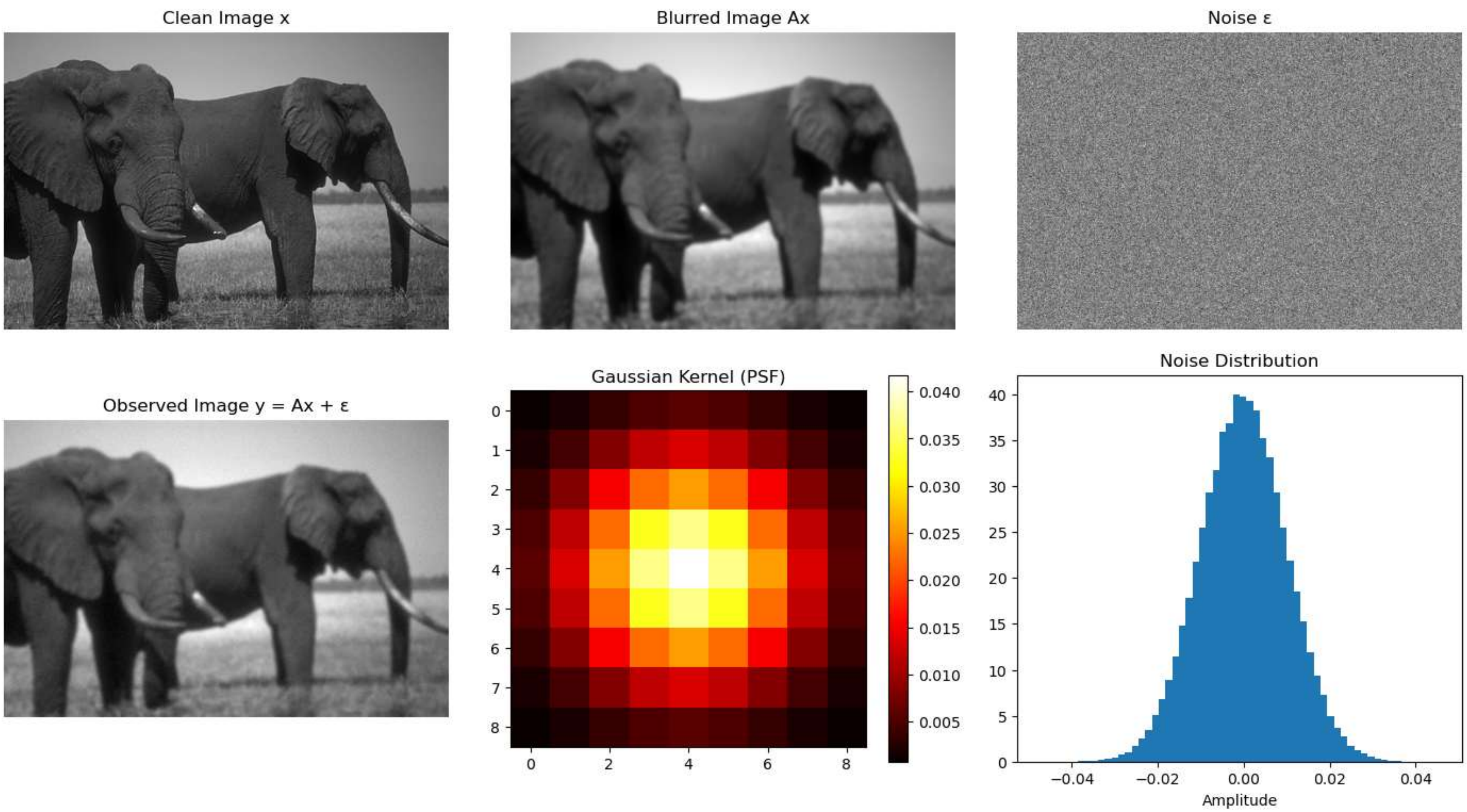
Given y , recover x , This is the problem , $x = A^{-1}y$, Noise ε is small,
but $|| \mathrm{A}^{-1} (y + \varepsilon) - \mathrm{A}^{-1} y || \gg ||\varepsilon||$, Small noise causes huge reconstruction error

The workflow starts from forward modeling, demonstrates the failure of naive inversion, and progressively introduces operator-based regularization methods culminating in NSIT for stable full-image reconstruction.

Image Reconstruction:



Visual Reconstruction Comparison



Non Stationary Iterated Tikhonov

Core Idea
- Improves solution using multiple Tikhonov steps
- Regularization strength changes at each iteration
- Starts smooth, then gradually adds fine details

Update Rule: $x_n = x_{n-1} + (A^T A + \alpha_n I)^{-1} A^T (y - A x_{n-1})$

- Uses residual to correct previous estimate
- Acts like repeated, controlled filtering

Why NSIT Works Better
- Avoids choosing a single fixed α
- Gradually improves conditioning during iterations
- Captures both smooth and detailed components

NSIT – Mathematical Foundation

Morozov Stopping in NSIT — Key Points

- Uses known noise level to decide when to stop iterations
- Measurement model:
  $y = Ax_{true} + \eta$ →data already contains noise
- No need to fit residual below noise level
- Stopping rule:

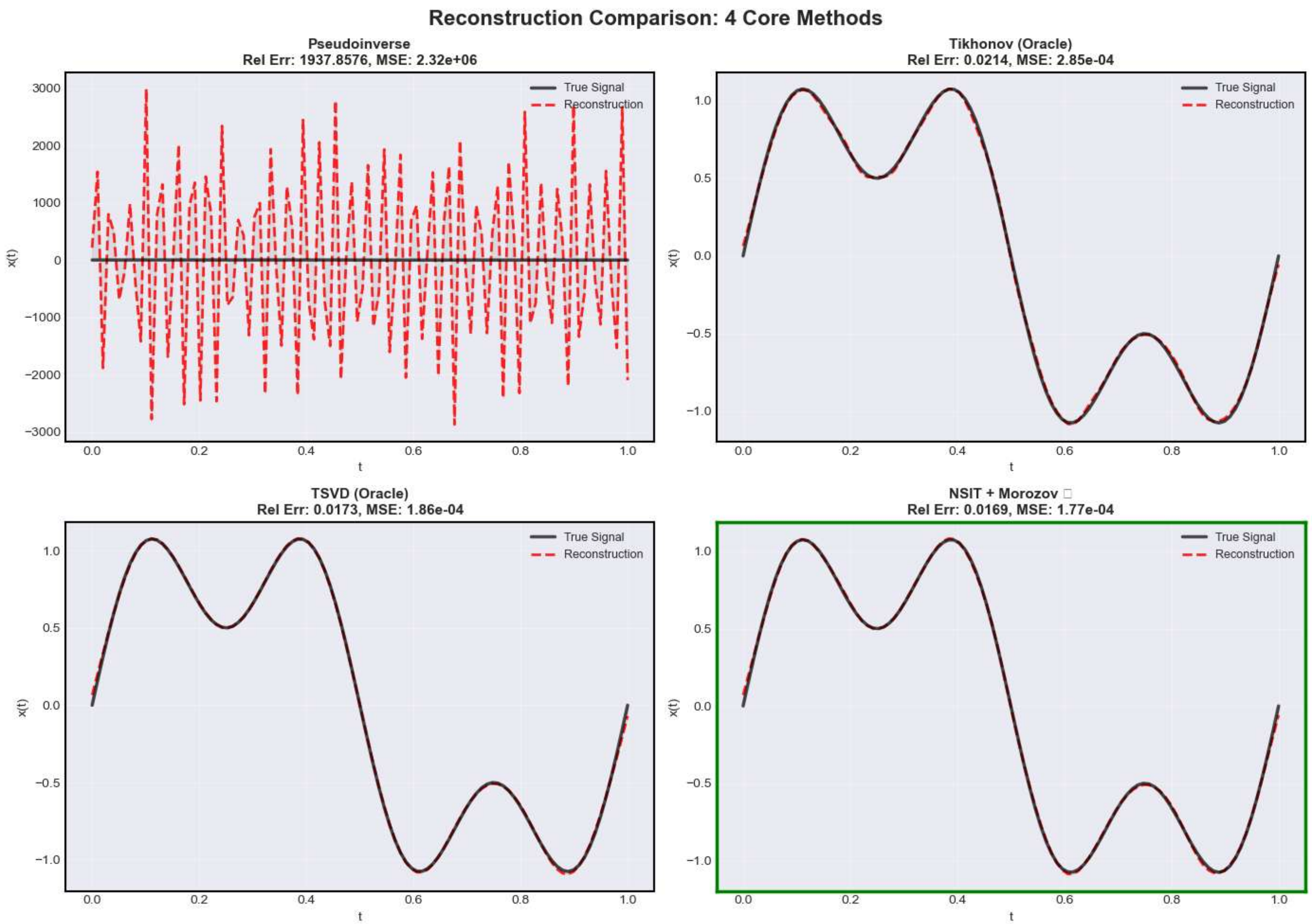
$$\| y - Ax_n \| \leq \tau \, \delta \, \| y \|$$

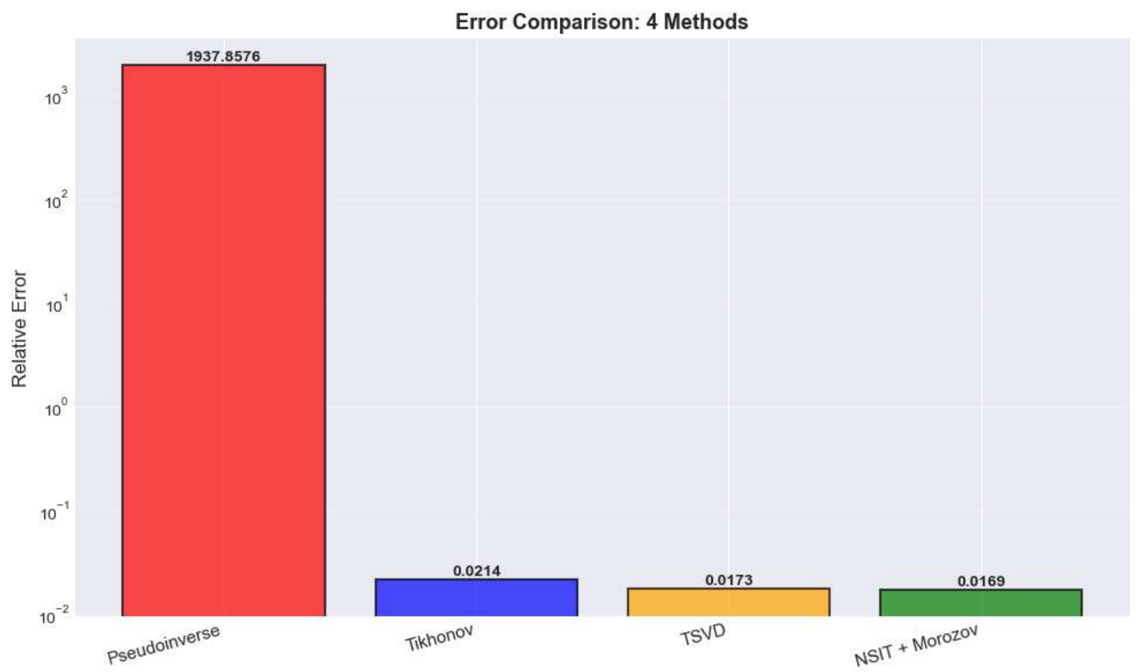
where

y— measured data (noisy)
A— forward operator (blur matrix)
x_n —current reconstruction at iteration n
$y - Ax_n$ — residual
$\| y - Ax_n \|$ —size of mismatch between model and measurements
δ— relative noise level in the data (e.g., 0.01 = 1% noise)
$\| y \|$— scale of the measurements (normalization factor)
τ— safety factor (usually = 1)

•If iterations continue after this point then noise will be amplified

Regularization Dramatically Improves Reconstruction




Error Comparison: 4 Methods
1937.8576
0.0214
0.0173
0.0169
Relative Error
Pseudoinverse
Tikhonov
TSVD
NSIT + Morozov

Key Takeaways

- Ill-posed inverse problems are highly unstable and cannot be solved by direct inversion
- The pseudoinverse fails due to severe noise amplification from small singular values
- Regularization is mandatory to obtain stable and meaningful reconstructions
- Tikhonov and TSVD effectively suppress noise and reduce reconstruction error by ~96%
- SVD is the core analytical tool, explaining ill-posedness and guiding all regularization methods
- Regularization parameters are critical, balancing noise suppression and signal fidelity
- Iterative methods (NSIT) provide automatic, adaptive regularization without manual tuning
- Proposed methods are robust across signals, operators, and noise levels

Limitations and Future Work

Current Limitations

- Experiments limited to 1D **signals** and 2D **images**
- Assumes **Gaussian noise only** (real data may have different noise types)
- Focus on **linear forward operators** (nonlinear inverse problems are harder)
- Requires careful parameter tuning for each method

Future Directions

- Using FNSIT(Fast Non Stationary Iterated Tikhonov) method for solving inverse problems
- LLM guided adaptive regularization and solver control
- Diffusion models as powerful learned priors for inverse problems
- Automatic parameter and stopping rule selection using LLMs

References and Resources

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Thank You !!