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ESTIMATION OF ANGULAR VELOCITY AND ACCELERATION FROM SHAFT ENCODER MEASUREMENTS

Pierre R. Bélanger
McGill Research Centre for Intelligent Machines and Department of Electrical Engineering
McGill University, Montréal, Québec, Canada

Abstract

The output of a shaft encoder is basically a quantized measurement of the shaft angle. One method of angular velocity and acceleration measurement is by finite difference of angle measurements (net pulse counts) at regular time samples. It is shown that these estimates become significantly degraded as sampling rates increase Kalman filtering of the angle measurements, based on some model of signal generation, is proposed. Asymptotic analysis at constant sampling rate as the intersample time tends to zero shows that the error variances tend to zero, but at decreasing rates for the angle, velocity and acceleration errors. The analysis also suggests the use of multiple-integrator systems to model the angle signal.

1 Introduction

An optical shaft encoder is basically a disk with two sets of regularly-spaced slots set along concentric circles. The passage of a slot in front of a light beam produces a pulse; the net number of pulses in a given direction, multiplied by the (constant) angle between slots, gives the angular displacement in that direction during the counting period. The two sets are in quadrature, so that it is possible to deduce direction of motion by knowing which pulse train leads the other.

In commercial robots, as well as in many research units, joint velocity feedback is used, either for PD control or for more complex state feedback. Tachometers are not always used in robots; the angular velocity is often derived from the joint angle measurements obtained by an optical shaft encoder. Joint acceleration is sometimes desired, for purposes of identification [1] or control [2,3].

There are some results on angular velocity estimation, fewer on angular acceleration. The velocity results fall in two categories: finite-difference and inverse-time. Finite-difference schemes count the net number of pulses during a fixed interval of time, multiply by the angle corresponding to successive pulses and divide by the duration of the interval. References [4, 5, 6] represent several physical implementations of the finite-difference scheme, with very few variations in the basic principle. The inverse-time method deduces velocity as the interpulse angle divided by the time between successive pulses [7, 8]. Here again, the focus in the literature is on the physical implementation rather than on the method.

In [9], the velocity estimation problem is posed in terms of Kalman filtering. A linear, noise-driven, time-invariant system is assumed to represent the angular motion to be measured. An exact position output is given at those time instants when a pulse is generated, but with a random error as to the time of occurrence. The random time error can be transformed to a position error (through multiplication by the velocity), and the problem then appears as a state estimation problem for a LTI system with non-uniform observation times and state (i.e. velocity)-dependent measurement noise covariance. The examples given in the paper lead one to believe that the results are applicable with relatively large interpulse intervals, so as to allow time to compute the complete time-varying Kalman filter equations.

To describe the nature of the problem, consider an encoder system with an interpulse angle θ_m. Let T_s be the sampling interval of the control system, and let N_p be the number of pulses measured during the interval T_s. The angular velocity estimate by finite difference is $N_p\theta_m/T_s$; this relation shows that the estimate is quantitized, with a quantitization step θ_m/T_s. To illustrate, let $\theta_m = 0.003$ deg (encoder resolution of 300 pulses/rev., $\times 4$ decoding, $100:1$ gear ratio) and $T_s = 0.01s$. Then $\theta_m/T_s = 0.3$ deg$/s$; that resolution may be appropriate at high angular velocity, but represents only 10% resolution at 3 deg$/s$, a figure that is not uncommonly low for a robot approaching steady state. Of course, at $1kHz$ sampling frequency, the resolution becomes a rather unacceptable 3 deg$/s$.

The acceleration estimates are worse. The application of the usual finite difference formula shows that the quantum is now θ_m/T^2, or 30 deg$/s^2$ in the above example.

2 The Finite Difference Method

The finite-difference method is reviewed in this section, for an encoder modeled by a quantizer with nominal step size θ_m. The nominal level i is assumed to be in error by ϵ_i, with ϵ_i having zero mean and variance r.

If the sign of the velocity is not known, then $\theta(t_s)$ can only be pinned down to be between levels $n-1$ and $n+1$. Then, Prob. $(\theta(t_s)|$ last level $= n, \epsilon_{n-1}, \epsilon_{n+1})$ = Uniform distribution, $(n-1)\theta_m + \epsilon_{n-1}$ to $(n+1)\theta_m + \epsilon_{n+1}$.

The estimate $\widehat{\theta}(t_s)$ turns out to be

$$\widehat{\theta}(t_s) = \mathrm{E}[\theta(t_s)|\text{ last level } = n] = n\theta_m \quad (1)$$

and the variance turns out to be

$$\text{var } (\widetilde{\theta}) = \mathrm{E}[\widetilde{\theta}^2|\text{ last level } = n] = \frac{\theta_m^2 + 2r}{3} \quad (2)$$

where $r = \mathrm{E}[\epsilon_i^2]$.

The angular velocity estimate is

$$\widehat{v}(t_s) = \frac{\widehat{\theta}(t_s) - \widehat{\theta}(t_{s-1})}{T}. \quad (3)$$

The estimation error $\widetilde{v}(t_s)$ has a deterministic component and a stochastic component, with

$$|\widetilde{v}_{\text{det}}| \leq \frac{1}{2}|a|_{\max}T \quad (4)$$

where $|a|_{\max} = \max_{t\in[t_s-T,T]}|a(t)|$.

$$\text{var } [\widetilde{v}_{\text{stoc}}] = \frac{2}{T^2}\text{ var } [\widetilde{\theta}]. \quad (5)$$

Equation (2) is used in Equation (5) to yield

$$\text{var } [\widetilde{v}_{\text{stoc}}] = \frac{2}{3}\frac{\theta_m^2 + 2r}{T^2}. \quad (6)$$

Comparing Equations (4) and (6), the deterministic error is proportional to T, while the standard deviation of the stochastic part goes as $1/T$. This suggests that there is an optimal T for the finite-difference estimate of velocity.

The acceleration estimate is

$$\widehat{a}(t_s) = \frac{\widehat{\theta}(t_s) - 2\widehat{\theta}(t_{s-1}) + \widehat{\theta}(t_{s-2})}{T^2}. \quad (7)$$

It follows that

$$|\widetilde{a}_{\text{det}}| \leq \frac{5}{3}|\dddot{\theta}|_{\max}T \quad (8)$$

with

$$|\dddot{\theta}|_{\max} = \max_{t\in[t_s,t_s-2T]}|\dddot{\theta}(t)|.$$

The stochastic component has a variance

$$\text{var } [\widetilde{a}_{\text{stoc}}] = \frac{4}{T^4}\text{ var } [\widetilde{\theta}] \quad (9)$$

3 Filtering Solution

Obviously, it is necessary to use filtering in order to reduce the variances; that is especially so at high sampling rates. It is useful to postulate a linear, stochastic model to represent $\theta(t)$:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \Gamma w(t)\\ y(t) &= \theta(t) = C\mathbf{x}(t) + e(t).\end{aligned} \quad (10)$$

In this model, the states x_1, x_2, x_3 are taken to be the angle, angular velocity and angular acceleration, respectively; if the acceleration is not of interest, only x_1 and x_2 need be so defined. Thus, $C = [\,1\ 0\ 0\cdots 0]$.

In Equation (10), $w(t)$ is the usual white Gaussian noise, zero mean with covariance q. It is recognized that robotic motion is not well characterized by such a stationary random process, and that q may be looked upon simply as a filter parameter to be adjusted.

The scalar output $y(t)$ is the measurement, equal to the latest encoder count times the resolution θ_m. The error $e(t)$ is the quantitization error, assumed white, zero mean and of variance R given by Equation (2).

The optimal filter is, of course, the Kalman filter. Because the observations are equally spaced in time, the discrete steady-state version is used; this is in contrast to [9], where the Kalman filter is time-varying because of the asynchronous observations.

4 Asymptotic Analysis

Since modern robots operate at high sampling rates, it is useful to examine the performance of the Kalman filter as the sampling time tends to zero. Let A_d and Q_d be the A and Q matrices for the discrete-time model that corresponds to the system of Equation (10) at the given sampling rate. Note that R does *not* vary with the sampling rate: that is a crucial feature of this analysis.

For small sampling period T,

$$\begin{aligned}A_d &= e^{AT} \approx I + AT\\ Q_d &= q\int_0^T e^{A(t-\tau)}\Gamma\Gamma^T e^{A^T(t-\tau)}d\tau \approx q\Gamma\Gamma^T.\end{aligned} \quad (11)$$

This is used in the discrete Riccati equation, where it is assumed (shown in [10]) that $P = P(t+1/t)$ is of the order of T to some positive power. If only terms of order up to T^2 are retained, there results

$$AP + PA^T - PC^T(RT)^{-1}CP + q\Gamma\Gamma^T = 0. \quad (12)$$

This is the continuous-time Kalman filter equation for the system of Equation (10), but with the measurement noise variance equal to RT. It is also the dual of the LQ control problem defined by

$$\dot{\mathbf{x}} = A^T\mathbf{x} + C^T u$$

$$J = \int_0^\infty [q\,\mathbf{x}^T\,\mathbf{\Gamma}\,\mathbf{\Gamma}^T\,\mathbf{x} + T\,R\,u^2]dt \qquad (13)$$

which is seen to be the so-called *cheap control problem.*

Pertinent results on this problem may be found in [11], and are summarized as follows:

1. $\lim_{T\to 0} P = P_0$
2. If dim $(\mathbf{\Gamma}^T\mathbf{x}) =$ dim $(\mathbf{u})$ and

$H(s) = \mathbf{\Gamma}^T(sI - A^T)C^T$ has all LHP zeros, then $P_0 = 0$.

Here, C^T is a column vector, and u is a scalar. Furthermore, there is no loss of generality with respect to second-order statistics if the model of Equation (10) is chosen to be minimum-phase. Therefore, Condition 2 is satisfied and $P_0 = 0$.

There remains to solve for P_0 as a function of T. Two authors [12, 13] present solutions, under the condition expressed in this case as

$$C\Gamma\Gamma^T C^T > 0$$

corresponding to $B^TQB > 0$ in the control problem The condition is reasonably practical in the control case; it states, roughly, that at least one of the state variables that are driven directly by u appear in the performance index. The condition is much less practical in filtering, since it requires the output to contain at least one variable driven directly by the plant noise. Here, since $C = [1\ 0\ 0\ \cdots\ 0]$, the condition $C\Gamma \neq 0$ requires that the model contain a white noise velocity component, which is not realistic.

The problem is solved here for the all-pole case, which is adequate to present purposes.

The state space model, in companion form, is

$$A = \begin{bmatrix} 0 & \vdots & I \\ \cdots & \cdots & \cdots \\ & \mathbf{a}^T & \end{bmatrix}$$
$$\mathbf{\Gamma}^T = [\ 0\ \ 0\ \ \cdots\ 0\ \ 1\ ]$$
$$C = [\ 1\ \ 0\ \ \cdots\cdots\ \ 0\ ]$$
$$\mathbf{a}^T = [\ 0\ \ 0\ \ -a_0\ \ -a_1\ \ \cdots\ \ -a_{m-1}\ ]$$

The solution is worked out for the dual control problem. The following result is proved in [10].

Theorem 1 The solution of the Riccati equation (15) for small T is $P^*_{ij}T^{\frac{2n-i-j+1}{2n}}$, where P^*_{ij} is the solution of the same Riccati equation, but with $T = 1$ and $a_0 = a_1 = a_{m-1} = 0$.

Remarks:

1. The diagonal elements of the covariance matrix are $P_{11} = P^*_{11}T^{\frac{2n-1}{2n}}, P_{22} = P^*_{22}T^{\frac{2n-3}{2n}}, \cdots P_{ii} = P^*_{ii}T^{\frac{2n-2i+1}{2n}} \cdots P_{nn} = P^*_{nn}T^{\frac{1}{2n}}$; the term in T gets progressively larger as the index i grows. Ignoring the relative magnitudes of the P^*_{ii}, this means that the covariance becomes larger with larger i. For instance, with $n = 3$ and $T = 0.01, T^{5/6} = 0.0215, T^{3/6} = 0.1, T^{1/6} = 0.464$.
2. The fact that the asymptotic gain and covariance do not depend on the coefficients $a_0, a_1, \cdots a_{m-1}$ is significant, in that it supports the choice of an n-integrator model if the sampling period is small.

5 Models To Be Used

If angular acceleration is to be estimated, the model

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$
$$y = [\ 1\ \ 0\ \ 0\ ]\,\mathbf{x} + e \qquad (14)$$

is postulated.

In Robotics, the behavior is perhaps better described by a deterministic system, which can be written as

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dddot{\theta}(t) \qquad (15)$$

The white plant noise of Equation (14) is a surrogate for the third derivative. Clearly, the more wide-band $\dddot{\theta}$ is, the better the correspondence will be between the two models.

For velocity estimation only, the model is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$
$$y = [\ 1\ \ 0\ ]\,\mathbf{x} + e \qquad (16)$$

6 Examples

We assume a single robot joint under PD control, modeled by

$$\frac{y}{y_d} = \frac{6s + 100}{s^2 + 6s + 100}. \qquad (17)$$

The pole pair has an ω_0 of 10 rad/s and a damping factor ζ of 0.3, leading to somewhat underdamped responses; that was thought to create a greater challenge for estimation than would have been provided by overdamped signals.

The desired acceleration $a_d(t)$ is shown in Fig. 1a, for an amplitude $A = 1$. The values $A = 1$ and $A = 10$ were used, corresponding to slow and fast trajectories. The responses $y(t), \dot{y}(t)$ and $\ddot{y}(t)$ are shown in Fig. 1b, c and d for $A = 1$.

Unless otherwise stated, the shaft encoder resolution is taken to be $\theta_m = 0.003$ deg. and ϵ_i assumed to have a triangular distribution with $\epsilon_m = 0.00075$ deg. The sampling period is $T = 0.01s$.

Example 1

This is to explore the effect of replacing the "true" model by a simpler triple or double integrator. The Kalman gain is computed for the system of Equation (17), in state form

Kalman filters are also derived for the simple models of Equations (15) and (16)). Since these models are not of the same order, it is appropriate to make the comparison on the basis of input-output behavior. Specifically, discrete bode plots of the transfer functions $\hat{x}_2/y$ and $\hat{x}_3/y$ (for the triple integrator of Equation (15), only the first for Equation (16)) are calculated and displayed in Fig. 2. The q's were adjusted until a good fit was obtained (by eye).

Example 2

This is a simulation, with $A = 10$ and a sampling period of $0.01s$. Figure 3 shows the angle error due to the quantization of the shaft encoder, and the angular velocity error of the finite difference estimate. (The angular acceleration estimate is a multiple of 30 deg/s^2, and is totally useless). Kalman filter estimates using the triple-integrator model, are shown in Fig. 4, for $q = 200$; that value corresponds roughly to the square of the maximum slope of the angular acceleration. Estimates were also computed for $q = 20$, but are not shown. All estimates are a priori estimates, i.e. $\hat{\mathbf{x}}(t+1|t)$.

Table I summarizes the means and standard deviations of several numerical experiments including this one. The results for $q = 200$ and $q = 20$ are similar, which seems to indicate that the choice of q is not crucial.

The Kalman filter estimates of the angle are marginally worse than the raw estimates; that is possible because the plant noise is not white, so that the Kalman filter need not be optimal. The Kalman filter estimates of angular velocity are better than the finite difference estimates by a factor of 2 in the standard deviation, the acceleration estimates are improved by an order of magnitude.

Example 3

This example is similar to the previous one, except that a double-integrator model is used. Acceleration estimates are not produced, of course. Results for $q = 20$ are summarized in Table I, along with results for $q = 2$. There is little difference between the results for the two values of q. The estimates are worse than the finite difference estimates, and significantly worse than those obtained with the triple-integrator model.

Example 4

This is a follow-up on Example 3. Since the error standard deviation goes as $1/T$ for the finite-difference estimate of velocity, and since Theorem 1 shows that the dependence is $T^{1/8}$ for the Kalman filter, it follows that a reduction in the sampling period T should give the edge to the Kalman filter. Table I shows that the improvement is as expected.

This example shows that there is a crossover point, in terms of the sampling period, below which the Kalman filter becomes better than the finite-difference estimate. This point occurs at relatively large sampling periods for angular acceleration, between $0.005s$ and $0.001s$ for angular velocity, and at some smaller value for the angle (recall that the covariance goes to zero as $T \to 0$).

Example 5

In this and the next example, a slow trajectory is used, with $A = 1$. The peak angular velocity is only 2 rad/s.

Since all signals are cut by a factor of 10, compared to the case where $A = 10$, it would make sense to decrease q by a factor of 100. The value $q = 2$ is used, with the triple-integrator model, to design the Kalman filter; the measurement noise covariance is as before. The means and standard deviations are given in Table II.

As expected, the finite-difference data are about the same as in the case where $A = 10$. The Kalman filter estimate means and standard deviations are given for $q = 0.2, 2$ and 20. While there appears to be a minimum at or near $q = 2$, it is a very shallow minimum, and performance does not depend heavily on q. Comparing the results with those obtained for $A = 10$, the standard deviation decreases by about 25% for the angle, by a factor of about 3 for the velocity and 5 for the acceleration. Since all signals are reduced by a factor of 10, the relative error is obviously larger for the smaller signals.

Example 6

The double integrator model is used in this example, with $q = 0.02$ and 0.2 as the Kalman filter design parameters. The means and standard deviations are also shown in Table II. Here again, the results are relatively insensitive to the choice of q. The standard deviation is greater than that obtained with the triple integrator model, but by a factor of less than 2.

7 Conclusions

It is permissible to draw certain conclusions from these results, albeit with some caution because of the limited nature of the simulations.

1. Kalman filtering results in only marginal improvement in the angle estimates, over those provided directly by the encoder. The velocity estimate

standard deviations are generally improved by a factor of 2 to 4, even more at shorter sampling periods. There is an order of magnitude improvement in the estimate of the angular acceleration.

2. If only velocity estimates are sought, the use of the double integrator is possible, but the standard deviation of the estimation error is 2-4 times worse than in the case of the triple integrator.

3. The results are relatively insensitive to the choice of q. It is reasonable to use the square of the maximum slope of the acceleration or velocity, as the case may be.

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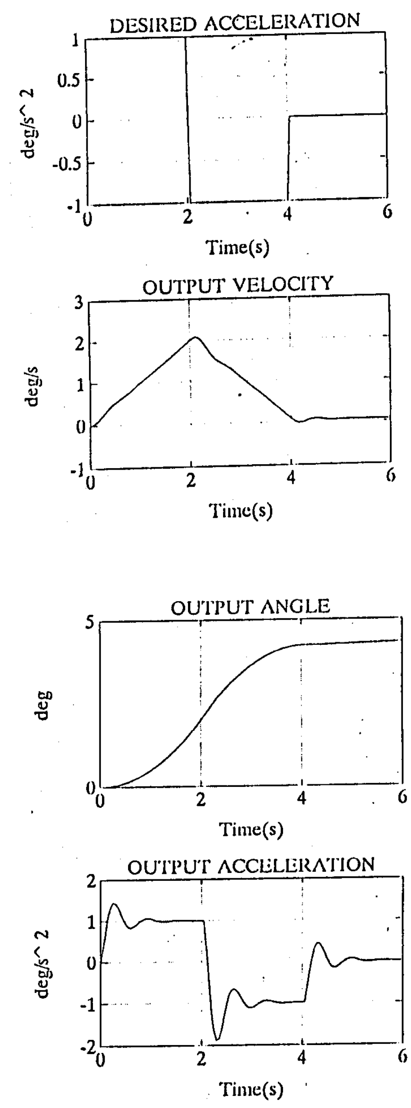

Figure 1 Angle, velocity and acceleration signals, $A = 1$.

Method	A	T	q	Angle (deg)		Angular Veloc. (deg/s)		Angular Accel. (deg/s^2)	
				Mean	Std	Mean	Std	Mean	Std
Finite Diff	10	.01	-	1.33e-3	1.16e-3 (9.01e-4)	2.10e-4	1.37e-1 (2.47e-1)	-8.33e-2	2.17e+1 (3.50e+1)
Kalman, Triple Integ.	"	"	200	1.33e-3	1.33e-3 (1.25e-3)	3.73e-4	6.08e-2 (7.51e-2)	1.62e-3	1.85 (3.10)
"	"	"	20	1.33e-3	1.27e-3 (9.39e-4)	3.13e-4	6.63e-2 (3.96fe-2)	2.44e-4	2.04 (1.14)
Kalman, Double Integ.	"	"	20	1.34e-3	2.22e-3 (4.86e-3)	7.06e-4	1.77e-1 (16.43e-1)	-	-
"	"	"	2	1.34e-3	2.10e-3 (2.10e-3)	6.92e-4	1.77e-1 (2.18e-1)	-	-
Finite Diff.	"	.005	-	1.19e-3	1.27e-3 (9.01e-4)	2.10e-4	2.60e-1 (4.95e-1)	-	-
Kalman, Double Integ.	" "	" "	2 -	1.19e-3	1.50e-3 (1.25e-3)	2.13e-4	1.31e-1 (1.74e-1)	-	-

TABLE I

Method	A	T	q	Angle (deg)		Angular Veloc. (deg/s)		Angular Accel. (deg/s)	
				Mean	Std	Mean	Std	Mean	Std
Finite Diff	1	.01	-	1.34e-3	1.40e-3	6.69e-4	1.28e-1	-4.8.3e-2	2.19e+1
Kalman, Triple Integ.	"	"	0.2	1.34e-3	1.12e-3	1.27e-4	2.80e-2	-1.11e-3	4.05e-1
"	"	"	2	1.34e-3	1.12e-3	3.88e-6	2.34e-2	-2.23e-3	3.57e-1
"	"	"	20	1.34e-3	1.12e-3	6.93e-5	3.16e-2	-7.24e-4	5.24e-1
Kalman, Double Integ.	"	"	.02	1.34e-3	1.28e-3	1.86e-4	4.00e-2	-	-
"	"	"	.2	1.34e-3	1.24e-3	1.67e-4	4.08e-2	-	-

TABLE II

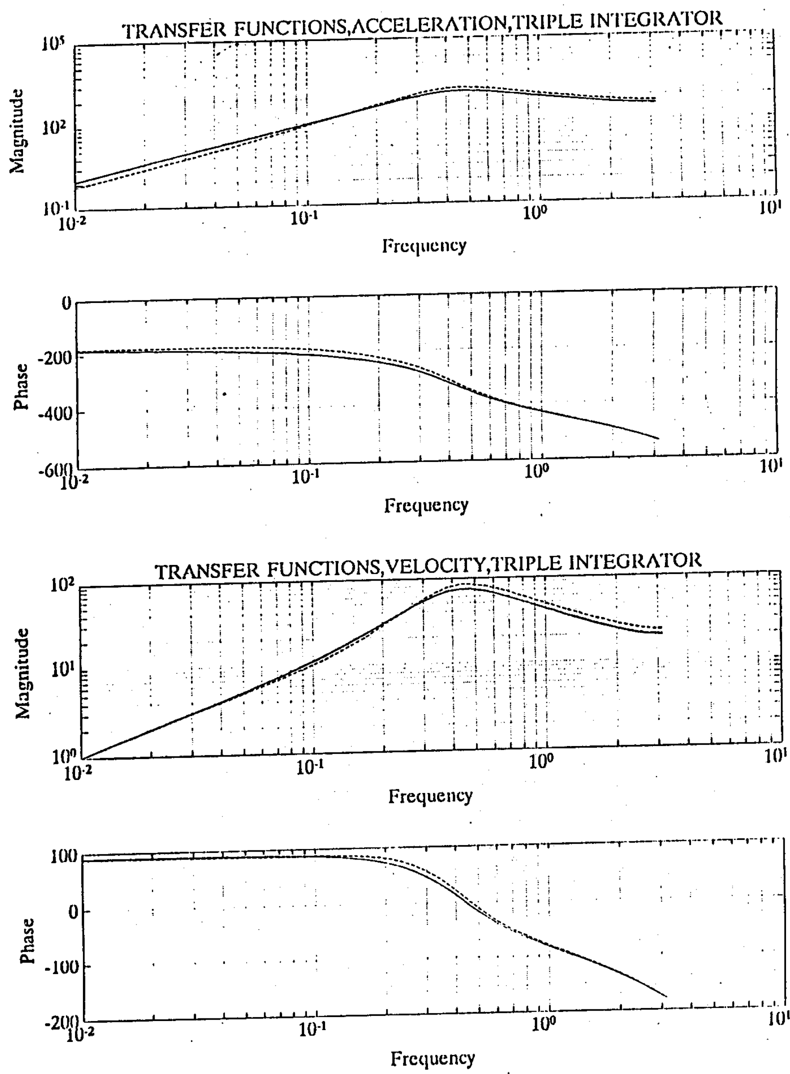

Figure 2 Measurement-to-estimate transfer functions (discrete time): true Kalman filter (dotted), based on triple integrator (full).

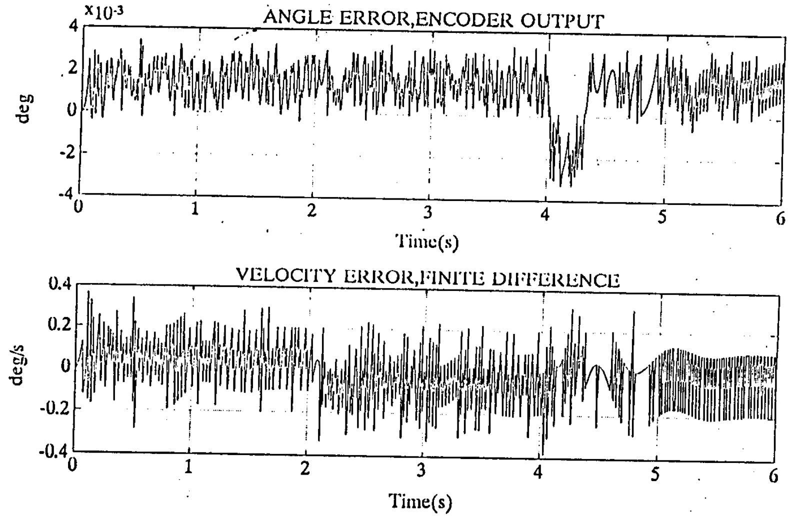

Figure 3 Fast trajectory ($A = 10$), encoder output error and finite-difference velocity error.

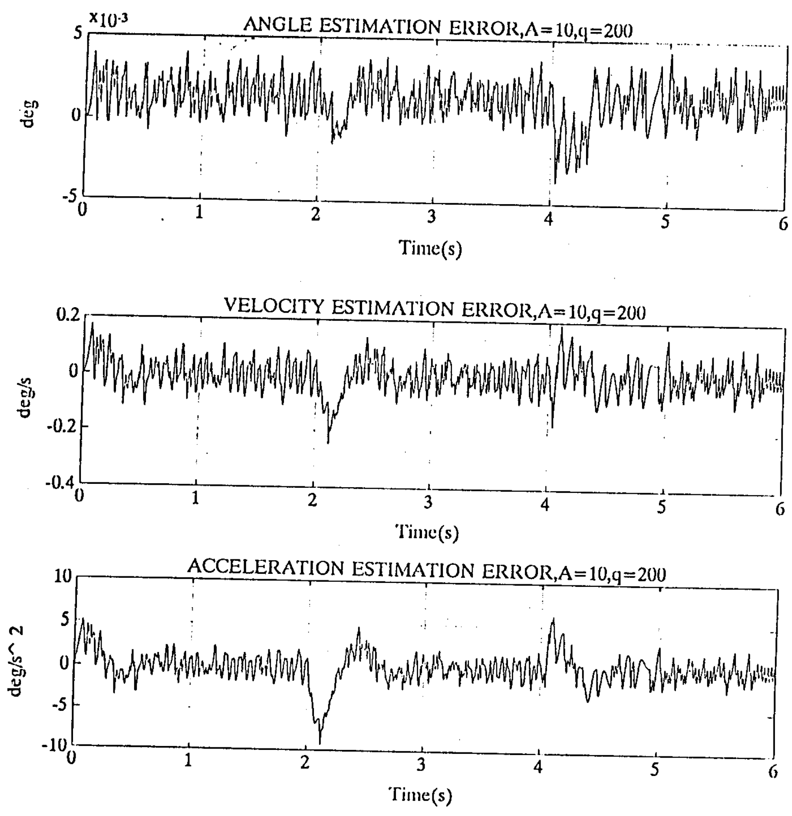

Figure 4 Fast trajectory, Kalman filter error for angle, velocity and accelerations, $q = 200$, triple integrator.