** D'Alembert's Ratio test **

If $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l$, then

Σu_n is

(i) convergent if $l < 1$

(ii) divergent if $l > 1$

Test fails if $l = 1$

** Raabe's test **

$$\text{If } \underset{n\to\infty}{Lt}\ n\left\{\frac{u_n}{u_{n+1}}-1\right\} = l$$

then Σu_n is

(i) convergent if $l>1$

(ii) divergent if $l<1$

Test fails if $l=1$

** Cauchy's root test **

If $\lim_{n \to \infty} (u_n)^{1/n} = l$, then

$\sum u_n$ is

(i) convergent if $l < 1$

(ii) divergent if $l > 1$

Test fails if $l = 1$

* NOTE *

(1) $\underset{x \to 0}{Lt} (1+x)^{1/x} = e$

- 1st term 1
- 2nd term $\to$ 0
- power $\longrightarrow \infty$
- 2nd term & power should be reciprocal of each other

$\{ a^{mn} = (a^m)^n \}$

<u>EX:</u> $\underset{x \to 0}{Lt} (1+2x)^{1/x}$

$= \underset{x \to 0}{Lt} (1+2x)^{2 \cdot \left(\frac{1}{2x}\right)}$

$= \underset{x \to 0}{Lt} \left[(1+2x)^{\frac{1}{2x}} \right]^2$

$= e^2$

tends to infinity

(2) $\lim_{n \to \infty} \left(1 + 1/n\right)^n = e$

tends to zero

** Problems **

Q. (1) Determine the nature of the following series

(a) $\sum_{n=1}^{\infty} \frac{10^n}{(n!)^2}$

(b) $\sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$

(c) $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+5}\right)^n$

(d) $\sum_{n=1}^{\infty} (\log n)^{-n}$

$\underline{\underline{\text{Sol}^n}}$:- $u_n = \dfrac{10^n}{(n!)^2}$

$$u_{n+1} = \frac{10^{n+1}}{[(n+1)!]^2} = \frac{10^n \cdot 10}{[(n+1) \cdot n!]^2}$$

$$\left\{ \because \; 6! = 6 \times 5! \right.$$
$$\left. = 6 \times 5 \times (4!) \right\}$$

$$u_{n+1} = \frac{10^n \cdot 10}{(n+1)^2 \cdot (n!)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{\cancel{10^n} \cdot 10}{(n+1)^2 \cdot \cancel{(n!)^2}} \cdot \frac{\cancel{(n!)^2}}{\cancel{10^n}}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{10}{(n+1)^2}$$

$$l = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{10}{\left[n\left(1+\frac{1}{n}\right)\right]^2}$$

$$= \lim_{n\to\infty} \frac{10}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$l = \frac{10}{\infty(1+0)^2} = 0 < 1$$

$\therefore$ By D'Allembert's ratio test,

Σu_n is convergent.

Q(2): $\sum_{n=1}^{\infty} \frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots(2n+1)}$

$$u_n = \frac{1\cdot 2\cdot 3\cdots n}{3\cdot 5\cdot 7\cdots(2n+1)}$$

$$u_{n+1} = \frac{1\cdot 2\cdot 3\cdots n\cdot(n+1)}{3\cdot 5\cdot 7\cdots(2n+1)\cdot(2n+3)}$$

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{2n+3}$$

$$l = \underset{n\to\infty}{Lt} \frac{u_{n+1}}{u_n} = \underset{n\to\infty}{Lt} \frac{n(1+1/n)}{n(2+3/n)}$$

$l = \frac{1}{2} < 1$

$\therefore$ By D'Alembert's ratio test

Σu_n is convergent.

(c) $\sum_{n=1}^{\infty} \left(\frac{n+1}{2n+5} \right)^n$

$$u_n = \left(\frac{n+1}{2n+5} \right)^n$$

$$u_n^{1/n} = \frac{n+1}{2n+5}$$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n+1}{2n+5}$$

$$l = \frac{1}{2} < 1$$

$\therefore$ By cauchy's root test, Σu_n is convergent.

(d) $\sum_{n=1}^{\infty} (\log n)^{-n}$

$$u_n = (\log n)^{-n}$$

$$u_n^{1/n} = (\log n)^{-1} = \frac{1}{\log n}$$

$$l = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log n}$$

$$= \frac{1}{\log \infty} = \frac{1}{\infty} = 0 < 1$$

$\therefore$ By cauchy's root test $\sum u_n$ is convergent.

Q (2) Determine the nature of the series $\sum \frac{1}{(2n+1)(2n+3)}$

method (1)

$$u_n = \frac{1}{(2n+1)(2n+3)}$$

$$u_{n+1} = \frac{1}{[2(n+1)+1][2(n+1)+3]}$$

$$u_{n+1} = \frac{1}{(2n+3)(2n+5)}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{1}{\cancel{(2n+3)}(2n+5)} \cdot (2n+1)\cancel{(2n+3)}$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+1}{2n+5}$$

$$l = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$$

$\therefore$ D'Alembert's ratio test fail.

Let us use Raabe's test

$$K = \lim_{n \to \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right]$$

$$= \lim_{n \to \infty} n \left[\frac{2n+5}{2n+1} - 1 \right]$$

$$= \lim_{n \to \infty} n \left[\frac{\cancel{2n}+5 - \cancel{2n}+1}{2n+1} \right]$$

$$= \lim_{n \to \infty} n \left[\frac{4}{2n+1} \right]$$

$$K = \lim_{n\to\infty} \frac{4\cancel{n}}{\cancel{n}\left(2+\frac{1}{n}\right)}$$

$$K = \frac{4}{2+0} = 2 > 1$$

$\therefore$ By Raabe's test $\sum u_n$ is convergent.

** method 2 **

$$\sum u_n = \sum \frac{1}{(2n+1)(2n+3)}$$

Let $\sum v_n = \sum \frac{1}{n^2}$

$$K = \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{(2n+1)(2n+3)}}{\frac{1}{n^2}}$$

$$K = \lim_{n\to\infty} \frac{n^2}{(2n+1)(2n+3)}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}}{n\left(2+\frac{1}{n}\right) n\left(2+\frac{3}{n}\right)}$$

$$K = \frac{1}{(2+0)(2+0)} = \frac{1}{4} \neq 0$$

∴ By limit form of comparison test, both Σu_n & Σv_n

behave alike.

W.K.T $\sum V_n = \sum \frac{1}{n^2}$ is a

p series with $p = 2 > 1$

& Hence it is convergent.

$\therefore$ $\sum U_n$ is also convergent.

Q(3) $1 + a + \frac{a \cdot (a+1)}{1 \cdot 2} + \frac{a(a+1)(a+2)}{1 \cdot 2 \cdot 3}$

$+ \cdots$

Let us consider a new series

$$\Sigma u_n = a + \frac{a(a+1)}{1 \cdot 2} + \frac{a \cdot (a+1) \cdot (a+2)}{1 \cdot 2 \cdot 3} + \cdots$$

by removing the first term.

clearly, nature of the given series

is same as the nature of Σu_n.

$$u_n = \frac{a \cdot (a+1) \cdot (a+2) \cdots [a+(n-1)]}{1 \cdot 2 \cdot 3 \cdots n}$$

$$\therefore u_n = \frac{a\cdot(a+1)\cdots[a+(n-1)]}{n!}$$

$$u_{n+1} = \frac{a\cdot(a+1)\cdots[a+(n-1)]\cdot(a+n)}{(n+1)!}$$

$\to (n+1)\cdot n!$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{a+n}{n+1}$$

$$l = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n+a}{n+1} = 1$$

$\therefore$ D'Alembert's ratio test fails.

Let us use Raabe's test

$$K = \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$K = \lim_{n\to\infty} n\left[\frac{n+1}{n+a} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(n+1)-(n+a)}{n+a}\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\cancel{n}+1-\cancel{n}-a}{n+a}\right]$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1-a)}{\cancel{n}\left(1+\frac{a}{n}\right)}$$

$$K = \frac{1-a}{1+0} = 1-a$$

$\therefore$ By Raabe's test, $\sum u_n$ is

(i) convergent, if $1-a > 1$

i.e $-a > 0$

i.e $a < 0$

(ii) divergent, if $1-a < 1$

i.e $-a < 0$

i.e $a > 0$.

Test fails if $1-a = 1$

i.e $\boxed{a=0}$.

when $a=0$,

$\sum u_n = 0+0+0+\cdots$

$\therefore \sum u_n$ is convergent.

* conclusion *

$\sum u_n$ is

(i) convergent if $\boxed{a \le 0}$

(ii) divergent if $\boxed{a > 0}$

Q(4) $\frac{2}{5}x + \frac{2 \cdot 4}{5 \cdot 8}x^2 + \frac{2 \cdot 4 \cdot 6}{5 \cdot 8 \cdot 11}x^3 + \dots$

where $x > 0$.

Numerator: $2, 4, 6, \dots\dots$ — AP

$a = 2, \quad d = 2$

$t_n = a + (n-1)d = 2 + (n-1)2$

$= \cancel{2} + 2n - \cancel{2} = 2n$

Denominator: $5, 8, 11, \dots\dots$

$a = 5, \quad d = 3$

$t_n = a + (n-1)d$

$= 5 + (n-1)3$

$= 5 + 3n - 3$

$= 3n + 2$

$$\therefore u_n = \frac{2\cdot 4\cdot 6 \cdots (2n)}{5\cdot 8\cdot 11 \cdots (3n+2)} x^n$$

$$u_{n+1} = \frac{2\cdot 4\cdot 6 \cdots (2n)\cdot(2n+2)}{5\cdot 8\cdot 11 \cdots (3n+2)\cdot(3n+5)} x^{n+1}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{2n+2}{3n+5} x$$

$$l = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(2n+2)}{(3n+5)} x$$

$$l = \frac{2x}{3}$$

$\therefore$ By D'Alembert ratio test,

$\sum u_n$ is

(i) convergent if $\frac{2x}{3} < 1$

i.e $x < \frac{3}{2}$

(ii) Divergent if $\frac{2x}{3} > 1$

i.e $x > \frac{3}{2}$

Test fails if $x = \frac{3}{2}$

x^n

When $x = \frac{3}{2}$

$$u_n = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{5 \cdot 8 \cdot 11 \cdots (3n+2)} \cdot \left(\frac{3}{2}\right)^n$$

$$\frac{u_{n+1}}{u_n} = \frac{2n+2}{3n+5}\left(\frac{3}{2}\right)$$

Let us use Raabe's test

$$K = \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{3n+5}{2n+2} \cdot \left(\frac{2}{3}\right) - 1\right]$$

$$K = \lim_{n\to\infty} n\left[\frac{6n+10}{6n+6} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{6n+10-6n-6}{6n+6}\right]$$

$$= \lim_{n\to\infty} \frac{4n}{6n+6}$$

$$K = \frac{4}{6} = \frac{2}{3} < 1$$

$\therefore$ By Raabe's test Σu_n is divergent.

* conclusim *

$\sum u_n$ is

(i) convergent if $x < 3/2$

(ii) Divergent if $x \geq 3/2$