CHAPTER 1

Review

1. Introduction

In this chapter, we shall first briefly review what you have studied in XII standard about complex numbers. Thereafter, we shall learn a few new results in Complex Numbers.

Euclid (300?-275? B.C.)

He taught at Alexandria and founded the Alexandrian School of Mathematics. Euclid is called **"the father of geometry"**. When the king of Alexandria (Egypt) Ptolemy I asked Euclid if there is an easy way to learn geometry, he replied "there is no royal road to geometry". Euclid wrote thirteen books on algebra and geometry mostly compiling the works of earlier mathematicians but had made also significant contributions of his own. Algebra and geometry that is taught at school level all over the world is based on his books. It is said that his books are read and studied only next to the Bible. More than 2000 editions of his book are published in various languages since the first print in 1482. Unfortunately very little is known about this great mathematician.

2. Standard Form of Complex Number

(i) Definition : A number of the form $x + iy$ where x and y are real and $i = \sqrt{-1}$ is called a **complex number**, x is called the **real part** and y is called the **imaginary part**. A complex number is generally denoted by z. If $x = 0$, the number is called purely imaginary and if $y = 0$, the number is called purely real. This form $x + iy$ is called the **standard form** of a Complex Number.

For example, $2 + 3i$ is a complex number, while 4 is purely real number and $-7i$ is purely imaginary number.

Further, $5 + 6i$ is in the standard form, while $\frac{2-3i}{1+i}$ is also a complex number but it is not in the standard form.

3. Conjugate of a Complex Number

Definition : Two complex numbers which differ only in the sign of imaginary parts are called **conjugates** of each other. Thus, $x + iy$ and $x - iy$ are conjugates of each other. The conjugate of z is denoted by $\bar{z}$.

The conjugate of a complex number is obtained by changing the sign of the imaginary part. Thus, the conjugate of $2 + 3i$ is $2 - 3i$, of $-3 - 4i$ is $-3 + 4i$, of $3 - 5i$ is $3 + 5i$, of $2i$ is $-2i$.

Sum and Product of Conjugate Numbers

If $z = x + iy$ then its conjugate $\bar{z} = x - iy$.

Now, $z + \bar{z} = (x + iy) + (x - iy) = 2x$

$\therefore$ Real Part of $z = x = \dfrac{z + \bar{z}}{2}$

Also $z - \bar{z} = (x + iy) - (x - iy) = 2iy$

$\therefore$ Imaginary part of $z = y = \dfrac{z - \bar{z}}{2}$.

Further, $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$.

Thus, the sum and product of two complex conjugates are real but their difference is imaginary.

4. Algebra of Complex Numbers

The arithmetic operations of addition, subtraction, multiplication and division, of complex numbers are performed as shown below.

(a) Addition : The sum of two complex numbers is given by

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

$$\boxed{z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)}$$

In other words, to add two complex numbers we **add their real parts and the imaginary parts separately.**

Properties of addition :

(i) $z_1 + z_2 = z_2 + z_1$

(ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

(iii) $z + 0 = 0 + z = z$

(iv) $\boxed{z + \bar{z} = 2\,\mathrm{Re}(z)}$

For, $z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}(z)$

(b) Subtraction : The difference of two complex numbers is given by

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2)$$

$$\boxed{z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)}$$

In other words, to subtract one complex number from another complex number, we **subtract their real parts and the imaginary parts separately.**

Also, $\boxed{z - \bar{z} = 2i \cdot \mathrm{Im}(z)}$

For, $z - \bar{z} = (x + iy) - (x - iy) = 2iy$

(c) Multiplication : While multiplying one complex number by another complex number we multiply them in usual manner and put $i^2 = -1$ as shown below.

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2)$$
$$= x_1 (x_2 + iy_2) + iy_1 (x_2 + iy_2)$$
$$= x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2$$

$$\therefore \quad \boxed{z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i (x_1 y_2 + x_2 y_1)}$$

Properties of Multiplication :

(i) $z_1 \cdot z_2 = z_2 \cdot z_1$

(ii) $(z_1 \cdot z_2) \cdot z_3 = z_1 - (z_2 \cdot z_3)$

(iii) $z \cdot 1 = 1 \cdot z = z$

(iv) $\boxed{z \cdot \bar{z} = x^2 + y^2}$

For, $z \cdot \bar{z} = (x + iy)(x - iy) = (x^2 - i^2 y^2) = x^2 + y^2$

(d) Division : The quotient $\dfrac{x_1 + iy_1}{x_2 + iy_2}$ is not in the standard form $x + iy$. To put the quotient in the standard form we multiply the numerator and denominator by the conjugate of the denominator as follows :

$$\frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1x_2 - ix_1y_2 + ix_2y_1 - i^2y_1y_2}{x_2^2 - i^2y_2^2}$$
$$= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$\therefore \quad \boxed{\frac{x_1 + iy_1}{x_2 + iy_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\right)}$$

5. Equality of Complex Numbers

Two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are said to be **equal if and only if their real parts are equal** *i.e.* $x_1 = x_2$ **and also their imaginary parts are equal** *i.e.* $y_1 = y_2$.

For example, if $x + iy = 2 + i\sqrt{3}$ then $x = 2$ and $y = \sqrt{3}$.

In general if $f(x, y) + i\Phi(x, y) = u + iv$ then $u = f(x, y)$ and $v = \Phi(x, y)$.

Order Relation : If x, y are two real numbers we know that there are three possibilities *viz.* either $x < y$ or $x = y$ or $x > y$. But if z_1 and z_2 are two complex numbers then there are only two possibilities, either $z_1 = z_2$ or $z_1 \neq z_2$.

Since a complex number consists of two numbers, real and imaginary, **there is no order relation** between two complex numbers such as $z_1 < z_2$ or $z_1 > z_2$. *e.g.*, $2 + 3i < 4 + 5i$ or $6 + 8i > 3 + 5i$ are **meaningless statements.**

6. Powers of *i*

In many problems we need various powers of i. We, therefore, note that

$$i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = i^2 \cdot i = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1, \quad i^5 = i^4 \cdot i = i$$

Similarly, $i^{47} = (i)^{46}\ i = (i^2)^{23} \cdot i = (-1)^{23} \cdot i = -i$

$i^{65} = i^{64}\ i = (i^2)^{32} \cdot i = (-1)^{32} \cdot i = i$

$i^{48} = (i^2)^{24} = (-1)^{24} = 1$

$i^{66} = (i^2)^{33} = (-1)^{33} = -1$

Thus, the powers of i are $+i$ or $-i$, $+1$ or -1 depending on the index of i.

7. Square Root of a Complex Number

To find the square root of a complex number $z = x + iy$ *i.e.* to find $\sqrt{x+iy}$, we put it equal to $a + ib$ where a and b are real and find a, b as shown below.

Let $\sqrt{x+iy} = a + ib$

$\therefore \quad x + iy = (a+ib)^2 = a^2 + 2i\,ab + i^2b^2 = (a^2 - b^2) + 2i\,ab$

Equating real and imaginary parts,

$\therefore \quad x = a^2 - b^2$ and $y = 2ab$.

By solving the two equations we find a and b.

8. Modulus and Argument of a Complex Number

In XIth standard, you have been introduced to **Argand diagram** and how a complex number $z = x + iy$ can be represented on the Argand diagram. Let the complex number $z = x + iy$ be represented on Argand diagram by the point P whose coordinates are x and y. Let r be the distance of P from the origin and θ be the angle made by seg. OP with the x-axis.

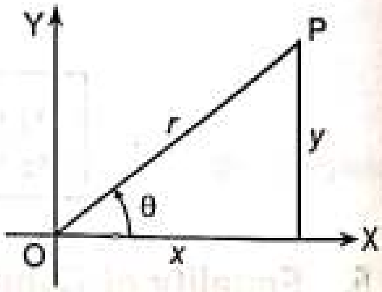


Fig. 1.1

Then $x = r\cos\theta$, $y = r\sin\theta$; $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$.

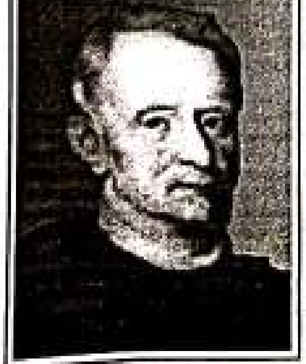
$$\therefore \quad r = \pm\sqrt{x^2 + y^2}$$

Since r is the distance of P from the origin, we take by convention positive sign before the radical sign.

$$\therefore \quad \boxed{r = \sqrt{x^2 + y^2}}$$

Further, $\dfrac{y}{x} = \dfrac{r\sin\theta}{r\cos\theta} = \tan\theta$ *i.e.*, $\boxed{\theta = \tan^{-1}\dfrac{y}{x}}$.

Jean Robert Argand (1768 - 1822)



A Swiss mathematician known for Argand diagram. While working in a book store in Paris he published the idea of representing a number geometrically known as Argand diagram. He later published a rigorous proof of fundamental theorem of algebra.

Definition : Every complex number $z = x + iy$ can be written in the form $z = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$.

$r = \sqrt{x^2 + y^2}$ is called the **modulus** of z denoted by $|z|$ and θ is called the **argument** or **amplitude** of z.

Hence, $\boxed{|z| = \sqrt{x^2 + y^2}}$ and $\boxed{\theta = \tan^{-1}(y/x)}$

Amplitude θ has infinite number of values. The value of θ lying between $-\pi$ and π is called the **principal value**.

Note ....

Note carefully that we take the value of θ which satisfies both equations $x = r\cos\theta$, $y = r\sin\theta$.

9. Polar Form of a Complex Number

As seen above since $x = r\cos\theta$, $y = r\sin\theta$.

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

This is called the **polar form** of the complex number z. Note that the polar co-ordinates of $P(z)$ are (r, θ) (with suitable co-ordinate system) and hence the name polar form.

10. Exponential Form of a Complex Number

We know that for any real value of x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.. \quad \ldots\ldots\ldots\ldots (1)$$

Also
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.. \quad \ldots\ldots\ldots\ldots (2)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots.. \quad \ldots\ldots\ldots\ldots (3)$$

Assuming that the series (1) is true for $x = i\theta$, we get,

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \ldots..$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \ldots..$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots..\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots..\right)$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad \ldots\ldots\ldots\ldots (4)$$

Changing the sign of i in (4),

$$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta} \quad \ldots\ldots\ldots\ldots (5)$$

Adding (4) and (5), we get,

$$\boxed{e^{i\theta} + e^{-i\theta} = 2\cos\theta} \quad \text{.................... (6)}$$

And subtracting (5) from (4), we get,

$$\boxed{e^{i\theta} - e^{-i\theta} = 2i\sin\theta} \quad \text{.................... (7)}$$

We have already seen that a complex number $z = x + iy$ can be written in polar form as $z = r(\cos\theta + i\sin\theta)$

Using (4) we can write it as $\boxed{z = re^{i\theta}}$

This is called **exponential form or Euler's form** of a complex number as it was first suggested by Euler.

Thus, we have three forms of a complex number,

$$\boxed{\begin{array}{ll} z = x + iy & \text{(Cartesian Form)} \\ z = r(\cos\theta + i\sin\theta) & \text{(Polar Form)} \\ z = re^{i\theta} & \text{(Exponential Form)} \end{array}}$$

Similarly, by changing the sign of i, we can have three expressions for the complex conjugate $\bar{z}$.

$$\boxed{\begin{array}{l} \bar{z} = x - iy \\ \bar{z} = r(\cos\theta - i\sin\theta) \\ \bar{z} = re^{-i\theta} \end{array}}$$

Corollary : If $z = \cos\theta + i\sin\theta$, then $\dfrac{1}{z} = \cos\theta - i\sin\theta$.

Proof : $z = e^{i\theta} \quad \therefore \quad \dfrac{1}{z} = (e^{i\theta})^{-1} = e^{-i\theta}$

$$\therefore \quad \frac{1}{z} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta.$$

In other words, $\dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta.$

Example : Prove that $e^{i\pi} + 1 = 0$.

Sol. : We have l.h.s. $= \cos\pi + i\sin\pi + 1$

$= -1 + 0 + 1 = 0$

$=$ r.h.s.

Note ....

It is said that while roaming on a bridge Euler struck at the above equation. It connects two basic real numbers 0 and 1 and two basic transcendental numbers e and π. He was so overjoyed by this discovery that he carved the above equation on a stone of the bridge.

Leonhard Euler (1707 - 1783)

One of the great mathematicians of Switzerland. His father wanted him to become a pastor (a priest). But Bernoulli persuaded his father to allow his son to pursue mathematics. He studied under his fellow countryman, mathematician Bernoulli and had published his first paper when he was 18. The word function first suggested by Leibnitz was generalised further by Bernoulli and Euler. Euler is supposed to be the most prolific mathematical writer in history. He has written a number of text books which are known for his clarity, detail and completeness. Although he had lost his eye-sight for the last 17 years of his life, he did not allow his work to be hampered because all the formulae from trigonometry and analysis (and many poems including the entire Latin epic-Aeneid) were on the tip of his tongue.

11. Product and Quotient of Two Complex Numbers in Exponential Form

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ be two complex numbers in exponential form.

Then their product is given by,

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \qquad \therefore \qquad \boxed{z_1 \cdot z_2 = r_1 \cdot r_2 \, e^{i(\theta_1+\theta_2)}}$$

and their quotient is given by,

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)} \qquad \therefore \qquad \boxed{\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}}$$

12. Meaning of Multiplication and Division

(a) If z_1 and z_2 are two non-zero complex numbers then

$$\boxed{\text{arg. } z_1 z_2 = \text{arg. } z_1 + \text{arg. } z_2}$$

Proof : Let $z_1 = r_1 (\cos\theta_1 + i \sin\theta_1)$, $z_2 = r_2 (\cos\theta_2 + i \sin\theta_2)$ be polar forms of z_1 and z_2.

Then arg. $z_1 = \theta_1$ and arg. $z_2 = \theta_2$.

$$\begin{aligned}
\text{Now, } z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1) \cdot (\cos\theta_2 + i\sin\theta_2) \\
&= r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) \\
&\qquad + i (\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)] \\
\therefore \quad z_1 z_2 &= r_1 r_2 [\cos(\theta_1+\theta_2) + i \sin(\theta_1+\theta_2)]
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ arg. } z_1 z_2 &= \theta_1 + \theta_2 \\
&= \text{arg. } z_1 + \text{arg. } z_2.
\end{aligned}$$

(b) If z_1 and z_2 are two non-zero complex numbers then

$$\boxed{\text{arg.}\left(\frac{z_1}{z_2}\right) = \text{arg. } z_1 - \text{arg. } z_2}$$

Proof : Taking z_1 and z_2 as above [in (a)]

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\therefore \quad \arg.\frac{z_1}{z_2} = \theta_1 - \theta_2 = \arg. z_1 - \arg. z_2$$

13. Properties of $|z|$

The following results about the modulus of z have been studied by you in the XIth standard. We list them below for ready reference.

1. $|z| \geq 0$
2. $|z| = 0$ if and only if $z = 0$ *i.e.* if $x = 0$, $y = 0$.
3. Re. $z \leq |z|$ $\quad (\because\ x \leq \sqrt{x^2 + y^2} = |z|)$
4. Im. $z \leq |z|$
5. $|z| = |\bar{z}|$ $\quad (\because |z| = |x + iy| = \sqrt{x^2 + y^2} = |x - iy| = |\bar{z}|)$
6. $z \cdot \bar{z} = |z|^2$ $\quad (\because z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2)$
7. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ $\quad (\because |z_1 \cdot z_2|^2 = (z_1 z_2)(\bar{z}_1 \bar{z}_2) = (z_1 \bar{z}_1) \cdot (z_2 \bar{z}_2) = |z_1|^2 \cdot |z_2|^2)$
8. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $|z_2| \neq 0$ $\quad (\because |z_1| = \left|\frac{z_1 z_2}{z_2}\right| = \left|\frac{z_1}{z_2}\right| \cdot |z_2|$. Hence.)

Summary

1. If $z = x + iy$, then $\bar{z} = x - iy$.
2. $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$; $\quad z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
3. $z + \bar{z} = 2x$, $\quad z - \bar{z} = 2iy$
4. $z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$

   $z \cdot \bar{z} = x^2 + y^2$; $\quad \frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$
5. If $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ then, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$.

☆☆☆

CHAPTER

2 Complex Numbers

1. Introduction

In the previous chapter, we have revised what you studied of complex numbers in XII standard. In this chapter, we shall learn how De Moiver's theorem can be used in various ways to obtain certain expansions and powers and roots of functions involving complex numbers.

Abraham De Moivre (1667 - 1754)

A French mathematician who made important contributions to statistics, theory of probability and trigonometry. The concept of statistically independent events was first developed by De Moivre. Through the use of complex number he transformed trigonometry from a branch of geometry to a branch of analysis. His treatise on probability has influenced the development of probability theory.

2. De Moivre's Theorem and its Corollaries

Theorem : We restate the De Moivre's Theorem and then deduce some corollaries from it.

De Moiver's Theorem : For any rational number n the value or one of the values of

$$\boxed{(\cos\theta + i\sin\theta)^n \text{ is } \cos n\theta + i\sin n\theta} \qquad \text{.......................... (1)}$$

For example,

(i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

(ii) $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^4 = \cos 4\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

$$= \cos\pi + i\sin\pi = -1 + 0 = -1$$

(iii) $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 = \cos 3\left(\frac{\pi}{6}\right) + i\sin 3\left(\frac{\pi}{6}\right)$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$$

Cor. 1 : If $z = \cos\theta + i\sin\theta$, then $\boxed{\frac{1}{z} = \cos\theta - i\sin\theta}$ .... **(2)**

For $\dfrac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

Cor. 2 : $\boxed{(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta}$ .... (3)

For, $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$

$$= \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta - i\sin n\theta.$$

For example,

(i) $(\cos\theta - i\sin\theta)^6 = \cos 6\theta - i\sin 6\theta$

(ii) $\left(\cos\dfrac{\pi}{8} - i\sin\dfrac{\pi}{8}\right)^8 = \cos 8\left(\dfrac{\pi}{8}\right) - i\sin 8\left(\dfrac{\pi}{8}\right)$

$$= \cos\pi - i\sin\pi = -1 - i(0) = -1$$

(iii) $\left(\cos\dfrac{\pi}{12} - i\sin\dfrac{\pi}{12}\right)^6 = \cos 6\left(\dfrac{\pi}{12}\right) - i\sin 6\left(\dfrac{\pi}{12}\right)$

$$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = 0 - i(1) = -i$$

Cor. 3 : $\boxed{(\cos\theta - i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-n}}$ .... (4)

For, $(\cos\theta - i\sin\theta)^n = [\cos(-\theta) + i\sin(-\theta)]^n$

$$= [(\cos\theta + i\sin\theta)^{-1}]^n = (\cos\theta + i\sin\theta)^{-n}$$

Cor. 4 : $\boxed{\dfrac{1}{(\cos\theta + i\sin\theta)^n} = \cos n\theta - i\sin n\theta}$ .... (5)

Cor. 5 : If $z = \cos\theta + i\sin\theta$, then

$\boxed{z^n = \cos n\theta + i\sin n\theta}$ .... (6)

$\boxed{z^{-n} = \cos n\theta - i\sin n\theta}$ .... (7)

By addition and subtraction

$\boxed{\cos n\theta = \dfrac{1}{2}(z^n + z^{-n})}$ .... (8)

and $\boxed{\sin n\theta = \dfrac{1}{2i}(z^n - z^{-n})}$ .... (9)

Cor. 5 : If $z_1 = \cos\theta + i\sin\theta$, $z_2 = \cos\phi + i\sin\phi$, then

$$\boxed{\begin{array}{c} z_1 \cdot z_2 = \cos(\theta + \Phi) + i\sin(\theta + \Phi) \\ \dfrac{z_1}{z_2} = \cos(\theta - \phi) + i\sin(\theta - \phi) \end{array}}$$ .... (10)

Sol. : Left to you.

Note ....

Note carefully that,

1. $(\sin\theta + i\cos\theta)^n \neq \sin n\theta + i\cos n\theta$.

   But $(\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n$

   $\therefore \quad (\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

2. $(\cos\theta + i\sin\Phi)^n \neq \cos n\theta + i\sin n\Phi$.

Type I : Class (a) : 3 Marks

Example 1 : Simplify $\dfrac{(\cos 3\theta + i\sin 3\theta)(\cos\theta - i\sin\theta)}{(\cos 5\theta - i\sin 5\theta)}$.

Sol. :

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$ [By DeMoivre's Theorem]

$\cos\theta - i\sin\theta = (\cos\theta + i\sin\theta)^{-1}$ [By Corollary 1]

$\cos 5\theta - i\sin 5\theta = (\cos\theta + i\sin\theta)^{-5}$ [By Corollary 1]

$$\therefore \quad \text{Expression} = \frac{(\cos\theta + i\sin\theta)^3(\cos\theta + i\sin\theta)^{-1}}{(\cos\theta + i\sin\theta)^{-5}}$$

$$= \frac{(\cos\theta + i\sin\theta)^2}{(\cos\theta + i\sin\theta)^{-5}}$$

$$= (\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta.$$

Example 2 : Simplify $\dfrac{(\cos 2\theta + i\sin 2\theta)(\cos\theta - i\sin\theta)^3}{(\cos 3\theta + i\sin 3\theta)^2(\cos 5\theta - i\sin 5\theta)^4}$.

Sol. :

$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$ [By De Moivre's Theorem]

$\cos\theta - i\sin\theta = (\cos\theta + i\sin\theta)^{-1}$ [By Corollary 1]

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$

$\cos 5\theta - i\sin 5\theta = (\cos\theta + i\sin\theta)^{-5}$

$$\therefore \quad \text{Expression} = \frac{(\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta + i\sin\theta)^6(\cos\theta + i\sin\theta)^{-20}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-1}}{(\cos\theta + i\sin\theta)^{-14}} = (\cos\theta + i\sin\theta)^{13}$$

$$= \cos 13\theta + i\sin 13\theta.$$

Example 3 : Simplify $\dfrac{(\cos 3\theta + i\sin 3\theta)^4(\cos 4\theta - i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^3(\cos 5\theta + i\sin 5\theta)^{-4}}$.

Sol. :

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$ [By De Moivre's Theorem]

$\cos 4\theta - i\sin 4\theta = (\cos\theta + i\sin\theta)^{-4}$ [By Corollary 1]

$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$

$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$$\therefore \quad \text{Expression} = \frac{(\cos\theta + i\sin\theta)^{12}(\cos\theta + i\sin\theta)^{-20}}{(\cos\theta + i\sin\theta)^{12}(\cos\theta + i\sin\theta)^{-20}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-8}}{(\cos\theta + i\sin\theta)^{-8}} = (\cos\theta + i\sin\theta)^{-8+8} = 1.$$

Example 4 : Simplify $\dfrac{(\cos 5\theta - i\sin 5\theta)^{2/5}[\cos(2\theta/7) + i\sin(2\theta/7)]^7}{(\cos 4\theta + i\sin 4\theta)^{1/4}[\cos(2\theta/3) - i\sin(2\theta/3)]^3}$.

Sol. : $(\cos 5\theta - i\sin 5\theta)^{2/5} = (\cos 5\theta + i\sin 5\theta)^{-2/5}$ [By Corollary 1]

$= (\cos\theta + i\sin\theta)^{5\times(-2/5)}$ [By De Moivre's Theorem]

$[\cos(2\theta/3) - i\sin(2\theta/3)]^3 = [\cos(2\theta/3) + i\sin(2\theta/3)]^{-3}$ [By Corollary 1]

$= (\cos\theta + i\sin\theta)^{(2/3)\times(-3)}$ [By De Moivre's Theorem]

$$\therefore \quad \text{Expression} = \frac{(\cos\theta + i\sin\theta)^{-2}(\cos\theta + \sin\theta)^{(2/7)\times 7}}{(\cos\theta + i\sin\theta)^{4\times(1/4)}(\cos\theta + i\sin\theta)^{-2}}$$

$$= (\cos\theta + i\sin\theta)^{-2+2-1+2} = (\cos\theta + i\sin\theta)^1$$

$$= \cos\theta + i\sin\theta$$

Type I : To Find the Sum $z^n + \bar{z}^n$

To find this sum, we first write z in the polar form $z = r(\cos\theta + i\sin\theta)$. Then $\bar{z} = r(\cos\theta - i\sin\theta)$. Then we use DeMoivers Theorem : Class (a) : 3 Marks

Example 1 (a) : If n is a positive integer show that

$$(a + ib)^n + (a - ib)^n = 2r^n \cos n\theta \quad \text{where } r^2 = a^2 + b^2 \text{ and } \theta = \tan^{-1}(b/a).$$

Hence, or otherwise deduce that $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$. **(M.U. 1980)**

Sol. : Let $a + ib = r(\cos\theta + i\sin\theta)$

$\therefore \quad a^2 + b^2 = r^2$ and $\theta = \tan^{-1}(b/a)$

l.h.s. $= [r(\cos\theta + i\sin\theta)]^n + [r(\cos\theta - i\sin\theta)]^n$

$= r^n(\cos n\theta + i\sin n\theta) + r^n(\cos n\theta - i\sin n\theta)$

$= 2r^n \cos n\theta.$ .... (1)

For the second part, we put $a = 1$, $b = \sqrt{3}$, $n = 8$ in (1)

$\therefore \quad r^2 = a^2 + b^2 = 1 + 3 = 4 \quad \therefore \quad r = 2$

and $\quad \theta = \tan^{-1}\dfrac{a}{b} = \dfrac{\sqrt{3}}{1} = 60° = \dfrac{\pi}{3}$ .... (2)

$$\therefore \quad (1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = 2\cdot 2^8 \cos\frac{8\pi}{3} = 2^9\cos\left(2\pi + \frac{2\pi}{3}\right)$$

$$= 2^9\cos\frac{2\pi}{3} = 2^9\cos\left(\pi - \frac{\pi}{3}\right)$$

$$= -2^9\cos\frac{\pi}{3} = -2^9\cdot\frac{1}{2} = -2^8.$$

Example 2 (a) : Prove that $(1+i\sqrt{3})^8+(1-i\sqrt{3})^8=-2^8$.

Sol. : Let us write $1+i\sqrt{3}$ in polar form. Since, $1+i\sqrt{3}=r(\cos\theta+i\sin\theta)$.

$$\therefore \quad r=\sqrt{x^2+y^2}=\sqrt{1+3}=2$$

and $\quad \cos\theta=\dfrac{x}{r}=\dfrac{1}{2},\quad \sin\theta=\dfrac{y}{r}=\dfrac{\sqrt{3}}{2} \quad \therefore\ \theta=\dfrac{\pi}{3}$

$$\therefore \quad (1+i\sqrt{3})=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \qquad \text{.....................(1)}$$

$$\text{l.h.s.}=\left[2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^8+\left[2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)\right]^8$$

$$=2^8\left(\cos\frac{8\pi}{3}+i\sin\frac{8\pi}{3}\right)+2^8\cos\left(\frac{8\pi}{3}-i\sin\frac{8\pi}{3}\right)$$

$$=2^8\cdot 2\cos\frac{8\pi}{3}=2^8 2\cos\left(3\pi-\frac{\pi}{3}\right)=-2^8.$$

Example 3 (a) : If n is a positive integer, prove that $(1+i)^n+(1-i)^n=2\cdot 2^{n/2}\cos\dfrac{n\pi}{4}$.

Hence, deduce that $(1+i)^{100}+(1-i)^{100}=-2^{51}$.

Sol. : We can prove the result by putting $a=1$, $b=1$ in the result of Ex. 1, so that $r=\sqrt{2}$ and $\theta=\pi/4$ or independently as follows.

Let use write $1+i$ in the polar form. Since, $\cos\dfrac{\pi}{4}=\sin\dfrac{\pi}{4}=\dfrac{1}{\sqrt{2}}$.

$$1+i=\sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right] \quad \text{and} \quad (1-i)=\sqrt{2}\left[\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right]$$

$$\therefore \quad (1+i)^n+(1-i)^n=(\sqrt{2})^n\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]^n+(\sqrt{2})^n\left[\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right]^n$$

$$=2^{n/2}\left[\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right]$$

$$=2^{n/2}\cdot 2\cos\frac{n\pi}{4}.$$

Now, put $n=100$,

$$\therefore \quad (1+i)^{100}+(1-i)^{100}=2^5\cdot 2\cos\frac{100\pi}{4}=-2^{51}.$$

Example 4 (a) : Evaluate $(1+i)^{100}+(1-i)^{100}$.

Sol. : We have $\quad 1+i=\sqrt{2}\left(\dfrac{1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)=\sqrt{2}\left(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}\right) \qquad$ ....(1)

$$1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)$$

$$\therefore \quad (1+i)^{100}+(1-i)^{100}=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{100}+\left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^{100}$$

$$=2^{50}(\cos 25\pi+i\sin 25\pi)+2^{50}(\cos 25\pi-i\sin 25\pi)$$

$$=2\cdot 2^{50}\cos 25\pi=2^{51}\cdot\cos(24\pi+\pi)=-2^{51}.$$

Example 5 (a) : Prove that $(\sqrt{3}+i)^n+(\sqrt{3}-i)^n=2^{n+1}\cos\dfrac{n\pi}{6}$.

Sol. : Let us write $\sqrt{3}+i$ in polar form. Since $\sqrt{3}+i=r(\cos\theta+i\sin\theta)$.

$$\therefore \quad r=\sqrt{x^2+y^2}=\sqrt{1+3}=2$$

and $\quad \cos\theta=\dfrac{x}{r}=\dfrac{\sqrt{3}}{2},\ \sin\theta=\dfrac{y}{r}=\dfrac{1}{2} \quad \therefore\ \theta=\dfrac{\pi}{6}$.

$$\therefore \quad \sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) \text{ and } \sqrt{3}-i=2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)$$

$$\therefore \quad (\sqrt{3}+i)^n+(\sqrt{3}-i)^n=\left[2\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]^n+\left[2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)\right]^n$$

$$=2^n\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)+2^n\left(\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right)$$

$$=2\cdot2^n\cos\frac{n\pi}{6}=2^{n+1}\cos\frac{n\pi}{6}.$$

Example 6 (a) : If $z=\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}$ and $\bar{z}$ is the conjugate of z, prove that $(z)^{10}+(\bar{z})^{10}=0$.

(M.U. 1989)

Sol. : We have $z=\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}=\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}$. $\quad \therefore \quad \bar{z}=\cos\dfrac{\pi}{4}-i\sin\dfrac{\pi}{4}$

$$\therefore \quad (z)^{10}+(\bar{z})^{10}=\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{10}+\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{10}$$

$$=\left(\cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}\right)+\left(\cos\frac{10\pi}{4}-i\sin\frac{10\pi}{4}\right)$$

$$=2\cos\frac{10\pi}{4}=2\cos\left(\frac{5\pi}{2}\right)=0$$

Example 7 (a) : Prove that $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^n+\left(\dfrac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1, if $n=3k\pm1$ and 2, if $n=3k$ where k is an integer.

Sol. : We have, $\quad \dfrac{-1+i\sqrt{3}}{2}=-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$

$$\therefore \quad \left(\frac{-1+i\sqrt{3}}{2}\right)^n=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^n=\cos\left(\frac{2n\pi}{3}\right)+i\sin\left(\frac{2n\pi}{3}\right)$$

Changing the sign of i

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^n=\cos\left(\frac{2n\pi}{3}\right)-i\sin\left(\frac{2n\pi}{3}\right)$$

$\therefore$ By addition, Expression $=2\cos\left(\dfrac{2n\pi}{3}\right)$

If $n = 3k \pm 1$, Expression $= 2\cos\left(2k\pi \pm \frac{2\pi}{3}\right) = 2\left(-\frac{1}{2}\right) = -1.$

If $n = 3k$, Expression $= 2\cos(2k\pi) = 2(1) = 2.$

Example 8 (a) : If $z = -\frac{1+i\sqrt{3}}{2}$ then prove that $z^{3k} + \bar{z}^{3k} = 2.$

Sol. : Left to you.

Example 9 (a) : If $z = -1 + i\sqrt{3}$ and n is an integer, prove that

$z^{2n} + 2^n \cdot z^n + 2^{2n} = 0$ if n is not a multiple of 3. **(M.U. 1999, 2015)**

Sol. : We have $z = -1 + i\sqrt{3} = 2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$$\therefore \quad z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \qquad \therefore \quad z^n = 2^n\left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right)$$

And $$\frac{1}{z} = \frac{1}{2[\cos(2\pi/3) + i\sin(2\pi/3)]} = \frac{\cos(2\pi/3) - i\sin(2\pi/3)}{2}$$ [By (2) of page 2-1]

$$\therefore \quad \frac{1}{z^n} = \frac{\cos(2n\pi/3) - i\sin(2n\pi/3)}{2^n}$$

$$\therefore \quad \frac{z^n}{2^n} + \frac{2^n}{z^n} = 2\cos\left(\frac{2n\pi}{3}\right)$$

If k is an integer and $n = 3k$, $\cos\left(\frac{2n\pi}{3}\right) = \cos 2k\pi = 1 \quad \therefore \quad \frac{z^n}{2^n} + \frac{2^n}{z^n} = 2$

$n = 3k + 1$, $\cos\left(\frac{2n\pi}{3}\right) = \cos\left(2k\pi + \frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} = -\frac{1}{2}$ .... (i)

$n = 3k + 2$, $\cos\left(\frac{2n\pi}{3}\right) = \cos\left(2k\pi + \frac{4\pi}{3}\right) = \cos\frac{4\pi}{3} = -\frac{1}{2}$ .... (ii)

$\therefore$ If n is not a multiple of 3, then from (i) and (ii),

$$\frac{z^n}{2^n} + \frac{2^n}{z^n} = 2 \cdot \left(-\frac{1}{2}\right) = -1 \quad i.e. \quad z^{2n} + 2^n \cdot z^n + 2^{2n} = 0.$$

Restatement : If $z = -1 + i\sqrt{3}$ then prove that

$$\left(\frac{z}{2}\right)^n + \left(\frac{2}{z}\right)^n = \begin{cases} 2, & \text{if n is a multiple of 3} \\ -1, & \text{if n is not a multiple of 3} \end{cases}$$ **(M.U. 2015)**

Sol. : Same as above.

Type III : To Find the Powers of z : Class (b) : 6 Marks

Example 1 (b) : Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}}$.

Sol. : We have $1 + i\sqrt{3} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ .... (1)

$$\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

$$\therefore \quad \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{2^{16}\left[\cos(\pi/3)+i\sin(\pi/3)\right]^{16}}{2^{17}\left[\cos(\pi/6)-i\sin(\pi/6)\right]^{17}}$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{16}\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^{-17}$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{16}\left[\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right]^{-17}$$

$$= \frac{1}{2}\left(\cos\frac{16\pi}{3}+i\sin\frac{16\pi}{3}\right)\left[\cos\left(\frac{17\pi}{6}\right)+i\sin\left(\frac{17\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{16}{3}+\frac{17}{6}\right)\pi+i\sin\left(\frac{16}{3}+\frac{17}{6}\right)\pi\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{49}{6}\right)\pi+i\sin\left(\frac{49}{6}\right)\pi\right] = \frac{1}{2}\left[\cos\left(8\pi+\frac{\pi}{6}\right)+i\sin\left(8\pi+\frac{\pi}{6}\right)\right]$$

$$\therefore \quad \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{1}{2}\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]$$

Hence, the modulus is $\frac{1}{2}$ and principal value of the argument is $\frac{\pi}{6}$.

Alternatively : We can use exponential form of a complex number.

$$\because \quad 1+i\sqrt{3} = 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = 2e^{i\pi/3}$$

$$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}-i\cdot\frac{1}{2}\right) = 2\left(\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right) = 2e^{-i\pi/6}$$

$$\therefore \quad \frac{(1+i\sqrt{3})^{16}}{(\sqrt{3}-i)^{17}} = \frac{(2e^{i\pi/3})^{16}}{(2e^{-i\pi/6})^{17}} = \frac{1}{2}e^{i16\pi/3}\cdot e^{i17\pi/6}$$

$$= \frac{1}{2}e^{i49\pi/6} = \frac{1}{2}\left(\cos\frac{49\pi}{6}+i\sin\frac{49\pi}{6}\right)$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) \text{ as before.}$$

Example 2 (b) : Express in the form $a+ib$, $\frac{(1+i)^6(1-i\sqrt{3})^4}{(1-i)^8(1+i\sqrt{3})^5}$.

Sol. : As in (1) in Ex. 4, page 2-5, we have

$$(1+i)^6 = (\sqrt{2})^6\left(\cos 6\cdot\frac{\pi}{4}+i\sin 6\cdot\frac{\pi}{4}\right) = \left\{\sqrt{2}\,e^{i\pi/4}\right\}^6$$

$$(1-i)^8 = (\sqrt{2})^8\left(\cos 8\cdot\frac{\pi}{4}-i\sin 8\cdot\frac{\pi}{4}\right) = \left\{\sqrt{2}\,e^{-i\pi/4}\right\}^8$$

As in (1) in the Ex. 2, page 2-5, we have

$$(1+i\sqrt{3})^5 = 2^5\left(\cos 5\cdot\frac{\pi}{3}+i\sin 5\cdot\frac{\pi}{3}\right) = \left\{2e^{i\pi/3}\right\}^5$$

$$(1-i\sqrt{3})^4 = 2^4\left(\cos 4\cdot\frac{\pi}{3}-i\sin 4\cdot\frac{\pi}{3}\right) = \left\{2e^{-i\pi/3}\right\}^4$$

$$\therefore \text{ Expression} = \frac{2^3 \cdot e^{i3\pi/2} \cdot 2^4 \cdot e^{-i4\pi/3}}{2^4 \cdot e^{-i2\pi} \cdot 2^5 \cdot e^{i5\pi/3}} = \frac{1}{4} \cdot \frac{e^{i\pi/6}}{e^{-i\pi/3}}$$

$$= \frac{1}{4} e^{i\pi/2} = \frac{1}{4}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{i}{4}.$$

Aliter : $\therefore$ Expression $= \dfrac{(\sqrt{2})^6 [\cos 6 \cdot (\pi/4) + i \sin 6 \cdot (\pi/4)] \cdot 2^4 [\cos 4 \cdot (\pi/3) - i \sin 4 \cdot (\pi/3)]}{(\sqrt{2})^8 [\cos 8 \cdot (\pi/4) - i \sin 8 \cdot (\pi/4)] \cdot 2^5 [\cos 5 \cdot (\pi/3) + i \sin 5 \cdot (\pi/3)]}$

$$= \frac{2^3 [\cos(3\pi/2) + i\sin(3\pi/2)] \cdot 2^4 [\cos(4\pi/3) - i\sin(4\pi/3)]}{2^4 [\cos 2\pi - i\sin 2\pi] \cdot 2^5 [\cos(5\pi/3) + i\sin(5\pi/3)]}$$

$$= \frac{1}{4} \cdot \frac{[\cos(3\pi/2) + i\sin(3\pi/2)] \cdot [\cos(-4\pi/3) + i\sin(-4\pi/3)]}{[\cos(-2\pi) + i\sin(-2\pi)][\cos(5\pi/3) + i\sin(5\pi/3)]}$$

$$= \frac{1}{4} \cdot \frac{\cos[(3\pi/2) - (4\pi/3)] + i\sin[(3\pi/2) - (4\pi/3)]}{\cos[(-2\pi) + (5\pi/3)] + i\sin[(-2\pi) + (5\pi/3)]}$$

$$= \frac{1}{4} \cdot \frac{\cos(\pi/6) + i\sin(\pi/6)}{\cos(-\pi/3) + i\sin(-\pi/3)}$$

$$= \frac{1}{4} \cdot \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]^{-1}$$

$$= \frac{1}{4} \cdot \left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$$= \frac{1}{4} \cdot \left[\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right]$$

$$= \frac{1}{4} \cdot \left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] = \frac{i}{4}.$$

Example 3 (b) : Show that $(4n)$th power of $\dfrac{1+7i}{(2-i)^2}$ is equal to $(-4)^n$ where n is a positive integer. **(M.U. 2004, 10)**

Sol. : We first express the given number in the standard form.

$$\frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} \cdot \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{9-16i^2}$$

$$\therefore \frac{1+7i}{(2-i)^2} = \frac{-25+25i}{25} = -1+i = -(1-i)$$

$$\therefore \left[\frac{1+7i}{(2-i)^2}\right]^{4n} = [-(1-i)]^{4n} = \left[-\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]^{4n}$$

$$= (-\sqrt{2})^{4n}\left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right]^{4n}$$

$$= (2^2)n\,[\cos n\pi - i\sin n\pi]$$

$$= 4^n[(-1)^n] = (-4)^n. \qquad [\because \cos n\pi = (-1)^n]$$

Example 4 (b) : Express $(1+7i)(2-i)^{-2}$ in the form of $r(\cos\theta + i\sin\theta)$ and prove that the second power is an imaginary number and the fourth power is a negative real number. **(M.U. 2010)**

Sol. : As proved above

$$z = \frac{1+7i}{(2-i)^2} = -(1-i) = -\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$\therefore$ Second power of z,

$$z^2 = (-\sqrt{2})^2\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]^2 = 2\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$$

$$= -2i, \text{ an imaginary number.}$$

And fourth power of z,

$$z^4 = (-2i)^2 = 4i^2$$

$$= -4, \text{ a negative real number.}$$

Type III : On $\cos\alpha + i\sin\alpha$, $\cos\beta + i\sin\beta$, etc. : Class (b) : 6 Marks

Example 1 (b) : If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, prove that

$$\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma).$$

Sol. : We have to consider,

$$(\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$$

$$\therefore\quad (\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma) = 0$$

Let $x = \cos\alpha + i\sin\alpha$, $y = 2(\cos\beta + i\sin\beta)$, $z = 3(\cos\gamma + i\sin\gamma)$.

We have, $(x+y+z)^3 = x^3 + y^3 + z^3 + 3(x+y+z)(xy+yz+zx) - 3xyz$

and if $x+y+z=0$, then $x^3+y^3+z^3 = 3xyz$ **[Note this]**

$$\therefore\quad (\cos\alpha + i\sin\alpha)^3 + 2^3(\cos\beta + i\sin\beta)^3 + 3^3(\cos\gamma + i\sin\gamma)^3$$
$$= 3(\cos\alpha + i\sin\alpha)\cdot 2\cdot(\cos\beta + i\sin\beta)\cdot 3\cdot(\cos\gamma + i\sin\gamma)$$

$\therefore$ By De Moivre's Theorem,

$$(\cos 3\alpha + i\sin 3\alpha) + 8(\cos 3\beta + i\sin 3\beta) + 27(\cos 3\gamma + i\sin 3\gamma)$$
$$= 18[\cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)]$$

Equating imaginary parts, we get the required result.

Example 2 (b) : If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, prove that

(i) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{3}{2}$ **(M.U. 2003, 08)**

(ii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0$ **(M.U. 1990, 2008)**

(iii) $\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha) = 0$ **(M.U. 2009)**

(iv) $\sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha) = 0$ **(M.U. 2009)**

Sol. : We have $(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

$$\therefore\quad (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma) = 0$$

$\therefore$ $a+b+c=0$, say, where $a = \cos\alpha + i\sin\alpha$, etc.

Also we can write

$$(\cos\alpha - i\sin\alpha) + (\cos\beta - i\sin\beta) + (\cos\gamma - i\sin\gamma) = 0$$

$$\therefore\quad (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1} = 0$$

i.e. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0 \quad \therefore \frac{bc+ca+ab}{abc}=0 \quad \therefore ab+bc+ca=0$

But $(a+b+c)^2=(a^2+b^2+c^2)+2(ab+bc+ca)$

$\therefore 0=a^2+b^2+c^2+0 \qquad \therefore a^2+b^2+c^2=0$

$\therefore (\cos\alpha+i\sin\alpha)^2+(\cos\beta+i\sin\beta)^2+(\cos\gamma+i\sin\gamma)^2=0$

$\therefore (\cos 2\alpha+i\sin 2\alpha)+(\cos 2\beta+i\sin 2\beta)+(\cos 2\gamma+i\sin 2\gamma)=0$

$\therefore (\cos 2\alpha+\cos 2\beta+\cos 2\gamma)+i(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)=0$

$\therefore \cos 2\alpha+\cos 2\beta+\cos 2\gamma=0$ .... (i)

$\therefore 2\cos^2\alpha-1+2\cos^2\beta-1+2\cos^2\gamma-1=0$

$\therefore \cos^2\alpha+\cos^2\beta+\cos^2\gamma=\frac{3}{2}$ .... (ii)

Further, $1-\sin^2\alpha+1-\sin^2\beta+1-\sin^2\gamma=\frac{3}{2}$

$\therefore \sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{3}{2}$ .... (iii)

Again consider $ab+bc+ca=0$,

$$\therefore (\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)+(\cos\beta+i\sin\beta)\times(\cos\gamma+i\sin\gamma)$$
$$+(\cos\gamma+i\sin\gamma)(\cos\alpha+i\sin\alpha)=0$$

$$\therefore [\cos(\alpha+\beta)+i\sin(\alpha+\beta)]+[\cos(\beta+\gamma)+i\sin(\beta+\gamma)]$$
$$+[\cos(\gamma+\alpha)+i\sin(\gamma+\alpha)]=0$$

$$\therefore [\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)]$$
$$+i[\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)]=0$$

Equating real and imaginary parts, we get

$\cos(\alpha+\beta)+\cos(\beta+\alpha)+\cos(\gamma+\alpha)=0$

$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$

Example 3 (b) : If $\cos\alpha+\cos\beta+\cos\gamma=0$, $\sin\alpha+\sin\beta+\sin\gamma=0$, prove that

(i) $\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin 3\alpha+\sin 3\beta+\sin 3\alpha=3\sin(\alpha+\beta+\alpha)$.

Sol. : Let us use a, b, c as in Ex. 2, then $a+b+c=0$.

As seen in Ex. 1, above $a^3+b^3+c^3=3abc$.

[Or Since $a+b+c=0$, $(a+b)=(-c)$

$\therefore (a+b)^3=-c^3 \qquad \therefore a^3+b^3+3ab(a+b)=-c^3$

$\therefore a^3\ b^3+3ab(-c)=-c^3 \qquad \therefore a^3+b^3+c^3=3abc.$]

$$\therefore (\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\alpha+i\sin\gamma)^3$$
$$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$$

$$\therefore (\cos 3\alpha+\cos 3\beta+\cos 3\gamma)+i(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)$$
$$=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$$

Equating the real and imaginary parts, we get the required results.

Example 4 (b) : If $a = \cos 2\alpha + i \sin 2\alpha$, $b = \cos 2\beta + i \sin 2\beta$, prove that

(i) $\sqrt{\dfrac{a}{b}} + \sqrt{\dfrac{b}{a}} = 2\cos(\alpha - \beta)$. (ii) $\sqrt{\dfrac{a}{b}} - \sqrt{\dfrac{b}{a}} = 2i\sin(\alpha - \beta)$.

Sol. : We have $\dfrac{a}{b} = \dfrac{\cos 2\alpha + i\sin 2\alpha}{\cos 2\beta + i \sin 2\beta} = (\cos 2\alpha + i \sin 2\alpha)(\cos 2\beta - i \sin 2\beta)$

$$= \cos 2(\alpha - \beta) + i \sin 2(\alpha - \beta)$$

$$\sqrt{\frac{a}{b}} = [\cos 2(\alpha - \beta) + i \sin 2(\alpha - \beta)]^{1/2} = \cos(\alpha - \beta) + i \sin(\alpha - \beta)$$

Similarly, $\sqrt{\dfrac{b}{a}} = \cos(\beta - \alpha) + i \sin(\beta - \alpha) = \cos(\alpha - \beta) - i\sin(\alpha - \beta)$

By addition and subtraction, we get

$$\therefore \quad \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 2\cos(\alpha - \beta) \quad \text{and} \quad \sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}} = 2i\sin(\alpha - \beta).$$

Type IV : On Roots of an Equation : Class (b) : 6 Marks

Example 1 (b) : If α, β are the roots of the equation $x^2 - 2 \cdot \sqrt{3} \cdot x + 4 = 0$, prove that

$\alpha^3 + \beta^3 = 0$. **(M.U. 2004, 16)**

Sol. : Solving the given quadratic

$$x = \frac{2\sqrt{3} \pm \sqrt{12 - 16}}{2} = \sqrt{3} \pm i = 2\left(\frac{\sqrt{3}}{2} \pm i \cdot \frac{1}{2}\right)$$

$\therefore \quad x = 2\left(\cos\dfrac{\pi}{6} \pm i \sin\dfrac{\pi}{6}\right)$ are the roots.

Let $\alpha = 2\left(\cos\dfrac{\pi}{6} + i \sin\dfrac{\pi}{6}\right)$, $\beta = 2\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)$

$$\therefore \quad \alpha^3 + \beta^3 = 2^3\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} + \cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right] = 2^3 \cdot 2\cos\frac{\pi}{2} = 0.$$

[Similarly, $\alpha^3 - \beta^3 = 2^3\left[\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} - \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right] = 2^3 \cdot 2i\sin\dfrac{\pi}{2} = 16i$.]

Example 2 (b) : If α, β are the roots of the equation $x^2 + 2x + 2 = 0$, prove that $\alpha^n \cdot \beta^n = 2^n$.

Sol. : Solving the given quadratic

$$x = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\therefore \quad \alpha = -1 + i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$$

$$\beta = -1 - i = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

$$\therefore \quad \alpha^n \cdot \beta^n = 2^{n/2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^n \cdot 2^{n/2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)^n$$

$$\therefore \quad \alpha^n \cdot \beta^n = 2^{n/2}\left(\cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}\right)\cdot 2^{n/2}\left(\cos\frac{5n\pi}{4} + i\sin\frac{5n\pi}{4}\right)$$

$$= 2^n\left(\cos\frac{8n\pi}{4} + i\sin\frac{8n\pi}{4}\right) = 2^n(\cos 2n\pi + i\sin 2n\pi)$$

$$= 2^n(1+0)^4 = 2^n.$$

Example 3 (b) : If α, β are the roots of $x^2 - 2x\cos\theta + 1 = 0$, find the equation whose roots are α^n, β^n.

Sol. : Solving $x^2 - 2x\cos\theta + 1 = 0$, we get $x = \dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$.

$\therefore \quad \alpha = \cos\theta + i\sin\theta, \quad \beta = \cos\theta - i\sin\theta$

$\therefore \quad \alpha^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ and $\beta^n = \cos n\theta - i\sin n\theta$.

$\therefore$ Sum of the roots $= \alpha^n + \beta^n = 2\cos n\theta$.

Product of the roots $= \alpha^n \cdot \beta^n = (\cos n\theta + i\sin\theta)(\cos n\theta - i\sin n\theta)$

Product of the roots $= \cos^2 n\theta + \sin^2 n\theta = 1$

$\therefore$ The required equation is

$x^2 - (\text{sum})\,x + (\text{product}) \qquad \therefore \quad x^2 - 2\cos n\theta\, x + 1 = 0.$

Example 4 (b) : If α, β are the roots of the equation $z^2\sin^2\theta - z\cdot\sin 2\theta + 1 = 0$, prove that

(i) $\alpha^n + \beta^n = 2\cos n\theta\,\mathrm{cosec}^n\,\theta$. **(M.U. 2011)**

(ii) $\alpha^n \cdot \beta^n = \mathrm{cosec}^{2n}\,\theta$. **(M.U. 2009)**

Sol. : (i) Solving the given quadratic equation in z,

$$z = \frac{\sin 2\theta \pm \sqrt{\sin^2 2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta \pm \sqrt{4\sin^2\theta\cos^2\theta - 4\sin^2\theta}}{2\sin^2\theta}$$

$$= \frac{\cos\theta \pm \sqrt{\cos^2\theta - 1}}{\sin\theta} = \frac{\cos\theta \pm \sqrt{-\sin^2\theta}}{\sin\theta}$$

$$= \frac{\cos\theta \pm i\sin\theta}{\sin\theta} = (\cos\theta \pm i\sin\theta)\,\mathrm{cosec}\,\theta$$

Let $\alpha = (\cos\theta + i\sin\theta)\,\mathrm{cosec}\,\theta, \quad \beta = (\cos\theta - i\sin\theta)\,\mathrm{cosec}\,\theta$

$\therefore \quad \alpha^n = (\cos\theta + i\sin\theta)^n\,\mathrm{cosec}^n\,\theta = (\cos n\theta + i\sin n\theta)\,\mathrm{cosec}^n\,\theta$

$\beta^n = (\cos\theta - i\sin\theta)^n\,\mathrm{cosec}^n\,\theta = (\cos n\theta - i\sin n\theta)\,\mathrm{cosec}^n\,\theta$

$\therefore \quad \alpha^n + \beta^n = 2\cos n\theta\,\mathrm{cosec}^n\,\theta$.

(ii) Further $\alpha^n \cdot \beta^n = \left(\dfrac{\cos\theta + i\sin\theta}{\sin\theta}\right)^n \left(\dfrac{\cos\theta - i\sin\theta}{\sin\theta}\right)^n$

$$= \left(\frac{\cos n\theta + i\sin n\theta}{\sin^n\theta}\right)\cdot\left(\frac{\cos n\theta - i\sin n\theta}{\sin^n\theta}\right)$$

$$= \frac{\cos^2 n\theta - i^2\sin^2 n\theta}{\sin^{2n}\theta} = \frac{\cos^2 n\theta + \sin^2 n\theta}{\sin^{2n}\theta} = \mathrm{cosec}^{2n}\,\theta.$$

Miscellaneous Examples

Class (a) : 3 Marks

Example 1 (a) : If $x+\dfrac{1}{x}=2\cos\theta$, prove that $x^r+\dfrac{1}{x^r}=2\cos r\theta$.

Sol. : Since $x+\dfrac{1}{x}=2\cos\theta$, $x^2-2x\cos\theta+1=0$

Solving the quadratic for x,

$$\therefore \quad x=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=\cos\theta\pm i\sin\theta$$

$$\therefore \quad x^r=(\cos\theta\pm i\sin\theta)^r=\cos r\theta\pm i\sin r\theta$$

$$\therefore \quad \frac{1}{x^r}=(\cos\theta\pm i\sin\theta)^{-r}=\cos r\theta\mp i\sin r\theta$$

$$\therefore \quad x^r+\frac{1}{x^r}=2\cos r\theta.$$

Example 2 (a) : If $x+\dfrac{1}{x}=2\cos\theta$, $y+\dfrac{1}{y}=2\cos\Phi$, prove that

$$x^2y^2+\frac{1}{x^2y^2}=2\cos(2\theta+2\Phi).$$

Sol. : As above, $x=\cos\theta\pm i\sin\theta$, $y=\cos\Phi\pm i\sin\Phi$

$$\begin{aligned}\therefore \quad x^2y^2 &= (\cos\theta\pm i\sin\theta)^2(\cos\Phi\pm i\sin\Phi)^2\\ &= (\cos 2\theta\pm i\sin 2\theta)(\cos 2\Phi\pm i\sin 2\Phi)\\ &= \cos 2(\theta+\Phi)\pm i\sin 2(\theta+\Phi)\end{aligned}$$

and $\dfrac{1}{x^2y^2}=\cos 2(\theta+\Phi)\mp i\sin 2(\theta+\Phi)$ $\quad\therefore\quad x^2y^2+\dfrac{1}{x^2y^2}=2\cos 2(\theta+\Phi).$

Note ....

In general, (i) If $x+\dfrac{1}{x}=2\cos\theta$, $y+\dfrac{1}{y}=\cos\Phi$, then

$$x^my^n+\frac{1}{x^my^n}=2\cos(m\theta+n\Phi)$$ **(M.U. 1991, 2004)**

(ii) If $x+\dfrac{1}{x}=2\cos\theta$, $y+\dfrac{1}{y}=2\cos\Phi$, $z+\dfrac{1}{z}=2\cos\Psi$, ..., then

$$xyz.....+\frac{1}{xyz.....}=2\cos(\theta+\Phi+\Psi+.....)$$ **(M.U. 2006)**

And $$x^my^nz^l....+\frac{1}{x^my^nz^l....}=2\cos(m\theta+n\Phi+l\psi+....)$$

Example 3 (a) : If $x-\dfrac{1}{x}=2i\sin\theta$, prove that $x^r-\dfrac{1}{x^r}=2i\sin r\theta$.

Sol. : Since $x-\dfrac{1}{x}=2i\sin\theta$, $\quad x^2-2ix\sin\theta-1=0$

Solving this quadratic for x,

$$x = \frac{2i\sin\theta \pm \sqrt{4i^2\sin\theta + 4}}{2} = i\sin\theta \pm \sqrt{-\sin^2\theta + 1}$$

$$= i\sin\theta \pm \cos\theta = \pm\cos\theta + i\sin\theta$$

$$\therefore \quad x^r = (\pm\cos\theta + i\sin\theta)^r = \pm\cos r\theta + i\sin r\theta$$

$$\therefore \quad \frac{1}{x^r} = (\pm\cos\theta + i\sin\theta)^{-r} = \pm\cos r\theta - ir\sin r\theta$$

$$\therefore \quad x^r - \frac{1}{x^r} = 2i\sin r\theta.$$

Example 4 (a) : If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ .....\ (\cos n\theta + i\sin n\theta) = 1$, then show that the general value of θ is $4r\pi / [n(n+1)]$.

Sol. : L.H.S. $= (\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ .....\ (\cos n\theta + i\sin n\theta)$

$$= \cos[1+2+3+.....+n]\theta + i\sin[1+2+3+.....+n]\theta$$

$$= \cos\left\{\frac{n(n+1)}{2}\theta\right\} + i\sin\left\{\frac{n(n+1)}{2}\theta\right\}$$

[$\because$ Sum of the first n natural numbers $(1+2+3+....+n) = \frac{n(n+1)}{2}$.]

Now, R.H.S. $= 1 = \cos 2r\pi + i\sin 2r\pi$ $\quad [r = 0, 1, 2, ...]$

Equating the two sides, we get $\frac{n(n+1)}{2}\theta = 2r\pi \quad \therefore \quad \theta = \frac{4r\pi}{n(n+1)}$.

Example 5 (a) : If $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ ....\ [\cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$, then show that the general value of θ is $\frac{2r\pi}{n^2}$.

Sol. : L.H.S. $= (\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ ....\ [\cos(2n-1)\theta + i\sin(2n-1)\theta]$

$$= \cos[1+3+....+(2n-1)]\theta + i\sin[1+3+....+(2n-1)]\theta$$

But $1 + 3 + ... + (2n-1)$ is an A.P. with first term 1, the number of terms n and common difference 2.

$$\therefore \text{ The sum, } S_n = \frac{n}{2}[2a + (n-1)\cdot d] = \frac{n}{2}[2 + (n-1)\cdot 2] = n^2$$

$$\therefore \quad \text{L.H.S.} = \cos(n^2\theta) + i\sin(n^2\theta)$$

R.H.S. $= 1 = \cos 2r\pi + i\sin 2r\pi$ $\quad [r = 0, 1, 2, ...]$

$$\therefore \quad n^2\theta = 2r\pi \quad \therefore \quad \theta = \frac{2r\pi}{n^2}.$$

Class (b) : 6 Marks

Example 1 (b) : If $x_n + iy_n = (1 + i\sqrt{3})^n$, prove that $x_{n-1}\cdot y_n - x_n\cdot y_{n-1} = 4^{n-1}\sqrt{3}$.

Sol. : As in (1), Ex. 2, page 2-5

$$x_n + iy_n = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^n = 2^n\left[\cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right)\right]$$

Equating real and imaginary parts,

$$\therefore \quad x_n = 2^n \cos\left(\frac{n\pi}{3}\right) \text{ and } y_n = 2^n \sin\left(\frac{n\pi}{3}\right)$$

$$\therefore \quad \text{l.h.s.} = 2^{n-1}\cos(n-1)\frac{\pi}{3}\cdot 2^n \sin\frac{n\pi}{3} - 2^n\cos\frac{n\pi}{3}\cdot 2^{n-1}\sin(n-1)\frac{\pi}{3}$$

$$= 2^{2n-1}\left[\sin\left\{n\frac{\pi}{3}-(n-1)\frac{\pi}{3}\right\}\right] = 2^{2n-1}\sin\frac{\pi}{3}$$

$$= 2^{2n-1}\cdot\frac{\sqrt{3}}{2} = 2^{2n-2}\cdot\sqrt{3} = 4^{n-1}\cdot\sqrt{3}.$$

Example 2 (b) : If $x_n = \cos\frac{\pi}{2^n} + i\sin\frac{\pi}{2^n}$, then show that $x_1 x_2 \ldots\ldots x_\infty = -1$ and $x_0 x_1 x_2 \ldots\ldots x_\infty = 1$.

Sol. : We have, putting $n = 1, 2, 3, \ldots\ldots$

$$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \quad x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2},$$

$$x_3 = \cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}, \ldots\ldots \text{ and so on.}$$

$$\therefore \quad x_1 x_2 x_3 \ldots\ldots x_\infty = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\cdot\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\ldots\left(\cos\frac{\pi}{2^n} + i\sin\frac{\pi}{2^n}\right)\ldots\ldots\infty$$

$$= \cos\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\frac{1}{2^n}+\ldots\right)\pi + i\sin\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\frac{1}{2^n}+\ldots\right)\pi$$

But $\quad \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots+\frac{1}{2^n}+\ldots\infty = \frac{1/2}{1-(1/2)} = 1 \qquad \left[\text{For a G.P. } S_\infty = \frac{a}{1-r}\right]$

$$\therefore \quad x_1 x_2 x_3 \ldots\ldots x_n \ldots\ldots x_\infty = \cos\pi + i\sin\pi = -1$$

Similarly, $\quad x_0 x_1 x_2 \ldots\ldots x_n \ldots\ldots x_\infty$

$$= \cos\left(\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\ldots\ldots\infty\right)\pi + i\sin\left(\frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\ldots\ldots\infty\right)\pi$$

$$= \cos\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\infty\right)\pi + i\sin\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\infty\right)\pi$$

But $\quad 1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots\infty = \frac{1}{1-(1/2)} = 2$

$$\therefore \quad x_0 x_1 x_2 x_3 \ldots\ldots x_n \ldots\ldots x_\infty = \cos(2\pi) + i\sin(2\pi) = 1.$$

Example 3 (b) : If $\alpha = 1 + i$, $\beta = 1 - i$ and $\cot\theta = x + 1$, prove that

$$(x+\alpha)^n - (x+\beta)^n = (\alpha-\beta)\sin n\theta \operatorname{cosec}^n\theta.$$

(M.U. 2000, 03, 11, 14)

Sol. : From data, we get, $x = \cot\theta - 1$ and $\alpha = 1 + i$

$$\therefore \quad x + \alpha = \cot\theta + i = \frac{\cos\theta + i\sin\theta}{\sin\theta}$$

$$\therefore \quad (x+\alpha)^n = \operatorname{cosec}^n\theta(\cos n\theta + i\sin n\theta)$$

Similarly, $\quad x + \beta = \cot\theta - i = \frac{\cos\theta - i\sin\theta}{\sin\theta}$

$$\therefore \quad (x+\beta)^n = \operatorname{cosec}^n\theta(\cos n\theta - i\sin n\theta)$$

By subtraction, $\quad (x+\alpha)^n - (x+\beta)^n = \operatorname{cosec}^n\theta(2i\sin n\theta)$

But $\alpha - \beta = 2i$ $\therefore$ $(x+\alpha)^n - (x+\beta)^n = (\alpha - \beta)\sin n\theta \operatorname{cosec}^n \theta$

[Similarly, we can prove by addition that

$$(x+\alpha)^n + (x+\beta)^n = (\alpha+\beta)\operatorname{cosec}^n \theta \cdot \cos n\theta \text{]}$$

Example 4 (b) : If $(a_1 + ib_1)(a_2 + ib_2) \ldots (a_n + ib_n) = A + iB$, prove that

$$(a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots (a_n^2 + b_n^2) = A^2 + B^2$$

$$\text{and} \quad \tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\frac{B}{A}.$$

Sol. : Let $a_i + ib_i = r_i(\cos\theta_i + i\sin\theta_i)$

$\therefore$ $a_i^2 + b_i^2 = r_i^2$ and $\theta_i = \tan^{-1}(b_i/a_i)$ for $i = 1, 2, \ldots, n$.

$\therefore$ $(a_1 + ib_1)(a_2 + ib_2) \ldots (a_n + ib_n)$

$$= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \ldots \cdot r_n(\cos\theta_n + i\sin\theta_n)$$

$$= r_1 r_2 \ldots r_n[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)]$$

But this is equal to $A + iB = R(\cos\Phi + i\sin\Phi)$, say where, $R = \sqrt{A^2 + B^2}$, $\Phi = \tan^{-1}\left(\frac{B}{A}\right)$

$\therefore$ $r_1 r_2 \ldots r_n = R$ or $r_1^2 r_2^2 \ldots r_n^2 = R^2$

and $\theta_1 + \theta_2 + \ldots + \theta_n = \Phi$

i.e. $(a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots (a_n^2 + b_n^2) = A^2 + B^2$

and $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\frac{B}{A}.$

Example 5 (b) : Evaluate $\left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n$. **(M.U. 1991, 2001, 04, 05)**

Sol. : We have $1 = \sin^2\alpha + \cos^2\alpha = \sin^2\alpha - i^2\cos^2\alpha$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha)$$

$\therefore$ $1 + \sin\alpha + i\cos\alpha = (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha) + (\sin\alpha + i\cos\alpha)$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha + 1)$$

$$\therefore \frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha} = \sin\alpha + i\cos\alpha \qquad \ldots\ldots\ldots\ldots \text{(A)}$$

$$= \cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)$$

$$\therefore \left(\frac{1+\sin\alpha + i\cos\alpha}{1+\sin\alpha - i\cos\alpha}\right)^n = \left\{\cos\left(\frac{\pi}{2} - \alpha\right) + i\sin\left(\frac{\pi}{2} - \alpha\right)\right\}^n \qquad \ldots\ldots\ldots\ldots \text{(1)}$$

$$= \cos n\left(\frac{\pi}{2} - \alpha\right) + i\sin n\left(\frac{\pi}{2} - \alpha\right).$$

Example 6 (b) : Prove that $\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$.

Hence, deduct that $\left(1 + \sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\left(1 + \sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5 = 0.$

Sol. : The first part is proved in the above example in (A).

Dividing the l.h.s. of second part by $\left[\left(1+\sin\frac{\pi}{5}\right)-i\cos\frac{\pi}{5}\right]^5$ and using (1) with $\alpha=\frac{\pi}{5}$, the

$$\text{l.h.s.}=\frac{\left[\left(1+\sin\frac{\pi}{5}\right)+i\cos\frac{\pi}{5}\right]^5}{\left[\left(1+\sin\frac{\pi}{5}\right)-i\cos\frac{\pi}{5}\right]^5}+i=\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)\right]^5+i$$

$$=\left[\cos 5\left(\frac{\pi}{2}-\frac{\pi}{5}\right)+i\sin 5\left(\frac{\pi}{2}-\frac{\pi}{5}\right)\right]+i$$

$$=\cos 5\left(\frac{3\pi}{10}\right)+i\sin 5\left(\frac{3\pi}{10}\right)+i$$

$$=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}+i=0-i+i=0$$

$\therefore$ l.h.s. = r.h.s.

Example 7 (b) : If n is a positive integer and

$(1+x)^n=p_0+p_1x+p_2x^2+\ldots.+p_nx^n$, prove that

(i) $p_0-p_2+p_4-\ldots.=2^{n/2}\cos\frac{n\pi}{4}$ **(M.U. 2002)**

(ii) $p_1-p_3+p_5-\ldots.=2^{n/2}\sin\frac{n\pi}{4}$

(iii) $p_0+p_4+p_8+\ldots.=2^{n-2}+2^{n/2}\cos\frac{n\pi}{4}$

Sol. : We have $(1+x)^n=p_0+p_1x+p_2x^2+\ldots.+p_nx^n$

Putting $x=i$, we get

$$(1+i)^n=p_0+p_1 i+p_2 i^2+p_3 i^3+p_4 i^4+p_5 i^5+\ldots..$$
$$=p_0+p_1 i-p_2-p_3 i+p_4+p_5 i+\ldots.$$
$$=(p_0-p_2+p_4-\ldots.)+i(p_1-p_3+p_5-\ldots.) \qquad \ldots\ldots(1)$$

But $$(1+i)^n=\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right]^n=(\sqrt{2})^n\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^n$$

$$=2^{n/2}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right) \qquad \ldots\ldots(2)$$

From (1) and (2),

$$2^{n/2}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)=(p_0-p_2+p_4-\ldots..)+i(p_1-p_3+p_5-\ldots..)$$

Equating real and imaginary parts,

$$p_0-p_2+p_4-\ldots.=2^{n/2}\cos\frac{n\pi}{4} \qquad \ldots\ldots(3)$$

$$p_1-p_3+p_5-\ldots.=2^{n/2}\sin\frac{n\pi}{4} \qquad \ldots\ldots(4)$$

Now, putting $x = 1$, in the given relation,

$$2^n = p_0 + p_1 + p_2 + p_3 + p_4 + \ldots\ldots \qquad \ldots\ldots (5)$$

Putting $x = -1$, in the given relation,

$$0 = p_0 - p_1 + p_2 - p_3 + p_4 + \ldots\ldots \qquad \ldots\ldots (6)$$

Adding (5) and (6), we get,

$$2^n = 2\,(p_0 + p_2 + p_4 + p_6 + \ldots\ldots)$$

$$\therefore \quad 2^{n-1} = p_0 + p_2 + p_4 + p_6 + p_8 + \ldots\ldots \qquad \ldots\ldots (7)$$

Adding (7) and (3), we get,

$$2^{n-1} + 2^{n/2}\cos\frac{n\pi}{4} = 2\,(p_0 + p_4 + p_8 + \ldots.)$$

$$\therefore \quad 2^{n-2} + 2^{(n/2)-1}\cdot\cos\frac{n\pi}{4} = p_0 + p_4 + p_8 + \ldots. \qquad \ldots\ldots (8)$$

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. Simplify

(i) $\left[\dfrac{1+\sin(\pi/8)+i\cos(\pi/8)}{1+\sin(\pi/8)-i\cos(\pi/8)}\right]^8$ **(M.U. 2009) [Ans. : –1]**

(ii) $\left(\dfrac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta}\right)^n$ **[Ans. :** $\cos n\theta + i\sin n\theta$ **]**

(iii) $\left[\dfrac{1+\cos(\pi/9)+i\sin(\pi/9)}{1+\cos(\pi/9)-i\sin(\pi/9)}\right]^{18}$ **[Ans. : 1]**

2. Prove that $(1+\cos\theta+i\sin\theta)^n + (1+\cos\theta-i\sin\theta)^n = 2^{n+1}\cos^n(\theta/2)\cos(n\theta/2)$.

3. If $z = 1+i\sqrt{3}$ and $\bar{z}$ is its conjugate, prove that $(z)^8 + (\bar{z})^8 = -2^8$.

4. If $z = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ and $\bar{z}$ is the conjugate of z, find the value of $(z)^{15} + (\bar{z})^{15}$. **[Ans. : – 2]**

5. Prove that, if n is a positive integer,

(i) $(a+ib)^{m/n} + (a-ib)^{m/n} = 2(a^2+b^2)^{m/2n}\cos\left(\dfrac{m}{n}\tan^{-1}\dfrac{b}{a}\right)$ **(M.U. 1980)**

(ii) $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\left(\dfrac{n\pi}{3}\right)$

(iii) $(3+4i)^{2/3} + (3-4i)^{2/3} = 2(5)^{2/3}\cos\left(\dfrac{2}{3}\tan^{-1}\dfrac{4}{3}\right)$

(iv) $(1+i)^n + (1-i)^n = 2^{(n/2)+1}\cos\left(\dfrac{n\pi}{4}\right)$ (v) $(-1+i\sqrt{3})^{3n} + (-1-i\sqrt{3})^{3n} = 2^{3n+1}$.

(vi) $\left(\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)^{4/3} + \left(\dfrac{1}{\sqrt{2}}-\dfrac{i}{\sqrt{2}}\right)^{4/3} = 1$ (vii) $\left(\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)^{10} + \left(\dfrac{1}{\sqrt{2}}-\dfrac{i}{\sqrt{2}}\right)^{10} = 0$

(viii) $\dfrac{(1+i)^8(\sqrt{3}-i)^4}{(1-i)^4(\sqrt{3}+i)^8} = -\dfrac{1}{4}$ (ix) $\dfrac{(1+i)^6(\sqrt{3}-i)^4}{(1-i)^8(\sqrt{3}+i)^5} = \dfrac{1}{4}$

(x) $\dfrac{(1+i)^8(1-i\sqrt{3})^6}{(1-i)^6(1+i\sqrt{3})^9}=\dfrac{i}{4}$ **(M.U. 2003)**

(xi) $\dfrac{(1-i)^4(\sqrt{3}+i)^8}{(1+i)^4(\sqrt{3}-i)^4}=-4$

(xii) $(1+i\sqrt{3})^{120}+(1-i\sqrt{3})^{120}=2^{121}$

(xiii) $(\sqrt{3}+i)^{120}+(\sqrt{3}-i)^{120}=2^{121}$

(xiv) $\dfrac{(1+i)^7(1-i\sqrt{3})^3}{(1-i)^7(1+i\sqrt{3})^6}=\dfrac{i}{8}$

(xv) $\dfrac{(1+i\sqrt{3})^9(1-i)^4}{(\sqrt{3}+i)^{12}(1+i)^4}=-\dfrac{1}{8}$.

6. Find the modulus and the principal value of the argument of

$$\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$$

[**Ans. :** Mod. = 4, Arg. = $\dfrac{\pi}{6}$]

7. Express (i) $\dfrac{(1+i)^{10}}{(1+i\sqrt{3})^5}$, (ii) $\dfrac{(\sqrt{3}-i)^7}{(1+i)^{10}}$ in the form $a+ib$.

[**Ans. :** (i) $-\dfrac{\sqrt{3}}{2}+i\dfrac{1}{2}$, (ii) $2-2\sqrt{3}\,i$]

8. If $x+\dfrac{1}{x}=2\cos\theta,\ y+\dfrac{1}{y}=2\cos\Phi,\ z+\dfrac{1}{z}=2\cos\Psi$, prove that

(i) $xyz+\dfrac{1}{xyz}=2\cos(\theta+\Phi+\Psi)$ and $\dfrac{x^m}{y^n}+\dfrac{y^n}{x^m}=2\cos(m\theta-n\Phi)$.

(M.U. 1996, 97, 2004)

(ii) $\sqrt{xyz}+\dfrac{1}{\sqrt{xyz}}=2\cos\left(\dfrac{\theta+\phi+\psi}{2}\right)$ and $\dfrac{\sqrt[m]{x}}{\sqrt[n]{y}}+\dfrac{\sqrt[n]{y}}{\sqrt[m]{x}}=2\cos\left(\dfrac{\theta}{m}-\dfrac{\phi}{n}\right)$ **(M.U. 2005)**

9. If $x-\dfrac{1}{x}=2i\sin\theta,\ y-\dfrac{1}{y}=2i\sin\phi,\ z-\dfrac{1}{z}=2i\sin\psi$, prove that

(i) $xyz+\dfrac{1}{xyz}=2\cos(\theta+\Phi+\psi)$ (ii) $\dfrac{\sqrt[m]{x}}{\sqrt[n]{y}}+\dfrac{\sqrt[n]{y}}{\sqrt[m]{x}}=2\cos\left(\dfrac{\theta}{m}-\dfrac{\Phi}{n}\right)$ **(M.U. 2005)**

Class (b) : 6 Marks

1. (a) If $a=\cos 2\alpha+i\sin 2\alpha,\ b=\cos 2\beta+i\sin 2\beta$ and $c=\cos 2\gamma+i\sin 2\gamma$, prove that

$\sqrt{\dfrac{ab}{c}}+\sqrt{\dfrac{c}{ab}}=2\cos(\alpha+\beta-\gamma)$. **(M.U. 1999)**

(b) If $a=\cos 3\alpha+i\sin 3\alpha,\ b=\cos 3\beta+i\sin 3\beta$ and $c=\cos 3\gamma+i\sin 3\gamma$, prove that

$\sqrt[3]{\dfrac{ab}{c}}+\sqrt[3]{\dfrac{c}{ab}}=2\cos(\alpha+\beta-\gamma)$.

2. If $a=\cos 2\alpha+i\sin 2\alpha,\quad b=\cos 2\beta+i\sin 2\beta,$
$c=\cos 2\gamma+i\sin 2\gamma,\quad d=\cos 2\delta+i\sin 2\delta,$ prove that

(i) $\sqrt{\dfrac{ab}{cd}}+\sqrt{\dfrac{cd}{ab}}=2\cos(\alpha+\beta-\gamma-\delta)$ (ii) $\sqrt{abcd}+\dfrac{1}{\sqrt{abcd}}=2\cos(\alpha+\beta+\gamma+\delta)$

3. (a) If $\cos\alpha+\cos\beta=0,\ \sin\alpha+\sin\beta=0$, prove that

(i) $\cos 2\alpha+\cos 2\beta=2\cos(\pi+\alpha+\beta)$ **(M.U. 1997)**

(ii) $\sin 2\alpha+\sin 2\beta=2\sin(\pi+\alpha+\beta)$ **(M.U. 1997)**

(iii) $\cos 3\alpha+\cos 3\beta=0,\ \sin 3\alpha+\sin 3\beta=0$

(b) If $\cos\alpha + \cos\beta + \cos\gamma = 0$, $\sin\alpha + \sin\beta + \sin\gamma = 0$, prove that

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

$\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$. **(M.U. 1990)**

(c) If a, b, c are three numbers such that $a + b + c = 0$, prove that

(i) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = 0$ and (ii) $a^2 + b^2 + c^2 = 0$.

4. If $a\cos\alpha + b\cos\beta + c\cos\gamma = a\sin\alpha + b\sin\beta + c\sin\gamma = 0$, prove that

$$a^3\cos 3\alpha + b^3\cos 3\beta + c^3\cos 3\gamma = 3abc\cos(\alpha + \beta + \gamma)$$

and $$a^3\sin 3\alpha + b^3\sin 3\beta + c^3\sin 3\gamma = 3abc\sin(\alpha + \beta + \gamma).$$

5. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$, prove that

$$\frac{(b+c)(c+a)(a+b)}{abc} = 8\cos\frac{(\alpha-\beta)}{2}\cos\frac{(\beta-\gamma)}{2}\cos\frac{(\gamma-\alpha)}{2}.$$ **(M.U. 2006)**

6. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$, $u = \cos\delta + i\sin\delta$, prove that

$$xy + zu = 2\cos\left(\frac{\alpha+\beta-\gamma-\delta}{2}\right)\cdot\left[\cos\left(\frac{\alpha+\beta+\gamma+\delta}{2}\right) + i\sin\left(\frac{\alpha+\beta+\gamma+\delta}{2}\right)\right]$$

7. If $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$, $z_3 = e^{i\theta_3}$ and $z_1 + z_2 + z_3 = 0$, prove that $\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3} = 0$ using De Moivre's Theorem.

8. If $(x_1 + iy_1)(x_2 + iy_2)(x_3 + iy_3) = x + iy$, prove that

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2)(x_3^2 + y_3^2) = x^2 + y^2$$

and $\tan^{-1}(y_1/x_1) + \tan^{-1}(y_2/x_2) + \tan^{-1}(y_3/x_3) = \tan^{-1}(y/x)$.

9. If $x_r = \cos\dfrac{\pi}{3^r} + i\sin\dfrac{\pi}{3^r}$, prove that

(i) $x_1\, x_2\, x_3$ ..... ad. inf. $= i$ (ii) $x_0\, x_1\, x_2$ ..... ad. inf. $= -i$ **(M.U. 2003)**

10. If $x_r = \cos\left(\dfrac{2}{3}\right)^r\pi + i\sin\left(\dfrac{2}{3}\right)^r\pi$, prove that

(i) $x_1\, x_2\, x_3 \ldots\ldots \infty = 1$, (ii) $x_0\, x_1\, x_2 \ldots\ldots \infty = -1$.

11. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, prove that

(i) $\alpha^n + \beta^n = 2^{n+1}\cos(n\pi/3)$. Hence, deduce that $\alpha^6 + \beta^6 = 128$.

(M.U. 1993, 96, 2003)

(ii) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$. **(M.U. 1981)**

12. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that

$\alpha^n + \beta^n = 2\cdot 2^{n/2}\cos(n\pi/4)$. Hence, deduce that $\alpha^8 + \beta^8 = 32$.

13. If α, β are the roots of the equation $x^2 - \sqrt{3}\cdot x + 1 = 0$, prove that

$\alpha^n + \beta^n = 2\cos(n\pi/6)$. Hence, deduce that $\alpha^{12} + \beta^{12} = 2$.

14. If α, β are the roots of the equation $x^2 + x + 1 = 0$, prove that

$\alpha^n + \beta^n = 2\cos(2n\pi/3)$. Hence, deduce that $\alpha^6 + \beta^6 = 2$.

15. **(a)** If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = i$, then show that the general value of $\theta = \left[2r + \dfrac{1}{n(n+1)}\right]\pi$.

(b) If $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ \ldots\ (\cos n\theta + i\sin n\theta) = 1$, then show that the general value of $\theta = \dfrac{4r\pi}{n(n+1)}$.

16. Prove that

(a) $1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta = \dfrac{1}{2} + \dfrac{\sin[n-(1/2)]\theta}{2\sin(\theta/2)}$

(b) $\sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta = \dfrac{1}{2}\cot\dfrac{\theta}{2} - \dfrac{\cos[n-(1/2)]\theta}{2\sin(\theta/2)}$

17. By using De Moivre's Theorem, show that

$$\sin\alpha + \sin 2\alpha + \ldots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{\sin(\alpha/2)}.$$ **(M.U. 1986)**

18. Evaluate $[(\cos\theta - \cos\Phi) + i(\sin\theta - \sin\Phi)]^n + [(\cos\theta - \cos\Phi) - i(\sin\theta - \sin\Phi)]^n$.

$$\left[\textbf{Ans. : } 2^{n+1}\sin^n\left(\frac{\theta-\Phi}{2}\right)\cos n\left(\frac{\pi+\theta+\Phi}{2}\right)\right]$$

3. Expansions of sin $n\theta$, cos $n\theta$ in powers of sin θ, cos θ

To obtain these expansions we write $(\cos\theta + i\sin\theta)^n$ in two ways (i) by De Moivre's theorem and (ii) by Binomial theorem and then equate the two results.

By De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \ldots\ldots\ldots (1)$$

By Binomial Theorem

$$(\cos\theta + i\sin\theta)^n = {}^nC_0\cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + \ldots$$
$$= ({}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots)$$
$$+ i({}^nC_1\cos^{n-1}\theta\sin\theta + \ldots) \quad \ldots\ldots\ldots (2)$$

Equating real and imaginary parts, from (1) and (2)

$$\boxed{\begin{aligned}\cos n\theta &= {}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots \\ \sin n\theta &= {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots\end{aligned}}$$

Remark ....

You are advised to learn the method rather than to remember the formula.

Class (b) : 6 Marks

Example 1 (b) : Express $\sin 6\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$.

Sol. : We have by De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^6 = (\cos 6\theta + i\sin 6\theta) \quad \ldots\ldots\ldots (1)$$

We have by Binomial Theorem

$$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6\cos^5\theta\,(i\sin\theta) + 15\cos^4\theta\,(i\sin\theta)^2$$
$$+ 20\cos^3\theta\,(i\sin\theta)^3 + 15\cos^2\theta\,(i\sin\theta)^4$$
$$+ 6\cos\theta\,(i\sin\theta)^5 + (i\sin\theta)^6$$

$$\therefore \quad (\cos\theta + i\sin\theta)^6 = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$+ i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta) \quad \ldots\ldots (2)$$

Equating imaginary parts, from (1) and (2)

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta.$$

(Equating real parts we get the expression for $\cos 6\theta$.)

Example 2 (b) : Use De Moivre's Theorem to show that

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}.$$

Hence deduce that $5\tan^4\frac{\pi}{10} - 10\tan^2\frac{\pi}{10} + 1 = 0.$ **(M.U. 1983, 90, 2017, 18)**

Sol. : We have by De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^5 = (\cos 5\theta + i\sin 5\theta) \quad \ldots\ldots (1)$$

We have by Binomial Theorem

$$(\cos\theta + i\sin\theta)^5$$
$$= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2$$
$$+ 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$
$$= \cos^5\theta + i\,5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$$
$$- i\cdot 10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \quad \ldots\ldots (2)$$

Equating real and imaginary parts

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$\therefore \quad \tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

Dividing the numerator and denominator by $\cos^5\theta$

$$\tan 5\theta = \frac{\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} \quad \ldots\ldots (1)$$

Now, put $\theta = \frac{\pi}{10}$. Then $\tan 5\theta = \tan\frac{\pi}{2} = \infty$ and hence the denominator of (1) must be zero

$$\therefore \quad 5\tan^4\frac{\pi}{10} - 10\tan^2\frac{\pi}{10} + 1 = 0.$$

Example 3 (b) : Show that $\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$. **(M.U. 1999, 2012)**

Sol. : We have by De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta \quad \ldots\ldots (1)$$

We have by Binomial Theorem,

$$(\cos\theta + i\sin\theta)^7 = \cos^7\theta + 7\cdot\cos^6\theta\, i\sin\theta + 21\cos^5\theta(i\sin\theta)^2$$
$$+ 35\cos^4\theta(i\sin\theta)^3 + 35\cos^3\theta(i\sin\theta)^4$$
$$+ 21\cos^2\theta(i\sin\theta)^5 + 7\cos\theta(i\sin\theta)^6 + (i\sin\theta)^7$$

$$\therefore \quad (\cos\theta + i\sin\theta)^7 = \cos^7\theta + 7i\cos^6\theta\sin\theta$$
$$- 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta$$
$$+ 35\cos^3\theta\sin^4\theta + 21i\cos^2\theta\sin^5\theta$$
$$- 7\cos\theta\sin^6\theta - i\sin^7\theta \qquad \ldots\ldots (2)$$

Equating real and imaginary parts from (1) and (2)

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \qquad \ldots\ldots (3)$$
$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta \qquad \ldots\ldots (4)$$

Dividing (4) by (3), we get

$$\tan 7\theta = \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta}$$

Now, dividing the numerator and denominator of r.h.s. by $\cos^7\theta$, we get the required result.

Example 4 (b) : Show that $\dfrac{\sin 5\theta}{\sin\theta} = 16\cos^4\theta - 12\cos^2\theta + 1.$ **(M.U. 1992, 2004)**

Sol. : As proved in the above Ex. 2, we have

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$\therefore \quad \frac{\sin 5\theta}{\sin\theta} = 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= 5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 16\cos^4\theta - 12\cos^2\theta + 1.$$

Example 5 (b) : Show that,

(i) $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$ **(M.U. 2003)**

(ii) $\cos 5\theta = 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta.$

Sol. : As proved in The above Ex. 2, we have

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$
$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.$$

and
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$
$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)$$
$$= 5\cos\theta - 20\cos^3\theta + 16\cos^5\theta.$$

Example 6 (b) : Using De Moivre's Theorem prove that

$2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2\cos\theta.$ **(M.U. 1998)**

Sol. : We have, $1 + \cos 8\theta = 2\cos^2 4\theta = 2(\cos 4\theta)^2$

$$\therefore \quad 2(1 + \cos 8\theta) = 4(\cos 4\theta)^2 = (2\cos 4\theta)^2 \qquad \ldots\ldots (1)$$

To find $\cos 4\theta$ in powers of $\cos\theta$, consider,

$$(\cos 4\theta + i\sin 4\theta) = (\cos\theta + i\sin\theta)^4$$
$$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta$$
$$+ 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$$

Equating real parts,

$$\begin{aligned} \cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1 \end{aligned}$$

$$\therefore \quad 2\cos 4\theta = 16\cos^4\theta - 16\cos^2\theta + 2$$

Putting this value in (1),

$$\begin{aligned} 2(1+\cos 8\theta) &= (16\cos^4\theta - 16\cos^2\theta + 2)^2 \\ &= [(2\cos\theta)^4 - 4(2\cos\theta)^2 + 2]^2 \\ &= (x^4 - 4x^2 + 2)^2 \quad \text{where } x = 2\cos\theta. \end{aligned}$$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. Using De Moivre's Theorem, prove that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

and $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$. **(M.U. 1984)**

2. Using De Moivre's Theorem, prove that

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$. **(M.U. 1984)**

3. Using De Moivre's Theorem express $\sin 3\theta$, $\cos 3\theta$, $\tan 3\theta$ in terms of powers of $\sin\theta$, $\cos\theta$, $\tan\theta$ respectively.

[**Ans. :** $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, $\tan 3\theta = (3\tan\theta - \tan^3\theta)/(1 - 3\tan^2\theta)$]

Class (b) : 6 Marks

1. Express $\sin 7\theta$ and $\cos 7\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$. **(M.U. 1984, 95)**

[**Ans. :** $\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$;
$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$]

2. (a) If $\sin 6\theta = a\cos^5\theta\sin\theta + b\cos^3\theta\sin^3\theta + c\cos\theta\sin^5\theta$, find the values of a, b, c.

(M.U. 1995, 2005) [**Ans. :** $a = 6$, $b = -20$, $c = 6$]

(b) If $\cos 6\theta = a\cos^6\theta + b\cos^4\theta\sin^2\theta + c\cos^2\theta\sin^4\theta + d\sin^6\theta$, find a, b, c, d.

[**Ans. :** $a = 1$, $b = -15$, $c = 15$, $d = -1$]

3. Prove that

$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$.

$\sin 8\theta = 8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta + 56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta$.

4. Prove that $\dfrac{\sin 6\theta}{\sin 2\theta} = 16\cos^4\theta - 16\cos^2\theta + 3$. **(M.U. 2004, 2014)**

5. Prove that $\dfrac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^4\theta$. **(M.U. 2015)**

6. Express $\tan 7\theta$ in terms of powers of $\tan\theta$.

Hence, deduce that $7\tan^6\dfrac{\pi}{14} - 35\tan^4\dfrac{\pi}{14} + 21\tan^2\dfrac{\pi}{14} - 1 = 0$.

7. Prove that

(i) $\dfrac{1+\cos 9A}{1+\cos A} = [16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1]^2$

(ii) $\dfrac{1-\cos 9A}{1-\cos A} = [16\cos^4 A + 8\cos^3 A - 12\cos^2 A - 4\cos A + 1]^2$

8. Prove that (i) $\dfrac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$ where $x = 2\cos\theta$

(ii) $\dfrac{1-\cos 7\theta}{1-\cos\theta} = (x^3 + x^2 - 2x - 1)^2$ where $x = 2\cos\theta$.

9. Prove that $\dfrac{1+\cos 6\theta}{1+\cos 2\theta} = 16\cos^4\theta - 24\cos^2\theta + 9$.

4. Expansions of $\cos^n\theta$, $\sin^n\theta$ in Terms of sines or cosines of Multiples of θ

If $x = \cos\theta + i\sin\theta$, then $\dfrac{1}{x} = \cos\theta - i\sin\theta$.

Also $x + \dfrac{1}{x} = 2\cos\theta$ and $x - \dfrac{1}{x} = 2i\sin\theta$.

$\therefore$ $x^n = \cos n\theta + i\sin n\theta$ and $\dfrac{1}{x^n} = \cos n\theta - i\sin n\theta$.

$$\therefore \boxed{x^n + \frac{1}{x^n} = 2\cos n\theta \text{ and } x^n - \frac{1}{x^n} = 2i\sin n\theta.}$$

We can use these results for expansions of $\cos^n\theta$, $\sin^n\theta$ in terms of sines and cosines of multiples of θ as illustrated below.

Class (b) : 6 Marks

Example 1 (b) : Expand $\sin^7\theta$ in a series of sines of multiples of θ. **(M.U. 2013, 19)**

Sol. : Let $x = \cos\theta + i\sin\theta$ $\therefore$ $\dfrac{1}{x} = \cos\theta - i\sin\theta$

Also $x + \dfrac{1}{x} = 2\cos\theta$ and $x - \dfrac{1}{x} = 2i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta$ and $\dfrac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore$ $x^n + \dfrac{1}{x^n} = 2\cos n\theta$ and $x^n - \dfrac{1}{x^n} = 2i\sin n\theta$

Now, by Binomial Theorem,

$$(2i\sin\theta)^7 = \left(x - \frac{1}{x}\right)^7$$

$$= x^7 - 7x^6\cdot\frac{1}{x} + 21x^5\cdot\frac{1}{x^2} - 35x^4\cdot\frac{1}{x^3} + 35x^3\cdot\frac{1}{x^4} - 21x^2\cdot\frac{1}{x^5} + 7x\cdot\frac{1}{x^6} - \frac{1}{x^7}$$

$$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 7\theta - 7\cdot(2i\sin 5\theta) + 21\cdot(2i\sin 3\theta) - 35\cdot(2i\sin\theta)$$

$$\therefore \quad -2^6\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$$

$$\therefore \quad \sin^7\theta = -\frac{1}{2^6}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$$

Example 2 (b) : Expand $\cos^7\theta$ in a series of cosines of multiples of θ. **(M.U. 2012, 14)**

Sol. : Let $x = \cos\theta + i\sin\theta \quad \therefore \quad \frac{1}{x} = \cos\theta - i\sin\theta$

Also $\quad x + \frac{1}{x} = 2\cos\theta \quad$ and $\quad x - \frac{1}{x} = 2i\sin\theta$

$x^n = \cos n\theta + i\sin n\theta \quad$ and $\quad \frac{1}{x^n} = \cos n\theta - i\sin n\theta$

$\therefore \quad x^n + \frac{1}{x^n} = 2\cos n\theta \quad$ and $\quad x^n - \frac{1}{x^n} = 2i\sin n\theta$

Now, by Binomial theorem

$$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7$$

$$= x^7 + 7x^6\cdot\frac{1}{x} + 21x^5\cdot\frac{1}{x^2} + 35x^4\cdot\frac{1}{x^3} + 35x^3\cdot\frac{1}{x^4} + 21x^2\cdot\frac{1}{x^5} + 7x\cdot\frac{1}{x^6} + \frac{1}{x^7}$$

$$= x^7 + 7x^5 + 21x^3 + 35x + 35\cdot\frac{1}{x} + 21\cdot\frac{1}{x^3} + 7\cdot\frac{1}{x^5} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

$$= (2\cos 7\theta) + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$$

$$\therefore \quad \cos^7\theta = \frac{1}{2^6}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta]$$

Note ....

Compare Ex. 1 with Ex. 2 and note that for cosine, we use $x + (1/x)$ and for sine we use $x - (1/x)$.

Example 3 (b) : Show that $\quad \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$. **(M.U. 1991, 2006, 18)**

Sol. : Let $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore \quad (2i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5 = x^5 - 5x^4\cdot\frac{1}{x} + 10x^3\cdot\frac{1}{x^2} - 10x^2\cdot\frac{1}{x^3} + 5x\cdot\frac{1}{x^4} - \frac{1}{x^5}$$

$$32i^5\sin^5\theta = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$32i\sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$\therefore \quad \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Alternatively the expansions of trigonometric functions of this type can also be obtained by using the result of § 10 obtained on page 1-5.

By (4), $e^{i\theta} = \cos\theta + i\sin\theta \quad \therefore \quad e^{in\theta} = \cos n\theta + i\sin n\theta$

and by (5), $e^{-i\theta} = \cos\theta - i\sin\theta \quad \therefore \quad e^{-in\theta} = \cos n\theta - i\sin n\theta$

$$\therefore \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

$$\therefore \quad \sin^5\theta = \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^5$$

$$= \frac{1}{32i}[e^{5i\theta} - 5e^{4i\theta}\cdot e^{-i\theta} + 10e^{3i\theta}\cdot e^{-2i\theta} - 10e^{2i\theta}\cdot e^{-3i\theta} + 5e^{i\theta}\cdot e^{-4i\theta} - e^{-5i\theta}]$$

$$= \frac{1}{32i}\left[(e^{5i\theta} - e^{-5i\theta}) - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta})\right]$$

$$= \frac{1}{32i}[2i\sin 5\theta - 5\cdot 2i\sin 3\theta + 10\cdot 2i\sin\theta]$$

$$\therefore \quad \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Example 4 (b) : Expand $\sin^5\theta\cos^3\theta$ in a series of sines of multiples of θ.

Sol. : As above we can write

$$(2i\sin\theta)^5(2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^5\left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)^2\left(x - \frac{1}{x}\right)^3\left(x + \frac{1}{x}\right)^3 = \left(x - \frac{1}{x}\right)^2\left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x^2 - 2 + \frac{1}{x^2}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= x^8 - 3x^4 + 3 - \frac{1}{x^4} - 2x^6 + 6x^2 - \frac{6}{x^2} + \frac{2}{x^6} + x^4 - 3 + \frac{3}{x^4} - \frac{1}{x^8}$$

$$= \left(x^8 - \frac{1}{x^8}\right) - 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) + 6\left(x^2 - \frac{1}{x^2}\right)$$

$$\therefore \quad 2^8 i\sin^5\theta\cos^3\theta = 2i\sin 8\theta - 2\cdot 2i\sin 6\theta - 2\cdot 2i\sin 4\theta + 6\cdot 2i\sin 2\theta$$

$$\therefore \quad \sin^5\theta\cos^3\theta = \frac{1}{128}[\sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta]$$

(For another method, see solved Ex. 6, page 8-13)

Example 5 (b) : If $\sin^4\theta\cos^3\theta = a_1\cos\theta + a_3\cos 3\theta + a_5\cos 5\theta + a_7\cos 7\theta$, prove that

$$a_1 + 9a_3 + 25a_5 + 49a_7 = 0. \qquad \textbf{(M.U. 1999, 2000, 02, 09, 13, 17)}$$

Sol. : Let $x = \cos\theta + i\sin\theta \quad \therefore \quad \frac{1}{x} = \cos\theta - i\sin\theta$

Also $x^n + \frac{1}{x^n} = 2\cos n\theta$ and $x^n - \frac{1}{x^n} = 2i\sin n\theta$

Now consider,

$$(2i\sin\theta)^4(2\cos\theta)^3 = \left(x - \frac{1}{x}\right)^4\left(x + \frac{1}{x}\right)^3$$

$$= \left(x - \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)^3 = \left(x^2 - \frac{1}{x^2}\right)^3 \left(x - \frac{1}{x}\right)$$

$$= \left(x^6 - 3x^2 + 3 \cdot \frac{1}{x^2} - \frac{1}{x^6}\right)\left(x - \frac{1}{x}\right)$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} - x^5 + 3x - \frac{3}{x^3} + \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) - \left(x^5 + \frac{1}{x^5}\right) - 3\left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$$

$$\therefore \quad (2i \sin\theta)^4 (2\cos\theta)^3 = 2\cos 7\theta - 2\cos 5\theta - 6\cos 3\theta + 6\cos\theta$$

$$\therefore \quad \sin^4\theta \cos^3\theta = \frac{\cos 7\theta}{2^6} - \frac{\cos 5\theta}{2^6} - \frac{3\cos 3\theta}{2^6} + \frac{3\cos\theta}{2^6}$$

Comparing this with the given equality,

$$a_1 = \frac{3}{2^6}, \quad a_3 = -\frac{3}{2^6}, \quad a_5 = -\frac{1}{2^6}, \quad a_7 = \frac{1}{2^6}$$

$$\therefore \quad a_1 + 9a_3 + 25a_5 + 49a_7 = \frac{3}{2^6} - \frac{27}{2^6} - \frac{25}{2^6} + \frac{49}{2^6} = \frac{52 - 52}{2^6} = 0.$$

Example 6 (b) : Using De Moivre's Theorem prove that,

$$\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5).$$ **(M.U. 1994, 95, 2001, 02, 15, 16)**

Sol. : Let as above $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$$(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6 = x^6 + 6x^5 \cdot \frac{1}{x} + 15x^4 \cdot \frac{1}{x^2} + 20x^3 \cdot \frac{1}{x^3} + 15x^2 \cdot \frac{1}{x^4} + 6x \cdot \frac{1}{x^5} + \frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + 15 \cdot \frac{1}{x^2} + 6 \cdot \frac{1}{x^4} + \frac{1}{x^6} \qquad \text{.................... (1)}$$

$$(2i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6 = x^6 - 6x^4 + 15x^2 - 20 + 15 \cdot \frac{1}{x^2} - 6 \cdot \frac{1}{x^4} + \frac{1}{x^6}$$

$$\therefore \quad (2\sin\theta)^6 = -x^6 + 6x^4 - 15x^2 + 20 - 15 \cdot \frac{1}{x^2} + 6 \cdot \frac{1}{x^4} - \frac{1}{x^6} \qquad \text{.................... (2)}$$

Adding (1) and (2), $\quad [\because \; i^6 = (i^2)^3 = -1]$

$$2^6(\cos^6\theta + \sin^6\theta) = 12x^4 + 40 + 12 \cdot \frac{1}{x^4} = 4\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$$

$$\therefore \quad \cos^6\theta + \sin^6\theta = \frac{1}{16}\left[3\left(x^4 + \frac{1}{x^4}\right) + 10\right]$$

But $\quad x^4 + \frac{1}{x^4} = 2\cos 4\theta$

$$\therefore \quad \cos^6\theta + \sin^6\theta = \frac{1}{16}[6\cos 4\theta + 10] = \frac{1}{8}[3\cos 4\theta + 5]$$

Restatement : If $\cos^6\theta + \sin^6\theta = \alpha\cos 4\theta + \beta$, then prove that $\alpha + \beta = 1$. **(M.U. 2015)**

Sol. : Same as above.

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

1. Expand $\cos^7 \theta$ in a series of cosines of multiples of θ.

   [**Ans. :** $\cos^7 \theta = \frac{1}{2^6}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$]

2. Express $\sin^8 \theta$ in a series of cosines of multiples of θ.

   [**Ans. :** $\sin^8 \theta = \frac{1}{2^7}(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$]

3. Prove that $\cos^8 \theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$ **(M.U. 1982, 90)**

4. Show that $2^5 \sin^4 \theta \cos^2 \theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.

5. Show that $2^6 \sin^4 \theta \cos^3 \theta = \cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta$.

6. Show that $\cos^6 \theta = \frac{1}{32}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$.

7. Prove that $\cos^8 \theta + \sin^8 \theta = \frac{1}{64}[\cos 8\theta + 28\cos 4\theta + 35]$. **(M.U. 1982, 90)**

8. Prove that $\cos^6 \theta - \sin^6 \theta = \frac{1}{16}[\cos 6\theta + 15\cos 2\theta]$. **(M.U. 1987, 2007, 16)**

9. Prove that

   $-2^{12}\cos^6 \theta \sin^7 \theta = \sin 13\theta - \sin 11\theta - 6\sin 9\theta + 6\sin 7\theta$
   $+ 15\sin 5\theta - 15\sin 3\theta - 20\sin\theta$.

10. Prove that $\sin^7 \theta \cos^3 \theta = -\frac{1}{256}[\sin 10\theta - 4\cos 8\theta + 3\sin 6\theta$
    $+ 8\sin 4\theta - 14\sin 2\theta]$

11. Prove that $\cos^5 \theta \sin^3 \theta = -\frac{1}{2^7}[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta]$ **(M.U. 2002, 04)**

5. Roots of a Complex Number

De Moivre's Theorem can be used to find all n-roots of a complex number.

Since, $\cos\theta = \cos(2k\pi + \theta)$ and $\sin\theta = \sin(2k\pi + \theta)$ where k is an integer, we have by De Moivre's Theorem.

$$(\cos\theta + i\sin\theta)^{1/n} = [\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]^{1/n}$$

$$\therefore \quad \boxed{(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)}$$

By putting $k = 0, 1, 2, \ldots\ldots, n-1$, we get n roots of the complex number.

Note ....

For the following examples we need the results given below.

$1 = \cos 0 + i\sin 0$, $\quad -1 = \cos\pi + i\sin\pi$

$i = \cos(\pi/2) + i\sin(\pi/2)$, $\quad -i = \cos(\pi/2) - i\sin(\pi/2)$

$1 + i = \sqrt{2}[\cos(\pi/4) + i\sin(\pi/4)]$, etc.

Type I : On Cube Roots of Unity : Class (b) : 6 Marks

Example 1 (b) : Find the cube roots of unity. If ω is a complex cube root of unity, prove that $(1-\omega)^6 = -27$. **(M.U. 2003, 11)**

Sol. : Consider $x^3 = 1 \quad \therefore \quad x = 1^{1/3}$

$$\therefore \quad x = (\cos 0 + i \sin 0)^{1/3} = (\cos 2k\pi + i \sin 2k\pi)^{1/3}$$

$$= \cos \frac{2k\pi}{3} + i \sin \frac{2k\pi}{3}$$

Putting $k = 0, 1, 2$, the cube roots of unity are

$$x_0 = 1, \quad x_1 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \omega, \text{ say}$$

and $$x_2 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = \left[\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right]^2 = \omega^2$$

Thus, the three cube-roots of unity are $1, \omega, \omega^2$.

Now, $$1 + \omega + \omega^2 = 1 + \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} + \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$$

$$\therefore \quad 1 + \omega + \omega^2 = 1 - \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} - \cos \frac{\pi}{3} - i \sin \frac{\pi}{3}$$

$$= 1 - 2\cos \frac{\pi}{3} = 1 - 2\left(\frac{1}{2}\right) = 0 \qquad \text{.................... (A)}$$

$$\therefore \quad 1 + \omega^2 = -\omega$$

Now, $$(1-\omega)^6 = [(1-\omega)^2]^3 = (1 - 2\omega + \omega^2)^3$$

$$= (-3\omega)^3 = -27\omega^3 = -27.$$

Example 2 (b) : If ω is a complex cube-root of unity, prove that

$$1 + \omega + \omega^2 = 0 \text{ and } \frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 0.$$ **(M.U. 2010)**

Sol. : (i) We have proved above that $1 + \omega + \omega^2 = 0$.

(ii) Now, $$\text{l.h.s.} = \frac{(2+\omega)(1+\omega) + (1+2\omega)(1+\omega) - (1+2\omega)(2+\omega)}{(1+2\omega)(2+\omega)(1+\omega)}$$

$$= \frac{(2 + 3\omega + \omega^2) + (1 + 3\omega + 2\omega^2) - (2 + 5\omega + 2\omega^2)}{(1+2\omega)(2+\omega)(1+\omega)}$$

$$= \frac{1 + \omega + \omega^2}{(1+2\omega)(2+\omega)(1+\omega)}$$

$$= 0 \qquad [\because 1 + \omega + \omega^2 = 0]$$

Example 3 (b) : If ω is a complex fourth root of unity, prove that $1 + \omega + \omega^2 + \omega^3 = 0$.

Sol. : Consider $x^4 = 1 \quad \therefore \quad x = 1^{1/4}$.

$$\therefore \quad x = (\cos 0 + i \sin 0)^{1/4} = (\cos 2k\pi + i \sin 2k\pi)^{1/4}$$

$$= \cos \frac{2k\pi}{4} + i \sin \frac{2k\pi}{4}$$

Putting $k = 0, 1, 2, 3$, the fourth-roots of unity are

$$x_0 = 1, \quad x_1 = \cos \frac{2\pi}{4} + i \sin \frac{2\pi}{4} = \omega, \text{ say}$$

$$x_2 = \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4} = \left[\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right]^2 = \omega^2$$

$$x_3 = \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4} = \left[\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right]^3 = \omega^3$$

$$\therefore \quad 1+\omega+\omega^2+\omega^3 = 1+\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right)+\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right)+\left(\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right)$$

$$= 1+\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)+(\cos\pi + i\sin\pi)+\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$$\therefore \quad 1+\omega+\omega^2+\omega^3 = 1+(0+i)+(-1+i(0))+(0+i(-1))$$

$$= 1+i-1-i = 0.$$

Example 4 (b) : Prove that the n nth roots of unity are in geometric progression.

Sol. : Consider the equation $x^n = 1 \quad \therefore \quad x = 1^{1/n}$

$$x = (\cos 0 + i\sin 0)^{1/n} = (\cos 2k\pi + i\sin 2k\pi)^{1/n}$$

$$\therefore \quad x = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$$

By putting $k = 0, 1, 2, \ldots\ldots, n-1$, we get the n roots.

When $k = 0$, $\quad x_0 = 1$

When $k = 1$, $\quad x_1 = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right) = \omega$, say

When $k = 2$, $\quad x_2 = \cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right) = \left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^2 = \omega^2$

When $k = 3$, $\quad x_3 = \cos\left(\frac{6\pi}{n}\right) + i\sin\left(\frac{6\pi}{n}\right) = \left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^3 = \omega^3$

....

....

When $k = m$, $\quad x_m = \cos\left(\frac{2m\pi}{n}\right) + i\sin\left(\frac{2m\pi}{n}\right) = \left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^m = \omega^m$

When $k = n-1$, $\quad x_n = \left[\cos\left(\frac{2(n-1)\pi}{n}\right) + i\sin\left(\frac{2(n-1)\pi}{n}\right)\right] = \omega^{n-1}$

Hence, the roots are $1, \omega, \omega^2, \omega^3, \ldots\ldots, \omega^m, \ldots\ldots \omega^{(n-1)}$ which are in G.P.

Example 5 (b) : Show that the sum of the n nth roots of unity is zero. **(M.U. 2007)**

Sol. : Now, the sum of the roots $= x_0 + x_1 + x_2 + \ldots\ldots + x_{n-1}$

$$= 1+\omega+\omega^2+\ldots\ldots+\omega^{n-1}$$

$$= \frac{1-\omega^n}{1-\omega} = \left[1-\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n\right]\Big/(1-\omega)$$

$$= \frac{1-(\cos 2\pi + i\sin 2\pi)}{1-\omega} = \frac{1-1}{1-\omega} = 0$$

Example 6 (b) : Prove that the product of the n nth roots of unity is $(-1)^{n-1}$.

Sol. : The product of the n nth roots

$$= 1 \cdot \omega \cdot \omega^2 \ldots\ldots \omega^{n-1} = \omega^{1+2+\ldots+(n-1)}$$

$$= \omega^{(n-1)(n)/2} \qquad \left[\because \ \Sigma r = \frac{n(n+1)}{2}\right]$$

$$= \left[\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right]^{(n-1)(n)/2} = \cos\left[\frac{(n-1)n}{2}\cdot\frac{2\pi}{n}\right] + i\sin\left[\frac{(n-1)n}{2}\cdot\frac{2\pi}{n}\right]$$

$$= \cos(n-1)\pi + i\sin(n-1)\pi$$

$$= [\cos\pi + i\sin\pi]^{n-1} = (-1)^{n-1}.$$

Example 7 (b) : If ω is a 7th root of unity prove that

$$S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$$

if n is a multiple of 7 and is equal to zero otherwise. **(M.U. 2008)**

Sol. : We have $x = 1^{1/7} = (\cos 2n\pi + i\sin 2n\pi)^{1/7}$

$$\therefore \quad x = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}, \quad n = 0, 1, 2, 3, 4, 5, 6.$$

Let $\quad \omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

Now, $\quad S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \ldots\ldots + \omega^{6n} = \dfrac{1-\omega^{7n}}{1-\omega^n}$

because nth roots of unity are in G.P.

Now, $\quad \omega^{7n} = (\omega^7)^n = \left\{\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7\right\}^n = 1$

$$\therefore \quad 1 - \omega^{7n} = 0$$

But ω is a 7th root of unity and hence $\omega^n \neq 1$ if n is not a multiple of 7. $1 - \omega^n \neq 0$ as n is not a multiple of 7 and hence ω^n is not a root.

$\therefore$ The sum $S = 0$ if n is not a multiple of 7.

If n is a multiple of 7, say $n = 7k$.

Then $\quad S = 1 + (\omega^7)^k + (\omega^7)^{2k} + \ldots\ldots + (\omega^7)^{6k}$

$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7.$$

Type II : On Solving Equations : Class (b) : 6 Marks

Example 1 (b) : Solve $x^6 + 1 = 0$. **(M.U. 2014)**

Sol. : We have $\quad x^6 = -1 = \cos\pi + i\sin\pi$

$$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$

$$x = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/6}$$

$$= \cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}; \qquad k = 0, 1, 2, 3, 4, 5$$

Putting $k = 0$, $\quad x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$

Putting $k=1$, $x_1 = \cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

Putting $k=2$, $x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = \cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)$

$$= -\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$$

Putting $k=3$, $x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = \cos\left(\pi + \frac{\pi}{6}\right) + i\sin\left(\pi + \frac{\pi}{6}\right)$

$$= -\cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

Putting $k=4$, $x_4 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$

$$= 0 - i\cdot(1) = -i$$

Putting $k=5$, $x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \cos\left(2\pi - \frac{\pi}{6}\right) + i\sin\left(2\pi - \frac{\pi}{6}\right)$

$$= \cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - i\cdot\frac{1}{2}.$$

Thus, the roots are $\pm i$, $\frac{\sqrt{3} \pm i}{2}$, $\frac{-\sqrt{3} \pm i}{2}$.

Example 2 (b) : Solve $x^6 - i = 0$. **(M.U. 1994, 95, 2013)**

Sol. : We have $x^6 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$\therefore \quad x^6 = \cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)$$

$$\therefore \quad x = \left[\cos(4k+1)\frac{\pi}{2} + i\sin(4k+1)\frac{\pi}{2}\right]^{1/6}$$

$$\therefore \quad x = \cos(4k+1)\frac{\pi}{12} + i\sin(4k+1)\frac{\pi}{12}$$

Putting $k = 0, 1, 2, 3, 4, 5$, we get the roots as

$$\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), \left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right), \left(\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right),$$

$$\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right), \left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right), \left(\cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12}\right)$$

i.e. $\pm\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right), \pm\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right),$

$\pm\left(\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right)$ *i.e.* $\pm\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

Example 3 (b) : Find the roots common to $x^4 + 1 = 0$ and $x^6 - i = 0$. **(M.U. 2001, 02, 2009)**

Sol. : We have solved the equation $x^6 - i = 0$.

Now consider $x^4 + 1 = 0 \qquad \therefore \quad x^4 = -1$

$$x^4 = \cos\pi + i\sin\pi = \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$$

$$\therefore \quad x = \cos\frac{(2k+1)}{4}\pi + i\sin\frac{(2k+1)}{4}\pi.$$

Putting $k = 0, 1, 2, 3$ we get the roots as

$$\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),\ \left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),\ \left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right),\ \left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$$

i.e., $\pm\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ and $\pm\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$.

Hence, common roots are $\pm\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$.

Example 4 (b) : Solve completely the equation $x^{10} + 11x^5 + 10 = 0$. **(M.U. 1995)**

Sol. : Let $x^5 = y$ $\quad\therefore\quad y^2 + 11y + 10 = 0$.

$\therefore\quad (y+10)(y+1) = 0 \qquad \therefore\quad y = -10$ or -1

$\therefore\quad x^5 = -10 = 10(\cos\pi + i\sin\pi)$

$\therefore\quad x = \sqrt[5]{10}(\cos\pi + i\sin\pi)^{1/5} = \sqrt[5]{10}\left[\cos(2k+1)\pi + i\sin(2k+1)\pi\right]^{1/5}$

$$= \sqrt[5]{10}\left(\cos\frac{(2k+1)\pi}{5} + i\sin\frac{(2k+1)\pi}{5}\right)$$

Putting $k = 0, 1, 2, 3, 4$, we get the roots.

Similarly, from $x^5 = -1 = (\cos\pi + i\sin\pi)$, we get the remaining five roots as

$$x = \cos\left(\frac{(2k+1)\pi}{5}\right) + i\sin\left(\frac{(2k+1)\pi}{5}\right) \text{ where } k = 0, 1, 2, 3, 4.$$

Note ....

If may be noted that the above forms of the roots are not unique. For example, the root $\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$ can be written as $\cos\left(\pi - \frac{7\pi}{12}\right) + i\sin\left(\pi - \frac{7\pi}{12}\right) = -\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}$.

Example 5 (b) : If $1 + 2i$ is a root of the equation $x^4 - 3x^3 + 8x^2 - 75x + 5 = 0$, find all the other roots of the equation.

Sol. : Since $1 + 2i$ is a root of the given equation we know that $1 - 2i$ must be one of the remaining roots. Hence, $(x - 1 - 2i)$ and $(x - 1 + 2i)$ are the factors of the l.h.s. *i.e.* the l.h.s is divisible by

$$\{(x-1)-2i\}\{(x-1)+2i\} \text{ i.e. by } (x-1)^2 - (4i)^2 = x^2 - 2x + 5.$$

Dividing the l.h.s. by $x^2 - 2x + 5$, we get $x^2 - x + 1$.

Solving the equation $x^2 - x + 1$, we get $x = \dfrac{1 \pm \sqrt{3}\cdot i}{2}$.

Hence, the remaining roots are $1 - 2i$, $\dfrac{(1 \pm \sqrt{3}\cdot i)}{2}$.

Aliter : Let the roots be $1 + 2i$, $1 - 2i$, $\alpha + i\beta$ and $\alpha - i\beta$.

Now, sum of the roots $= -\dfrac{\text{coefficient of } x^3}{\text{coefficient of } x^4}$

$\therefore\quad (1+2i) + (1-2i) + (\alpha + i\beta) + (\alpha - i\beta) = 3$

$\therefore\quad 2 + 2\alpha = 3 \qquad \therefore\quad \alpha = 1/2$.

Further, product of the roots $= \dfrac{\text{constant term}}{\text{coefficient of } x^4}$

$\therefore \quad (1+2i)(1-2i)+(\alpha+i\beta)(\alpha-i\beta)=5$

$\therefore \quad (1-4i^2)(\alpha^2-i^2\beta^2)=5 \qquad \therefore \quad 5(\alpha^2+\beta^2)=5$

$\therefore \quad \alpha^2+\beta^2=1 \qquad \therefore \quad \beta^2=1-\alpha^2=1-\dfrac{1}{4}=\dfrac{3}{4} \qquad \therefore \quad \beta=\pm\dfrac{\sqrt{3}}{2}.$

$\therefore$ The other roots are $1-2i,\ \dfrac{1\pm i\sqrt{3}}{2}$.

Example 6 (b) : Find all the values of $\left(\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1. **(M.U. 1990, 2002)**

Sol. : Let $\dfrac{1}{2}=r\cos\theta$ and $\dfrac{\sqrt{3}}{2}=r\sin\theta$.

$\therefore \quad r^2(\cos^2\theta+\sin^2\theta)=\dfrac{1}{4}+\dfrac{3}{4}=1 \qquad \therefore \quad r=1$

$\therefore \quad \cos\theta=\dfrac{1}{2}$ and $\sin\theta=\dfrac{\sqrt{3}}{2} \qquad \therefore \quad \theta=\dfrac{\pi}{3}$

$$\therefore \quad \left\{\frac{1}{2}+i\frac{\sqrt{3}}{2}\right\}^{3/4}=\left\{\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^3\right\}^{1/4}=(\cos\pi+i\sin\pi)^{1/4}$$

$$=[\cos(2k+1)\pi+i\sin(2k+1)\pi]^{1/4}$$

$$=\cos(2k+1)\frac{\pi}{4}+i\sin(2k+1)\frac{\pi}{4}.$$

Putting $k=0, 1, 2, 3$, we get the four roots as,

$$\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),\ \left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),\ \left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right),\ \left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right).$$

Their continued product,

$$=\cos\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)+i\sin\left(\frac{\pi}{4}+\frac{3\pi}{4}+\frac{5\pi}{4}+\frac{7\pi}{4}\right)$$

$$=\cos 4\pi+i\sin 4\pi=1.$$

Example 7 (b) : Solve the equation $x^{4/3}=\left(\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)$. **(M.U. 2002, 10)**

Sol. : As above.

Example 8 (b) : Solve $\left(\dfrac{1+x}{1-x}\right)^6=1$.

Sol. : We have $\left(\dfrac{1+x}{1-x}\right)^6=1=\cos 2\pi+i\sin 2\pi$

$$\therefore \quad \frac{1+x}{1-x}=1^{1/6}=\frac{(\cos 2k\pi+i\sin 2k\pi)^{1/6}}{1}$$

$$=\left(\cos\frac{2k\pi}{6}+i\sin\frac{2k\pi}{6}\right)\Big/1=\left(\cos\frac{k\pi}{3}+i\sin\frac{k\pi}{3}\right)\Big/1.$$

[If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{b-a}=\frac{c+d}{d-c}$. This is called componendo and dividendo.]

By componendo and dividendo

$$\frac{(1+x)+(1-x)}{(1-x)-(1+x)}=\frac{[\cos(k\pi/3)+i\sin(k\pi/3)]+1}{1-[\cos(k\pi/3)+i\sin(k\pi/3)]}$$

$$\therefore\quad \frac{1}{-x}=\frac{[\cos(k\pi/3)+i\sin(k\pi/3)]+1}{1-[\cos(k\pi/3)+i\sin(k\pi/3)]}\qquad \therefore\quad x=\frac{[\cos(k\pi/3)+i\sin(k\pi/3)]-1}{[\cos(k\pi/3)+i\sin(k\pi/3)]+1}$$

Putting $k=0, 1, 2, 3, 4, 5$, we get the six roots.

Example 9 (b) : Show that the roots of $(x+1)^7=(x-1)^7$ are given by

$$\pm i\cot\frac{r\pi}{7},\quad r=1, 2, 3. \qquad \textbf{(M.U. 2008)}$$

Sol. : We have $\left(\frac{x+1}{x-1}\right)^7=1=\cos 2\pi+i\sin 2\pi$

$$\therefore\quad \frac{x+1}{x-1}=1^{1/7}=(\cos 2k\pi+i\sin 2k\pi)^{1/7}$$

$$=\cos\frac{2k\pi}{7}+i\sin\frac{2k\pi}{7},\qquad k=0, 1, 2, 3, 4, 5, 6.$$

Let $\frac{2k\pi}{7}=\theta\quad\therefore\quad \frac{x+1}{x-1}=\frac{\cos\theta+i\sin\theta}{1}$.

By componendo and dividendo

$$\frac{x}{-1}=\frac{\cos\theta+i\sin\theta+1}{1-\cos\theta-i\sin\theta}=\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}$$

$$\therefore\quad \frac{x}{-1}=\frac{2\cos^2(\theta/2)+2i\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)-2i\sin(\theta/2)\cos(\theta/2)}$$

$$\therefore\quad x=\cot\frac{\theta}{2}\cdot\left(\frac{\cos(\theta/2)+i\sin(\theta/2)}{-\sin(\theta/2)+i\cos(\theta/2)}\right)$$

$$=\cot\frac{\theta}{2}\cdot\left(\frac{\cos(\theta/2)+i\sin(\theta/2)}{\cos[(\pi/2)+(\theta/2)]+i\sin[(\pi/2)+(\theta/2)]}\right)$$

$$=\cot\frac{\theta}{2}\cdot\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\cdot\left[\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right]^{-1}$$

$$=\cot\frac{\theta}{2}\cdot\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\cdot\left[\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right]$$

$$=\cot\frac{\theta}{2}\left[\cos\frac{\theta}{2}\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)-i\cos\frac{\theta}{2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right.$$
$$\left.+i\sin\frac{\theta}{2}\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)+\sin\frac{\theta}{2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right]$$

$$=\cot\frac{\theta}{2}\left[\left\{\cos\frac{\theta}{2}\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)+\sin\frac{\theta}{2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right\}\right.$$
$$\left.+i\left\{\sin\frac{\theta}{2}\cos\left(\frac{\pi}{2}+\frac{\theta}{2}\right)-\cos\frac{\theta}{2}\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)\right\}\right].$$

$$= \cot\frac{\theta}{2}\left[\cos\left(\frac{\theta}{2}-\frac{\pi}{2}-\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}-\frac{\pi}{2}-\frac{\theta}{2}\right)\right]$$

$$= \cot\frac{\theta}{2}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$$

$$\therefore \quad x = -i\cot\frac{\theta}{2} = -i\cot\left(\frac{k\pi}{7}\right); \quad k = 0, 1, 2, 3, 4, 5, 6. \qquad \left[\because \frac{\theta}{2}=\frac{k\pi}{7}\right]$$

We have to neglect $k = 0$ as it gives the root cot 0 which is infinite. Further, the given equation though appears to be of 7th degree it is actually of 6th degree, since on expansion of $(x+1)^7 = (x-1)^7$, x^7 will be cancelled from both sides. Hence, the roots are

$$x = -i\cot\frac{n\pi}{7}, \quad n = 1, 2, 3, 4, 5, 6$$

which can be expressed as

$$x = \pm i\cot\frac{r\pi}{7}, \quad r = 1, 2, 3.$$

Example 10 (b) : Show that the roots of the equation $(x+1)^6 + (x-1)^6 = 0$ are given by

$$-i\cot\frac{(2k+1)\pi}{12}, \qquad k = 0, 1, 2, 3, 4, 5.$$ **(M.U. 2012, 17)**

Sol. : We have $(x+1)^6 = -(x-1)^6$

$$\therefore \quad \left(\frac{x+1}{x-1}\right)^6 = -1 = \cos\pi + i\sin\pi = \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$\therefore \quad \frac{x+1}{x-1} = \cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}$$

Let $(2k+1)\frac{\pi}{6} = \theta \qquad \therefore \quad \frac{x+1}{x-1} = \frac{\cos\theta + i\sin n\theta}{1}.$

By componendo and dividendo, $\quad \dfrac{x}{-1} = \dfrac{\cos\theta + i\sin\theta + 1}{1-\cos\theta - i\sin\theta}$

Then as above, $\quad x = -i\cot\frac{\theta}{2} = -i\cot\left((2k+1)\cdot\frac{\pi}{12}\right); \quad k = 0, 1, 2, 3, 4, 5.$

Example 11 (b) : Solve $x^5 = 1 + i$ and find the continued product of the roots.

(M.U. 2004, 05, 14)

Sol. : We have $x^5 = 1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\therefore \quad x = 2^{1/10}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{1/5} = 2^{1/10}\left[\cos\left(2k\pi + \frac{\pi}{4}\right)\frac{1}{5} + i\sin\left(2k\pi + \frac{\pi}{4}\right)\frac{1}{5}\right]$$

$$= 2^{1/10}\left[\cos(8k+1)\frac{\pi}{20} + i\sin(8k+1)\frac{\pi}{20}\right]$$

The roots are obtained by putting $k = 0, 1, 2, 3, 4$. (Write down the roots)

The product of the roots $= 2^{5/10}(\cos\Phi + i\sin\Phi)$

where, $\Phi = \frac{\pi}{20} + \frac{9\pi}{20} + \frac{17\pi}{20} + \frac{25\pi}{20} + \frac{33\pi}{20} = \frac{85}{20}\cdot\pi = \frac{17}{4}\cdot\pi = 4\pi + \frac{\pi}{4}.$

$\therefore$ The product of the roots

$$= \sqrt{2}\left[\cos\left(4\pi + \frac{\pi}{4}\right) + i\sin\left(4\pi + \frac{\pi}{4}\right)\right] = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i.$$

Example 12 (b) : Prove that $\sqrt[n]{a+bi} + \sqrt[n]{a-bi}$ has n real values.

Hence, find those of $\sqrt[3]{1+i\sqrt{3}} + \sqrt[3]{1-i\sqrt{3}}$.

Sol. : Let $a = r\cos\theta$, $b = r\sin\theta$. $\therefore$ $r = \sqrt{a^2+b^2}$ and $\theta = \tan^{-1}(b/a)$

$$\therefore \quad a + bi = r(\cos\theta + i\sin\theta) = r[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)]$$

$$\therefore \quad (a+bi)^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

Changing the sign of i,

$$(a-bi)^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) - i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$$

$$\therefore \quad \sqrt[n]{a+bi} + \sqrt[n]{a-bi} = 2r^{1/n}\cos\left(\frac{2k\pi+\theta}{n}\right)$$

Putting $k = 0, 1, 2, \ldots, (n-1)$, we get n values which are all real.

For the second part put $a = 1$, $b = \sqrt{3}$, $n = 3$.

$$\therefore \quad r = \sqrt{1+3} = 2, \quad \theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}.$$

$$\therefore \quad \sqrt[3]{1+i\sqrt{3}} + \sqrt[3]{1-i\sqrt{3}} = 2 \cdot 2^{1/3}\cos\left(\frac{2k\pi+(\pi/3)}{3}\right), \quad k = 0, 1, 2.$$

Note ....

If $\sqrt[n]{a+ib} = \alpha + i\beta$ then $\sqrt[n]{a-ib} = \alpha - i\beta$. Note this carefully.

Example 13 (b) : Solve the equation $z^3 = (z+1)^3$ and show that the real part of all roots is $-1/2$. **(M.U. 2013)**

Sol. : We have $\left(\frac{z}{z+1}\right)^3 = 1 = \cos 0 + i\sin 0 = \cos 2n\pi + i\sin 2n\pi$

$$\therefore \quad \frac{z}{z+1} = (\cos 2n\pi + i\sin 2n\pi)^{1/3} = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

$$\therefore \quad \frac{z}{z+1} = \frac{\cos\theta + i\sin\theta}{1} \quad \text{where } \theta = \frac{2n\pi}{3}. \qquad \text{.......................(A)}$$

By subtracting the numerator from the denominator on both sides, we get,

$$\frac{z}{z+1-z} = \frac{\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$$

$$\therefore \quad z = \frac{\cos\theta + i\sin\theta}{2\sin^2(\theta/2) - 2i\sin(\theta/2)\cdot\cos(\theta/2)}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin(\theta/2)[\sin(\theta/2) - i\cos(\theta/2)]} \cdot \frac{[\sin(\theta/2) + i\cos(\theta/2)]}{[\sin(\theta/2) + i\cos(\theta/2)]}$$ [Note this

$$= \frac{(\cos\theta + i\sin\theta)[\sin(\theta/2) + i\cos(\theta/2)]}{2\sin(\theta/2)[\sin^2(\theta/2) + \cos^2(\theta/2)]}$$

$$= \frac{\cos\theta\sin(\theta/2) + i\cos\theta\cos(\theta/2) + i\sin\theta\sin(\theta/2) - \sin\theta\cos(\theta/2)}{2\sin(\theta/2)}$$

$$= \frac{-[\sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2)] + i[\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)]}{2\sin(\theta/2)}$$

$$\therefore \quad z = \frac{-\sin(\theta/2) + i\cos(\theta/2)}{2\sin(\theta/2)} = -\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}, \quad \text{where } \theta = 2n\frac{\pi}{3}.$$

For $n = 0, 1, 2$, we get the three roots.

It is clear that, all these roots have the real part $-1/2$.

Alternatively : We may find from (A)

$$\frac{z + (z+1)}{z - (z+1)} = \frac{\cos\theta + i\sin\theta + 1}{(\cos\theta + i\sin\theta) - 1} \qquad \therefore \quad \frac{2z+1}{-1} = \frac{(1+\cos\theta) + i\sin\theta}{(-1+\cos\theta) + i\sin\theta}$$

$$-(2z+1) = \frac{2\cos^2(\theta/2) + 2i\sin(\theta/2)\cos(\theta/2)}{-2\sin^2(\theta/2) + 2i\sin(\theta/2)\cos(\theta/2)}$$

$$= \frac{\cos^2(\theta/2) + i\sin(\theta/2)\cos(\theta/2)}{i^2\sin^2(\theta/2) + i\sin(\theta/2)\cos(\theta/2)}$$

$$= \frac{\cos(\theta/2)[\cos(\theta/2) + i\sin(\theta/2)]}{i\sin(\theta/2)[\cos(\theta/2) + i\sin(\theta/2)]}$$

$$= \frac{1}{i}\cot\left(\frac{\theta}{2}\right) \qquad \left[\text{But } \frac{1}{i} = -\frac{i^2}{i} = -i\right]$$

$$\therefore \quad -(2z+1) = -i\cot\left(\frac{\theta}{2}\right) \qquad \therefore \quad 2z+1 = i\cot\left(\frac{\theta}{2}\right)$$

$$\therefore \qquad z = -\frac{1}{2} + \frac{i}{2}\cot\left(\frac{\theta}{2}\right).$$

Example 14 (b) : Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$.

Sol. : We have $\left(\frac{z}{z-1}\right)^4 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$\therefore \quad \frac{z}{z-1} = \left[\cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{1/4}$$

$$= \cos(4n+1)\frac{\pi}{8} + i\sin(4n+1)\frac{\pi}{8}$$

$$\therefore \quad \frac{z}{z-1} = \frac{\cos\theta + i\sin\theta}{1} \qquad \text{where } \theta = (4n+1)\frac{\pi}{8}$$

$$\therefore \quad \frac{z}{z-1-z} = \frac{\cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta}$$

Simplifying as in the above Ex. 13, we get,

$$\therefore \quad \frac{z}{-1} = \frac{-\sin(\theta/2) + i\cos(\theta/2)}{2\sin(\theta/2)}$$

$$\therefore \quad z = \frac{1}{2} - i\cot\frac{\theta}{2}, \quad \text{where } \theta = (4n+1)\frac{\pi}{8}.$$

For $n = 0, 1, 2$, we get the three roots.

It is clear that, all these roots have the real part $1/2$.

Restatement : Show that the points representing the roots of $z^4 = i(z-1)^4$ on rhe Argand diagram are collinear.

Sol. : Same as above.

Example 15 (b) : If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them and show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$. **(M.U. 1997, 2007, 11, 16, 19)**

Show that the roots of $x^5 - 1 = 0$ can be written as $1, \alpha, \alpha^2, \alpha^3, \alpha^4$. Hence, show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^5) = 5$. **(M.U. 2016)**

Sol. : We have $\quad x^5 = 1 = \cos 0 + i\sin 0$

$$\therefore \quad x^5 = \cos(2k\pi) + i\sin(2k\pi)$$

$$\therefore \quad x = (\cos 2k\pi + i\sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

Putting $k = 0, 1, 2, 3, 4$, we get the five roots as

$$x_0 = \cos 0 + i\sin 0 = 1, \quad x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \quad x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$$

$$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \quad x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}.$$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$, we see that $x_2 = \alpha^2, \; x_3 = \alpha^3, \; x_4 = \alpha^4$.

$\therefore$ The roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ and hence

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore \quad \frac{x^5-1}{x-1} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore \quad (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$$

Putting $x = 1$, we get

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^5) = 5.$$

Example 16 (b) : If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + x^3 + x^2 + x + 1 = 0$, find them and show that $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = 5$. **(M.U. 2011, 16)**

Sol. : Multiplying the given equation by $x - 1$, we get

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = 0 \quad \therefore \quad x^5 - 1 = 0 \quad \therefore \quad x = 1^{1/5}.$$

Now, $\quad x = (\cos\theta + i\sin\theta)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}.$

Putting $k = 0, 1, 2, 3, 4$, we get the roots of the equation

$$x_0 = \cos 0 + i\sin 0 = 1, \quad x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \quad x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$$

$$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \quad x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}.$$

It is clear that 1 is the root of $x - 1 = 0$ and the remaining roots are the roots of

$$x^4 + x^3 + x^2 + x + 1 = 0$$

By data, $\quad x_1 = \alpha, \; x_2 = \beta, \; x_3 = \gamma, \; x_2 = \delta$

$$\therefore \quad (x^4 + x^3 + x^2 + x + 1) = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$$

Putting $x = 1$, we get

$$(1 - \alpha)(1 - \beta)(1 - \gamma)(1 - \delta) = 1 + 1 + 1 + 1 + 1 = 5.$$

Example 17 (b) : Prove that

$$x^5 - 1 = (x - 1)\left(x^2 + 2x\cos\frac{\pi}{5} + 1\right)\left(x^2 + 2x\cos\frac{3\pi}{5} + 1\right) = 0. \qquad \textbf{(M.U. 2000)}$$

Sol. : As seen in the above two examples, the roots of the equation $x^5 - 1 = 0$ are

$$x_1 = \cos 0 + i\sin 0 = 1,$$

$$x_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$x_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = -\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$$

$$x_4 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = -\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$$

$$x_5 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

Now consider

$$x = \cos\frac{2\pi}{5} \pm i\sin\frac{2\pi}{5} \qquad \therefore \left(x - \cos\frac{2\pi}{5}\right)^2 = \left(\pm i\sin\frac{2\pi}{5}\right)^2$$

$$\therefore \quad x^2 - 2x\cos\frac{2\pi}{5} + \cos^2\frac{2\pi}{5} = -\sin^2\frac{2\pi}{5}$$

$$\therefore \quad x^2 - 2x\cos\frac{2\pi}{5} + 1 = 0 \qquad \therefore \quad x^2 - 2x\cos\left(\pi - \frac{3\pi}{5}\right) + 1 = 0$$

$$\therefore \quad x^2 + 2x\cos\frac{3\pi}{5} + 1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly consider,

$$x = -\cos\frac{\pi}{5} \pm i\sin\frac{\pi}{5} \qquad \therefore \left(x + \cos\frac{\pi}{5}\right)^2 = \left(\pm i\sin\frac{\pi}{5}\right)^2$$

$$\therefore \quad x^2 + 2x\cos\frac{\pi}{5} + \cos^2\frac{\pi}{5} = -\sin^2\frac{\pi}{5}$$

$$\therefore \quad x^2 + 2x\cos\frac{\pi}{5} + 1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Since 1 is also a root, $x = 1 \quad \therefore \; x - 1 = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$

From (1), (2) and (3),

$$(x - 1)\left(x^2 + 2\cos\frac{\pi}{5} + 1\right)\left(x^2 + 2x\cos\frac{3\pi}{5} + 1\right) = 0.$$

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

1. Find all the value of the following :

(i) $(-1)^{1/5}$ (ii) $(1)^{1/4}$ (iii) $(-i)^{1/3}$ (iv) $(1+i)^{2/3}$

(v) $(32)^{1/5}$ (vi) $(1-i)^{2/3}$ **(M.U. 1992)**

(vii) $\sqrt[3]{1+i\sqrt{3}}+\sqrt[3]{1-i\sqrt{3}}$, (viii) $\sqrt[3]{(1+i)/\sqrt{2}}+\sqrt[3]{(1-i)/\sqrt{2}}$

[**Ans. :** (i) -1, $\cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5}$, $\cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$, (ii) $\pm 1, \pm i$,

(iii) $-i, \pm\frac{\sqrt{3}}{2}-\frac{i}{2}$, (iv) $2^{1/3}\left(\cos\frac{r\pi}{6}-i\sin\frac{r\pi}{6}\right)$, $r=1, 5, 9.$

(v) 2, $2\left(\cos\frac{r\pi}{5}\pm i\sin\frac{r\pi}{5}\right)$, $r=2, 4.$ (vi) $2^{1/3}\left(\cos\frac{r\pi}{6}-i\sin\frac{r\pi}{6}\right)$, $r=1, 5, 9.$

(vii) $2\cdot 2^{1/3}\cos\frac{r\pi}{9}$, $r=1, 7, 13$ (viii) $2\cos\frac{r\pi}{12}$, $r=1, 9, 17$.]

2. Find the continued product of all the values of $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^{3/4}$. [**Ans. :** 1]

3. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$. **(M.U. 2002)** [**Ans. :** 1]

4. Solve the equation $x^4+x^3+x^2+x+1=0$.

[**Ans. :** $\cos\frac{2\pi}{5}\pm i\sin\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}\pm i\sin\frac{4\pi}{5}$]

5. Find all the values of $(1-i\sqrt{3})^{1/4}$.

[**Ans. :** $2^{1/4}\left[\cos\frac{(6k+5)\pi}{12}+i\sin\frac{(6k+5)\pi}{12}\right]$ where $k=0, 1, 2, 3$]

6. Find all the values of $(1+i)^{2/3}$ and find the continued product of these values.

(M.U. 1992) [**Ans. :** $2^{1/3}\left(\cos\frac{8\pi k+\pi}{6}+i\sin\frac{8\pi k+\pi}{6}\right)$, $k=0, 1, 2$. Product $=2i$.]

7. Find all the values of $(1+i)^{1/3}$. **(M.U. 2017)**

Also show that their continued product is $(1+i)$. **(M.U. 2018)**

[**Ans. :** $2^{1/6}\left(\cos\frac{r\pi}{12}+i\sin\frac{r\pi}{12}\right)$ where $r=1, 9, 17$.]

8. Find the four fourth roots of unity. [**Ans. :** $\pm 1, \pm i$]

9. Solve the equations

(i) $x^9+8x^6+x^3+8=0$

(ii) $x^5+1=0$ **(M.U. 2005)**

(iii) $x^4-x^3+x^2-x+1=0$ **(M.U. 1988, 2003, 18)**

(iv) $(x+1)^8+x^8=0$ **(M.U. 1990, 97, 2003)**

(v) $x^3=i(x-1)^3$ **(M.U. 2008)**

(vi) $x^6-x^5+x^4-x^3+x^2-x+1=0$ **(M.U. 2012)**

[**Ans. :** (i) $\cos(2k+1)\dfrac{\pi}{6}+i\sin(2k+1)\dfrac{\pi}{6}$, $k=0,1,2,3,4,5$ and

$$2^{1/3}\left[\cos(2k+1)\frac{\pi}{3}+i\sin(2k+1)\frac{\pi}{3}\right],\quad k=0,1,2.$$

(ii) $\cos(2k+1)\dfrac{\pi}{5}+i\sin(2k+1)\dfrac{\pi}{5}$, $k=0,1,2,3,4$.

(iii) Multiply by $x+1$ $\therefore$ $x^5+1=0$. Answer as above, *i.e.*, (ii).

(iv) $x=\dfrac{1}{[\cos(2k+1)(\pi/8)+i\sin(2k+1)(\pi/8)-1]}$

where $k=0,1,2,3,4,5,6,7$.

(v) $x=\dfrac{\cos(4k+1)\pi/6+i\sin(4k+1)\pi/6}{\cos(4k+1)\pi/6+i\sin(4k+1)\pi/6-1}$.

(vi) Multiply by $x+1$; $x^7+1=0$

$$\cos(2k+1)\frac{\pi}{7}+i\sin(2k+1)\frac{\pi}{7};\quad k=0,1,2,3,4,5,6.\ ]$$

10. Show that the continued product of all the values of

(a) $i^{2/3}$ is -1, (b) $(-i)^{2/3}$ is -1

11. If one root of $x^4-6x^3+15x^2-18x+10=0$ is $1+i$, find all other roots.

[**Ans. :** $1-i$, $2\pm i$]

12. If one root of $x^4-6x^3+18x^2-24x+16=0$ is $1+i$, find all other roots.

[**Ans. :** $1-i$, $2(1\pm i)$]

13. Solve the equations

(i) $x^{12}-1=0$ **(M.U. 2003)**

(ii) $x^7+x^4+x^3+1=0$ **(M.U. 1988, 95, 2002, 15)**

(iii) $x^7-x^4+x^3-1=0$

(iv) $x^7-x^4-x^3+1=0$

(v) $x^7+x^4+ix^3+i=0$ **(M.U. 1999, 2015)**

(vi) $x^9-x^5+x^4-1=0$ **(M.U. 2000, 01)**

(vii) $x^7+64x^4+x^3+64=0$ **(M.U. 1985, 95)**

[**Ans. :** (i) ±1, $\pm i$, $\pm\left(\cos\dfrac{\pi}{6}\pm i\sin\dfrac{\pi}{6}\right)$, $\pm\left(\cos\dfrac{\pi}{3}\pm i\sin\dfrac{\pi}{3}\right)$

(ii) $\pm\left(\dfrac{1}{\sqrt2}\pm i\cdot\dfrac{1}{\sqrt2}\right)$, $\dfrac12\pm i\cdot\dfrac{\sqrt3}{2}$, -1

(iii) 1, $-\dfrac12(1\pm i\sqrt3)$, $\pm\left(\dfrac{1}{\sqrt2}\pm i\cdot\dfrac{1}{\sqrt2}\right)$ (iv) ±1, $\pm i$, $-\dfrac12(1\pm i\sqrt3)$

(v) $\pm\left(\cos\dfrac{\pi}{8}+i\sin\dfrac{\pi}{8}\right)$, $\pm\left(\cos\dfrac{\pi}{8}-i\sin\dfrac{\pi}{8}\right)$, $\dfrac12\pm i\dfrac{\sqrt3}{2}$, -1

(vi) $\cos(2k+1)\dfrac{\pi}{5}+i\sin(2k+1)\dfrac{\pi}{5}$; $k=0,1,2,3,4$

$$\cos\frac{2k\pi}{4}+i\sin\frac{2k\pi}{4},\quad nk=0,1,2,3.$$

(vii) -4, $2(1\pm i\sqrt3)$, $\pm\left(\dfrac{1}{\sqrt2}\right)(1\pm i)$.]

14. Find all the value of $\left(\dfrac{2+3i}{1+i}\right)^{1/4}$. **(M.U. 1985)**

[**Ans. :** $(13)^{1/4}\left[\cos\left(\dfrac{2\pi k+\theta}{4}\right)+i\sin\left(\dfrac{2\pi k+\theta}{4}\right)\right]$

where, $\theta=\tan^{-1}(1/5)$, $k=0, 1, 2, 3.$]

15. If $\alpha, \alpha^2, \alpha^3, \ldots, \alpha^6$ are the roots of $x^7-1=0$, find them and prove that $(1-\alpha)(1-\alpha^2)\ldots(1-\alpha^6)=7$. **(M.U. 2010)**

16. Use De Moivre's Theorem to solve the equation

(i) $x^4-x^2+1=0$ **(M.U. 1996)** (ii) $x^4+x^2+1=0$

[**Ans. :** (i) $\cos(2k+1)\dfrac{\pi}{6}+i\sin(2k+1)\dfrac{\pi}{6}$, $k=0, 1, 2, 3, 4, 5.$

(ii) $x=\cos\dfrac{2k\pi}{6}+i\sin\dfrac{2k\pi}{6}$, $k=0, 1, 2, 3, 4, 5.$]

17. Solve $x^{14}+127x^7-128=0$. **(M.U. 1998)**

[**Ans. :** $x=2\left[\cos(2k+1)\dfrac{\pi}{7}+i\sin(2k+1)\dfrac{\pi}{7}\right]$, $x=0, 1, 2, 3, 4, 5, 6$

and $x=\cos\dfrac{2k\pi}{7}+i\sin\dfrac{2k\pi}{7}$, $k=0, 1, 2, 3, 4, 5, 6.$]

18. Show that the roots of $(x-1)^5=32(x+1)^5$ are given by

$$x=\left(-3+4i\sin\frac{2n\pi}{5}\right)\Big/\left(5-4\cos\frac{2n\pi}{5}\right).$$

19. Find all the roots of $x^{12}-1=0$ and identify the roots which are also the roots of $x^4-x^2+1=0$.

[**Ans. :** $x=\cos(2m+1)\dfrac{\pi}{6}+i\sin(2m+1)\dfrac{\pi}{6}$, $m=0, 2, 3, 5$ give common roots.]

20. If $(1+x)^6+x^6=0$ show that $x=-\dfrac{1}{2}-i\cot\dfrac{\theta}{2}$ where $\theta=(2k+1)\dfrac{\pi}{6}$, $k=0, 1, 2, 3, 4, 5.$ **(M.U. 2003)**

21. If $(x+1)^6=x^6$, show that $x=-\dfrac{1}{2}-i\cot\dfrac{\theta}{2}$, where $\theta=\dfrac{2k\pi}{6}$, $k=0, 1, 2, 3, 4, 5.$

22. Show that the continued product of all the values of

(i) $(1+i)^{1/8}$ is $-(1+i)$ (ii) $(1-i)^{1/8}$ is $-(1-i)$

(iii) $(1+i)^{1/5}$ is $1+i$ (iv) $(1-i)^{1/5}$ is $1-i$.

23. Find the cube roots of unity. Prove further that if α, β are complex roots then $\alpha^{3n}+\beta^{3n}=2$ where n is any integer.

24. Find the cube roots of $1-\cos\theta-i\sin\theta$.

[**Ans. :** $\left(2\sin\dfrac{\theta}{2}\right)^{1/3}\left[\cos\left(\dfrac{(2n-1)\pi-\theta}{6}\right)+i\sin\left(\dfrac{(2n-1)\pi-\theta}{6}\right)\right]$

25. Find all the roots of the equation $z^n=(z+1)^n$ and show that the real part of all the roots is $-1/2$.

6. Use of Exponential Form of a Complex Number

We shall now solve some problems based on Exponential form of a Complex Number. We know that

$z = x + iy$	(Cartesian Form)
$z = r(\cos\theta + i\sin\theta)$	(Polar Form)
$z = re^{i\theta}$	(Exponential Form)

Note ....

Note the following representations :

(i) $1 = \cos 2n\pi + i\sin 2n\pi = e^{i2n\pi}$ (ii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

(iii) $\sqrt{i} = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{1/2} = \left[e^{i\pi/2}\right]^{1/2} = e^{i\pi/4}$

Class (a) : 3 Marks

Example 1 (a) : Prove that i^i is real and find the value of $\sin\log_e i^i$.

Sol. : We have $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$\therefore \quad i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2} \quad \therefore \quad \boxed{i^i = e^{-\pi/2}}$

This shows that i^i is real.

Now, $\sin\log i^i = \sin\log_e(e^{-\pi/2}) = \sin\left(-\frac{\pi}{2}\right) = -1 \quad \therefore \; 1 + \sin\log i^i = 0.$

Similarly, we get $\sin\log i^{-i} = 1$. **(M.U. 2003)**

Cor. : $(i^i)^n = (e^{-\pi/2})^n = e^{-n\pi/2}$ e.g., $i^{3i} = e^{-3\pi/2}$; $i^{-5i} = e^{5\pi/2}$.

(For another method, see Ex. 6 on page 4-12.)

Example 2 (a) : Separate into real and imaginary parts $(\sqrt{i})^{\sqrt{i}}$. **(M.U. 2004)**

Sol. : We shall use exponential form for base $\sqrt{i}$ and standard form for exponent $\sqrt{i}$.

We have $\quad \sqrt{i} = i^{1/2} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}$

Also $\quad \sqrt{i} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = (e^{i\pi/2})^{1/2} = e^{i\pi/4}$

$$\therefore \quad (\sqrt{i})^{\sqrt{i}} = \left\{e^{i\pi/4}\right\}^{\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right)} = e^{i\pi/4\sqrt{2} - \pi/4\sqrt{2}}$$

$$= e^{-\pi/4\sqrt{2}} \cdot e^{i\pi/4\sqrt{2}} = e^{-\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$$

$\therefore \quad$ Real part $= e^{-\pi/4\sqrt{2}}\cos\left(\frac{\pi}{4\sqrt{2}}\right)$; $\quad$ Imaginary part $= e^{-\pi/4\sqrt{2}}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$.

(For another method, see Ex. 3 (iii) on page 4-11.)

Example 3 (a) : Separate into real and imaginary parts z^z where $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Sol. : We have, $z = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos 60° + i\sin 60° = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = e^{i\pi/3}$

$$\therefore \quad z^z = (e^{i\pi/3})^{(1/2)+i(\sqrt{3}/2)} = e^{(i\pi/6)-(\pi/2\sqrt{3})}$$

$$= e^{-(\pi/2\sqrt{3})+i(\pi/6)} = e^{-(\pi/2\sqrt{3})} \cdot e^{i\pi/6}$$

$$= e^{-\pi/2\sqrt{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = e^{-\pi/2\sqrt{3}}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right).$$

$\therefore$ Real part $= e^{-\pi/2\sqrt{3}} \cdot \frac{\sqrt{3}}{2}$; Imaginary part $= e^{-\pi/2\sqrt{3}} \cdot \frac{1}{2}$.

Example 4 (a) : If $i^{i^{\cdot^{\cdot^{\cdot}}}\text{ad. inf.}} = A + iB$, prove that $A^2 + B^2 = e^{-\pi B}$ and $\tan\left(\frac{\pi}{2}A\right) = B$.

(M.U. 1999, 2002, 15)

Sol. : We have by data $i^{A+iB} = A + iB$. But $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$.

$$\therefore \quad \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{A+iB} = A + iB \qquad \left[\because \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i(\pi/2)}\right]$$

$$\therefore \quad e^{(i\pi/2)(A+iB)} = A + iB \quad \therefore \quad e^{-\pi(B/2)} \cdot e^{i\pi A/2} = A + iB$$

$$\therefore \quad e^{-\pi B/2}\left[\cos\left(\frac{\pi}{2}A\right) + i\sin\left(\frac{\pi}{2}A\right)\right] = A + iB.$$

$$\therefore \quad e^{-\pi B/2} \cdot \cos\left(\frac{\pi}{2}A\right) = A \text{ and } e^{-\pi B/2} \cdot \sin\left(\frac{\pi}{2}A\right) = B$$

$$\therefore \quad A^2 + B^2 = e^{-\pi B} \text{ and } \tan\left(\frac{\pi}{2}A\right) = \frac{B}{A}.$$

Alternatively : $i^{A+iB} = A + iB$

Since $x = e^{\log_e x}$, we get

$$i^{A+iB} = e^{\log i^{(A+iB)}} = e^{(A+iB)\log i}$$

$$\therefore \quad A + iB = e^{(A+iB)\log i} = e^{(A+iB)\left[\log\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]}$$

$$= e^{(A+iB)\log(e^{i\pi/2})} = e^{(A+iB)(i\pi/2)}$$

$$= e^{-\pi B/2} \cdot e^{iA\pi/2} = e^{-B\pi/2}\left(\cos\frac{\pi A}{2} + i\sin\frac{\pi A}{2}\right)$$

Equating real and imaginary parts,

$$A = e^{-B\pi/2}\cos\frac{\pi A}{2}, \quad B = e^{-B\pi/2}\sin\frac{\pi A}{2}$$

Dividing B by A, $\frac{B}{A} = \tan\frac{\pi A}{2}$

Squaring and adding, $A^2 + B^2 = e^{-B\pi}$.

Example 5 (a) : Prove that $\sqrt{1+\operatorname{cosec}(\theta/2)} = (1-e^{i\theta})^{-1/2} + (1-e^{-i\theta})^{-1/2}$.

(M.U. 2004, 05, 06)

Sol. : We have to show that

$$\sqrt{\left(1+\operatorname{cosec}\frac{\theta}{2}\right)} = \frac{1}{\sqrt{(1-e^{i\theta})}} + \frac{1}{\sqrt{(1-e^{-i\theta})}}$$

Squaring both sides, we get

$$1+\operatorname{cosec}\frac{\theta}{2} = \frac{1}{1-e^{i\theta}} + \frac{1}{1-e^{-i\theta}} + \frac{2}{\sqrt{(1-e^{i\theta})(1-e^{-i\theta})}}$$

We shall prove this result.

Now, $\quad \text{r.h.s.} = \dfrac{1-e^{-i\theta}+1-e^{i\theta}}{1-e^{-i\theta}-e^{i\theta}+1} + \dfrac{2}{\sqrt{1-e^{-i\theta}-e^{i\theta}+1}}$

$$= 1 + \frac{2}{\sqrt{2-(e^{i\theta}+e^{-i\theta})}} = 1 + \frac{2}{\sqrt{2-2\cos\theta}} \qquad \text{[See (6), page 1-6]}$$

$$= 1 + \frac{2}{\sqrt{2(1-\cos\theta)}} + \frac{2}{\sqrt{4\sin^2(\theta/2)}}$$

$$= 1 + \frac{2}{2\sin(\theta/2)} = 1+\operatorname{cosec}\frac{\theta}{2} = \text{l.h.s.}$$

Example 6 (a) : If $i^z = z$ where $z = x+iy$, prove that $|i^z|^2 = e^{-(4n+1)\pi y}$, where $n = 0, 1, 2, \ldots$

(M.U. 1989, 95)

Sol. : $\quad i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = \cos\left(2n\pi+\dfrac{\pi}{2}\right) + i\sin\left(2n\pi+\dfrac{\pi}{2}\right)$

$$= e^{i(2n\pi+(\pi/2))}, \qquad n = 0, 1, 2, \ldots$$

$$\therefore \quad i^z = e^{i(2n\pi+(\pi/2))\cdot z} = e^{i(2n\pi+(\pi/2))(x+iy)}$$

$$= e^{i(2n\pi+(\pi/2))\cdot x} \cdot e^{-(2n\pi+(\pi/2))y}$$

$$\therefore \quad |i^z| = e^{-(4n+1)\pi y/2} \qquad \therefore \quad |i^z|^2 = e^{-(4n+1)\pi y}.$$

EXERCISE - V

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. Prove that i^{-i} is real and hence show that $\sin\log(i^{-i}) = 1$. **(M.U. 2003)**
2. Separate into real and imaginary parts $(\sqrt{-i})^{\sqrt{-i}}$.

$$\left[\textbf{Ans. :}\ e^{-\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}} - i\sin\frac{\pi}{4\sqrt{2}}\right)\right]$$

3. Prove that
   (i) $\sqrt{1-\operatorname{cosec}(\theta/2)} = (1-e^{i\theta})^{-1/2} - (1-e^{-i\theta})^{-1/2}$
   (ii) $\sqrt{1+\sec(\theta/2)} = (1+e^{i\theta})^{-1/2} + (1+e^{-i\theta})^{-1/2}$
   (iii) $\sqrt{1-\sec(\theta/2)} = (1+e^{i\theta})^{-1/2} - (1+e^{-i\theta})^{-1/2}$

EXERCISE - VI

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Short Answer Questions : Class (a) : 3 Marks

1. Find the value of i^i. [**Ans. :** $e^{-\pi/2}$]
2. Find the roots of $x^4 = 1$. [**Ans. :** $\pm 1, \pm i$]
3. If $x = \cos\theta + i\sin\theta$, find the value of $x^n - \dfrac{1}{x^n}$. [**Ans. :** $2i \sin n\theta$]
4. If $x = \cos\theta + i\sin\theta$, then find the value of $x^6 + \dfrac{1}{x^6}$. [**Ans. :** $2\cos 6\theta$]
5. If $x = e^{i\theta}$, $y = e^{-i\theta}$, then find the value of $x^n - y^n$. [**Ans. :** $2i \sin n\theta$]
6. If $x = \cos\theta + i\sin\theta$, $y = \cos\theta - i\sin\theta$, then find the value of $x^n + y^n$. [**Ans. :** $2\cos n\theta$]
7. Find the modulus and amplitude of $1 + i$. [**Ans. :** $\sqrt{2}, \pi/4$]
8. Find the modulus of $\tan\alpha + i$. [**Ans. :** $\sec\alpha$]
9. Find the modulus and amplitude of $\sqrt{3} + i$. [**Ans. :** $2, \pi/6$]
10. Find $\left[\sin\dfrac{\pi}{3} + i\cos\dfrac{\pi}{3}\right]^2$. [**Ans. :** $\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$]
11. Find $\left(\sqrt{i}\right)^i$. [**Ans. :** $e^{-\pi/4}$]
12. Find the real part of $e^{2+i\pi}$. [**Ans. :** $-e^2$]
13. Find the real part of $\sqrt{i}$. [**Ans. :** $1/\sqrt{2}$]
14. Find the value of $\log(i^i)$. [**Ans. :** $-\dfrac{\pi}{2}$]
15. Find the value of $i^{60} + i^{62}$. [**Ans. :** 0]

Summary

1. De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

$$\frac{1}{(\cos\theta + i\sin\theta)^n} = \cos n\theta - i\sin n\theta$$

$$(\cos\theta - i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-n}$$

(2) $$(\cos\theta + i\sin\theta)^n = {}^nC_0 \cos^n\theta + {}^nC_1 \cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2 \cos^{n-2}\theta\,(i\sin\theta)^2 + \ldots\ldots$$
$$= [\,{}^nC_0 \cos^n\theta - {}^nC_2 \cos^{n-2}\theta \sin^2\theta + \ldots\ldots\,] + i\,[\,{}^nC_1 \cos^{n-1}\theta \sin\theta - {}^nC_3 \cos^{n-3}\theta \sin^3\theta + \ldots\,]$$

(3) $$(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)$$

☆ ☆ ☆

CHAPTER

3 Hyperbolic Functions

1. Introduction

In this chapter, we shall define circular functions of complex numbers and then define new functions viz. hyperbolic functions. Later on we shall learn how to separate real and imaginary parts of functions of complex variables.

2. Circular Functions of Complex Numbers

By addition and subtraction of (4) and (5) given on page 1-5, we get,

$$\boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}} \quad \text{and} \quad \boxed{\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}}$$

These forms are known as **Euler's Exponential Forms of circular functions.**

Further, if $z = x + iy$ is a complex number we define

$$\boxed{\cos z = \frac{e^{iz}+e^{-iz}}{2}} \quad \text{and} \quad \boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$$

These are called **circular functions** of complex numbers.

Type I : On Euler's Form of Circular Functions : Class (a) : 3 Marks

Example 1 (a) : Using Euler's Form, prove that $\sin^2\theta + \cos^2\theta = 1$.

Sol. : l.h.s. $= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2 + \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^2$

$$= \frac{e^{2i\theta}-2+e^{-2i\theta}}{4i^2} + \frac{e^{2i\theta}+2+e^{-2i\theta}}{4}$$

$$= \frac{1}{4}[-e^{2i\theta}+2-e^{-2i\theta}+e^{2i\theta}+2+e^{-2i\theta}] = \frac{1}{4}(4) = 1.$$

Example 2 (a) : Prove that $[\sin(\alpha+\theta) - e^{i\alpha}\sin\theta]^n = \sin^n\alpha\, e^{-in\theta}$.

Sol. : Using $e^{\pm i\alpha} = \cos\alpha \pm i\sin\alpha$

l.h.s. $= [\sin\alpha\cos\theta + \cos\alpha\sin\theta - (\cos\alpha + i\sin\alpha)\sin\theta]^n$

$= [\sin\alpha\cos\theta - i\sin\alpha\sin\theta]^n$

$= \sin^n\alpha\,(\cos\theta - i\sin\theta)^n = \sin^n\alpha\cdot(e^{-i\theta})^n$

$= \sin^n\alpha\cdot e^{-in\theta}$.

Example 3 (a) : Prove that $\cos^{-1} z = -i\log(z \pm \sqrt{z^2 - 1})$, $\tan^{-1} z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$.

(M.U. 1996)

Sol. : (i) Let $\cos^{-1} z = u \quad \therefore \quad z = \cos u$

But $\cos u = \frac{e^{iu} + e^{-iu}}{2} \quad \therefore \quad \frac{e^{iu} + e^{-iu}}{2} = z \quad \therefore \quad e^{iu} + \frac{1}{e^{iu}} = 2z$

$\therefore \quad e^{2iu} - 2z\,e^{iu} + 1 = 0$

Solving this as a quadratic in e^{iu}.

$$e^{iu} = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z \pm \sqrt{z^2 - 1}$$

$$\therefore \quad iu = \log(z \pm \sqrt{z^2 - 1}) \quad \therefore \quad u = -i\log(z \pm \sqrt{z^2 - 1})$$

(ii) Let $\tan^{-1} z = u \quad \therefore \quad z = \tan u$

$$\therefore \quad z = \frac{\sin u}{\cos u} = \frac{e^{iu} - e^{-iu}}{i(e^{iu} + e^{-iu})} \quad \therefore \quad \frac{z}{i} = \frac{e^{iu} - e^{-iu}}{i^2(e^{iu} + e^{-iu})} = \frac{e^{-iu} - e^{iu}}{e^{iu} + e^{-iu}}$$

By componendo and dividendo,

$$\therefore \quad \frac{i+z}{i-z} = \frac{2e^{-iu}}{2e^{iu}} = e^{-2iu}$$

$$\therefore \quad e^{-2iu} = \frac{i+z}{i-z} \quad \therefore \quad -2iu = \log\left(\frac{i+z}{i-z}\right)$$

$$u = \frac{1}{-2i}\log\left(\frac{i+z}{i-z}\right) \quad \therefore \quad u = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right).$$

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 marks

1. Prove the following results for circular functions of complex numbers.
   (a) $1 + \cos 2z = 2\cos^2 z$, (b) $1 - \cos 2z = 2\sin^2 z$,
   (c) $\sin 3z = 3\sin z - 4\sin^3 z$, (d) $\cos 3z = 4\cos^3 z - 3\cos z$.

2. Prove that (i) $[\sin(\alpha - \theta) + e^{i\alpha}\sin\theta]^n = \sin^n\alpha \cdot e^{in\theta}$
   (ii) $[\cos(\alpha + \theta) + i\,e^{-i\alpha}\sin\theta]^n = \cos^n\alpha \cdot e^{in\theta}$

3. Prove that (i) $\frac{1 + \cos\theta + i\sin\theta}{1 - \cos\theta + i\sin\theta} = \cot\frac{\theta}{2} \cdot e^{i(\theta - \pi/2)}$
   (ii) $\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = e^{in(\pi/2 - \theta)}$ **(M.U. 2002)**
   (iii) $\frac{1 + \cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta} = i\cot\left(\frac{\theta}{2}\right)$

4. Prove that $\sin^{-1} z = -i\log(iz \pm \sqrt{1 - z^2})$. **(M.U. 2006)**

3. Hyperbolic Functions

In this article we shall learn a new set of functions called hyperbolic functions. **The hyperbolic functions have the same relationship with hyperbolas, which the circular functions have with circles.** (See note in § 4, page 3-4).

Definition : If x is real or complex $\dfrac{e^x + e^{-x}}{2}$ is called **hyperbolic cosine** of x and is denoted by $\cos h\, x$ and $\dfrac{e^x - e^{-x}}{2}$ is called **hyperbolic sine** of x and is denoted by $\sin h\, x$. (h standing for hyperbola.)

Thus, $$\boxed{\sin h\, x = \frac{e^x - e^{-x}}{2}, \quad \cos h\, x = \frac{e^x + e^{-x}}{2}}$$

Other hyperbolic functions are defined as

$$\tan h\, x = \frac{\sin h\, x}{\cos h\, x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \sec h\, x = \frac{1}{\cos h\, x},$$

$$\operatorname{cosec} h\, x = \frac{1}{\sin h\, x}, \qquad \cot h\, x = \frac{1}{\tan h\, x}.$$

4. Relationship between Hyperbolic and Circular Functions

(i) $\boxed{\sin ix = i \sin h\, x}$ and $\boxed{\sin h\, ix = i \sin x}$ .... (1)

We have $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$$\therefore \quad \sin i x = \frac{e^{i^2 x} - e^{-i^2 x}}{2i} = \frac{e^{-x} - e^{x}}{2i^2} \cdot i = i\left(\frac{e^x - e^{-x}}{2}\right) = i \sin h\, x$$

Now, $$\sin h\, ix = \frac{e^{ix} - e^{-ix}}{2} = i \cdot \frac{e^{ix} - e^{-ix}}{2i} = i \sin x$$

Similarly, we can prove that

and $$\boxed{\begin{array}{ll} \cos i x = \cos hx, & \cos h\, i x = \cos x, \\ \tan i x = i \tan hx, & \tan h\, i x = i \tan x \end{array}}$$ .... (2)

and three more results for remaining three functions.

The above relationship can also be stated as

$\boxed{\sin hx = -i \sin ix}$ and $\boxed{\cos hx = \cos ix}$ .... (2A)

From 1 (i) above, we have

$$\sin h\, x = \frac{1}{i} \sin ix = \frac{i}{i^2} \sin ix = -i \sin ix$$

and the second result follows from (2).

From (2A), we have

$$\boxed{\tan h\, x = -i \tan ix = \frac{1}{i} \tan ix}$$ .... (2B)

Note ....

Hyperbolic identities can be obtained very easily from identities of circular functions, first by replacing x by ix and then replacing **sin ix** by **i sin hx** and **cos ix** by **cos hx**.

List of Formulae

1. $\sin ix = i \sin hx$; $\sin h\, ix = i \sin x$
2. $\cos ix = \cos hx$; $\cos h\, ix = \cos x$
3. $\tan ix = i \tan hx$; $\tan h\, ix = i \tan x$

Also we have

1. $\sin hx = -i \sin ix$
2. $\cos hx = \cos ix$
3. $\tan hx = -i \tan ix = \frac{1}{i} \tan ix$

5. Hyperbolic Identities

We prove some hyperbolic identities.

(a) $\boxed{\sin h(-x) = -\sin hx}$ .... (3)

Since $\sin hx = \frac{e^x - e^{-x}}{2}$

$$\therefore \quad \sin h(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} \qquad \therefore \quad \sin h(-x) = -\sin hx$$

(b) Similarly, $\boxed{\cos h(-x) = \cos hx}$

(c) $\boxed{e^x = \cos hx + \sin hx}$ .... (4)

$$\cos hx + \sin hx = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = e^x$$

By adding (3) and (4),

(d) $\boxed{e^{-x} = \cos hx - \sin hx}$ .... (5)

(e) $\boxed{\cos h^2 x - \sin h^2 x = 1}$ .... (6)

This result and many more can be proved very easily by converting hyperbolic functions into circular functions or by using the definitions, as illustrated below :

Since, $\cos hx = \cos ix$ and $\sin hx = -i \sin ix$

$$\begin{aligned} \therefore \quad \cos h^2 x - \sin h^2 x &= (\cos ix)^2 - (-i \sin ix)^2 \\ &= \cos^2 (ix) - i^2 \sin^2 (ix) \\ &= \cos^2 (ix) + \sin^2 (ix) = 1 \end{aligned}$$

Aliter : $\cos h^2 x - \sin h^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$

$$= \frac{1}{4}[(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})]$$

$$= \frac{1}{4} \cdot (4) = 1$$

Note ....

If $x = a\cos h\, t$, $y = b\sin h\, t$, then $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = \cos h^2 t - \sin h^2 t = 1$.

$\therefore$ $x = a\cos h\, t$, $y = b\sin h\, t$ are parametric equations of a hyperbola. Hence, the name **hyperbolic functions**.

We give below the complete list of formulae of Hyperbolic Functions.

(a) Square Relations

1. $\cos h^2 x - \sin h^2 x = 1$
2. $\sec h^2 x + \tan h^2 x = 1$
3. $\cot h^2 x - \operatorname{cosec} h^2 x = 1$

(b) Addition Formulae

1. $\sin h(x \pm y) = \sin h\, x \cos h\, y \pm \cos h\, x \sin h\, y$
2. $\cos h(x \pm y) = \cos h\, x \cos h\, y \pm \sin h\, x \sin h\, y$
3. $\tan h(x \pm y) = \dfrac{\tan hx \pm \tan hy}{1 \pm \tan hx \tan hy}$

(c) Formulae for 2x and 3x

1. $\sin h\, 2x = 2\sin h\, x \cos h\, x$
2. $\cos h\, 2x = \cos h^2 x + \sin h^2 x = 2\cos h^2 x - 1 = 2\sin h^2 x + 1$
3. $\tan h\, 2x = \dfrac{2\tan hx}{1 + \tan h^2 x}$
4. $\sin h\, 3x = 3\sin h\, x + 4\sin h^3 x$
5. $\cos h\, 3x = 4\cos h^3 x - 3\cos h\, x$
6. $\tan h\, 3x = \dfrac{3\tan h\, x + \tan h^3 x}{1 + 3\tan h^2 x}$
7. $\sin h\, x = \dfrac{2\tan h(x/2)}{1 - \tan h^2(x/2)}$
8. $\cos h\, x = \dfrac{1 + \tan h^2(x/2)}{1 - \tan h^2(x/2)}$
9. $\tan h\, x = \dfrac{2\tan h(x/2)}{1 + \tan h^2(x/2)}$

(d) Product Formulae

1. $\sin h(x + y) + \sin h(x - y) = 2\sin h\, x \cos h\, y$
2. $\sin h(x + y) - \sin h(x - y) = 2\cos h\, x \sin h\, y$
3. $\cos h(x + y) + \cos h(x - y) = 2\cos h\, x \cos h\, y$
4. $\cos h(x + y) - \cos h(x - y) = 2\sin h\, x \sin h\, y$
5. $\sin hx + \sin hy = 2\sin h\left(\dfrac{x+y}{2}\right)\cos h\left(\dfrac{x-y}{2}\right)$
6. $\sin hx - \sin hy = 2\cos h\left(\dfrac{x+y}{2}\right)\sin h\left(\dfrac{x-y}{2}\right)$
7. $\cos hx + \cos hy = 2\cos h\left(\dfrac{x+y}{2}\right)\cos h\left(\dfrac{x-y}{2}\right)$
8. $\cos hx - \cos hy = 2\sin h\left(\dfrac{x+y}{2}\right)\sin h\left(\dfrac{x-y}{2}\right)$

Example : Prove that $\tan h(u+iv) = \frac{1}{i}\tan(iu-v)$ and $\tan h(u-iv) = \frac{1}{i}\tan(iu+v)$.

Sol. : We have $\tan h(u+iv) = \frac{\tan hu + \tan h\, iv}{1+\tan hu \tan h\, iv} = \frac{\frac{1}{i}\tan iu + i\tan v}{1+\frac{1}{i}\tan iu \cdot i\tan v}$

$$= \frac{\frac{1}{i}\tan iu - \frac{1}{i}\tan v}{1+\tan iu \tan v} = \frac{1}{i}\tan(iu-v)$$

Similarly, $\tan h(u-iv) = \frac{\tan h\, u - \tan h\, iv}{1-\tan h\, u\ \tan h\, iv} = \frac{\frac{1}{i}\tan iu - i\tan v}{1-\frac{1}{i}\tan iu \cdot i\tan v}$

$$= \frac{\frac{1}{i}\tan iu + \frac{1}{i}\tan v}{1-\tan iu\ \tan v} = \frac{1}{i}\cdot\frac{\tan iu + \tan v}{1-i\tan u \tan v}$$

$$= \frac{1}{i}\tan(iu+v)$$

6. Differentiation and Integration

If $y = \sin hx$, $y = \frac{e^x - e^{-x}}{2}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{2}\right) = \frac{e^x+e^{-x}}{2} = \cos hx$$

$$\therefore \quad \boxed{\textbf{If } y = \sin hx,\ \frac{dy}{dx} = \cos hx}$$

If $y \cos hx$, $y = \frac{e^x + e^{-x}}{2}$,

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{e^x-e^{-x}}{2} = \sin hx.$$

$$\therefore \quad \boxed{\textbf{If } y = \cos hx,\ \frac{dy}{dx} = \sin hx}$$

If $y = \tan hx$, $y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right) = \frac{(e^x+e^{-x})(e^x+e^{-x})-(e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2} = \left(\frac{2}{e^x+e^{-x}}\right)^2$$

$$= \left(\frac{1}{\cos hx}\right)^2 = \sec h^2 x.$$

[Or $y = \dfrac{\sin hx}{\cos hx}$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{\cos hx \cdot \cos hx - \sin hx \cdot \sin hx}{\cos h^2 x} = \dfrac{1}{\cos h^2 x} = \sec h^2 x$]

$$\therefore \quad \boxed{\text{If } y = \tan hx, \ \frac{dy}{dx} = \sec h^2 x}$$

Hence, we get the following three results.

$$\boxed{\begin{aligned} &\int \cos hx \, dx = \sin hx \\ &\int \sin hx \, dx = \cos hx \\ &\int \sec h^2 \, dx = \tan hx \end{aligned}}$$

7. Periods of Hyperbolic Functions

Since $\quad \sin hx = \dfrac{1}{i} \sin ix = \dfrac{1}{i} \sin(2n\pi + ix)$

$= \dfrac{1}{i} \sin i(x - 2n\pi i)$

$= -i \sin i(x - 2n\pi i)$

$= \sin h(x - 2n\pi i)$ [By (2A), page 3-3]

Thus, $\sin hx$ is a periodic function of **period $2\pi i$**.

Similarly, $\quad \cos hx = \cos ix = \cos(ix + 2n\pi)$

$= \cos\{i(x - 2n\pi i)\}$

$= \cos h(x - 2n\pi i)$ [By (2), page 3-3]

Hence, $\cos hx$ is a periodic function of **period $2\pi i$**.

In the same way we can prove that $\tan hx$ periodic with **period $2\pi i$**.

Alternatively to prove the periodicity of hyperbolic functions we can proceed as follows :

(i) $\quad \sin h(x + 2\pi i) = \sin hx \cos h 2\pi i + \cos hx \sin h 2\pi i$

[By (1), page 3-3, $\quad \sin h 2\pi i = i \sin 2\pi$

and by (2), page 3-3, $\quad \cos h 2\pi i = \cos 2\pi$.]

$\therefore \quad \sin h(x + 2\pi i) = \sin hx \cos 2\pi + i \cos hx \sin 2\pi$

$= \sin hx$

Hence, $\sin hx$ is periodic with period $2\pi i$.

(ii) $\quad \cos h(x + 2\pi i) = \cos hx \cos h 2\pi i + \sin hx \sin h 2\pi i$

$= \cos hx \cos 2\pi + i \sin hx \sin 2\pi$

$= \cos hx$

Hence, $\cos hx$ is periodic with period $2\pi i$.

Note that the period of hyperbolic functions is imaginary.

8. Table of Values of Hyperbolic Functions

From the definitions of sin hx, cos hx, tan hx we can obtain the following table of values.

x	0	$-\infty$	∞
sin hx	0	$-\infty$	∞
cos hx	1	∞	∞
tan hx	0	-1	1

Further, we note that, since

$$\tan h(-\infty) = -1, \quad \tan h(0) = 0, \quad \tan h(\infty) = 1$$

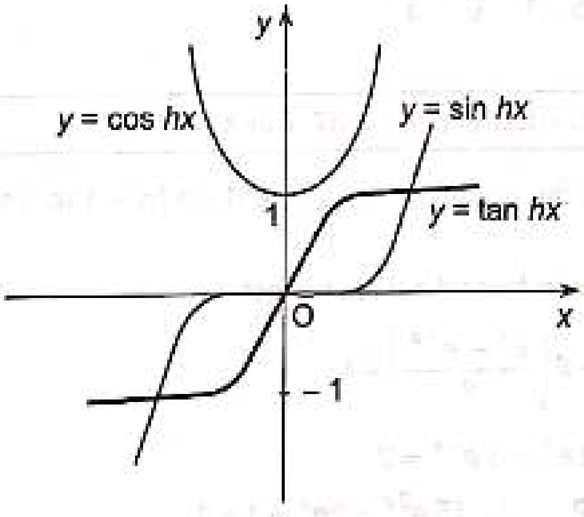
$$\therefore \qquad \boxed{|\tan hx| \leq 1}$$

Note ....

While sin x lies between -1 and $+1$, sin hx lies between $-\infty$ and ∞. But tan hx lies between -1 and $+1$.

Graphs

We give below the rough sketches of these three hyperbolic functions.



Type II : On use of Definitions of Hyperbolic Functions : Class (a) : 3 Marks

Example 1 (a) : If tan $hx = 1/2$, find the value of x and sin $h\,2x$. **(M.U. 1999, 2008)**

Sol. : We can obtain sin $h\,2x$ from the definition.

$$\tan hx = \frac{e^x - e^{-x}}{e^x + e^x} = \frac{1}{2} \qquad \therefore \quad \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2}$$

$$\therefore \quad 2e^{2x} - 2 = e^{2x} + 1 \qquad \therefore \quad e^{2x} = 3$$

Now, $$\sin h\,2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3-(1/3)}{2} = \frac{4}{3}$$

Aliter : We can also use the formula for sin $h\,2x$.

$$\sin h\,2x = \frac{2\tan hx}{1-\tan^2 hx} = \frac{2(1/2)}{1-(1/2)^2} = \frac{4}{3}.$$

Example 2 (a) : If $\tan hx = \dfrac{2}{3}$, find the value of x and then $\cos h\,2x$. **(M.U. 2014)**

Sol. : Left to you. Do as Ex. 1 above.

$$x = \frac{1}{2}\log_e 5 \quad \text{and} \quad \cos h\,2x = \frac{13}{5}.$$

Example 3 (a) : Find the value of $\tan h \log x$ if $x = \sqrt{3}$. Also find the value of $\tan h \log \sqrt{5}$. **(M.U. 2019)**

Sol. : By definition of $\tan hx$, we get

$$\tan h(\log x) = \frac{e^{\log x} - e^{-\log x}}{e^{\log x} + e^{-\log x}}. \qquad \text{But } e^{\log x} = x.$$

$$= \frac{x - x^{-1}}{x + x^{-1}} = \frac{x^2 - 1}{x^2 + 1} = \frac{3 - 1}{3 + 1} = \frac{1}{2}.$$

As proved above

$$\tan h \log x = \frac{x^2 - 1}{x^2 + 1}$$

Now, put $x = \sqrt{5}$.

$$\therefore \quad \tan h \log \sqrt{5} = \frac{5 - 1}{5 + 1} = \frac{4}{6} = \frac{2}{3}.$$

Note ....

For another method, see Ex. 9, page 3-37. Put $k = 3$.

Example 4 (a) : Solve the equation $7 \cos hx + 8 \sin hx = 1$ for real values of x. **(M.U. 1989, 2008, 16)**

Sol. : Putting the values of $\cos hx$ and $\sin hx$, we get

$$7\left(\frac{e^x + e^{-x}}{2}\right) + 8\left(\frac{e^x - e^{-x}}{2}\right) = 1$$

$$\therefore \quad 7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$$

$$\therefore \quad 15e^x - e^{-x} = 2 \qquad \therefore \quad 15e^{2x} - 2e^x - 1 = 0$$

Solving it as a quadratic equation in e^x,

$$e^x = \frac{2 \pm \sqrt{4 - 4(15)(-1)}}{2(15)} = \frac{2 \pm 8}{30} = \frac{1}{3} \quad \text{or} \quad -\frac{1}{5}$$

$$\therefore \quad x = \log\left(\frac{1}{3}\right) \quad \text{or} \quad x = \log\left(-\frac{1}{5}\right)$$

[Or by factorisation of $15e^{2x} - 2e^x - 1$, we get $(5e^x + 1)(3e^x - 1) = 0$]

Since x is real, $\quad x = \log\left(\dfrac{1}{3}\right) = -\log 3.$

Remark ....

Logarithm of a negative number is complex [See Ex. 1 (i), page 4-2].

Example 5 (a) : If 5 sin hx – cos hx = 5, find tan hx. **(M.U. 2004, 17)**

Sol. : Again putting the values of sin hx and cos hx, we get,

$$5\left(\frac{e^x - e^{-x}}{2}\right) - \frac{e^x + e^{-x}}{2} = 5$$

$$5e^x - 5e^{-x} - e^x - e^{-x} = 10 \quad \therefore \quad 4e^x - 10 - 6e^{-x} = 0$$

$$\therefore \quad 4e^{2x} - 10e^x - 6 = 0 \quad \therefore \quad 2e^{2x} - 5e^x - 3 = 0$$

$$\therefore \quad (2e^x + 1)(e^x - 3) = 0 \quad \therefore \quad e^x = -\frac{1}{2} \text{ or } 3.$$

$$\therefore \quad \tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{(-1/2)-(-2)}{(-1/2)+(-2)} = \frac{2-(1/2)}{-2-(1/2)} = \frac{3/2}{-5/2} = -\frac{3}{5}.$$

$$\text{or} \quad \tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{3-(1/3)}{3+(1/3)} = \frac{8/3}{10/3} = \frac{4}{5}.$$

Aliter : Dividing the equation by cos hx, we get 5 tan hx – 1 = 5 sec hx.

Squaring both sides, we get,

$$25\tan h^2 x - 10\tan hx + 1 = 25\sec h^2 x = 25(1 - \tan h^2 x)$$

$$\therefore \quad 50\tan h^2 x - 10\tan hx - 24 = 0$$

$$\therefore \quad 25\tan h^2 x - 5\tan hx - 12 = 0$$

$$(5\tan hx - 4)(5\tan hx + 3) = 0$$

$$\therefore \quad \tan hx = 4/5 \text{ or } -3/5.$$

Example 6 (a) : Prove that $(\cos hx - \sin hx)^n = \cos h\, nx - \sin h\, nx$. **(M.U. 2001, 02)**

Sol. : By definition,

$$\text{l.h.s.} = \left[\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right]^n = (e^{-x})^n = e^{-nx}$$

$$\text{r.h.s.} = \frac{e^{nx} + e^{-nx}}{2} - \frac{e^{nx} - e^{-nx}}{2} = e^{-nx}$$

Hence, l.h.s. = r.h.s.

Aliter : By De Moiver's Theorem

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx$$

Change x to ix,

$$\therefore \quad (\cos ix + i\sin ix)^n = \cos nix + i\sin nix$$

But $\cos ix = \cos hx$ and $\sin ix = i\sin hx$

$$\therefore \quad (\cos hx - \sin hx)^n = \cos h\, nx - i\sin h\, nx.$$

[Similarly, we can prove that $(\cos hx + \sin hx)^n = \cos h\, nx + \sin h\, nx$.]

Example 7 (a) : Prove that

(i) $\left(\frac{1+\tan hx}{1-\tan hx}\right)^n = \cos h\, 2nx + \sin h\, 2nx.$ **(M.U. 1998, 99, 2001)**

(ii) $\left(\frac{1+\tan hx}{1-\tan hx}\right)^3 = \cos h\, 6x + \sin h\, 6x$ **(M.U. 1998, 201, 2009)**

(iii) $\left(\dfrac{\cos hx + \sin hx}{\cos hx - \sin hx}\right)^n = \cos h\,2n\,x + \sin h\,2nx.$

Sol. : (i) $\text{l.h.s.} = \left[\dfrac{1 + \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}{1 - \dfrac{e^x - e^{-x}}{e^x + e^{-x}}}\right] = \left(\dfrac{2e^x}{2e^{-x}}\right)^n = (e^{2x})^n = e^{2nx}$

$$\text{r.h.s.} = \frac{e^{2nx} + e^{-2nx}}{2} + \frac{e^{2nx} - e^{-2nx}}{2} = e^{2nx}$$

$\therefore$ l.h.s. = r.h.s.

(ii) Putting $n = 3$, we get the required result.

(iii) Since $\tan hx = \dfrac{\cos hx}{\sin hx}$, the result follows from (i).

Type III : Expansions of $\sin h^n x$, $\cos h^n x$: Class (b) : 6 Marks

Example 1 (b) : Prove that $16 \cos h^5 x = \cos h\,5x + 5 \cos h\,3x + 10 \cos hx$. **(M.U. 1996)**

Sol. : Putting the value of $\cos hx$ in the l.h.s.

$$\text{l.h.s.} = 16\left(\frac{e^x + e^{-x}}{2}\right)^5$$

$$= \frac{16}{32}\left[e^{5x} + 5e^{4x}\cdot e^{-x} + 10e^{3x}\cdot e^{-2x} + 10e^{2x}\cdot e^{-3x} + 5e^x\cdot e^{-4x} + e^{-5x}\right]$$

[By Binomial Theorem]

$$= \frac{(e^{5x} + e^{-5x})}{2} + 5\frac{(e^{3x} + e^{-3}x)}{2} + 10\frac{(e^x + e^{-x})}{2}$$

$$= \cos h\,5x + 5\cos h\,3x + 10\cos hx = \text{r.h.s.}$$

Example 2 (b) : Express $\sin h^7 x$ in terms of multiples of hyperbolic sines of x.

Sol. : We have

$$\sin h^7 x = \left(\frac{e^x - e^{-x}}{2}\right)^7$$

$$= \frac{1}{2^7}\Big[e^{7x} - 7e^{6x}\cdot e^{-x} + 21e^{5x}\cdot e^{-2x} - 35e^{4x}\cdot e^{-3x}$$
$$+ 35e^{3x}\cdot e^{-4x} - 21e^{2x}\cdot e^{-5x} + 7e^x\cdot e^{-6x} - e^{0x}\cdot e^{-7x}\Big]$$

[By Binomial Theorem]

$$= \frac{1}{2^6}\left[\frac{e^{7x} - e^{-7x}}{2} - 7\left(\frac{e^{5x} - e^{-5x}}{2}\right) + 21\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 35\left(\frac{e^x - e^{-x}}{2}\right)\right]$$

$$= \frac{1}{64}\left[\sin h\,7x - 7\sin h\,5x + 21\sin h\,3x - 35\sin hx\right]$$

Example 3 (b) : Express $\cos h^8 x$ in terms of hyperbolic cosines of multiples of x.

Sol. : We have

$$\cos h^8 x = \left(\frac{e^x + e^{-x}}{2}\right)^8$$

$$= \frac{1}{2^8}\Big[e^{8x} + 8e^{7x} \cdot e^{-x} + 28 \cdot e^{6x} \cdot e^{-2x} + 56 \cdot e^{5x} \cdot e^{-3x} + 70 \cdot e^{4x} \cdot e^{-4x} + 56 \cdot e^{3x} \cdot e^{-5x} + 28 \cdot e^{2x} \cdot e^{-6x} + 8 \cdot e^{x} \cdot e^{-7x} + e^{-8x}\Big]$$

[By Binomial Theorem]

$$= \frac{1}{2^8}\left[e^{8x} + 8e^{6x} + 28e^{4x} + 56e^{2x} + 70 + 56e^{-2x} + 28e^{-4x} + 8e^{-6x} + e^{-8x}\right]$$

$$= \frac{1}{2^7}\left[\frac{e^{8x} + e^{-8x}}{2} + 8 \cdot \frac{e^{6x} + e^{-6x}}{2} + 28 \cdot \frac{e^{4x} + e^{-4x}}{2} + 56 \cdot \frac{e^{2x} + e^{-2x}}{2} + 35\right]$$

$$= \frac{1}{2^7}\left[\cos h\,8x + 8\cos h\,6x + 28\cos h\,4x + 56\cos h\,2x + 35\right]$$

Type IV : On Use of Formulae : Class (a) : 3 Marks

Example 1 (a) : If $\sin\alpha\cos h\beta = \frac{x}{2}$, $\cos\alpha\sin h\beta = \frac{y}{2}$, show that

(i) $\operatorname{cosec}(\alpha - i\beta) + \operatorname{cosec}(\alpha + i\beta) = \frac{4x}{x^2 + y^2}$ **(M.U. 1997, 99, 2016)**

(ii) $\operatorname{cosec}(\alpha - i\beta) - \operatorname{cosec}(\alpha + i\beta) = \frac{4iy}{x^2 + y^2}$

Sol. : We have

$$\operatorname{cosec}(\alpha + i\beta) = \frac{1}{\sin(\alpha + i\beta)} = \frac{1}{\sin\alpha\cos i\beta + \cos\alpha\sin i\beta}$$

$$= \frac{1}{\sin\alpha\cos h\beta + i\cos\alpha\sin h\beta}$$

$$= \frac{1}{(x/2) + i(y/2)} = \frac{2}{x + iy} \quad \text{[By data]}$$

Similarly (or by changing the sign of i)

$$\operatorname{cosec}(\alpha - i\beta) = \frac{2}{x - iy}$$

$$\therefore \quad \operatorname{cosec}(\alpha - i\beta) + \operatorname{cosec}(\alpha + i\beta) = \frac{2}{x - iy} + \frac{2}{x + iy} = \frac{4x}{x^2 + y^2}$$

And $$\operatorname{cosec}(\alpha - i\beta) - \operatorname{cosec}(\alpha + i\beta) = \frac{2}{x - iy} - \frac{2}{x + iy} = \frac{4iy}{x^2 + y^2}$$

Example 2 (a) : Prove that $\cos h^2 x = \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \cos h^2 x}}}$. **(M.U. 2012)**

Sol. : r.h.s. $= \dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{-\sin h^2 x}}} = \dfrac{1}{1-\dfrac{1}{1+\operatorname{cosec} h^2 x}}$

$$= \frac{1}{1-\dfrac{1}{\cot h^2 x}} = \frac{1}{1-\tan h^2 x} = \frac{1}{1-\dfrac{\sin h^2 x}{\cos h^2 x}}$$

$$= \frac{\cos h^2 x}{\cos h^2 x - \sin h^2 x} = \cos h^2 x.$$

Type V : Class (c) : 8 Marks

Example 1 (c) : If $\log \tan x = y$, prove that $\sin h\, ny = \frac{1}{2}(\tan^n x - \cot^n x)$

and $\cos h\,(n+1)\,y + \cos h\,(n-1)\,y = 2\cos h\, ny \operatorname{cosec} 2x$. **(M.U. 1997, 2004, 05)**

Sol. : By data, $y = \log \tan x$ $\therefore$ $e^y = \tan x$ and $e^{-y} = \cot x$

Now, $\sin h\, ny = \dfrac{e^{ny} - e^{-ny}}{2} = \dfrac{1}{2}(\tan^n x - \cot^n x)$

Again $\cos h(n+1)\,y + \cos h(n-1)\,y = 2\cos h\, ny \cos h\, y$

$$= 2\cos h\, ny \cdot \left(\frac{e^y + e^{-y}}{2}\right) = 2\cos h\, ny \cdot \left(\frac{\tan x + \cot x}{2}\right)$$

$$= 2\cos h\, xy \cdot \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right) \Big/ 2$$

$$= 2\cos h\, ny \cdot \frac{1}{2\sin x \cos x} = 2\cos h\, ny \operatorname{cosec} 2x.$$

Example 2 (c) : If $u = \log \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, prove that

(i) $\cos h\, u = \sec\theta$ **(ii)** $\sin h\, u = \tan\theta$

(iii) $\tan h\, u = \sin\theta$ **(iv)** $\tan h \dfrac{u}{2} = \tan\dfrac{\theta}{2}$. **(M.U. 1986, 2005, 07, 08, 09)**

Sol. : We have $u = \log \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$

$$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\tan(\pi/4) + \tan(\theta/2)}{1 - \tan(\pi/4)\tan(\theta/2)} = \frac{1+\tan(\theta/2)}{1-\tan(\theta/2)} \qquad \text{.................... (1)}$$

$$= \frac{\cos(\theta/2) + \sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)} \cdot \frac{\cos(\theta/2) + \sin(\theta/2)}{\cos(\theta/2) + \sin(\theta/2)}$$

$$= \frac{1 + 2\sin(\theta/2)\cos(\theta/2)}{\cos^2(\theta/2) - \sin^2(\theta/2)} = \frac{1+\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$$

$\therefore$ $e^u = \sec\theta + \tan\theta$

$\therefore$ $e^{-u} = \dfrac{1}{\sec\theta + \tan\theta} \cdot \dfrac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta} = \sec\theta - \tan\theta$

(i) By addition, $e^u + e^{-u} = 2\sec\theta$. $\quad \therefore \quad \cos hu = \dfrac{e^u + e^{-u}}{2} = \sec\theta$

(ii) By subtraction, $e^u - e^{-u} = 2\tan\theta$ $\quad \therefore \quad \sin hu = \dfrac{e^u - e^{-u}}{2} = \tan\theta$

(iii) $\quad \tan hu = \dfrac{\sin hu}{\cos hu} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$

(iv) $$\tan h\left(\frac{u}{2}\right) = \frac{\sin h(u/2)}{\cos h(u/2)} = \frac{2\sin h(u/2)\cdot\cos h(u/2)}{2\cos h(u/2)\cos h(u/2)}$$

$$= \frac{\sin hu}{1+\cos hu} = \frac{\tan\theta}{1+\sec\theta} \qquad [\text{ By (ii) and (i) }]$$

$$\therefore \quad \tan h\left(\frac{u}{2}\right) = \frac{\sin\theta/\cos\theta}{(\cos\theta+1)/\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)}$$

$$= \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}.$$

[For another method see the next example.]

Example 3 (c) : If $u = \log_e \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$, prove that $\tan h\dfrac{u}{2} = \tan\dfrac{\theta}{2}$ and $\cos hu\cos\theta = 1$ (or $\cos hu = \sec\theta$). **(M.U. 1990, 2005, 07, 08, 18)**

Sol. : We have $e^u = \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$ .... (1)

$$\therefore \quad e^u = \frac{\tan(\pi/4) + \tan(\theta/2)}{1 - \tan(\pi/4)\cdot\tan(\theta/2)} = \frac{1+\tan(\theta/2)}{1-\tan(\theta/2)}$$

By componendo and dividendo

$$\frac{e^u+1}{e^u-1} = \frac{2}{2\tan(\theta/2)} \qquad \therefore \quad \frac{e^u-1}{e^u+1} = \tan\frac{\theta}{2}. \qquad \text{..................... (2)}$$

$$\text{Now,} \quad \tan h\frac{u}{2} = \frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}}\cdot\frac{e^{u/2}}{e^{u/2}} = \frac{e^u - 1}{e^u + 1} = \tan\frac{\theta}{2} \qquad [\text{ By (2) }]$$

$$\text{Further,} \quad \cos hu = \frac{e^u + e^{-u}}{2} = \frac{1}{2}\left[\frac{1+\tan(\theta/2)}{1-\tan(\theta/2)} + \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}\right]$$

$$= \frac{1}{2}\cdot\frac{[1+\tan(\theta/2)]^2 + [1-\tan(\theta/2)]^2}{1-\tan^2(\theta/2)} = \frac{1+\tan^2(\theta/2)}{1-\tan^2(\theta/2)}$$

$$= \frac{1+\sin^2(\theta/2)/\cos^2(\theta/2)}{1-\sin^2(\theta/2)/\cos^2(\theta/2)} = \frac{\cos^2(\theta/2)+\sin^2(\theta/2)}{\cos^2(\theta/2)-\sin^2(\theta/2)}$$

$$= \frac{1}{\cos^2(\theta/2)-\sin^2(\theta/2)} = \frac{1}{\cos\theta} \qquad [\,\therefore \ \cos hu = \sec\theta\,]$$

$\therefore \cos hu\cdot\cos\theta = 1.$

Example 4 (c) : If $\tan\dfrac{x}{2} = \tan h\dfrac{u}{2}$, prove that

(i) $\sin hu = \tan x$, **(ii)** $\cos hu = \sec x$ **(M.U. 1984)**

(iii) $u = \log\left(\tan\dfrac{\pi}{4} + \dfrac{x}{2}\right)$. **(M.U. 2015, 18)**

Sol. : (i) We have [By C(7), page 3-5]

$$\sin hu = \frac{2\tan h(u/2)}{1-\tan h^2(u/2)} = \frac{2\tan(x/2)}{1-\tan^2(x/2)} = \tan\left(\frac{x}{2}+\frac{x}{2}\right) = \tan x$$

(ii) Now, $\cos hu = \sqrt{1+\sin h^2 u} = \sqrt{1+\tan^2 x} = \sec x$.

(iii) $\because \tan h\frac{u}{2} = \tan\frac{x}{2}$ $\therefore \frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \frac{\tan(x/2)}{1}$

By componendo and dividendo

$$\frac{2e^{u/2}}{2e^{-u/2}} = \frac{1+\tan(x/2)}{1-\tan(x/2)}$$

$$\therefore \quad e^u = \frac{1+\tan(x/2)}{1-\tan(x/2)} = \frac{\tan(\pi/4)+\tan(x/2)}{1-\tan(\pi/4)\tan(x/2)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$\therefore \quad u = \log\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$$

[For another proof, see Ex. 12, page 3-38.]

Example 5 (c) : If $\cos hx = \sec\theta$, prove that **(i)** $x = \log(\sec\theta+\tan\theta)$, **(ii)** $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$. **(iii)** $\tan h\frac{x}{2} = \tan\frac{\theta}{2}$. **(iv)** $\tan hx = \sin\theta$. **(M.U. 1997, 2003, 05, 13)**

Sol. : (i) By definition $\cos hx = \frac{e^x+e^{-x}}{2}$ $\therefore \frac{e^x+e^{-x}}{2} = \sec\theta$

$$\therefore \quad e^x - 2\sec\theta + e^{-x} = 0 \quad \therefore \quad (e^x)^2 - 2e^x\sec\theta + 1 = 0$$

Solving the quadratic in e^x,

$$e^x = \sec\theta \pm \sqrt{\sec^2\theta - 1} = \sec\theta \pm \tan\theta \quad \text{.................... (1)}$$

$$\therefore \quad x = \log(\sec\theta \pm \tan\theta) = \pm\log(\sec\theta+\tan\theta)$$

[Prove that $\log(\sec\theta-\tan\theta) = -\log(\sec\theta+\tan\theta)$.]

(ii) For (ii) let $\tan^{-1}e^{-x} = \alpha$ $\therefore e^{-x} = \tan\alpha$ $\therefore e^x = \cot\alpha$.

Now, by data $\sec\theta = \cos hx = \frac{e^x+e^{-x}}{2} = \frac{\cot\alpha+\tan\alpha}{2}$

$$\therefore \quad 2\sec\theta = \cot\alpha+\tan\alpha = \frac{\cos\alpha}{\sin\alpha}+\frac{\sin\alpha}{\cos\alpha} = \frac{2}{\sin 2\alpha}$$

$$\therefore \quad \cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2}-2\alpha\right) \quad \therefore \quad \theta = \frac{\pi}{2}-2\alpha = \frac{\pi}{2}-2\tan^{-1}(e^{-x})$$

[See also Ex. 6 (ii), page 3-42]

(iii) Now, $\tan h\frac{x}{2} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{e^x-1}{e^x+1}$

$$= \frac{\sec\theta+\tan\theta-1}{\sec\theta+\tan\theta+1} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} \quad \text{[By (1)]}$$

$$= \frac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta} = \frac{2\sin^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)}$$

$$= \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}.$$

(For another method, see Ex. 6 (iii), page 3-42.)

(iv) By data, $\cos hx = \sec\theta$

$\therefore \sqrt{1+\sin h^2 x} = \sqrt{1+\tan^2\theta} \quad \therefore \sin h^2 x = \tan^2\theta \quad \therefore \sin hx = \tan\theta$

Now, $\tan hx = \dfrac{\sin hx}{\cos hx} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$.

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. If $\tan hx = \dfrac{2}{3}$, find the value of x and then $\cos h\,2x$. **(M.U. 2014)** [**Ans. :** $\dfrac{1}{2}\log 5, \ \dfrac{13}{5}$]
2. If $\tan hx = \dfrac{1}{2}$, find the value of $\tan h\,3x$. [**Ans. :** $\dfrac{13}{14}$]
3. Find the value of $\tan h \log\sqrt{5}$. [**Ans. :** $\dfrac{2}{3}$]
4. Solve the following equation for real values of x
   $17\cos hx + 18\sin hx = 1$. [**Ans. :** $-\log 5$]
5. If $\sin hx - \cos hx = 5$, find $\tan hx$. [**Ans. :** $-\dfrac{12}{13}$]
6. If $6\sin hx + 2\cos hx + 7 = 0$, find $\tan hx$. [**Ans. :** $\dfrac{3}{5}, \ -\dfrac{15}{17}$]
7. If $\tan hx = \dfrac{1}{2}$, find $\sin h\,2x$ and $\cos h\,2x$. **(M.U. 1999)** [**Ans. :** $\dfrac{4}{3}, \ \dfrac{5}{3}$]
8. If $\cos\alpha\cos h\beta = \dfrac{x}{2}$, $\sin\alpha\sin h\beta = \dfrac{y}{2}$, show that
   (i) $\sec(\alpha - i\beta) + \sec(\alpha + i\beta) = \dfrac{4x}{x^2+y^2}$ **(M.U. 2016)**
   (ii) $\sec(\alpha - i\beta) - \sec(\alpha + i\beta) = -\dfrac{4iy}{x^2+y^2}$
9. Prove that $\operatorname{cosec} hx + \cot hx = \cot h\dfrac{x}{2}$.
10. Prove that $(\cos h\,x + \sin h\,x)^n = \cos h\,nx + \sin h\,nx$. **(M.U. 2003)**
11. If $\log\tan x = y$, prove that $\cos h\,ny = \dfrac{1}{2}[\tan^n x + \cot^n x]$ and
    $\sin h(n+1)y + \sin h(n-1)y = 2\sin h\,ny\operatorname{cosec} 2x$. **(M.U. 2015)**
12. Prove that $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1+\sin h^2 x}}} = -\sin h^2 x$. **(M.U. 1998, 2001)**

Class (b) : 6 Marks

1. If $\cos h^6 x = a\cos h\,6x + b\cos h\,4x + c\cos h\,2x + d$, prove that $5a - 5b + 3c - 2d = 0$.
2. Prove that $16\sin h^5 x = \sin h\,5x - 5\sin h\,3x + 10\sin hx$.
3. Prove that $32\cos h^6 x - 10 = \cos h\,6x + 6\cos h\,4x + 15\cos h\,3x$.
4. Prove that $\cos h^7 x = \dfrac{1}{64}[\cos h\,7x + 7\cos h\,5x + 21\cos h\,3x + 35\cos hx]$.

Class (c) : 8 Marks

1. If $\cos hu = \sec\theta$, prove that

(i) $\sin hu = \tan\theta$, (ii) $\tan h\,u = \sin\theta$, (iii) $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$. (M.U. 1985)

9. Separation of Real and Imaginary Parts

A complex function $w = f(z)$ is a function of z. Sometimes we need to write it in the form $f(z) = f_1(x, y) + i\, f_2(x, y)$ where real and imaginary parts $f_1(x, y)$ and $f_2(x, y)$ are separated. The process of obtaining $f_1(x, y)$ and $f_2(x, y)$ is called seperation of real and imaginary parts.

Real and imaginary parts of standard circular and hyperbolic functions $\sin z$, $\cos z$, $\tan z$, $\sin hz$, $\cos hz$ and $\tan hz$ are obtained easily by expanding the functions of $x + iy$ as illustrated below.

(I) sin z

$$\sin(x+iy) = \sin x\cos iy + \cos x\sin iy$$
$$= \sin x\cos hy + i\cos x\sin hy$$

$\therefore$ Real part $= \sin x\cos hy$

Imaginary part $= \cos x\cos hy$

Similarly, $\sin(x-iy) = \sin x\cos hy - i\cos\ \sin hy$

(ii) cos z

$$\cos(x+iy) = \cos x\cos iy - \sin x\sin iy$$
$$= \cos x\cos hy - i\sin x\sin hy$$

$\therefore$ Real part $= \cos x\cos hy$

Imaginary part $= -\sin x\sin hy$

Similarly, $\cos(x-iy) = \cos x\cos hy + i\sin x\sin hy$

(iii) tan z

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

We multiply the numerator and the denominator by the conjugate of the denominator *i.e.*, by $\cos(x-iy)$.

$$\therefore \quad \tan(x+iy) = \frac{2\sin(x+iy)}{2\cos(x+iy)}\cdot\frac{\cos(x-iy)}{\cos(x-iy)}$$
$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sin h2y}{\cos 2x + \cos h2y}$$

$\therefore$ Real part $= \dfrac{\sin 2x}{\cos 2x + \cos h2y}$

Imaginary part $= \dfrac{\sin h2y}{\cos 2x + \cos h2y}$.

(iv) cot z

$$\cot z = \frac{\cos(x+iy)}{\sin(x+iy)}$$

We multiply the numerator and denominator by conjugate of the denominator.

$$\therefore \quad \cot z = \frac{2\cos(x+iy)\sin(x-iy)}{2\sin(x+iy)\sin(x-iy)} = \frac{\sin(2x)-\sin(2iy)}{-\cos(2x)+\cos(2iy)}$$

$$= \frac{\sin 2x - i\sin h2y}{-\cos 2x + \cos h2y}$$

$$\therefore \quad \text{Real part} = \frac{\sin 2x}{\cos h2y - \cos 2x}$$

$$\text{and} \quad \text{Imaginary part} = -\frac{\sin h2y}{\cos h2y - \cos 2x}$$

(v) sec *z*

$$\sec z = \frac{1}{\cos(x+iy)}$$

As above

$$\sec z = \frac{1}{2\cos(x+iy)} \cdot \frac{2\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{2\cos x\cos(iy) + 2\sin x\sin(iy)}{\cos 2x + \cos 2iy}$$

$$= \frac{2\cos x\cos h\,y + i\,2\sin x\sin h\,y}{\cos 2x + \cos h2y}$$

$$\therefore \quad \text{Real part} = \frac{2\cos x\cos hy}{\cos 2x + \cos h2y}$$

$$\text{and} \quad \text{Imaginary part} = \frac{2\sin x\sin hy}{\cos 2x + \cos h2y}$$

(vi) cosec *z*

Sol. : Left to you.

$$\text{Real part} = \frac{2\sin x\cos hy}{-\cos 2x + \cos h2y}$$

$$\text{and} \quad \text{Imaginary part} = \frac{-2\cos x\sin hy}{-\cos 2x + \cos h2y}$$

(vii) sin *hz*

$$\sin h(x+iy) = \sin hx\cos h\,iy + \cos hx\sin h\,iy$$

$$= \sin hx\cos y + i\cos hx\sin y$$

$$\therefore \quad \text{Real part} = \sin hx\cos y$$

$$\text{Imaginary part} = \cos hx\sin y$$

(viii) cos *hz*

$$\cos h(x+iy) = \cos hx\cos h\,iy + \sin hx\sin h\,iy$$

$$= \cos hx\cos y + i\sin hx\sin y$$

$$\therefore \quad \text{Real part} = \cos hx\cos y$$

$$\text{Imaginary part} = \sin hx\sin y$$

(ix) tan hz **(M.U. 2015)**

$$\tan h(x+iy) = \frac{\sin h(x+iy)}{\cos h(x+iy)}$$

$$= \frac{2\sin h(x+iy)}{2\cos h(x+iy)} \cdot \frac{\cos h(x-iy)}{\cos h(x-iy)}$$

$$= \frac{\sin h2x + \sin h2iy}{\cos h2x + \cos h2iy} = \frac{\sin h2x + i\sin 2y}{\cos h2x + \cos 2y}$$

$$\therefore \quad \text{Real part} = \frac{\sin h2x}{\cos h2x + \cos 2y}$$

$$\text{Imaginary part} = \frac{\sin 2y}{\cos h2x + \cos 2y}$$

Type I : On sin $(\alpha + i\beta) = x + iy$: Class (a) : 3 Marks

Example 1 (a) : If $\sin(\alpha + i\beta) = x + iy$, then prove that

$$\frac{x^2}{\cos h^2 \beta} + \frac{y^2}{\sin h^2 \beta} = 1 \text{ and } \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = 1.$$

Sol. : We have $\sin\alpha \cos h\beta + i\cos\alpha \sin h\beta = x + iy$

Equating real and imaginary parts, we get,

$$\sin\alpha\cos h\beta = x \text{ and } \cos\alpha \sin h\beta = y$$

$$\therefore \quad \frac{x^2}{\cos h^2 \beta} + \frac{y^2}{\sin h^2 \beta} = \sin^2\alpha + \cos^2\alpha = 1$$

$$\text{and} \quad \frac{x^2}{\sin^2 \alpha} - \frac{y^2}{\cos^2 \alpha} = \cos h^2\beta - \sin h^2 \beta = 1.$$

Example 2 (a) : If $\sin(\alpha + i\beta) = x + iy$, prove that

$$x^2 \sec h^2 \beta + y^2 \operatorname{cosec} h^2 \beta = 1$$

$$\text{and} \quad x^2 \operatorname{cosec}^2 \alpha - y^2 \sec^2 \alpha = 1.$$

Sol. : Same as above.

Example 3 (a) : If $\cos h(u + iv) = x + iy$, prove that

$$\text{(i)} \ \frac{x^2}{\cos h^2 u} + \frac{y^2}{\sin h^2 u} = 1 \qquad \text{(ii)} \ \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$

Sol. : We have $x + iy = \cos h(u + iv)$

$$= \cos hu \cos h iv + \sin hu \sin h iv$$

$$= \cos hu \cos v + i \sin u \sin v$$

$$\therefore \quad x = \cos hu \cos v, \ \ y = \sin hu \sin v.$$

$$\therefore \quad \frac{x^2}{\cos h^2 u} + \frac{y^2}{\sin h^2 u} = \cos^2 v + \sin^2 v = 1$$

$$\text{and} \quad \frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \cos h^2 u - \sin^2 u = 1.$$

Example 4 (a) : If $x+iy=c\sin(u+iv)$, prove that u = constant represents a family of confocal hyperbolas and v = constant represents a family of confocal ellipses.

Sol. : As above $c\sin u\cos hv=x$ and $c\cos u\sin hv=y$

We now eliminate v.

$$\because\ \cos h^2 v-\sin h^2 v=1 \quad \therefore\ \frac{x^2}{c^2\sin^2 u}-\frac{y^2}{c^2\cos^2 u}=1$$

which represents a family of confocal hyperbolas for u = constant.

We now eliminate u.

$$\text{Further, } \sin^2 u+\cos^2 u=1 \quad \therefore\ \frac{x^2}{c^2\cos h^2 v}+\frac{y^2}{c^2\sin h^2 v}=1$$

which represents a family of confocal ellipses for v = constant.

Type II : On $\sin(\theta+i\Phi)=r(\cos\alpha+i\sin\alpha)$ etc : Class (b) : 6 Marks

Example 1 (b) : If $e^z=\sin(u+iv)$ and $z=x+iy$, prove that

$2e^{2x}=\cos h\,2v-\cos 2u$ and $\tan y=\tan hv\cdot\cot u$. **(M.U. 1984, 2006)**

Sol. : We have $e^z=\sin(u+iv)$

$$\therefore\ e^{x+iy}=\sin(u+iv) \qquad \therefore\ e^x\cdot e^{iy}=\sin u\cos iv+\cos u\sin iv.$$

$$\therefore\ e^x(\cos y+i\sin y)=\sin u\cos hv+i\cos u\sin hv.$$

Equating real and imaginary parts,

$$e^x\cos y=\sin u\cos hv \quad\text{and}\quad e^x\sin y=\cos u\sin hv.$$

Squaring and adding,

$$e^{2x}(\cos^2 y+\sin^2 y)=\sin^2 u\cos h^2 v+\cos^2 u\sin h^2 v$$

$$\therefore\ e^{2x}=(1-\cos^2 u)\cos h^2 v+\cos^2 u(\cos h^2 v-1)$$

$$=\cos h^2 v-\cos^2 u=\frac{1}{2}(1+\cos h2v)-\frac{1}{2}(1+\cos 2u)$$

$$\therefore\ e^{2x}=\frac{1}{2}(\cos h2v-\cos 2u)$$

Dividing the imaginary part by the real part, we get

$$\frac{e^x\sin y}{e^x\cos y}=\frac{\cos u\sin hv}{\sin u\cos hv}$$

$$\therefore\ \tan y=\tan hv\cdot\cot u.$$

Example 2 (b) : If $\log\cos(x-iy)=\alpha+i\beta$ [or $\cos(x-iy)=e^{\alpha+i\beta}$], prove that

$$\alpha=\frac{1}{2}\log\left(\frac{\cos h2y+\cos 2x}{2}\right) \text{ and } \beta=\tan^{-1}(\tan hy\tan x).$$

Sol. : We have $\log\cos(x-iy)=\alpha+i\beta$

$$\therefore\ \cos(x-iy)=e^{\alpha+i\beta}=e^{\alpha}\cdot e^{i\beta}=e^{\alpha}(\cos\beta+i\sin\beta)$$

$$\therefore\ \cos x\cos hy+i\sin x\sin hy=e^{\alpha}\cos\beta+ie^{\alpha}\sin\beta$$

Equating real and imaginary parts on both sides, we get

$$e^{\alpha}\cos\beta=\cos x\cos hy \qquad \text{.................... (i)}$$

$$e^{\alpha}\sin\beta=\sin x\sin hy \qquad \text{.................... (ii)}$$

Squaring and adding (i) and (ii), we get

$$e^{2\alpha}(\cos^2\beta+\sin^2\beta)=\cos^2 x\cos h^2 y+\sin^2 x\sin h^2 y$$

Since in the required result x and y are required to be separated, we write,

$$e^{2\alpha}=\cos^2 x\cos h^2 y+(1-\cos^2 x)(\cos h^2 y-1)$$
$$=\cos^2 x\cos^2 y+\cos h^2 y-1-\cos^2 x\cos h^2 y+\cos^2 x$$
$$=\cos h^2 y-1+\cos^2 x$$

$$\therefore\quad e^{2\alpha}=\frac{2\cos h^2 y-2+2\cos^2 x}{2}=\frac{2\cos h^2 y-1+2\cos^2 x-1}{2}$$
$$=\frac{\cos h2y+\cos 2x}{2}$$

$$\therefore\quad 2\alpha=\log\left(\frac{\cos h2y+\cos 2x}{2}\right)\qquad\therefore\quad \alpha=\frac{1}{2}\log\left(\frac{\cos h2y+\cos 2x}{2}\right)$$

Dividing (ii) by (i), we get

$$\tan\beta=\frac{\sin x\sin hy}{\cos x\cos hy}\qquad\therefore\quad \beta=\tan^{-1}(\tan x\tan hy).$$

Example 3 (b) : If $\sin(\theta+i\Phi)=r(\cos\alpha+i\sin\alpha)$ (or $=re^{i\alpha}$), prove that

$$r^2=\frac{1}{2}[\cos h2\Phi-\cos 2\theta]\quad\text{and}\quad\tan\alpha=\tan h\Phi\cot\theta$$ **(M.U. 1986)**

Sol. : We have

$$\sin\theta\cos h\Phi+i\cos\theta\sin h\Phi=r\cos\alpha+ir\sin\alpha$$

Equating real and imaginary parts,

$$\sin\theta\cos h\Phi=r\cos\alpha\quad\text{and}\quad\cos\theta\sin h\Phi=r\sin\alpha$$

Squaring and adding

$$\therefore\quad r^2(\cos^2\alpha+\sin^2\alpha)=\sin^2\theta\cos h^2\Phi+\cos^2\theta\sin h^2\Phi$$
$$\therefore\quad r^2=(1-\cos^2\theta)\cos h^2\Phi+\cos^2\theta(\cos h^2\Phi-1)=\cos h^2\Phi-\cos^2\theta$$
$$=\left(\frac{1+\cos h2\Phi}{2}\right)-\left(\frac{1+\cos 2\theta}{2}\right)=\frac{1}{2}[\cos h2\Phi-\cos 2\theta]$$

Dividing imaginary part by the real part, $\tan\alpha=\cot\theta\tan h\Phi$.

Example 4 (b) : If $\sin(\theta+i\Phi)=\cos\alpha+i\sin\alpha$ (or $=e^{i\alpha}$), prove that

$$\cos^4\theta=\sin^2\alpha=\sin h^4\Phi.\ (\cos^2\theta=\pm\sin\alpha)$$ **(M.U. 1981, 82, 91, 2003, 04)**

Sol. : As in Ex. 3, $\sin\theta\cos h\Phi=\cos\alpha$ and $\cos\theta\sin h\Phi=\sin\alpha$ .... (1)

But $\cos h^2\Phi-\sin h^2\Phi=1$

Putting the values of $\cos h\Phi$ and $\sin h\Phi$ from (1), we get

$$\therefore\quad\frac{\cos^2\alpha}{\sin^2\theta}-\frac{\sin^2\alpha}{\cos^2\theta}=1$$
$$\therefore\quad\cos^2\alpha\cos^2\theta-\sin^2\alpha\sin^2\theta=\sin^2\theta\cos^2\theta$$

To prove the first part, we have to eliminate $\cos^2\alpha$ and $\sin^2\theta$.

$$\therefore\quad(1-\sin^2\alpha)\cos^2\theta-\sin^2\alpha\sin^2\theta=(1-\cos^2\theta)\cos^2\theta$$
$$\therefore\quad\cos^2\theta-\sin^2\alpha(\cos^2\theta+\sin^2\theta)=\cos^2\theta-\cos^4\theta$$
$$\therefore\quad\sin^2\alpha=\cos^4\theta.\qquad[\because\ \cos^2\theta=\pm\sin\alpha]$$

Again $\sin^2\theta + \cos^2\theta = 1$

Putting the values of $\sin\theta$ and $\cos\theta$, from (1), we get

$$\therefore \quad \frac{\cos^2\alpha}{\cos h^2\Phi} + \frac{\sin^2\alpha}{\sin h^2\Phi} = 1$$

$$\therefore \quad \cos^2\alpha \sin h^2\Phi + \sin^2\alpha \cos h^2\Phi = \sin h^2\Phi \cos h^2\Phi$$

To prove the second result we have to eliminate $\cos^2\alpha$ and $\cos h^2\Phi$.

$$(1-\sin^2\alpha)\sin h^2\Phi + \sin^2\alpha(1+\sin h^2\Phi) = \sin h^2\Phi(1+\sin h^2\Phi)$$

$$\sin h^2\Phi - \sin^2\alpha \sin h^2\Phi + \sin^2\alpha + \sin^2\alpha \sin h^2\Phi = \sin h^2\Phi + \sin h^4\Phi$$

$$\therefore \quad \sin^2\alpha = \sin^4 h\,\Phi.$$

Example 5 (b) : If $\cos(\theta + i\Phi) = r(\cos\alpha + i\sin\alpha)$, prove that

$$\Phi = \frac{1}{2}\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right].$$ **(M.U. 1989, 92, 97)**

Sol. : We have

$$r(\cos\alpha + i\sin\alpha) = \cos\theta\cos h\Phi - i\sin\theta\sin h\Phi$$

$$\therefore \quad r\cos\alpha = \cos\theta\cos h\Phi, \quad r\sin\alpha = -\sin\theta\sin h\Phi$$

$$\therefore \quad \tan h\Phi = -\frac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta} \qquad \therefore \quad \frac{e^{\Phi}-e^{-\Phi}}{e^{\Phi}+e^{-\Phi}} = -\frac{\sin\alpha\cos\theta}{\cos\alpha\sin\theta}$$

By componendo and dividendo [See (G) in Appendix, page A-4]

$$\frac{e^{\Phi}}{e^{-\Phi}} = \frac{\cos\alpha\sin\theta - \sin\alpha\cos\theta}{\cos\alpha\sin\theta + \sin\alpha\cos\theta} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$\therefore \quad e^{2\Phi} = \frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$$

$$\therefore \quad 2\Phi = \log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right] \qquad \therefore \quad \Phi = \frac{1}{2}\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right].$$

Type III : On other functions such as tan z, tan hz, etc. : Class (b) : 6 marks

Example 1 (b) : If $\alpha + i\beta = \tan h\left(x + \frac{i\pi}{4}\right)$, prove that $\alpha^2 + \beta^2 = 1$. **(M.U. 1997)**

Sol. : To prove the required result, we have to separate real part α from imaginary part β.

$$\alpha + i\beta = \frac{\sin h[x+(i\pi/4)]}{\cos h[x+(i\pi/4)]}$$

$$= \frac{2\sin h[x+(i\pi/4)]}{2\cos h[x+(i\pi/4)]}\cdot\frac{\cos h[x-(i\pi/4)]}{\cos h[x-(i\pi/4)]}$$

$$= \frac{\sin h\,2x + \sin h(i\pi/2)}{\cos h\,2x + \cos h(i\pi/2)} \quad \text{[By (d), page 3-5]}$$

$$= \frac{\sin h\,2x + i\sin(\pi/2)}{\cos h\,2x + \cos(\pi/2)} = \frac{\sin h\,2x + i}{\cos h\,2x}$$

Equating real and imaginary parts.

$$\therefore \quad \alpha = \frac{\sin h\,2x}{\cos h\,2x}, \quad \beta = \frac{1}{\cos h\,2x}$$

$$\therefore \quad \frac{\sqrt{2}}{\cos hx + i \sin hx} = u + iv \qquad \therefore \quad \frac{\sqrt{2}\,(\cos hx - i \sin hx)}{\cos h^2 x + \sin h^2 x} = u + iv$$

$$\therefore \quad \frac{\sqrt{2}\,(\cos hx - i \sin hx)}{\cos h\, 2x} = u + iv$$

Equating real and imaginary parts,

$$\therefore \quad u = \frac{\sqrt{2} \cos hx}{\cos h\, 2x}, \quad v = -\frac{\sqrt{2} \sin hx}{\cos h\, 2x}$$

$$\therefore \quad u^2 + v^2 = \frac{2}{\cos h^2\, 2x}(\cos h^2 x + \sin h^2 x) = \frac{2 \cos h\, 2x}{\cos h^2 (2x)} = \frac{2}{\cos h\, 2x}$$

$$\therefore \quad (u^2 + v^2)^2 = \left(\frac{2}{\cos h\, 2x}\right)^2 = \frac{4}{\cos h^2\, 2x} \qquad \text{.......................... (1)}$$

Also $\quad u^2 - v^2 = \frac{2}{\cos h^2\, 2x}(\cos h^2 x - \sin h^2 x) = \frac{2}{\cos h^2\, 2x}$ .... (2)

From (1) and (2), $\quad (u^2 + v^2)^2 = 2\,(u^2 - v^2)$.

Example 7 (b) : If $\sin(\theta + i\Phi) = \tan \alpha + i \sec \alpha$, prove that $\cos 2\theta \cos h\, 2\Phi = 3$.

(M.U. 1987, 91, 96, 2005, 10)

Sol. : We have $\quad \sin(\theta + i\Phi) = \tan\alpha + i\sec\alpha$

$$\therefore \quad \sin\alpha \cos i\Phi + \cos\theta \sin i\Phi = \tan\alpha + i \sec\alpha$$

$$\sin\theta \cos h\Phi + i \cos\theta \sin h\Phi = \tan\alpha + i\sec\alpha$$

Equating real and imaginary parts,

$$\tan\alpha = \sin\theta \cos h\Phi\ ; \qquad \sec\alpha = \cos\theta \sin h\Phi$$

But $\quad \sec^2\alpha - \tan^2\alpha = 1$

$$\therefore \quad \cos^2\theta \sin h^2\Phi - \sin^2\theta \cos^2 h\Phi = 1$$

$$\left(\frac{1+\cos 2\theta}{2}\right)\left(\frac{\cos h2\Phi - 1}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos h2\Phi}{2}\right) = 1$$

$$\therefore \quad \cos h\, 2\Phi - 1 + \cos 2\theta \cos h\, 2\Phi - \cos 2\theta - 1$$
$$- \cos h\, 2\Phi + \cos 2\theta + \cos 2\theta \cos h\, 2\Phi = 4$$

$$\therefore \quad 2\cos 2\theta \cos h\, 2\Phi = 6 \qquad \therefore \quad \cos 2\theta \cos h\, 2\Phi = 3.$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : If $\tan(\alpha + i\beta) = \sin(x + iy)$, prove that $\frac{\tan x}{\tan hy} = \frac{\sin 2\alpha}{\sin h 2\beta}$. **(M.U. 1988, 2004)**

Sol. : Since $\tan(\alpha + i\beta) = \sin(x + iy)$; $\quad \tan(\alpha - i\beta) = \sin(x - iy)$.

Let $A = \alpha + i\beta,\ B = \alpha - i\beta,\ C = x + iy,\ D = x - iy$

$$\therefore \quad \tan A = \sin C \quad \text{and} \quad \tan B = \sin D$$

$$\therefore \quad \tan A + \tan B = \sin C + \sin D$$

And $\tan A - \tan B = \sin C - \sin D$.

$$\therefore \quad \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\sin C + \sin D}{\sin C - \sin D}$$

$$\frac{\dfrac{\sin A}{\cos A}+\dfrac{\sin B}{\cos B}}{\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}}=\frac{\sin C+\sin D}{\sin C-\sin B}$$

$$\therefore \quad \frac{\sin A\cos B+\cos A\sin B}{\sin A\cos B-\cos A\sin B}=\frac{2\sin\left(\dfrac{C+D}{2}\right)\cos\left(\dfrac{C-D}{2}\right)}{2\cos\left(\dfrac{C+D}{2}\right)\sin\left(\dfrac{C-D}{2}\right)}$$

[By Formula (F), See Appendix, page A-2]

$$\therefore \quad \frac{\sin(A+B)}{\sin(A-B)}=\frac{\tan\left(\dfrac{C+D}{2}\right)}{\tan\left(\dfrac{C-D}{2}\right)}$$

Resubstituting the values of A, B, C, D

$$\frac{\sin 2\alpha}{\sin 2i\beta}=\frac{\tan x}{\tan iy} \qquad \therefore \quad \frac{\sin 2\alpha}{\sin h\, 2\beta}=\frac{\tan x}{\tan hy}.$$

Aliter : Since both sides of the data are complex functions we shall use (iii) of § 9, page 3-17.

$$\tan(\alpha+i\beta)=\frac{\sin 2\alpha}{\cos 2\alpha+\cos h\, 2\beta}+\frac{i\sin h\, 2\beta}{\cos 2\alpha+\cos h\, 2\beta} \qquad \text{.................... (1)}$$

and $\quad \sin(x+iy)=\sin x\cos hy+i\cos x\sin hy$ .... (2)

Equating real and imaginary parts, from (1) and (2)

$$\frac{\sin 2\alpha}{\cos 2\alpha+\cos h\, 2\beta}=\sin x\cos hy \qquad \text{.................... (3)}$$

and

$$\frac{\sin h\, 2\beta}{\cos 2\alpha+\cos h\, 2\beta}=\cos x\sin hy \qquad \text{.................... (4)}$$

Dividing (3) by (4)

$$\frac{\sin 2\alpha}{\sin h\, 2\beta}=\frac{\sin x\cos hy}{\cos x\sin hy}=\frac{\sin x/\cos x}{\sin hy/\cos hy}=\frac{\tan x}{\tan hy}.$$

Example 2 (c) : If $\sin h(\theta+i\Phi)=e^{i\alpha}$, (or $=\cos\alpha+i\sin\alpha$), prove that

$$\sin h^4\theta=\cos^2\alpha=\cos^4\Phi.$$ **(M.U. 1985, 2000, 02)**

Sol. : We have $\quad \sin h(\theta+i\Phi)=\cos\alpha+i\sin\alpha$

$\therefore \quad \sin h\,\theta\cos h\, i\Phi+\cos h\,\theta\sin h\, i\Phi=\cos\alpha+i\sin\alpha$

But $\quad \sin h\, i\Phi=i\sin\Phi \quad$ and $\quad \cos h\, i\Phi=\cos\Phi$

$\therefore \quad \sin h\,\theta\cos\Phi+i\cos h\,\theta\sin\Phi=\cos\alpha+i\sin\alpha$

Equating real and imaginary parts,

$$\sin h\,\theta\cos\Phi=\cos\alpha, \quad \cos h\,\theta\sin\Phi=\sin\alpha \qquad \text{.................... (1)}$$

First we eliminate α,

$$1=\cos^2\alpha+\sin^2\alpha=\sin h^2\theta\cos^2\Phi+\cos h^2\theta\sin^2\Phi$$

$$=\sin h^2\theta\cos^2\Phi+(1+\sin h^2\theta)(1-\cos^2\Phi)$$

$\therefore \quad 1=\sin h^2\cos^2\Phi+1-\cos^2\Phi+\sin h^2\theta-\sin h^2\theta\cos^2\Phi$

$\therefore \quad \sin h^2\theta=\cos^2\Phi$

Now, we elminate Φ,

$$1 = \cos^2\Phi + \sin^2\Phi = \frac{\cos^2\alpha}{\sin h^2\theta} + \frac{\sin^2\alpha}{\cos h^2\theta} \qquad [\text{From (1)}]$$

$$\therefore \quad 1 = \frac{\cos^2\alpha}{\sin h^2\theta} + \frac{1-\cos^2\alpha}{1+\sin h^2\theta} = \frac{\cos^2\alpha + \sin h^2\theta}{\sin h^2\theta + \sin h^4\theta}$$

$\therefore \quad \sin h^4\theta = \cos^2\alpha$ (after simplification)

(Compare this example with Example 4, page 3-21.)

Example 3 (c) : If $\tan(A + iB) = \alpha + i\beta$, show that $\dfrac{1-(\alpha^2+\beta^2)}{1+(\alpha^2+\beta^2)} = \dfrac{\cos 2A}{\cos h\, 2B}$. **(M.U. 1997, 2011)**

Sol. : Let $A + iB = x$ and $A - iB = y$

$\therefore \quad \tan x = \tan(A + iB) = \alpha + i\beta \quad$ and $\quad \tan y = \tan(A - iB) = \alpha - i\beta$

$\therefore \quad \alpha^2 + \beta^2 = \tan x \tan y$

$$\therefore \quad \text{l.h.s.} = \frac{1-\tan x\tan y}{1+\tan x \tan y} = \frac{\cos x\cos y - \sin x \sin y}{\cos x \cos y + \sin x \sin y}$$

$$= \frac{\cos(x+y)}{\cos(x+y)} = \frac{\cos[(A+iB)+(A-iB)]}{\cos[(A+iB)-(A-iB)]}$$

$$= \frac{\cos 2A}{\cos 2iB} = \frac{\cos 2A}{\cos h\, 2B} = \text{r.h.s.}$$

Example 4 (c) : Separate into real and imaginary parts

$\tan^{-1} e^{i\theta}$ [or $\tan^{-1}(\cos\theta + i\sin\theta)$]. **(M.U. 1993, 2008, 09, 13)**

Sol. : Let $\tan^{-1} e^{i\theta} = x + iy \quad \therefore \quad e^{i\theta} = \tan(x + iy)$

$\therefore \quad \cos\theta + i\sin\theta = \tan(x+iy) \quad \therefore \quad \cos\theta - i\sin\theta = \tan(x - iy)$

Now, $\tan 2x = \tan[(x+iy)+(x-iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(\cos\theta + i\sin\theta)+(\cos\theta - i\sin\theta)}{1-(\cos\theta+i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= \frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)} = \frac{2\cos\theta}{1-1} = \frac{2\cos\theta}{0} = \infty$$

$$\therefore \quad 2x = \frac{\pi}{2} \qquad \therefore \quad x = \frac{\pi}{4}.$$

Also $\tan 2iy = \tan[(x+iy)-(x-iy)]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)}{1+(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}$$

$$= \frac{2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)} = \frac{2i\sin\theta}{2}$$

$\therefore \quad i \tan h\, 2y = i \sin \theta \qquad \therefore \quad \tan h\, 2y = \sin \theta$

$\therefore \quad 2y = \tan h^{-1} \sin \theta \qquad \therefore \quad y = \dfrac{1}{2} \tan h^{-1} \sin \theta.$

Example 5 (c) : If $x + iy = c \cot (u + iv)$, show that

$$\frac{x}{\sin 2u} = -\frac{y}{\sin h\, 2v} = \frac{c}{\cos h\, 2v - \cos 2u}.$$

(M.U. 1994, 2007, 08)

Sol. : We have $\quad x + iy = c \cot (u + iv) \qquad \therefore \quad x - iy = c \cot (u - iv)$

By adding the two events, we get

$$2x = c[\cot (u + iv) + \cot (u - iv)] = c\left[\frac{\cos (u + iv)}{\sin (u + iv)} + \frac{\cos (u - iv)}{\sin (u - iv)}\right]$$

$$= c\frac{[\cos (u + iv) \sin (u - iv) + \sin (u + iv) \cos (u - iv)]}{\sin (u + iv) \sin (u - iv)}$$

$$\therefore \quad 2x = \frac{c \sin [(u - iv) + (u + iv)]}{-[\cos (u + iv + u - iv) - \cos (u - iv - u + iv)]/2}$$

$$\therefore \quad x = \frac{c \sin 2u}{-[\cos 2u - \cos 2iv]} = \frac{c \sin 2u}{\cos h\, 2v - \cos 2u}$$

Now, by subtracting the same results, we get

$$2iy = c[\cot (u + iv) - \cot (u - iv)] = c\left[\frac{\cos (u + iv)}{\sin (u + iv)} - \frac{\cos (u - iv)}{\sin (u - iv)}\right]$$

$$= c\left[\frac{\cos (u + iv) \sin (u - iv) - \cos (u - iv) \sin (u + iv)}{\sin (u + iv) \sin (u - iv)}\right]$$

$$\therefore \quad 2iy = \frac{c \sin [(u - iv) - (u + iv)]}{-[\cos (u + iv + u - iv) - \cos (u + iv - u + iv)]/2}$$

$$\therefore \quad iy = \frac{c \sin (-2iv)}{-[\cos 2u - \cos 2iv]} = -\frac{ic \sin h\, 2v}{\cos h\, 2v - \cos 2u}.$$

Hence, the result.

Example 6 (c) : If $\cos (\alpha + i\beta) = x + iy$, prove that

$$(\cos \alpha + \cos h\, \beta)^2 = (1 + x)^2 + y^2 \quad \text{and} \quad (\cos \alpha - \cos h\, \beta)^2 = (1 - x)^2 + y^2.$$

Sol. : We have $\quad \cos \alpha \cos i\beta - \sin \alpha \sin i\beta = x + iy$

$\therefore \quad \cos \alpha \cos h\, \beta - i \sin \alpha \sin h\, \beta = x + iy$

Now, consider $\; 1 + x + iy = 1 + \cos \alpha \cos h\, \beta - i \sin \alpha \sin h\, \beta$

Equating the squares of the moduli on both sides,

$$\begin{aligned}(1 + x)^2 + y^2 &= (1 + \cos \alpha \cos h\, \beta)^2 + \sin^2 \alpha \sin h^2 \beta \\ &= 1 + 2 \cos \alpha \cos h\, \beta + \cos^2 \alpha \cos h^2 \beta + (1 - \cos^2 \alpha)(\cos h^2 \beta - 1) \\ &= 1 + 2 \cos \alpha \cos h\, \beta + \cos^2 \alpha \cos h^2 \beta + \cos h^2 \beta - 1 \\ &\qquad - \cos^2 \alpha \cos h^2 \beta + \cos^2 \alpha \\ &= \cos^2 \alpha + 2 \cos \alpha \cos h\, \beta + \cos h^2 \beta \\ &= (\cos \alpha + \cos h\, \beta)^2\end{aligned}$$

Now, consider $\; 1 - x - iy = 1 - \cos \alpha \cos h\, \beta + i \sin \alpha \sin h\, \beta$

Equating the squares of the moduli on both sides,

$$\begin{aligned}(1-x)^2+y^2 &= (1-\cos\alpha\cos h\beta)^2+\sin^2\alpha\sin h^2\beta \\ &= 1-2\cos\alpha\cos h\beta+\cos^2\alpha\cos h^2\beta+\sin^2\alpha\sin h^2\beta \\ &= 1-2\cos\alpha\cos h\beta+\cos^2\alpha\cos h^2\beta+(1-\cos^2\alpha)(\cos h^2\beta-1) \\ &= 1-2\cos\alpha\cos h\beta+\cos^2\alpha\cos h^2\beta+\cos h^2\beta-1 \\ &\qquad -\cos^2\alpha\cos h^2\beta+\cos^2\alpha \\ &= \cos^2\alpha-2\cos\alpha\cos h\beta+\cos h^2\beta \\ &= (\cos\alpha-\cos h\beta)^2\end{aligned}$$

Example 7 (c) : If $x+iy=\cos(\alpha+i\beta)$ [or if $\cos^{-1}(x+iy)=\alpha+i\beta$] express x and y in terms of α and β. Hence, show that $\cos^2\alpha$ and $\cos h^2\beta$ are the roots of the equation

$$\lambda^2-(x^2+y^2+1)\lambda+x^2=0. \qquad \textbf{(M.U. 1990, 2002, 04)}$$

Sol. : We have, $\cos\alpha\cos i\beta-\sin\alpha\sin i\beta=x+iy$

$\therefore \quad \cos\alpha\cos h\beta-i\sin\alpha\sin h\beta=x+iy$

Equating real and imaginary parts,

$\therefore \quad \cos\alpha\cos h\beta=x$ and $\sin\alpha\sin h\beta=-y.$

We know that, in terms of the roots, the quadratic equation is given by

λ^2- (sum of the roots) $\lambda+$ (product of the roots) $=0$

[By Formula (D), See Appendix, page A-3]

Hence, the equation whose roots are $\cos^2\alpha$ and $\cos^2\beta$ is

$$\lambda^2-(\cos^2\alpha+\cos^2\beta)\lambda+(\cos^2\alpha\cdot\cos^2\beta)=0$$

This means we have to prove that $x^2+y^2+1=\cos^2\alpha+\cos^2\beta$ and $x^2=\cos^2\alpha\cdot\cos^2\beta$.

Now, from (1),

$$\begin{aligned}x^2+y^2+1 &= \cos^2\alpha\cos h^2\beta+\sin^2\alpha\sin h^2\beta+1 \\ &= \cos^2\alpha\cos h^2\beta+(1-\cos^2\alpha)(\cos h^2\beta-1)+1 \\ &= \cos^2\alpha\cos h^2\beta+\cos h^2\beta-1-\cos^2\alpha\cos h^2\beta+\cos^2\alpha+1 \\ &= \cos^2\alpha+\cos h^2\beta = \text{sum of the roots.}\end{aligned}$$

And also from (1), $x^2=\cos^2\alpha\cos h^2\beta=$ product of the roots.

Hence, the equation whose roots are $\cos^2\alpha$, $\cos h^2\beta$ is

$$\lambda^2-(x^2+y^2+1)\lambda+x^2=0.$$

Example 8 (c) : If $\tan(u+iv)=x+iy$, prove that the curves $u=$ constant, $v=$ constant are a family of circles which are mutually orthogonal. **(M.U. 1994, 95)**

Sol. : We have $\tan(u+iv)=x+iy \qquad \therefore \tan(u-iv)=x-iy$

To separate real and imaginary parts, consider,

$$\tan 2u=\tan[(u+iv)+(u-iv)]=\frac{\tan(u+iv)+\tan(u-iv)}{1-\tan(u+iv)\tan(u-iv)}$$

$$\therefore \quad \tan 2u=\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}=\frac{2x}{1-x^2-y^2}$$

And $\quad \tan 2iv=\tan[(u+iv)-(u-iv)]$

$$=\frac{\tan(u+iv)-\tan(u-iv)}{1+\tan(u+iv)\tan(u-iv)}=\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$\therefore \quad i \tan h 2v = \dfrac{2iy}{1+x^2+y^2} \qquad \therefore \quad \tan h 2v = \dfrac{2y}{1+x^2+y^2}$

Since u = constant, $\tan 2u$ = constant.

$$\therefore \quad \frac{2x}{1-x^2-y^2} = \frac{1}{c_1}, \text{ a constant} \qquad \therefore \quad x^2+y^2+2c_1x-1=0 \qquad \ldots\ldots (1)$$

Since v = constant, $\tan h 2v$ = constant

$$\therefore \quad \frac{2y}{1+x^2+y^2} = \frac{1}{c_2}, \text{ a constant} \qquad \therefore \quad x^2+y^2-2c_2y+1=0 \qquad \ldots\ldots (2)$$

Therefore, equations (1) and (2) represent family of circles.

Now, differentiating (1) and (2),

$$2x+2y\frac{dy}{dx}+2c_1=0 \qquad \therefore \quad \frac{dy}{dx} = -\frac{x+c_1}{y}$$

and $\quad 2x+2y\dfrac{dy}{dx}-2c_2\dfrac{dy}{dx}=0 \qquad \therefore \quad \dfrac{dy}{dx} = -\dfrac{x}{y-c_2}$

But $\dfrac{dy}{dx}$ represents the slope of the tangent to the curve *i.e.* the slope of the curve.

Product of the slopes $= \dfrac{x+c_1}{y} \cdot \dfrac{x}{y-c_2} = \dfrac{x(x+c_1)}{y(y-c_2)}$.

By adding (2) to (1), we get,

$$x^2+y^2+c_1x-c_2y=0 \qquad \therefore \quad x(x+c_1) = -y(y-c_2)$$

$$\therefore \quad \frac{x(x+c_1)}{y(y-c_2)} = -1$$

$\therefore$ Product of the slopes $= -1$.

Hence, families of circles (1) and (2) are orthogonal.

Example 9 (c) : If $\dfrac{x+iy-c}{x+iy+c} = e^{u+iv}$, prove that $x = -\dfrac{c\sin hu}{\cos hu - \cos v}$, $\quad y = \dfrac{c \sin hv}{\cos hu - \cos v}$.

(M.U. 1998)

Further, if $v = (2n+1)\dfrac{\pi}{2}$ and n is an integer, prove that $x^2+y^2=c^2$.

Sol. : Let $z = x+iy$ and $w = u+iv$ $\therefore$ By data $\dfrac{z-c}{z+c} = e^w$.

By componendo and dividendo,

$$\therefore \quad \frac{z}{c} = \frac{1+e^w}{1-e^w} \qquad \therefore \quad \frac{z}{c} = \frac{1+e^w}{1-e^w}\cdot\frac{e^{-w/2}}{e^{-w/2}} \qquad \therefore \quad \frac{z}{c} = \frac{e^{-w/2}+e^{w/2}}{e^{-w/2}-e^{w/2}}$$

$$\therefore \quad \frac{x}{c}+\frac{y}{c}i = \frac{\cos h(w/2)}{-\sin h(w/2)}$$

$$\therefore \quad -\frac{x}{c}-i\frac{y}{c} = \frac{\cos h[(u/2)+(iv/2)]}{\sin h[(u/2)+(iv/2)]}$$

$$= \frac{2\cos h[(u/2)+(iv/2)]}{2\sin h[(u/2)+(iv/2)]} \cdot \frac{\sin h[(u/2)-(iv/2)]}{\sin h[(u/2)-(iv/2)]}$$

$$= \frac{\sin hu - \sin h(iv)}{\cos hu - \cos h(-iv)}$$ [By § 5 (d), page 3-5]

$$\therefore \quad -\frac{x}{c} - i\frac{y}{c} = \frac{\sin hu - i\sin v}{\cos hu - \cos v}$$

Equating real and imaginary parts, we get,

$$\frac{x}{c} = -\frac{\sin hu}{\cos hu - \cos v}, \qquad \frac{y}{c} = \frac{\sin v}{\cos hu - \cos v}$$

$$\therefore \quad x = -\frac{c\sin hu}{\cos hu - \cos v}, \qquad y = \frac{c\sin v}{\cos hu - \cos v}$$

Further, if $v = (2n+1)\frac{\pi}{2}$ and n is an integer, then $\cos v = 0$ and $\sin v = \pm 1$.

$$\therefore \quad x = -\frac{c\sin hu}{\cos hu}, \quad y = \pm\frac{c}{\cos hu} \qquad \therefore \quad x^2 + y^2 = c^2\frac{(1+\sin h^2 u)}{\cos h^2 u} = c^2.$$

Example 10 (c) : If $\frac{u-1}{u+1} = \sin(x+iy)$, where $u = \alpha + i\beta$, show that the argument of u is $\theta - \Phi$ where $\tan\theta = \frac{\cos x \sin hy}{1+\sin x\cos hy}$ and $\tan\Phi = \frac{-\cos x\sin hy}{1-\sin x\sin hy}$. **(M.U. 1995)**

Sol. : By data, $\frac{u-1}{u+1} = \frac{\sin(x+iy)}{1}$

By componendo and dividendo, we get

$$\therefore \quad u = \frac{1+\sin(x+iy)}{1-\sin(x+iy)} = \frac{(1+\sin x\cos hy) + i\cos x\sin hy}{(1-\sin x\cos hy) - i\cos x\sin hy} \qquad \text{.................... (1)}$$

Let $(1 + \sin x\cos hy) + i\cos x \sin hy = r_1 e^{i\theta}$

and $(1 - \sin x\cos hy) - i\cos x\sin hy = r_2 e^{i\Phi}$

$$\therefore \quad \tan\theta = \frac{\text{Imaginary part}}{\text{Real part}} = \frac{\cos x\sin hy}{1+\sin x\cos hy},$$

$$\tan\Phi = \frac{\text{Imaginary part}}{\text{Real part}} = \frac{-\cos x\sin hy}{1-\sin x\cos hy}$$

Now from (1), $u = \frac{r_1 e^{i\theta}}{r_2 e^{i\Phi}} = \frac{r_1}{r_2}e^{i(\theta-\Phi)}$.

Therefore, argument of u is $\theta - \Phi$ when θ and Φ are given as above.

[From $\tan\Phi = -\frac{\cos x\sin hy}{1-\sin x\cos hy}$, we get

$$-\tan\Phi = \frac{\cos x\cos hy}{1-\sin x\sin hy} \qquad \therefore \quad \tan(-\Phi) = \frac{\cos x\cos hy}{1-\sin x\sin hy}$$

Putting $-\Phi = \psi$, we see that the argument of u is

$$\theta + \psi \qquad \text{where } \tan\theta = \frac{\cos x\sin hy}{1+\sin x\cos hy}$$

and $\tan\psi = \frac{\cos x\cos hy}{1-\sin x\sin hy}$.]

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 marks

1. Separate into real and imaginary parts.

(i) $\cos h\,(x+iy)$ (ii) $\cot h\,(x+iy)$ (iii) $\sec h\,(x+iy)$

(iv) $\cot h\, i\,(x+iy)$ (v) $\cos\,(x+iy)$ (vi) $\tan\,(x+iy)$

(vii) $\cot\,(x+iy)$ (viii) $\sec\,(x+iy)$

[**Ans. :** (i) $\cos h\,x\cos y+i\sin h\,x\sin y$

(ii) $(\sin h\,2x-i\sin 2y)\,/\,(\cos h\,2x-\cos 2y)$

(iii) $(2\cos hx\cos y-i\,2\sin hx\sin y)\,/\,(\cos h\,2x+\cos 2y)$

(iv) $(-\sin h\,2y-i\sin 2x)\,/\,(\cos h\,2x-\cos 2y)$

(v) $\cos x\cos hy-i\sin x\sin h\,y$

(vi) $(\sin 2x+i\sin h\,2y)\,/\,(\cos 2x+\cos h\,2y)$

(vii) $(\sin 2x-i\sin h\,2y)\,/\,(-\cos 2x+i\cos h\,2y)$

(viii) $\dfrac{2(\cos x\cos hy+i\sin x\sin hy)}{\cos 2x+\cos h\,2y}$]

2. Separate into real and imaginary parts : $\tan^{-1}(\alpha+i\beta)$. **(M.U. 1995, 2002)**

$$\left[\textbf{Ans. :}\ \frac{1}{2}\tan^{-1}\left(\frac{2\alpha}{1-\alpha^2-\beta^2}\right),\ \frac{1}{2}\tan h^{-1}\left(\frac{2\beta}{1+\alpha^2+\beta^2}\right)\right]$$

3. If $A+iB=C\tan(x+iy)$, prove that $\tan 2x=\dfrac{2CA}{C^2-A^2-B^2}$.

4. If $\cos(\theta+i\Phi)=r(\cos\alpha+i\sin\alpha)$, prove that

$$r^2=\frac{1}{2}[\cos h\,2\Phi+\cos 2\theta],\quad \tan\alpha=-\tan\theta\tan h\,\Phi.$$

5. If $\cos(\alpha+i\beta)=x+iy$, prove that

$$\frac{x^2}{\cos^2\alpha}-\frac{y^2}{\sin^2\alpha}=1,\quad \frac{x^2}{\cos h^2\beta}+\frac{y^2}{\sin h^2\beta}=1.$$ **(M.U. 1991)**

Class (b) : 6 Marks

1. If $\cot\left(\dfrac{\pi}{6}+i\alpha\right)=x+iy$, prove that $x^2+y^2-\dfrac{2x}{\sqrt{3}}=1$.

2. If $\tan\left(\dfrac{\pi}{3}+i\alpha\right)=x+iy$, prove that $x^2+y^2-\dfrac{2x}{\sqrt{3}}-1=0$.

3. If $\tan\left(\dfrac{\pi}{8}+i\alpha\right)=x+iy$, prove that $x^2+y^2+2x=1$.

4. If $\tan h\left(\alpha+\dfrac{i\pi}{6}\right)=x+iy$, prove that $x^2+y^2+\dfrac{2y}{\sqrt{3}}=1$.

5. If $\cot\left(\dfrac{\pi}{8}+i\alpha\right)=x+iy$, prove that $x^2+y^2-2x=1$.

6. If $\cot h\left(\alpha+i\dfrac{\pi}{8}\right)=x+iy$, prove that $x^2+y^2+2y=1$.

7. If $\tan h\left(\alpha+\frac{i\pi}{8}\right)=x+iy$, prove that $x^2+y^2+2y=1$.

8. If $\tan(\alpha+i\beta)=x+iy$, prove that

$x^2+y^2+2x\cot 2\alpha=1$ and $x^2+y^2-2y\cot h\,2\beta=-1$. **(M.U. 1995, 2000, 02)**

9. If $\cot(\alpha+i\beta)=x+iy$, prove that

$x^2+y^2-2x\cot 2\alpha=1$, and $x^2+y^2+2y\cot h\,2\beta+1=0$.

10. If $\tan h(\alpha+i\beta)=x+iy$, prove that

$x^2+y^2+1=2x\cot h\,2\alpha$ and $x^2+y^2+2y\cot 2\beta=1$.

11. If $\sin h(x+iy)=e^{i\pi/3}$, prove that

(i) $3\cos^2 y-\sin^2 y=4\sin^2 y\cos^2 y$

(ii) $3\sin h^2 x+\cos h^2 x=4\sin h^2 x\cos h^2 x$.

12. If $\cos(x+iy)=e^{i\pi/6}$, prove that

(i) $3\sin^2 x-\cos^2 x=4\sin^2 x\cos^2 x$

(ii) $3\sin h^2 y+\cos h^2 y=4\sin h^2 y\cos h^2 y$.

13. If $x+iy=2\sin h\left(\alpha+\frac{i\pi}{4}\right)$, prove that $y^2-x^2=2$.

14. If $x+iy=2\cos h\left(\alpha+\frac{i\pi}{3}\right)$, prove that $3x^2-y^2=3$.

15. If $\cos h(\theta+i\Phi)=e^{i\alpha}$, prove that $\sin^2\alpha=\sin^4\Phi=\sin h^4\theta$.

16. If $\cot(u+iv)=\operatorname{cosec}(x+iy)$, prove that $\cot hy\sin h\,2v=\cot x\sin 2u$.

17. If $\sin h(a+ib)=x+iy$, prove that

$x^2\operatorname{cosec} h^2 a+y^2\sec h^2 a=1$ and $y^2\operatorname{cosec}^2 b-x^2\sec^2 b=1$.

18. If $\cos(u+iv)=x+iy$, prove that

$(1+x)^2+y^2=(\cos hv+\cos u)^2$ and $(1-x)^2+y^2=(\cos hv-\cos u)^2$ **(M.U. 1996)**

19. If $\tan(x+iy)=\sin(u+iv)$ then prove that $\frac{\sin 2x}{\sin h\,2y}=\frac{\tan u}{\tan hv}$. **(M.U. 1986)**

20. If $\tan y=\tan\alpha\tan h\beta$, $\tan z=\cot\alpha\tan h\beta$, prove that

$\tan(y+z)=\sin h\,2\beta\operatorname{cosec}2\alpha$.

21. Show that $\tan\left(\frac{u+iv}{2}\right)=\frac{\sin u+i\sin hv}{\cos u+\cos hv}$.

22. If $\sin^{-1}(\alpha+i\beta)=x+iy$, show that $\sin^2 x$ and $\cos h^2 y$ are the roots of the equation

$\lambda^2-(\alpha^2+\beta^2+1)\lambda+\alpha^2=0$.

Class (c) : 8 Marks

1. If $\cos(x+iy)=\cos\alpha+i\sin\alpha$, prove that

(i) $\sin\alpha=\pm\sin^2 x=\pm\sin h^2 y$, **(ii)** $\cos 2x+\cos h\,2y=2$ **(M.U. 1984)**

2. If $\sin(x+iy)=\cos\alpha+i\sin\alpha$, prove that

(i) $\cos h\,2y-\cos 2x=2$ **(ii)** $y=\frac{1}{2}\log\left[\frac{\cos(x-\alpha)}{\cos(x+\alpha)}\right]$

(iii) $\sin\alpha=\pm\cos^2 x=\pm\sin h^2 y$ **(M.U. 1985)**

3. If $\frac{1}{p} = \frac{1}{Lpi} + Cpi + \frac{1}{R}$ where L, p, R are real, prove that $p = Ae^{i\theta}$, where

$$A = \frac{1}{\sqrt{\frac{1}{R^2} + \left(\frac{1}{Lp} - Cp\right)^2}} \quad \text{and} \quad \tan\theta = R\left(\frac{1}{Lp} - Cp\right).$$

10. Inverse Hyperbolic Functions

Definition : If $\sin hu = z$ then u is called **inverse hyperbolic sine** of z and is denoted by $u = \sin h^{-1} z$.

Similarly, we can define **inverse hyperbolic cosine** and **inverse hyperbolic tangent** which we denote as $\cos h^{-1} z$, $\tan h^{-1} z$, etc.

The inverse hyperbolic functions are many valued but we will consider their principal values only.

If z is real we can show that

1. $\sin h^{-1} z = \log(z + \sqrt{z^2+1})$ **(M.U. 2004, 09)**
2. $\cos h^{-1} z = \log(z + \sqrt{z^2-1})$ **(M.U. 2009)**
3. $\tan h^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ **(M.U. 2004)**

Proof : (1) Let $\sin h^{-1} z = y$ $\quad\therefore\quad \sin h\, y = z$

$$\therefore\quad z = \sin hy = \frac{e^y - e^{-y}}{2} = \frac{e^{2y}-1}{2e^y} \qquad \therefore\quad e^{2y} - 2e^y z - 1 = 0$$

This is a quadratic in e^y.

$$\therefore\quad e^y = \frac{2z \pm \sqrt{4z^2+4}}{2} = z \pm \sqrt{z^2+1}$$

Conventionally we take positive sign.

$$\therefore\quad e^y = z + \sqrt{z^2+1} \qquad \therefore\quad y = \log(z + \sqrt{z^2+1})$$

$$\therefore\quad \sin h^{-1} z = \log(z + \sqrt{z^2+1})$$

(2) We leave this as an exercise.

(3) Let $\tan h^{-1} z = y$ $\qquad \therefore\quad z = \tan hy = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

By componendo and dividendo,

$$\therefore\quad \frac{1+z}{1-z} = \frac{(e^y + e^{-y}) + (e^y - e^{-y})}{(e^y + e^{-y}) - (e^y - e^{-y})} = \frac{2e^y}{2e^{-y}} = e^{2y}$$

$$\therefore\quad 2y = \log\left(\frac{1+z}{1-z}\right) \quad i.e.\quad y = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).$$

Note ....

Note carefully that z must be real.

Integration Formulae

If $y=\sin h^{-1}\left(\frac{x}{a}\right)$, then $y=\log(x+\sqrt{x^2+a^2})-\log a$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\left[1+\frac{x}{\sqrt{x^2+a^2}}\right]=\frac{1}{\sqrt{x^2+a^2}}$$

$$\therefore \quad \int\frac{dx}{\sqrt{x^2+a^2}}=\sin h^{-1}\frac{x}{a}$$

Also $$\int\frac{dx}{\sqrt{x^2-a^2}}=\cos h^{-1}\frac{x}{a}$$

and $$\int\frac{dx}{a^2-x^2}=\frac{1}{a}\tan h^{-1}\frac{x}{a}$$

Type I : On Inverse Representation : Class (a) : 3 Marks

Example 1 (a) : Prove that $\cos h^{-1}\sqrt{1+x^2}=\sin h^{-1}x$ **(M.U. 2002, 04, 07, 09)**

Sol. : To solve problems of this type, we start with one side l.h.s. or r.h.s., express the given function as direct function and obtain the other direct function. Then express it as an inverse function.

Let $\cos h^{-1}\sqrt{1+x^2}=y \quad \therefore \quad \sqrt{1+x^2}=\cos hy$

$\therefore \quad 1+x^2=\cos h^2y \quad \therefore \quad x^2=\cos h^2y-1=\sin h^2y$

$\therefore \quad x=\sin hy \quad \therefore \quad y=\sin h^{-1}x$

$\therefore \quad \cos h^{-1}\sqrt{1+x^2}=\sin h^{-1}x$

Example 2 (a) : Prove that $\cos h^{-1}(\sqrt{1+x^2})=\tan h^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$. **(M.U. 2002)**

Sol. : Let $\cos h^{-1}\sqrt{1+x^2}=y$, then as above, $\cos hy=\sqrt{1+x^2}$ and $\sin hy=x$.

$$\therefore \quad \tan hy=\frac{\sin hy}{\cos hy}=\frac{x}{\sqrt{1+x^2}} \quad \therefore \quad y=\tan h^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

$$\therefore \quad \cos h^{-1}\sqrt{1+x^2}=\tan h^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Example 3 (a) : Prove that $\tan h^{-1}x=\sin h^{-1}\frac{x}{\sqrt{1-x^2}}$. **(M.U. 2001, 02, 06, 09, 11)**

Sol. : Let $\tan h^{-1}x=y \quad \therefore \quad x=\tan hy$

$$\text{Now, } \frac{x}{\sqrt{1-x^2}}=\frac{\tan hy}{\sqrt{1-\tan^2h^2y}}=\frac{\tan hy}{\sqrt{\cos^2hy-\sin h^2y}\big/\cos hy}$$

$$=\frac{\sin hy}{\cos hy}\times\frac{\cos hy}{1}=\sin hy$$

$$\therefore \quad y=\sin h^{-1}\frac{x}{\sqrt{1-x^2}} \quad \therefore \quad \tan h^{-1}x=\sin h^{-1}\frac{x}{\sqrt{1-x^2}}.$$

Example 4 (a) : Prove that $\tan h^{-1}(\sin\theta) = \cos h^{-1}(\sec\theta)$. **(M.U. 1996, 2003, 11, 17)**

Sol. : Since $\tan h^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$\therefore \quad \tan h^{-1}(\sin\theta) = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) \qquad \text{.................... (i)}$$

Again since, [By (2), page 3-34] $\cos h^{-1}x = \log\left(x+\sqrt{x^2-1}\right)$

$$\cos h^{-1}(\sec\theta) = \log\left(\sec\theta+\sqrt{\sec^2\theta-1}\right) = \log(\sec\theta+\tan\theta)$$

$$\therefore \quad \cos h^{-1}(\sec\theta) = \log\left(\frac{1+\sin\theta}{\cos\theta}\right) = \frac{1}{2}\log\frac{(1+\sin\theta)^2}{\cos^2\theta} \qquad \text{[Note this]}$$

$$= \frac{1}{2}\log\frac{(1+\sin\theta)^2}{(1-\sin^2\theta)} = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right). \qquad \text{.................... (ii)}$$

From (i) and (ii) the result follows.

Example 5 (a) : Prove that $\cot h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right)$.

Sol. : Let $\cot h^{-1}\left(\frac{x}{a}\right) = y$ .... (1)

$$\therefore \quad \frac{x}{a} = \cot h\,y \qquad \therefore \quad \tan h\,y = \frac{1}{\cot h\,y} = \frac{1}{x/a} = \frac{a}{x}$$

$$\therefore \quad y = \tan h^{-1}\left(\frac{a}{x}\right) = \frac{1}{2}\log\left(\frac{1+(a/x)}{1-(a/x)}\right) \qquad \text{[By (3), page 3-34]}$$

$$\therefore \quad y = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right) \qquad \text{.................... (2)}$$

From (1) and (2) the required result follows.

Example 6 (a) : Prove that $\sec h^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$. **(M.U. 2014, 18)**

Sol. : Let $\sec h^{-1}(\sin\theta) = x \qquad \therefore \quad \sin\theta = \sec hx$

$$\therefore \quad \sin\theta = \frac{1}{\cos hx} = \frac{2}{e^x+e^{-x}} = \frac{2e^x}{e^{2x}+1}$$

$$\therefore \quad \sin\theta \cdot e^{2x} - 2e^x + \sin\theta = 0$$

This is a quadratic in e^x.

$$\therefore \quad e^x = \frac{2\pm\sqrt{4-4\sin^2\theta}}{2\sin\theta} = \frac{1\pm\cos\theta}{\sin\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = \cot\frac{\theta}{2}$$

$$\therefore \quad x = \log\left(\cot\frac{\theta}{2}\right)$$

$$\therefore \quad \sec h^{-1}(\sin\theta) = \log\left(\cot\frac{\theta}{2}\right)$$

Example 7 (a) : Prove that $\sin h^{-1}(\tan\theta) = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$. **(M.U. 1996)**

Sol. : Let $\sin h^{-1}(\tan\theta) = x$

$$\therefore \quad \tan\theta = \sin hx = \frac{e^x - e^{-x}}{2} = \frac{e^{2x}-1}{2e^x}$$

$$\therefore \quad e^{2x} - 2e^x\tan\theta - 1 = 0$$

Solving this quadratic in e^x.

$$e^x = \frac{2\tan\theta \pm \sqrt{4\tan^2\theta + 4}}{2} = \tan\theta \pm \sqrt{\tan^2\theta + 1} = \tan\theta + \sec\theta$$

$$\therefore \quad x = \log(\tan\theta + \sec\theta) = \log\left[\frac{1+\sin\theta}{\cos\theta}\right]$$

$$= \log\left[\frac{1+\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)}\right] = \log\left[\frac{2\cos^2(\pi/4-\theta/2)}{2\sin(\pi/4-\theta/2)\cos(\pi/4-\theta/2)}\right]$$

$$= \log\left[\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right] = \log\left[\tan\left(\frac{\pi}{2}-\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$= \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

Alternatively : By (1), page 3-34, we get

$$\sin^{-1}(\tan\theta) = \log(\tan\theta + \sqrt{\tan^2\theta + 1})$$

$$= \log(\tan\theta + \sec\theta) = \log\left(\frac{\sin\theta + 1}{\cos\theta}\right)$$

Then proceed as above.

Example 8 (a) : If $\cos hx = \sec\theta$, prove that $x = \log(\sec\theta + \tan\theta)$. **(M.U. 2003, 08)**

Sol. : Since $\cos hx = \sec\theta$, we have $x = \cos h^{-1}\sec\theta$.

By (2), page 3-34,

$$x = \log\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right) = \log(\sec\theta + \tan\theta).$$

Example 9 (a) : Find the value of $\tan h(\log\sqrt{k})$ in terms of k. Hence find $\tan h\log\sqrt{7}$.

Sol. : Let $z = \tan h(\log\sqrt{k})$

$$\therefore \quad \tan h^{-1} z = \log\sqrt{k} \qquad \therefore \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \frac{1}{2}\log k$$

$$\therefore \quad \log\left(\frac{1+z}{1-z}\right) = \log k \qquad \therefore \quad \frac{1+z}{1-z} = k$$

By componendo and dividendo

$$\frac{2}{-2z} = \frac{k+1}{1-k} \qquad \therefore \quad z = \frac{k-1}{k+1}. \qquad \text{.................... (A)}$$

Putting $k = 7$, we get $\tan h\log\sqrt{7} = \frac{7-1}{7+1} = \frac{3}{4}$.

[For another method, see Ex. 3, page 3-9.]

Example 10 (a) : Find the value of $\tan h(\log\sqrt{11})+\tan h(\log\sqrt{13})$.

Sol. : As proved above in (A) if $z=\tan(\log\sqrt{k})$ then $z=\dfrac{k-1}{k+1}$.

Let $z_1=\tan h(\log\sqrt{11})$ $\quad\therefore\quad z_1=\dfrac{11-1}{11+1}=\dfrac{10}{12}=\dfrac{5}{6}$

Let $z_2=\tan h(\log\sqrt{13})$ $\quad\therefore\quad z_2=\dfrac{13-1}{13+1}=\dfrac{12}{14}=\dfrac{6}{7}$

$$\therefore\quad z_1+z_2=\frac{5}{6}+\frac{6}{7}=\frac{35+36}{42}=\frac{71}{42}.$$

Example 11 (a) : Prove that $\tan h\log\sqrt{z}=\dfrac{z-1}{z+1}$.

Hence, deduce that $\tan h(\log\sqrt{5/3})+\tan h(\log\sqrt{7})=1$.

Sol. : As proved above in (A), $\tan h\log\sqrt{z}=\dfrac{z-1}{z+1}$.

Put $z=5/3$ and $z=7$ and add,

$$\log h(\log\sqrt{5/3})+\tan h(\log\sqrt{7})=\frac{(5/3)-1}{(5/3)+1}+\frac{7-1}{7+1}=\frac{2}{8}+\frac{6}{8}=1.$$

Example 12 (a) : If $\tan\left(\dfrac{x}{2}\right)=\tan h\left(\dfrac{u}{2}\right)$, prove that $u=\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$.

Sol. : By data $\tan h\dfrac{u}{2}=\tan\dfrac{x}{2}$

$$\therefore\quad \frac{u}{2}=\tan h^{-1}\left(\tan\frac{x}{2}\right)=\frac{1}{2}\log\left[\frac{1+\tan(x/2)}{1-\tan(x/2)}\right]$$

$$=\frac{1}{2}\log\left[\frac{\tan(\pi/4)+\tan(x/2)}{1-\tan(\pi/4)\ \tan(x/2)}\right]=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

[For another proof, see Ex. 4 (iii), page 3-14.]

Example 13 (a) : If $\cos h^{-1}a+\cos h^{-1}b=\cos h^{-1}x$, prove that

$$a\sqrt{b^2-1}+b\sqrt{a^2-1}=\sqrt{x^2-1}.$$

Sol. : Let $\cos h^{-1}a=\alpha$ $\quad\therefore\quad a=\cos h\,\alpha$

$\cos h^{-1}b=\beta$ $\quad\therefore\quad b=\cos h\,\beta$

$\cos h^{-1}x=\gamma$ $\quad\therefore\quad \gamma=\cos hx.$

Hence, by data $\alpha+\beta=\gamma$

Taking sin h of both sides, $\therefore\ \sin h(\alpha+\beta)=\sin\gamma$

$$\therefore\quad \sin h\,\alpha\cos h\,\beta+\cos h\,\alpha\sin h\,\beta=\sin h\,\gamma \qquad \text{.....................} (1)$$

Now, $\cos h^2\alpha-\sin h^2\alpha=1$ $\quad\therefore\quad \sin h^2\alpha=\cos h^2\alpha-1$

$$\therefore\quad \sin h\alpha=\sqrt{a^2-1}$$

Similarly, $\sin h\beta=\sqrt{b^2-1}$ and $\sin h\gamma=\sqrt{x^2-1}$.

Hence, putting these values in (1), we get $b\sqrt{a^2-1}+a\sqrt{b^2-1}=\sqrt{x^2-1}$.

Example 14 (a) : If $\sin h^{-1} a + \sin h^{-1} b = \sin h^{-1} x$, then prove that

$$x = a\sqrt{1+b^2} + b\sqrt{1+a^2}.$$

Sol. : Let $\sin h^{-1} a = \alpha,\ \sin h^{-1} b = \beta$ and $\sin h^{-1} x = \gamma$.

We are given $\alpha + \beta = \gamma$ $\quad\therefore\ \sin h(\alpha+\beta) = \sin h\gamma$

$\therefore\ \sin h\alpha \cos h\beta + \cos h\alpha \sin h\beta = \sin h\gamma$ .... (2)

But $\sin h\alpha = a,\ \sin h\beta = b,\ \sin h\gamma = x.$

$\therefore\ \cos h\alpha = \sqrt{1+\sin h^2\alpha} = \sqrt{1+a^2}$

$\therefore\ \cos h\beta = \sqrt{1+\sin h^2\beta} = \sqrt{1+b^2}$

Putting these values in (2), we get $a\sqrt{1+b^2} + b\sqrt{1+a^2} = x.$

Type II : On Separation of Real and Imaginary Parts : Class (c) : 8 Marks

Example 1 (c) : If $\tan z = \dfrac{i}{2}(1-i)$, prove that $z = \dfrac{1}{2}\tan^{-1}2 + \dfrac{i}{4}\log 5.$

Sol. : The required relation suggests that we have to separate real and imaginary parts.

By data, $\quad \tan z = \dfrac{1}{2}(i - i^2) = \dfrac{1}{2}i + \dfrac{1}{2}$

Let $z = x + iy$ $\quad\therefore\ \tan(x+iy) = \dfrac{i}{2} + \dfrac{1}{2}$

$\therefore\ \tan(x+iy) = \dfrac{1}{2} + \dfrac{i}{2},\quad \tan(x-iy) = \dfrac{1}{2} - \dfrac{i}{2}$

$$\therefore\ \tan(2x) = \tan[(x+iy)+(x-iy)] = \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\ \tan(x-iy)}$$

$$= \frac{[(1/2)+(i/2)]+[(1/2)-(i/2)]}{1-[(1/2)+(i/2)]\ [(1/2)-(i/2)]}$$

$$\therefore\ \tan(2x) = \frac{1}{1-[(1/4)+(1/4)]} = \frac{1}{1/2} = 2$$

$$\therefore\ 2x = \tan^{-1}2 \quad \therefore\ x = \frac{1}{2}\tan^{-1}2$$

$$\text{Now, } \tan(2iy) = \tan[(x+iy)-(x-iy)] = \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{[(1/2)+(i/2)]-[(1/2)-(i/2)]}{1+[(1/2)+(i/2)]\ [(1/2)-(i/2)]}$$

$$= \frac{i}{1+[(1/4)+(1/4)]} = \frac{i}{1+(1/2)} = \frac{2}{3}i$$

$$\therefore\ i\tan h2y = \frac{2}{3}i \quad \therefore\ \tan h2y = \frac{2}{3} \quad \therefore\ 2y = \tan h^{-1}\left(\frac{2}{3}\right)$$

$$\therefore\ 2y = \frac{1}{2}\log\left[\frac{1+(2/3)}{1-(2/3)}\right] = \frac{1}{2}\log 5$$ [By (3) of § 10, page 3-34]

$$\therefore\ y = \frac{1}{4}\log 5 \quad \therefore\ z = x+iy = \frac{1}{2}\tan^{-1}2 + \frac{i}{4}\log 5.$$

Example 2 (c) : Show that $\tan^{-1}\left(\dfrac{x+iy}{x-iy}\right)=\dfrac{\pi}{4}+\dfrac{i}{2}\log\left(\dfrac{x+y}{x-y}\right)$. **(M.U. 2015)**

Sol. : Let $\tan^{-1}\left(\dfrac{x+iy}{x-iy}\right)=\alpha+i\beta$

$$\therefore \quad \frac{x+iy}{x-iy}=\tan(\alpha+i\beta) \quad \text{and} \quad \frac{x-iy}{x+iy}=\tan(\alpha-i\beta)$$

Now, $\tan(2\alpha)=\tan[(\alpha+i\beta)+(\alpha-i\beta)]$

$$=\frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}=\frac{\left(\frac{x+iy}{x-iy}\right)+\left(\frac{x-iy}{x+iy}\right)}{1-\left(\frac{x+iy}{x-iy}\right)\left(\frac{x-iy}{x+iy}\right)}$$

$$=\frac{\left(\frac{x+iy}{x-iy}\right)+\left(\frac{x-iy}{x+iy}\right)}{1-1}=\frac{\left(\frac{x+iy}{x-iy}\right)+\left(\frac{x-iy}{x+iy}\right)}{0}=\infty$$

$$\therefore \quad \tan(2\alpha)=\tan\frac{\pi}{2}$$

$$\therefore \quad 2\alpha=\frac{\pi}{2} \qquad \therefore \quad \alpha=\frac{\pi}{4}.$$

Now, $\tan(2i\beta)=\tan[(\alpha+i\beta)-(\alpha-i\beta)]$

$$=\frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$=\frac{\left(\frac{x+iy}{x-iy}\right)-\left(\frac{x-iy}{x+iy}\right)}{1+\left(\frac{x+iy}{x-iy}\right)\left(\frac{x-iy}{x+iy}\right)}=\frac{\frac{(x+iy)^2-(x-iy)^2}{x^2+y^2}}{2}$$

$$=\frac{4ixy}{2(x^2+y^2)}=\frac{2ixy}{x^2+y^2}$$

$$\therefore \quad i\tanh 2\beta=\frac{2ixy}{x^2+y^2} \qquad \therefore \quad \tanh 2\beta=\frac{2xy}{x^2+y^2}$$

$$\therefore \quad 2\beta=\tanh^{-1}\left(\frac{2xy}{x^2+y^2}\right)$$

$$=\frac{1}{2}\log\left[\frac{1+[2xy/(x^2+y^2)]}{1-[2xy/(x^2+y^2)]}\right]$$ [By (3) of § 10, page 3-34]

$$=\frac{1}{2}\log\left(\frac{x^2+y^2+2xy}{x^2+y^2-2xy}\right)=\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)^2$$

$$\therefore \quad \beta=\log\left(\frac{x+y}{x-y}\right)=\frac{1}{2}\log\left(\frac{x+y}{x-iy}\right)$$

$$\therefore \quad \tan^{-1}\left(\frac{x+iy}{x-iy}\right)=\alpha+i\beta=\frac{\pi}{4}+\frac{i}{2}\log\left(\frac{x+y}{x-y}\right).$$

Example 3 (c) : Prove that $\sin^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} + i\log\cot\frac{\theta}{2}$. **(M.U. 1999)**

Sol. : The required result suggests that we have to separate real and imaginary parts of l.h.s.

Let $\sin^{-1}(\text{cosec}\,\theta) = \alpha + i\beta$

$\therefore\quad \text{cosec}\,\theta = \sin(\alpha + i\beta) = \sin\alpha\cos i\beta + \cos\alpha\sin i\beta$

$\quad = \sin\alpha\cos h\beta + i\cos\alpha\sin h\beta$

Equating real and imaginary parts,

$\text{cosec}\,\theta = \sin\alpha\cos h\beta$ and $\cos\alpha\sin h\beta = 0$

$\therefore\quad \cos\alpha = 0 \quad \therefore\quad \alpha = \pi/2$ and $\sin\alpha = 1$.

$\therefore\quad \text{cosec}\,\theta = \cos h\beta$

$\therefore\quad \beta = \cos h^{-1}(\text{cosec}\,\theta)$

$= \log(\text{cosec}\,\theta + \sqrt{\text{cosec}^2\theta - 1})$ [By (2) of § 10, page 3-34]

$= \log(\text{cosec}\,\theta + \cot\theta) = \log\left(\frac{1+\cos\theta}{\sin\theta}\right)$

$\therefore\quad \beta = \log\left(\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right) = \log\cot\left(\frac{\theta}{2}\right)$

$\therefore\quad \sin^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} + i\log\cot\left(\frac{\theta}{2}\right)$.

Example 4 (c) : Show that the principal value of $\sin^{-1} x$ is $i\log(x - \sqrt{1+x^2})$. **(M.U. 1998)**

Sol. : Let $\sin^{-1} ix = \alpha + i\beta \quad \therefore\quad ix = \sin(\alpha + i\beta)$

$\therefore\quad ix = \sin\alpha\cos i\beta + \cos\alpha\sin i\beta$

$\quad = \sin\alpha\cos h\beta + i\cos\alpha\sin h\beta$

Equating real and imaginary parts,

$\sin\alpha\cos h\beta = 0 \quad \therefore\quad \alpha = 0$ [or $\alpha = 2n\pi$, (General value)]

and $x = \cos\alpha\sin h\beta = \sin h\beta$ [$\because$ $\cos\alpha = 1$]

$\beta = \sin h^{-1} x = \log(x + \sqrt{1+x^2})$ [By (1) of § 10, page 3-34]

$\therefore\quad \sin ix = \alpha + i\beta = i\log(x + \sqrt{1+x^2})$

[General value of $\sin^{-1} ix = 2n\pi + i\log(x + \sqrt{1+x^2})$.]

Example 5 (c) : If $\cos\left(\frac{\pi}{4} + ia\right)\cdot\cos h\left(b + \frac{i\pi}{4}\right) = 1$ where a, b are real, prove that

$$2b = \log(2+\sqrt{3}).$$

Sol. : We have

$$\left(\cos\frac{\pi}{4}\cos ia - \sin\frac{\pi}{4}\sin ia\right)\cdot\left(\cos h b\cos h i\frac{\pi}{4} + \sin hb\sin h i\frac{\pi}{4}\right) = 1$$

$$\therefore\quad \left(\cos\frac{\pi}{4}\cos ha - \sin\frac{\pi}{4} i\sin ha\right)\cdot\left(\cos hb\cos\frac{\pi}{4} + \sin hb\cdot i\sin\frac{\pi}{4}\right) = 1$$

$\therefore \quad \left(\frac{1}{\sqrt{2}}\cos ha - \frac{1}{\sqrt{2}}\cdot i\sin ha\right)\cdot\left(\cos hb\cdot\frac{1}{\sqrt{2}} + \sin hb\cdot i\frac{1}{\sqrt{2}}\right) = 1$

Equating real and imaginary parts, (after expansion)

$\frac{1}{2}(\cos ha\cos hb + \sin ha\sin hb) = 1 \quad \therefore \quad \cos h\,(a+b) = 2$ .... (i)

and $\quad -\frac{1}{2}\sin ha\cos hb + \frac{1}{2}\cos ha\sin hb = 0$

$\therefore \quad \frac{1}{2}[\sin hb\cos ha - \cos hb\sin ha] = 0$

$\therefore \quad \sin h\,(b-a) = 0 \quad \therefore \quad b - a = 0 \quad \therefore \quad b = a$

Hence, from (i), $\quad \cos h\,2b = 2$

$\therefore \quad 2b = \cos h^{-1}(2) = \log(2+\sqrt{3}).$ [By (2) of § 10, page 3-34]

Aliter : We have $\quad \cos\left(\frac{\pi}{4} + ia\right)\cos h\left(b + \frac{i\pi}{4}\right) = 1$

$\therefore \quad \cos\left(\frac{\pi}{4} + ia\right)\cos i\left(b + \frac{i\pi}{4}\right) = 1 \quad \therefore \quad \cos\left(\frac{\pi}{4} + ia\right)\cos\left(-\frac{\pi}{4} + i\,b\right) = 1$

$\therefore \quad 2\cos\left(\frac{\pi}{4} + ia\right)\cos\left(-\frac{\pi}{4} + i\,b\right) = 2$

$\therefore \quad \cos\left[\left(\frac{\pi}{4} + ia\right) + \left(-\frac{\pi}{4} + i\,b\right)\right] + \cos\left[\left(\frac{\pi}{4} + ia\right) - \left(-\frac{\pi}{4} + i\,b\right)\right] = 2$

$\therefore \quad \cos i\,(a+b) + \cos\left[\frac{\pi}{2} + i\,(a-b)\right] = 2$

$\therefore \quad \cos h\,(a+b) - \sin i\,(a-b) = 2$

$\quad \cos h\,(a+b) - i\sin h\,(a-b) = 2$

Equating real and imaginary parts,

$\quad \cos h\,(a+b) = 2 \quad$ and $\quad \sin h\,(a-b) = 0$

$\therefore \quad a + b = \cos h^{-1}\,2 \quad$ and $\quad a - b = 0 \quad \therefore \quad b = a$

$\therefore \quad 2b = \cos h^{-1}\,2 = \log\,(2+\sqrt{3}).$ [By (2) of § 10, page 3-34]

Example 6 (c) : If $\cos hx = \sec\theta$, prove that

(i) $x = \log\,(\sec\theta + \tan\theta)$, **(ii)** $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

(iii) $\tan h\,(x/2) = \tan\,(\theta/2)$ **(M.U. 1994, 97, 2001, 03, 04, 17)**

Sol. : We have $\quad x = \cos h^{-1}\sec\theta$

$$= \log\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right)$$ [By (2) of § 10, page 3-34]

$\therefore \quad x = \log(\sec\theta + \tan\theta)$ .... (i)

$\therefore \quad e^x = \sec\theta + \tan\theta \quad \therefore \quad e^{-x} = \frac{1}{\sec\theta + \tan\theta} = \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta}$

$\therefore \quad e^{-x} = \sec\theta - \tan\theta = \frac{1-\sin\theta}{\cos\theta} = \frac{1-\cos\alpha}{\sin\alpha}$ where $\alpha = \frac{\pi}{2} - \theta$

$$\therefore \quad e^{-x} = \frac{2\sin^2(\alpha/2)}{2\sin(\alpha/2)\cos(\alpha/2)} = \tan\frac{\alpha}{2} \quad \therefore \quad \frac{\alpha}{2} = \tan^{-1}(e^{-x})$$

$$\therefore \quad \alpha = 2\tan^{-1}(e^{-x}) \quad \therefore \quad \frac{\pi}{2} - \theta = 2\tan^{-1}(e^{-x})$$

$$\therefore \quad \theta = (\pi/2) - 2\tan(e^{-x}) \qquad \text{......... (ii)}$$

Further, $\tan h\left(\frac{x}{2}\right) = \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} = \frac{e^x - 1}{e^x + 1}$

$$\therefore \quad \tan h\left(\frac{x}{2}\right) = \frac{\sec\theta + \tan\theta - 1}{\sec\theta + \tan\theta + 1} = \frac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}$$

$$= \frac{(1 - \cos\theta) + \sin\theta}{(1 + \cos\theta) + \sin\theta} = \frac{2\sin^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2) + 2\sin(\theta/2)\cos(\theta/2)}$$

$$= \frac{\sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}. \qquad \text{......... (iii)}$$

(See also Ex. 5 (ii), page 3-15.)

Example 7 (c) : If $\tan(\theta + i\Phi) = \tan\alpha + i\sec\alpha$ $\left\{\text{or } \theta + i\Phi = \tan^{-1}\left(\frac{i + \sin\alpha}{\cos\alpha}\right)\right\}$, prove that

$$e^{2\Phi} = \cot\frac{\alpha}{2} \quad \text{and} \quad 2\theta = n\pi + \frac{\pi}{2} + \alpha. \qquad \textbf{(M.U. 1986, 92, 2016)}$$

Sol. : We have $\tan(\theta + i\Phi) = \tan\alpha + i\sec\alpha$ $\quad \therefore \quad \tan(\theta - i\Phi) = \tan\alpha - i\sec\alpha$

$$\therefore \quad \tan 2\theta = \tan[(\theta + i\Phi) + (\theta - i\Phi)] = \frac{\tan(\theta + i\Phi) + \tan(\theta - i\Phi)}{1 - \tan(\theta + i\Phi)\tan(\theta - i\Phi)}$$

$$= \frac{(\tan\alpha + i\sec\alpha) + (\tan\alpha - i\sec\alpha)}{1 - (\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)} = \frac{2\tan\alpha}{1 - (\tan^2\alpha + \sec^2\alpha)}$$

$$= \frac{2\tan\alpha}{-2\tan^2\alpha} = -\cot\alpha = \tan\left(\frac{\pi}{2} + \alpha\right)$$

$$\therefore \quad 2\theta = n\pi + \frac{\pi}{2} + \alpha \ \text{(General value)} \qquad \text{......... (i)}$$

Again $\tan(2i\Phi) = \tan[(\theta + i\Phi) - (\theta - i\Phi)]$

$$= \frac{(\tan\alpha + i\sec\alpha) - (\tan\alpha - i\sec\alpha)}{1 + (\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$$

$$\therefore \quad i\tan h 2\Phi = \frac{2i\sec\alpha}{2\sec^2\alpha} = i\cos\alpha \quad \therefore \quad \tan h 2\Phi = \cos\alpha$$

$$\therefore \quad 2\Phi = \tan h^{-1}(\cos\alpha) = \frac{1}{2}\log\left[\frac{1 + \cos\alpha}{1 - \cos\alpha}\right] \qquad \text{[By (3) of § 10, page 3-34]}$$

$$= \frac{1}{2}\log\left[\frac{2\cos^2(\alpha/2)}{2\sin^2(\alpha/2)}\right] = \log\cot\frac{\alpha}{2}$$

$$\therefore \quad e^{2\Phi} = \cot(\alpha/2) \qquad \text{......... (ii)}$$

Example 8 (c) : If $\tan(x + iy) = i$ and x, y are real, prove that x is indeterminate and y is infinite.

Sol. : We have $\tan(x + iy) = i \quad \therefore \quad \tan(x - iy) = -i$

$\therefore \quad \tan(2x) = \tan[(x+iy)+(x-iy)] = \dfrac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$

$\therefore \quad \tan(2x) = \dfrac{i-i}{1-1} = \dfrac{0}{0} = \text{indeterminate}$

$\tan(2iy) = \tan[(x+iy)-(x-iy)] = \dfrac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$

$\therefore \quad \tan(2iy) = \dfrac{i-(-i)}{1+1} = \dfrac{2i}{2} = i$

$\therefore \quad i\tan h\, 2y = i \quad \therefore \quad \tan h\, 2y = 1$

$2y = \tan h^{-1}(1) = \dfrac{1}{2}\log\left(\dfrac{1+1}{1-1}\right) = \infty$ [By (3) of § 10, page 3-34]

$\therefore \quad y$ is infinite.

Example 9 (c) : If $\tan(\alpha+i\beta) = \cos\theta + i\sin\theta$ (or $= e^{i\theta}$), prove that

$$\alpha = \frac{n\pi}{2}+\frac{\pi}{4} \quad \text{and} \quad \beta = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$ **(M.U. 1983, 92, 2001, 19)**

Sol. : We have $\tan(\alpha+i\beta) = \cos\theta + i\sin\theta \quad \therefore \quad \tan(\alpha-i\beta) = \cos\theta - i\sin\theta$

$\therefore \quad \tan 2\alpha = \tan[(\alpha+i\beta)+(\alpha-i\beta)]$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\cdot\tan(\alpha-i\beta)} = \frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)}$$

$\therefore \quad \tan 2\alpha = \dfrac{2\cos\theta}{0}$

$\therefore \quad 2\alpha = \dfrac{\pi}{2}$ and $2\alpha = n\pi + \dfrac{\pi}{2}$ (General value)

$\therefore \quad \alpha = \dfrac{n\pi}{2}+\dfrac{\pi}{4}$ .... (i)

Also $\tan(2i\beta) = \tan[(\alpha+i\beta)-(\alpha-i\beta)]$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\cdot\tan(\alpha-i\beta)} = \frac{2i\sin\theta}{1+1} = i\sin\theta$$

$\therefore \quad i\tan h\, 2\beta = i\sin\theta \quad \therefore \quad \tan h\, 2\beta = \sin\theta$

$\therefore \quad 2\beta = \tan h^{-1}(\sin\theta) = \dfrac{1}{2}\log\left(\dfrac{1+\sin\theta}{1-\sin\theta}\right)$ [By (3) of § 10, page 3-34]

But $\quad 1+\sin\theta = \left(\sin^2\dfrac{\theta}{2}+\cos^2\dfrac{\theta}{2}\right)+2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \left(\cos\dfrac{\theta}{2}+\sin\dfrac{\theta}{2}\right)^2$

and $\quad 1-\sin\theta = \left(\sin^2\dfrac{\theta}{2}+\cos^2\dfrac{\theta}{2}\right)-2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2} = \left(\cos\dfrac{\theta}{2}-\sin\dfrac{\theta}{2}\right)^2$

$\therefore \quad 2\beta = \dfrac{1}{2}\log\left[\dfrac{\cos(\theta/2)+\sin(\theta/2)}{\cos(\theta/2)-\sin(\theta/2)}\right]^2 = \log\left[\dfrac{\cos(\theta/2)+\sin(\theta/2)}{\cos(\theta/2)-\sin(\theta/2)}\right]$

$\therefore \quad \beta = \dfrac{1}{2}\log\left[\dfrac{1+\tan(\theta/2)}{1-\tan(\theta/2)}\right] = \dfrac{1}{2}\log\tan\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)$ .... (ii)

Example 10 (c) : Separate into real and imaginary parts $\tan^{-1} e^{i\theta}$. **(M.U. 1985, 87, 93)**

Sol. : Let $\tan^{-1}(e^{i\theta}) = \alpha + i\beta \quad \therefore \ e^{i\theta} = \tan(\alpha + i\beta)$

$\therefore \quad \tan(\alpha + i\beta) = \cos\theta + i\sin\theta$

Then proceed as in the above example and get,

$$\alpha = \frac{n\pi}{2} + \frac{\pi}{4} \quad \text{and} \quad \beta = \frac{1}{2}\tan h^{-1}\sin\theta.$$

Example 11 (c) : Separate into real and imaginary parts $\tan^{-1}(x + iy)$. **(M.U. 1995, 2002)**

Sol. : Let $\tan^{-1}(x + iy) = \alpha + i\beta \qquad \therefore \ \tan^{-1}(x - iy) = \alpha - i\beta.$

$$\therefore \quad 2\alpha = \tan^{-1}(x + iy) + \tan^{-1}(x - iy)$$

$$= \tan^{-1}\left\{\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}\right\} = \tan^{-1}\left\{\frac{2x}{1-(x^2+y^2)}\right\}$$

$$\therefore \quad \alpha = \frac{1}{2}\tan^{-1}\left\{\frac{2x}{1-(x^2+y^2)}\right\}$$

And $\quad 2i\beta = \tan^{-1}(x + iy) - \tan^{-1}(x - iy) = \tan^{-1}\left\{\dfrac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}\right\}$

$$= \tan^{-1}\left\{\frac{2iy}{1+x^2+y^2}\right\} = i\tan h^{-1}\left\{\frac{2y}{1+x^2+y^2}\right\}$$

$$\therefore \quad \beta = \frac{1}{2}\tan h^{-1}\left\{\frac{2y}{1+x^2+y^2}\right\}$$

Example 12 (c) : Separate into real and imaginary parts $\tan h^{-1}(x + iy)$.

Sol. : Let $\tan h^{-1}(x + iy) = \alpha + i\beta$

$$x + iy = \tan h(\alpha + i\beta) = \frac{1}{i}\tan(i\alpha - \beta) \qquad \therefore \ ix - y = \tan(i\alpha - \beta)$$

and $\quad x - iy = \tan h(\alpha - i\beta) = \dfrac{1}{i}\tan(i\alpha + \beta) \qquad \therefore \ ix + y = \tan(i\alpha + \beta)$

(By 2 (B), page 3-3)

$$\therefore \quad \tan 2i\alpha = \tan(\overline{i\alpha + \beta} + \overline{i\alpha - \beta}) = \frac{\tan(i\alpha+\beta)+\tan(i\alpha-\beta)}{1-\tan(i\alpha+\beta)\tan(i\alpha-\beta)}$$

$$= \frac{ix+y+ix-y}{1-(ix+y)(ix-y)}$$

$$\therefore \quad i\tan h\, 2\alpha = \frac{2ix}{1+x^2+y^2} \qquad \therefore \ \alpha = \frac{1}{2}\tan h^{-1}\left(\frac{2x}{1+x^2+y^2}\right)$$

Also, $\quad \tan 2\beta = \tan(\overline{i\alpha + \beta} - \overline{i\alpha - \beta}) = \dfrac{\tan(i\alpha+\beta)-\tan(i\alpha-\beta)}{1+\tan(i\alpha+\beta)\tan(i\alpha-\beta)}$

$$= \frac{ix+y-ix+y}{1+(ix+y)(ix-y)} = \frac{2y}{1-x^2-y^2}$$

$$\therefore \quad 2\beta = \tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right) \qquad \therefore \ \beta = \frac{1}{2}\tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right)$$

Example 13 : Separate into real and imaginary parts $\sin^{-1} e^{i\theta}$ [or $\sin^{-1}(\cos\theta + i\sin\theta)$] where θ is an acute angle. **(M.U. 2003)**

Sol. : Let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

$$\therefore \quad \cos\theta + i\sin\theta = \sin(x+iy) = \sin x\cos iy + \cos x\sin iy$$
$$= \sin x\cosh y + i\cos x\sinh y$$

Equating real and imaginary parts,

$$\cos\theta = \sin x\cosh y \quad \text{and} \quad \sin\theta = \cos x\sinh y \qquad \text{.................... (1)}$$

$$\therefore \quad 1 = \cos^2\theta + \sin^2\theta = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$$
$$= \sin^2 x(1+\sinh^2 y) + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + \sin^2 x\sinh^2 y + \cos^2 x\sinh^2 y$$
$$= \sin^2 x + \sinh^2 y(\sin^2 x + \cos^2 x)$$
$$= \sin^2 x + \sinh^2 y$$

$$\therefore \quad 1 - \sin^2 x = \sinh^2 y \qquad \therefore \quad \cos^2 x = \sinh^2 y \qquad \text{.................... (2)}$$

Now, from (1), we get

$$\sin^2\theta = \cos^2 x\sinh^2 y = \cos^2 x\cos^2 x \qquad [\text{ By (2) }]$$

$$\sin\theta = \cos^2 x \qquad \therefore \quad \cos x = \sqrt{\sin\theta} \qquad \text{.................... (3)}$$

(Since θ is an acute angle, we take + sign before the radical.)

$$\therefore \quad x = \cos^{-1}\sqrt{\sin\theta}$$

Now, again from (i),

$$\sin\theta = \cos x\sinh y = \sqrt{\sin\theta}\,\sinh y \qquad [\text{ By (3) }]$$

$$\therefore \quad \sqrt{\sin\theta} = \sinh y \qquad \therefore \quad y = \sinh^{-1}\sqrt{\sin\theta}.$$

Example 14 (c) : Show that $\tan^{-1} i\left(\dfrac{x-a}{x+a}\right) = \dfrac{i}{2}\log\dfrac{x}{a}$. **(M.U. 1986, 96, 2002, 06, 11)**

Sol. : Let $\tan^{-1}\left[i\left(\dfrac{x-a}{x+a}\right)\right] = \theta$ .... (1)

$$\therefore \quad i\left(\frac{x-a}{x+a}\right) = \tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

$$\therefore \quad \frac{x-a}{x+a} = \frac{e^{-i\theta} - e^{i\theta}}{e^{i\theta} + e^{-i\theta}} \qquad [\because i^2 = -1]$$

By componendo and dividendo,

$$\frac{x-a+x+a}{x+a-x+a} = \frac{e^{-i\theta} - e^{i\theta} + e^{i\theta} + e^{-i\theta}}{e^{i\theta} + e^{-i\theta} - e^{-i\theta} + e^{i\theta}}$$

$$\frac{x}{a} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta} \qquad \therefore \quad -2i\theta = \log\frac{x}{a}.$$

Multiplying by i throughout,

$$2\theta = i\log\frac{x}{a} \qquad \therefore \quad \theta = \frac{i}{2}\log\left(\frac{x}{a}\right). \qquad \text{.................... (2)}$$

From (1) and (2) the result follows.

Aliter : We shall now separate the real and imaginary parts.

Let $\tan^{-1} i\left(\dfrac{x-a}{x+a}\right) = u + iv \qquad \therefore \tan(u+iv) = i\left(\dfrac{x-a}{x+a}\right)$

and $\tan(u-iv) = -i\left(\dfrac{x-a}{x+a}\right)$

$$\therefore \tan(2u) = \tan[(u+iv)+(u-iv)] = \frac{\tan(u+iv)+\tan(u-iv)}{1-\tan(u+iv)\cdot\tan(u-iv)}$$

But $\tan(u+iv)+\tan(u-iv) = i\left(\dfrac{x-a}{x+1}\right) - i\left(\dfrac{x-a}{x+a}\right) = 0$

$\therefore \tan 2u = 0 \qquad \therefore 2u = 0 \qquad \therefore u = 0$

Again $\tan(2iv) = \tan[(u+iv)-(u-iv)] = \dfrac{\tan(u+iv)-\tan(u-iv)}{1+\tan(u+iv)\cdot\tan(u-iv)}$

$$\therefore i\tanh 2v = \frac{2i\left(\dfrac{x-a}{x+a}\right)}{1+\dfrac{(x-a)^2}{(x+a)^2}} = \frac{2i(x^2-a^2)}{2(x^2+a^2)}$$

$$\therefore \tanh 2v = \frac{x^2-a^2}{x^2+a^2} \qquad \therefore 2v = \tanh^{-1}\left(\frac{x^2-a^2}{x^2+a^2}\right)$$

$$\therefore 2v = \frac{1}{2}\log\left[\frac{1+\left(\dfrac{x^2-a^2}{x^2+a^2}\right)}{1-\left(\dfrac{x^2-a^2}{x^2+a^2}\right)}\right] = \frac{1}{2}\log\left(\frac{x^2}{a^2}\right) = \log\frac{x}{a}$$

$$\therefore \tan^{-1} i\left(\frac{x-a}{x+a}\right) = u + iv = \frac{i}{2}\log\left(\frac{x}{a}\right).$$

Aliter : Let $\log\dfrac{x}{a} = y \quad \therefore \dfrac{x}{a} = e^y \quad \therefore x = ae^y$, then we have to show that

$$\tan^{-1}\left[i\cdot\frac{e^y-1}{e^y+1}\right] = \frac{i}{2}\log e^y = \frac{iy}{2}$$

i.e. we have to show that $i\cdot\dfrac{e^y-1}{e^y+1} = \tan\dfrac{iy}{2} = i\tanh\dfrac{y}{2}$

i.e. we have to show that $\dfrac{e^y-1}{e^y+1} = \tanh\dfrac{y}{2}$.

But $\tanh\dfrac{y}{2} = \dfrac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}} = \dfrac{e^y-1}{e^y+1}$. Hence, the result.

Example 15 (c) : Separate into real and imaginary parts $\cos^{-1}\left(\dfrac{3i}{4}\right)$.

(M.U. 1992, 95, 2003, 11, 17)

Sol. : Let $\cos^{-1}\left(\dfrac{3i}{4}\right) = x + iy \quad \therefore \dfrac{3i}{4} = \cos(x+iy)$

$$\therefore \frac{3i}{4} = \cos x\cosh y - i\sin x\sinh y$$

$\therefore \quad \cos x \cos hy = 0 \quad \therefore \quad \cos x = 0 \quad \therefore \quad x = \dfrac{\pi}{2} \quad [\because \cos hy \neq 0]$

And $\quad -\sin x \sin hy = \dfrac{3}{4}$. But $\sin x = \sin\left(\dfrac{\pi}{2}\right) = 1 \quad \therefore \quad \sin hy = -\dfrac{3}{4}$

$$\therefore \quad y = \log\left(-\frac{3}{4} + \sqrt{1 + \frac{9}{16}}\right) \qquad \text{[By (1) of § 10, page 3-34]}$$

$$\therefore \quad y = \log\left(-\frac{3}{4} + \frac{5}{4}\right) = \log\left(\frac{1}{2}\right) = -\log 2$$

$\therefore$ Real Part, $x = \pi / 2$. Imaginary part, $y = -\log 2$.

Example 16 (c) : If $\tan\left(\dfrac{\pi}{4} + iv\right) = re^{i\theta}$, show that

(i) $r = 1$, (ii) $\tan\theta = \sin h\, 2v$, (iii) $\tan hv = \tan\dfrac{\theta}{2}$. **(M.U. 1998, 99, 2012)**

Sol. : We have $\quad \tan\left(\dfrac{\pi}{4} + iv\right) = \dfrac{2\sin\left(\dfrac{\pi}{4} + iv\right)}{2\cos\left(\dfrac{\pi}{4} + iv\right)} \cdot \dfrac{\cos\left(\dfrac{\pi}{4} - iv\right)}{\cos\left(\dfrac{\pi}{4} - iv\right)}$

$$\therefore \quad \tan\left(\frac{\pi}{4} + iv\right) = \frac{\sin\dfrac{\pi}{2} + \sin(2iv)}{\cos\dfrac{\pi}{2} + \cos(2iv)} = \frac{1 + i\sin h\, 2v}{\cos h\, 2v}$$

$\therefore$ By data, $\quad r(\cos\theta + i\sin\theta) = \dfrac{1}{\cos h\, 2v} + i\dfrac{\sin h\, 2v}{\cos h\, 2v}$

Equating real and imaginary parts,

$$r\cos\theta = \frac{1}{\cos h\, 2v} \quad \text{and} \quad r\sin\theta = \frac{\sin h\, 2v}{\cos h\, 2v}$$

Squaring and adding,

(i) $$r^2 = \frac{1}{\cos h^2\, 2v} + \frac{\sin h^2\, 2v}{\cos h^2\, 2v} = \frac{\cos h^2\, 2v}{\cos h^2\, 2v} = 1$$

And by division

$$\tan\theta = \frac{\sin h\, 2v / \cos h\, 2v}{1 / \cos h\, 2v} = \sin h\, 2v$$

(ii) $\quad \therefore \quad \sin h\, 2v = \tan\theta$

Now, $\quad 2v = \sin h^{-1}(\tan\theta) = \log(\tan\theta + \sqrt{1 + \tan^2\theta})$

$$= \log(\sec\theta + \tan\theta) = \log\left(\frac{1 + \sin\theta}{\cos\theta}\right) = \log\left[\frac{\sin(\theta/2) + \cos(\theta/2)^2}{\cos^2(\theta/2) - \sin^2(\theta/2)}\right]$$

$$= \log\left[\frac{\cos(\theta/2) + \sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)}\right] = \log\left[\frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)}\right]$$

$$\therefore \quad 2v = 2\tan h^{-1}[\tan(\theta/2)] \qquad \left[\because \frac{1}{2}\log\frac{(1+z)}{(1-z)} = \tan h^{-1} z\right]$$

$$\therefore \quad v = \tan h^{-1}[\tan(\theta/2)]$$

(iii) $\therefore \tan hv = \tan(\theta/2)$

Aliter : We have, by (ii), $\tan\theta = \sin h\, 2v$

$$\therefore \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \sin h 2v$$

$$\therefore 2\tan\left(\frac{\theta}{2}\right) = \sin h 2v - \sin h 2v \tan^2\left(\frac{\theta}{2}\right)$$

$$\therefore \sin h 2v \tan^2\left(\frac{\theta}{2}\right) + 2\tan\left(\frac{\theta}{2}\right) - \sin h 2v = 0$$

This is a quadratic in $\tan(\theta/2)$.

$$\therefore \tan\left(\frac{\theta}{2}\right) = \frac{-2 \pm \sqrt{4 + 4\sin h^2 2v}}{2\sin h 2v} = \frac{-1 \pm \cos h 2v}{\sin h 2v}$$

$$= \frac{\cos h 2v - 1}{\sin h 2v} = \frac{2\sin h^2 v}{2\sin hv \cos hv} = \tan hv.$$

Example 17 : Separate into real and imaginary part **(i)** $\cos^{-1}(i\theta)$, **(ii)** $\sin^{-1}(\operatorname{cosec}\theta)$. **(M.U. 2016)**

Sol. : (i) Let $\cos^{-1}(i\theta) = x + iy$ $\quad\therefore \cos(x + iy) = i\theta$

$\therefore \cos x \cos iy - \sin x \sin iy = i\theta$

$\therefore \cos x \cos hy - i \sin x \sin hy = i\theta$

Equating real and imaginary parts on both sides,

$\cos x \cos hy = 0$ .... (1)

and $\sin x \sin hy = -\theta$ .... (2)

From (1), we get

$\cos x = 0 \quad \therefore x = \pi/2$

From (2), we get

$\sin hy = -\theta \quad [\because \sin x = \sin(\pi/2) = 1]$

$$\therefore y = \sin h^{-1}(-\theta) = -\sin h^{-1}\theta = -\log\left(\theta + \sqrt{\theta^2+1}\right)$$

$$\therefore \cos^{-1}(i\theta) = x + iy = \frac{\pi}{2} - i\log\left(\theta + \sqrt{\theta^2+1}\right)$$

(ii) Let $\sin^{-1}(\operatorname{cosec}\theta) = x + iy$

$\therefore \operatorname{cosec}\theta = \sin(x + iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cos hy + i \cos x \sin hy$

Equating real and imaginary parts on both sides,

$\operatorname{cosec}\theta = \sin x \cos hy$ .... (1)

$0 = \cos x \sin hy$ .... (2)

From (2), we get $\cos x = 0 \quad \therefore x = \pi/2$

Now, from (1), we get

$$\operatorname{cosec}\theta = \sin\frac{\pi}{2}\cos hy = \cos hy$$

$$\therefore \quad y = \cos h^{-1}(\text{cosec}\,\theta) = \log\left(\text{cosec}\,\theta + \sqrt{\text{cosec}^2\theta - 1}\right)$$

$$= \log(\text{cosec}\,\theta + \cot\theta)$$

$$= \log\left(\frac{1+\cos\theta}{\sin\theta}\right) = \log\left(\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right) = \log\cot\frac{\theta}{2}$$

$$\therefore \quad \sin^{-1}(\text{cosec}\,\theta) = \frac{\pi}{2} + i\log\cot\frac{\theta}{2}$$

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. Prove that $\text{cosec}\,h^{-1} z = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right)$. Is it defined for all values of z?

[**Ans. :** No, not defined for $z < 0$.]

2. Prove that $\tan h \log\sqrt{5} + \tan h \log\sqrt{7} = \frac{17}{12}$.

3. Separate into real and imaginary parts

(i) $\cos^{-1}(i)$ (ii) $\sin^{-1}(3i/4)$ (iii) $\sin h^{-1}(ix)$

(iv) $\cos h^{-1}(ix)$ (v) $\sin^{-1}(i)$ (vi) $\tan h^{-1}(i)$

(vii) $\cos^{-1}(5i/12)$ (viii) $\cos^{-1}(16i/63)$

[**Ans. :** (i) $\frac{\pi}{2} + i\log(\sqrt{2}+1)$, (ii) $i\log 2$, (iii) $\cos h^{-1}x + \frac{i\pi}{2}$,

(iv) $\sin h^{-1}x + \frac{i\pi}{2}$, (v) $i\log(1+\sqrt{2})$, (vi) $\frac{i\pi}{4}$,

(vii) $\frac{\pi}{2} + i\log\frac{2}{3}$, (viii) $\frac{\pi}{2} + i\log\frac{49}{63}$.]

4. Prove that $\cos h^{-1}\left(\frac{3i}{4}\right) = \log 2 + \frac{i\pi}{2}$.

5. Prove that $\cos^{-1} ix = \frac{\pi}{2} - i\log(x + \sqrt{x^2+1})$. **(M.U. 1996)**

6. If $\tan z = \frac{1}{2}(1-i)$, prove that $z = \frac{1}{2}\tan^{-1}2 + \frac{i}{4}\log\left(\frac{3}{5}\right)$.

7. If $\sin h^{-1}(x+iy) + \sin h^{-1}(x-iy) = \sin h^{-1} a$, prove that

$$2(x^2+y^2)\sqrt{a^2+1} = a^2 - 2x^2 + 2y^2.$$

8. If $\tan(x+iy) = a+ib$, prove that $\tan h\,2y = \frac{2b}{1+a^2+b^2}$. **(M.U. 2015)**

9. Prove that $\cos^{-1}(\sec\theta) = i\log(\sec\theta + \tan\theta)$.

10. Find all the roots of the equation $\cos z = 2$. [**Ans. :** $z = 2n\pi \pm i\log(2+\sqrt{3})$]

11. Show that $\sin^{-1}(ix) = 2n\pi + i\log(x + \sqrt{1+x^2})$. **(M.U. 2001)**

12. If $\cos h^{-1} a + \cos h^{-1} b = \cos h^{-1} c$, prove that $a^2 + b^2 + c^2 = 2abc + 1$.

Class (b) : 6 Marks

1. Prove that
   (i) $\tan h^{-1} \cos \theta = \cos h^{-1} \operatorname{cosec} \theta$ **(M.U. 1996)**
   (ii) $\sin h^{-1} \tan \theta = \log (\sec \theta + \tan \theta)$
   (iii) $\sin h^{-1} \tan \theta = \log \tan \left(\frac{\theta}{2} + \frac{\pi}{4} \right)$ **(M.U. 1996, 2010)**

2. Separate into real and imaginary parts,
   (i) $\sin^{-1} e^{i\theta}$ or $\sin^{-1} (\cos \theta + i \sin \theta)$ **(M.U. 1993, 2003, 14)**
   (ii) $\cos^{-1} e^{i\theta}$ or $\cos^{-1} (\cos \theta + i \sin \theta)$

[**Ans. :** (i) $\cos^{-1} \sqrt{\sin \theta} + i \log (\sqrt{\sin \theta} + \sqrt{1 + \sin \theta})$
(ii) $\sin^{-1} \sqrt{\sin \theta} + i \log (\sqrt{1 + \sin \theta} - \sqrt{\sin \theta})$]

Class (c) : 8 Marks

1. If $\cos h^{-1} (x + i y) + \cos h^{-1} (x - i y) = \cos h^{-1} a$, prove that
   $2 (a - 1) x^2 + 2 (a + 1) y^2 = a^2 - 1$.

EXERCISE - V

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. Define $\sin hx$ and $\cos hx$.
   Prove that $\sin h\, ix = i \sin x$ and $\cos h\, ix = \cos x$. **(M.U. 1989)**
2. Prove that $\cos h\, iu = \cos u$ and $\sin h\, iu = i \sin u$. **(M.U. 1989)**
3. Define hyperbolic tan x and prove that $\tan ix = i \tan hx$. **(M.U. 195)**
4. Show that $\sin hx$ is periodic and $| \tan hx | \le 1$.
5. Prove that :
   (i) $\cos h^2 x = 1 + \sin h^2 x$; (ii) $\sec h^2 x = 1 - \tan h^2 x$; (iii) $\tan h (\log \sqrt{5}) = \frac{2}{3}$.
6. Define inverse hyperbolic sin of x and prove that
   $\sin h^{-1} x = \log (x + \sqrt{x^2 + 1})$ **(M.U. 1989, 92, 95, 2009)**
7. Define inverse hyperbolic functions and prove that
   (i) $\tan h^{-1} = \frac{1}{2} \log \left[\frac{1 + x}{1 - x} \right]$
   (ii) $\cos h^{-1} \sqrt{1 + x^2} = \sin h^{-1} x$ **(M.U. 1982, 83, 95, 2002, 04, 09)**
   (iii) $\operatorname{cosec} h^{-1} = \log \left(\frac{1 + \sqrt{1 + x^2}}{x} \right)$ **(M.U. 2004)**
   (iv) $\sec h^{-1} x = \log \left(\frac{1 + \sqrt{1 - x^2}}{x} \right)$

8. Prove that,

(1) $\tan h^{-1}x = \sin h^{-1}\dfrac{x}{\sqrt{1-x^2}}$ **(M.U. 2001, 02, 06, 09)**

(2) $\sin h^{-1}x = \text{cosec}\, h^{-1}\dfrac{x}{2x\sqrt{1-x^2}}$ **(M.U. 2009)**

(3) $\tan h^{-1}x = \cos h^{-1}\dfrac{1}{\sqrt{1-x^2}}$

(4) $\cot h^{-1}x = \dfrac{1}{2}\log\left(\dfrac{x+1}{x-1}\right)$

EXERCISE - VI

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Short Answer Questions : Class (a) : 3 Marks

1. Prove that $\cos h^2 x - \sin h^2 x = 1$.
2. Find the period of $\sin hx$.
3. If $\tan hx = \dfrac{2}{3}$ find the value of $\sin h\, 2x$. **[Ans. :** $\dfrac{12}{5}$ **]**
4. Find the value of $\tan h \log x$ if $x = \sqrt{2}$. **[Ans. :** $\dfrac{1}{3}$ **]**
5. Prove that $(\cos hx - \sin hx)^3 = \cos h\, 3x - \sin h\, 3x$.
6. Prove that $\left(\dfrac{1+\tan hx}{1-\tan hx}\right)^3 = \cos h\, 6x + \sin h\, 6x$.
7. Prove that $\tan h^2 x = \dfrac{1}{1-\dfrac{1}{1-\cos h^2 x}}$.
8. If $\tan hx = \dfrac{2}{3}$, find the value of x. **[Ans. :** $\dfrac{1}{2}\log 5$ **]**
9. If $\sin hx - \cos hx = 3$, find the value of $\tan hx$. **[Ans. :** $-\dfrac{4}{5}$ **]**
10. Prove that $\cos h^2 x = \dfrac{1}{1-\dfrac{1}{1+\text{cosec}\, h^2 x}}$.
11. Prove that $\dfrac{1}{1-\dfrac{1}{1-\sec h^2 x}} = -\sin h^2 x$.
12. State real and imaginary parts of $\cos(x+iy)$.

    [Ans. : R.P. $= \cos x \cos hy$, I.P. $= -\sin x \sin hy$ **]**
13. State real and imaginary parts of $\tan(\alpha + i\beta)$.

    [Ans. : R.P. $= \dfrac{\sin 2\alpha}{\cos 2\alpha + \cos h 2\beta}$; I.P. $= \dfrac{\sin h 2\beta}{\cos 2\alpha + \cos h 2\beta}$ **]**

14. If $\tan(\alpha + i\beta) = x + iy$, prove that $\alpha = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-x^2-y^2}\right]$.

15. If $\sin(\theta + i\Phi) = r e^{i\alpha}$, prove that $\tan\alpha = \cot\theta \tan h\,\Phi$.

16. If $\sin(\alpha + i\beta) = x + iy$, prove that $\frac{x^2}{\cos h^2\beta} + \frac{y^2}{\sin h^2\beta} = 1$.

17. If $\tan\left(\frac{\pi}{8} + i\alpha\right) = x + iy$, prove that $x^2 + y^2 + 2x = 1$.

18. If $x + iy = 2\cos\left(\frac{\pi}{4} + i\alpha\right)$, prove that $x^2 - y^2 = 2$.

19. If $x + iy = 2\cos\left(\frac{\pi}{3} + i\alpha\right)$, find the value of $3x^2 - y^2$. [**Ans.** : 3]

20. If $\sin(\alpha + i\beta) = x + iy$, then prove that $\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\beta} = 1$.

21. Find the value of $\tan h \log\sqrt{6}$. [**Ans.** : $\frac{5}{7}$]

22. Find the real and imaginary parts of $\sin^{-1}(i)$.

[**Ans.** : R.P. = 0, I.P. = $\log(1+\sqrt{2})$]

23. Find the real and imaginary parts of $\sin h^{-1} 2i$.

[**Ans.** : R.P. = $\log(2+\sqrt{3})$, I.P. = $\frac{\pi}{2}$]

Summary

1. $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

2. $\sin h\,x = \frac{e^x - e^{-x}}{2}$, $\cos h\,x = \frac{e^x + e^{-x}}{2}$

3. $\sin ix = i\sin h\,x$, $\sin h\,ix = i\sin x$
   $\cos ix = \cos hx$, $\cos h\,ix = \cos x$
   $\tan ix = i\tan hx$, $\tan h\,ix = i\tan x$

4. $\sin h(-x) = -\sin hx$
   $\cos h(-x) = \cos hx$

5. $e^x = \cos hx + \sin hx$
   $e^{-x} = \cos hx - \sin hx$

6. $\cos h^2 x - \sin h^2 x = 1$
   $\sec h^2 x + \tan h^2 x = 1$
   $\cot h^2 x - \operatorname{cosec} h^2 x = 1$

7. $\sin h\,2x = 2\sin hx \cos hx$
   $\cos h\,2x = \cos h^2 x + \sin h^2 x$
   $= 2\cos h^2 x - 1 = 2\sin h^2 x + 1$
   $\tan h\,2x = \frac{2\tan hx}{1 + \tan h^2 x}$

8. $\frac{d}{dx}(\sin hx) = \cos hx$, $\frac{d}{dx}(\cos hx) = \sin hx$

9.

x	0	$-\infty$	∞
$\sin hx$	0	$-\infty$	∞
$\cos hx$	1	∞	∞
$\tan hx$	0	-1	1

10. $\sin h^{-1} z = \log(z + \sqrt{z^2 + 1})$; $\cos h^{-1} z = \log(z + \sqrt{z^2 - 1})$

$\tan h^{-1} z = \dfrac{1}{2} \log\left(\dfrac{1+z}{1-z}\right)$

☆ ☆ ☆

CHAPTER

4 Logarithms of Complex Numbers

1. Introduction

In this chapter, we shall first learn how to obtain logarithms of complex numbers and then solve problems based on them.

2. Logarithm of a Complex Number

Let $z = \log(x + iy)$ and also let $x = r\cos\theta$, $y = r\sin\theta$, so that

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

Hence, $z = \log r(\cos\theta + i\sin\theta) = \log(r \cdot e^{i\theta})$

$$= \log r + \log e^{i\theta} = \log r + i\theta$$

$$\therefore \quad \boxed{\log(x + iy) = \log r + i\theta} \qquad \text{.................... (1)}$$

$$i.e. \quad \boxed{\log(x + iy) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)} \qquad \text{................ (1.A)}$$

This is called **principal value** of $\log(x + iy)$.

The general value of $\log(x + iy)$ is denoted with l capital by $\text{Log}(x + iy)$ and is given by

$$\boxed{\text{Log}(x + iy) = 2n\pi i + \log(x + iy)} \qquad \text{................ (1.B)}$$

$$\boxed{\text{Log}(x + iy) = \log r + i(2n\pi + \theta)} \qquad \text{.................... (2)}$$

Cor. 1 : Putting $x = 0$, $y = 1$, in (1.A), we get

$$\log(0 + i) = \frac{1}{2}\log(0^2 + 1^2) + i\tan^{-1}\left(\frac{1}{0}\right)$$

$$\therefore \quad \log i = i\tan^{-1}(\infty) = \frac{i\pi}{2}$$

$$\therefore \quad \boxed{\log i = \frac{i\pi}{2}} \qquad \text{.................... (3)}$$

$$\text{and} \quad \boxed{\text{Log } i = i\left(2n\pi + \frac{\pi}{2}\right)} \qquad \text{.............. (3.A)}$$

Now, $\log(i^i) = i\log i = i\left(\frac{i\pi}{2}\right) = -\frac{\pi}{2}$ [By (3)]

$$\therefore \quad \boxed{\log(i^i) = -\frac{\pi}{2}} \quad \text{and} \quad \boxed{i^i = e^{-\pi/2}} \qquad \text{.................... (4)}$$

From this, we get two results.

$$\sin(\log i^i) = \sin\left(-\frac{\pi}{2}\right) = -1$$

$\therefore$ $\boxed{\sin(\log i^i) = -1}$ or $\boxed{\sin(i \log i) = -1}$ .... (5)

$$\cos(\log i^i) = \cos\left(-\frac{\pi}{2}\right) = 0$$

$\therefore$ $\boxed{\cos(\log i^i) = 0}$ or $\boxed{\cos(i \log i) = 0}$ .... (6)

Remember these results. We will need them again and again in the examples that follow.

Cor. 2 : Putting the value of r in (2),

$$\boxed{\text{Log}\,(x + i\,y) = \frac{1}{2}\log(x^2 + y^2) + i(2n\pi + \theta)}$$

Caution ....

$\theta = \tan^{-1}(y/x)$ only when x and y are both positive. In any other case θ is to be determined from $x = r\cos\theta$, $y = r\sin\theta$, $-\pi \le \theta \le \pi$.

Cor. 3 : Changing the sign of i in (1.A), we get

$$\boxed{\log(x - i\,y) = \frac{1}{2}\log(x^2 + y^2) - i\tan^{-1}\frac{y}{x}} \quad \text{..................... (7)}$$

Type I : On Logarithm of Complex or Negative Real Number — Class (a) : 3 Marks

Example 1 (a) : Find the value of log (– 5). **(M.U. 1981)**

Sol. : Now we shall obtain the logarithms of negative as well as complex numbers. Also we shall see how to obtain logarithm of a number to the negative base.

Since $\log(x + i\,y) = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x}$, we have

(i) putting $x = -5$, $y = 0$

$$\log(-5) = \frac{1}{2}\log(25) + i\tan^{-1}\left(\frac{0}{-5}\right) = \frac{1}{2}\log 5^2 + i\pi = \log 5 + i\pi.$$

Aliter : Let $-5 = x + iy$ $\quad \therefore\ x = -5,\ y = 0$

$\therefore\ r = \sqrt{x^2 + y^2} = \sqrt{25 + 0} = 5$

$\because\ x = r\cos\theta,\ -5 = 5\cos\theta \quad \therefore\ \cos\theta = -1$

$\because\ y = r\sin\theta,\ \ 0 = 5\sin\theta \quad \therefore\ \sin\theta = 0.$

Since $\sin\theta = 0$, $\cos\theta = -1$; $\theta = \pi$.

$\therefore\ \log(-5) = \log r + i\theta = \log 5 + i\pi.$

Remark ....

We see that if a is a positive integer then logarithm of $-a$ to the base e is given by

$\boxed{\log(-a) = \log a + i\pi}$ .... (1)

Example 2 (a) : Find the general value of (i) log (– i), (ii) log (– 5), (iii) log (3 + 4i).

Sol. : (i) We have

$$\log(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}$$

Putting $x=0$, $y=-1$

$$\log(-i)=\frac{1}{2}\log(1)+i\tan^{-1}(-\infty)=i\left(-\frac{\pi}{2}\right)=-\frac{\pi}{2}i$$

The general value is

$$\text{Log}(-i)=2n\pi i+\log(-i)=2n\pi i-\frac{\pi}{2}i=(4n-1)\frac{\pi}{2}i$$

(ii) We have seen above

$$\log(-5)=\log 5+\pi i$$

The general value is

$$\text{Log}(-i)=2n\pi+\log(-5)=2n\pi i+\pi i+\log 5$$
$$=(2n+1)\pi i+\log 5$$

(iii) For (iii), see Ex. 1, page 4-10.

Example 3 (a) : Prove that $\log(1+e^{2i\theta})=\log 2\cos\theta+i\theta$. **(M.U. 1996)**

Sol. : We have

$$\begin{aligned}\log(1+e^{2i\theta})&=\log(1+\cos 2\theta+i\sin 2\theta)\\&=\log(1+2\cos^2\theta-1+i\sin 2\theta)\\&=\log[2\cos^2\theta+i\,2\sin\theta\cos\theta]\\&=\log[2\cos\theta(\cos\theta+i\sin\theta)]\\&=\log 2\cos\theta\, e^{i\theta}\\&=\log 2\cos\theta+\log e^{i\theta}\\&=\log 2\cos\theta+i\theta.\end{aligned}$$

Example 4 (a) : Find the value of log (1 + i).

Sol. : $$\log(1+i)=\frac{1}{2}\log(1^2+1^2)+i\tan^{-1}\left(\frac{1}{1}\right)=\frac{1}{2}\log 2+i\cdot\frac{\pi}{4}.$$

Example 5 (a) : Find the value of $\log_{-2}(-3)$.

Sol. : We first change the base to e.

We know that $\log_a x=\dfrac{\log_b x}{\log_b a}$, where b is the new base.

$$\therefore\quad \log_{(-2)}(-3)=\frac{\log_e(-3)}{\log_e(-2)}=\frac{\log 3+i\pi}{\log 2+i\pi}\qquad [\text{ By (1), above }]$$

Example 6 (a) : Prove that $\log_2(-3)=\dfrac{\log 3+i\pi}{\log 2}$. **(M.U. 2001)**

Sol. : As in Ex. 5 (a) changing the base to e,

$$\log_2(-3)=\frac{\log_e(-3)}{\log_e 2}=\frac{\log 3+i\pi}{\log 2}\qquad [\text{ By (1), above }]$$

Example 7 (a) : Show that $\text{Log}_i\, i = \dfrac{2n+1}{4m+1}$ when n, m are integers. **(M.U. 2001)**

Sol. : We have $\quad \text{Log}_i\, i = \dfrac{\text{Log}_e\, i}{\text{Log}_e\, i} = \dfrac{i[2n\pi + (\pi/2)]}{i[2m\pi + (\pi/2)]}$ [By (3.A)]

$$\therefore \quad \text{Log}_i\, i = \frac{(4n+1)}{(4m+1)\pi} = \frac{4n+1}{4m+1}.$$

(Since i in the numerator and i in the denominator need not be same, we use two integers n and m.)

Example 8 (a) : Find the general value of Log $(1+i)$ + Log $(1-i)$. **(M.U. 2002)**

Sol. : As in Ex. 2 (a), $\quad \log(1+i) = \dfrac{1}{2}\log 2 + i\dfrac{\pi}{4} = \log\sqrt{2} + i\dfrac{\pi}{4}$

$$\therefore \quad \text{Log}\,(1+i) = \log\sqrt{2} + i\left(2n\pi + \frac{\pi}{4}\right)$$

Changing the sign of i,

$$\text{Log}\,(1-i) = \log\sqrt{2} - i\left(2n\pi + \frac{\pi}{4}\right)$$

By addition, we get

$$\text{Log}\,(1+i) + \text{Log}\,(1-i) = 2\log\sqrt{2} = 2\cdot\frac{1}{2}\log 2 = \log 2.$$

Example 9 (a) : Show that $\log(-\log i) = \log\dfrac{\pi}{2} - i\dfrac{\pi}{2}$.

Sol. : By (3), page 4-1,

$$\log i = \frac{i\pi}{2} \qquad \therefore \quad -\log i = -\frac{i\pi}{2} \qquad \therefore \quad \log(-\log i) = \log\left(-i\frac{\pi}{2}\right)$$

But $\quad \log(x+iy) = \dfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\dfrac{y}{x}$

Putting $x = 0$, $y = -\dfrac{\pi}{2}$, we get,

$$\log\left(-\frac{i\pi}{2}\right) = \frac{1}{2}\log\frac{\pi^2}{4} + i\tan^{-1}(-\infty) = \log\frac{\pi}{2} - i\frac{\pi}{2}$$

$$\therefore \quad \log(-\log i) = \log\frac{\pi}{2} - i\frac{\pi}{2}.$$

Tyep II : On Direct Use of $\log(x+iy) = \dfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\dfrac{y}{x}$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Prove that $\log\left(\dfrac{x+iy}{x-iy}\right) = 2i\tan^{-1}\left(\dfrac{y}{x}\right)$. **(M.U. 1999, 2016)**

Sol. : By definition $\quad \log(x+iy) = \dfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\dfrac{y}{x}\right)$

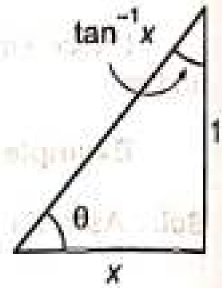
and $\quad \log(x-iy) = \dfrac{1}{2}\log(x^2+y^2) - i\tan^{-1}\left(\dfrac{y}{x}\right)$

Now, $\log\left[\dfrac{x+iy}{x-iy}\right]=\log(x+iy)-\log(x-iy)=2i\tan^{-1}\dfrac{y}{x}$.

Example 2 (a) : Show that $i\log\left(\dfrac{x-i}{x+i}\right)=\pi-2\tan^{-1}x$. **(M.U. 1982, 2016)**

Sol. : We have $\log(x+i)=\dfrac{1}{2}\log(x^2+1)+i\tan^{-1}\dfrac{1}{x}$

and $\log(x-i)=\dfrac{1}{2}\log(x^2+1)-i\tan^{-1}\dfrac{1}{x}$



$$\therefore \log\left(\frac{x-i}{x+i}\right)=\log(x-i)-\log(x+i)$$

$$=-2i\tan^{-1}\frac{1}{x}=-2i\left(\frac{\pi}{2}-\tan^{-1}x\right) \qquad \left[\theta=\tan^{-1}\frac{1}{x}=\frac{\pi}{2}-\tan^{-1}x\right]$$

$$\therefore \log\left(\frac{x-i}{x+i}\right)=-i(\pi-2\tan^{-1}x) \qquad \therefore i\log\left(\frac{x-i}{x+i}\right)=\pi-2\tan^{-1}x.$$

Example 3 (a) : Find the value of $\log[\sin(x+iy)]$. **(M.U. 2002)**

Sol. : We have, $\sin(x+iy)=\sin x\cos hy+i\cos x\sin hy$

$$\therefore \log\sin(x+iy)=\frac{1}{2}\log(\sin^2x\cos h^2y+\cos^2x\sin h^2y)+i\tan^{-1}\left(\frac{\cos x\sin hy}{\sin x\cos hy}\right)$$

[By (1.A), page 4-1]

Now, $\sin^2x\cos h^2y+\cos^2x\sin h^2y=(1-\cos^2x)\cos h^2y+\cos^2x(\cos h^2y-1)$

$$=\cos h^2y-\cos^2x$$

$$=\left(\frac{1+\cos h2y}{2}\right)-\left(\frac{1+\cos 2x}{2}\right)=\frac{1}{2}(\cos h2y-\cos 2x)$$

$$\therefore \log\sin(x+iy)=\frac{1}{2}\log\left(\frac{\cos h2y-\cos 2x}{2}\right)+i\tan^{-1}(\cot x\tan hy).$$

Example 4 (a) : Prove that $\log\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right]=2i\tan^{-1}(\cot x\tan hy)$.

(M.U. 1997, 2005, 07, 17)

Sol. : As above

$$\log\sin(x+iy)=\frac{1}{2}\log\left(\frac{\cos h2y-\cos 2x}{2}\right)+i\tan^{-1}(\cot x\tan hy)$$

Changing the sign of i,

$$\therefore \log\sin(x-iy)=\frac{1}{2}\log\left(\frac{\cos h2y-\cos 2x}{2}\right)-i\tan^{-1}(\cot x\tan hy)$$

Now, $\log\left[\dfrac{\sin(x+iy)}{\sin(x-iy)}\right]=\log\sin(x+iy)-\log\sin(x-iy)$

$$=2i\tan^{-1}(\cot x\tan hy).$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Show that $\tan\left\{i\log\frac{a-bi}{a+bi}\right\}=\frac{2ab}{a^2-b^2}$. **(M.U. 1981, 95, 2000, 02, 14)**

Sol. : We have $\log(a-bi)=\frac{1}{2}\log(a^2+b^2)-i\tan^{-1}\frac{b}{a}$

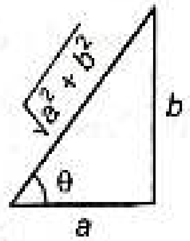
and $\log(a+bi)=\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\frac{b}{a}$

$\therefore\ \log\left(\frac{a-bi}{a+bi}\right)=\log(a-bi)-\log(a+bi)=-2i\tan^{-1}\frac{b}{a}$ **(M.U. 2018)**

$\therefore\ i\log\left(\frac{a-bi}{a+bi}\right)=-2i^2\tan^{-1}\frac{b}{a}-2\tan^{-1}\frac{b}{a}$ .... (A)

$\therefore\ \tan\left\{i\log\left(\frac{a-bi}{a+bi}\right)\right\}=\tan\left(2\tan^{-1}\frac{b}{a}\right)$ .... (1)

Let $\tan^{-1}\frac{b}{a}=\theta\quad\therefore\ \frac{b}{a}=\tan\theta$



$\tan\left(2\tan^{-1}\frac{b}{a}\right)=\tan2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2(b/a)}{1-(b^2/a^2)}=\frac{2ab}{a^2-b^2}$ .... (2)

From (1) and (2) the result follows.

Aliter : Let $a+ib=re^{i\theta}$

$\therefore\ a+ib=re^{i\theta}$ where $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\frac{b}{a}$.

$\therefore\ \log\left(\frac{a-ib}{a+ib}\right)=\log\left(\frac{re^{-i\theta}}{re^{i\theta}}\right)=\log e^{-2i\theta}=-2i\theta$

$\therefore\ \tan\left\{i\log\frac{a-ib}{a+ib}\right\}=\tan[i(-2i\theta)]$

$=\tan2\theta=\frac{2ab}{a^2-b^2}$. [As above]

Example 2 (b) : Prove that

(i) $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]=\frac{a^2-b^2}{a^2+b^2}$ **(ii)** $\sin\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]=\frac{2ab}{a^2+b^2}$

(M.U. 1990, 2018)

Sol. : As above [from (A)]

(i) $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]=\cos\left(2\tan^{-1}\frac{b}{a}\right)$

$\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right]=\cos2\theta$ where $\tan^{-1}\frac{b}{a}=\theta$

$$\therefore \quad \cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \cos^2\theta - \sin^2\theta$$

$$= \frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2} \qquad \text{[From the above figure]}$$

$$= \frac{a^2-b^2}{a^2+b^2}.$$

(ii) Left to students as an exercise.

Example 3 (b) : If $(1+i)(1+2i)\,.....\,(1+ni) = x+iy$, prove that

$$2\cdot 5\cdot 10\,.....\,(1+n^2) = x^2+y^2.$$

Sol. : By data $\quad (1+i)(1+2i)\,.....\,(1+ni) = x+iy$ .... (i)

Taking logarithm of both sides,

$$\log(1+i) + \log(1+2i) + ... + \log(1+ni) = \log(x+iy)$$

But $\quad \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

$$\therefore \quad \left[\frac{1}{2}\log(1^2+1^2) + i\tan^{-1}\frac{1}{1}\right] + \left[\frac{1}{2}\log(1^2+2^2) + i\tan^{-1}\frac{2}{1}\right] + ...$$

$$.... + \left[\frac{1}{2}\log(1^2+n^2) + i\tan^{-1}\frac{n}{1}\right] = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$$

Equating real parts,

$$\frac{1}{2}\log(2) + \frac{1}{2}\log(5) + \frac{1}{2}\log(10) + ... + \frac{1}{2}\log(1+n^2) = \frac{1}{2}\log(x^2+y^2)$$

$$\therefore \quad \log[\,2\cdot 5\cdot 10\,.....\,(1+n^2)\,] = \log(x^2+y^2)$$

$$\therefore \quad 2\cdot 5\cdot 10\,.....\,(1+n^2) = x^2+y^2.$$

Example 4 (b) : Prove that $\quad \log(e^{i\alpha} - e^{i\beta}) = \log\left[2\sin\frac{\alpha-\beta}{2}\right] + i\left(\frac{\pi+\alpha+\beta}{2}\right)$

Sol. : $\log(e^{i\alpha} - e^{i\beta}) = \log[(\cos\alpha + i\sin\alpha) - (\cos\beta + i\sin\beta)]$

$$= \log[(\cos\alpha - \cos\beta) + i(\sin\alpha - \sin\beta)] \qquad \text{..................... (1)}$$

$$= \frac{1}{2}\log[(\cos\alpha-\cos\beta)^2 + (\sin\alpha-\sin\beta)^2] + i\tan^{-1}\frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}$$

$$\therefore \quad \log(e^{i\alpha} - e^{i\beta}) = \frac{1}{2}\log[\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta]$$

$$+ i\tan^{-1}\left[\frac{2\cos\frac{(\alpha+\beta)}{2}\cdot\sin\frac{(\alpha-\beta)}{2}}{-2\sin\frac{(\alpha+\beta)}{2}\cdot\sin\frac{(\alpha-\beta)}{2}}\right]$$

$$= \frac{1}{2}\log[2 - 2\cos(\alpha-\beta)] + i\tan^{-1}\left[-\cot\left(\frac{\alpha+\beta}{2}\right)\right]$$

$$= \frac{1}{2}\log\left[4\sin^2\left(\frac{\alpha-\beta}{2}\right)\right] + i\tan^{-1}\tan\left(\frac{\pi}{2} + \frac{\alpha+\beta}{2}\right)$$

$$\therefore \quad \log(e^{i\alpha} - e^{i\beta}) = \frac{1}{2}\log\left[2\sin\left(\frac{\alpha-\beta}{2}\right)\right]^2 + i\left(\frac{\pi}{2} + \frac{\alpha+\beta}{2}\right) \qquad [\because \log m^n = n\log m]$$

$$= \log 2\sin\left(\frac{\alpha-\beta}{2}\right) + i\left(\frac{\pi}{2} + \frac{\alpha+\beta}{2}\right)$$

Aliter : From (1)

$$\log(e^{i\alpha} - e^{i\beta}) = \log\left[-2\sin\frac{(\alpha+\beta)}{2}\sin\frac{(\alpha-\beta)}{2} + i\,2\cos\frac{(\alpha-\beta)}{2}\sin\frac{(\alpha-\beta)}{2}\right]$$

$$\therefore \quad \log(e^{i\alpha} - e^{i\beta}) = \log\left[2\sin\frac{(\alpha-\beta)}{2}\cdot\left(-\sin\frac{(\alpha+\beta)}{2} + i\cos\frac{(\alpha+\beta)}{2}\right)\right]$$

$$= \log\left[2\sin\frac{(\alpha-\beta)}{2}\left\{\cos\left(\frac{\pi}{2} + \frac{(\alpha+\beta)}{2}\right) + i\sin\left(\frac{\pi}{2} + \frac{(\alpha+\beta)}{2}\right)\right\}\right]$$

$$= \log\left[2\sin\frac{(\alpha-\beta)}{2}\cdot e^{i(\pi+\alpha+\beta)/2}\right]$$

$$= \log\left[2\sin\frac{(\alpha-\beta)}{2}\right] + \log e^{i(\pi+\alpha+\beta)/2}$$

$$= \log\left[2\sin\frac{(\alpha-\beta)}{2}\right] + \frac{i(\pi+\alpha+\beta)}{2}.$$

Similarly, we canprove that

$$\log(e^{i\alpha} + e^{i\beta}) = \log\left[2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\right] + \frac{i(\alpha+\beta)}{2}. \qquad \textbf{(M.U. 2004, 10, 16)}$$

[Similarly, prove that

$$\log(e^{i\alpha} + e^{i\beta}) = \log 2\left(\frac{\cos\alpha-\beta}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right).\,]$$

Example 5 (b) : Prove that $\log\left(\dfrac{1}{1-e^{i\theta}}\right) = \log\left(\dfrac{1}{2}\operatorname{cosec}\dfrac{\theta}{2}\right) + i\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$.

(M.U. 1995, 98, 2000, 01, 02)

Sol. :

$$\log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left[\frac{1}{1-(\cos\theta + i\sin\theta)}\right] = \log\left[\frac{1}{(1-\cos\theta) - i\sin\theta}\right]$$

$$= \log\left[\frac{1}{2\sin(\theta/2)[\sin(\theta/2) - i\cos(\theta/2)]}\right]$$

$$= -\log\left(2\sin\frac{\theta}{2}\right) - \log\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right) \qquad \text{......................... (1)}$$

Now,

$$-\log\left(2\sin\frac{\theta}{2}\right) = \log\left(2\sin\frac{\theta}{2}\right)^{-1} \qquad [\because \log m^n = n\log m]$$

$$= \log\left(\frac{1}{2\sin(\theta/2)}\right) = \log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right) \qquad \text{......................... (2)}$$

And

$$\log\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right) = \frac{1}{2}\log\left(\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\right) - i\tan^{-1}\left(\frac{\cos(\theta/2)}{\sin(\theta/2)}\right)$$

[By putting $x = \sin\frac{\theta}{2}$, $y = \cos\frac{\theta}{2}$ in (7), page 4-2]

$$\therefore \quad \log\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right) = \frac{1}{2}\log 1 - i\tan^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$= -i\tan^{-1}\left[\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right] = -i\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \qquad \text{.................... (3)}$$

Hence, from (1), (2) and (3),

$$\log\left(\frac{1}{1-e^{i\theta}}\right) = \log\left(\frac{1}{2}\operatorname{cosec}\frac{\theta}{2}\right) + i\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Example 6 (b) : If $u + iv = \frac{1}{i}\log\left(\frac{1+ie^{i\theta}}{1-ie^{i\theta}}\right)$, prove that $u = \frac{\pi}{2}$ and $v = \log(\sec\theta + \tan\theta)$.

Sol. : We have by data, $\quad i(u+iv) = \log(1+e^{i\theta}) - \log(1-e^{i\theta})$

$$\therefore \quad -v + iu = \log(1+ie^{i\theta}) - \log(1-ie^{i\theta})$$

Now, $\quad \log(1+ie^{i\theta}) = \log[1 + i(\cos\theta + i\sin\theta)] = \log[(1-\sin\theta) + i\cos\theta]$

$$= \frac{1}{2}\log[(1-\sin\theta)^2 + \cos^2\theta] + i\tan^{-1}\left(\frac{\cos\theta}{1-\sin\theta}\right)$$

$$= \frac{1}{2}\log[2(1-\sin\theta)] + i\tan^{-1}\left(\frac{\cos\theta}{1-\sin\theta}\right)$$

Similarly, $\quad \log(1-ie^{i\theta}) = \log[1 - i(\cos\theta + i\sin\theta)] = \log[(1+\sin\theta) - i\cos\theta]$

$$= \frac{1}{2}\log[(1+\sin\theta)^2 + \cos^2\theta] - i\tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right)$$

$$= \frac{1}{2}\log[2(1+\sin\theta)] - i\tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right)$$

By subtraction, we get

$$-v + iu = \frac{1}{2}\log\left[\frac{2(1-\sin\theta)}{2(1+\sin\theta)}\right] + i\left[\tan^{-1}\left(\frac{\cos\theta}{1-\sin\theta}\right) + \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right)\right]$$

$$\therefore \quad -v = \frac{1}{2}\log\left(\frac{1-\sin\theta}{1+\sin\theta}\right)$$

$$\therefore \quad v = -\frac{1}{2}\log\left(\frac{1-\sin\theta}{1+\sin\theta}\right) = \frac{1}{2}\log\left(\frac{1-\sin\theta}{1+\sin\theta}\right)^{-1} \qquad [\because \log m^{-1} = -\log m]$$

$$\therefore \quad v = \frac{1}{2}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right) = \frac{1}{2}\log\left[\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\cdot\frac{(1+\sin\theta)}{(1+\sin\theta)}\right]$$

$$= \frac{1}{2}\log\frac{(1+\sin\theta)^2}{\cos^2\theta} = \log\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$\therefore \quad v = \log(\sec\theta + \tan\theta)$$

Now, $\quad u = \tan^{-1}\left(\frac{\cos\theta}{1-\sin\theta}\right) + \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right) = \tan^{-1}\left[\frac{\left(\frac{\cos\theta}{1-\sin\theta}\right) + \left(\frac{\cos\theta}{1+\sin\theta}\right)}{1 - \left(\frac{\cos\theta}{1-\sin\theta}\right)\left(\frac{\cos\theta}{1-\sin\theta}\right)}\right]$

$$\therefore \quad u = \tan^{-1}\infty = \frac{\pi}{2} \qquad \left[\because \quad \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

Aliter : We can also solve the problem without using logarithm.

By data $\quad iu - v = \log\left(\dfrac{1+ie^{i\theta}}{1-ie^{i\theta}}\right)$

$$\therefore \quad e^{iu-v} = \frac{1+ie^{i\theta}}{1-ie^{i\theta}} = \frac{1+i(\cos\theta + i\sin\theta)}{1-i(\cos\theta+i\sin\theta)}$$

$$= \frac{(1-\sin\theta)+i\cos\theta}{(1+\sin\theta)-i\cos\theta}\cdot\frac{(1+\sin\theta)+i\cos\theta}{(1+\sin\theta)+i\cos\theta}$$

$$= \frac{(1-\sin\theta+i\cos\theta+\sin\theta-\sin^2\theta+i\sin\theta\cos\theta+i\cos\theta-i\sin\theta\cos\theta-\cos^2\theta)}{2(1+\sin\theta)}$$

$$= \frac{2i\cos\theta}{2(1+\sin\theta)} = i\,\frac{\cos\theta}{1+\sin\theta}$$

$$\therefore \quad e^{-v}(\cos u + i\sin u) = i\,\frac{\cos\theta}{1+\sin\theta}$$

Equating real and imaginary parts,

$$\therefore \quad e^{-v}\cos u = 0 \quad \text{and} \quad e^{-v}\sin u = \frac{\cos\theta}{1+\sin\theta}.$$

$$\therefore \quad \cos u = 0 \qquad \therefore \quad u = \frac{\pi}{2}$$

$$\therefore \quad e^{-v} = \frac{\cos\theta}{1+\sin\theta} \qquad [\because \quad \sin u = 1] \qquad \therefore \quad e^{v} = \frac{1+\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$$

$$\therefore \quad v = \log(\sec\theta + \tan\theta).$$

Type III : On Separation of Real and Imaginary Parts by taking Logarithm

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Separate into real and imaginary parts Log $(3 + 4i)$. **(M.U. 2016)**

Sol. : Let $\quad 3 + 4i = x + iy = r(\cos\theta + i\sin\theta)$

$\therefore \quad 3 = x, \quad 4 = y, \quad r = \sqrt{x^2+y^2} = 5$

$x = r\cos\theta \qquad \therefore \quad 3 = 5\cos\theta \qquad \therefore \quad \cos\theta = 3/5$

$y = r\sin\theta \qquad \therefore \quad 4 = 5\sin\theta \qquad \therefore \quad \sin\theta = 4/5$

$\therefore \quad \tan\theta = 4/3 \qquad$ *i.e.* $\qquad \theta = \tan^{-1}(4/3)$

$$\therefore \quad \text{Log}(3+4i) = \log r + i(2n\pi + \theta)$$
$$= \log 5 + i[2n\pi + \tan^{-1}(4/3)]$$

$\therefore \quad$ Real Part $= \log 5 \qquad$ and $\qquad$ Imaginary Part $= 2n\pi + \tan^{-1}(4/3)$.

Example 2 (a) : Prove that i^i is wholly real. Find its principal value. Show that the values of i^i form a G.P.

Sol. : We know that $a^x = e^{x\,\text{Log}\,a}$. $\qquad$ Hence, $\quad i^i = e^{i\,\text{Log}\,i}$

But by corollary (3.A), page 4-1, $\text{Log } i = i\left(2n\pi + \frac{\pi}{2}\right)$

$\therefore \quad i^i = e^{i^2[2n\pi + (\pi/2)]} = e^{-[2n\pi + (\pi/2)]} = e^{-(4n+1)(\pi/2)}$, real.

The principal value of i^i is obtained by putting $n = 0$

$$\therefore \qquad \boxed{i^i = e^{-\pi/2}}$$

Putting $n = 0, 1, 2, 3, \ldots$, we get the values of i^i, $\quad e^{-\pi/2}, e^{-5\pi/2}, e^{-9\pi/2}, \ldots$

These values are in G.P. with common ratio $= e^{-2\pi}$.

Example 3 (a) : Separate into real and imaginary parts **(i)** i^i, **(ii)** $i^{n!}$, **(iii)** $\sqrt{i}^{\sqrt{i}}$. **(M.U. 2004, 08)**

Sol. : We have solved these problems earlier by using exponential form of a complex number. We shall now use logarithmic method.

(i) Since $a^x = e^{x\log a}$, $i^i = e^{i\log i}$ $\quad [x = a = i]$

But, by (3), page 4-1, $\quad \log i = \frac{i\pi}{2}$.

$\therefore \quad i^i = e^{i(i\pi/2)} = e^{-\pi/2}$, a real number.

(ii) Further, $\quad (i^i)^n = (e^{-\pi/2})^n \qquad \therefore \quad i^{n!} = e^{-n\pi/2}$.

(iii) $\sqrt{i}^{\sqrt{i}} = e^{\log \sqrt{i}^{\sqrt{i}}} = e^{\sqrt{i}\log\sqrt{i}}$ $\quad [\because x = e^{\log_e x}$. Put $x = \sqrt{i}^{\sqrt{i}}$] .... (1)

Again by the same Cor. $\quad \log\sqrt{i} = \frac{1}{2}\log i = \frac{1}{2}\left(i\frac{\pi}{2}\right) = \frac{i\pi}{4}$

Also $\quad \sqrt{i} = (i)^{1/2} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$\therefore \quad \sqrt{i}\log\sqrt{i} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\frac{i\pi}{4}\right)$

$\therefore \quad \sqrt{i}\log\sqrt{i} = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)i\frac{\pi}{4} = i\frac{\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$

$\therefore \quad \sqrt{i}^{\sqrt{i}} = e^{(i\pi/4\sqrt{2}) - (\pi/4\sqrt{2})} = e^{-(\pi/4\sqrt{2})} \cdot e^{(i\pi/4\sqrt{2})}$ $\qquad$ [From (1)]

$$= e^{-(\pi/4\sqrt{2})} \cdot \left[\cos\left(\frac{\pi}{4\sqrt{2}}\right) + i\sin\left(\frac{\pi}{4\sqrt{2}}\right)\right].$$

Now, separate the real and imaginary parts. $\qquad$ [See also Ex. 2, page 2-46]

Example 4 (a) : Separate into real and imaginary part $i^{(1-i)}$.

Sol. : Let $\alpha + i\beta = i^{(1-i)}$

$\therefore \quad \log(\alpha + i\beta) = (1-i)\log i = (1-i)\left(i\frac{\pi}{2}\right) = \frac{\pi}{2} + i\frac{\pi}{2}$ $\qquad \left[\because \log i = \frac{i\pi}{2}\right]$

$\therefore \quad \alpha + i\beta = e^{(\pi/2) + i(\pi/2)} = e^{\pi/2} \cdot e^{i\pi/2} = e^{\pi/2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i\,e^{\pi/2}$

$\therefore \quad \alpha = 0, \quad \beta = e^{\pi/2}$.

Example 5 (a) : Separate into real and imaginary parts $(1+i)^i$.

Sol. : Let $\alpha + i\beta = (1+i)^i$

Taking logarithms of both the sides,

$$\log(\alpha + i\beta) = i\log(1+i) = i\left[\frac{1}{2}\log 2 + i\tan^{-1}1\right]$$

$$= \frac{i}{2}\log 2 + i^2\tan^{-1}1 = \frac{i}{2}\log 2 - \frac{\pi}{4}$$

$$\therefore \quad \alpha + i\beta = e^{(i/2)\log 2 - (\pi/4)} = e^{-\pi/4} \cdot e^{i\log\sqrt{2}}$$

$$= e^{-\pi/4}(\cos\log\sqrt{2} + i\sin\log\sqrt{2})$$

$$\therefore \quad \text{Real part, } \alpha = e^{-\pi/4}\cos\log\sqrt{2}$$

$$\therefore \quad \text{Imaginary part, } \beta = e^{-\pi/4}\sin\log\sqrt{2}$$

Example 6 (a) : Show that $\sin\log i^i = -1$.

Sol. : We have $\sin(\log i^i) = \sin(i\log i)$

But $\log i = \dfrac{i\pi}{2}$ [By (3), page 4-1]

$$\therefore \quad \sin(\log i^i) = \sin\left(i \cdot \frac{i\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1.$$

(See also Ex. 1, page 2-46.)

Example 7 (a) : Find $\sin h\log i$.

Sol. : By (3), page 4-1

$$\log i = \frac{i\pi}{2} \qquad \therefore \quad \sin h\log i = \sin h\frac{i\pi}{2} = i\sin\frac{\pi}{2} = i.$$

Example 8 (a) : Prove that $a^i = e^{-2n\pi}[\cos(\log a) + i\sin(\log a)]$.

Sol. : We have $a^i = e^{i\log a}$

$$\therefore \quad a^i = e^{i[\log a + 2n\pi i]} = e^{i^2 2n\pi} \cdot e^{i\log a}$$

$$= e^{-2n\pi}[\cos(\log a) + i\sin(\log a)].$$

Example 9 (a) : Find the principal value of $i^{\log(1+i)}$ and show that its real part is

$$e^{-\pi^2/8} \cdot \cos\left(\frac{\pi}{4}\log 2\right).$$ **(M.U. 1984, 92, 2016)**

Sol. : Let $z = i^{\log(1+i)}$ $\quad \therefore \quad \log z = \log(1+i) \cdot \log i$

But $\log(1+i) = \log\sqrt{2} + i\tan^{-1}1 = \log\sqrt{2} + i\dfrac{\pi}{4}$, and $\log i = i\dfrac{\pi}{2}$

$$\therefore \quad \log z = \left(\log\sqrt{2} + i\frac{\pi}{4}\right) \cdot i\frac{\pi}{2} = \left[\frac{1}{2}\log 2 + \frac{i\pi}{4}\right]\frac{i\pi}{2} = -\frac{\pi^2}{8} + i \cdot \frac{\pi}{4}\log 2$$

$$\therefore \quad z = e^{-\pi^2/8 + i\theta} = e^{-\pi^2/8} \cdot e^{i\theta} \quad \text{where } \theta = \frac{\pi}{4}\log 2$$

$$= e^{-\pi^2/8}[\cos\theta + i\sin\theta]$$

$$\therefore \quad \text{Real part of } z = e^{-\pi^2/8}\cos\theta = e^{-\pi^2/8}\cos\left(\frac{\pi}{4}\log 2\right).$$

Example 10 (a) : Find the principal value of $(1+i)^{1-i}$. **(M.U. 2019)**

Sol. : Let $z=(1+i)^{1-i}$ $\therefore$ $\log z=(1-i)\log(1+i)$

$$\therefore \quad \log z=(1-i)\left[\log\sqrt{1+1}+i\tan^{-1}1\right]$$

$$=(1-i)\left[\frac{1}{2}\log 2+i\cdot\frac{\pi}{4}\right]=\frac{1}{2}\log 2+\frac{i\pi}{4}-\frac{i}{2}\log 2+\frac{\pi}{4}$$

$$=\left(\frac{1}{2}\log 2+\frac{\pi}{4}\right)+i\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)=x+iy \text{ say.}$$

$$\therefore \quad z=e^{x+iy}=e^x\cdot e^{iy}=e^x(\cos y+i\sin y)$$

$$=e^{(1/2)\log 2+(\pi/4)}\left[\cos\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)+i\sin\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)\right]$$

$$=\sqrt{2}\,e^{\pi/4}\left[\cos\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)+i\sin\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)\right] \qquad \left[\because\ e^{(1/2)\log 2}=e^{\log\sqrt{2}}=\sqrt{2}\right]$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Separate into real and imaginary parts $\tan h^{-1}(x+iy)$. **(M.U. 2015)**

Sol. : We know that $\tan h^{-1} z=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

$$\therefore \quad \tan h^{-1}(x+iy)=\frac{1}{2}\log\left(\frac{1+x+iy}{1-x-iy}\right)$$

$$=\frac{1}{2}\left[\log(1+x+iy)-\log(1-x-iy)\right]$$

$$=\frac{1}{2}\left[\left\{\frac{1}{2}\log\left\{(1+x)^2+y^2\right\}+i\tan^{-1}\frac{y}{1+x}\right\}-\left\{\frac{1}{2}\log\left\{(1-x)^2+y^2\right\}-i\tan^{-1}\frac{y}{1-x}\right\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\}+i\left\{\tan^{-1}\frac{y}{1+x}+\tan^{-1}\frac{y}{1-x}\right\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\}+i\tan^{-1}\left\{\frac{\frac{y}{1+x}+\frac{y}{1-x}}{1-\frac{y}{1+x}\cdot\frac{y}{1-x}}\right\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}\log\left\{\frac{(1+x)^2+y^2}{(1-x)^2+y^2}\right\}+i\tan^{-1}\left(\frac{2y}{1-x^2-y^2}\right)\right]$$

Example 2 (b) : If $(a+ib)^p=q^{x+iy}$, then prove that $\frac{y}{x}=\frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}$.

OR If $p\log(a+ib)=(x+iy)\log q$, then prove that $\frac{y}{x}=\frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}$. **(M.U. 1995, 2011)**

Sol. : We have $(a+ib)^p=q^{x+iy}$

Taking logarithms of both sides,

$$p\log(a+ib)=(x+iy)\log q$$

Scanned by CamScanner

$$\therefore \quad p\left[\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\left(\frac{b}{a}\right)\right]=(x+iy)\cdot\log q$$

$$\therefore \quad \frac{p}{\log q}\left[\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\left(\frac{b}{a}\right)\right]=x+iy$$

Equating real and imaginary parts on both sides, we get

$$x=\frac{p}{2\log q}\cdot\log(a^2+b^2);\quad y=\frac{p}{\log q}\cdot\tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \quad \frac{y}{x}=\frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}$$

Example 3 (b) : If $\dfrac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}}=\alpha+i\beta$, find α and β. **(M.U. 2012)**

Sol. : Taking logarithms of both sides,

$$\log(\alpha+i\beta)=(x+iy)\log(a+ib)-(x-iy)\log(a-ib)$$

$$\log(\alpha+i\beta)=(x+iy)\left[\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$-(x-iy)\left[\frac{1}{2}\log(a^2+b^2)-i\tan^{-1}\left(\frac{b}{a}\right)\right]$$

$$\log(\alpha+i\beta)=2i\left[x\tan^{-1}\frac{b}{a}+\frac{y}{2}\log(a^2+b^2)\right]$$

$$=2ik \text{ say where } k=\left[x\tan^{-1}\frac{b}{a}+\frac{y}{2}\log(a^2+b^2)\right]$$

$$\therefore \quad \alpha+i\beta=e^{2ik}=\cos 2k+i\sin 2k \qquad \therefore \quad \alpha=\cos 2k,\ \beta=\sin 2k.$$

Example 4 (b) : If $\dfrac{(1+i)^{x+iy}}{(1-i)^{x-iy}}=a+ib$ then considering the principle values only show that $\tan^{-1}\left(\dfrac{b}{a}\right)=\dfrac{\pi x}{2}+y\log 2.$ **(M.U. 1994, 96)**

Sol. : Taking logarithm of both sides,

$$(x+iy)\log(1+i)-(x-iy)\log(1-i)=\log(a+ib)$$

$$\therefore \quad (x+iy)[\log\sqrt{2}+i\tan^{-1}1]-(x-iy)[\log\sqrt{2}-i\tan^{-1}1]=\log\sqrt{a^2+b^2}+i\tan^{-1}\frac{b}{a}.$$

$$\therefore \quad (x+iy)\left[\frac{1}{2}\log 2+i\cdot\frac{\pi}{4}\right]-(x-iy)\left[\frac{1}{2}\log 2-i\cdot\frac{\pi}{4}\right]=\frac{1}{2}\log(a^2+b^2)+i\tan^{-1}\frac{b}{a}.$$

Equating imaginary parts,

$$\frac{y}{2}\log 2+x\cdot\frac{\pi}{4}+\frac{y}{2}\log 2+x\cdot\frac{\pi}{4}=\tan^{-1}\frac{b}{a} \qquad \left[\text{Real part } \frac{1}{2}\log(a^2+b^2)=0\right]$$

$$\therefore \quad \tan^{-1}\frac{b}{a}=\frac{\pi x}{2}+y\log 2.$$

Remark ....

Example 4 is a particular case of Ex. 3 where in l.h.s. $a=b=1$.

Miscellaneous Examples : Class (b) : 6 Marks

Example 1 (b) : If $\log(x+iy) = e^{p+iq}$, prove that $y = x\tan\theta$, where $2\theta = \tan q \log(x^2+y^2)$ or if $\log\log(x+iy) = p+iq$, prove that $y = x\tan(\tan q \cdot \log\sqrt{x^2+y^2})$.

OR If $\log\log(x+iy) = p+iq$, then prove that $y = x\tan\left[\tan q \log\sqrt{x^2+y^2}\right]$.

Sol. : We have $\log(x+iy) = e^p \cdot e^{iq} = e^p(\cos q + i\sin q)$

$$\therefore \quad \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x} = e^p(\cos q + i\sin q)$$

Equating real and imaginary parts

$$e^p\cos q = \frac{1}{2}\log(x^2+y^2) \qquad \text{.................... (1)}$$

and $\quad e^p\sin q = \tan^{-1}\dfrac{y}{x}$

$$\therefore \quad \frac{y}{x} = \tan(e^p\sin q) \quad \therefore \quad y = x\tan\theta \quad \text{where, } \theta = e^p \cdot \sin q = \sin q \cdot e^p$$

$$\therefore \quad \theta = \sin q \cdot \frac{1}{\cos q} \cdot \frac{1}{2} \cdot \log(x^2+y^2) \qquad [\text{ From (1) }]$$

i.e. $\quad 2\theta = \tan q \cdot \log(x^2+y^2)$.

Example 2 (b) : Considering only the principle values, prove that the real part of

$$(1+i\sqrt{3})^{(1+i\sqrt{3})} \text{ is } 2e^{-\pi/\sqrt{3}}\left(\cos\frac{\pi}{3} + \sqrt{3}\cdot\log 2\right). \qquad \textbf{(M.U. 2004, 05, 11)}$$

Sol. : Let $x+iy = (1+i\sqrt{3})^{(1+i\sqrt{3})}$

$$\therefore \quad \log(x+iy) = (1+i\sqrt{3})\log(1+i\sqrt{3}) = (1+i\sqrt{3})[\log\sqrt{1+3} + i\tan^{-1}\sqrt{3}]$$

$$\therefore \quad \log(x+iy) = (1+i\sqrt{3})[\log 2 + i\tan^{-1}\sqrt{3}] = (1+i\sqrt{3})[\log 2 + i\pi/3]$$

$$= \left(\log 2 - \frac{\pi}{\sqrt{3}}\right) + i\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)$$

$$\therefore \quad x+iy = e^{\log 2} \cdot e^{-\pi/\sqrt{3}} \cdot e^{i[(\pi/3)+\sqrt{3}\log 2]}$$

$$= 2e^{-\pi/\sqrt{3}}\left[\cos\left(\frac{\pi}{3} + \sqrt{3}\log 2\right) + i\sin\left(\frac{\pi}{3} + \sqrt{3}\log 2\right)\right] \qquad [\because e^{\log 2} = 2]$$

$$\therefore \text{ Real Part } x = 2e^{-\pi/\sqrt{3}}\cos\left(\frac{\pi}{3} + \sqrt{3}\cdot\log 2\right).$$

Example 3 (b) : Prove that $i^{i^i} = \cos\theta + i\sin\theta$ where $\theta = \left(2n+\dfrac{1}{2}\right)\pi e^{-\left(2m+\frac{1}{2}\right)\pi}$.

(M.U. 2002)

Sol. : Let $z = i^{i^i}$ $\therefore$ $\text{Log } z = i^i \text{ Log } i$

But $\quad i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = \cos\left(2m+\dfrac{1}{2}\right)\pi + i\sin\left(2m+\dfrac{1}{2}\right)\pi = e^{i\left(2m+\frac{1}{2}\right)\pi}$

$$\therefore \quad i^i = \left[e^{i\left(2m+\frac{1}{2}\right)\pi}\right]^i = e^{-\left(2m+\frac{1}{2}\right)\pi}$$

Also $\quad \text{Log}\, i = i\left(2n+\frac{1}{2}\right)\pi$ [By (3.A), page 4-1]

$$\text{Log}\, z = i^i[\text{Log}\, i] = e^{-\left(2m+\frac{1}{2}\right)\pi} \cdot i\left[\left(2n+\frac{1}{2}\right)\pi\right]$$

$$= i\theta \text{ say} \qquad \text{where } \theta = \left(2n+\frac{1}{2}\right)\pi \cdot e^{-\left(2m+\frac{1}{2}\right)\pi}$$

$$\therefore \quad z = e^{i\theta} = (\cos\theta + i\sin\theta) \qquad \text{where } \theta = \left(2n+\frac{1}{2}\right)\pi \cdot e^{-\left(2m+\frac{1}{2}\right)\pi}$$

Example 4 (b) : If $\tan[\log(x+iy)] = a+ib$, prove that

$$\tan[\log(x^2+y^2)] = \frac{2a}{1-a^2-b^2} \text{ when } a^2+b^2 \neq 1.$$ **(M.U. 1986, 95, 97, 2006, 13)**

Sol. : We have $\tan[\log(x+iy)] = a+ib \qquad \therefore \log(x+iy) = \tan^{-1}[a+ib]$

Let $\quad \tan^{-1}(a+ib) = \alpha + i\beta$ .... (i)

$$\therefore \quad \log(x+iy) = \alpha+i\beta \qquad \therefore \quad \log\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} = \alpha + i\beta$$

Equating real parts, we have

$$\log\sqrt{x^2+y^2} = \alpha \quad i.e. \quad \frac{1}{2}\log(x^2+y^2) = \alpha$$

$$\therefore \quad \log(x^2+y^2) = 2\alpha \qquad \text{.................... (ii)}$$

We now find the real part α of (i)

$$\tan 2\alpha = \tan[(\alpha+i\beta)+(\alpha-i\beta)] = \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)} \qquad \text{.................... (iii)}$$

Since $\tan^{-1}(a+ib) = \alpha+i\beta$

$\therefore \quad \tan(\alpha+i\beta) = a+ib$ and $\tan(\alpha-i\beta) = a-ib$

Hence, from (iii), we get,

$$\tan 2\alpha = \frac{(a+ib)+(a-ib)}{1-(a+ib)+(a-ib)} = \frac{2a}{1-a^2-b^2} \qquad \text{.................... (iv)}$$

Hence, by putting the values of 2α from (ii) in (iv),

$$\therefore \quad \tan\{\log(x^2+y^2)\} = \frac{2a}{1-a^2-b^2} \text{ when } a^2+b^2 \neq 1. \qquad \text{[By (iv)]}$$

Example 5 (b) : Show that for real values of a and b,

$$e^{2ai\cot^{-1}b} \cdot \left[\frac{bi-1}{bi+1}\right]^{-a} = 1.$$ **(M.U. 2004, 08, 14)**

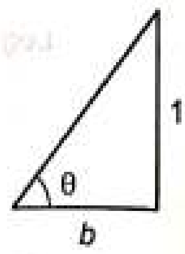
Sol. : Consider $\quad \dfrac{bi-1}{bi+1} = \dfrac{bi+i^2}{bi-i^2} = \dfrac{b+i}{b-i} \qquad \therefore \left(\dfrac{bi-1}{bi+1}\right)^{-a} = \left(\dfrac{b+i}{b-i}\right)^{-a}$

Now, $\quad \log\left(\dfrac{b+i}{b-i}\right)^{-a} = -a[\log(b+i) - \log(b-i)]$

Scanned by CamScanner

$$\therefore \quad \log\left(\frac{b+i}{b-i}\right)^{-a} = -a\left[\left(\log\sqrt{b^2+1} + i\tan^{-1}\frac{1}{b}\right) - \left(\log\sqrt{b^2+1} - i\tan^{-1}\frac{1}{b}\right)\right]$$

$$= -2ai\tan^{-1}\frac{1}{b}.$$



$$\therefore \quad \left(\frac{b+i}{b-i}\right)^{-a} = e^{-2ai\tan^{-1}(1/b)}$$

But, $\cot^{-1} b = \tan^{-1}\left(\frac{1}{b}\right)$ $\left[\text{If } \cot\theta = b,\ \tan\theta = \frac{1}{b}\right]$

$$\therefore \quad \text{l.h.s.} = e^{2ai\tan^{-1}(1/b)} \cdot e^{-2ai\tan^{-1}(1/b)} = e^0 = 1.$$

Alternatively : We may use exponential form of a complex number, instead of taking logarithms as follows.

As before $\quad \dfrac{bi-1}{bi+1} = \dfrac{bi+i^2}{bi-i^2} = \dfrac{b+i}{b-i}$

Now, let $b+i = r\,e^{i\theta} = r\cos\theta + ir\sin\theta$. $\quad \because \; b = r\cos\theta,\ 1 = r\sin\theta.$

$$\therefore \quad r^2 = b^2+1 \qquad i.e., \qquad r = \sqrt{b^2+1}$$

and $\quad \tan\theta = \dfrac{1}{b} \qquad \therefore \quad \theta = \tan^{-1}\dfrac{1}{b} = \cot^{-1} b$

Further, $b - i = r\,e^{-i\theta}$.

$$\therefore \quad \frac{bi-1}{bi+1} = \frac{b+i}{b-i} = \frac{r\,e^{i\theta}}{r\,e^{-i\theta}} = e^{2i\theta} = e^{2i\cot^{-1}b}$$

Hence, $e^{2ai\cot^{-1}b} \cdot \left(\dfrac{bi-1}{bi+1}\right)^{-a} = e^{2ai\cot^{-1}b} \cdot e^{(2i\cot^{-1}b)^{-a}}$

$$= e^{2ai\cot^{-1}b} \cdot e^{-2ai\cot^{-1}b} = e^0 = 1.$$

Example 6 (b) : Considering only the principal value, if $(1+i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{\sec^2\beta}$. **(M.U. 1998, 2000, 07, 2012, 13)**

Sol. : Let $\quad z = (1+i\tan\alpha)^{1+i\tan\beta}$

Taking logarithms of both sides,

$$\log z = (1+i\tan\beta)\log(1+i\tan\alpha)$$

$$= (1+i\tan\beta)\left[\frac{1}{2}\log(1+\tan^2\alpha) + i\tan^{-1}\tan\alpha\right]$$

$$= (1+i\tan\beta)\ [\log\sec\alpha + i\alpha]$$

$$\therefore \quad \log z = (\log\sec\alpha - \alpha\tan\beta) + i(\alpha + \tan\beta\log\sec\alpha)$$

$$= x + iy \text{ say}$$

where $x = \log\sec\alpha - \alpha\tan\beta$ and $y = \alpha + \tan\beta\log\sec\alpha$ .... (1)

Now, $\quad z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

Since by data z is real $e^x\sin y = 0 \quad \therefore \quad y = 0 \quad \therefore \quad \cos y = 1$

$$\therefore \quad z = e^x\cos y = e^x = e^{\log\sec\alpha - \alpha\tan\beta}$$

$$\therefore \quad z = e^{\log\sec\alpha} \cdot e^{-\alpha\tan\beta} = \sec\alpha \cdot e^{-\alpha\tan\beta} \qquad \text{..................... (2)}$$

But since $y = 0$, from (1), $\alpha + \tan\beta \log\sec\alpha = 0$

$\therefore \quad -\alpha = \tan\beta \log\sec\alpha$

$\therefore \quad -\alpha\tan\beta = \tan^2\beta \cdot \log\sec\alpha = \log(\sec\alpha)^{\tan^2\beta}$

$\therefore \quad e^{-\alpha\tan\beta} = (\sec\alpha)^{\tan^2\beta}$

$\therefore \quad z = \sec\alpha \cdot (\sec\alpha)^{\tan^2\beta} = (\sec\alpha)^{(1+\tan^2\beta)}$ [From (2)]

$= (\sec\alpha)^{\sec^2\beta}$

Example 7 (b) : If $x^{x^{x\cdots\infty}} = a(\cos\alpha + i\sin\alpha)$, prove that the general value of x is given by $r(\cos\theta + i\sin\theta)$ where

$$\log r = \frac{(2n\pi+\alpha)\sin\alpha + \cos\alpha \cdot \log a}{a} \text{ and } \theta = \frac{(2n\pi+\alpha)\cos\alpha - \sin\alpha \cdot \log a}{a}.$$

Sol. : We have $\quad x^{a(\cos\alpha + i\sin\alpha)} = a(\cos\alpha + i\sin\alpha) = a\,e^{i\alpha}$

Taking logarithms of both sides,

$a(\cos\alpha + i\sin\alpha)\log x = \log a + i(2n\pi + \alpha)$ [$\because \log e^{i\alpha} = i\alpha = i(2n\pi+\alpha)$]

Since we are required to find the general value, we take $(2n\pi + \alpha)$ is the second bracket.

$\therefore \quad a(\cos\alpha + i\sin\alpha)(\log r + i\theta) = \log a + i(2n\pi + \alpha)$ [$\because x = r e^{i\theta}$]

Equating real and imaginary parts,

$a[\cos\alpha \cdot \log r - \theta\sin\alpha] = \log \alpha$

$a[\sin\alpha \cdot \log r + \theta\cos\alpha] = 2n\pi + \alpha$

$$\therefore \quad \cos\alpha\log r - \theta\sin\alpha = \frac{\log a}{a} \qquad \text{.................... (i)}$$

$$\text{and} \quad \sin\alpha\log r + \theta\cos\alpha = \frac{2n\pi+\alpha}{a} \qquad \text{.................... (ii)}$$

We solve these two equations for $\log r$ and θ.

To find $\log r$ multiply (i) by $\cos\alpha$ and (ii) by $\sin\alpha$ and add

$$\therefore \quad \log r = \frac{\log a \cdot \cos\alpha + (2n\pi+\alpha)\sin\alpha}{a}$$

To find θ multiply (i) by $\sin\alpha$ and (ii) by $\cos\alpha$ and subtract.

$$\therefore \quad \theta = \frac{(2n\pi+\alpha)\cos\alpha - \log a \cdot \sin\alpha}{a}.$$

Example 8 (b) : Prove that the general value of $(1 + i\tan\alpha)^{-i}$ is

$$e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)] \qquad \textbf{(M.U. 2001, 04, 09)}$$

Sol. : Let $\quad 1 + i\tan\alpha = r\,e^{i\theta} = r(\cos\theta + i\sin\theta)$

$\therefore \quad r\cos\theta = 1, \quad r\sin\theta = \tan\alpha.$

Squaring, we get

$\therefore \quad r^2 = 1 + \tan^2\alpha = \sec^2\alpha \qquad \therefore \quad r = \sec\alpha.$

and $\dfrac{r\sin\theta}{r\cos\theta} = \dfrac{\tan\alpha}{1} \quad \therefore \quad \tan\theta = \tan\alpha \quad \therefore \quad \theta = \alpha.$

Now, $\quad \text{Log}(1 + i\tan\alpha) = \log(r\,e^{i\theta}) = \log r + (2m\pi + \theta)\,i$

$= \log\sec\alpha + (2m\pi + \alpha)\,i$ [Taking the general value $2m\pi + \alpha$]

$$\therefore \quad 1 + i\tan\alpha = e^{[\log\sec\alpha + (2m\pi+\alpha)i]}$$

$$\therefore \quad (1+i\tan\alpha)^{-i} = e^{-i[\log\sec\alpha + (2m\pi+\alpha)i]}$$

$$= e^{2m\pi+\alpha} \cdot e^{-i\log\sec\alpha} = e^{2m\pi+\alpha} \cdot e^{i\log\cos\alpha}$$

$$= e^{2m\pi+\alpha} \cdot [\cos\log\cos\alpha + i\sin\log\cos\alpha]$$

Example 9 (a) : If $\sin^{-1}(x+iy) = \log(A+iB)$, prove that

$$\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = 1 \text{ where } A^2+B^2 = e^{2\alpha}.$$ **(M.U. 2002)**

Sol. : Let $\sin^{-1}(x+iy) = \alpha + i\beta$

$\therefore \quad x + iy = \sin(\alpha+i\beta) = \sin\beta\cos h\beta + i\cos\alpha\sin h\beta.$

Equating real and imaginary parts,

$x = \sin\alpha\cos h\beta$ and $y = \cos\alpha\sin h\beta$

$$\therefore \quad \frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = \cos h^2\beta - \sin h^2\beta = 1 \qquad \text{.......... (i)}$$

Now, by data $\quad \alpha + i\beta = \log(A+iB) = \log\sqrt{A^2+B^2} + i\tan^{-1}\frac{B}{A}$

Equating real and imaginary parts,

$$\alpha = \frac{1}{2}\log(A^2+B^2) \quad \therefore \quad 2\alpha = \log(A^2+B^2)$$

$$e^{2\alpha} = A^2 + B^2 \qquad \text{.......... (ii)}$$

From (i) and (ii), we get the required result.

Example 10 (b) : If $i^{\alpha+i\beta} = \alpha + i\beta$, prove that $\alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$ where n is any positive integer. **(M.U. 1993, 95, 2002, 03)**

Sol. : Taking logarithms of both sides $\quad (\alpha+i\beta)\,\text{Log}\, i = \text{Log}\,(\alpha+i\beta)$

But $i = e^{i\left(2n\pi+\frac{\pi}{2}\right)} \quad \therefore \quad \text{Log}\, i = i(4n+1)\frac{\pi}{2}$ [See 3(A), page 4-1]

$$\therefore \quad (\alpha+i\beta)\,i(4n+1)\frac{\pi}{2} = \text{Log}\,(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

Equating real parts,

$$-(4n+1)\frac{\pi}{2}\beta = \frac{1}{2}\log(\alpha^2+\beta^2) \quad \therefore \quad (\alpha^2+\beta^2) = e^{-(4n+1)\pi\beta}.$$

Aliter : Since $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = \cos\left(2n\pi+\frac{\pi}{2}\right) + i\sin\left(2n\pi+\frac{\pi}{2}\right)$, we have

$$\left[\cos\left(2n\pi+\frac{\pi}{2}\right) + i\sin\left(2n\pi+\frac{\pi}{2}\right)\right]^{\alpha+i\beta} = \alpha + i\beta$$

$$\therefore \quad e^{i\left(2n\pi+\frac{\pi}{2}\right)(\alpha+i\beta)} = \alpha + i\beta \qquad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$\therefore \quad e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta + i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$$

$$\therefore \quad e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta} \cdot e^{i\left(2n\pi+\frac{\pi}{2}\right)\alpha} = \alpha + i\beta$$

$$\therefore \quad e^{-\left(2n\pi+\frac{\pi}{2}\right)\beta}\left[\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha + i\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha\right] = \alpha + i\beta$$

Equating real and imaginary parts,

$$e^{-(4n+1)\frac{\pi}{2}\beta}\cos\left(2n\pi+\frac{\pi}{2}\right)\alpha = \alpha; \quad e^{-(4n+1)\frac{\pi}{2}\beta}\sin\left(2n\pi+\frac{\pi}{2}\right)\alpha = \beta$$

$\therefore$ Squaring and adding, we get, $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$.

Example 11 (b) : Prove that $\log\tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = i\tan^{-1}(\sin h\, x)$. **(M.U. 1990, 98, 2004, 17)**

Sol. : We have

$$\log\tan\left(\frac{\pi}{4} + \frac{ix}{2}\right) = \log\left\{\frac{1+\tan(ix/2)}{1-i\tan(ix/2)}\right\} = \log\left\{\frac{1+i\tan h(x/2)}{1-i\tan h(x/2)}\right\}$$

$$= \log[1+i\tan h(x/2)] - \log[1-i\tan h(x/2)]$$

$$= \frac{1}{2}\log\left(1+\tan^2 h\frac{x}{2}\right) + i\tan^{-1}\tan\left(\frac{x}{2}\right) - \frac{1}{2}\left[\log\left(1+\tan h^2\frac{x}{2}\right) - i\tan^{-1}\tan h\frac{x}{2}\right]$$

$$= 2i\tan^{-1}\left(\tan h\frac{x}{2}\right) \qquad \left[\text{ Putting } \alpha = \tan h\frac{x}{2} \text{ in (A) given below. }\right]$$

$$= i \cdot \tan^{-1}\left\{\frac{2\tan h(x/2)}{1-\tan h^2(x/2)}\right\}$$

$$= i\tan^{-1}(\sin hx). \qquad [\text{ By C(7), page 3-5 }]$$

[To prove that $\tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$

Proof : Let $\tan^{-1}\alpha = \theta$, $\tan^{-1}\beta = \Phi$ $\quad\therefore\quad \tan\theta = \alpha$ and $\tan\Phi = \beta$.

Consider $\tan(\theta+\Phi) = \dfrac{\tan\theta + \tan\Phi}{1-\tan\theta\tan\Phi}$

$$\therefore \quad \tan(\tan^{-1}\alpha + \tan^{-1}\beta) = \frac{\alpha+\beta}{1-\alpha\beta}$$

$$\therefore \quad \tan^{-1}\alpha + \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

Similarly, we can have

$$\tan^{-1}\alpha - \tan^{-1}\beta = \tan^{-1}\left(\frac{\alpha-\beta}{1+\alpha\beta}\right)$$

If we put $\beta = \alpha$, then

$$2\tan^{-1}\alpha = \tan^{-1}\left(\frac{2\alpha}{1-\alpha^2}\right). \qquad \text{.................... (A)}$$

Aliter : The problem suggests that, we have to separate real and imaginary parts of l.h.s.

Let $\log\tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = a + ib \quad\therefore\quad \tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = e^{a+ib} = e^a \cdot e^{ib}$

$$\therefore \quad \frac{1+\tan i(x/2)}{1-\tan i(x/2)} = e^a(\cos b + i\sin b)$$

Now, $$\frac{1+\tan(ix/2)}{1-\tan(ix/2)} = \frac{\cos(ix/2)+\sin(ix/2)}{\cos(ix/2)-\sin(ix/2)}$$

$$= \frac{[\cos(ix/2)+\sin(ix/2)]^2}{\cos^2(ix/2)-\sin^2(ix/2)} = \frac{1+\sin ix}{\cos ix}$$

$$= \sec ix + \tan ix = \sec hx + i\tan hx$$

$$\therefore \quad \sec hx + i\tan hx = e^a(\cos b + i\sin b)$$

Equating real and imaginary parts,

$$\sec hx = e^a\cos b \text{ and } \tan hx = e^a \sin b.$$

Squaring and adding. $\quad e^{2a} = 1 \qquad \therefore \ a = 0$

Also by division $\quad \tan b = \sin hx \qquad \therefore \ b = \tan^{-1}(\sin hx)$

$$\therefore \quad \log\tan\left(\frac{\pi}{4} + i\frac{x}{2}\right) = a + ib = i\tan^{-1}(\sin hx).$$

Example 12 (b) : Prove that $\text{Log}\left[\dfrac{(a-b)+i(a+b)}{(a+b)+i(a-b)}\right] = i\left(2n\pi + \tan^{-1}\dfrac{2ab}{a^2-b^2}\right)$.

(M.U. 2006, 07)

Sol. : Let $a+b=A,\ a-b=B.$

$$\therefore \quad \text{Log}\left[\frac{B+iA}{A+iB}\right] = 2n\pi i + \log\left[\frac{B+iA}{A+iB}\right] \qquad \text{[By (1.B), page 4-1]}$$

$$= 2n\pi i + \log(B+iA) - \log(A+iB)$$

$$= 2n\pi i + \left[\log\sqrt{B^2+A^2} + i\tan^{-1}\frac{A}{B}\right] - \left[\log\sqrt{A^2+B^2} + i\tan^{-1}\frac{B}{A}\right]$$

$$= 2n\pi i + i\left[\tan^{-1}\frac{A}{B} - \tan^{-1}\frac{B}{A}\right]$$

$$\left[\text{But } \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$\therefore \quad \text{Log}\left[\frac{B+iA}{A+iB}\right] = 2n\pi i + i\tan^{-1}\left[\frac{(A/B)-(B/A)}{1+(A/B)\cdot(B/A)}\right]$$

$$= 2n\pi i + i\tan^{-1}\left(\frac{A^2-B^2}{2AB}\right)$$

[But $A^2 - B^2 = (a+b)^2 - (a-b)^2 = 4ab$ and $AB = (a+b)(a-B) = a^2 - b^2$.]

$$\therefore \quad \text{Log}\left[\frac{B+iA}{A+iB}\right] = 2n\pi i + i\cdot\tan^{-1}\frac{4ab}{2(a^2-b^2)}$$

$$= i\left[2n\pi + \tan^{-1}\frac{2ab}{a^2-b^2}\right]$$

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. Express the following in the form $a + ib$

   (i) Log $(-i)$ **(ii)** Log (-1) **(iii)** Log $(3+4i)$

   (iv) Log (-3) **(v)** Log $(1+i)$

   [**Ans. : (i)** $i\left(2n-\frac{1}{2}\right)\pi$, **(ii)** $i(2n+1)\pi$, **(iii)** $\log 5 + i(2n\pi+\theta),\ \theta = \tan^{-1}\left(\frac{4}{3}\right)$.

   (iv) $\log 3 + i(2n+1)\pi$, **(v)** $\frac{1}{2}\log 2 + i\left(2n+\frac{1}{4}\right)\pi$.]

2. Show that $\sin\log_e(i^i) = -1$.

3. Prove that $\log(1+e^{i\theta}) = \log\left(2\cos\frac{\theta}{2} + i\frac{\theta}{2}\right)$.

4. Prove that $\log\left(\frac{1}{1+e^{i\theta}}\right) = \log\left(\frac{1}{2}\sec\frac{\theta}{2}\right) - i\frac{\theta}{2}$.

5. Find the value of log $(1+i)$. **(M.U. 1995)** [**Ans. :** $\frac{1}{2}\log 2 + i\frac{\pi}{4}$]

6. Separate into real and imaginary part $\log_{(1-i)}(1+i)$. **(M.U. 1998)**

   [**Ans. :** $\dfrac{4(\log 2)^2 - \pi^2 + i\,4\log 2\pi}{4(\log 2)^2 + \pi^2}$]

7. Find all the roots of the equation $\tan h\,z + 2 = 0$. [**Ans. :** $-\frac{1}{2}\log 3 - i\left(n+\frac{1}{2}\right)\pi$]

8. Separate into real and imaginary parts.

   (i) i^i **(ii)** $(-i)^{(i-1)}$ **(iii)** $i^{(i+1)}$ **(iv)** $(\sin\theta + i\cos\theta)^i$.

   [**Ans. : (i)** $e^{-[2n+(1/2)]\pi}$, **(ii)** $i\,e^{\pi/2}$, **(iii)** $i\,e^{-\pi/2}$, **(iii)** $e^{\theta-\pi/2}$.]

9. Prove that $\log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta) = \log(2\cos\theta) + i\theta$. **(M.U. 1996)**

10. Find the general value of $\text{Log}(1+i\sqrt{3}) + \text{Log}(1-i\sqrt{3})$. [**Ans. :** 2 log 2]

11. Prove that $\log(1+i\tan\alpha) = \log\sec\alpha + i\alpha$.

12. If $e^{i\alpha} = i^{\beta}$, prove that $\frac{\alpha}{\beta} = 2n\pi + \frac{\pi}{2}$.

13. If $(1+i)^{x+iy} = \alpha + i\beta$, prove that $\tan^{-1}\frac{\beta}{\alpha} = \frac{\pi}{4}x + \frac{y}{2}\log 2$.

14. Prove that principal value of $(1+\tan\alpha)^{-i}$ is e^{α} [$\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)$]. **(M.U. 2001, 04)**

15. If $i^{i^{i\ldots\infty}} = A + iB$, considering the principal value, prove that

    $\tan\left(\frac{\pi A}{2}\right) = \frac{B}{A}$ and $A^2 + B^2 = e^{-\pi B}$. **(M.U. 2002, 15)**

16. If $\sqrt{i}^{\sqrt{i}^{\ldots\infty}} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-\pi\beta/2}$. **(M.U. 2015)**

17. If $p\log(a+ib) = (x+iy)\log m$, prove that $\frac{y}{x} = \frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}$. **(M.U. 1995)**

18. Prove that the real part of the principal value of $(1+i)^{\log i}$ is

$$e^{-\pi^2/8}\cos\left(\frac{\pi}{4}\log 2\right).$$ (M.U. 2002)

Class (b) : 6 Marks

1. Prove that $\log[\cos(x+iy)] = \frac{1}{2}\log\left(\frac{\cos h2y+\cos 2x}{2}\right) - i\tan^{-1}(\tan x \tan hy)$ (M.U. 1984, 92)

2. Prove that $\log\left[\frac{\cos(x-iy)}{\cos(x+iy)}\right] = 2i\tan^{-1}(\tan x + \tan hy)$.

3. If $\frac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \alpha + i\beta$, find α and β.

[**Ans. :** $\alpha = \cos k$, $\beta = \sin k$ where $k = (x \cdot \frac{\pi}{2} + y \cdot \log 2)$]

4. If $\log\sin(x+iy) = a+ib$, prove that
   (i) $2e^{2a} = \cos h2y - \cos 2x$, (ii) $\tan b = \cot x \tan hy$. (M.U. 2009)

5. If $\log\cos(x+iy) = a+ib$, prove that
   (i) $2e^{2a} = \cos h2y + \cos 2x$, (ii) $\tan b = -\tan x \tan hy$.

6. Prove that $\log(e^{i\alpha}+e^{i\beta}) = \log\left\{2\cdot\cos\left(\frac{\alpha-\beta}{2}\right)\right\} + i\frac{(\alpha+\beta)}{2}$. (M.U. 2004, 10, 16)

7. If $\log[\log(x+iy)] = p+iq$, prove that $y = x\tan[\tan q \cdot \log\sqrt{x^2+y^2}]$. (M.U. 2001, 02, 03)

8. Find all the values of $\sin^{-1}2$ treating 2 as a complex number.

[**Ans. :** $\sin^{-1}2 = -i\cos h^{-1}2 + (4n+1)\frac{\pi}{2}$]

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (b) : 6 Marks

Prove that $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$.

Hence, deduce that if $(a_1+ib_1)(a_2+ib_2)\ldots\ldots(a_n+ib_n) = A+iB$, then

$$(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots\ldots(a_n^2+b_n^2) = A^2+B^2.$$

and $$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots\ldots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right).$$

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Short Answer Questions : Class (a) : 3 Marks

1. Find $\log(-7)$. [**Ans. :** $\log 7 + i\pi$]

2. Find $\log(1+i)$. [**Ans. :** $\frac{1}{2}\log 2 + \frac{i\pi}{4}$]

3. Find the principal value of $\log i$. [**Ans. :** $\frac{i\pi}{2}$]
4. Find the value of i^{2i}. [**Ans. :** $e^{-\pi}$]
5. Find the value of $\sin h \log i$. [**Ans. :** i]
6. Find the value of $\cos h \log i$. [**Ans. :** 0]
7. Find $\log_3(-5)$. [**Ans. :** $\frac{\log 5 + i\pi}{\log 3}$]
8. Find $\log i^i$. [**Ans. :** $-\frac{\pi}{2}$]
9. Find the value of i^i and hence find i^{4i}. [**Ans. :** $e^{-\pi/2}$, $e^{-2\pi}$]
10. Find the value of i^i and hence find i^{3i}. [**Ans. :** $e^{-\pi/2}$, $e^{-3\pi/2}$]

Summary

1. $\log(x+iy) = \log r + i\theta$

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

2. $\text{Log}\,(x+iy) = 2n\pi i + \log(x+iy)$

$\text{Log}\,(x+iy) = \log r + i(2n\pi + \theta)$

3. $\log i = \frac{i\pi}{2}$, $\quad \text{Log}\ i = i\left(2n\pi + \frac{\pi}{2}\right)$

4. $\log(i^i) = -\frac{\pi}{2}$, $\quad i^i = e^{-\pi/2}$

5. $\sin(\log i^i) = -1$

$\cos(\log i^i) = 0s$

☆ ☆ ☆

CHAPTER

5 Partial Differentiation

1. Introduction

In many scientific and engineering problems we find that a dependent variable depends for its values on two or more independent variables. For example, the velocity of a boat moving in the stream of a river depends upon the initial velocity of the boat, velocity of water, velocity of wind, time, etc. Such a dependence is denoted by

$$y = f(x_1, x_2, \ldots\ldots, x_n)$$

where y is the dependent variable and $x_1, x_2, \ldots\ldots, x_n$ are all independent variables.

However, we shall restrict our discussion to functions of two (or three) independent variables only. We shall denote the dependent variable by z, the independent variables by x and y and the function by

$$z = f(x, y) \quad \text{or} \quad z = \Phi(x, y).$$

2. Partial Derivatives of the First Order

Definition : Let $z = f(x, y)$ be a function of two independent variables x and y. Suppose now x changes to $x + \delta x$, y remaining constant. Then z will change to $z + \delta z$.

$$\therefore \quad z + \delta z = f(x + \delta x, y) \qquad \therefore \quad \delta z = f(x + \delta x, y) - f(x, y)$$

$$\frac{\delta z}{\delta x} = \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

The limit of $\frac{\delta z}{\delta x}$ as $\delta x \to 0$, if it exists, is called the **partial derivative of z with respect to x** and it is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or by f_x.

$$\therefore \quad \boxed{\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}}$$

If on the other hand y changes to $y + \delta y$, x remaining constant, then

$$z + \delta z = f(x, y + \delta y) - f(x, y) \qquad \therefore \quad \frac{\delta z}{\delta y} = \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

The limit of $\frac{\delta z}{\delta y}$ as $\delta y \to 0$, if it exists, is called the **partial derivative of z with respect to y** and it is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or by f_y.

$$\therefore \quad \boxed{\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}}$$

Partial Derivatives of $z = f(x, y)$ at a given point (a, b)

Let x change from a to $a + \delta x$, y remaining at b. As a consequence let z change to $z + \delta z$.

Then $\delta z = f(a + \delta x, b) - f(a, b)$ is called the increment in z. $\frac{\delta z}{\delta x}$ is called the incrementary ratio.

The limit of the incrementary ratio $\frac{\delta z}{\delta x}$ as $\delta x \to 0$ is called the partial derivative of z w.r.t. x at (a, b).

$$\therefore \quad \frac{\delta z}{\delta x} = \frac{f(a+\delta x,\ b) - f(a,b)}{\delta x}$$

$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{\delta z}{\delta x} = \lim_{\delta x \to 0} \frac{f(a+\delta x,\ b) - f(a,b)}{\delta x}$$

$$\text{Similarly,} \quad \frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{\delta z}{\delta y} = \lim_{\delta y \to 0} \frac{f(a, b+\delta x) - f(a,b)}{\delta y}$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : If $z = x^2 y^2$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ from the definition.

Sol. : Let x change to $x + \delta x$, then z will change to $z + \delta z$.

$$\therefore \quad z + \delta z = (x + \delta x)^2 y^2 \qquad \therefore \quad \delta z = (x+\delta x)^2 y^2 - x^2 y^2$$

$$\therefore \quad \frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{\delta z}{\delta x} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 y^2 - x^2 y^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2x\,\delta x\, y^2 + (\delta x)^2 y^2}{\delta x} = \lim_{\delta x \to 0} 2xy^2 + y^2(\delta x)$$

$$\therefore \quad \frac{\partial z}{\partial x} = 2xy^2$$

To find $\frac{\partial z}{\partial y}$, let y change $y + \delta y$ (x, remaining constant). Then z will change to $z + \delta z$.

$$\therefore \quad z + \delta z = x^2 (y + \delta y)^2 \qquad \therefore \quad z = x^2 (y+\delta y)^2 - x^2 y^2$$

$$\therefore \quad \frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{2x^2 y\, \delta y + x^2 (\delta y)^2}{\delta y} = \lim_{\delta y \to 0} 2x^2 y + x^2 (\delta y) = 2x^2 y.$$

Example 2 (a) : If $w = f(x, y, z)$ be a function of three independent variables, write the formal definition of the partial derivatives of for $\frac{\partial f}{\partial z}$ at (x_0, y_0, z_0). Using this definition find $\frac{\partial f}{\partial z}$ at (1, 2, 3) for $f(x, y, z) = x^2 y z^2$.

Sol. : Let $w = f(x, y, z)$ and $\delta x, \delta y, \delta z$ denote the increments in x, y, z. Let z change from z_0 to $z_0 + \delta z$, x and y remaining at x_0, y_0. $\delta w = f(x_0, y_0, z_0 + \delta z) - f(x_0, y_0, z_0)$ is called the increment of $f(x, y, z)$.

Then $\frac{\delta w}{\delta z}$ is called the incrementary ratio and the limit of $\frac{\delta w}{\delta z}$ as $\delta z \to 0$ is called the partial derivative of $f(x, y, z)$ w.r.t. z at (x_0, y_0, z_0).

$$\left(\frac{\partial f}{\partial z}\right)_{(x_0,\, y_0,\, z_0)} = \lim_{\delta z \to 0} \frac{f(1,\, 2,\, 3+\delta z) - f(1,\, 2,\, 3)}{\delta z}$$

$$\therefore \left(\frac{\partial f}{\partial z}\right)_{(x_0,\, y_0,\, z_0)} = \lim_{\delta z \to 0} \frac{1^2 \cdot 2 \cdot (3+\delta z)^2 - 1^2 \cdot 2 \cdot 3^2}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{2(9 + 6\,\delta z + \delta z^2) - 2 \cdot 9}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{12\,\delta z + (\delta z)^2}{\delta z} = \lim_{\delta z \to 0} 12 + (\delta z) = 12.$$

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

1. Use the definition of partial derivative and find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ for the following functions.

   (1) $z = x^2 y$ (2) $z = xy^2$ (3) $z = x^3 y^2$ (4) $z = x^2 y^3$

2. Using the definition of partial derivative, find

   (1) $\frac{\partial u}{\partial x}$ if $u = x^2 y^2 z$ at (1, 1, 1). (2) $\frac{\partial u}{\partial z}$ if $u = xyz^2$ at (1, 1, 1).

Note ....

It is not convenient to find partial derivatives as shown above. In general, to find the derivative of $z = f(x, y)$, we use the methods of differentiation of a function of one variable, treating x or y as a constant at a time.

3. Geometrical Interpretation of Partial Derivatives

We know that $z = f(x, y)$ represents a surface in space. If $x = k$, $z = f(k, y)$ represents a curve of intersection of $z = f(x, y)$ and the plane $x = k$.

Similarly, if $y = k$, $z = f(x, k)$ represents a curve of intersection of $z = f(x, y)$ and the plane $y = k$.

$\therefore \frac{\partial z}{\partial x}$ represents the **slope of the tangent to the curve of intersection** of the surface $z = f(x, y)$ and a plane $y = k$. (See the figure)

Similarly, $\frac{\partial z}{\partial y}$ represent **the slope of the tangent to the curve of interaction** of the surface $z = f(x, y)$ and a plane $x = k$. (Not shown in the figure)

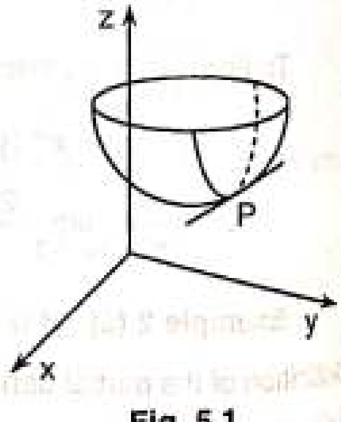


Fig. 5.1

4. Partial Derivatives of Higher Order

The partial derivatives of higher order, if they exist, can be obtained from partial derivatives of the first order by using the above definitions again. Thus, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ is the second order partial derivative of z w.r.t. x and is denoted by $\frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

Similarly, we have,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial y^2}=f_{yy}, \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial y\,\partial x}=f_{yx},$$

and $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\,\partial y}=f_{xy}.$$

It may be noted that although $\frac{\partial^2 z}{\partial x\,\partial y}$ and $\frac{\partial^2 z}{\partial y\,\partial x}$ are equal in general, they need not be equal always.

5. Partial Derivatives of Some Standard Functions

Using the above definition *i.e.* treating y constant while partially differentiating z w.r.t. x and treating x constant while partially differentiating z w.r.t. y, we can write down partial derivatives of some standard functions.

1. If $z=k$, $\frac{\partial z}{\partial x}=0$, $\frac{\partial z}{\partial y}=0$.

   If $z=f(y)$, $\frac{\partial z}{\partial x}=0$ because $f(y)$ is constant for partial differentiation w.r.t. x.

   If $z=f(x)$, $\frac{\partial z}{\partial y}=0$ because $f(x)$ is constant for partial differentiation w.r.t. y.

2. If $z=x^n y^m$, $\quad \frac{\partial z}{\partial x}=nx^{n-1}\cdot y^m$; $\frac{\partial z}{\partial y}=x^n\cdot my^{m-1}$

   For example, if $z=x^2y^3$, $\quad \frac{\partial z}{\partial x}=2xy^3$, $\frac{\partial z}{\partial y}=3x^2y^2$.

3. If $z=\sin(x+y)$, $\quad \frac{\partial z}{\partial x}=\cos(x+y)$; $\frac{\partial z}{\partial y}=\cos(x+y)$

4. If $z=e^{x+y}$, $\quad \frac{\partial z}{\partial x}=e^{x+y}$; $\frac{\partial z}{\partial y}=e^{x+y}$

5. If $z=\log(x+y)$, $\quad \frac{\partial z}{\partial x}=\frac{1}{x+y}$; $\frac{\partial z}{\partial y}=\frac{1}{x+y}$

6. If $z=\sin^{-1}\left(\frac{x}{y}\right)$, $\quad \frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-(x^2/y^2)}}\cdot\left(\frac{1}{y}\right)=\frac{1}{\sqrt{y^2-x^2}}$

   $$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-(x^2/y^2)}}\cdot\left(-\frac{x}{y^2}\right)=-\frac{x}{y\sqrt{y^2-x^2}}$$

7. If $z=\tan^{-1}\left(\frac{x}{y}\right)$, $\quad \frac{\partial z}{\partial x}=\frac{1}{1+(x^2/y^2)}\cdot\frac{1}{y}=\frac{y}{x^2+y^2}$

   $$\frac{\partial z}{\partial y}=\frac{1}{1+(x^2/y^2)}\cdot\left(-\frac{x}{y^2}\right)=-\frac{x}{y^2+x^2}$$

8. If $z=x^y$, $\quad \frac{\partial z}{\partial x}=yx^{y-1}$; $\frac{\partial z}{\partial y}=x^y\cdot\log x$. **[Note this]**

Standard Rules

If u and v are functions of x and y possessing partial derivatives of the first order, then we can use standard rules of differentiation of sum, difference, product and quotient of u and v as stated below.

1. If $z = u \pm v$, $\quad \dfrac{\partial z}{\partial x} = \dfrac{\partial u}{\partial x} \pm \dfrac{\partial v}{\partial x}, \quad \dfrac{\partial z}{\partial y} = \dfrac{\partial u}{\partial y} \pm \dfrac{\partial v}{\partial y}$

2. If $z = uv$, $\quad \dfrac{\partial z}{\partial x} = u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x}, \quad \dfrac{\partial z}{\partial y} = u\dfrac{\partial v}{\partial y} + v\dfrac{\partial u}{\partial y}$

3. If $z = \dfrac{u}{v}$, $\quad \dfrac{\partial z}{\partial x} = \dfrac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2}, \quad \dfrac{\partial z}{\partial y} = \dfrac{v\dfrac{\partial u}{\partial y} - u\dfrac{\partial v}{\partial y}}{v^2}$

Type I : Partial Differentiation using Standard Rules

Example 1 : If $z = ax^2 + by^2 + 2ab\,xy$, find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

Sol. : We have $\dfrac{\partial z}{\partial x} = 2ax + 2aby \quad ; \quad \dfrac{\partial z}{\partial y} = 2by + 2abx$.

Example 2 : If $u = e^x \sin x \sin y$, find $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$.

Sol. :

$$\frac{\partial u}{\partial x} = e^x \sin x \sin y + e^x \cos x \sin y$$

$$\frac{\partial u}{\partial y} = e^x \sin x \cos y$$

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : If $z\,(x+y) = x - y$, find $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2$. **(M.U. 2016)**

Sol. : We have $z = \dfrac{x-y}{x+y}$

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{(x+y)(1) - (x-y)(1)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)(-1) - (x-y)(1)}{(x+y)^2} = -\frac{2x}{(x+y)^2}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{2y+2x}{(x+y)^2}\right]^2 = \frac{4}{(x+y)^2}$$

Example 2 (a) : If $z\,(x+y) = (x^2 + y^2)$, prove that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$ **(M.U. 1991, 98, 2002)**

Sol. : Since $z = \dfrac{(x^2+y^2)}{x+y}$,

$$\frac{\partial z}{\partial x}=\frac{(x+y)2x-(x^2+y^2)}{(x+y)^2}=\frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y}=\frac{(x+y)2y-(x^2+y^2)}{(x+y)^2}=\frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$\therefore \quad \text{l.h.s.}=\left[\frac{x^2+2xy-y^2+x^2-2xy-y^2}{(x+y)^2}\right]^2=\left[2\cdot\frac{(x^2-y^2)}{(x+y)^2}\right]^2$$

$$=\left[2\cdot\left(\frac{x-y}{x+y}\right)\right]^2=4\frac{(x-y)^2}{(x+y)^2}$$

$$\therefore \quad \text{r.hs.}=4\left[1-\frac{x^2+2xy-y^2}{(x+y)^2}-\frac{-x^2+2xy+y^2}{(x+y)^2}\right]$$

$$=4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right]=4\frac{(x-y)^2}{(x+y)^2}$$

$\therefore$ l.h.s. = r.h.s.

Example 3 (a) : If $u=\tan^{-1}\dfrac{y}{x}$, find the value of $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}$. **(M.U. 2010, 14)**

Sol. : We have

$$\frac{\partial u}{\partial x}=\frac{1}{1+(y^2/x^2)}\cdot\left(\frac{-y}{x^2}\right)=-\frac{y}{x^2+y^2};\quad \frac{\partial u}{\partial y}=\frac{1}{1+(y^2/x^2)}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$

Now, $$\frac{\partial^2 u}{\partial x^2}=-y\cdot\frac{-1}{(x^2+y^2)^2}(2x);\qquad \frac{\partial^2 u}{\partial y^2}=x\cdot\frac{-1}{(x^2+y^2)}\cdot(2y)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

Example 4 (a) : If $x=\cos\theta-r\sin\theta$, $y=\sin\theta+r\cos\theta$, prove that $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$. **(M.U. 2014)**

Sol. : Squaring, we get

$$x^2+y^2=\cos^2\theta+r^2\sin^2\theta-2r\sin\theta\cos\theta+\sin^2\theta+r^2\cos^2\theta+2r\sin\theta\cos\theta$$

$$=(\cos^2\theta+\sin^2\theta)+r^2(\cos^2\theta+\sin^2\theta)=1+r^2.$$

$$\therefore \quad r^2=x^2+y^2-1$$

Differentiating this partially w.r.t. x,

$$2r\frac{\partial r}{\partial x}=2x \qquad \therefore \quad \frac{\partial r}{\partial x}=\frac{x}{r}.$$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. Find the partial derivatives of the following functions.

1. $x^2y^3+x^3y^2$ **2.** $2x^2+3xy+y^2$ **3.** $\log x\cdot\sin y$ **4.** $\sin x\cos y$

5. $\dfrac{\sin x}{\cos y}$ 6. $e^x \sin y$ 7. $10^x \cdot \cos y$ 8. $3^x \cdot \tan y$

9. $\dfrac{x}{x^2+y^2}$ 10. $\dfrac{y}{x^2+y^2}$ 11. $2^x \sin y \cos z$ 12. $e^x y^3 z^2$.

[Ans. : (1) $\dfrac{\partial u}{\partial x} = 2xy^3 + 3x^2y^2$, $\dfrac{\partial u}{\partial y} = 3x^2y^2 + 2x^3y$; (2) $\dfrac{\partial u}{\partial x} = 4x + 3y$, $\dfrac{\partial u}{\partial y} = 3x + 2y$;

(3) $\dfrac{\partial u}{\partial x} = \dfrac{\sin y}{x}$, $\dfrac{\partial u}{\partial y} = \log x \cos y$; (4) $\dfrac{\partial u}{\partial x} = \cos x \cos y$, $\dfrac{\partial u}{\partial y} = -\sin x \sin y$;

(5) $\dfrac{\partial u}{\partial x} = \dfrac{\cos x}{\cos y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\sin x}{\cos^2 y} \cdot \sin y$; (6) $\dfrac{\partial u}{\partial x} = e^x \sin y$, $\dfrac{\partial u}{\partial y} = e^x \cos y$;

(7) $\dfrac{\partial u}{\partial x} = 10^x \log 10 \cdot \cos y$, $\dfrac{\partial u}{\partial y} = -10^x \sin y$; (8) $\dfrac{\partial y}{\partial x} = 3^x \log 3 \tan y$, $\dfrac{\partial u}{\partial y} = 3^x \sec^2 y$;

(9) $\dfrac{\partial u}{\partial x} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$, $\dfrac{\partial u}{\partial y} = -\dfrac{2xy}{(x^2+y^2)^2}$; (10) $\dfrac{\partial u}{\partial x} = -\dfrac{-2xy}{(x^2+y^2)^2}$, $\dfrac{\partial u}{\partial y} = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$;

(11) $\dfrac{\partial u}{\partial x} = 2^x \log 2 \cdot \sin y \cos z + 2^x \cos y \cos z - 2^x \sin y \sin z$;

(12) $\dfrac{\partial u}{\partial x} = e^x y^3 z^2 + 3e^x y^2 z^2 + 2e^x y^3 z$.]

2. Find the second order partial derivatives $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial y^2}$ of the following functions.

1. $x^3 y + xy^3$ 2. $e^x \cdot y^2$
3. $x^2 - 4x^2 y + 5y^2$ 4. $e^x \log y + \sin y \log x$.

[Ans. : (1) $\dfrac{\partial^2 u}{\partial x^2} = 6xy$, $\dfrac{\partial^2 u}{\partial y^2} = 6xy$; (2) $\dfrac{\partial^2 u}{\partial x^2} = e^x \cdot y^2$, $\dfrac{\partial^2 u}{\partial y^2} = 2e^x$;

(3) $\dfrac{\partial^2 u}{\partial x^2} = 2 - 8y$, $\dfrac{\partial^2 u}{\partial y^2} = 10$;

(4) $\dfrac{\partial^2 u}{\partial x^2} = e^x \log y - \dfrac{\sin y}{x^2}$, $\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{e^x}{y^2} - \sin y \cdot \log x$.]

Class (a) : 3 Marks

1. If $u = e^{ax} \sin by$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.

2. If $u = \sin^{-1} \dfrac{x}{y}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$.

3. If $u = \dfrac{x}{y} + \dfrac{y}{x}$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0$.

4. If $u = x^2 y + y^2 z + z^2 x$, prove that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = 3u$.

(Examples 2, 3 and 4 can also be solved by using Eulers theorem. See Chapter 6.)

5. If $u = e^{xyz}$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}$. **(M.U. 2004)**

6. Differentiation of a Function of a Function

Let $z = f(u)$ and $u = \Phi(x, y)$ so that z is function of u, and u itself is a function of two independent variables x and y. The two relations define z as a function of x and y. In such cases z may be called a **function of a function** of x and y.

e.g. (i) $z = \dfrac{1}{u}$ and $u = \sqrt{x^2 + y^2}$. (ii) $z = \tan u$ and $u = x^2 + y^2$

define z as a function of a function of x and y.

Differentiation : If $z = f(u)$ is differentiable function of u and $u = \Phi(x, y)$ possesses first order partial derivatives, then

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \qquad i.e. \qquad \frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x}$$

Similarly, $$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = f'(u)\frac{\partial u}{\partial y}.$$

e.g. If $z = (ax + by)^n$, then

$$\frac{\partial z}{\partial x} = n(ax + by)^{n-1} \cdot a \quad \text{and} \quad \frac{\partial z}{\partial y} = n(ax + by)^{n-1} \cdot b$$

The rule can be easily remembered with the help of the tree diagram given on the right.

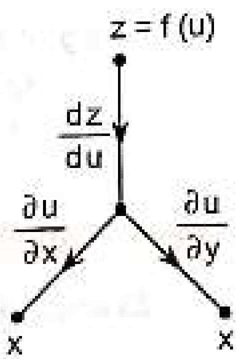


Fig. 5.2

We consider below some standard functions of the type $z = f(u)$.

1. If $z = u^n$, then $\dfrac{\partial z}{\partial x} = nu^{n-1}\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = nu^{n-1}\dfrac{\partial u}{\partial y}$.

e.g., if $z = (2x + 3y)^5$, then

$$\frac{\partial z}{\partial x} = 5(2x + 3y)^4 \cdot 2 \quad \text{and} \quad \frac{\partial z}{\partial y} = 5(2x + 3y)^4 \cdot 3$$

2. If $z = \sqrt{u}$, then $\dfrac{\partial z}{\partial x} = \dfrac{1}{2\sqrt{u}} \cdot \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \dfrac{1}{2\sqrt{u}} \cdot \dfrac{\partial u}{\partial y}$

e.g., if $z = \sqrt{(4x - 5y)}$, then

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{(4x - 5y)}} \cdot 4 \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{(4x - 5y)}} \cdot (-5)$$

3. If $z = \sin u$, then $\dfrac{\partial z}{\partial x} = \cos u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \cos u\dfrac{\partial u}{\partial y}$.

e.g., if $z = \sin(2x - y)$, then

$$\frac{\partial z}{\partial x} = \cos(2x - y) \cdot 2 \quad \text{and} \quad \frac{\partial z}{\partial y} = \cos(2x - y)(-1)$$

4. If $z = \cos u$, then $\dfrac{\partial z}{\partial x} = -\sin u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = -\sin u\dfrac{\partial u}{\partial y}$.

e.g. if $z = \cos(3x - 2y)$, then

$$\frac{\partial z}{\partial x} = -\sin(3x - 2y) \cdot (3) \quad \text{and} \quad \frac{\partial z}{\partial y} = -\sin(3x - 2y)(-2)$$

5. If $z = \tan u$, then $\dfrac{\partial z}{\partial x} = \sec^2 u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y} = \sec^2 u\dfrac{\partial u}{\partial y}$.

e.g., If $z = \tan(3x+2y)$, then

$$\frac{\partial z}{\partial x} = \sec^2(3x+2y)\cdot 3 \quad \text{and} \quad \frac{\partial z}{\partial y} = \sec^2(3x+2y)\cdot 2$$

6. If $z = e^u$, then $\frac{\partial z}{\partial x} = e^u \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = e^u \frac{\partial u}{\partial y}$.

e.g., if $z = e^{3x-4y}$, then

$$\frac{\partial z}{\partial x} = e^{3x-4y}\cdot 3 \quad \text{and} \quad \frac{\partial z}{\partial y} = e^{3x-4y}(-4)$$

7. If $z = \log u$, then $\frac{\partial z}{\partial x} = \frac{1}{u}\cdot\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \frac{1}{u}\cdot\frac{\partial u}{\partial y}$.

e.g., if $z = \log(3x+7y)$, then

$$\frac{\partial z}{\partial x} = \frac{1}{(3x+7y)}\cdot 3 \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{1}{(3x+7y)}\cdot 7$$

Type II : Partial Derivatives of First Order of a Function of a Function : Class (a) : 3 Marks

Example 1 (a) : If $u = \cos\left(\sqrt{x}+\sqrt{y}\right)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right) = 0.$$

Sol. : We have $\frac{\partial u}{\partial x} = -\sin\left(\sqrt{x}+\sqrt{y}\right)\frac{1}{2\sqrt{x}}$; $\quad \frac{\partial u}{\partial y} = -\sin\left(\sqrt{x}+\sqrt{y}\right)\frac{1}{2\sqrt{y}}$.

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x}+\sqrt{y}\right)\cdot\frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\sin\left(\sqrt{x}+\sqrt{y}\right) = 0.$$

Example 2 (a) : If $u = \sin\left(\sqrt{x}+\sqrt{y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)\cos\left(\sqrt{x}+\sqrt{y}\right)$.

Sol. : Prove it.

Example 3 (a) : If $z = e^{ax+by} f(ax-by)$, prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2ab\,z$.

Sol. : Let $ax+by = u$ and $ax-by = v$

$$\therefore \quad \frac{\partial u}{\partial x} = a, \quad \frac{\partial u}{\partial y} = b, \quad \frac{\partial v}{\partial x} = a, \quad \frac{\partial v}{\partial y} = -b$$

Hence, $z = e^u \cdot f(v)$.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[e^u\right]f(v) + e^u\cdot\frac{\partial}{\partial x}f(v) = e^u\frac{\partial u}{\partial x}\cdot f(v) + e^u\cdot f'(v)\frac{\partial v}{\partial x}$$

$$= e^u\cdot a\cdot f(v) + e^u\cdot f'(v)\cdot a$$

Also, $\frac{\partial z}{\partial y} = e^u\frac{\partial u}{\partial y}\cdot f(v) + e^u\cdot f'(v)\frac{\partial v}{\partial y} = e^u\cdot b\cdot f(v) + e^u\cdot f'(v)\cdot(-b)$

$$\therefore \quad b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abe^u f(v) = 2ab\,z.$$

Example 4 (a) : If $u = (1 - 2xy + y^2)^{-1/2}$, prove that

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2 u^3.$$ **(M.U. 1991, 99, 2004, 05, 08)**

Sol. : Since $u = (1 - 2xy + y^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2y) = yu^3 \qquad \therefore \quad x\frac{\partial u}{\partial x} = xy\, u^3$$

Also, $$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2}(-2x + 2y) = (x - y)\, u^3 \quad \therefore \quad y\frac{\partial u}{\partial y} = (xy - y^2)\, u^3$$

$$\therefore \quad x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy\, u^3 - xy\, u^3 + y^2\, u^3 = y^2 u^3.$$

Example 5 (a) : If $u = \log(\tan x + \tan y)$, prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = 2.$$ **(M.U. 1991, 2003, 05, 10, 12, 15)**

Sol. : We have $$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y}\sec^2 x$$

$$\therefore \quad \sin 2x\frac{\partial u}{\partial x} = 2\sin x\cos x\frac{1}{(\tan x + \tan y)}\cdot\sec^2 x = 2\cdot\frac{\tan x}{\tan x + \tan y}$$

Similarly, $$\sin 2y\frac{\partial u}{\partial y} = 2\cdot\frac{\tan y}{\tan x + \tan y}.$$

$$\therefore \quad \sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} = 2\cdot\frac{\tan x + \tan y}{\tan x + \tan y} = 2.$$

[Similarly, if $u = \log(\tan x + \tan y + \tan z)$, prove that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$]

Example 6 (a) : If $u = f[x^2 + y^2 + z^2]$, $x = r\cos\alpha\cos\beta$, $y = r\cos\alpha\sin\beta$, $z = r\sin\alpha$, show that

$$\frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial \beta} = 0.$$ **(M.U. 1988)**

Sol. : From data,

$$x^2 + y^2 + z^2 = r^2\cos^2\alpha\cos^2\beta + r^2\cos^2\alpha\sin^2\beta + r^2\sin^2\alpha$$

$$\therefore \quad x^2 + y^2 + z^2 = r^2\cos^2\alpha + r^2\sin^2\alpha = r^2$$

$$\therefore \quad u = f[x^2 + y^2 + z^2] = f[r^2]$$

$$\therefore \quad \frac{\partial u}{\partial \alpha} = \frac{\partial u}{\partial \beta} = 0 \qquad (\because u \text{ is independent of } \alpha \text{ and } \beta)$$

Example 7 (a) : If $u = \dfrac{e^{x+y}}{e^x + e^y}$, prove that $u_x + u_y = u$. **(M.U. 1996, 2000)**

Sol. : We have $$\frac{\partial u}{\partial x} = \frac{(e^x + e^y)\cdot e^{x+y} - e^{x+y}\cdot e^x}{(e^x + e^y)^2} = \frac{e^{x+y}(e^y)}{(e^x + e^y)^2}$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{e^{x+y}(e^x)}{(e^x + e^y)^2}$$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\frac{e^{x+y}\cdot(e^x+e^y)}{(e^x+e^y)^2}=\frac{e^{x+y}}{e^x+e^y}=u.$$

[Similarly, prove that if $u=\dfrac{e^{x+y+z}}{e^x+e^y+e^z}$, then $u_x+u_y+u_z=2u$.] **(M.U. 2002)**

Class (b) : 6 Marks

Example 1 (b) : If $\theta=t^n e^{-r^2/4t}$, find n which will make

$$\frac{\partial\theta}{\partial t}=\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right).$$ **(M.U. 1986, 93, 2000, 02, 06, 19)**

Sol. :
$$\frac{\partial\theta}{\partial t}=n t^{n-1}\cdot e^{-r^2/4t}+t^n e^{-r^2/4t}\cdot\left(-\frac{r^2}{4}\right)\left(-\frac{1}{t^2}\right)$$

$$=\frac{n}{t}\cdot t^n\cdot\frac{\theta}{t^n}+t^n\cdot\frac{\theta}{t^n}\left(\frac{r^n}{4t^2}\right)$$

$$=\frac{n}{t}\theta+\frac{r^2}{4t^2}\theta=\left(\frac{n}{t}+\frac{r^2}{4t^2}\right)\theta \qquad \left[\because\ e^{-r^2/4t}=\frac{\theta}{t^n}\right] \qquad \text{.......... (1)}$$

Also, $$\frac{\partial\theta}{\partial r}=t^n e^{-r^2/4t}\cdot\left(-\frac{2r}{4t}\right)=-\frac{r\theta}{2t} \qquad \therefore\ r^2\frac{\partial\theta}{\partial r}=-\frac{r^3\theta}{2t}$$

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right)=-\frac{1}{2t}\cdot\frac{\partial}{\partial r}(r^3\theta)=-\frac{1}{2t}\left[r^3\frac{\partial\theta}{\partial r}+3r^2\theta\right]$$

$$=-\frac{1}{2t}\left[-\frac{r^4\theta}{2t}+3r^2\theta\right]=r^2\left(\frac{r^2}{4t^2}-\frac{3}{2t}\right)\theta$$

$$\therefore \quad \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\left(\frac{r^2}{4t^2}-\frac{3}{2t}\right)\theta \qquad \text{.......... (2)}$$

$\therefore$ Equating (1) and (2), we get

$$\frac{n}{t}=-\frac{3}{2t} \qquad \therefore\ n=-\frac{3}{2}.$$

Example 2 (b) : Find the value of n so that $V=r^n(3\cos^2\theta-1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0.$$ **(M.U. 1995, 2001, 02, 04, 06)**

Sol. : We have by differentiating partially w.r.t. r,

$$\frac{\partial V}{\partial r}=n r^{n-1}(3\cos^2\theta-1) \qquad \therefore\ r^2\frac{\partial V}{\partial r}=n r^{n+1}(3\cos^2\theta-1)$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=n(n+1)r^n(3\cos^2\theta-1) \qquad \text{.......... (1)}$$

Further differentiating partially w.r.t. θ,

$$\frac{\partial V}{\partial\theta}=r^n(-6\cos\theta\sin\theta) \qquad \therefore\ \sin\theta\frac{\partial V}{\partial\theta}=-6r^n\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -6\,r^n\,[2\sin\theta\cos^2\theta - \sin^3\theta]$$

$$\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = -6\,r^n\,(2\cos^2\theta - \sin^2\theta)$$

$$= -6\,r^n\,(3\cos^2\theta - 1) \qquad \text{.....................} (2)$$

Adding (1) and (2) and equating the result to zero, (by data) we get,

$$\therefore \quad n(n+1)\,r^n\,(3\cos^2\theta - 1) - 6\,r^n\,(3\cos^2\theta - 1) = 0$$

$$\therefore \quad [\,n(n+1) - 6\,]\,r^n\,(3\cos^2\theta - 1) = 0$$

$$\therefore \quad n^2 + n - 6 = 0 \quad \therefore \quad (n+3)(n-2) = 0 \quad \therefore \quad n = 2 \text{ or } -3.$$

Type III : Partial Derivatives of Second Order of a Function of a Function

Class (a) : 3 Marks

Example 1 (a) : If $u = \log(x^2 + y^2)$, prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$. **(M.U. 2013)**

Sol. : We have $\quad \dfrac{\partial u}{\partial x} = \dfrac{1}{x^2 + y^2}\cdot 2x \quad$ and $\quad \dfrac{\partial u}{\partial y} = \dfrac{1}{x^2 + y^2}\cdot 2y$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = 2x\left[-\frac{1}{(x^2+y^2)^2}\right]\cdot 2y = -\frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = 2y\left[-\frac{1}{(x^2+y^2)^2}\right]\cdot 2x = -\frac{4xy}{(x^2+y^2)^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

Example 2 (a) : If $u = 2(ax + by)^2 - k(x^2 + y^2)$ and $a^2 + b^2 = k$, evaluate $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

Sol. : We have $\quad \dfrac{\partial u}{\partial x} = 4(ax + by)a - 2kx \quad \therefore \quad \dfrac{\partial^2 u}{\partial x^2} = 4a^2 - 2k$

and $\quad \dfrac{\partial u}{\partial y} = 4(ax + by)b - 2ky \quad \therefore \quad \dfrac{\partial^2 u}{\partial y^2} = 4b^2 - 2k$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4(a^2 + b^2) - 4k = 4k - 4k = 0 \quad [\because a^2 + b^2 = k]$$

Example 3 (a) : If $z = \tan(y + ax) + (y - ax)^{3/2}$, show that $\dfrac{\partial^2 z}{\partial x^2} = a^2\dfrac{\partial^2 z}{\partial y^2}$.

(M.U. 2002, 03, 09, 11, 17)

Sol. : We have $\quad \dfrac{\partial z}{\partial x} = a\cdot\sec^2(y + ax) - a\cdot\dfrac{3}{2}(y - ax)^{1/2}$

and $\quad \dfrac{\partial^2 z}{\partial x^2} = a^2\cdot 2\sec^2(y + ax)\cdot\tan(y + ax) + a^2\cdot\dfrac{3}{4}(y - ax)^{-1/2} \qquad \text{.....................} (1)$

Also, $\dfrac{\partial z}{\partial y} = \sec^2(y + ax) + \dfrac{3}{2}(y - ax)^{1/2}$

and $\dfrac{\partial^2 z}{\partial y^2} = 2\sec^2(y + ax) \cdot \tan(y + ax) + \dfrac{3}{4}(y - ax)^{-1/2}$ .... (2)

From (1) and (2), we see that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$.

Example 4 (a) : If $z = \log(e^x + e^y)$, show that $rt - s^2 = 0$ where $r = \dfrac{\partial^2 z}{\partial x^2}$, $t = \dfrac{\partial^2 z}{\partial y^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$.

(M.U. 2016)

Sol. : We have $\dfrac{\partial z}{\partial x} = \dfrac{e^x}{e^x + e^y}$ $\quad\therefore\quad \dfrac{\partial^2 z}{\partial x^2} = \dfrac{(e^x + e^y)e^x - e^x(e^x)}{(e^x + e^y)^2} = \dfrac{e^{x+y}}{(e^x + e^y)^2}$

$\dfrac{\partial z}{\partial y} = \dfrac{e^y}{e^x + e^y}$ $\quad\therefore\quad \dfrac{\partial^2 z}{\partial y^2} = \dfrac{(e^x + e^y)e^y - e^y(e^y)}{(e^x + e^y)^2} = \dfrac{e^{x+y}}{(e^x + e^y)^2}$

Now, $\dfrac{\partial^2 z}{\partial x \partial y} = e^x\left[-\dfrac{1}{(e^x + e^y)^2} \cdot e^y\right] = -\dfrac{e^{x+y}}{(e^x + e^y)^2}$

$$\therefore\quad rt - s^2 = \frac{e^{x+y}}{(e^x + e^y)^2} \cdot \frac{e^{x+y}}{(e^x + e^y)^2} - \left(-\frac{e^{x+y}}{(e^x + e^y)^2}\right)^2$$

$$= \left[\frac{e^{x+y}}{(e^x + e^y)^2}\right]^2 - \left[\frac{e^{x+y}}{(e^x + e^y)^2}\right]^2 = 0$$

Class (b) : 6 Marks

Example 1 (b) : If $u = e^{ax}\sin(x + bt)$ is the solution of $\dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2}$ with the condition that $u \to 0$ as $x \to \infty$, find the values of a and b.

Sol. : We have, by differentiating partially w.r.t. t,

$$\frac{\partial u}{\partial t} = be^{ax}\cos(x + bt)$$

Now, differentiating partially w.r.t. x,

$$\frac{\partial u}{\partial x} = ae^{ax}\sin(x + bt) + e^{ax}\cos(x + bt)$$

Differentiating again w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = a^2 e^{ax}\sin(x + bt) + 2ae^{ax}\cos(x + bt) - e^{ax}\sin(x + bt)$$

Putting these values in $\dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2}$,

$$be^{ax}\cos(x + bt) = \mu a^2 e^{ax}\sin(x + bt) + 2\mu a e^{ax}\cos(x + bt) - \mu e^{ax}\sin(x + bt)$$

$$\therefore\quad \mu(a^2 - 1)e^{ax}\sin(x + bt) + (2\mu a - b)e^{ax}\cos(x + bt) = 0$$

The equality will hold good only if the coefficients of $\sin(x + bt)$ and $\cos(x + bt)$ are equal to zero.

$\therefore$ Equating to zero the coefficients of sine and cosine,

$$\mu(a^2 - 1) = 0 \quad \text{and} \quad 2\mu a - b = 0$$

$$\therefore \quad a^2 = 1 \quad i.e. \quad a = \pm 1 \quad \text{and} \quad b = 2\mu a.$$

Since by data $u \to 0$ as $x \to \infty$, we get, from $u = e^{ax}\sin(x + bt)$, $a = -1$ $\therefore$ $b = -2\mu$.

[If $a = 1$, u does not tend to zero as $x \to \infty$. $\because$ $e^{-x} = \dfrac{1}{e^x} \to 0$ as $x \to \infty$ and $e^x \to \infty$ as $x \to \infty$.]

Example 2 (b) : If $u = e^{x^2+y^2+z^2}$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial z} = 8\,xyz\,u$.

Sol. : We have $\quad \dfrac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot 2z$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = 2z \cdot e^{x^2+y^2+z^2} \cdot 2y = 4yz \cdot e^{x^2+y^2+z^2}$$

$$\therefore \quad \frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = 4yz \cdot e^{x^2+y^2+z^2} \cdot 2x$$

$$= 8xyz \cdot e^{x^2+y^2+z^2} = 8xyzu.$$

Example 3 (b) : If $u = f\left(\dfrac{x^2}{y}\right)$, prove that

$$x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x \partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0.$$ **(M.U. 1994, 97, 99, 2004)**

Sol. : We have $\dfrac{\partial u}{\partial x} = f'\left(\dfrac{x^2}{y}\right) \cdot \dfrac{2x}{y}, \quad \dfrac{\partial u}{\partial y} = f'\left(\dfrac{x^2}{y}\right) \cdot \left(-\dfrac{x^2}{y^2}\right)$

$$\therefore \quad x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = f'\left(\frac{x^2}{y}\right)\left[\frac{2x^2}{y} - \frac{2x^2}{y}\right] = 0 \quad \text{..................... (1)}$$

Differentiating (1) partially w.r.t. x,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + 2y\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{..................... (2)}$$

Differentiating (1) partially w.r.t. y, now

$$x\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial u}{\partial y} + 2y\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{..................... (3)}$$

Multiply (2) by x, (3) by y and add,

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + 2xy\frac{\partial^2 u}{\partial x \partial y} + xy\frac{\partial^2 u}{\partial x \partial y} + 2y\frac{\partial u}{\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0$$

But $\quad x\dfrac{\partial u}{\partial x} + 2y\dfrac{\partial u}{\partial y} = 0$. $\quad$ Hence, $\quad x^2\dfrac{\partial^2 u}{\partial x^2} + 3xy\dfrac{\partial^2 u}{\partial x \partial y} + 2y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

Example 4 (b) : If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, [OR if $u = \log r$ and $r = x^3 + y^3 + z^3 - 3xyz$], prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$. **(M.U. 1999, 2002, 09)**

Sol. : We have $\quad$ l.h.s. $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$ $\quad$ **[Note this]**

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \qquad \text{.................... (1)}$$

Now, $$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz)$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}, \quad \frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\,\frac{x^2 + y^2 + z^2 - xy - yz - zx}{x^3 + y^3 + z^3 - 3xyz} = \frac{3}{(x+y+z)}$$

[$\because$ $(x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z) = x^3 + y^3 + z^3 - 3xyz$. **(Note this)**]

Hence, from (1),

$$\text{l.h.s.} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \cdot \frac{3}{(x+y+z)}$$

$$= 3\left[\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2}\right]$$

$$= -\frac{9}{(x+y+z)} = \text{r.h.s.}$$

Example 5 (b) : If $u = (1 - 2xy + y^2)^{-1/2}$, prove that

$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[y^2 \frac{\partial u}{\partial y}\right] = 0. \qquad \textbf{(M.U. 1986, 88, 99, 2004, 05)}$$

Sol. : We have, $\quad$ l.h.s. $= -2x\frac{\partial u}{\partial x} + (1 - x^2)\frac{\partial^2 u}{\partial x^2} + 2y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ .... (1)

But as in the Ex. 4, page 5-10,

$$\frac{\partial u}{\partial x} = u^3 y \qquad \therefore \quad \frac{\partial^2 u}{\partial x^2} = 3u^2\frac{\partial u}{\partial x} \cdot y = 3u^5 y^2$$

Also, $$\frac{\partial u}{\partial y} = (x - y)u^3 \quad \therefore \quad \frac{\partial^2 u}{\partial y^2} = (x - y) \cdot 3u^2\frac{\partial u}{\partial y} - u^3 = (x - y)^2 3u^5 - u^3$$

Putting these values in (1),

$$\text{l.h.s.} = -2xy\,u^3 + (1 - x^2) \cdot 3u^5 y^2 + 2y(x - y)\,u^3 + y^2 (x - y)^2\, 3u^5 - u^3 y^2$$
$$= 3u^5 y^2 [1 - x^2 + x^2 - 2xy + y^2] - 3u^3 y^2$$
$$= 3u^5 y^2 (1 - 2xy + y^2) - 3u^3 y^2.$$

But by data $1 - 2xy + y^2 = u^{-2}$

$$\therefore \quad \text{l.h.s.} = 3u^5 y^2 u^{-2} - 3u^3 y^2$$
$$= 3u^3 y^2 - 3y^3 y^2 = 0.$$

Example 6 (b) : If $u = x^y$, show that $\dfrac{\partial^3 u}{\partial x^2 \partial y} = \dfrac{\partial^3 u}{\partial x \partial y \partial x}$. **(M.U. 2010, 15)**

Sol. : Since $u = x^y$, treating y constant $\dfrac{\partial u}{\partial x} = y\,x^{y-1}$ .... (1)

Treating x constant, $\dfrac{\partial u}{\partial y} = x^y \log x$ .... (2)

Differentiating (2) partially w.r.t. x, we get,

$$\frac{\partial^2 u}{\partial x \partial y} = x^y \cdot \frac{1}{x} + yx^{y-1} \log x = x^{y-1} + yx^{y-1} \log x$$

$$= x^{y-1}(1 + y \log x)$$

Differentiating again partially w.r.t. x, we get,

$$\frac{\partial^3 u}{\partial x^2 \partial y} = (y-1)\,x^{y-2} \cdot (1 + y \log x) + x^{y-1} \cdot \frac{y}{x}$$

$$\therefore \quad \frac{\partial^3 u}{\partial x^2 \partial y} = x^{y-2}[\,y - 1 + y(y-1)\log x + y\,]$$

$$= x^{y-2}[2y - 1 + y(y-1)\log x]$$

Now, differentiating (1) partially w.r.t. y, we get

$$\frac{\partial^2 u}{\partial y \partial x} = x^{y-1} + yx^{y-1} \log x = x^{y-1}(1 + y \log x)$$

Differentiating again w.r.t. x, we get

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = (y-1)\,x^{y-2}(1 + y \log x) + x^{y-1} \cdot \frac{y}{x}$$

$$= x^{y-2}[\,y - 1 + y(y-1)\log x + y\,]$$

$$= x^{y-2}[2y - 1 + y(y-1)\log x] \quad \text{..................... (3)}$$

Hence, from (2) and (3) the result follows.

Example 7 (b) : If $z = x^y + y^x$, verify that $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$. **(M.U. 1996, 2003, 04, 05)**

Sol. : Differentiating z partially w.r.t. y, we get,

$$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$

Differentiating this partially w.r.t. x, we get,

$$\frac{\partial^2 z}{\partial x \partial y} = yx^{y-1} \cdot \log x + x^y \cdot \frac{1}{x} + 1 \cdot y^{x-1} + xy^{x-1} \log y$$

$$= yx^{y-1} \cdot \log x + x^{y-1} + y^{x-1} + xy^{x-1} \log y \quad \text{..................... (1)}$$

Now, differentiating z partially w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiating this again partially w.r.t. y, we get,

$$\frac{\partial^2 z}{\partial y\,\partial x} = x^{y-1} + y \cdot x^{y-1}\log x + \frac{y^x}{y} + xy^{x-1}\log y$$

$$= yx^{y-1}\log x + x^{y-1} + y^{x-1} + xy^{x-1}\log y \qquad \text{.................... (2)}$$

From (1) and (2), the result follows.

Example 8 (b) : If $u = f(r)$ and $r = \sqrt{x^2 + y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r). \qquad \textbf{(M.U. 1993, 97)}$$

Sol. : Since $r^2 = x^2 + y^2 \quad \therefore \quad 2r\frac{\partial r}{\partial x} = 2x$

$\therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now, $\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = f'(r) \cdot \frac{x}{r}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{x}{r} \cdot \frac{\partial r}{\partial x} + f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{x}{r^2}\frac{\partial r}{\partial x}$$

Putting the value of $\frac{\partial r}{\partial x}$,

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = f''(r) \cdot \frac{x^2}{r^2} + f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{x^2}{r^3}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \frac{y^2}{r^2} + f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{y^2}{r^3}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \cdot \frac{(x^2 + y^2)}{r^2} + 2f'(r) \cdot \frac{1}{r} - f'(r) \cdot \frac{(x^2 + y^2)}{r^3}$$

$$= f''(r) + \frac{f'(r)}{r} \qquad [\because \ x^2 + y^2 = r^2]$$

Example 9 (b) : If $u = f(r)$ and $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r). \qquad \textbf{(M.U. 1991, 93, 97, 2002)}$$

Sol. : Left to you.

Example 10 (b) : If $u = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2). \qquad \textbf{(M.U. 1992, 97)}$$

Sol. : We have $2r\frac{\partial r}{\partial x} = 2x \quad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \ \frac{\partial r}{\partial z} = \frac{z}{r}$.

Now, since $u = f(r^2)$,

$$\frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = f'(r^2) \cdot 2r \cdot \frac{x}{r} = 2 \cdot f'(r^2) \cdot x$$

Similarly, $\frac{\partial u}{\partial y} = 2f'(r^2) \cdot y, \quad \frac{\partial u}{\partial z} = 2f'(r^2) \cdot z$

Now, $\frac{\partial^2 u}{\partial x^2} = 2\cdot\left[f'(r^2) + x\cdot f''(r^2)\cdot 2r\cdot\frac{\partial r}{\partial x}\right] = 2\left[f'(r^2) + x\cdot f''(r^2)\cdot 2\cdot r\cdot\frac{x}{r}\right]$

$\therefore\quad \frac{\partial^2 u}{\partial x^2} = 2f'(r^2) + 4f''(r^2)\cdot x^2$

Similarly, $\frac{\partial^2 u}{\partial y^2} = 2f'(r^2) + 4f''(r^2)\cdot y^2$ and $\frac{\partial^2 u}{\partial z^2} = 2f'(r^2) + 4f''(r^2)\cdot z^2$

$$\therefore\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6f'(r^2) + 4f''(r^2)[x^2 + y^2 + z^2]$$

$$= 6f'(r^2) + 4r^2\cdot f''(r^2)$$

Example 11 (b) : If $u = r^m$, $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)\,r^{m-2}.$$ **(M.U. 1988, 95, 2001)**

Sol. : Since, $r^2 = x^2 + y^2 + z^2$, $\quad 2r\frac{\partial r}{\partial x} = 2x$

$\therefore\quad \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\because\quad u = r^m$, $\quad \frac{du}{dr} = m\,r^{m-1}$

Now, $\frac{\partial u}{\partial x} = \frac{du}{dr}\cdot\frac{\partial r}{\partial x} = m\,r^{m-1}\cdot\frac{x}{r} = m\,x\,r^{m-2}$

$\therefore\quad \frac{\partial^2 u}{\partial x^2} = m\,r^{m-2} + mx(m-2)\cdot r^{m-3}\,\frac{\partial r}{\partial x}$

$\therefore\quad \frac{\partial^2 u}{\partial x^2} = m\,r^{m-2} + m(m-2)\,r^{m-3}\cdot x\cdot\frac{x}{r}$

$$= m\,r^{m-2} + m(m-2)\,r^{m-4}\cdot x^2$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = m\,r^{m-2} + m(m-2)\,r^{m-4}\cdot y^2$

and $\quad \frac{\partial^2 u}{\partial z^2} = m\,r^{m-2} + m(m-2)\,r^{m-4}\cdot z^2$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 3\,m\,r^{m-2} + m(m-2)\,r^{m-4}(x^2 + y^2 + z^2)$

$$= 3\,m\,r^{m-2} + m(m-2)\,r^{m-2}$$

$$= m(m+1)\,r^{m-2}.$$

Example 12 (b) : Show that $z = f(x + at) + \Phi(x - at)$ is a solution of

$a^2\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$ for all f and Φ (a, being constant). **(M.U. 1982, 91)**

Sol. : We have $z = f(x+at) + \Phi(x-at)$

$$\therefore \quad \frac{\partial z}{\partial x} = f'(x+at) + \Phi'(x-at)$$

and $$\frac{\partial^2 z}{\partial x^2} = f''(x+at) + \Phi''(x-at) \qquad \text{.................... (1)}$$

Further, $$\frac{\partial z}{\partial t} = af'(x+at) - a\Phi'(x-at)$$

and $$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x+at) + a^2 \Phi''(x-at) \qquad \text{.................... (2)}$$

From (1) and (2), we get $a^2 \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial t^2}$ for all f and Φ. Hence, the required result.

Example 13 (b) : If $u = Ae^{-gx} \sin(nt - gx)$ satisfies the equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}, \text{ prove that } g = \sqrt{\frac{n}{2\mu}}.$$ **(M.U. 1998, 2004, 07)**

[OR If $u = Ae^{-gx}\sin(nt - gx)$ satisfies the equation $\dfrac{\partial u}{\partial t} = \mu^2 \dfrac{\partial^2 u}{\partial x^2}$, prove that $g = \dfrac{1}{\mu}\sqrt{\dfrac{n}{2}}$.]

Sol. : We have

$$\frac{\partial u}{\partial x} = A[-ge^{-gx}\sin(nt - gx) - ge^{-gx}\cos(nt - gx)]$$

$$= -Age^{-gx}[\sin(nt - gx) + \cos(nt - gx)]$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = -Ag[-g \cdot e^{-gx}\{\sin(nt - gx) + \cos(nt - gx)\}$$
$$+ e^{-gx}\{-g\cos(nt - gx) + g\sin(nt - gx)\}]$$
$$= 2Ag^2 e^{-gx}\cos(nt - gx)$$

Further, $\dfrac{\partial u}{\partial t} = Ane^{-gx}\cos(nt - gx)$. But $\dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2}$ [By data]

$$\therefore \quad Ane^{-gx}\cos(nt - gx) = \mu \cdot 2A \cdot g^2 e^{-gx}\cos(nt - gx)$$

$$\therefore \quad n = 2\mu g^2 \qquad \therefore \quad g = \sqrt{\frac{n}{2\mu}}.$$

Example 14 (b) : If $u = (ar^n + br^{-n})\cos(n\theta - \alpha)$ or

[$u = (ar^n + br^{-n})(\cos n\theta + \sin n\theta)$], prove that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = 0.$$ **(M.U. 1994, 96)**

Sol. : We have $\dfrac{\partial u}{\partial r} = (nar^{n-1} - nbr^{-n-1})\cos(n\theta - \alpha)$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} = [n(n-1)ar^{n-2} + n(n+1)br^{-n-2}]\cos(n\theta - \alpha)$$

Further $\dfrac{\partial u}{\partial \theta} = (a r^n + b r^{-n})[-n \sin(n\theta - \alpha)]$

$\therefore \quad \dfrac{\partial^2 u}{\partial \theta^2} = (a r^n + b r^{-n})[-n^2 \cos(n\theta - \alpha)]$

Putting these values in the l.h.s.

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2}$$

$$= n(n-1) a r^{n-2} \cos(n\theta - \alpha) + n(n+1) b r^{-n-2} \cos(n\theta - \alpha) + n a r^{n-2} \cos(n\theta - \alpha) - n b r^{-n-2} \cos(n\theta - \alpha) - n^2 a r^{n-2} \cos(n\theta - \alpha) - n^2 b r^{-n-2} \cos(n\theta - \alpha)$$

$$= 0$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : If $z = u(x, y)\, e^{ax+by}$ where $u(x, y)$ is such that $\dfrac{\partial^2 u}{\partial x \partial y} = 0$, find the constants a, b such that $\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} + z = 0$.

Sol. : We have, from $z = u(x, y)\, e^{ax+by}$ .... (1)

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \cdot e^{ax+by} + u \cdot e^{ax+by} \cdot a = e^{ax+by}\left(\frac{\partial u}{\partial x} + au\right) \qquad \text{..................... (2)}$$

And $$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \cdot e^{ax+by} + u \cdot e^{ax+by} \cdot b = e^{ax+by}\left(\frac{\partial u}{\partial y} + bu\right) \qquad \text{..................... (3)}$$

Differentiating (3) partially w.r.t. x

$$\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by} \cdot a \cdot \left(\frac{\partial u}{\partial y} + bu\right) + e^{ax+by}\left(\frac{\partial^2 u}{\partial x \partial y} + b \cdot \frac{\partial u}{\partial x}\right) \qquad \text{..................... (4)}$$

But since by data $\dfrac{\partial^2 u}{\partial x \partial y} = 0$, we get

$$\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by}\left(a \cdot \frac{\partial u}{\partial y} + b \cdot \frac{\partial u}{\partial x} + abu\right) \qquad \text{..................... (5)}$$

Further by data $$\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0 \qquad \text{..................... (6)}$$

Putting the values from (1), (2), (3) and (5) in (6), we get,

$$e^{ax+by}\left[a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} + abu - \frac{\partial u}{\partial x} - au - \frac{\partial u}{\partial y} - bu + u\right] = 0$$

$$\therefore \quad e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + au(b-1) - u(b-1)\right] = 0$$

$$\therefore \quad e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (b-1) \cdot u(a-1)\right] = 0$$

Since $u \neq 0$, $\dfrac{\partial u}{\partial x} \neq 0$ and $\dfrac{\partial u}{\partial y} \neq 0$, we should have

$$a - 1 = 0, \quad b - 1 = 0 \quad i.e. \quad a = 1, \quad b = 1.$$

Example 2 (c) : If $u = e^{xyz} f\left(\dfrac{xy}{z}\right)$, prove that

$$x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = 2\,xy\,zu; \quad y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\,xy\,zu.$$

Hence, show that $x\dfrac{\partial^2 u}{\partial z \partial x} = y\dfrac{\partial^2 u}{\partial z \partial y}$. **(M.U. 1992, 99, 2018)**

Sol. : We have $\dfrac{\partial u}{\partial x} = e^{xyz} \cdot yz \cdot f\left(\dfrac{xy}{z}\right) + e^{xyz} \cdot f'\left(\dfrac{xy}{z}\right) \cdot \dfrac{y}{z}$

Similarly, $\dfrac{\partial u}{\partial y} = e^{xyz} \cdot xz \cdot f\left(\dfrac{xy}{z}\right) + e^{xyz} \cdot f'\left(\dfrac{xy}{z}\right) \cdot \dfrac{x}{z}$

and $\dfrac{\partial u}{\partial z} = e^{xyz} \cdot xy \cdot f\left(\dfrac{xy}{z}\right) + e^{xyz} \cdot f'\left(\dfrac{xy}{z}\right) \cdot \left(-\dfrac{xy}{z^2}\right)$

$$\therefore \quad x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = e^{xyz} \cdot xyz \cdot f\left(\frac{xy}{z}\right) + e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \cdot \left(\frac{xy}{z}\right)$$
$$+ e^{xyz} \cdot xyz \cdot f\left(\frac{xy}{z}\right) + e^{xyz} \cdot f'\left(\frac{xy}{z}\right) \cdot \left(-\frac{xy}{z}\right)$$
$$= 2e^{xyz} \cdot xyz \cdot f\left(\frac{xy}{z}\right) = 2\,xy\,zu.$$

Similarly, it can be easily proved that $y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2\,xy\,zu$

Now, differentiating both sides of $x\dfrac{\partial u}{\partial x} + z\dfrac{\partial u}{\partial z} = 2\,xy\,zu$ partially w.r.t. z,

$$x\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial u}{\partial z} + z\frac{\partial^2 u}{\partial z^2} = 2\,xyu + 2\,xyz\frac{\partial u}{\partial z} \quad \text{.................... (1)}$$

Further differentiating both sides of $y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 2\,xy\,zu$ partially w.r.t. z

$$y\frac{\partial^2 u}{\partial z \partial y} + \frac{\partial u}{\partial z} + z\frac{\partial^2 u}{\partial z^2} = 2\,xy\,u + 2\,xyz\frac{\partial u}{\partial z} \quad \text{.................... (2)}$$

From (1) and (2) it is clear that $x\dfrac{\partial^2 u}{\partial z \partial x} = y\dfrac{\partial^2 u}{\partial z \partial y}$.

Example 3 (c) : If $z = x \log(x + r) - r$ where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{x + r}, \quad \frac{\partial^3 z}{\partial x^3} = -\frac{x}{r^3}.$$ **(M.U. 1983, 91, 2002, 04, 08, 09, 11)**

Sol. : Since $r^2 = x^2 + y^2$ as seen before $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ and $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Differentiating $z = x \log(x + r) - r$ partially w.r.t. x,

$$\frac{\partial z}{\partial x}=\left[\frac{x}{x+r}\left(1+\frac{\partial r}{\partial x}\right)+\log(x+r)\cdot 1\right]-\frac{\partial r}{\partial x}$$

$$=\left[\frac{x}{x+r}\left(1+\frac{x}{r}\right)+\log(x+r)\right]-\frac{x}{r}$$

$$=\frac{x}{r}+\log(x+r)-\frac{x}{r}=\log(x+r)$$

$$\therefore\quad \frac{\partial^2 z}{\partial x^2}=\frac{1}{x+r}\left(1+\frac{\partial r}{\partial x}\right)=\frac{1}{x+r}\left(1+\frac{x}{r}\right)=\frac{1}{r} \qquad \text{.................... (1)}$$

Now, differentiating $z = x\log(x+r)-r$ partially w.r.t. y,

$$\frac{\partial z}{\partial y}=x\cdot\frac{1}{x+r}\left(\frac{\partial r}{\partial y}\right)-\frac{\partial r}{\partial y}=\frac{x}{x+r}\cdot\frac{y}{r}-\frac{y}{r}$$

$$=\frac{y}{r}\left(\frac{x}{x+r}-1\right)=-\frac{y}{x+r}$$

$$\frac{\partial^2 z}{\partial y^2}=-\frac{(x+r)(1)-y(\partial r/\partial y)}{(x+r)^2}=-\frac{(x+r)-y\cdot(y/r)}{(x+r)^2}$$

$$=-\frac{rx+r^2-y^2}{r(x+r)^2}=-\frac{rx+x^2}{r(x+r)^2}\qquad [\because r^2-y^2=x^2]$$

$$\therefore\quad \frac{\partial^2 z}{\partial y^2}=-\frac{x(r+x)}{r(x+r)^2}=-\frac{x}{r(x+r)} \qquad \text{.................... (2)}$$

From (1) and (2),

$$\frac{\partial^2 z}{\partial z^2}+\frac{\partial^2 z}{\partial y^2}=\frac{1}{r}-\frac{x}{r(x+r)}=\frac{x+r-x}{r(x+r)}=\frac{1}{x+r}$$

Now from (1), $\dfrac{\partial^3 z}{\partial x^3}=-\dfrac{1}{r^2}\cdot\dfrac{\partial r}{\partial x}=-\dfrac{1}{r^2}\cdot\dfrac{x}{r}=-\dfrac{x}{r^3}$.

Example 4 (c) : If $x=e^{r\cos\theta}\cos(r\sin\theta)$, $y=e^{r\cos\theta}\sin(r\sin\theta)$,

prove that $\dfrac{\partial x}{\partial r}=\dfrac{1}{r}\cdot\dfrac{\partial y}{\partial\theta}$, $\dfrac{\partial y}{\partial r}=-\dfrac{1}{r}\cdot\dfrac{\partial x}{\partial\theta}$. **(M.U. 2004, 06)**

Hence, deduce that $\dfrac{\partial^2 x}{\partial r^2}+\dfrac{1}{r}\cdot\dfrac{\partial x}{\partial r}+\dfrac{1}{r^2}\cdot\dfrac{\partial^2 x}{\partial\theta^2}=0.$ **(M.U. 1999)**

Sol. : **Since** $x=e^{r\cos\theta}\cos(r\sin\theta)$,

$$\frac{\partial x}{\partial r}=e^{r\cos\theta}\cdot\cos\theta\cos(r\sin\theta)-e^{r\cos\theta}\cdot\sin(r\sin\theta)\sin\theta$$

$$=e^{r\cos\theta}[\cos\theta\cos(r\sin\theta)-\sin\theta\sin(r\sin\theta)]$$

$$=e^{r\cos\theta}\cos(r\sin\theta+\theta) \qquad \text{.................... (i)}$$

And $$\frac{\partial x}{\partial\theta}=e^{r\cos\theta}(-r\sin\theta)\cos(r\sin\theta)+e^{r\cos\theta}[-\sin(r\sin\theta)][r\cos\theta]$$

$$=-re^{r\cos\theta}[\sin\theta\cos(r\sin\theta)+\cos\theta\sin(r\sin\theta)]$$

$$=-re^{r\cos\theta}\sin(r\sin\theta+\theta) \qquad \text{.................... (ii)}$$

Similarly, $\dfrac{\partial y}{\partial r} = e^{r\cos\theta} \sin(r\sin\theta + \theta)$ .... (iii)

and $\dfrac{\partial y}{\partial \theta} = r\, e^{r\cos\theta} \cos(r\sin\theta + \theta)$ .... (iv)

From (i) and (iv), we get $\dfrac{\partial x}{\partial r} = \dfrac{1}{r} \cdot \dfrac{\partial y}{\partial \theta}$ .... (v)

From (ii) and (iii), we get $\dfrac{\partial y}{\partial r} = -\dfrac{1}{r} \cdot \dfrac{\partial x}{\partial \theta}$ .... (vi)

Now, differentiating (v) w.r.t. r, we get

$$\frac{\partial^2 x}{\partial r^2} = -\frac{1}{r^2} \cdot \frac{\partial y}{\partial \theta} + \frac{1}{r} \cdot \frac{\partial^2 y}{\partial r \partial \theta} \quad \text{..................... (vii)}$$

From (vi), we get $\dfrac{\partial x}{\partial \theta} = -r \dfrac{\partial y}{\partial r}$

Differentiating this w.r.t. θ, we get

$$\frac{\partial^2 x}{\partial \theta^2} = -r \frac{\partial^2 y}{\partial r \partial \theta} \qquad \therefore \frac{1}{r^2} \cdot \frac{\partial^2 x}{\partial \theta^2} = -\frac{1}{r} \cdot \frac{\partial^2 y}{\partial r \partial \theta} \quad \text{..................... (viii)}$$

Adding (vii) and (viii), $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r^2} \cdot \dfrac{\partial^2 x}{\partial \theta^2} = -\dfrac{1}{r^2} \cdot \dfrac{\partial y}{\partial \theta}$ .... (ix)

But from (v), $\dfrac{1}{r^2} \cdot \dfrac{\partial y}{\partial \theta} = \dfrac{1}{r} \cdot \dfrac{\partial x}{\partial r}$

Hence, (ix) becomes $\dfrac{\partial^2 x}{\partial r^2} + \dfrac{1}{r} \cdot \dfrac{\partial x}{\partial r} + \dfrac{1}{r^2} \cdot \dfrac{\partial^2 x}{\partial \theta^2} = 0.$

Type III : Examples Satisfying Laplace Equation : Class (b) : 6 marks

Example 1 (b) : If $u = \cos 4x \cos 3y \sin h\, 5z$, prove that u satisfies Laplace equation *i.e.* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0.$

Sol. : We have

$$\frac{\partial u}{\partial x} = -4 \cdot \sin 4x \cos 3y \sin h\, 5z \qquad \therefore \frac{\partial^2 u}{\partial x^2} = -16 \cdot \cos 4x \cos 3y \sin h 5z = -16u$$

Similarly, $\dfrac{\partial u}{\partial y} = -3 \cdot \cos 4x \sin 3y \sin h\, 5z \qquad \therefore \dfrac{\partial^2 u}{\partial y^2} = -9 \cdot \cos 4x \cos 3y \sin h\, 5z = -9u$

And $\dfrac{\partial u}{\partial z} = 5 \cdot \cos 4x \cos 3y \cos h\, 5z \qquad \therefore \dfrac{\partial^2 u}{\partial z^2} = 25 \cdot \cos 4x \cos 3y \sin h\, 5z = 25u$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -16u - 9u + 25u = 0.$$

Example 2 (b) : If $u = \frac{1}{r}$, $r = \sqrt{x^2 + y^2 + z^2}$, [Or if $u = (x^2 + y^2 + z^2)^{-1/2}$] prove that u statisfies Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. **(M.U. 1996)**

Sol. : We have $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = -\frac{1}{r^2} \cdot \frac{x}{r} = -\frac{x}{r^3}$$

and $$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{r^3} + \frac{3x}{r^4} \cdot \frac{\partial r}{\partial x} = -\frac{1}{r^3} + \frac{3x^2}{r^5}$$

Similarly, $\frac{\partial^2 u}{\partial y^2} = -\frac{1}{r^3} + \frac{3y^2}{r^5}$ and $\frac{\partial^2 u}{\partial z^2} = -\frac{1}{r^3} + \frac{3z^2}{r^5}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{r^3} + \frac{3(x^2 + y^2 + z^2)}{r^5} = -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

(A) Class (b) : 6 Marks

1. If $u = e^{ax} \sin by$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

2. If $u = e^{xyz}$, prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}$. **(M.U. 1993, 2002)**

3. If $u = x^3 y + e^{xy^2}$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$. **(M.U. 2013)**

4. If $u = e^{ax} \tan by \log_e z$, prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial^3 u}{\partial z \partial y \partial x}$.

5. If $u = \log(x^2 + y^2 + z^2)$, prove that $x \frac{\partial^2 u}{\partial y \partial z} = z \frac{\partial^2 u}{\partial x \partial y} = y \frac{\partial^2 u}{\partial z \partial x}$. **(M.U. 1987, 94)**

6. If $z = x^3 + y^3 - 3a\,xy$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

7. If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$. **(M.U. 2001, 11, 17)**

8. If $z = \tan^{-1}\left[xy / \sqrt{(1 + x^2 + y^2)}\right]$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(1 + x^2 + y^2)^{3/2}}$.

9. If $z = \frac{1}{\sqrt{y}} e^{-(x-a)^2/4y}$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

10. If $z = (3xy - y^3) - (y^2 - 2x)^{3/2}$, prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

11. If $u = \dfrac{xy}{2x+z}$, prove that $\dfrac{\partial^3 u}{\partial y\,\partial z^2} = \dfrac{\partial^3 u}{\partial z^2\,\partial y}$.

12. If $u = x^m y^n$, prove that $\dfrac{\partial^3 u}{\partial x \partial y \partial x} = \dfrac{\partial^3 u}{\partial y \partial x^2}$.

13. If $z = c\,t^{-1/2}\,e^{-x^2/4a^2 t}$, prove that $\dfrac{\partial z}{\partial t} = a^2 \dfrac{\partial^2 z}{\partial x^2}$. **(M.U. 1994)**

14. If $u = x^3 - 3xy^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(B) Class (b) : 6 Marks

1. If $z = \tan^{-1}\left(\dfrac{y}{x}\right)$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

2. If $z = (x^2 + y^2)^{1/2}$, prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

3. If $u = x^2 y + y^2 z + z^2 x$, prove that

(i) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x+y+z)^2$ (ii) $\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)^2 u = 6(x+y+z)$

(iii) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 2(x+y+z)$

(iv) $\left((x+y)\dfrac{\partial^2 u}{\partial x^2} + (y+z)\dfrac{\partial^2 u}{\partial y^2} + (z+x)\dfrac{\partial^2 u}{\partial z^2}\right) - \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}\right) = x^2 + y^2 + z^2$

(v) $\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}\right) - \left(x\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial y^2} + z\dfrac{\partial^2 u}{\partial z^2}\right) = x^2 + y^2 + z^2$.

(C) Class (b) : 6 Marks

1. If $u = x^3 - 3xy^2$, $v = 3x^2 y - y^3$, prove that

(i) $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0$ (ii) $\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0$

(iv) $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ (iv) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

2. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\left[x\left(\dfrac{\partial x}{\partial r}\right) + y\left(\dfrac{\partial y}{\partial r}\right)\right]^2 = x^2 + y^2$. **(M.U. 1982)**

3. $u = f(ax^2 + 2h\,xy + by^2)$, $v = \Phi\,(ax^2 + 2h\,xy + by^2)$, prove that $\dfrac{\partial}{\partial x}\left(u\dfrac{\partial v}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(u\dfrac{\partial v}{\partial x}\right)$.

(D) Class (b) : 6 Marks

1. If $u = xf\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

2. If $z = f(x+y) + \Phi\,(x-y)$ then show that $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial x^2}$.

3. If $u = xy\, f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$.

Further if u is constant, prove that $\dfrac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)} = \dfrac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}$.

4. If $u = \log r$, $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

5. If $\log \theta = r - x$, $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 \theta}{\partial y^2} = \frac{\theta}{r^3}(x^2 + r y^2)$.

6. If $u = \log r$, $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{r^2}$.

7. If $u = a \cos(bx + c) \cos h\,(by + d)$ where a, b, c, d are constants,

prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

8. If $u^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{u}$.

9. If $u = (ax + by)^2 - (x^2 + y^2)$ where $a^2 + b^2 = 2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

10. If $u = r$, $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{r}$.

11. If $u = \frac{1}{r}$, $r^2 = x^2 + y^2 + z^2$, prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + u = 0$, (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

12. If $z = x f(x + y) + y g(x + y)$, [OR if $u = \log r$ and $r = x^3 + y^3 - x^2 y - y^2 x$], prove that

$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$. **(M.U. 1997)**

13. If $u = \log(x^3 + y^3 - x^2 y - xy^2)$, prove that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$ (ii) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x + y)^2}$. **(M.U. 1993, 97, 2002, 07)**

14. If $u = \log r$, $r = x^3 + y^3 - x^2 y - xy^2$, prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$ and $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x + y)^2}$. **(M.U. 1994)**

15. If $u = (x^2 - y^2) f(x, y)$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (x^4 - y^4) f''(x, y)$.

16. If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ and $a^2 + b^2 + c^2 = 1$,

prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. **(M.U. 2004)**

7. Variable to be Treated as Constant

We have seen that $\frac{\partial u}{\partial x}$ means partial derivative of u w.r.t. x treating y constant and $\frac{\partial u}{\partial y}$ means partial derivative of u w.r.t. y treating x constant. In some problems there is possibility of confusion as to which variable is to be treated as constant. Consider $x = r\cos\theta$, $y = r\sin\theta$.

Since $x = r\cos\theta$, we can obtain the partial derivative of x w.r.t. r *i.e.* $\frac{\partial x}{\partial r} = \cos\theta$. But we may also consider r as a function of x, y and we may want find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$. To obtain these partial derivatives, we eliminate θ from the relation $x = r\cos\theta$, $y = r\sin\theta$ and get $r = x^2 + y^2$. From this relation, we get $2r\frac{\partial r}{\partial x} = 2x$ *i.e.* $\frac{\partial r}{\partial x} = \frac{x}{r}$.

This is called the partial derivative of r w.r.t. x treating y constant. It is denoted by $\left(\frac{\partial r}{\partial x}\right)_y$.

In such cases we write the variable to be treated as constant at the bottom of the bracket. Thus, $\left(\frac{\partial r}{\partial x}\right)_\theta$ means partial derivative of r w.r.t. x treating θ constant in a function expressing r as a function of x and θ. Similarly, $\left(\frac{\partial x}{\partial \theta}\right)_y$ will mean partial derivative of x w.r.t. θ treating y constant in a function expressing x as a function of y and θ.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : If $x = \frac{r}{2}(e^\theta + e^{-\theta})$, $y = \frac{r}{2}(e^\theta - e^{-\theta})$, prove that $\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$ where suffixes denote the variables to be treated constants. **(M.U. 2006)**

Sol. : We are given $x = r\cos h\,\theta$, $y = r\sin h\,\theta$.

$$\therefore \quad \left(\frac{\partial x}{\partial r}\right)_\theta = \cos h\,\theta \qquad \text{.......................... (1)}$$

The find $\left(\frac{\partial r}{\partial x}\right)_y$ we must express r as a function of x and y by eliminating θ between $x = r\cos h\,\theta$, $y = r\sin h\,\theta$.

Since $\cos h^2\,\theta - \sin h^2\,\theta = 1$, we get $x^2 - y^2 = r^2$.

$$\therefore \quad 2r\left(\frac{\partial r}{\partial x}\right) = 2x \qquad \therefore \quad \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \cos h\,\theta \qquad \text{.......................... (2)}$$

From (1) and (2) the result follows.

Example 2 (a) : If $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\left[x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = x^2 + y^2,$$

where the suffixes denote the variables kept constant.

Sol. : Since $x = r\cos\theta$, $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos\theta$ and since $y = r\sin\theta$, $\left(\frac{\partial y}{\partial r}\right)_\theta = \sin\theta$

$$\therefore \quad x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta = x\cos\theta + y\sin\theta = x\left(\frac{x}{r}\right) + y\left(\frac{y}{r}\right) = \frac{x^2+y^2}{r} = r$$

$$\therefore \quad \left[x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = r^2 = x^2 + y^2.$$

Example 3 (a) : If $x = \cos\theta - r\sin\theta$, $y = \sin\theta + r\cos\theta$, prove that $\left(\frac{\partial r}{\partial x}\right)_\theta = \frac{x}{r}$. **(M.U. 2004)**

Sol. : Since we want $\left(\frac{\partial r}{\partial x}\right)$, we express r as a function of x and y.

Squaring and adding the given functions, we get

$$\begin{aligned} x^2 + y^2 &= \cos^2\theta + r^2\sin^2\theta - 2r\sin\theta\cos\theta + \sin^2\theta + r^2\cos^2\theta + 2r\sin\theta\cos\theta \\ &= (\sin^2\theta + \cos^2\theta) + r^2(\sin^2\theta + \cos^2\theta) \\ &= 1 + r^2 \end{aligned}$$

$$\therefore \quad r^2 = x^2 + y^2 - 1$$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \qquad \therefore \quad \left(\frac{\partial r}{\partial x}\right) = \frac{x}{r}.$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : If $x = \frac{\cos\theta}{u}$, $y = \frac{\sin\theta}{u}$, prove that $\left(\frac{\partial x}{\partial u}\right)_\theta\left(\frac{\partial u}{\partial x}\right)_y = \cos^2\theta$.

Sol. : Since, x is given as a function of u and θ, differentiating $x = \frac{\cos\theta}{u}$ w.r.t. u treating θ constant, we get

$$\left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2} \qquad \text{.......................... (1)}$$

To find $\left(\frac{\partial u}{\partial x}\right)_y$, we must express u in terms of x and y *i.e.* we must eliminate θ from the given relations. From data,

$$x^2 + y^2 = \frac{\cos^2\theta}{u^2} + \frac{\sin^2\theta}{u^2} = \frac{1}{u^2} \qquad \therefore \quad u^2 = \frac{1}{x^2+y^2}$$

$$\therefore \quad 2u\frac{\partial u}{\partial x} = -\frac{1}{(x^2+y^2)^2}\cdot 2x \qquad \therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = -\frac{1}{u}\cdot\frac{x}{(x^2+y^2)^2}$$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_\theta\cdot\left(\frac{\partial u}{\partial x}\right)_y = \left(-\frac{\cos\theta}{u^2}\right)\cdot\left(-\frac{x}{u(x^2+y^2)^2}\right)$$

But $x^2 + y^2 = \frac{1}{u^2}$ and $ux = \cos\theta$.

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_\theta\cdot\left(\frac{\partial u}{\partial x}\right)_y = \frac{\cos\theta}{u^2}\cdot\frac{x}{u}\cdot u^4 = \cos\theta\cdot ux = \cos^2\theta.$$

Example 2 (b) : If $x = \dfrac{\cos\theta}{u}$, $y = \dfrac{\sin\theta}{u}$, then show that $\left(\dfrac{\partial x}{\partial u}\right)_\theta \cdot \left(\dfrac{\partial u}{\partial x}\right)_y + \left(\dfrac{\partial y}{\partial u}\right)_\theta \cdot \left(\dfrac{\partial u}{\partial y}\right)_x = 1$.

Sol. : As in Example 1 above $\quad \left(\dfrac{\partial x}{\partial u}\right)_\theta \cdot \left(\dfrac{\partial u}{\partial x}\right)_y = \cos^2\theta$

Further, since $\quad y = \dfrac{\sin\theta}{u} \quad \therefore \left(\dfrac{\partial y}{\partial u}\right)_\theta = -\dfrac{\sin\theta}{u^2}$

Since $\quad u^2 = \dfrac{1}{x^2 + y^2} \quad \therefore \ 2u\dfrac{\partial u}{\partial y} = -\dfrac{1}{(x^2+y^2)} \cdot 2y$

$$\therefore \quad \frac{\partial u}{\partial y} = -\frac{y}{u(x^2+y^2)^2}$$

$$\therefore \quad \left(\frac{\partial y}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial y}\right)_x = \left(-\frac{\sin\theta}{u^2}\right)\left(-\frac{y}{u(x^2+y^2)}\right)$$

But $x^2 + y^2 = \dfrac{1}{u^2}$ and $uy = \sin\theta$.

$$\left(\frac{\partial y}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial y}\right)_x = \frac{\sin\theta}{u^2} \cdot \frac{y \cdot u^4}{u} = uy\sin\theta = \sin^2\theta$$

$$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial y}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial y}\right)_x = \cos^2\theta + \sin^2\theta = 1.$$

Example 3 (b) : If $x = e^u \tan v$, $y = e^u \sec v$, find

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \cdot \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right).$$

(M.U. 1999, 2010, 15)

Sol. : As we want $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ we must express u, v as functions of x and y.

Now $x^2 = e^{2u}\tan^2 v$, $y^2 = e^{2u}\sec^2 v$.

$$\therefore \quad y^2 - x^2 = e^{2u} \quad \therefore \quad u = \frac{1}{2}\log(y^2 - x^2)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{y^2 - x^2} \cdot (-2x) = -\frac{x}{y^2 - x^2}$$

$$\text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{y^2 - x^2} \cdot (2y) = \frac{y}{y^2 - x^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{x^2}{y^2 - x^2} + \frac{y^2}{y^2 - x^2} = 1 \quad \text{.................... (1)}$$

Further, $\dfrac{x}{y} = \dfrac{\tan v}{\sec v} = \sin v \quad \therefore \ v = \sin^{-1}\left(\dfrac{x}{y}\right)$

$$\therefore \quad \frac{\partial v}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \frac{1}{y} = \frac{1}{\sqrt{y^2 - x^2}}$$

$$\text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{y\sqrt{y^2 - x^2}}$$

$$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial x} = 0 \qquad \text{.................... (2)}$$

$\therefore$ From (1) and (2), $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\cdot\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = (1)\cdot(0) = 0.$

(For another method, see solved Ex. 1, page 6-5.)

Example 4 (b) : If $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1$, show that $\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2 + y^2}{y^2 - x^2}$ where suffixes denote the variables to be treated as constants. **(M.U. 2013)**

Sol. : Since $\left(\frac{\partial u}{\partial x}\right)_y$ means partial derivative of u w.r.t. x treating y constant, we have to express u as a function of x and y *i.e.* we have to eliminate v between the given relations.

$$\because \quad vy = -ux \quad \therefore \quad v = -\frac{ux}{y}$$

Putting this value of v in $\frac{u}{x} + \frac{v}{y} = 1$, we get

$$\therefore \quad \frac{u}{x} - \frac{(ux/y)}{y} = 1 \quad \therefore \quad u\left(\frac{1}{x} - \frac{x}{y^2}\right) = 1 \quad \therefore \quad u = \frac{xy^2}{y^2 - x^2}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = y^2\left[\frac{(y^2 - x^2)\cdot 1 - x(-2x)}{(y^2 - x^2)^2}\right] = \frac{y^2(x^2 + y^2)}{(y^2 - x^2)^2}$$

We now eliminate u between the given relations.

$$\because \quad ux = -vy \quad \therefore \quad u = -\frac{vy}{x}.$$

Putting this value of u in $\frac{u}{x} + \frac{v}{y} = 1$, we get,

$$\therefore \quad -\frac{(vy/x)}{x} + \frac{v}{y} = 1 \quad \therefore \quad v\left(-\frac{y}{x^2} + \frac{1}{y}\right) = 1 \quad \therefore \quad v = -\frac{x^2 y}{y^2 - x^2}$$

$$\therefore \quad \left(\frac{\partial v}{\partial y}\right)_x = -x^2\left[\frac{(y^2 - x^2)\cdot 1 - y(2y)}{(y^2 - x^2)^2}\right] = \frac{x^2(x^2 + y^2)}{(y^2 - x^2)^2}$$

$$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{y^2(x^2 + y^2) - x^2(x^2 + y^2)}{(y^2 - x^2)^2}$$

$$= \frac{(x^2 + y^2)(y^2 - x^2)}{(y^2 - x^2)^2} = \frac{x^2 + y^2}{y^2 - x^2}.$$

Example 5 (b) : If $u = ax + by$, $v = bx - ay$, find the value of $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u$ where suffixes denote the variables to be treated as constants.

Sol. : Since u is given as a function of x, y we have $\left(\frac{\partial u}{\partial x}\right)_y = a$ .... (1)

Now, $v = bx - ay$ $\quad \therefore \quad y = \dfrac{bx - v}{a}$

$\therefore$ y is a function of x, v $\quad \therefore \quad \left(\dfrac{\partial y}{\partial v}\right)_x = -\dfrac{1}{a}$ .... (2)

To find $\left(\dfrac{\partial x}{\partial u}\right)_v$, we must express x in terms of u and v *i.e.* we must eliminate y from the given relations. From data,

$$y = \frac{u - ax}{b} \quad \text{and} \quad y = \frac{bx - v}{a} \quad \therefore \quad \frac{u - ax}{b} = \frac{bx - v}{a}$$

$$\therefore \quad au - a^2x = b^2x - bv \quad \therefore \quad x = \frac{au + bv}{a^2 + b^2}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{a^2 + b^2} \quad \text{.................... (3)}$$

To find $\left(\dfrac{\partial v}{\partial y}\right)_u$, we must express v in terms of y and u *i.e.* we must eliminate x from the given relations. From data,

$$x = \frac{u - by}{a} \quad \text{and} \quad x = \frac{v + ay}{b} \quad \therefore \quad \frac{u - by}{a} = \frac{v + ay}{b}$$

$$\therefore \quad bu - b^2y = av + a^2y \quad \therefore \quad v = \frac{bu - a^2y - b^2y}{a}$$

$$\left(\frac{\partial v}{\partial y}\right)_u = -\frac{a^2 + b^2}{a} \quad \text{.................... (4)}$$

From (1), (2), (3), (4) we get,

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = a \cdot \frac{a}{a^2 + b^2} \cdot \left(-\frac{1}{a}\right) \cdot \left(-\frac{a^2 + b^2}{a}\right) = 1.$$

(a) Where elimination is difficult or not possible

However, in some cases we cannot use the above method. In such cases we obtain partial derivatives from the given two relations and substitute one result in the other. This is illustrated below.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : If $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$ find $\left(\dfrac{\partial w}{\partial x}\right)_y$.

Sol. : We get treating y **as independent variable and** z **as a dependent variable,** the following partial derivatives

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial x} = 2x$$

$$\therefore \quad \frac{\partial w}{\partial x} = 2x + 2z \cdot 2x = 2x + 4zx$$

(You can verify the result by substituting $z^2 = x^2 + y^2$ in w.)

Example 2 (b) : Find $\frac{\partial w}{\partial x}$ from $w = x^2 + y^2 + z^2$ and $z^3 - xy + yz + y^3 = 1$ treating x, y as independent variables and z as another dependent variable.

Sol. : Differentiating w partially w.r.t. x treating y as independent variable (*i.e.* constant) and z as another dependent variable,

$$\frac{\partial w}{\partial x} = 2x + 2z \frac{\partial z}{\partial x} \quad \text{.................... (1)}$$

Differentiating the other function treating z as a function of x, y, we get

$$3z^2 \frac{\partial z}{\partial x} - y + y \frac{\partial z}{\partial x} = 0 \quad \therefore \quad \frac{\partial z}{\partial x} = \frac{y}{3z^2 + y}$$

Putting this value in (1), we get

$$\frac{\partial w}{\partial x} = 2x + 2z \cdot \frac{y}{3z^2 + y}.$$

Example 3 (b) : If $w = x^2 y^2 + yz - z^3$ and $x^2 + y^2 + z^2 = 6$, find $\left(\frac{\partial w}{\partial y}\right)_x$.

Sol. : Differentiating the first equation w.r.t. y treating x constant and z as another dependent variable, we get

$$\frac{\partial w}{\partial y} = 2x^2 y + z + y \frac{\partial z}{\partial y} - 3z^2 \frac{\partial z}{\partial y} \quad \text{.................... (1)}$$

Differentiating the second equation w.r.t. y treating x constant and z as another dependent variable, we get

$$2y + 2z \frac{\partial z}{\partial y} = 0 \quad \therefore \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$$

Putting this value in (1),

$$\frac{\partial w}{\partial y} = 2x^2 y + z + y\left(-\frac{y}{z}\right) - 3z^2\left(-\frac{y}{z}\right) = 2x^2 y + z - \frac{y^2}{z} + 3yz$$

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 3 Marks

1. If $x = r\cos\theta$, $y = r\sin\theta$, prove that

(i) $\left(\frac{\partial r}{\partial x}\right)_y = \left(\frac{\partial x}{\partial r}\right)_\theta$ (ii) $\left(\frac{\partial x}{\partial \theta}\right)_r = r^2 \left(\frac{\partial \theta}{\partial x}\right)_y$ (iii) $\left(\frac{\partial y}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial y}\right)_x$

(iv) $\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$ (v) $\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$

(M.U. 1989, 2004, 05) **(M.U. 1999)**

Class (b) : 6 Marks

1. If $ux + vy = 0$, $\frac{u}{x} + \frac{v}{y} = 1$, prove that $\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = 0$.

2. If $u = l\,x + my,\ v = mx - l\,y$, prove that

(i) $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_y = \frac{l^2}{l^2+m^2}$; (ii) $\left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2+m^2}{l^2}$ **(M.U. 2012)**

3. If $x^2 = au + bv,\ y^2 = au - bv$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u$.

(M.U. 2011)

4. If $x^2 = a\sqrt{u} + b\sqrt{v},\ y^2 = a\sqrt{u} - b\sqrt{v}$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial v}\right)_u$.

5. If $x = u\tan v,\ y = u\sec v$, prove that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_x$.

6. If $x = \cos\theta - r\sin\theta,\ y = \sin\theta + r\cos\theta$, prove that

(i) $\left(\frac{\partial \theta}{\partial x}\right)_y = -\frac{\cos\theta}{r}$ (ii) $\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r}$ (iii) $\left(\frac{\partial^2 \theta}{\partial x^2}\right)_y = \frac{\cos\theta}{r^3}(\cos\theta - 2r\sin\theta)$

---

8. Composite Functions

(a) Let $z = f(x, y)$ and $x = \Phi(t),\ y = \Psi(t)$ so that z is a function of x, y and x, y are themselves functions of a third variable t. These three relations define z as a function of t. In such cases z is called a **Composite Function** of t.

e.g. (i) $z = x^2 + y^2,\ x = at^2,\ y = 2\,at$ (ii) $z = x^2 y + xy^2,\ x = a\cos t,\ y = b\sin t$

define z as a composite function of t.

Chain Rule : Let $z = f(x, y)$ possess continuous first order partial derivatives and $x = \Phi(t)$, $y = \Psi(t)$ possess continuous first order derivatives then the derivative of z w.r.t. t is given by the **chain rule**,

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}}$$

We accept this result without proof.

The above chain rule can be easily remembered with the help of the tree diagram given in Fig. 5.3 (a).

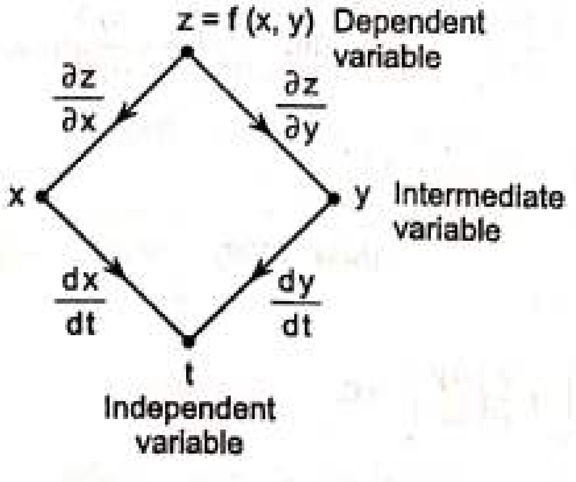


Fig. 5.3 (a)

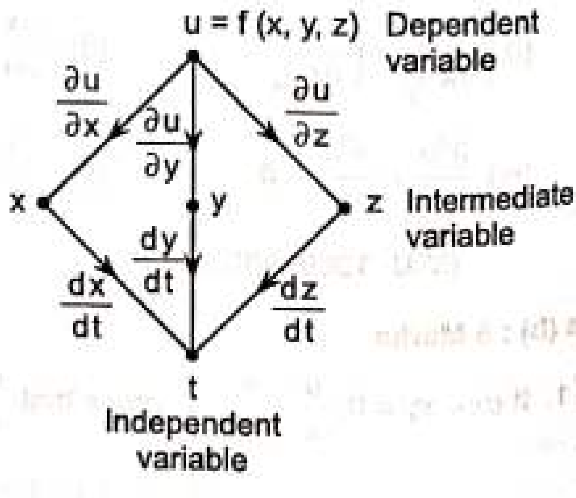


Fig. 5.3 (b)

Extension of the above chain rule to three functions

If $u = f(x, y, z)$ possesses continuous first order partial derivatives and if $x = g_1(t)$, $y = g_2(t)$, $z = g_3(t)$ possess continuous first order derivatives then the derivative of u w.r.t. t is given by the chain rule

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}}$$

The above rule can be easily remembered with the help of the tree diagram given in the Fig. 6.3 (b).

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : If $u = x^2 + y^2$ and $x = e^t \cos t$, $y = e^t \sin t$, find $\frac{du}{dt}$ in two different ways.

Sol. : We have by the above formula

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Now, $\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$

$$\frac{dx}{dt} = e^t \cos t - e^t \sin t; \quad \frac{dy}{dt} = e^t \sin t + e^t \cos t$$

$$\therefore \quad \frac{du}{dt} = 2x(e^t \cos t - e^t \sin t) + 2y(e^t \sin t + e^t \cos t)$$

$$= 2x(x - y) + 2y(y + x) = 2x^2 - 2xy + 2y^2 + 2xy$$

$$= 2(x^2 + y^2) = 2u.$$

Now, $u = (x^2 + y^2) = (e^t \cos t)^2 + (e^t \sin t)^2 = e^{2t}(\cos^2 t + \sin^2 t) = e^{2t}$.

$$\therefore \quad \frac{du}{dt} = 2e^{2t} = 2u.$$

Example 2 (b) : Find $\frac{dw}{dz}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$.

Sol. : We have $\quad \frac{dw}{dz} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$

But $\quad \frac{\partial w}{\partial x} = y, \quad \frac{\partial w}{\partial y} = x, \quad \frac{\partial w}{\partial z} = 1$

$$\therefore \quad \frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t, \quad \frac{dz}{dt} = 1$$

$$\therefore \quad \frac{dw}{dz} = (y)(-\sin t) + x \cdot \cos t + 1 \cdot 1$$

$$= -y \sin t + x \cos t + 1 = -\sin t \cdot \sin t + \cos t \cos t + 1$$

$$= 1 + \cos^2 t - \sin^2 t = 1 + \cos 2t.$$

Example 3 (b) : If $u = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$, find $\frac{du}{dt}$ by chain rule.

Sol. : We have $\quad \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$

$$\frac{du}{dt}=\frac{1}{z}\cdot(-2\cos t\sin t)+\frac{1}{z}(2\sin t\cos t)+\left(-\frac{x}{z^2}-\frac{y}{z^2}\right)\left(-\frac{1}{t^2}\right)$$

Putting the values of x, y, z in terms of t,

$$\frac{du}{dt}=t(-2\sin t\cos t)+t(2\sin t\cos t)+(t^2\cos^2 t+t^2\sin^2 t)\cdot\frac{1}{t^2}$$

$$=0+(\sin^2 t+\cos^2 t)=1.$$

Example 4 (b) : If $z=\sin^{-1}(x-y)$, $x=3t$, $y=4t^3$, prove that $\dfrac{dz}{dt}=\dfrac{3}{\sqrt{1-t^2}}$.

(M.U. 1995, 2003, 04, 11)

Sol. : We have

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$=\frac{1}{\sqrt{1-(x-y)^2}}\cdot(3)+\frac{-1}{\sqrt{1-(x-y)^2}}\cdot(12t^2)$$

$$=\frac{3(1-4t^2)}{\sqrt{1-(x-y)^2}}=\frac{3(1-4t^2)}{\sqrt{1-x^2-y^2+2xy}}$$

$$=\frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$=\frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4)-t^2-16t^6+8t^4}}\qquad\textbf{[Note this]}$$

$$=\frac{3(1-4t^2)}{\sqrt{(1-8t^2+16t^4)-t^2(1-8t^2+16t^4)}}$$

$$=\frac{3(1-4t^2)}{\sqrt{(1-4t^2)^2-t^2(1-4t^2)^2}}=\frac{3(1-4t^2)}{(1-4t^2)\sqrt{1-t^2}}$$

$$\therefore\quad\frac{dz}{dt}=\frac{3}{\sqrt{1-t^2}}.$$

Example 5 (b) : If $u=x^2+y^2+z^2-2xyz=1$, prove that $\dfrac{dx}{\sqrt{1-x^2}}+\dfrac{dy}{\sqrt{1-y^2}}+\dfrac{dz}{\sqrt{1-z^2}}=0.$

Sol. : Since $u=x^2+y^2+z^2-2xyz-1=0$, we have

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0$$

$$\therefore\quad(2x-2yz)\,dx+(2y-2xz)\,dy+(2z-2xy)\,dz=0$$

$$\therefore\quad(x-yz)\,dx+(y-zx)\,dy+(z-xy)\,dz=0\qquad\text{.......... (1)}$$

Now, consider $x^2+y^2+z^2-2xyz=1$ $\quad\therefore\quad x^2-2xyz=1-y^2-z^2$

Now, we add the term y^2z^2 to both sides,

$$\therefore\quad x^2-2xyz+y^2z^2=1-y^2-z^2+y^2z^2\qquad\textbf{[Note this]}$$

$$\therefore\quad(x-yz)^2=(1-y^2)(1-z^2)\qquad\therefore\quad(x-yz)=\sqrt{1-y^2}\sqrt{1-z^2}$$

Similarly, $(y - zx) = \sqrt{1-z^2}\sqrt{1-x^2}$ and $(z - xy) = \sqrt{1-x^2}\sqrt{1-y^2}$.

Putting these values in (1), we get

$$\sqrt{1-y^2}\sqrt{1-z^2}\,dx + \sqrt{1-z^2}\sqrt{1-x^2}\,dy + \sqrt{1-x^2}\sqrt{1-y^2}\,dz = 0$$

Dividing throughout by $\sqrt{1-x^2}\sqrt{1-y^2}\sqrt{1-z^2}$, we get

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} + \frac{dz}{\sqrt{1-z^2}} = 0.$$

EXERCISE - V

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

1. If $z = x^2 + y^2$, $x = \cos t$, $y = \sin t$, find $\frac{dz}{dt}$ at $t = \pi$. **[Ans. : 0]**

2. If $z = x^2 + y^2$, $x = at^2$, $y = 2at$, verify that $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$. **(M.U. 2002)** **[Ans. : $4a^2 t^3 + 8a^2 t$]**

3. If $z = xy^2 + x^2 y$, $x = at^2$, $y = 2at$, find $\frac{dz}{dt}$. **(M.U. 1982) [Ans. : $a^3(16t^3 + 10t^4)$]**

4. If $z = \tan^{-1}\left(\frac{x}{y}\right)$, $x = 2t$, $y = 1 - t^2$, prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$. **(M.U. 2017)**

5. If $z = xe^y$, $x = 2t$, $y = 1 - t^2$, prove that $\frac{dz}{dt} = 2e^y\cdot(1 - 2t^2)$.

6. If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$. **[Ans. : $-\frac{\pi^3}{8}$]**

7. If $z = x^2 y + y^2$, $x = \log t$, $y = e^t$, find $\frac{dz}{dt}$ at $t = 1$. **[Ans. : $2e^2$]**

8. If $u = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$, $z = e^t \cos t$, find $\frac{du}{dt}$. **(M.U. 2004m 16)** **[Ans. : $4e^{2t}$.]**

9. $u = \log(x^2 + y^2 + z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, find at $t = 3$. **[Ans. : 16 / 49]**

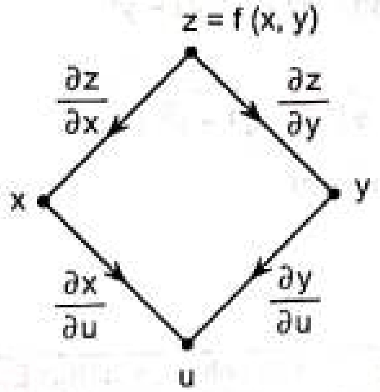
9. Partial Differentiation of Composite Functions

Let $z = f(x, y)$ possess continuous first order partial derivatives and let $x = \Phi(u, v)$, $y = \Psi(u, v)$ possess continuous first order partial derivatives, then, the partial derivatives of z w.r.t. u and v are given by the following **chain rule**

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}}$$

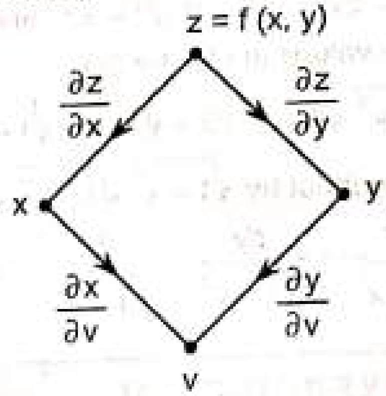
We shall accept this result without proof.

The chain rule can be easily remembered with the help of the tree diagram given in the figure.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

Fig. 5.4 (a)



$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Fig. 5.4 (b)

Extension of the above rule to three variables

Let $w = f(x, y, z)$ possess continuous first order partial derivatives and let $x = g_1(u, v)$, $y = g_2(u, v)$, $z = g_1(u, v)$ possess continuous first order partial derivatives then the partial derivatives are given by

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u}; \quad \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v}$$

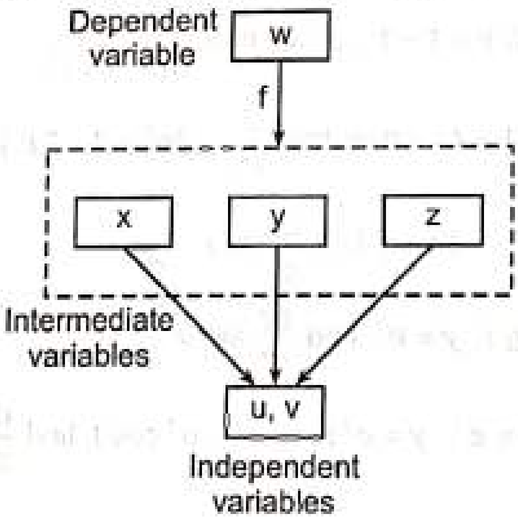


Fig. 5.5

The partial derivatives can be remembered with the help of the tree diagram given below.

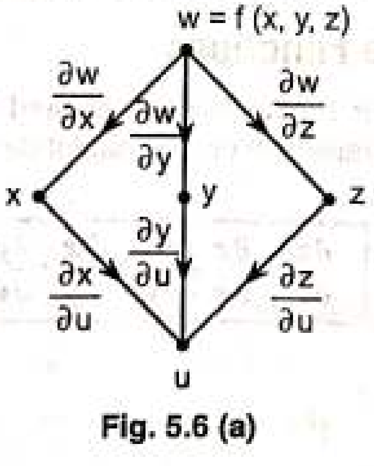


Fig. 5.6 (a)

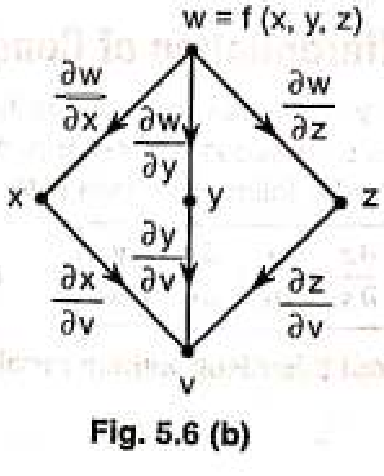


Fig. 5.6 (b)

Type I : First Order Partial Derivatives : Class (b) : 6 Marks

Example 1 (b) : Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \log s$, $z = 2r$.

Sol. : We have

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r} = 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2$$

$$= \frac{1}{s} + 4r + 2(2r) \cdot 2 = \frac{1}{s} + 12r \qquad [\because z = 2r]$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$\therefore \quad \frac{\partial w}{\partial s} = 1 \cdot \left(-\frac{r}{s^2}\right) + 2 \cdot \frac{1}{s} + 2z \cdot (0) = -\frac{r}{s^2} + \frac{2}{s}.$$

Example 2 (b) : If $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, prove that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$ **(M.U. 2001, 03, 09, 13, 18)**

Sol. : We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} \cdot e^u - \frac{\partial z}{\partial y} \cdot e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} = -\frac{\partial z}{\partial x} \cdot e^{-v} - \frac{\partial z}{\partial y} \cdot e^v$$

By substraction,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Example 3 (b) : If $z = f(u, v)$, $u = x\cos\theta - y\sin\theta$, $v = x\sin\theta + y\cos\theta$,

prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$, θ being constant.

Sol. : We have $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot \cos\theta + \frac{\partial z}{\partial v} \cdot \sin\theta \qquad \text{.................... (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \cdot (-\sin\theta) + \frac{\partial z}{\partial v} \cdot \cos\theta \qquad \text{.................... (2)}$$

Multiply (1) by x, (2) by y and add

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(x\cos\theta - y\sin\theta) + \frac{\partial z}{\partial v}(x\sin\theta + y\cos\theta)$$

$$= u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}.$$

Example 4 (b) : If $z = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$ **(M.U. 2008, 17)**

Sol. : We have $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial z}{\partial x} \cdot \cos\theta + \frac{\partial z}{\partial y} \cdot \sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r \sin \theta) + \frac{\partial z}{\partial y}(r \cos \theta)$$

$$\therefore \quad \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \cos^2 \theta \left(\frac{\partial z}{\partial x}\right)^2 + 2\cos\theta \sin\theta \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + \sin^2\theta \left(\frac{\partial z}{\partial y}\right)^2$$

$$+ \sin^2\theta \left(\frac{\partial z}{\partial x}\right)^2 - 2\cos\theta \sin\theta \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + \cos^2\theta \left(\frac{\partial z}{\partial y}\right)^2$$

$$\therefore \quad \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

Example 5 (b) : If $u = f(ax - by, by - cz, cz - ax)$, prove that

$$\frac{1}{a} \cdot \frac{\partial u}{\partial x} + \frac{1}{b} \cdot \frac{\partial u}{\partial y} + \frac{1}{c} \cdot \frac{\partial u}{\partial z} = 0.$$

Sol. : Let $X = ax - by$, $Y = by - cz$ and $Z = cz - ax$. Then $u = f(X, Y, Z)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x}$$

But $\dfrac{\partial X}{\partial x} = a$, $\dfrac{\partial Y}{\partial x} = 0$, $\dfrac{\partial Z}{\partial x} = -a$ $\quad \therefore \quad \dfrac{\partial u}{\partial x} = a\dfrac{\partial u}{\partial X} + 0 + \dfrac{\partial u}{\partial Z}(-a)$ .... (1)

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial X}\dfrac{\partial X}{\partial y} + \dfrac{\partial u}{\partial Y}\dfrac{\partial Y}{\partial y} + \dfrac{\partial u}{\partial Z}\dfrac{\partial Z}{\partial y}$

But $\dfrac{\partial X}{\partial y} = -b$, $\dfrac{\partial Y}{y} = b$, $\dfrac{\partial Z}{\partial y} = 0$ $\quad \therefore \quad \dfrac{\partial u}{\partial y} = -b\dfrac{\partial u}{\partial X} + b\dfrac{\partial u}{\partial Y} + 0$ .... (2)

And $\dfrac{\partial u}{\partial z} = 0 + (-c)\dfrac{\partial u}{\partial Y} + c\left(\dfrac{\partial u}{\partial Z}\right).$ .... (3)

$\therefore$ Multiply these results (1), (2), (3) by $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ and add to get the required result.

Example 6 (b) : If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\dfrac{\partial u}{\partial x} + 4\dfrac{\partial u}{\partial y} + 3\dfrac{\partial u}{\partial z} = 0.$

(M.U. 2003, 07)

Sol. : Left to you.

Example 7 (b) : If $u = f(x - y, y - z, z - x)$, prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

(M.U. 1989, 91, 95, 2014)

Sol. : Let $X = x - y$, $Y = y - z$, $Z = z - x$. Then $u = f(X, Y, Z)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z}$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial Y} - \dfrac{\partial u}{\partial X}$, $\dfrac{\partial u}{\partial Z} = \dfrac{\partial u}{\partial Z} - \dfrac{\partial u}{\partial Y}$

$\therefore$ Adding the three results $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

(Note that this is a particular case of Ex. 5 where $a = b = c = 1$).

Example 8 (b) : If $u = f(e^{x-y}, e^{y-z}, e^{z-x})$, then prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ **(M.U. 2001, 08, 09, 11, 16, 19)**

Sol. : Let $X = e^{x-y}$, $Y = e^{y-z}$, $Z = e^{z-x}$. Then $u = f(X, Y, Z)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x}$$

$$= \frac{\partial u}{\partial X} e^{x-y}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z} e^{z-x}(-1)$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} e^{x-y} - \frac{\partial u}{\partial Z} e^{z-x} \qquad \text{.................... (1)}$$

Now, $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} e^{x-y}(-1) + \frac{\partial u}{\partial Y} \cdot e^{y-z}(1) + \frac{\partial u}{\partial Z}(0)$$

$$= -\frac{\partial u}{\partial X} \cdot e^{x-y} + \frac{\partial u}{\partial Y} \cdot e^{y-z} \qquad \text{.................... (2)}$$

And $$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial z}$$

$$= \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y} \cdot e^{y-z}(-1) + \frac{\partial u}{\partial Z} e^{z-x}(1)$$

$$\therefore \quad \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y} e^{y-z} + \frac{\partial u}{\partial Z} \cdot e^{z-x} \qquad \text{.................... (3)}$$

Adding (1), (2) and (3), we get $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0.$

Example 9 (b) : If $u = f\left(\dfrac{x-y}{xy}, \dfrac{z-x}{zx}\right)$, prove that

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0.$$ **(M.U. 2014, 17, 18)**

Sol. : Let $r = \dfrac{x-y}{xy}$, $s = \dfrac{z-x}{zx}$. Then $u = f(r, s)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} \qquad \text{.................... (1)}$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} \qquad \text{.................... (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} \qquad \text{.................... (3)}$$

But $$r = \frac{x-y}{xy} = \frac{1}{y} - \frac{1}{x}$$ **[Note this]**

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{1}{x^2}, \quad \frac{\partial r}{\partial y} = -\frac{1}{y^2}, \quad \frac{\partial r}{\partial z} = 0$$

and $$s = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$$ **[Note this]**

$$\therefore \quad \frac{\partial s}{\partial x}=-\frac{1}{x^2}, \quad \frac{\partial s}{\partial y}=0, \quad \frac{\partial s}{\partial z}=\frac{1}{z^2}$$

Putting these values in (1), (2), (3), we get

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{1}{x^2}+\frac{\partial u}{\partial s}\cdot\left(-\frac{1}{x^2}\right); \quad \frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\left(-\frac{1}{y^2}\right)+\frac{\partial u}{\partial s}(0)$$

and $$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}(0)+\frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right)$$

Multiplying these equalities by x^2, y^2, z^2 and adding we get,

$$x^2\frac{\partial y}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}-\frac{\partial u}{\partial r}+\frac{\partial u}{\partial s}=0.$$

Example 10 (b) : If $z=f(x,y)$, $x=\log u$, $y=\log v$, prove that $\dfrac{\partial^2 z}{\partial x\partial y}=uv\dfrac{\partial^2 z}{\partial u\partial v}$.

Sol. : We have $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{1}{u}+\frac{\partial z}{\partial y}\cdot 0=\frac{1}{u}\cdot\frac{\partial z}{\partial x}.$$

Now, $\dfrac{\partial z}{\partial x}$ is also a function of x, y and note that u, v are independent variables.

Let $\dfrac{\partial z}{\partial x}=w$ where w is a function of x, y.

$$\therefore \quad \frac{\partial^2 z}{\partial u\partial v}=\frac{\partial}{\partial v}\left[\frac{1}{u}\cdot w\right]=\frac{1}{u}\left[\frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v}\right]$$

$$=\frac{1}{u}\left[\frac{\partial w}{\partial x}\cdot 0+\frac{\partial w}{\partial y}\cdot\frac{1}{v}\right]=\frac{1}{uv}\cdot\frac{\partial w}{\partial y}=\frac{1}{uv}\cdot\frac{\partial^2 z}{\partial y\partial x}$$

$$\therefore \quad \frac{\partial^2 z}{\partial x\partial y}=uv\frac{\partial^2 z}{\partial u\partial v}.$$

Example 11 (b) : If $x=\sqrt{vw}$, $y=\sqrt{wu}$, $z=\sqrt{uv}$, prove that

$$x\frac{\partial \Phi}{\partial x}+y\frac{\partial \Phi}{\partial y}+z\frac{\partial \Phi}{\partial z}=u\frac{\partial \Phi}{\partial u}+v\frac{\partial \Phi}{\partial v}+w\frac{\partial \Phi}{\partial w}$$

where Φ is the function of x, y, z. **(M.U. 1991, 96, 2002, 05, 07, 12)**

Sol. : Since Φ is the function of x, y, z and x, y, z themselves are functions of u, v, w, Φ is a composite function of u, v, w.

$$\therefore \quad \frac{\partial \Phi}{\partial u}=\frac{\partial \Phi}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial \Phi}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial \Phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$=\frac{\partial \Phi}{\partial x}\cdot 0+\frac{\partial \Phi}{\partial y}\cdot\sqrt{w}\cdot\frac{1}{2\sqrt{u}}+\frac{\partial \Phi}{\partial z}\cdot\sqrt{v}\cdot\frac{1}{2\sqrt{u}}$$

$$\therefore \quad u\frac{\partial \Phi}{\partial u}=\frac{1}{2}\sqrt{uw}\frac{\partial \Phi}{\partial y}+\frac{1}{2}\sqrt{uv}\frac{\partial \Phi}{\partial z}=\frac{1}{2}\left(y\frac{\partial \Phi}{\partial y}+z\frac{\partial \Phi}{\partial z}\right) \quad \text{.................. (i)}$$

Similarly, $$v\frac{\partial \Phi}{\partial v}=\frac{1}{2}\left(z\frac{\partial \Phi}{\partial z}+x\frac{\partial \Phi}{\partial x}\right) \quad \text{.................. (ii)}$$

and $$w\frac{\partial \Phi}{\partial w}=\frac{1}{2}\left(x\frac{\partial \Phi}{\partial x}+y\frac{\partial \Phi}{\partial y}\right) \quad \text{.................. (iii)}$$

Adding the three results, we get,

$$u\frac{\partial \Phi}{\partial u}+v\frac{\partial \Phi}{\partial v}+w\frac{\partial \Phi}{\partial w}=x\frac{\partial \Phi}{\partial x}+y\frac{\partial \Phi}{\partial y}+z\frac{\partial \Phi}{\partial z}.$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : If $u=f(x^2+2yz,\ y^2+2zx)$, prove that

$$(y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}=0.$$

Sol. : Let $r=x^2+2yz,\ \ s=y^2+2zx$

$$\therefore\quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}=\frac{\partial u}{\partial r}\cdot(2x)+\frac{\partial u}{\partial s}\cdot(2z) \qquad \text{.................... (1)}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}=\frac{\partial u}{\partial r}\cdot(2z)+\frac{\partial u}{\partial s}\cdot(2y) \qquad \text{.................... (2)}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}=\frac{\partial u}{\partial r}\cdot(2y)+\frac{\partial u}{\partial s}\cdot(2x) \qquad \text{.................... (3)}$$

Now, multiply (1) by (y^2-zx), (2) by (x^2-yz) and (3) by (z^2-xy) and add.

$$\therefore\quad (y^2-zx)\frac{\partial u}{\partial x}+(x^2-yz)\frac{\partial u}{\partial y}+(z^2-xy)\frac{\partial u}{\partial z}$$

$$=(y^2-zx)\cdot 2x\cdot\frac{\partial u}{\partial r}+(y^2-zx)\cdot 2z\cdot\frac{\partial u}{\partial s}+(x^2-yz)\cdot 2z\cdot\frac{\partial u}{\partial r}$$

$$+(x^2-yz)\cdot 2y\cdot\frac{\partial u}{\partial s}+(z^2-xy)\cdot 2y\cdot\frac{\partial u}{\partial r}+(z^2-xy)\cdot 2x\cdot\frac{\partial u}{\partial s}$$

$$=(2xy^2-2x^2z+2zx^2-2yz^2+2yz^2-2xy^2)\frac{\partial u}{\partial r}$$

$$+(2zy^2-2z^2x+2x^2y-2y^2z+2z^2x-2x^2y)\frac{\partial u}{\partial s}$$

$$=(0)\frac{\partial u}{\partial r}+(0)\frac{\partial u}{\partial s}$$

Example 2 (c) : If $f(u,v)=0$ where $u=l\,x+my+nz,\ v=x^2+y^2+z^2$, prove that

$$(l\,y-mx)+(ny-mz)\frac{\partial z}{\partial x}+(l\,z-nx)\frac{\partial z}{\partial y}=0.$$

Sol. : We have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}\cdot\left(l+n\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\cdot\left(2x+2z\frac{\partial z}{\partial x}\right) \qquad \text{................ (1)}$$

and $$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial f}{\partial u}\cdot\left(m+n\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\cdot\left(2y+2z\frac{\partial z}{\partial y}\right) \qquad \text{................ (2)}$$

But $f(u,v)=0$ can be considered a function of x, y which is identically equal to zero.

$$\therefore\quad \frac{\partial f}{\partial x}=0 \quad\text{and}\quad \frac{\partial f}{\partial y}=0.$$

Hence, from (1) and (2), we have

$$\frac{\partial f}{\partial u}\left(l+n\frac{\partial z}{\partial x}\right)+\frac{\partial f}{\partial v}\left(2x+2z\frac{\partial z}{\partial x}\right)=0 \quad \text{and} \quad \frac{\partial f}{\partial u}\left(m+n\frac{\partial z}{\partial y}\right)+\frac{\partial f}{\partial v}\left(2y+2z\frac{\partial z}{\partial y}\right)=0$$

$$\therefore \quad \left(\frac{\partial f}{\partial u}\right)\Big/\left(\frac{\partial f}{\partial v}\right)=-\frac{2x+2z(\partial z/\partial x)}{l+n(\partial z/\partial x)} \quad \text{and} \quad \left(\frac{\partial f}{\partial u}\right)\Big/\left(\frac{\partial f}{\partial v}\right)=-\frac{2y+2z(\partial z/\partial y)}{m+n(\partial z/\partial y)}$$

Equating the two results,

$$\frac{x+z(\partial z/\partial x)}{l+n(\partial z/\partial x)}=\frac{y+z(\partial z/\partial y)}{m+n(\partial z/\partial y)}$$

$$\therefore \quad mx+mz\left(\frac{\partial z}{\partial x}\right)+nx\left(\frac{\partial z}{\partial y}\right)+nz\left(\frac{\partial z}{\partial x}\right)\cdot\left(\frac{\partial z}{\partial y}\right)=ly+lz\left(\frac{\partial z}{\partial y}\right)+ny\left(\frac{\partial z}{\partial x}\right)+nx\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)$$

$$\therefore \quad (ly-mx)+(ny-mz)\frac{\partial z}{\partial x}+(lz-nx)\frac{\partial z}{\partial y}=0.$$

Example 3 (c) : If $z=\Phi(x, y)$, $x=uv$, $y=u/v$, prove that

$$\frac{\partial z}{\partial x}=\frac{1}{2v}\cdot\frac{\partial z}{\partial u}+\frac{1}{2u}\cdot\frac{\partial z}{\partial v} \quad \text{and} \quad \frac{\partial z}{\partial y}=\frac{v}{2}\cdot\frac{\partial z}{\partial u}-\frac{v^2}{2u}\cdot\frac{\partial z}{\partial v}.$$

Sol. : We have $\quad \dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u}=\dfrac{\partial z}{\partial x}v+\dfrac{\partial z}{\partial y}\cdot\dfrac{1}{v}$ .... (1)

And $\quad \dfrac{\partial z}{\partial v}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v}=\dfrac{\partial z}{\partial x}u+\dfrac{\partial z}{\partial y}\cdot\left(-\dfrac{u}{v^2}\right)$ .... (2)

Multiply (1) by $\dfrac{1}{2v}$ and (2) by $\dfrac{1}{2u}$ and add.

$$\therefore \quad \frac{1}{2v}\cdot\frac{\partial z}{\partial u}+\frac{1}{2u}\cdot\frac{\partial z}{\partial v}=\frac{1}{2}\cdot\frac{\partial z}{\partial x}+\frac{1}{2v^2}\cdot\frac{\partial z}{\partial y}+\frac{1}{2}\cdot\frac{\partial z}{\partial x}-\frac{1}{2v^2}\cdot\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}$$

Multiply (1) by $\dfrac{v}{2}$ and (2) by $\dfrac{v^2}{2u}$ and subtract.

$$\therefore \quad \frac{v}{2}\cdot\frac{\partial z}{\partial u}-\frac{v^2}{2u}\cdot\frac{\partial z}{\partial v}=\frac{v^2}{2}\cdot\frac{\partial z}{\partial x}+\frac{1}{2}\cdot\frac{\partial z}{\partial y}-\frac{v^2}{2}\cdot\frac{\partial z}{\partial x}+\frac{1}{2}\cdot\frac{\partial z}{\partial y}=\frac{\partial z}{\partial y}.$$

Example 4 (c) : If $z=f(x, y)$, $x=u\cos h\, v$, $y=u\sin hv$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2-\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial u}\right)^2-\frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2$$ **(M.U. 1998, 2001, 10)**

Sol. : We have

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}=\frac{\partial z}{\partial x}\cos hv+\frac{\partial z}{\partial y}\sin hv \quad \text{.................... (1)}$$

And $\quad \dfrac{\partial z}{\partial v}=\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v}+\dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v}=\dfrac{\partial z}{\partial x}u\sin hv+\dfrac{\partial z}{\partial y}u\cos hv$

$$\therefore \quad \frac{1}{u}\cdot\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\sin hv+\frac{\partial z}{\partial y}\cos hv \quad \text{.................... (2)}$$

Now, squaring (1) and (2) and subtracting, we get

$$\therefore \quad \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos h^2 v + \left(\frac{\partial z}{\partial y}\right)^2 \sin h^2 v$$

$$+ 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos hv \cdot \sin hv - \left(\frac{\partial z}{\partial x}\right)^2 \sin h^2 v$$

$$- 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos hv \cdot \sin hv - \left(\frac{\partial z}{\partial y}\right)^2 \cos h^2 v$$

$$\therefore \quad \left(\frac{\partial z}{\partial u}\right)^2 - \frac{1}{u^2}\left(\frac{\partial z}{\partial v}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2.$$

Example 5 (c) : If $z = f(x, y)$, $x = e^u \cos v$, $y = e^u \sin v$, prove that

(i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$ (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

(M.U. 1995, 97, 2008, 09)

Sol. : We have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u \cos v + \frac{\partial z}{\partial y}\cdot e^u \sin v$$

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} \qquad \text{.................... (1)}$$

And $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}(-e^u \sin v) + \frac{\partial z}{\partial y}e^u \cdot \cos v$$

$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} \qquad \text{.................... (2)}$$

Multiply (2) by x, (1) by y and add.

$$\therefore \quad x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = -xy\frac{\partial z}{\partial x} + x^2\frac{\partial z}{\partial y} + xy\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y}$$

$$= (x^2 + y^2)\frac{\partial z}{\partial y} = (e^{2u}\cos^2 v + e^{2u}\sin^2 v)\frac{\partial z}{\partial y} = e^{2u}\frac{\partial z}{\partial y}$$

Squaring (1) and (2) and adding, we get,

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = x^2\left(\frac{\partial z}{\partial x}\right)^2 + y^2\left(\frac{\partial z}{\partial y}\right)^2 + 2xy\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

$$+ y^2\left(\frac{\partial z}{\partial x}\right)^2 + x^2\left(\frac{\partial z}{\partial y}\right)^2 - 2xy\cdot\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$$

$$\therefore \quad \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = (x^2 + y^2)\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

$$= e^{2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

Example 6 (c) : If $x = u + v + w,\ \ y = uv + vw + wu,\ z = uvw$, prove that

$$x\frac{\partial \Phi}{\partial x} + 2y\frac{\partial \Phi}{\partial y} + 3z\frac{\partial \Phi}{\partial z} = u\frac{\partial \Phi}{\partial u} + v\frac{\partial \Phi}{\partial v} + w\frac{\partial \Phi}{\partial w}$$

where Φ is a function of x, y, z. **(M.U. 1998, 2016)**

Sol. : Φ is a function of x, y, z and x, y, z are themselves functions of u, v, w.

$$\therefore \quad \frac{\partial \Phi}{\partial u} = \frac{\partial \Phi}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial \Phi}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial \Phi}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= \frac{\partial \Phi}{\partial x}\cdot 1 + \frac{\partial \Phi}{\partial y}(v+w) + \frac{\partial \Phi}{\partial z}\cdot vw \qquad \text{.................... (1)}$$

And $$\frac{\partial \Phi}{\partial v} = \frac{\partial \Phi}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial \Phi}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial \Phi}{\partial z}\cdot\frac{\partial z}{\partial v}$$

$$= \frac{\partial \Phi}{\partial x}\cdot 1 + \frac{\partial \Phi}{\partial y}(u+w) + \frac{\partial \Phi}{\partial z}\cdot uw \qquad \text{.................... (2)}$$

And $$\frac{\partial \Phi}{\partial w} = \frac{\partial \Phi}{\partial x}\cdot\frac{\partial x}{\partial w} + \frac{\partial \Phi}{\partial y}\cdot\frac{\partial y}{\partial w} + \frac{\partial \Phi}{\partial z}\cdot\frac{\partial z}{\partial w}$$

$$= \frac{\partial \Phi}{\partial x}\cdot 1 + \frac{\partial \Phi}{\partial y}(v+u) + \frac{\partial \Phi}{\partial z}\cdot uv \qquad \text{.................... (3)}$$

Multiply (1) by u, (2) by v, (3) by w and add

$$\therefore \quad u\frac{\partial \Phi}{\partial u} + v\frac{\partial \Phi}{\partial v} + w\frac{\partial \Phi}{\partial w}$$

$$= (u+v+w)\frac{\partial \Phi}{\partial x} + [(uv+uw)+(vu+vw)+(wv+wu)]\frac{\partial \Phi}{\partial y} + 3uvw\frac{\partial \Phi}{\partial z}$$

$$= (u+v+w)\frac{\partial \Phi}{\partial x} + [2(uv+vw+wu)]\frac{\partial \Phi}{\partial y} + 3uvw\frac{\partial \Phi}{\partial z}$$

$$= x\frac{\partial \Phi}{\partial x} + 2y\frac{\partial \Phi}{\partial y} + 3z\frac{\partial \Phi}{\partial z}$$

$$\therefore \quad u\frac{\partial \Phi}{\partial x} + v\frac{\partial \Phi}{\partial y} + w\frac{\partial \Phi}{\partial w} = x\frac{\partial \Phi}{\partial x} + 2y\frac{\partial \Phi}{\partial y} + 3z\frac{\partial \Phi}{\partial z}.$$

Type II : Second Order Partial Derivatives of Function of a Function : Class (c) : 8 Marks

Example 1 (c) : If $x = p\cos\theta - q\sin\theta,\ \ y = p\sin\theta + q\cos\theta$ and u is a function of x, y,

prove that $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial p^2} + \frac{\partial^2 u}{\partial q^2}.$$

Sol. : We have $\dfrac{\partial x}{\partial p} = \cos\theta,\quad \dfrac{\partial y}{\partial p} = \sin\theta$

$$\therefore \quad \frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial p} = \frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta$$

$$\therefore \quad \frac{\partial^2 u}{\partial p^2} = \cos\theta\cdot\frac{\partial}{\partial p}\left(\frac{\partial u}{\partial x}\right) + \sin\theta\cdot\frac{\partial}{\partial p}\left(\frac{\partial u}{\partial y}\right)$$

$$= \cos\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\frac{\partial x}{\partial p} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\frac{\partial y}{\partial p}\right] + \sin\theta\left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\frac{\partial x}{\partial p} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial y}{\partial p}\right]$$

$$= \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\cdot\cos\theta + \frac{\partial^2 u}{\partial y\,\partial x}\cdot\sin\theta\right] + \sin\theta\left[\frac{\partial^2 u}{\partial x\,\partial y}\cdot\cos\theta + \frac{\partial^2 u}{\partial y^2}\cdot\sin\theta\right]$$

$$\therefore\quad \frac{\partial^2 u}{\partial p^2} = \cos^2\theta\,\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 u}{\partial y^2}$$

Similarly, we can obtain,

$$\frac{\partial^2 u}{\partial q^2} = \sin^2\theta\,\frac{\partial^2 u}{\partial x^2} - 2\sin\theta\cos\theta\,\frac{\partial^2 u}{\partial x\,\partial y} + \cos^2\theta\,\frac{\partial^2 u}{\partial y^2}$$

Adding the two results, we get,

$$\frac{\partial^2 u}{\partial p^2} + \frac{\partial^2 u}{\partial q^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Example 2 (c) : If $x = r\cos\theta$, $y = r\sin\theta$ show that the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ transforms into $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2} = 0.$ **(M.U. 1990, 91, 99)**

Sol. : We have $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$.

$$\therefore\quad 2r\frac{\partial r}{\partial x} = 2x \quad i.e.\quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta.\quad \text{Similarly,}\quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta.$$

Further,
$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y^2/x^2)}\cdot\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y^2/x^2)}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

Now,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}(\cos\theta) + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right) \qquad \text{.....................} (1)$$

$$\therefore\quad \frac{\partial}{\partial x} \equiv \cos\theta\cdot\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta} \qquad \textbf{[Note this]} \qquad \text{.....................} (2)$$

Again
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\cdot\sin\theta + \frac{\partial u}{\partial \theta}\cdot\frac{\cos\theta}{r} \qquad \text{.....................} (3)$$

$$\therefore\quad \frac{\partial}{\partial y} \equiv \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial}{\partial \theta} \qquad \textbf{[Note this]} \qquad \text{.....................} (4)$$

Now,
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\cdot\left(\frac{\partial u}{\partial x}\right)$$

By operating by (2) on (1),

$$\frac{\partial^2 u}{\partial x^2} = \left(\cos\theta\cdot\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\right)\left(\cos\theta\cdot\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\cdot\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta \cdot \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) - \sin\theta\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}\right)$$

$$- \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(\cos\theta\cdot\frac{\partial u}{\partial r}\right) + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial u}{\partial\theta}\right)$$

$$= \cos^2\theta\cdot\frac{\partial^2 u}{\partial r^2} - \sin\theta\cos\theta\cdot\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}\right)$$

$$- \frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\left(\cos\theta\cdot\frac{\partial u}{\partial r}\right) + \frac{\sin\theta\cos\theta}{r^2}\cdot\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}$$

$$= \cos^2\theta\cdot\frac{\partial^2 u}{\partial r^2} + \frac{\sin\theta\cos\theta}{r^2}\cdot\frac{\partial u}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial r\,\partial\theta}$$

$$+ \frac{\sin^2\theta}{r}\cdot\frac{\partial u}{\partial r} - \frac{\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial\theta\,\partial r} + \frac{\sin\theta\cos\theta}{r^2}\cdot\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \cos^2\theta\cdot\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial\theta\,\partial r}$$

$$+ \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial u}{\partial\theta} + \frac{\sin^2\theta}{r}\cdot\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} \quad \text{.....................} (5)$$

Comparing (2) with (4) we find that for differentiating with respect to y, we have to replace cos θ by sin θ and sin θ by $-$ cos θ. Hence, making these changes we get from (5)

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta\cdot\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\cdot\frac{\partial^2 u}{\partial\theta\,\partial r} - \frac{2\sin\theta\cos\theta}{r^2}\cdot\frac{\partial u}{\partial\theta}$$

$$+ \frac{\cos^2\theta}{r}\cdot\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2} \quad \text{.....................} (6)$$

Adding (5) and (6), we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial\theta^2}.$$

Note ....

The L.H.S. is known as **Laplace Equation in cartesian coordinates** and R.H.S. as **Laplace Equation in polar coordinates.**

Example 3 (c) : Transform $\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$ into polar form. **(M.U. 1998, 2006)**

Sol. : We have for polar form, $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \quad x^2 + y^2 = r^2 \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \quad 2x = 2r\frac{\partial r}{\partial x} \qquad \therefore \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$$

$$\text{And} \quad 2y = 2r\frac{\partial r}{\partial y} \qquad \therefore \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

Further, $\dfrac{\partial \theta}{\partial x} = \dfrac{1}{1+(y^2/x^2)} \cdot \left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2+y^2} = -\dfrac{r\sin\theta}{r^2} = -\dfrac{\sin\theta}{r}$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y^2/x^2)} \cdot \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

But $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial x} = \cos\theta \dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r} \cdot \dfrac{\partial u}{\partial \theta}$ .... (1)

$\therefore \quad \dfrac{\partial}{\partial x} \equiv \cos\theta \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r} \cdot \dfrac{\partial}{\partial \theta}$ **[Note this]** .... (2)

Again $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r} \cdot \dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial \theta} \cdot \dfrac{\partial \theta}{\partial y} = \sin\theta \dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r} \cdot \dfrac{\partial u}{\partial \theta}$ .... (3)

$\therefore \quad \dfrac{\partial}{\partial y} = \sin\theta \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r} \cdot \dfrac{\partial}{\partial \theta}$ **[Note this]** .... (4)

Operating by (4) on (1),

$$\frac{\partial^2 u}{\partial y \partial x} = \left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial}{\partial \theta}\right)\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial u}{\partial \theta}\right)$$

$$= \sin\theta \frac{\partial}{\partial r}\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial u}{\partial \theta}\right) + \frac{\cos\theta}{r} \cdot \frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial u}{\partial \theta}\right)$$

$$= \sin\theta\cos\theta \frac{\partial^2 u}{\partial r^2} - \sin^2\theta \frac{\partial}{\partial r} \cdot \left(\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}\right)$$

$$+ \frac{\cos\theta}{r} \cdot \frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial u}{\partial r}\right) - \frac{\cos\theta}{r^2} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial y \partial x} = \sin\theta\cos\theta \frac{\partial^2 u}{\partial r^2} - \sin^2\theta \cdot \frac{1}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} + \sin^2\theta \cdot \frac{1}{r^2} \cdot \frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r} \cdot \frac{\partial^2 u}{\partial \theta \partial r}$$

$$- \frac{\cos\theta\sin\theta}{r} \cdot \frac{\partial u}{\partial r} - \frac{\cos^2\theta}{r^2} \cdot \frac{\partial u}{\partial \theta} - \frac{\cos\theta\sin\theta}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} \qquad \text{..................... (5)}$$

Now, operating by (2) on (3),

$$\frac{\partial^2 u}{\partial x \partial y} = \left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta}\right)\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta \frac{\partial}{\partial r}\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial u}{\partial \theta}\right)$$

$$= \sin\theta\cos\theta \frac{\partial^2 u}{\partial r^2} + \cos^2\theta \frac{\partial}{\partial r}\left(\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial r}\right)$$

$$- \frac{\sin\theta}{r^2} \cdot \frac{\partial}{\partial \theta}\left(\cos\theta \frac{\partial u}{\partial \theta}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x \partial y} = \sin\theta\cos\theta \frac{\partial^2 u}{\partial r^2} - \frac{\cos^2\theta}{r^2} \cdot \frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\sin\theta\cos\theta}{r} \cdot \frac{\partial u}{\partial r}$$

$$- \frac{\sin^2\theta}{r} \cdot \frac{\partial^2 u}{\partial \theta \partial r} + \frac{\sin^2\theta}{r^2} \cdot \frac{\partial u}{\partial \theta} - \frac{\sin\theta\cos\theta}{r^2} \cdot \frac{\partial^2 u}{\partial \theta^2} \qquad \text{..................... (6)}$$

Equating (5) and (6), we get, (terms in $\frac{\partial^2 u}{\partial r \partial \theta}$ and $\frac{\partial^2 u}{\partial \theta \partial r}$ remain.)

$$-\frac{\sin^2\theta}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r} \cdot \frac{\partial^2 u}{\partial \theta \partial r}$$

$$= \frac{\cos^2\theta}{r} \cdot \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\sin^2\theta}{r} \cdot \frac{\partial^2 u}{\partial \theta \partial r}$$

$$\therefore \quad (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial \theta \partial r} = (\cos^2\theta + \sin^2\theta)\frac{\partial^2 u}{\partial r \partial \theta} \qquad \therefore \quad \frac{\partial^2 u}{\partial \theta \partial r} = \frac{\partial^2 u}{\partial r \partial \theta}$$

$\therefore$ The equation $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ transforms to $\frac{\partial^2 u}{\partial \theta \partial r} = \frac{\partial^2 u}{\partial y \partial \theta}$.

Example 4 (c) : If $x = r\cos\theta$, $y = r\sin\theta$, prove that $r\frac{\partial^2 u}{\partial r \partial \theta} - \frac{\partial u}{\partial \theta} = 0$, changes to

$$xy\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) - (x^2 - y^2)\frac{\partial^2 u}{\partial x \partial y} = 0.$$

Sol. : Since $x = r\cos\theta$, $\quad \frac{\partial x}{\partial r} = \cos\theta$, $\quad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$\because \quad y = r\sin\theta$, $\quad \frac{\partial y}{\partial r} = \sin\theta$, $\quad \frac{\partial y}{\partial \theta} = r\cos\theta$

Now, $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y} \qquad \text{.................... (1)}$$

$$\therefore \quad \frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \qquad \text{.................... (2)}$$

Also $$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} \qquad \text{.................... (3)}$$

By operating by (2) on (3),

$$\frac{\partial^2 u}{\partial r \partial \theta} = \left(\cos\theta \cdot \frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial x}\right)\left(-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}\right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial r \partial \theta} = \cos\theta\frac{\partial}{\partial x}\left(-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}\right) + \sin\theta\frac{\partial}{\partial y}\left(-y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}\right)$$

$$= -\cos\theta \cdot y \cdot \frac{\partial^2 u}{\partial x^2} + \cos\theta \cdot \frac{\partial u}{\partial y} + x\cos\theta\frac{\partial^2 u}{\partial x \partial y}$$

$$-\sin\theta \cdot \frac{\partial u}{\partial x} - y\sin\theta\frac{\partial^2 u}{\partial x \partial y} + x\sin\theta\frac{\partial^2 u}{\partial y^2} \qquad \text{.................... (4)}$$

Multiply (4) by r and then subtract (3) from it

$$r\frac{\partial^2 u}{\partial r \partial \theta} - \frac{\partial u}{\partial \theta} = -r\cos\theta \cdot y \cdot \frac{\partial^2 u}{\partial x^2} + r\cos\theta\frac{\partial u}{\partial y} + r\cos\theta \cdot x\frac{\partial^2 u}{\partial x \partial y}$$

$$-r\sin\theta\frac{\partial u}{\partial x} - y \cdot r\sin\theta\frac{\partial^2 u}{\partial x \partial y} + x \cdot r\sin\theta \cdot \frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y}$$

$$= -xy\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x \partial y} - y\frac{\partial u}{\partial x} - y^2\frac{\partial^2 u}{\partial x \partial y} + xy\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y}$$

$$= -xy\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}\right) + (x^2 - y^2)\frac{\partial^2 u}{\partial x \partial y}$$

$\therefore$ $r\dfrac{\partial^2 u}{\partial r \partial \theta} - \dfrac{\partial u}{\partial \theta} = 0$ becomes $xy\left(\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial y^2}\right) - (x^2 - y^2)\dfrac{\partial^2 u}{\partial x \partial y} = 0.$

Example 5 (c) : If $x + y = 2e^{\theta} \cos \Phi$, $x - y = 2\,i\,e^{\theta} \sin \Phi$, show that

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \Phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}.$$ **(M.U. 1999, 2003, 14)**

Sol. : Adding the given results, $2x = 2e^{\theta}(\cos \Phi + i \sin \Phi)$

$\therefore$ $x = e^{\theta} \cdot e^{i\Phi} = e^{\theta + i\Phi}$ **[Note this]**

and subtracting the results. $2y = 2e^{\theta}(\cos \Phi - i \sin \Phi)$

$\therefore$ $y = e^{\theta} \cdot e^{-i\Phi} = e^{\theta - i\Phi}$ **[Note this]**

Now, u is a function of x, y and x, y are functions of θ and Φ.

$$\therefore \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x} \cdot e^{\theta + i\Phi} + \frac{\partial u}{\partial y} \cdot e^{\theta - i\Phi}$$

$$\therefore \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot x + \frac{\partial u}{\partial y} \cdot y \quad \text{.................... (1)}$$

$$\therefore \quad \frac{\partial}{\partial \theta} \equiv x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \quad \text{.................... (2)}$$

$$\therefore \quad \frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial \theta}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \quad \text{[From (1)]}$$

$$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) \quad \text{[From (2)]}$$

$$= x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \quad \text{.................... (3)}$$

Also $$\frac{\partial u}{\partial \Phi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \Phi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \Phi} = \frac{\partial u}{\partial x} \cdot e^{\theta + i\Phi} \cdot i + \frac{\partial u}{\partial y} \cdot e^{\theta - i\Phi} \cdot (-i)$$

$$\therefore \quad \frac{\partial u}{\partial \Phi} = i\frac{\partial u}{\partial x} \cdot x - i\frac{\partial u}{\partial y} \cdot y \quad \text{.................... (4)}$$

$$\therefore \quad \frac{\partial}{\partial \Phi} \equiv i\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) \quad \text{.................... (5)}$$

$$\therefore \quad \frac{\partial^2 u}{\partial \Phi^2} = \frac{\partial}{\partial \Phi} \cdot \left(\frac{\partial u}{\partial \Phi}\right) = \frac{\partial}{\partial \Phi}\left[i\left(x\frac{\partial u}{\partial x} - u\frac{\partial u}{\partial y}\right)\right] \quad \text{[From (4)]}$$

$$= i\left[i\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y}\right)\right] \quad \text{[From (5)]}$$

$$= -\left[x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$\therefore \quad \frac{\partial^2 u}{\partial \Phi^2} = -x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} - y^2 \frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} \qquad \text{.................... (6)}$$

Adding the two results, (3) and (6), we get,

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \Phi^2} = 4xy\frac{\partial^2 u}{\partial x \partial y}.$$

Example 6 (c) : If $f(x, y) = \Phi(u, v)$ where $u = x - y$ and $v = xy$, prove that

$$x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y^2} = (x+y)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$

Sol. : We have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\cdot 1 + \frac{\partial f}{\partial v}\cdot y$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u} + u\frac{\partial f}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v}\right)\cdot\frac{\partial v}{\partial x} \qquad \textbf{[Note this]}$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = \left(\frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial u \partial v}\right)\cdot 1 + \left(\frac{\partial^2 f}{\partial v \partial u} + y\frac{\partial^2 f}{\partial v^2}\right)\cdot y \qquad \text{.................... (1)}$$

Further, $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}(-1) + \frac{\partial f}{\partial v}\cdot x$

$$\therefore \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}\right)\cdot\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial u} + x\frac{\partial f}{\partial v}\right)\cdot\frac{\partial v}{\partial y} \qquad \textbf{[Note this]}$$

$$\therefore \quad \frac{\partial^2 f}{\partial y^2} = \left(-\frac{\partial^2 f}{\partial u^2} + x\frac{\partial^2 f}{\partial u \partial v}\right)\cdot(-1) + \left(-\frac{\partial^2 f}{\partial v \partial u} + x\frac{\partial^2 f}{\partial v^2}\right)\cdot x \qquad \text{.................... (2)}$$

Multiply (1) by x and (2) by y and add.

$$\therefore \quad x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y^2} = x\frac{\partial^2 f}{\partial u^2} + xy\frac{\partial^2 f}{\partial u \partial v} + xy\frac{\partial^2 f}{\partial u \partial v}$$

$$+ xy^2\frac{\partial^2 f}{\partial v^2} + y\frac{\partial^2 f}{\partial u^2} - xy\frac{\partial^2 f}{\partial u \partial v} - xy\frac{\partial^2 f}{\partial u \partial v} + x^2 y\frac{\partial^2 f}{\partial v^2}$$

$$= (x+y)\frac{\partial^2 f}{\partial u^2} + xy(x+y)\frac{\partial^2 f}{\partial v^2}$$

$$\therefore \quad x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial y^2} = (x+y)\left(\frac{\partial^2 f}{\partial u^2} + xy\frac{\partial^2 f}{\partial v^2}\right).$$

Example 7 (c) : If z is a function of x and y and u, v are two other variables such that $u = l\,x + my$, $v = l\,y - mx$ then show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

Sol. : We have $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}$

But $u=l\,x+my$ $\quad\therefore\ \dfrac{\partial u}{\partial x}=l,\ \dfrac{\partial u}{\partial y}=m$

and $v=l\,y-mx$ $\quad\therefore\ \dfrac{\partial v}{\partial x}=-m,\ \dfrac{\partial v}{\partial y}=l$ $\quad\therefore\ \dfrac{\partial z}{\partial x}=l\dfrac{\partial z}{\partial u}-m\dfrac{\partial z}{\partial v}.$

Differentiating again,

$$\frac{\partial^2 z}{\partial x^2}=l\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]-m\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=l\left[\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v\,\partial u}\cdot\frac{\partial v}{\partial x}\right]-m\left[\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right]$$

$$=l\left[\frac{\partial^2 z}{\partial u^2}\cdot(l)+\frac{\partial^2 z}{\partial v\,\partial u}(-m)\right]-m\left[\frac{\partial^2 z}{\partial u\,\partial v}(l)+\frac{\partial^2 z}{\partial v^2}(-m)\right]$$

$$\therefore\ \frac{\partial^2 z}{\partial x^2}=l^2\frac{\partial^2 z}{\partial u^2}-2lm\frac{\partial^2 z}{\partial u\,\partial v}+m^2\frac{\partial^2 z}{\partial v^2}\qquad\text{.................... (1)}$$

Now, $\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}=m\dfrac{\partial z}{\partial u}+l\dfrac{\partial z}{\partial v}$

$$\therefore\ \frac{\partial^2 z}{\partial y^2}=m\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+l\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=m\left[\frac{\partial^2 z}{\partial u^2}(m)+\frac{\partial^2 z}{\partial u\,\partial v}(l)\right]+l\left[\frac{\partial^2 z}{\partial u\,\partial v}(m)+\frac{\partial^2 z}{\partial v^2}(l)\right]$$

$$=m^2\frac{\partial^2 z}{\partial u^2}+2lm\frac{\partial^2 z}{\partial u\,\partial v}+l^2\frac{\partial^2 z}{\partial v^2}\qquad\text{.................... (2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=(l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial y^2}\right).$$

Example 8 (c) : If $f(x, y)=\Phi(u, v)$ and $u=x^2-y^2,\ v=2xy$, prove that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right).\qquad\textbf{(M.U. 1995, 2015)}$$

Sol. : We have

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial f}{\partial u}\cdot 2x+\frac{\partial f}{\partial v}\cdot 2y=2\left(x\frac{\partial f}{\partial u}+y\frac{\partial f}{\partial v}\right)$$

$$\therefore\ \frac{\partial^2 f}{\partial x^2}=2\left[\left\{x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)+\frac{\partial f}{\partial u}\right\}+\left\{y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)\right\}\right]$$

$$\therefore\ \frac{\partial^2 f}{\partial x^2}=2\left[\left\{x\left(\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial u\,\partial v}\cdot\frac{\partial v}{\partial x}\right)+\frac{\partial f}{\partial u}\right\}+\left\{y\left(\frac{\partial^2 f}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right)\right\}\right]$$

[Note this]

$$= 2\left[\left\{x\left(\frac{\partial^2 f}{\partial x^2}\cdot 2x + \frac{\partial^2 f}{\partial u\,\partial v}\cdot 2y\right) + \frac{\partial f}{\partial u}\right\} + \left\{y\left(\frac{\partial^2 f}{\partial u\,\partial v}\cdot 2x + \frac{\partial^2 f}{\partial v^2}\cdot 2y\right)\right\}\right]$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2 f}{\partial u^2} + 8xy\frac{\partial^2 f}{\partial u\,\partial v} + 4y^2\frac{\partial^2 f}{\partial v^2} + 2\frac{\partial f}{\partial u} \qquad \text{.................... (1)}$$

Further, $\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\cdot\dfrac{\partial v}{\partial y} = \dfrac{\partial f}{\partial u}(-2y) + \dfrac{\partial f}{\partial v}(2x) = 2\left(x\dfrac{\partial f}{\partial v} - y\dfrac{\partial f}{\partial u}\right)$

$$\therefore \quad \frac{\partial^2 f}{\partial y^2} = 2\left[\left\{x\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right)\right\} - \left\{y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\right) + \frac{\partial f}{\partial u}\right\}\right]$$

$$= 2\left[\left\{x\left(\frac{\partial^2 f}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\cdot\frac{\partial v}{\partial y}\right)\right\} - \left\{y\left(\frac{\partial^2 f}{\partial u^2}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial u\,\partial v}\cdot\frac{\partial v}{\partial y}\right) + \frac{\partial f}{\partial u}\right\}\right]$$

[Note this]

$$= 2\left[x\left\{-2y\frac{\partial^2 f}{\partial u\,\partial v} + 2x\frac{\partial^2 f}{\partial v^2}\right\} - \left\{y\left(-2y\frac{\partial^2 f}{\partial u^2} + 2x\frac{\partial^2 f}{\partial u\,\partial v}\right) + \frac{\partial f}{\partial u}\right\}\right]$$

$$\therefore \quad \frac{\partial^2 f}{\partial y^2} = 4y^2\frac{\partial^2 f}{\partial u^2} - 8xy\frac{\partial^2 f}{\partial u\,\partial v} + 4x^2\frac{\partial^2 f}{\partial v^2} - 2\frac{\partial f}{\partial u} \qquad \text{.................... (2)}$$

Adding the two results (1) and (2), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\frac{\partial^2 f}{\partial u^2} + 4(x^2 + y^2)\frac{\partial^2 f}{\partial v^2}$$

$$\therefore \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2 + y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Deduction : Since $u^2 + v^2 = (x^2 - y^2) + 4x^2 y^2 = (x^2 + y^2)^2$

$\therefore \quad x^2 + y^2 = (u^2 + v^2)^{1/2}$.

From the above result, we get,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(u^2 + v^2)^{1/2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$

(M.U. 2015)

EXERCISE - VI

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

1. If $z = f(u, v)$, $u = x^2 - y^2$, $v = y^2 - x^2$, prove that $x\dfrac{\partial z}{\partial y} + y\dfrac{\partial z}{\partial x} = 0$.

2. If $x^2 = au + bv$, $y^2 = au - bv$ and $z = f(x, y)$, prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

(M.U. 1998)

3. If $w = \Phi(u, v)$, $u = x^2 - y^2 - 2xy$, $v = y$, prove that $\dfrac{\partial w}{\partial v} = 0$ is equivalent to

$$(x + y)\frac{\partial w}{\partial x} + (x - y)\frac{\partial w}{\partial y} = 0.$$

(M.U. 2006)

4. If $z = u^2 + v^2$, $x = u^2 - v^2$, $y = uv$, prove that $\dfrac{\partial z}{\partial x} = \dfrac{x}{z}$.

5. If $z = f(u, v)$, $u = l\,x + my$, $v = l\,y - mx$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right).$$ **(M.U. 1997)**

6. If $z = f(x, y)$ and $x = e^u \sec v$, $y = e^u \tan v$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \cos^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$

7. If $x = e^u \operatorname{cosec} v$, $y = e^u \cot v$ and Φ is a function of x and y, prove that

$$\left(\frac{\partial \Phi}{\partial x}\right)^2 - \left(\frac{\partial \Phi}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial \Phi}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial \Phi}{\partial u}\right)^2\right]$$

8. If $u = x^2 - y^2$, $v = 2xy$ and $z = f(u, v)$, prove that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\sqrt{u^2 + v^2}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$$ **(M.U. 2015)**

9. If $z = f(x, y)$, $u = e^x$, $v = e^y$, prove that $\dfrac{\partial^2 z}{\partial x \partial y} = uv\dfrac{\partial^2 z}{\partial u \partial v}$.

10. If $u = \log(x^2 + y^2)$, $v = \dfrac{y}{x}$, prove that $x\dfrac{\partial z}{\partial y} - y\dfrac{\partial z}{\partial x} = (1 + v^2)\dfrac{\partial z}{\partial v}$. **(M.U. 1992)**

11. If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that

$$\frac{1}{x}\cdot\frac{\partial u}{\partial x} + \frac{1}{y}\cdot\frac{\partial u}{\partial y} + \frac{1}{z}\cdot\frac{\partial u}{\partial z} = 0.$$ **(M.U. 1992, 2003)**

12. If $u = f(x^n - y^n, y^n - z^n, z^n - x^n)$, prove that

$$\frac{1}{x^{n-1}}\cdot\frac{\partial u}{\partial x} + \frac{1}{y^{n-1}}\cdot\frac{\partial u}{\partial y} + \frac{1}{z^{n-1}}\cdot\frac{\partial u}{\partial z} = 0.$$

13. If $z = f(x, y)$, $x = uv$, $y = \dfrac{u + v}{u - v}$, prove that $u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v} = 2x\dfrac{\partial z}{\partial x}$.

14. If $u = x^2 + y^2$, $v = 2xy$, $z = f(u, v)$, prove that $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2\sqrt{u^2 - v^2}\cdot\dfrac{\partial z}{\partial u}$. **(M.U. 2015)**

15. If $u = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2.$$

16. If $z = f(x, y)$, $x = u + v$, $y = uv$, prove that

$$\frac{\partial^2 z}{\partial x^2} - y\frac{\partial^2 z}{\partial y^2} = \frac{1}{u - v}\left(u\frac{\partial^2 z}{\partial u^2} - v\frac{\partial^2 z}{\partial v^2}\right).$$

17. If $x = u + v$, $y = uv$ and V is a function of x, y, prove that

$$\frac{\partial^2 V}{\partial u^2} - 2\frac{\partial^2 V}{\partial u \partial v} + \frac{\partial^2 V}{\partial u^2} = (x^2 - 4y)\frac{\partial^2 V}{\partial y^2} - 2\frac{\partial V}{\partial y}.$$

Class (c) : 8 Marks

1. If $x = e^v \sec u$, $y = e^v \tan u$ and Φ is a function of x and y, prove that

$$\cos u\left(\frac{\partial^2\Phi}{\partial u\,\partial v} - \frac{\partial\Phi}{\partial u}\right) = xy\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + (x^2 + y^2)\frac{\partial^2\Phi}{\partial x\,\partial y}.$$

2. If $x = \cos h\,\xi \cos\eta$, $y = \sin h\,\xi \sin\eta$, and u is a function of x, y,

prove that $\dfrac{\partial^2 u}{\partial\xi^2} + \dfrac{\partial^2 u}{\partial\eta^2} = (\sin h^2\xi + \sin^2\eta)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$.

3. If $u = f(x, y, z)$, $x = r\sin\theta\cos\Phi$, $y = r\sin\theta\sin\Phi$, $z = r\cos\theta$, prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial f}{\partial\Phi}\right)^2.$$

EXERCISE - VII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (a) : 3 Marks

1. If $z = f(x, y)$ is a differentiable function, define $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

2. If $z = f(u, v)$, $u = \Phi(x, y)$ and $v = \Psi(x, y)$ are differentiable functions, state the formulae for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

EXERCISE - VIII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Short Answer Questions : Class (a) : 3 Marks

1. If $u = xyz$, find $\dfrac{\partial^3 u}{\partial x\,\partial y\,\partial z}$. [Ans. : 1]

2. If $z = x^y$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. [Ans. : $y\,x^{y-1}$, $x^y\log x$]

3. If $z = y^x$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. [Ans. : $y^x\log y$, xy^{x-1}]

4. If $z = \sin^{-1}\left(\dfrac{x}{y}\right)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. $\left[\text{Ans. : } \dfrac{1}{\sqrt{y^2 - x^2}};\ \dfrac{1}{\sqrt{y^2 - x^2}}\left(-\dfrac{x}{y}\right)\right]$

5. If $z = \tan^{-1}\left(\dfrac{x}{y}\right)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. $\left[\text{Ans. : } \dfrac{y}{x^2 + y^2};\ -\dfrac{x}{x^2 + y^2}\right]$

6. If $z = e^{xy}$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. [Ans. : $y\,e^{xy}$, $x\,e^{xy}$]

7. If $z = e^{x+y} f(x - y)$, find $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$. [Ans. : $2z$]

8. If $z = \log(x^2 + y^2)$, find $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$. [Ans. : 2]

9. If $z = \sqrt{x^2 + y^2}$, find $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$. [Ans. : z]

10. If $z = f(r)$ and $r = \sqrt{x^2 + y^2}$, find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$. [**Ans. :** $r f'(r)$]

11. If $z = x^2 + y^2$, $x = at$, $y = bt$, find $\frac{dz}{dt}$. [**Ans. :** $2(ax + by)$]

12. If $z = e^{x+y}$, $x = 2t$, $y = 3t$, find $\frac{dz}{dt}$. [**Ans. :** $5z$]

13. If $z = x^2 + y^2$, $x = \sin t$, $y = \cos t$, find $\frac{dz}{dt}$. [**Ans. :** 0]

14. If $z = u + v$, $u = x + y$, $v = x - y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. [**Ans. :** 2, 0]

15. If $z = u^2 + v^2$, $u = 2x + 3y$, $v = 3x + 2y$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. [**Ans. :** $4u + 6v$, $6u + 4v$]

16. If $u = x^2 + y^2$, find $\nabla^2 u$. [**Ans. :** 4]

17. If $u = x^2 + y^2 + z^2$, find $\nabla^2 u$. [**Ans. :** 6]

18. If $e^{x+y} = k$, find $\frac{dy}{dx}$. [**Ans. :** -1]

19. If $x^x + y^y = k$, find $\frac{dy}{dx}$ at $x = y$. [**Ans. :** -1]

20. If $z = x^y$, find $\frac{\partial^2 z}{\partial x \partial y}$ at (1, 2). [**Ans. :** 1]

21. If $z = x^y$, find $\frac{\partial^2 z}{\partial x^2}$. [**Ans. :** $y(y-1)x^{y-2}$]

22. If $z = x^y$, find $\frac{\partial^2 z}{\partial y^2}$. [**Ans. :** $x^y (\log x)^2$]

23. If $z = e^{x^2+y^2}$, find $\frac{\partial^2 z}{\partial x \partial y}$ at (1, 1). [**Ans. :** $4e^2$]

24. If $z = \frac{x}{y} + \frac{y}{x}$, find $\frac{\partial^2 z}{\partial x \partial y}$. [**Ans. :** $-\frac{1}{x^2} - \frac{1}{y^2}$]

25. If $z = e^{\sin(x/y)}$, find $\frac{\partial^2 z}{\partial x^2}$ at $\left(\frac{\pi}{2}, 1\right)$. [**Ans. :** e^{-1}]

26. If $x = r\cos\theta$, $y = r\sin\theta$, find $\left(\frac{\partial r}{\partial x}\right)_y$. [**Ans. :** $\frac{x}{r}$]

27. If $x = r\cos\theta$, $y = r\sin\theta$, find $\left(\frac{\partial r}{\partial y}\right)_x$. [**Ans. :** $\frac{y}{r}$]

28. If $u = f(x + ay) + \Phi(x - ay)$, find the value of $a^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$. [**Ans. :** 0]

29. If $z = \log(x \tan^{-1} y)$, find the value of $\frac{\partial^2 z}{\partial x \partial y}$. [**Ans. :** 0]

30. If $u = f(r, s)$, $r = x + y$, $s = x - y$, find $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$. [**Ans. :** $2\frac{\partial u}{\partial r}$]

Summary

1. If $z = f(u)$ and $u = \Phi(x, y)$ then

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = f'(u)\frac{\partial u}{\partial x}; \qquad \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

2. If $z = f(x, y)$, $x = \Phi(t)$, $y = \Psi(t)$ then $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

3. If $z = f(u, v)$, $u = \Phi(x, y)$, $v = \Psi(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}; \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

4. If $f(x, y) = 0$, then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

5. If $f(x, y, z) = 0$, then $\frac{\partial z}{\partial x} = -\frac{f_x}{f_z}$, $\quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}$.

☆☆☆

CHAPTER 6

Homogeneous Functions

1. Introduction

In this chapter we shall learn a very important and elegant theorem due to Euler. We shall also learn a few deductions from it and also how to apply them to obtain partial derivatives of functions apparently in complex form in a very simple way.

2. Homogeneous Functions

A function $f(x, y, z)$ is called a **homogeneous function of degree n** if by putting $X = xt$, $Y = yt$, $Z = zt$ the function $f(X, Y, Z)$ becomes $t^n \cdot f(x, y, z)$ i.e. $f(xt, yt, zt) = t^n f(x, y, z)$.

For example, $f(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$ is a homogeneous function of degree 2.

For, by putting $X = xt$, $Y = yt$, $Z = zt$, we get,

$$\begin{aligned} f(X, Y, Z) &= X^2 + Y^2 + Z^2 + 2XY + 2YZ + 2ZX \\ &= x^2t^2 + y^2t^2 + z^2t^2 + 2xyt^2 + 2yzt^2 + 2zxt^2 \\ &= t^2 \cdot (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) \\ &= t^2 f(x, y, z). \end{aligned}$$

Also $g(x, y, z) = x^3 \sin(y/x) + xyz \log(x/y) + xz^2 \tan^{-1}(z/y)$ is a homogeneous function of degree 3.

For, by putting $X = xt$, $Y = yt$, $Z = zt$, we get

$$\begin{aligned} g(X, Y, Z) &= X^3 \sin\left(\frac{Y}{X}\right) + XYZ \log\left(\frac{X}{Y}\right) + XZ^2 \tan^{-1}\left(\frac{Z}{Y}\right) \\ &= x^3t^3 \sin\left(\frac{yt}{xt}\right) + xyzt^3 \log\left(\frac{xt}{yt}\right) + xz^2t^3 \tan^{-1}\left(\frac{zt}{yt}\right) \\ &= t^3 \cdot \left[x^3 \sin\left(\frac{y}{x}\right) + xyz \log\left(\frac{x}{y}\right) + xz^2 \tan^{-1}\left(\frac{z}{y}\right)\right] \\ &= t^3 g(x, y, z) \end{aligned}$$

Notice that $f(x, y, z)$ is an algebraic polynomial while $g(x, y, z)$ is not an algebraic polynomial and still both are homogeneous functions.

But $f(x, y, z) = \log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$, $g(x, y, z) = \sin^{-1}\left(\dfrac{x^2 + y^2}{x^3 + y^3}\right)$ are not homogeneous functions because

$$f(X, Y, Z) = \log\left(\frac{x^3 t^3 + y^3 t^3}{x^2 t^2 + y^2 t^2}\right) = \log\left[t \cdot \left(\frac{x^3 + y^3}{x^2 + y^2}\right)\right] \neq t^n f(x, y, z) \quad \text{for any } n$$

and $$g(X, Y, Z) = \sin^{-1}\left[\frac{x^2 t^2 + y^2 t^2}{x^3 t^3 + y^3 t^3}\right] = \sin^{-1}\left[t^{-1} \cdot \left(\frac{x^2 + y^2}{x^3 + y^3}\right)\right] \neq t^n g(x, y, z) \quad \text{for any } n$$

3. Euler's Theorem

(M.U. 2016)

If z is a homogeneous function of two variables x and y of degree n then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$ (M.U. 2012, 13, 14, 15, 16)

Proof : Let $z = f(x, y)$ be the given function. Since it is a homogeneous function of degree n, on putting $X = x t$, $Y = y t$, we get

$$f(X, Y) = t^n f(x, y) \qquad \text{.................... (i)}$$

Differentiating l.h.s. w.r.t. t,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial X} \cdot \frac{\partial X}{\partial t} + \frac{\partial f}{\partial Y} \cdot \frac{\partial Y}{\partial t} = x\frac{\partial f}{\partial X} + y\frac{\partial f}{\partial Y} \qquad \left[\because \frac{\partial X}{\partial t} = x, \frac{\partial Y}{\partial t} = y\right]$$

If we put $t = 1$, *i.e.* $X = x$, $Y = y$, we get,

$$\frac{\partial f}{\partial t} = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \qquad \text{.................... (ii)}$$

Differentiating r.h.s. of (i) w.r.t. t, we get,

$$\frac{\partial f}{\partial t} = n t^{n-1} f(x, y)$$

If we put $t = 1$, we get $\dfrac{\partial f}{\partial t} = n f(x, y)$ .... (iii)

From (ii) and (iii), we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \quad \text{i.e.} \quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nz$$

Generalisation : If u is a homogeneous function of $x, y, z, \ldots\ldots t$ of degree n then Euler's Theorem states that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + \ldots\ldots + t\frac{\partial u}{\partial t} = nu.$$

Cor : If $u = f\left(\dfrac{y}{x}\right)$, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Proof : Since $n = 0$, the result follows.

Note ....

Before applying Euler's Theorem, it is absolutely necessary to see that the function is homogeneous. If the function is not homogeneous Euler's theorem cannot be applied.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Explain whether we can apply Euler's theorem for the function

$$u = f(x, y) = \frac{x^2 + y^2 + 1}{x + y}$$

Sol. : The numerator of u is not homogeneous because the degree of a constant is zero. Hence, Euler's Theorem cannot be applied.

Example 2 (a) : If $u = \frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}}$, then verify Euler's theorem for u. **(M.U. 1992, 2012, 14)**

Sol. : (a) We have $u = \sqrt{y} \cdot \left(\frac{\sqrt{x}}{\sqrt{x} + \sqrt{y}}\right)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\sqrt{y}\left[\left(\sqrt{x} + \sqrt{y}\right) \cdot \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2\sqrt{x}}\right]}{\left(\sqrt{x} + \sqrt{y}\right)^2}$$

$$= \frac{\sqrt{y} \cdot \sqrt{y}}{2\sqrt{x}\left(\sqrt{x} + \sqrt{y}\right)^2} = \frac{y}{2\sqrt{x} \cdot \left(\sqrt{x} + \sqrt{y}\right)^2}$$

By symmetry, (changing x to y and y to x)

$$\frac{\partial u}{\partial y} = \frac{x}{2\sqrt{y}\left(\sqrt{x} + \sqrt{y}\right)^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x \cdot y}{2\sqrt{x}\left(\sqrt{x} + \sqrt{y}\right)^2} + \frac{x \cdot y}{2\sqrt{y}\left(\sqrt{x} + \sqrt{y}\right)^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2\left(\sqrt{x} + \sqrt{y}\right)^2}\left[\sqrt{x} \cdot y + \sqrt{y} \cdot x\right]$$

$$= \frac{\sqrt{xy}}{2\left(\sqrt{x} + \sqrt{y}\right)^2} \cdot \left(\sqrt{x} + \sqrt{y}\right) = \frac{\sqrt{xy}}{2\left(\sqrt{x} + \sqrt{y}\right)}.$$

(b) Putting $X = xt$, $Y = yt$, we get,

$$f(X, Y) = \frac{\sqrt{XY}}{\sqrt{X} + \sqrt{Y}} = \frac{\sqrt{xt \cdot yt}}{\sqrt{xt} + \sqrt{yt}}$$

$$= t^{1/2}\frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}} = t^{1/2} f(x, y)$$

Thus, u is a homogeneous function of degree 1/2.

Hence, by Euler's Theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}u = \frac{1}{2} \cdot \frac{\sqrt{xy}}{\sqrt{x} + \sqrt{y}}.$$

From (a) and (b), Euler's theorem is verified.

Example 3 (a) : If $u = \sin^{-1}\left(\frac{x}{y}\right) + \cos^{-1}\left(\frac{y}{x}\right)$ then find $\frac{\partial u / \partial x}{\partial u / \partial y}$.

Sol. : Putting $X = xt$, $Y = yt$, we see that

$$f(X, Y) = \sin^{-1}\frac{X}{Y} + \cos^{-1}\frac{Y}{X} = \sin^{-1}\left(\frac{xt}{yt}\right) + \cos^{-1}\left(\frac{yt}{xt}\right)$$

$$= \sin^{-1}\frac{x}{y} + \cos^{-1}\frac{y}{x} = t^0 f(x, y)$$

$\therefore$ u is a homogeneous function of degree zero.

Hence, by Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \cdot u = 0$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -y\frac{\partial u}{\partial y} \qquad \therefore \quad \frac{\partial u / \partial x}{\partial u / \partial y} = -\frac{y}{x}.$$

Example 4 (a) : If $u = \log\frac{x}{y} + \log\frac{y}{x}$ find $\frac{\partial u / \partial x}{\partial u / \partial y}$.

Sol. : u is homogeneous function of x, y of degree zero.

Hence, by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \cdot u = 0 \qquad \therefore \quad \frac{\partial u / \partial x}{\partial u / \partial y} = -\frac{y}{x}.$$

Example 5 (a) : If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Sol. : Putting $X = x\,t$, $Y = y\,t$

$$f(X, Y) = \sin^{-1}\frac{X}{Y} + \tan^{-1}\frac{Y}{X} = \sin^{-1}\left(\frac{xt}{yt}\right) + \tan^{-1}\left(\frac{yt}{xt}\right)$$

$$\therefore \quad f(X, Y) = t^0\left[\sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}\right] = t^0 f(x, y)$$

Thus, u is a homogeneous function of degree zero.

Hence, by Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \cdot u = 0.$

Example 6 (a) : If $u = \frac{x+y}{x^2+y^2}$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. **(M.U. 2018)**

Sol. : Putting $X = x\,t$, $Y = y\,t$

$$f(X, Y) = \frac{X+Y}{X^2+Y^2} = \frac{xt+yt}{x^2t^2+y^2t^2} = \frac{1}{t}\left(\frac{x+y}{x^2+y^2}\right) = t^{-1} f(x, y)$$

Thus, u is a homogeneous function of x, y of degree -1.

Hence, by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n(u) = -1(u) = -u.$$

Example 7 (a) : If $u = x^2 \log\left[\dfrac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right]$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = 2x^2\log\left[\dfrac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right]$.

Sol. : Putting $X = xt$, $Y = yt$, we get

$$f(X,Y) = X^2\log\left[\frac{\sqrt[3]{Y}-\sqrt[3]{X}}{\sqrt[3]{Y}+\sqrt[3]{X}}\right] = x^2t^2\log\left[\frac{\sqrt[3]{yt}-\sqrt[3]{xt}}{\sqrt[3]{yt}+\sqrt[3]{xt}}\right]$$

$$\therefore\quad f(X,Y) = t^2\left\{x^2\log\left[\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right]\right\} = t^2\cdot f(x,y)$$

Thus, u is a homogeneous function of degree two.

Hence, by Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2u = 2x^2\log\left[\frac{\sqrt[3]{y}-\sqrt[3]{x}}{\sqrt[3]{y}+\sqrt[3]{x}}\right].$$

Example 8 (a) : If $u = \dfrac{\sqrt{x}+\sqrt{y}}{x+y}$, find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$. **(M.U. 2013)**

Sol. : Putting $X = xt$, $Y = yt$, we see that

$$f(X,Y) = \frac{\sqrt{X}+\sqrt{Y}}{X+Y} = \frac{\sqrt{xt}+\sqrt{yt}}{xt+yt}$$

$$= \frac{\sqrt{t}\,(\sqrt{x}+\sqrt{y})}{t\,(x+y)} = \frac{1}{\sqrt{t}}\cdot f(x,y)$$

$\therefore$ u is a homogeneous function of degree $-1/2$.

Hence, by Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = -\frac{1}{2}u = -\frac{1}{2}\left(\frac{\sqrt{x}+\sqrt{y}}{x+y}\right)$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : If $x = e^u\tan v$, $y = e^u\sec v$, prove that

$$\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\right) = 0$$ **(M.U. 2015)**

Sol. : We first see that in the first bracket Euler's theorem is applied to u and in the second bracket Euler's theorem is applied to v.

We have $\quad y^2-x^2 = e^{2u}\sec^2 v - e^{2u}\tan^2 v$

$$= e^{2u}(\sec^2 v-\tan^2 v) = e^{2u}$$

$$\therefore\quad 2u = \log(y^2-x^2)\qquad\therefore\quad u = \frac{1}{2}\log(y^2-x^2)$$

Now, $\quad f(X,Y,Z) = \dfrac{1}{2}\log(t^2y^2-t^2x^2) = \dfrac{1}{2}\log t^2(y^2-x^2)$

$$\neq t^n f(x,y,z)$$

However, the modified Euler's theorem given in Cor. 2, page 6-17 can be applied.

Here, $n = 2$, $f(u) = e^{2u}$ $\therefore$ $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)} = 2 \cdot \dfrac{e^{2u}}{2e^{2u}} = 1.$

But $\dfrac{x}{y} = \dfrac{e^u \tan v}{e^u \sec v} = \sin v$ $\therefore$ $v = \sin^{-1}\left(\dfrac{x}{y}\right)$

$\therefore$ $f(X, Y, Z) = \sin^{-1}\left(\dfrac{xt}{yt}\right) = \sin^{-1}\left(\dfrac{x}{y}\right) = t^0 f(x, y)$

Hence, v is a homogeneous function of degree $n = 0$ and Euler's theorem can be applied

$\therefore$ $\left(x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y}\right) = 0 \cdot v = 0$

Hence, $\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right)\left(x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y}\right) = 0$

(For another method, see solved Ex. 3, page 5-29.)

Example 2 (b) : If $u = \dfrac{f(\theta)}{r}$, $x = r\cos\theta$, $y = r\sin\theta$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -u.$

(M.U. 1992)

Sol. : We first note that $\theta = \tan^{-1}\dfrac{y}{x}$ and $r = \sqrt{x^2 + y^2}$.

$$\therefore \quad u = \frac{f(\theta)}{r} = \frac{f\left[\tan^{-1}(y/x)\right]}{\sqrt{x^2 + y^2}}$$

Putting $X = xt$, $Y = yt$, we have

$$u = \frac{f[\tan^{-1}(Y/X)]}{\sqrt{X^2 + Y^2}} = \frac{f[\tan^{-1}(yt/xt)]}{t\sqrt{x^2 + y^2}} = t^{-1}\frac{f[\tan^{-1}(y/x)]}{\sqrt{x^2 + y^2}}$$

$\therefore$ u is a homogeneous function of degree -1.

Hence, by Euler's Theorem $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu = -u.$

Example 3 (b) : If $u = f\left(\dfrac{y}{x}\right) + \sqrt{x^2 + y^2}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sqrt{x^2 + y^2}$.

Sol. : Let $v = f\left(\dfrac{y}{x}\right)$ and $w = \sqrt{x^2 + y^2}$. $\therefore$ $u = v + w$

Putting $X = xt$, $Y = yt$ in v, we get

$$f_1(X, Y) = f_1\left(\frac{yt}{xt}\right) = t^0 f_1\left(\frac{y}{x}\right).$$

Thus, v is a homogeneous function of degree 0.

Similarly, putting $X = xt$, $Y = yt$ in w, we get

$$f_2(X, Y) = \sqrt{X^2 + Y^2} = \sqrt{x^2 t^2 + y^2 t^2} = t\sqrt{x^2 + y^2} = t\, f_2(x, y)$$

Thus, w is a homogeneous function of degree 1.

Hence, by Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0\cdot v=0; \qquad x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=1\cdot w=w$$

By adding the two results, we get,

$$\therefore \quad x\left(\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}\right)=w \qquad \therefore \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sqrt{x^2+y^2}.$$

Example 4 (b) : If $z=\log(x^2+y^2)+\dfrac{x^2+y^2}{x+y}-2\log(x+y)$, prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{x^2+y^2}{x+y}.$$

(M.U. 2012)

Sol. : We have $\quad z=\dfrac{x^2+y^2}{x+y}+\log\left(\dfrac{x^2+y^2}{(x+y)^2}\right)$ **[Note this]**

Let $\quad u=\dfrac{x^2+y^2}{x+y}$ and $v=\log\left[\dfrac{x^2+y^2}{(x+y)^2}\right] \qquad \therefore \quad z=u+v$

Putting $X=xt,\ Y=yt$ in u, we get

$$u=f_1(X,Y)=\frac{X^2+Y^2}{X+Y}=\frac{x^2t^2+y^2t^2}{xt+yt}=t\,\frac{(x^2+y^2)}{(x+y)}=t\,f_1(x,y)$$

Thus, u is a homogeneous function of x, y of degree 1.

Similarly, putting $X=xt,\ Y=yt$ in v, we get,

$$v=f_2(X,Y)=\log\left[\frac{X^2+Y^2}{(X+Y)^2}\right]=\log\left[\frac{x^2t^2+y^2t^2}{(xt+yt)^2}\right]=t^0\log\left[\frac{x^2+y^2}{(x+y)^2}\right]$$

Thus, v is a homogeneous function of x, y of degree zero.

Hence, by Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1\cdot u=u; \qquad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0\cdot v=0$$

Adding the two results, we get

$$x\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)+y\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\right)=u \qquad \therefore \quad x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{x^2+y^2}{x+y}.$$

Example 5 (b) : If $u=\dfrac{x^2y^2}{x^2+y^2}+\cos\left(\dfrac{xy}{x^2+y^2}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\left(\dfrac{x^2y^2}{x^2+y^2}\right)$.

Sol. : Let $v=\dfrac{x^2y^2}{x^2+y^2}$ and $w=\cos\left(\dfrac{xy}{x^2+y^2}\right) \qquad \therefore \quad u=v+w$

Putting $X=xt,\ Y=yt$, in v, we get

$$f_1(X,Y)=\frac{X^2Y^2}{X^2+Y^2}=\frac{x^2t^2\cdot y^2t^2}{x^2t^2+y^2t^2}=\frac{t^4x^2y^2}{t^2(x^2+y^2)}$$

$$=t^2\left(\frac{x^2y^2}{x^2+y^2}\right)=t^2f_1(x,y)$$

Thus, v is homogeneous function of degree 2.

Putting $X = xt$, $Y = yt$, in w, we get,

$$f_2(X, Y) = \cos\left(\frac{XY}{X^2+Y^2}\right) = \cos\left(\frac{xt \cdot yt}{x^2t^2+y^2t^2}\right)$$

$$= \cos\left(\frac{xy}{x^2+y^2}\right) = t^0 f_2(x, y)$$

Thus, w is a homogeneous function of degree zero.

Hence, by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2v \quad \text{and} \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$$

By adding the two results, we get,

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = 2v \quad \therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\left(\frac{x^2y^2}{x^2+y^2}\right).$$

Example 6 (b) : If $u = \dfrac{x^3y^3}{x^3+y^3} + \log\left(\dfrac{xy}{x^2+y^2}\right)$, find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

Sol. : Let $v = \dfrac{x^3y^3}{x^3+y^3}$ and $w = \log\left(\dfrac{xy}{x^2+y^2}\right)$ $\quad \therefore \quad u = v + w$

Putting $x = Xt$, $y = Yt$ in v, we get

$$f_1(X, Y) = \frac{X^3Y^3}{X^3+Y^3} = \frac{x^3t^3 \cdot y^3t^3}{x^3t^3+y^3t^3} = \frac{t^6 \cdot x^3y^3}{t^3(x^3+y^3)}$$

$$= \frac{t^3 \cdot x^3y^3}{x^3+y^3} = t^3 f_1(x, y)$$

Thus, v is homogeneous function of degree 3.

Putting $x = Xt$, $y = Yt$ in w, we get,

$$f_2(X, Y) = \log\left(\frac{XY+YZ}{X^2+Y^2}\right) = \log\left(\frac{xt \cdot yt}{x^2t^2+y^2t^2}\right)$$

$$= \log\left[\frac{t^2(xy)}{t^2(x^2+y^2)}\right] = \log\left(\frac{xy}{x^2+y^2}\right) = t^0 f_2(x, y)$$

Thus, w is a homogeneous function of degree zero.

Hence, by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3v \quad \text{and} \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0$$

By adding the two results, we get,

$$x\left(\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right) = 3v \quad \therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3 \cdot \frac{x^3y^3}{x^3+y^3}.$$

Example 7 (b) : If u is a function of a homogeneous function v, *i.e.* if $u = f(v)$ where v is a homogeneous function x, y of degree n, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n v f'(v)$$

Hence, deduce that, if $u = \log v$, $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n.$ **(M.U. 2003)**

Sol. : Since, v is a homogeneous function of x, y of degree n, by Euler's Theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv \quad \text{.................... (1)}$$

But since $u = f(v)$,

$$\frac{\partial u}{\partial x} = f'(v)\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} = f'(v)\frac{\partial v}{\partial y}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = f'(v)\left\{x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right\}$$

$$= f'(v) \cdot nv$$

$$= nv \cdot f'(v) \qquad \text{[By (1)]}$$

Now, if $u = f(v) = \log v$, $\quad f'(v) = \dfrac{1}{v}$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nv \cdot \frac{1}{v} = \frac{1}{v}.$$

Example 8 (b) : If $z = f(x, y)$ and u, v are homogeneous functions of degree n in x, y then show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = n\left(u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v}\right).$ **(M.U. 1998)**

Sol. : Since u, v are functions of x, y; x, y, themselves can be considered as functions of u, v. Therefore, we can consider z as a function of u, v and u, v as functions of x, y.

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \quad \text{.................... (1)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \quad \text{.................... (2)}$$

Multiply (1) by x and (2) by y and add.

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + \frac{\partial z}{\partial v}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) \quad \text{.................... (3)}$$

Since u and v are homogeneous functions of degree n, by Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{and} \quad x\frac{\partial v}{\partial x} + u\frac{\partial v}{\partial y} = nv$$

Hence, from (3), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nu\frac{\partial z}{\partial u} + nv\frac{\partial z}{\partial v} = n\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right).$$

Example 9 (b) : Verify Euler's theorem for $u = x^2 \tan^{-1}\dfrac{y}{x} - y^2 \tan^{-1}\dfrac{x}{y}$ and also prove that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}. \qquad \textbf{(M.U. 1990, 2008)}$$

Sol. : We have

$$\frac{\partial u}{\partial x} = 2x \cdot \tan^{-1}\frac{y}{x} + x^2 \cdot \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) - y^2 \cdot \frac{1}{1 + (x^2/y^2)}\left(\frac{1}{y}\right)$$

$$= 2x \tan^{-1}\frac{y}{x} - \frac{y \cdot x^2}{x^2+y^2} - \frac{y \cdot y^2}{x^2+y^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y \cdot \frac{(x^2+y^2)}{(x^2+y^2)} = 2x \tan^{-1}\left(\frac{y}{x}\right) - y \qquad \text{.................... (1)}$$

Further, $$\frac{\partial u}{\partial y} = x^2 \cdot \frac{1}{1+(y^2/x^2)}\left(\frac{1}{x}\right) - 2y \tan^{-1}\left(\frac{x}{y}\right) - y^2 \cdot \frac{1}{1+(x^2/y^2)}\left(-\frac{x}{y^2}\right)$$

$$= -2y \tan^{-1}\left(\frac{x}{y}\right) + x \cdot \frac{(x^2+y^2)}{(x^2+y^2)} = -2y \tan^{-1}\left(\frac{x}{y}\right) + x$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2 \tan^{-1}\frac{y}{x} - xy - 2y^2 \tan^{-1}\frac{x}{y} + xy$$

$$= 2x^2 \tan^{-1}\frac{y}{x} - 2y^2 \tan^{-1}\frac{x}{y} = 2u \qquad \text{.................... (2)}$$

Now, putting $X = xt$, $Y = yt$, we get,

$$f(X, Y) = X^2 \tan^{-1}\frac{Y}{X} - Y^2 \tan^{-1}\frac{X}{Y} = x^2 t^2 \tan^{-1}\frac{yt}{xt} - y^2 t^2 \tan^{-1}\frac{xt}{yt}$$

$$\therefore \quad f(X, Y) = t^2\left[x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y}\right] = t^2 f(x, y)$$

Hence, u is a homogeneous function of degree two.

By Euler's Theorem, we get $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u. \qquad \text{.................... (3)}$$

From (2) and (3), we see that the Euler's theorem is verified.

Now, differentiating (1) again partially w.r.t. y, we get,

$$\frac{\partial^2 u}{\partial x \partial y} = 2x \cdot \frac{1}{1+(y^2/x^2)} \cdot \left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2}$$

Solved Example : Class (c) : 8 Marks

Example 1 (c) : If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} = 1$ where u is a homogeneous function of degree n in x, y prove that $u_x^{\,2} + u_y^{\,2} + u_z^{\,2} = 2nu$. **(M.U. 2002, 07)**

Sol. : Since u is a homogeneous function of degree n, we have, by Euler's theorem

$$x\,u_x + y\,u_y = nu \qquad \text{.................... (1)}$$

Differentiating the given equation partially w.r.t. x and remembering that u is a function of x, y, z, we get,

$$\left[x^2 \cdot \frac{-1}{(a^2+u)^2} \cdot u_x + \frac{2x}{a^2+u}\right] + y^2 \cdot \frac{-1}{(b^2+u)^2} \cdot u_x = 0$$

$$\therefore \quad u_x\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2}\right] = \frac{2x}{a^2+u}$$

Let us put $$F = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} \qquad \therefore \quad u_x \cdot F = \frac{2x}{a^2+u}$$

Similarly, $u_y \cdot F = \dfrac{2y}{b^2+u}$ .... (2)

Squaring these three equalities and adding, we get

$$\therefore \quad (u_x^2 + u_y^2)\cdot F^2 = 4\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2}\right]$$

$$\therefore \quad (u_x^2 + u_y^2)\cdot F^2 = 4F$$

$$\therefore \quad u_x^2 + u_y^2 = \frac{4}{F} \qquad \text{..................... (3)}$$

Now, putting the values of u_x, u_y from (2) in (1), we get,

$$x \cdot \frac{2x}{(a^2+u)F} + y \cdot \frac{2y}{(b^2+u)F} = nu$$

$$\therefore \quad \frac{2}{F}\left[\frac{x^2}{(a^2+u)} + \frac{y^2}{(b^2+u)}\right] = nu$$

$$\therefore \quad \frac{2}{F}(1) = nu \qquad [\text{ By data }] \qquad \text{..................... (4)}$$

Hence, from (3) and (4), $\quad u_x^2 + u_y^2 = 2nu.$

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 marks

1. If $u = e^{x/y}\sin\left(\dfrac{x}{y}\right) + e^{y/x}\cos\left(\dfrac{y}{x}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$ **(M.U. 2001)**

2. If $u = \sin^{-1}\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

3. If $u = x^3\sin^{-1}\left[\dfrac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}\right]$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3x^3\sin^{-1}\left[\dfrac{\sqrt{y}+\sqrt{x}}{\sqrt{y}-\sqrt{x}}\right].$

4. If $u = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{\log x - \log y}{x^2+y^2}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + 2u = 0.$ **(M.U. 2005)**

5. If $u = x^4y^2\sin^{-1}\left(\dfrac{x}{y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 6u.$

6. If $u = \dfrac{x^2}{y}f\left(\dfrac{y}{x}\right) + \dfrac{y^2}{x}g\left(\dfrac{x}{y}\right)$, prove that $x^2\left[y\dfrac{\partial u}{\partial x} - x\cdot f\left(\dfrac{y}{x}\right)\right] + y^2\left[x\dfrac{\partial u}{\partial y} - y\cdot g\left(\dfrac{x}{y}\right)\right] = 0.$

7. If $u = \left(\dfrac{x}{y}\right)^{y/x}$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$ **(M.U. 2001)**

8. If $u = [8x^2 + y^2](\log x - \log y)$, find $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$. **(M.U. 2004)**

[**Ans. :** $2[8x^2 + y^2](\log x/y)$]

9. Verify Euler's Theorem for

(1) $u = ax^2 + 2h\,xy + by^2$ (M.U. 1995, 2002)

(2) $u = \dfrac{x(x^3 - y^3)}{(x^3 + y^3)}$ (M.U. 1997) (3) $u = \dfrac{x^2 + y^2}{x + y}$ (M.U. 1992)

(4) $u = x^4\, y^2 \sin^{-1}\left(\dfrac{y}{x}\right)$ (M.U. 1984) (5) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Cor. 1 : If z is a homogeneous function of two variables x and y of degree n then

$$\boxed{x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)\, z.}$$

Proof : Since z is a homogeneous function of degree n in x and y, by Euler's Theorem.

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz \quad \text{.................... (1)}$$

Differentiating (1) partially w.r.t. x,

$$\left(x \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot 1\right) + y \frac{\partial^2 z}{\partial x \partial y} = n \frac{\partial z}{\partial x}$$

$$\therefore \quad x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} = (n-1) \frac{\partial z}{\partial x} \quad \text{.................... (2)}$$

Differentiating (1) partially w.r.t. y,

$$x \frac{\partial^2 z}{\partial y \partial x} + y \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} = n \frac{\partial z}{\partial y}$$

$$\therefore \quad x \frac{\partial^2 z}{\partial y \partial x} + y \frac{\partial^2 z}{\partial y^2} = (n-1) \frac{\partial z}{\partial y} \quad \text{.................... (3)}$$

Multiplying (2) by x and (3) by y and adding, we get,

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (n-1)\left[x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}\right] = (n-1)\, nz \quad \text{[By (1)]}$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : If $u = \dfrac{x^3 y + y^3 x}{y - x}$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 6\left(\frac{x^3 y + y^3 x}{y - x}\right).$$

Sol. : Putting $X = xt$, $Y = yt$, we get,

$$f(X, Y) = \frac{X^3 Y + Y^3 X}{Y - X} = \frac{x^3 y t^4 + y^3 x t^4}{yt - xt}$$

$$\therefore \quad f(X, Y) = t^3\left(\frac{x^3 y + y^3 x}{y - x}\right) = t^3 \cdot f(x, y)$$

Thus, u is a homogeneous function of degree three. Hence, by the above corollary (1), page 6-12, we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)z=3\cdot 2\cdot u=6u=6\left(\frac{x^3y+y^3x}{y-x}\right)$$

Example 2 (b) : If $u=x^2\tan^{-1}\frac{y}{x}+y^2\sin^{-1}\frac{x}{y}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2u.$$

Sol. : Putting $X=xt,\ Y=yt$, we get,

$$f(X,Y)=X^2\tan^{-1}\frac{Y}{X}+Y^2\sin^{-1}\frac{X}{Y}$$

$$=x^2t^2\tan^{-1}\left(\frac{yt}{xt}\right)+y^2t^2\sin^{-1}\left(\frac{xt}{yt}\right)$$

$$=t^2\left[x^2\tan^{-1}\frac{y}{x}+y^2\sin^{-1}\frac{x}{y}\right]=t^2f(x,y)$$

Thus, u is a homogeneous function of degree two. Hence, by the above corollary (1), page 6-12, we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)z=2\cdot 1\cdot u=2u.$$

Example 3 (b) : If $z=x^n f\left(\frac{y}{x}\right)+y^n g\left(\frac{x}{y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)z.$$

Sol. : Putting $X=xt,\ Y=yt$, we get,

$$f(X,Y)=X^n f\left(\frac{Y}{X}\right)+Y^n g\left(\frac{X}{Y}\right)=x^nt^nf\left(\frac{yt}{xt}\right)+y^nt^ng\left(\frac{xt}{yt}\right)$$

$$=t^n\left[x^nf\left(\frac{y}{x}\right)+y^ng\left(\frac{x}{y}\right)\right]=t^nf(x,y)$$

Thus, z is a homogeneous function of x, y of degree n. Hence, by the above corollary (1), page, 6-12, we get

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z.$$

Note ....

Example 2 is a particular case of Example 3.

Example 4 (b) : If $u=x\Phi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$, find the value of

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}.$$ **(M.U. 1994)**

Sol. : Let $u = v + w$ where $v = x\Phi\left(\frac{y}{x}\right)$ and $w = \psi\left(\frac{y}{x}\right)$.

Since V is a homogeneous function of degree $n = 1$,

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)v = 0$$

Also W is a homogeneous function of degree $n = 0$.

$$\therefore \quad x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} = n(n-1)w = 0$$

Adding the two results, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0.$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : If $z = x^n f\left(\frac{y}{x}\right) + y^{-n} f\left(\frac{x}{y}\right)$, prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n^2 z.$$ **(M.U. 1990)**

Sol. : Let $u = x^n f\left(\frac{y}{x}\right)$ and $v = y^{-n} f\left(\frac{x}{y}\right)$ so that $z = u + v$.

Now, u and v are homogeneous functions of degree n and $-n$, hence by Euler's Theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{.................... (1)}$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -nv \qquad \text{.................... (2)}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u \qquad \text{.................... (3)}$$

and
$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = (-n)(-n-1)v$$

$$= n(n+1)v \qquad \text{.................... (4)}$$

Adding (1) and (2), we get,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] + \left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right]$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nu - nv = n(u - v) \qquad \text{.................... (5)}$$

Adding (3) and (4), we get,

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \left[x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}\right]$$
$$+ \left[x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2}\right]$$

$$\therefore \quad x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)u + n(n+1)v$$

$$= n^2(u+v) - n(u-v) \qquad \text{.......... (6)}$$

From (5) and (6), we get r.h.s. of the required result

$$= n^2(u+v) - n(u-v) + n(u-v)$$

$$= n^2(u+v) = n^2 z.$$

Example 2 (c) : If $u = \dfrac{x^3+y^3}{y\sqrt{x}} + \dfrac{1}{x^7}\sin^{-1}\left(\dfrac{x^2+y^2}{x^2+2xy}\right)$, find the value of

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \text{ at } x = 1, y = 2.$$

Sol. : Let $u = v + w$ where $v = \dfrac{x^3+y^3}{y\sqrt{x}}$, which is a homogeneous of degree $n_1 = \dfrac{3}{2}$ and $w = \dfrac{1}{x^7}\sin^{-1}\left(\dfrac{x^2+y^2}{x^2+2xy}\right)$, which is a homogeneous of degree $n_2 = -7$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n_1 v + n_2 w = \frac{3}{2}v - 7w \qquad \text{.......... (1)}$$

By corollary 1 of Euler's Theorem

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n_1(n_1-1)v + n_2(n_2-1)w$$

$$= \frac{3}{2}\cdot\frac{1}{2}v + (-7)(-8)w = \frac{3}{4}v + 56w \qquad \text{.......... (2)}$$

Adding the two results (1) and (2), we get

$$\text{l.h.s.} = \left(\frac{3}{2}+\frac{3}{4}\right)v + (-7+56)w = \frac{9}{4}v + 49w$$

At $x = 1, y = 2$, $v = \dfrac{1+8}{2} = \dfrac{9}{2}$, $\quad w = 1\cdot\sin^{-1}1 = \dfrac{\pi}{2}$

$$\therefore \quad \text{l.h.s.} = \frac{9}{4}\cdot\frac{9}{2} + 49\cdot\frac{\pi}{2} = \frac{81}{8} + \frac{49}{2}\cdot\pi.$$

Example 3 (c) : If $u = x^3\sin^{-1}\dfrac{y}{x} + x^4\tan^{-1}\dfrac{y}{x}$, find the value of

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \text{ at } x = 1, y = 1. \qquad \textbf{(M.U. 2001, 10)}$$

Sol. : Let $u = v + w$ where $v = x^3\sin^{-1}\dfrac{y}{x}$, $\quad w = x^4\tan^{-1}\dfrac{y}{x}$.

Clearly v is a homogeneous function of degree 3 and w is a homogeneous function of degree 4.

$$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 3v; \qquad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = nw = 4w$$

$$x^2 \frac{\partial^2 v}{\partial x^2} + 2xy \frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2} = n(n-1)\, v = 3 \cdot 2 \cdot v = 6v$$

$$x^2 \frac{\partial^2 w}{\partial x^2} + 2xy \frac{\partial^2 w}{\partial x \partial y} + y^2 \frac{\partial^2 w}{\partial y^2} = n(n-1)\, w = 4 \cdot 3 \cdot w = 12\, w$$

Adding all these results,

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= 6v + 12w + 3v + 4w = 9v + 16w \qquad \text{.................... (1)}$$

At $x = 1,\ y = 1,\ v = \sin^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2}$ and $w = \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4}$

Hence, from (1), we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{9\pi}{2} + \frac{16\pi}{4} = \frac{17\pi}{2}.$$

Example 4 (c) : If $u = \dfrac{(x^2+y^2)^m}{2m(2m-1)} + x f\left(\dfrac{y}{x}\right) + \Phi\left(\dfrac{x}{y}\right)$, find the value of

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}.$$ **(M.U. 1993, 98)**

Sol. : Let $u = u_1 + u_2 + u_3$ where $u_1 = \dfrac{(x^2+y^2)^m}{2m(2m-1)}$ which is a homogeneous function of degree $2m$, $u_2 = x f\left(\dfrac{y}{x}\right)$ which is a homogeneous function of degree 1, $u_3 = \Phi\left(\dfrac{x}{y}\right)$ which is a homogeneous function of degree zero.

Hence, by the above corollary

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2}$$

$$= 2m(2m-1)\,u_1 + 1(1-1)\,u_2 + 0(0-1)\,u_3$$

$$= 2m(2m-1) \cdot \frac{(x^2+y^2)^m}{2m(2m-1)} + 0 + 0$$

$$= (x^2+y^2)^m.$$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

1. If $u = \dfrac{x^2 y + xy^2}{x^2+y^2}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0.$

2. If $u = x^2 \sin^{-1}\left(\dfrac{y}{x}\right) - y^2 \cos^{-1}\left(\dfrac{x}{y}\right)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2u.$

3. If $u = \left(\dfrac{x}{y}\right)^{y/x}$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 0.$

4. If $u = x\sin^{-1}\left(\dfrac{y}{x}\right) + y\tan^{-1}\left(\dfrac{y}{x}\right)$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

[Ans. : 0]

5. If $u = x^3 e^{-x/y}$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 6u$. **(M.U. 1996)**

6. If $u = \tan^{-1}\left[\dfrac{\sqrt{x^2+y^2}}{x+y}\right]$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

7. If $u = \log\left[\dfrac{\sqrt{x^2+y^2}}{x+y}\right]$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$.

8. If $u = x^3 e^{-x/y}$, find $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$. **(M.U. 1996, 2002)**

[Ans. : $6x^3e^{-x/y}$]

9. If $y = x\cos u$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0$. **(M.U. 2003, 16)**

Class (c) : 8 Marks

1. If $u = \dfrac{x^4+y^4}{x^2y^2} + x^6\tan^{-1}\left(\dfrac{x^2+y^2}{x^2+2xy}\right)$, find the value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \text{ at } x = 1,\ y = 1.$$

[Ans. : 9π]

2. If $u = x^4\sin^{-1}\dfrac{y}{x} + x^6\tan^{-1}\dfrac{y}{x}$, find the value of

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \text{ at } x = 1,\ y = 1.$$

[Ans. : 17π]

3. If $u = x^3\left[\tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{y}{x}e^{-y/x}\right] + y^{-3}\left(\sin^{-1}\dfrac{x}{y} + \dfrac{x}{y}\log\dfrac{x}{y}\right)$, prove that

$$x^2\frac{\partial^2 x}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 9u.$$ **(M.U. 1999)**

Cor 2 : If $u = \sin^{-1}\Phi(x, y)$, $\cos^{-1}\Phi(x, y)$, $\tan^{-1}\Phi(x, y)$, $\text{cosec}^{-1}\Phi(x, y)$, $\sec^{-1}\Phi(x, y)$, $\cot^{-1}\Phi(x, y)$, $\log\Phi(x, y)$ where $\Phi(x, y)$ is a homogeneous function, then u is not a homogeneous function of x, y even through $\Phi(x, y)$ is a homogeneous function. But then $\sin u = \Phi(x, y)$, $\cos u = \Phi(x, y)$, $\tan u = \Phi(x, y)$, $\text{cosec}\, u\,\Phi(x, y)$, $\sec u = \Phi(x, y)$, $\cot u = \Phi(x, y)$ and $e^u = \Phi(x, y)$ are homogeneous functions. In other words, $z = f(u) = \sin u$, $\cos u$, $\tan u$, $\text{cosec}\, u$, $\sec u$, $\cot u$, e^u are a homogeneous functions of x . In this case, Euler's theorem can be applied. We have the following corollary.

If z is homogeneous function of degree n in x and y and $z = f(u)$ then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot\frac{f(u)}{f'(u)}}$$

The above result is sometimes referred to as **Modified Euler's Theorem.**

Sol. : By Euler's Theorem, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz=n\,f(u)$ .... (i)

Since, $z=f(u)$, $\dfrac{\partial z}{\partial x}=f'(u)\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y}=f'(u)\dfrac{\partial u}{\partial y}$.

Putting these values in (i), we get,

$$x\,f'(u)\frac{\partial u}{\partial x}+y\,f'(u)\frac{\partial u}{\partial y}=n\,f(u)\quad\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\cdot\frac{f(u)}{f'(u)}.$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Verify Euler's theorem for $u=\log\left(\dfrac{xy}{x^2+y^2}\right)$.

Sol. : (a) Since $u=\log\left(\dfrac{xy}{x^2+y^2}\right)$ $\therefore$ $\dfrac{xy}{x^2+y^2}=e^u=f(u)$

Putting $X=xt$, $Y=yt$,

$$F(X,Y)=\frac{XY}{X^2+Y^2}=\frac{t^2(xy)}{t^2(x^2+y^2)}=t^0\,f(x,y)=F(X,Y)\text{ say}$$

Now, $f(u)$ is a homogeneous function of x, y of zero degree ($n=0$).

Hence, by Euler's theorem *i.e.*, the above corollary,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\cdot\frac{f(u)}{f'(u)}=0 \qquad \text{..................... (1)}$$

(b) Now, $u=\log(xy)-\log(x^2+y^2)$

$$\therefore\quad \frac{\partial u}{\partial x}=\frac{y}{xy}-\frac{2x}{x^2+y^2}$$

$$\therefore\quad x\frac{\partial u}{\partial x}=\frac{xy}{xy}-\frac{2x^2}{x^2+y^2}\quad\text{and}\quad y\frac{\partial u}{\partial y}=\frac{yx}{xy}-\frac{2y^2}{x^2+y^2}.$$

By adding these results,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2(xy)}{xy}-\frac{2(x^2+y^2)}{x^2+y^2}$$

$$\therefore\quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2-2=0 \qquad \text{..................... (2)}$$

From (1) and (2), Euler's theorem is verified.

Example 2 (c) : If $u=\log\left[\dfrac{x^3+y^3}{x^2+y^2}\right]$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1$.

Sol. : We first note that u is not a homogeneous function of x, y.

Since, $u=\log\left[\dfrac{x^3+y^3}{x^2+y^2}\right]$, $e^u=\dfrac{x^3+y^3}{x^2+y^2}$

Now, let $z = e^u = \dfrac{x^3 + y^3}{x^2 + y^2} = f(u) = F(X, Y)$ say

By putting $X = x\,t,\ \ Y = y\,t,$ we get,

$$F(X,Y) = \frac{X^3 + Y^3}{X^2 + Y^2} = \frac{x^3 t^3 + y^3 t^3}{x^2 t^2 + y^2 t^2} = t \cdot \frac{x^3 + y^3}{x^2 + y^2} = t\, f(x, y)$$

Thus, $z = f(u) = e^u$ is a homogeneous function of x, y of degree one.

Hence, by corollary 2, we have $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = n \cdot \dfrac{f(u)}{f'(u)} = 1 \cdot \dfrac{e^u}{e^u} = 1.$

Example 3 (c) : If $u = \dfrac{x^2 y^2}{x^2 + y^2} + \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

Sol. : Let $u = v + w$ where $v = \dfrac{x^2 y^2}{x^2 + y^2}$ and $w = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$.

Putting $x = X\,t,\ \ y = Y\,t$ in v

$$f(X,Y) = \frac{x^2 t^2\ y^2 t^2}{x^2 t^2 + y^2 t^2} = \frac{t^4 (x^2 y^2)}{t^2 (x^2 + y^2)} = t^2\left[\frac{x^2 y^2}{x^2 + y^2}\right] = t^2 f(x, y).$$

$\therefore\ \ v$ is homogeneous function of degree 2.

$$\therefore \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial v} = 2v = 2 \cdot \frac{x^2 y^2}{x^2 + y^2} \qquad \text{.......................... (1)}$$

Note that w is not a homogeneous function of x, y, z.

Since $w = \cos^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, let $p = \cos w = \dfrac{x + y}{\sqrt{x} + \sqrt{y}} = F(X, Y)$

Now, $\quad F(X,Y) = \dfrac{X + Y}{\sqrt{X} + \sqrt{Y}} = \dfrac{xt + yt}{\sqrt{xt} + \sqrt{yt}} = t^{1/2}\left[\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right] = t^{1/2} f(x, y)$

Hence, $p = \cos w$, is a homogeneous function of x, y of degree 1/2.

$$\therefore \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \frac{n f(w)}{f'(w)} = \frac{1}{2} \cdot \frac{(\cos w)}{(-\sin w)} = -\frac{1}{2}\cot w \qquad \text{.......................... (2)}$$

From (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\frac{x^2 y^2}{x^2 + y^2} - \frac{1}{2}\cot\left[\cos^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right)\right]$$

Example 4 (c) : If $u = \sin^{-1}\left[\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right]$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}\tan u$.

Sol. : Note that u is not a homogeneous function of x, y.

Since, $\ u = \sin^{-1}\left[\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right]$, $\sin u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Let $\quad z = \sin u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} = f(u) = F(X, Y)$ say.

Putting $X = x\,t,\ \ Y = y\,t,$ we get,

$$F(X,Y) = \frac{X^{1/4} + Y^{1/4}}{X^{1/5} + Y^{1/5}} = \frac{x^{1/4} \cdot t^{1/4} + y^{1/4} \cdot t^{1/4}}{x^{1/5} \cdot t^{1/5} + y^{1/5} \cdot t^{1/5}}$$

$$= t^{1/20}\left(\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right) = t^{1/20} \cdot F(x, y)$$

Thus, z is a homogeneous function of x, y of degree $1/20$.

Hence, by corollary 2, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)} = \frac{1}{20} \cdot \frac{\sin u}{\cos u} = \frac{1}{20}\tan u.$$

Example 5 (c) : If $u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\sin 2u.$

Sol. : Here u is not a homogeneous function of x and y.

Since $\quad u = \tan^{-1}\left(\dfrac{x^2 + y^2}{x - y}\right)$, $\tan u = \dfrac{x^2 + y^2}{x - y}$

Let $\quad z = \tan u = \dfrac{x^2 + y^2}{x - y} = f(u) = f(X, Y)$

Putting $X = x\,t,\ \ Y = y\,t,$ we get,

$$F(X,Y) = \frac{X^2 + Y^2}{X - Y} = \frac{x^2\,t^2 + y^2\,t^2}{xt - yt} = t \cdot \left(\frac{x^2 + y^2}{x - y}\right) = t^1 \cdot F(x, y)$$

Thus, $z = f(u) = \tan u$ is a homogeneous function of x, y of degree one.

Hence, by corollary (2), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 1 \cdot \frac{\tan u}{\sec^2 u} = \sin u \cos u = \frac{1}{2}\sin 2u.$$

Example 6 (c) : If $u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u.$

Sol. : Here u is not a homogeneous function of x, y.

Since $\quad u = \sin^{-1}\left(\dfrac{x^2 + y^2}{x + y}\right)$, $\quad \sin u = \dfrac{x^2 + y^2}{x + y}$

Let $\quad z = \sin u = \dfrac{x^2 + y^2}{x + y} = f(u) = f(X, Y)$

Putting $X = x\,t,\ \ Y = y\,t,$ we get,

$$F(X,Y) = \frac{X^2 + Y^2}{X + Y} = \frac{x^2\,t^2 + y^2\,t^2}{xt + yt} = t\left(\frac{x^2 + y}{x + y}\right) = t^1 \cdot F(X, Y)$$

Thus, $z = f(u) = \sin u$ is a homogeneous function of x, y of degree one.

Hence, by corollary (2), we have,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)}=1\cdot\frac{\sin u}{\cos u}=\tan u.$$

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (c) : 8 Marks

1. If $u=\sin^{-1}\left(\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=-\dfrac{1}{12}\tan u.$

2. If $u=\tan^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{4}\sin 2u.$

3. If $u=\cos^{-1}\left(\dfrac{x^3+y^3}{x^2+y^2}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=-\cot u.$

4. If $u=\tan^{-1}\left(\dfrac{x^4+y^4}{x^2+y^2}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin 2u.$

5. If $u=\operatorname{cosec}^{-1}\left(\dfrac{x+y}{x^2+y^2}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\tan u.$

6. If $u=\sec^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\cot u.$

7. If $u=\log x+\log y$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2.$

8. If $u=\log\left(\dfrac{x^2+y^2}{x^3+y^3}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+1=0.$

9. If $u=\dfrac{1}{3}\log\left(\dfrac{x^3+y^3}{x^2+y^2}\right)$, find the value of $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$. [Ans. : $\dfrac{1}{3}$]

10. If $u=\log\left(\dfrac{x^2+y^2}{x-y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1.$

11. If $u=e^{x^2 f(y/x)}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2u\log u.$

12. If $u=\sin^{-1}(x^2+y^2)^{2/5}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{2}{5}\tan u.$

13. If $u=\cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+\dfrac{1}{2}\cot u=0.$

14. If $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin 2u.$ **(M.U. 1981, 84)**

15. If $u=\sin^{-1}(x\,y)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2\tan u.$ **(M.U. 1999)**

16. If $z = \log(x^2+y^2) + \dfrac{x^2+y^2}{x+y} - 2\log(x+y)$, find the value of $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}$. (M.U. 1992)

[Ans. : $\dfrac{x^2+y^2}{x+y}$]

Cor. 3 : If z is a homogeneous function of degree n in x and y and $z = f(u)$ then

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]}$$

where $g(u) = n \cdot \dfrac{f(u)}{f'(u)}$.

Proof : By corollary (2), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot\frac{f(u)}{f'(u)} = g(u) \qquad \text{.................... (i)}$$

Differentiating (i) partially w.r.t. x,

$$\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot 1\right) + y\frac{\partial^2 u}{\partial x\,\partial y} = g'(u)\frac{\partial u}{\partial x}$$

$$\therefore \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = [g'(u)-1]\frac{\partial u}{\partial x} \qquad \text{.................... (ii)}$$

Similarly, differentiating (i) partially w.r.t. y, we get,

$$y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial y\,\partial x} = [g'(u)-1]\frac{\partial u}{\partial y} \qquad \text{.................... (iii)}$$

Multiplying (ii) by x and (iii) by y and adding, we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = [g'(u)-1]\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$\therefore \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1] \qquad \text{[By (i)]}$$

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : If $u = \sin^{-1}\sqrt{x^2+y^2}$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

Sol. : Note that u is not a homogeneous function of x, y.

Let $z = \sin u = \sqrt{x^2+y^2} = f(u) = F(x, y)$

Putting $X = xt$, $Y = yt$, we get

$$F(X, Y) = \sqrt{X^2+Y^2} = \sqrt{x^2t^2+y^2t^2} = t\sqrt{x^2+y^2} = t\,f(x, y)$$

Hence, $z = f(u) = \sin u$ is a homogeneous function of x, y of degree one. Hence, by corollary 3, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where $\quad g(u)=n\dfrac{f(u)}{f'(u)}=1\cdot\dfrac{\sin u}{\cos u}=\tan u \quad$ and $\quad g'(u)=\sec^2 u.$

$$\therefore\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan u\cdot[\sec^2 u-1]=\tan^3 u.$$

Example 2 (c) : If $u=\log\left[\dfrac{x^3+y^3}{x^2+y^2}\right]$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=-1.$

Sol. : Note that u is not a homogeneous function of x and y.

Let $\quad z=e^u=\dfrac{x^3+y^3}{x^2+y^2}=f(u).$

Then as seen in Ex. 2, page 6-18, $z=f(u)=e^u$ is a homogeneous function of x, y of degree one.

$$\therefore\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where $\quad g(u)=n\cdot\dfrac{f(u)}{f'(u)}=1\cdot\dfrac{e^u}{e^u}=1 \quad$ and $\quad g'(u)=0.$

$$\therefore\quad x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-1.$$

Example 3 (c) : If $u=\log(x^3+y^3-x^2y-xy^2)$, prove that

(i) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3,$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=-3.$ **(M.U. 2002, 04, 09)**

Sol. : Note that u is not a homogeneous function of x and y.

Let $z=e^u=x^3+y^3-x^2y-xy^2=f(u)=F(X,Y)$ say.

Putting $X=xt,\ \ Y=yt,$ we get,

$$\begin{aligned}F(X,Y)&=X^3+Y^3+X^2Y-XY^2\\&=x^3t^3+y^3t^3-x^2yt^3-xy^2t^3\\&=t^3(x^3+y^3-x^2y-xy^2)=t^3F(x,y)\end{aligned}$$

Thus, $z=f(u)=e^u$ is a homogeneous function of x, y of degree 3.

Hence, by the previous corollary,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\cdot\frac{f(u)}{f'(u)}=3\cdot\frac{e^u}{e^u}=3$$

Again by the above corollary,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1] \qquad \text{.....................} (1)$$

where $\quad g(u) = n \cdot \dfrac{f(u)}{f'(u)} = 3 \cdot \dfrac{e^u}{e^u} = 3$ and $g'(u) = 0$.

$$\therefore \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 3(0-1) = -3.$$

Example 4 (c) : If $u = \tan^{-1}\left[\dfrac{x^3 + y^3}{x - y}\right]$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \sin u \cos 3u.$$ **(M.U. 2004)**

$$= \sin 2u [1 - 4 \sin^2 u]$$

$$= \sin 4u - \sin 2u.$$ **(M.U. 1984, 95, 2002, 04)**

Sol. : Note that u is not a homogeneous function of x, y.

Let $\quad z = \tan u = \dfrac{x^3 + y^3}{x - y} = f(u) = F(x, y)$

Putting $X = xt$, $Y = yt$, we get,

$$F(X, Y) = \frac{X^3 + Y^3}{X - Y} = \frac{x^3 t^3 + y^3 t^3}{xt - yt} = t^2 \left(\frac{x^3 + y^3}{x - y}\right) = t^2 F(x, y)$$

Thus, $z = f(u) = \tan u$ is a homogeneous function of x, y of degree 2.

Hence, by the above corollary.

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

where $\quad g(u) = n \dfrac{f(u)}{f'(u)} = 2 \cdot \dfrac{\tan u}{\sec^2 u} = 2 \sin u \cos u$

$\therefore \quad g'(u) = 2 \cos^2 u - 2 \sin^2 u = 4 \cos^2 u - 2.$

$$\begin{aligned} \therefore \quad g(u)[g'(u) - 1] &= 2 \sin u \cos u [4 \cos^2 u - 3] \qquad \text{.................. (i)} \\ &= \sin 2u [1 - 4 \sin^2 u] \\ &= 2 \sin u [4 \cos^3 u - 3 \cos u] \qquad [\text{From (i)}] \\ &= 2 \sin u \cos 3u \\ &= \sin 4u - \sin 2u. \\ [2 \cos A \sin B &= \sin(A + B) - \sin(A - B)] \end{aligned}$$

Hence, the required result.

Example 5 (c) : If $u = \tan^{-1}(x^2 + 2y^2)$, prove that

(i) $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \sin 2u.$

(ii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 2 \sin u \cos 3u$

$$= \sin 2u [1 - 4 \sin^2 u]$$

$$= \sin 4u - \sin 2u$$

Sol. : Note that u is not a homogeneous function of x and y.

Let $z = \tan u = (x^2 + 2y^2) = f(u) = F(X, Y)$ say.

Putting $X = xt$, $Y = yt$, we get,

$$F(X,Y) = X^2 + 2Y^2 = x^2 t^2 + 2y^2 t^2 = t^2 (x^2 + 2y^2) = t^2 f(x, y).$$

Thus, $z = f(u) = \tan u$ is a homogeneous function of x, y of degree 2.

Hence, by the above corollary,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = g(u)$$

where, $g(u) = n \dfrac{f(u)}{f'(u)} = 2 \cdot \dfrac{\tan u}{\sec^2 u} = 2 \sin u \cos u$

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u.$$

Further, $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = n \cdot \dfrac{f(u)}{f'(u)} = 2 \cdot \dfrac{\tan u}{\sec^2 u} = 2 \sin u \cos u.$

$$\therefore \quad g'(u) = 2\cos^2 u - 2\sin^2 u = 4\cos^2 u - 2$$

$$\therefore \quad g(u)[g'(u-1)] = 2 \sin u \cos u [4\cos^2 u - 3] \qquad \text{.................... (i)}$$

$$= \sin 2u [1 - 4\sin^2 u]$$

$$= 2 \sin u [4 \cos^3 u - 3 \cos u] \qquad \text{[From (i)]}$$

$$= 2 \sin u \cos 3u$$

$$= \sin 4u - \sin 2u.$$

Hence, the required result.

Example 6 (c) : If $u = \tan^{-1}\left[\dfrac{x^3 + y^3}{2x + 3y}\right]$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u. \qquad \textbf{(M.U. 1992)}$$

Sol. : Note that u is not a homogeneous function of x, y.

Let $z = \tan u = \dfrac{x^3 + y^3}{2x + 3y} = f(u) = F(x, y)$

Putting $X = xt$, $Y = yt$, we get,

$$F(X,Y) = \frac{X^3 + Y^3}{2X + 3Y} = \frac{x^3 t^3 + y^3 t^3}{2xt + 3yt} = t^2 \left(\frac{x^3 + y^3}{2x + 3y}\right) = t^2 F(x, y)$$

Thus, $z = f(u) = \tan u$ is a homogeneous function of x, y of degree 2.

Hence, by the above corollary,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$$

where $g(u) = n \dfrac{f(u)}{f'(u)} = 2 \cdot \dfrac{\tan u}{\sec^2 u} = 2 \sin u \cos u = \sin 2u$; $\quad g'(u) = 2 \cos 2u$

$$\therefore \quad g(u)[g'(u)-1] = \sin 2u[2\cos 2u - 1]$$
$$= 2\sin 2u \cos 2u - \sin 2u$$
$$= \sin 4u - \sin 2u$$

Hence, the required result.

(Notice the similarity between the examples 4, 5 and 6.)

Example 7 (c) : If $u = \operatorname{cosec}^{-1}\sqrt{\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u).$$ **(M.U. 2001, 04, 07, 08, 09, 16)**

Sol. : Note that u is not a homogeneous function of x and y.

Let $\quad z = \operatorname{cosec} u = \sqrt{\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)} = f(u) = F(X,Y)$ say.

Putting $X = xt,\ Y = yt$, we get,

$$F(X,Y) = \sqrt{\left(\frac{X^{1/2}+Y^{1/2}}{X^{1/3}+Y^{1/3}}\right)} = \sqrt{\left(\frac{x^{1/2}t^{1/2}+y^{1/2}t^{1/2}}{x^{1/3}t^{1/3}+y^{1/3}t^{1/3}}\right)}$$
$$= \sqrt{t^{(1/2)-(1/3)}}\cdot f(x,y) = t^{1/12} f(x,y)$$

Thus, $z = f(u) = \operatorname{cosec} u$ is a homogeneous function of x, y of degree 1 / 12.

Hence, by the above corollary,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1]$$

where $\quad g(u) = n\cdot\dfrac{f(u)}{f'(u)} = -\dfrac{1}{12}\dfrac{\operatorname{cosec} u}{\operatorname{cosec} u\cot u} = -\dfrac{1}{12}\tan u$

$$g'(u) - 1 = -\frac{1}{12}\sec^2 u - 1 = -\frac{1}{12}(1+\tan^2 u) - 1$$
$$= -\frac{1}{12}\tan^2 u - \frac{13}{12} = -\frac{1}{12}(\tan^2 u + 13)$$
$$\therefore \quad g(u)[g'(u-1)] = \frac{1}{12}\tan u\cdot\frac{1}{12}(\tan^2 u + 13)$$
$$= \frac{1}{144}\tan u(13+\tan^2 u).$$

Example 8 (c) : If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x^2 u_{xx} + 2xy\,u_{xy} + u_{yy} = -\dfrac{\sin u\cos 2u}{4\cos^3 u}$.

(M.U. 1986, 91, 2003, 17, 18)

Sol. : Note that u is not a homogeneous function of x and y.

Let $z = \sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = f(u) = F(X, Y)$, say.

Putting $X = xt,\ Y = yt$, we get,

$$F(X,Y)=\frac{X+Y}{\sqrt{X}+\sqrt{Y}}=\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}$$

$$=\frac{t}{\sqrt{t}}\cdot\frac{(x+y)}{\sqrt{x}+\sqrt{y}}=t^{1/2}\cdot\frac{x+y}{\sqrt{x}+\sqrt{y}}=t^{1/2}F(x,y)$$

Thus, $z=\sin u$ is a homogeneous function of x, y of degree 1/2.

Hence, by the above corollary,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 y}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)\bullet[g'(u)-1]$$

where, $g(u)=n\cdot\dfrac{f(u)}{f'(u)}=\dfrac{1}{2}\cdot\dfrac{\sin u}{\cos u}=\dfrac{1}{2}\tan u.$

$$\therefore\quad g'(u)=\frac{1}{2}\sec^2 u$$

$$\therefore\quad g(u)[g'(u)-1]=\frac{1}{2}\tan u\cdot\left[\frac{1}{2}\sec^2 u-1\right]$$

$$=\frac{1}{4}\tan u\left[\frac{1-2\cos^2 u}{\cos^2 u}\right]=\frac{1}{4\cos^3 u}\cdot\sin u(-\cos 2u)$$

$$=-\frac{\sin u\cos 2u}{4\cos^3 u}.$$

Example 9 (c) : If $u=\sin^{-1}(x^2+y^2)^{1/5}$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{2}{25}\cdot\tan u\cdot(2\tan^2 u-3).$$ **(M.U. 1985, 92)**

Sol. : Note that u is not a homogeneous function of x and y.

Let $\quad z=\sin u=(x^2+y^2)^{1/5}=f(u)=F(X,Y)$ say.

Putting $X=xt,\ Y=yt$, we get,

$$F(X,Y)=(X^2+Y^2)^{1/5}=\left[x^2t^2+y^2t^2\right]^{1/5}=t^{2/5}(x^2+y^2)=t^{2/5}f(x,y)$$

Thus, $z=f(u)=\sin u$ is a homogeneous function of x, y of degree $2/5$.

Hence, by the above corollary,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

where $\quad g(u)=n\cdot\dfrac{f(u)}{f'(u)}=\dfrac{2}{5}\cdot\dfrac{\sin u}{\cos u}=\dfrac{2}{5}\cdot\tan u\quad$ and $\quad g'(u)=\dfrac{2}{5}\sec^2 u.$

$$\therefore\quad g(u)[g'(u)-1]=\frac{2}{5}\tan u\left[\frac{2}{5}\sec^2 u-1\right]=\frac{2}{25}\cdot\tan u\cdot\left[2\sec^2 u-5\right]$$

$$=\frac{2}{25}\cdot\tan u\cdot\left[2(1+\tan^2 u)-5\right]$$

$$=\frac{2}{25}\cdot\tan u\cdot\left[2\tan^2 u-3\right].$$

Hence, the required result.

Example 10 (c) : If $z = e^{x/y} + \log(x^3 + y^3 - x^2y - xy^2)$, find the value of

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}.$$ **(M.U. 2013)**

Sol. : Let $u = e^{x/y}$ and $v = \log(x^3 + y^3 - x^2y - xy^2)$

Since $u = e^{x/y}$ is of degree n = zero by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0 \quad \text{.................... (1)}$$

and by corollary (1), page 6-12,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = 0 \quad \text{.................... (2)}$$

We have proved in (1), Ex. 3, page 6-23

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 3 \quad \text{.................... (3)}$$

and $$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2\frac{\partial^2 v}{\partial y^2} = -3 \quad \text{.................... (4)}$$

Hence, using the above results (1), (2), (3) and (4), we get

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0 + 0 + 3 - 3 = 0$$

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (c) : 8 Marks

1. If $u = \tan^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -2\sin^3 u \cos u.$

(M.U. 1995, 2004)

2. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u\,(1 - 4\sin^2 u).$$ **(M.U. 2009)**

3. If $u = \sec^{-1}\left(\dfrac{x^2+y^2}{x-y}\right)$, find the value of $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2}$.

[**Ans. :** $-\cot u\,(2 + \cot^2 u)$]

4. If $u = \sin^{-1}\left[\dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right]$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{400}\tan u\,(\tan^2 u - 19).$$

5. If $u = \sin^{-1}\left(\dfrac{x^{1/4} + y^{1/4}}{x^{1/6} + y^{1/6}}\right)$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{144} \tan u (\tan^2 u - 11).$$

6. If $u = \sin h^{-1}\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\tan h^3 u.$

(M.U. 1995)

7. If $3u = \log\left(\dfrac{x^3 + y^3}{x^2 + y^2}\right)$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{1}{3}.$

8. If $y = \sin^{-1}\left[\dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}\right]^{1/2}$, prove that

$$x^2 \frac{\partial^2 y}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}[\tan^2 u + 13].$$ **(M.U. 2018)**

9. If $u = \log r$ and $r^2 = x^2 + y^2$, prove that $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} + 1 = 0.$

10. If $u = \log \dfrac{x + y}{\sqrt{x^2 + y^2}} + \sin^{-1} \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin w \cos 2w}{4\cos^3 w} \quad \text{where } w = \sin^{-1}\left(\frac{x + y}{\sqrt{x} + \sqrt{y}}\right).$$

EXERCISE - V

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (a) : 3 Marks

1. If u is a homogeneous function of x, y, z of degree n then show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu.$$ **(M.U. 1989, 91)**

2. State and prove Euler's Theorem for function of two (three) variables.

(M.U. 2001, 02, 03, 04, 07, 10, 14, 15, 16, 18)

3. If $u(x, y)$ is a homogeneous function of degree n then show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$ **(M.U. 1998)**

4. If z is homogeneous function of degree n in x and y and if $z = f(u)$ then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}.$$

5. State Euler's Theorem on homogeneous function of two variables and hence, deduce that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \quad \text{where } g(u) = n \cdot \frac{F(u)}{F'(u)}.$$

(M.U. 1993, 95, 2001)

6. If $u = f(x, y)$ is a homogeneous function of x, y of degree n then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{and} \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$ **(M.U. 2002, 03)**

EXERCISE - VI

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Short Answer Questions : Class (a) : 3 Marks

1. If $u = x\sin^{-1}\frac{y}{x} + y\sin^{-1}\frac{x}{y}$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2}$. **[Ans. : 0]**

2. If $u = x\log\left(\frac{y}{x}\right) + y\log\left(\frac{x}{y}\right)$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2}$. **[Ans. : 0]**

3. If $u = x e^{y/x} + y e^{x/y}$, then find the value of $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2}$. **[Ans. : 0]**

4. If $u = \log\left(\frac{x^2 + y^2}{x - y}\right)$, then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. **[Ans. : 1]**

Summary

1. **Euler's Theorem :**

   If z is a homogeneous function of two variables x and y of degree n then

   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz.$$

2. If z is a homogeneous function of two variables x and y of degree n then

   $$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z.$$

3. If z is homogeneous function of degree n in x and y and $z = f(u)$ then

   $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot \frac{f(u)}{f'(u)}.$$

4. If z is a homogeneous function of degree n in x and y and $z = f(u)$ then

   $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

   where $g(u) = n \cdot \frac{f(u)}{f'(u)}$.

☆ ☆ ☆

CHAPTER

7 Maxima and Minima

1. Introduction

In this chapter, we are going study how to find maxima and minima of functions of the type

$$z = f(x, y).$$

2. Maxima and Minima of z = f (x, y)

The conditions for maxima and minima given below can be derived from Taylor's Theorem.

Conditions for $f(x, y)$ to be Maximum and Minimum

A function $f(x, y)$ is said to be **maximum** at (a, b) if $f(a, b) > f(a+h, y+k)$ for small values of h and k, positive or negative.

A function $f(x, y)$ is said to be **minimum** at (a, b) if $f(a, b) < f(a+h, y+k)$ for small values of h and k, positive or negative.

It can be shown that the necessary conditions for $f(x, y)$ to be maximum or minimum at (a, b) are $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Further, for maxima or minima we should have at (a, b).

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $\dfrac{\partial^2 f}{\partial x^2}$ or $\left(\dfrac{\partial^2 f}{\partial y^2}\right) < 0$, then $f(x, y)$ is maximum at (a, b).

If $\dfrac{\partial^2 f}{\partial x^2}$ or $\left(\dfrac{\partial^2 f}{\partial y^2}\right) > 0$, then $f(x, y)$ is minimum at (a, b).

Method of Finding Maxima and Minima

We shall state here the method of finding the maximum and minimum values (also called stationary or extreme or turning values) of a function of two variables $f(x, y)$.

(i) First find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

(ii) Then solve $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$ simultaneously for x and y.

(iii) See that $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} > \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$ for the values of x, y obtained in (ii).

(iv) When this condition is satisfied $f(x, y)$ is maximum if

$$\frac{\partial^2 f}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial y^2} < 0 \text{ and } f(x, y) \text{ is minimum if } \frac{\partial^2 f}{\partial x^2} \text{ or } \left(\frac{\partial^2 f}{\partial y^2}\right) > 0.$$

At $x = a$ and $y = b$ obtained in (ii) let us denote $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$.

For maxima or minima we must have

$$rt - s^2 > 0$$

$f(x, y)$ is maximum if r (or t) < 0

$f(x, y)$ is minimum if r (or t) > 0

Notes ....

1. If $f_x = 0$ and $f_y = 0$ or one or both do not exist at a point then the point is called a **critical point.**
2. If $rt - s^2 < 0$, $f(x, y)$ is neither maximum nor minimum. Such a point is called a **Saddle Point.**
3. If $rt - s^2 = 0$, further investigations are necessary.

Working Rule to Solve a Problem on Maxima, Minima

We shall follow the following procedure to solve a problem on maxima, minima.

Step I : Find f_x, f_y, f_{xx}, f_{xy} and f_{yy}.

Step II : Solve the equations $f_x = 0$, $f_y = 0$.

Step III : Find the values of $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$ at the roots obtained in step II.

If $rt - s^2 > 0$ and $r < 0$ then $f(x, y)$ is maximum.

If $rt - s^2 > 0$ and $r > 0$ then $f(x, y)$ is minimum.

If $r = 0$ (or $t = 0$) $f(x, y)$ is neither maximum nor minimum.

Type I : f (x, y) given : Class (a) : 3 Marks

Example 1 (a) : Verify whether $(-1, 0)$ is a stationary value of $z = x^3 + 3xy^2 - 3x$.

Sol. : We have $f(x, y) = x^3 + 3xy^2 - 3x$.

Step I : $f_x = 3x^2 + 3y^2 - 3$, $f_y = 6xy$;

$r = f_{xx} = 6x$, $s = f_{xy} = 6y$, $t = f_{yy} = 6x$.

Step II : When $x = -1$, $y = 0$, $f_x = 0$, $f_y = 0$

$r = f_{xx} = -6$, $s = f_{xy} = 0$, $t = f_{yy} = -6$.

$\therefore \quad rt - s^2 = (-6)(-6) - 0 = 36 > 0$.

Hence, $(-1, 0)$ is a stationary value of z.

Class (b) : 6 Marks

Example 1 (b) : Discuss the maxima and minima of $x^2 + y^2 + 8x + 6y + 6$.

Sol. : We have $f(x, y) = x^2 + y^2 + 8x + 6y + 6$.

Step I : $f_x = 2x + 8,\quad f_y = 2y + 6\ ;\quad f_{xx} = 2,\quad f_{xy} = 0,\quad f_{yy} = 2.$

Step II : We now solve $f_x = 0,\ f_y = 0$

$\therefore\ 2x + 8 = 0 \quad \therefore\ x = -4\ ; \quad 2y + 6 = 0 \quad \therefore\ y = -3$

$\therefore$ $(-4, -3)$ is a stationary point.

Step III : For $x = -4,\ y = -3,\quad r = f_{xx} = 2,\quad s = f_{xy} = 0,\quad t = f_{yy} = 2.$

$\therefore\ rt - s^2 = 4 - 0 = 4 > 0.$ And $r = f_{xx} = 2 > 0$

$\therefore$ $(-4, -3)$ is a minima.

The minimum value of $f(x, y) = 16 + 9 - 32 - 18 + 6 = -19.$

Example 2 (b) : Discuss the maxima and minima of $x^3 + 6x^2 - y^2$.

Sol. : We have $f(x, y) = x^3 + 6x^2 - y^2$

Step I : $f_x = 3x^2 + 12x,\quad f_y = -2y,\quad f_{xx} = 6x + 12,\quad f_{xy} = 0,\quad f_{yy} = -2.$

Step II : We now solve $f_x = 0,\ f_y = 0$

$\therefore\ 3x^2 + 12x = 0 \quad \therefore\ 3x(x + 4) = 0 \quad \therefore\ x = 0, -4.$

$f_y = 0 \quad \therefore\ -2y = 0 \quad \therefore\ y = 0$

$\therefore$ $(0, 0)$ and $(-4, 0)$ are stationary values.

Step III : (i) For $x = 0,\ y = 0.\quad r = f_{xx} = 12,\quad s = f_{xy} = 0,\quad t = f_{yy} = -2.$

$\therefore\ rt - s^2 = -24 < 0$

$\therefore$ $f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(ii) For $x = -4,\ y = 0.\quad r = f_{xx} = -12,\quad s = f_{xy} = 0,\quad t = f_{yy} = -2$

$\therefore\ rt - s^2 = 24 > 0$ and $r = f_{xx} = -12 < 0$

$\therefore$ $f(x, y)$ is maximum at $(-4, 0)$.

$\therefore$ The maximum value $= -64 + 96 - 0 = 32.$

Example 3 (b) : Discuss the maxima and minima of $x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10.$ **(M.U. 2016)**

Sol. : We have $f(x, y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10.$

Step I : $f_x = 3x^2 + y^2 - 24x + 21,\quad f_y = 2xy - 4y,$

$f_{xx} = 6x - 24,\quad f_{xy} = 2y,\quad f_{yy} = 2x - 4.$

Step II : We now solve the equations $f_x = 0,\ f_y = 0$

$\therefore\ 3x^2 + y^2 - 24x + 21 = 0$ and $2xy - 4y = 0.$

The second equation gives $2y(x - 2) = 0 \quad \therefore\ x = 2$ or $y = 0.$

When $x = 2$, the first equation $3x^2 + y^2 - 24x + 21 = 0$ gives

$12 + y^2 - 48 + 21 = 0 \quad \therefore\ y^2 - 15 = 0 \quad \therefore\ y^2 = 15 \quad \therefore\ y = \pm\sqrt{15}.$

$\therefore$ The stationary values are $(2, \sqrt{15}), (2, -\sqrt{15})$.

When $y = 0$, the first equation $3x^2 + y^2 - 24x + 21 = 0$ gives

$3x^2 - 24x + 21 = 0 \quad \therefore\ x^2 - 8x + 7 = 0$

$\therefore\ (x - 7)(x - 1) = 0 \quad \therefore\ x = 1, 7.$

$\therefore$ The stationary values are $(1, 0), (7, 0)$.

Step III : (i) For $x = 2,\ y = \sqrt{15}$

$r = f_{xx} = 12 - 24 = -12,\quad s = f_{xy} = 2\sqrt{15},\quad t = f_{yy} = 4 - 4 = 0.$

$\therefore \quad rt - s^2 = 0 - 60 = -60 < 0.$

$\therefore \quad f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(ii) For $x = 2,\ y = -\sqrt{15}$

$r = f_{xx} = 12 - 24 = -12,\ \ s = f_{xy} = -2\sqrt{15},\ \ t = f_{yy} = 4 - 4 = 0.$

$\therefore \quad rt - s^2 = 0 - 60 = -60 < 0.$

$\therefore \quad f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(iii) For $x = 1,\ y = 0$

$r = f_{xx} = 6 - 24 = -18,\ \ s = f_{xy} = 0,\ \ t = f_{yy} = 2 - 4 = -2.$

$\therefore \quad rt - s^2 = 36 - 0 = 36 > 0.$ And $r = -18$, negative.

$\therefore \quad (1, 0)$ is a maxima.

$\therefore$ The maximum value $= 1 + 0 - 12 - 0 + 21 + 10 = 20.$

(iv) For $x = 7,\ y = 0$

$r = f_{xx} = 42 - 24 = 18,\ \ s = f_{xy} = 0,\ \ t = f_{yy} = 14 - 4 = 10.$

$\therefore \quad rt - s^2 = 180 - 0 = 180 > 0.$ And $r = 18$, positive.

$\therefore \quad (7, 0)$ is a minima.

$\therefore$ The minimum value $= 343 + 0 - 588 - 0 + 147 + 10 = -88.$

Example 4 (b) : Discuss the maxima and minima of $x^3 + y^3 - 3x - 12y + 40.$

Sol. : We have $f(x, y) = x^3 + y^3 - 3x - 12y + 40.$

Step I : $f_x = 3x^2 - 3,\ \ f_y = 3y^2 - 12,\ \ f_{xx} = 6x,\ \ f_{xy} = 0,\ \ f_{yy} = 6y.$

Step II : We now solve $f_x = 0,\ f_y = 0$

$\therefore \quad 3x^2 - 3 = 0 \qquad \therefore \quad 3(x^2 - 1) = 0 \qquad \therefore \quad x = 1, -1.$

and $\quad 3y^2 - 12 = 0 \qquad \therefore \quad 3(y^2 - 4) = 0 \qquad \therefore \quad y = 2, -2.$

$\therefore$ Stationary values are $(1, 2), (1, -2), (-1, 2), (-1, -2).$

Step III : (i) For $x = 1,\ y = 2,$

$r = f_{xx} = 6, \qquad s = f_{xy} = 0, \qquad t = f_{yy} = 12$

$\therefore \quad rt - s^2 = 72 - 0 = 72 > 0.$ And $r = 6 > 0$

$\therefore \quad (1, 2)$ is a minima.

The minimum value $= 1 + 8 - 3 - 24 + 40 = 22.$

(ii) For $x = 1,\ y = -2;\quad r = f_{xx} = 6,\quad s = f_{xy} = 0,\quad t = f_{yy} = -12$

$\therefore \quad rt - s^2 = -72 < 0.$

$\therefore \quad f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(iii) For $x = -1,\ y = 2;\quad r = f_{xx} = -6,\quad s = f_{xy} = 0,\quad t = f_{yy} = 12.$

$\therefore \quad rt - s^2 = -72 < 0.$

$\therefore \quad f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(iv) For $x = -1,\ y = -2;\quad r = f_{xx} = -6,\quad s = f_{xy} = 0,\quad t = f_{yy} = -12$

$\therefore \quad rt - s^2 = 72 - 0 = 72 > 0.$ And $r = -6 < 0$

$\therefore \quad (-1, -2)$ is a maxima.

The maximum value $= -1 - 8 + 3 + 24 + 40 = 58.$

Example 5 (b) : Find the stationary values of $x^3 + y^3 - 3a\,xy,\ a > 0.$

(M.U. 1991, 99, 2003, 04, 13, 19)

Sol. : We have $f(x, y) = x^3 + y^3 - 3a\,xy.$

Step I : $f_x = 3x^2 - 3ay,\ \ f_y = 3y^2 - 3ax$

$r = f_{xx} = 6x,\ \ s = f_{xy} = -3a,\ \ t = f_{yy} = 6y.$

Step II : We now solve, $f_x = 0,\ f_y = 0.$

$\therefore\ \ x^2 - ay = 0$ and $y^2 - ax = 0$

To eliminate y, we put $y = x^2 / a$ in the second equation.

$\therefore\ \ x^4 - a^3 x = 0 \quad \therefore\ \ x(x^3 - a^3) = 0$

Hence, $x = 0$ or $x = a.$

When $x = 0,\ y = 0$ and when $x = a,\ y = a.$

$\therefore$ (0, 0) and (a, a) are stationary points.

Step III : (i) For $x = 0,\ y = 0,\ r = f_{xx} = 0,\ s = f_{xy} = -3a$ and $t = f_{yy} = 0.$

Hence, $rt - s^2 = 0 - 9a^2 < 0$

$\therefore\ \ f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(ii) For $x = a,\ y = a,\ r = f_{xx} = 6a,\ s = f_{xy} = -3a,\ t = f_{yy} = 6a$

$\therefore\ \ rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$\therefore\ \ f(x, y)$ is stationary at $x = a,\ y = a.$

And $\quad r = f_{xx} = 6a > 0$, since $a > 0$

$\therefore\ \ f(x, y)$ is minimum at $x = a,\ y = a.$

Putting $x = a,\ y = a$ in $x^3 + y^3 - 3a\,xy$ the minimum value of

$f(x, y) = a^3 + a^3 - 3a^3 = -a^3.$

Example 6 (b) : Find the maximum and minimum values of $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$

(M.U. 1995, 2006, 08, 14, 15, 17)

Sol. : We have $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$

Step I : $f_x = 3x^2 + 3y^2 - 6x\ \ ;\ \ f_y = 6xy - 6y$

$f_{xx} = 6x - 6\ \ ;\ \ f_{xy} = 6y\ ;\ f_{yy} = 6x - 6.$

Step II : We now solve $f_x = 0,\ f_y = 0.$

$\therefore\ \ x^2 + y^2 - 2x = 0$

And $\quad xy - y = 0 \quad \therefore\ \ y(x - 1) = 0 \quad \therefore\ \ y = 0$ or $x = 1.$

When $y = 0,\ x^2 - 2x = 0$ gives $x = 0$ or 2.

$\therefore$ (0, 0), (2, 0) are stationary points.

When $x = 1,\ 1 + y^2 - 2 = 0$

$\therefore\ \ y^2 - 1 = 0 \quad \therefore\ \ y = 1$ or $-1.$

$\therefore$ (1, 1), (1, −1) are stationary points.

Step III : (i) At $x = 0, y = 0\ ;\ \ r = f_{xx} = -6,\ \ s = f_{xy} = 0,\ \ t = f_{yy} = -6$

$\therefore\ \ rt - s^2 = 36 - 0 = 36 > 0 \quad$ And $\quad r = -6 < 0.$

$\therefore\ \ f(x, y)$ is maximum at (0, 0).

Maximum Value = 4.

(ii) At $x = 2, y = 0$; $r = f_{xx} = 12 - 6 = 6,\ s = f_{xy} = 0,\ t = f_{yy} = 6$

$\therefore\ rt - s^2 = (6)(6) - 0 = 36 > 0$ And $r = 6 > 0$.

$\therefore\ f(x, y)$ is minimum at (2, 0).

Minimum Value $= 8 - 12 + 4 = 0$.

(iii) At $x = 1, y = 1$; $r = f_{xx} = 6 - 6 = 0,\ s = f_{xy} = 6,\ t = f_{yy} = 6 - 6 = 0$

$\therefore\ rt - s^2 = 0 - 36 = -36 < 0$.

$\therefore\ f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(iv) At $x = 1, y = -1$; $r = f_{xx} = 6 - 6 = 0,\ s = f_{xy} = -6,\ t = f_{yy} = 0$

$\therefore\ rt - s^2 = 0 - (-6)^2 = -36 < 0$

$\therefore\ f(x, y)$ again is neither maximum nor minimum. It is also a saddle point.

Example 7 (b) : Find all stationary values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

(M.U. 2004, 05, 07, 10, 12, 15, 17, 18)

Sol. : We have $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

Step I : $f_x = 3x^2 + 3y^2 - 30x + 72,\quad f_y = 6xy - 30y$

$f_{xx} = 6x - 30,\ f_{xy} = 6y,\ f_{yy} = 6x - 30$.

Step II : We now solve, $f_x = 0,\ f_y = 0$.

$\therefore\ 3x^2 + 3y^2 - 30x + 72 = 0$ *i.e.* $x^2 + y^2 - 10x + 24 = 0$

and $6xy - 30y = 0$ $\therefore\ 6y(x - 5) = 0$ $\therefore\ y = 0$ or $x = 5$

(i) When $y = 0,\ x^2 - 10x + 24 = 0$,

$(x - 6)(x - 4) = 0$ $\therefore\ x = 6,\ x = 4$.

$\therefore$ (6, 0), (4, 0) are stationary points.

(ii) When $x = 5$, $25 + y^2 - 50 + 24 = 0$

$\therefore\ y^2 - 1 = 0$ $\therefore\ y = 1$ or -1.

$\therefore$ (5, 1) and (5, − 1) are stationary points.

Step III : (i) When $x = 6,\ y = 0$

$r = f_{xx} = 36 - 30 = 6,\ s = f_{xy} = 0$ and $t = f_{yy} = 36 - 30 = 6$.

$\therefore\ rt - s^2 = 36 > 0$ and $r = f_{xx} = 6 > 0$

$\therefore\ f(x, y)$ is minimum at (6, 0).

The minimum value $= 216 - 540 + 432 = 108$.

(II) When $x = 4,\ y = 0$

$r = f_{xx} = 24 - 30 = -6,\ s = f_{xy} = 0,\ t = f_{yy} = 24 - 30 = -6$

$\therefore\ rt - s^2 = 36 > 0$ and $r = f_{xx} = -6 < 0$

$\therefore\ f(x, y)$ is maximum at (4, 0).

The maximum value $= 64 - 240 + 288 = 108$.

(iii) When $x = 5,\ y = 1$; $r = f_{xx} = 6 \times 5 - 30 = 0$.

$\therefore\ f(x, y)$ is neither maximum nor minimum.

(iv) When $x = 5,\ y = -1$; $r = f_{xx} = 6 \times 5 - 30 = 0$

$\therefore\ f(x, y)$ is neither maximum nor minimum.

Example 8 (b) : Examine the function $f(x, y) = y^2 + 4xy + 3x^2 + x^3$ for extreme values. **(M.U. 2008)**

Sol. : We have $f(x, y) = y^2 + 4xy + 3x^2 + x^3$

Step I : $f_x = 4y + 6x + 3x^2$, $f_y = 2y + 4x$, $f_{xx} = 6 + 6x$, $f_{xy} = 4$, $f_{yy} = 2$.

Step II : We now solve $f_x = 0$, $f_y = 0$.

$\therefore$ $4y + 6x + 3x^2 = 0$ and $2y + 4x = 0$

Putting $2y = -4x$ in the first equation

$3x^2 - 2x = 0$ $\therefore$ $x(3x - 2) = 0$ $\therefore$ $x = 0$ or $x = 2/3$.

When $x = 0$, $y = 0$ and when $x = 2/3$, $y = -4/3$.

$\therefore$ (0, 0), (2/3, – 4/3) are stationary points.

Step III : (i) When $x = 0$, $y = 0$

$r = f_{xx} = 6$, $s = f_{xy} = 4$, $t = f_{yy} = 2$

$\therefore$ $rt - s^2 = 12 - 16 < 0$

$\therefore$ $f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(ii) When $x = 2/3$, $y = -4/3$; $r = f_{xx} = 10$, $s = 4$, $t = 2$.

$\therefore$ $rt - s^2 = 20 - 16 = 4 > 0$

But $r = f_{xx} = 10 > 0$.

Hence, $f(x, y)$ is minimum at $x = 2/3$, $y = -4/3$.

The minimum value $= \dfrac{16}{9} - \dfrac{32}{9} + \dfrac{12}{9} + \dfrac{8}{27} = -\dfrac{4}{27}$.

Example 9 (b) : Find the extreme value of $xy(3 - x - y)$. **(M.U. 1988, 2005)**

Sol. : We have $f(x, y) = 3xy - x^2y - xy^2$

Step I : $f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy$,

$f_{xx} = -2y$, $f_{xy} = 3 - 2x - 2y$, $f_{yy} = -2x$.

Step II : We now solve $f_x = 0$, $f_y = 0$.

(i) $y(3 - 2x - y) = 0$ and $x(3 - x - 2y) = 0$

$\therefore$ $y = 0$ or $y = 3 - 2x$

When $y = 0$, $x(3 - x - 2y) = 0$, gives, $x = 0$ or $x = 3$.

$\therefore$ (0, 0), (3, 0) are stationary points.

(ii) When $y = 3 - 2x$, $x(3 - x - 2y) = 0$, gives,

$x[3 - x - 2(3 - 2x)] = 0$ *i.e.* $x(-3 + 3x) = 0$

i.e. $x = 0$ or $x = 1$.

When $x = 0$, from $y = 3 - 2x$, we get $y = 3$; When $x = 1$, $y = 1$.

$\therefore$ (0, 3), (1, 1) are stationary points.

Step III : (i) When $x = 0$, $y = 0$; $r = f_{xx} = 0$, $s = f_{xy} = 3$, $t = f_{yy} = 0$

$\therefore$ $rt - s^2 = 0 - 9 < 0$

$\therefore$ $f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(ii) When $x = 3$, $y = 0$; $r = f_{xx} = 0$, $s = f_{xy} = -3$, $t = f_{yy} = -6$

$\therefore$ $rt - s^2 = 0 - 9 < 0$

$\therefore$ $f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(iii) When $x = 0,\ y = 3$; $r = f_{xx} = -6,\quad s = f_{xy} = -3,\quad t = f_{yy} = 0$

$\therefore\quad rt - s^2 = 0 - 9 < 0$

$\therefore\quad f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(iv) When $x = 1,\ y = 1$; $r = f_{xx} = -2,\quad s = f_{xy} = -1,\quad t = f_{yy} = -2$

$\therefore\quad rt - s^2 = 4 - 1 = 3 > 0$ And $r = f_{xx} = -2 < 0.$

$\therefore\quad f(x, y)$ is maximum at $x = 1,\ y = 1$.

Maximum value $= 1\,(3 - 2) = 1.$

(Similarly, show that the stationary value of $xy\,(a - x - y)$ is $a^3 / 27.$)

Example 10 (b) : Discuss the maxima and minima of $x^3\,y^2\,(1 - x - y)$.

(M.U. 1996, 2004, 16)

Sol. : We have $f(x, y) = x^3\,y^2\,(1 - x - y)$.

Step I :
$$f_x = y^2\,[3x^2(1 - x - y) - x^3] = y^2(3x^2 - 4x^3 - 3x^2y)$$
$$= 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$
$$f_y = x^3\,[2y\,(1 - x - y) - y^2] = x^3(2y - 2xy - 3y^2)$$
$$= 2x^3y - 2x^4y - 3x^3y^2$$
$$\therefore\quad f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3,\quad f_{xy} = 6x^2y - 8x^3y - 9x^2y^2,$$
$$f_{yy} = 2x^3 - 2x^4 - 6x^3y$$

Step II : We now solve $f_x = 0,\ f_y = 0.$

$\therefore\quad 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0\quad$ *i.e.* $\quad x^2y^2(3 - 4x - 3y) = 0$

and $\quad 2x^3y - 2x^4y - 3x^3y^2 = 0\quad$ *i.e.* $\quad x^3y\,(2 - 2x - 3y) = 0$

$\therefore\quad x = 0,\ y = 0$ and $3 - 4x - 3y = 0,\ 2 - 2x - 3y = 0.$

Subtracting, we get $1 - 2x = 0$

$\therefore x = 1/2\qquad \therefore\quad 3y = 3 - 4\,(1/2) = 1\qquad \therefore\quad y = 1/3$

$\therefore$ (0, 0) and (1/ 2, 1/ 3) are stationary points.

Step III : (i) At $x = 0,\ y = 0,\ r = 0,\ s = 0,\ t = 0 \therefore\ rt - s^2 = 0.$

Hence, our method fails and we reject this pair.

(ii) At $x = 1/2,\ y = 1/3.$

$$r = f_{xx} = 6\left(\frac{1}{2}\right)\left(\frac{1}{9}\right) - 12\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) - 6\left(\frac{1}{2}\right)\left(\frac{1}{27}\right) = \frac{1}{3} - \frac{1}{3} - \frac{1}{9} = -\frac{1}{9}$$

$$s = f_{xy} = 6\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) - 8\left(\frac{1}{8}\right)\left(\frac{1}{3}\right) - 9\left(\frac{1}{4}\right)\left(\frac{1}{9}\right) = \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = -\frac{1}{12}$$

$$t = f_{yy} = 2\left(\frac{1}{8}\right) - 2\left(\frac{1}{16}\right) - 6\left(\frac{1}{8}\right)\left(\frac{1}{3}\right) = \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$$

$$\therefore\quad rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(\frac{1}{12}\right)^2 = \frac{1}{72} - \frac{1}{144} = \frac{1}{144} > 0$$

And $r = -\dfrac{1}{9} < 0\quad \therefore\quad f(x, y)$ is a maxima.

$$\text{Maximum Value} = \frac{1}{8}\cdot\frac{1}{9}\left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{72}\cdot\frac{1}{6} = \frac{1}{432}.$$

Example 11 (b) : Find the minimum and maximum values of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Sol. : We have $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Step I : $f_x = 4x^3 - 4x + 4y$, $\quad f_y = 4y^3 + 4x - 4y$

$f_{xx} = 12x^2 - 4$, $f_{xy} = 4$, $f_{yy} = 12y^2 - 4$.

Step II : We now solve $f_x = 0$, $f_y = 0$.

$\therefore \quad x^3 - x + y = 0$ and $y^3 + x - y = 0$

Adding the two, we get $\quad x^3 + y^3 = 0 \qquad \therefore \; x = -y$

Putting $y = -x$ in $x^3 - x + y = 0$, we get

$x^3 - 2x = 0 \qquad \therefore \; x(x^2 - 2) = 0 \qquad \therefore \; x = 0, \sqrt{2}, -\sqrt{2}$.

Since, $y = -x$, we have $\qquad y = 0, -\sqrt{2}, \sqrt{2}$.

Hence, $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$ are stationary points.

Step III : (i) At $x = 0, y = 0$, $r = f_{xx} = -4$, $s = f_{xy} = 4$, $t = f_{yy} = -4$.

$\therefore \quad rt - s^2 = (-4)(-4) - 4^2 = 0$

Hence, our method fails and we reject this pair.

(ii) At $x = \sqrt{2}, y = -\sqrt{2}$,

$r = f_{xx} = 12(2) - 4 = 20$, $s = f_{xy} = 4$, $t = f_{yy} = 12(2) - 4 = 20$

$\therefore \quad rt - s^2 = 400 - 16 = 384 > 0$

But $r = 20 > 0$. $\qquad \therefore \; f(x, y)$ is minimum.

Minimum value $= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 = -8$.

(iii) At $x = -\sqrt{2}, y = \sqrt{2}$,

$r = f_{xx} = 12(2) - 4 = 20$, $s = f_{xy} = 4$, $t = f_{yy} = 12(2) - 4 = 20$.

$\therefore \quad rt - s^2 = 400 - 16 = 384 > 0$

But $r = 20 > 0$. $\qquad \therefore \; f(x, y)$ is minimum.

Minimum value $= -8$.

Example 12 (b) : Show that the minimum value of $u = xy + a^3\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$ is $3a^2$.

(M.U. 1989, 96, 2002, 09)

Sol. : We have $\quad f(x, y) = xy + a^3\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

Step I : $f_x = y - \dfrac{a^3}{x^2}$, $\quad f_y = x - \dfrac{a^3}{y^2}$; $\quad f_{xx} = \dfrac{2a^3}{x^3}$, $\quad f_{xy} = 1$, $\quad f_{yy} = \dfrac{2a^3}{y^3}$.

Step II : We now solve $f_x = 0$, $f_y = 0$.

$\therefore \quad x^2 y = a^3$ and $x = \dfrac{a^3}{y^2} \qquad \therefore \; \dfrac{a^6}{y^3} = a^3 \qquad \therefore \; y = a$

Hence, $x^2 = a^2 \qquad \therefore \; x = +a$ or $-a$.

$\therefore$ (a, a), $(-a, a)$ are stationary points.

Step III : (i) When $x = -a$, $y = a$

$r = f_{xx} = -2$, $\quad r = f_{xy} = 1$, $\quad t = f_{yy} = 2$

$\therefore \quad rt - s^2 = -4 - 1 < 0$

$\therefore \quad f(x, y)$ is neither maximum nor minimum. It is a saddle point.

(ii) When $x = a, \ y = a$; $\quad r = f_{xx} = 2, \quad r = f_{xy} = 1, \quad t = f_{yy} = 2$

$\therefore \quad rt - s^2 = 4 - 1 > 0$

And $\quad r = f_{xx} = 2 > 0 \quad \therefore \ f(x, y)$ is minimum at (a, a).

At (a, a), $\quad u = a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right) = 3a^2.$

$\therefore \quad$ Minimum value $= 3a^2$.

Example 13 (b) : Find the stationary values of $\sin x \cdot \sin y \cdot \sin(x + y)$.

(M.U. 1989, 97, 2013)

Sol. : We have $\quad f(x, y) = \sin x \cdot \sin y \cdot \sin(x + y)$

Spte I :

$$\begin{aligned} f_x &= \sin y\,[\cos x \cdot \sin(x + y) + \sin x \cdot \cos(x + y)] \\ &= \sin y \cdot \sin(2x + y) \end{aligned}$$

Similarly,

$$\begin{aligned} f_y &= \sin x \cdot \sin(x + 2y) \\ f_{xx} &= 2 \sin y \cdot \cos(2x + y) \\ f_{xy} &= \cos y \cdot \sin(2x + y) + \sin y \cdot \cos(2x + y) \\ &= \sin(2x + 2y) \\ f_{yy} &= 2 \sin x \cdot \cos(x + 2y) \end{aligned}$$

Step II : Now, we solve $f_x = 0, \ f_y = 0$.

$\therefore \quad \sin y \sin(2x + y) = 0$ and $\sin x \sin(x + 2y) = 0$

$\therefore \quad x = 0, \ y = 0$ or $x + 2y = \pi$ and $2x + y = \pi$ and hence

$$x = \frac{\pi}{3}, \ y = \frac{\pi}{3}.$$

$\therefore \quad (0, 0)$ and $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ are stationary points.

Step III : (i) When $x = 0, \ y = 0$; $\quad r = f_{xx} = 0, \ s = f_{xy} = 0, \ t = f_{yy} = 0$

$\therefore \quad rt - s^2 = 0$

$\therefore \quad$ Out method fails. We reject this pair.

(ii) When $x = \frac{\pi}{3}, \ y = \frac{\pi}{3}$,

$$r = f_{xx} = 2 \cdot \frac{\sqrt{3}}{2} \cdot (-1) = -\sqrt{3}, \quad s = f_{xy} = -\frac{\sqrt{3}}{2}, \quad t = f_{yy} = -\sqrt{3}$$

$\therefore \quad rt - s^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0 \qquad$ And $\qquad r = f_{xx} = -\sqrt{3} < 0$

$\therefore \quad x = \frac{\pi}{3}, \ y = \frac{\pi}{3}$ is a maxima.

$$\text{Maximum value} = \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}.$$

Example 14 (b) : Find the maximum value of $\cos A \cos B \cos C$, where A, B, C are angles of a triangle. **(M.U. 1998)**

Sol. : We have $f(x, y) = \cos x \cos y \cos z$

But $x + y + z = \pi$ $\therefore$ $\cos z = \cos[\pi - (x + y)] = -\cos(x + y)$

$\therefore$ $f(x, y) = -\cos x \cos y \cos(x + y)$

Step I : $f_x = \cos y[\sin x \cos(x + y) + \cos x \sin(x + y)]$

$= \cos y \cdot \sin(2x + y)$

$\therefore$ $f_y = \cos x[\sin y \cos(x + y) + \cos y \sin(x + y)]$

$= \cos x \cdot \sin(x + 2y)$

Now, $r = f_{xx} = 2\cos y \cos(2x + y)$

$s = f_{xy} = -\sin y \sin(2x + y) + \cos y \cos(2x + y)$

$= \cos(2x + 2y)$

$t = f_{yy} = 2\cos x \cos(x + 2y)$

Step II : $f_x = 0$ gives $\cos y \sin(2x + y) = 0$

$\therefore$ $\cos y = 0$ or $\sin(2x + y) = 0$ $\therefore$ $y = \pi/2$ or $2x + y = \pi$ .... (1)

$f_y = 0$ gives $\cos x \sin(2x + y) = 0$

$\therefore$ $\cos x = 0$ or $\sin(x + 2y) = 0$ $\therefore$ $x = \pi/2$ or $x + 2y = \pi$ .... (2)

Solving (1) and (2), we get $x = y = \pi/3$.

Step III : When $x = y = \pi/3$

$$r = f_{xx} = 2\cos\left(\frac{\pi}{3}\right)\cos\pi = -1, \quad s = f_{xy} = \cos\left(\frac{2\pi}{3} + \frac{2\pi}{3}\right) = -\frac{1}{2},$$

$$t = f_{yy} = 2\cos\left(\frac{\pi}{3}\right)\cos\pi = -1.$$

$$\therefore \quad rt - s^2 = 1 - \left(\frac{1}{4}\right) = \frac{3}{4} > 0$$

$\therefore$ $f(x, y)$ is stationary at $x = \frac{\pi}{3}$, $y = \frac{\pi}{3}$.

But $$r = f_{xx} = 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right) = -1 < 0$$

$\therefore$ $f(x, y)$ is maximum at $x = \frac{\pi}{3}$, $y = \frac{\pi}{3}$.

$\therefore$ Maximum value of $f(x, y) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{2\pi}{3}\right) = \frac{1}{8}$.

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Find the extreme values of the following functions : Class (b) : 6 Marks

(1) $x^2 + y^2 + 6x + 12$

(2) $x^4 + y^4 - x^2 - y^2 + 1$

(3) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$ **(M.U. 2006, 11)**

(4) $x^2y^3(1 - x - y)$ **(M.U. 2001)**

(5) $x^3 + y^3 - 63(x + y) + 12xy$ (M.U. 2002)

(6) $x^3 + xy^2 + 21x - 12x^2 - 2y^2$ (M.U. 2002, 04)

(7) $x^4 + y^4 - 2(x - y)^2$ (M.U. 2002)

(8) $xy(3a - x - y)$ (M.U. 1992, 2018)

(9) $2(x^2 - y^2) - x^4 + y^4$ (M.U. 1990)

(10) $y^2 + 4xy + 3x^2 + x^3$ (M.U. 1993)

(11) $xy(a - x - y)$ (M.U. 1997)

(12) $x^4 + y^4 - 4a^2 xy$

(13) $x^2 y - 3x^2 - 2y^2 - 4y + 3$ (M.U. 2002)

(14) $x^3 y^2 (12 - 3x - 4y)$ (M.U. 2002, 03)

(15) $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ (M.U. 2002)

[**Ans. :** (1) Min. value 3 at $(-3, 0)$.

(2) Max. value 1 at $(0, 0)$; min. value 1/2 at four points $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$

(3) Max. value 7 at $(0, 0)$, Min. value 3 at $(2, 0)$.

(4) Max. value 1 / 432 at $(1/3, 1/2)$.

(5) Min. value -216 at $(3, 3)$, Max. value 784 at $(-7, -7)$.

(6) Max value 10 at $(1, 0)$, Min. value -98 at $(7, 0)$.

(7) Min. value -8 at $(-\sqrt{2}, \sqrt{2})$ and at $(\sqrt{2}, -\sqrt{2})$

(8) Max. value a^3 at (a, a)

(9) No stationary value.

(10) No stationary value.

(11) Max. value $\dfrac{a^3}{27}$ at $\left(\dfrac{a}{3}, \dfrac{a}{3}\right)$.

(12) Min. value $-8a^4$ at (a, a) and $(-a, -a)$.

(13) Max. at $(0, -1)$, Max. value 5.

(14) Max. at $(2, 1)$, Max. value 16.

(15) Min. at $(\pm\sqrt{2}, \pm\sqrt{2})$.]

3. Lagrange's Method of Undetermined Multipliers

Consider the function $u = f(x, y, z)$ .... (1)

with the condition $\Phi(x, y, z) = 0$ .... (2)

Consider the following function, called Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\,\Phi(x, y, z)$$

$$\therefore \quad dF = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz + \lambda\left[\frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \frac{\partial \Phi}{\partial z}dz\right]$$

$$= \left[\frac{\partial f}{\partial x} + \lambda\frac{\partial \Phi}{\partial x}\right]dx + \left[\frac{\partial f}{\partial y} + \lambda\frac{\partial \Phi}{\partial y}\right]dy + \left[\frac{\partial f}{\partial z} + \lambda\frac{\partial \Phi}{\partial z}\right]dz$$

Now, $dF = 0$ gives

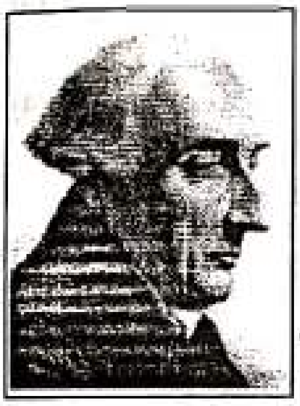
$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \Phi}{\partial x} = 0 \; ; \; \frac{\partial f}{\partial y} + \lambda \frac{\partial \Phi}{\partial y} = 0 \; ; \; \frac{\partial f}{\partial z} + \lambda \frac{\partial \Phi}{\partial z} = 0 \qquad \text{.....................} (3)$$

We have to eliminate x, y, z and λ from (1), (2) and (3).

The values so obtained are the values of x, y, z where u may be stationary.

It may be noted that Lagrange's method does not enable us to find a maxima or a minima. It gives us only a stationary value. Whether the value obtained is a maxima or a minima is determined from other physical considerations.

Joseph Louis Lagrange (1736 - 1813)



A French-Italian mathematician who developed a new branch of mathematics called the **calculus of variations.** His father wanted him to be a lawyer but he was attracted to mathematics after reading a memoir by the great astronomer Halley. At the age of 16 he began to study mathematics on his own and at the age of 19 he was appointed as a professor. By the age of 25, he was regarded by many as the greatest living mathematician. In 1766, Fredrick the great wrote Lagrange that "the greatest king in Europe would like to have the greatest mathematician of Europe" at his court. Langrange accepted this invitation and remained in Berlin for the next 20 years. He made significant contributions to analysis, analytic mechanics, calculus, probability and number theory. He is known for his memoir on analysis of mechanics. He was buried in Pantheon - a temple in Rome build in 27B.C. where famous persons are buried.

Procedure of Lagrange's Method

Step I : Write the function to be maximised or minimised

$$u = f(x, y, z) \qquad \text{.....................} (1)$$

with the condition

$$\Phi(x, y, z) = 0 \qquad \text{.....................} (2)$$

Form a new function called Lagrange's function.

$$F(x, y, z) = f(x, y, z) + \lambda\, \Phi(x, y, z) \qquad \text{.....................} (3)$$

For stationary values, we have $dF = 0$.

$$\therefore \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \lambda \left[\frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz \right] = 0$$

$$\therefore \quad \left[\frac{\partial f}{\partial x} + \lambda \frac{\partial \Phi}{\partial x} \right] dx + \left[\frac{\partial f}{\partial y} + \lambda \frac{\partial \Phi}{\partial y} \right] dy + \left[\frac{\partial f}{\partial z} + \lambda \frac{\partial \Phi}{\partial z} \right] dz = 0$$

$$\therefore \quad \frac{\partial f}{\partial x} + \lambda \frac{\partial \Phi}{\partial x} = 0 \; ; \; \frac{\partial f}{\partial y} + \lambda \frac{\partial \Phi}{\partial y} = 0 \; ; \; \frac{\partial f}{\partial z} + \lambda \frac{\partial \Phi}{\partial z} = 0 \qquad \text{.....................} (4)$$

Step II : We have to solve the equations (1), (2) and (4). We eliminate x, y, z and λ from these five equations and find the value of u.

Type I : Two Independent Variables : Class (b) : 6 Marks

Example 1 (b) : Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange's multipliers.

Sol. : Step I : We have to find the stationary values of

$$u = f \equiv 3x + 4y \quad \text{.................... (1)}$$

with the condition $\Phi \equiv x^2 + y^2 - 1 = 0$ .... (2)

Consider the Lagrange's function

$$F(x, y) = f(x, y) + \lambda \Phi(x, y) = 3x + 4y + \lambda(x^2 + y^2 - 1)$$

$$\therefore \quad dF = (3 + 2\lambda x)\, dx + (4 + 2\lambda y)\, dy$$

$dF = 0$ gives $3 + 2\lambda x = 0,\ 4 + 2\lambda y = 0$ .... (3)

Step II : We have to eliminate x, y and λ from (1), (2) and (3).

From (3), we get $x = -3/2\lambda,\ y = -4/2\lambda$ .... (4)

Putting these values in (2),

$$\frac{9}{4\lambda^2} + \frac{16}{4\lambda^2} - 1 = 0 \quad \therefore \quad 4\lambda^2 = 25 \quad \therefore \quad \lambda = \pm\frac{5}{2}.$$

Hence, from (4), we get

$$\text{when } \lambda = \frac{5}{2}, \quad x = -\frac{3}{5},\ y = -\frac{4}{5}$$

$$\text{when } \lambda = -\frac{5}{2}, \quad x = \frac{3}{5},\ y = \frac{4}{5}$$

$$\therefore \quad \text{When } x = -\frac{3}{5},\ y = -\frac{4}{5}, \quad f(x, y) = -\frac{9}{5} - \frac{16}{5} = -5$$

$$\text{When } x = \frac{3}{5},\ y = \frac{4}{5} \quad f(x, y) = \frac{9}{5} + \frac{16}{5} = 5$$

Further, $\dfrac{\partial f}{\partial x} = 3$, $\dfrac{\partial f}{\partial y} = 4$ and second order partial derivatives are zero. Hence, we cannot use usual criterion for determining maxima or minima.

But $f(x, y) = z = 3x + 4y$ is a plane passing through the origin. Hence, $f(x, y) = -5$ is the minimum value and $f(x, y) = 5$ is the maximum value of $f(x, y)$ on the circle $x^2 + y^2 = 1$.

Example 2 (b) : Find the greatest and the least values the function $f(x, y) = xy$ takes on the ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{2} = 1$ by Lagrange's method.

Sol. : Step I : We have to find the stationary values of

$$u = f \equiv xy \quad \text{.................... (1)}$$

with the condition that $\Phi = \dfrac{x^2}{8} + \dfrac{y^2}{2} - 1 = 0$ .... (2)

Consider the Lagrange's function

$$F(x, y) = f(x, y) + \lambda \Phi(x, y) = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$$

$$\therefore \quad dF = x\,dy + y\,dx + \frac{2x}{8}\lambda\, dx + \frac{2y}{2}\lambda\, dy = \left(y + \frac{\lambda x}{4}\right)dx + (x + \lambda y)\, dy$$

$\therefore \quad dF = 0$ gives $\quad y + \dfrac{\lambda x}{4} = 0$ and $x + \lambda y = 0$ .... (3)

Step II : We have to eliminate x, y, λ from (1), (2) and (3).

Putting the value of λ from the first equation in the second equation of (3), we get,

$$x - \frac{4y}{x} \cdot y = 0 \quad \therefore \quad x^2 - 4y^2 = 0 \quad \therefore \quad y^2 = \frac{x^2}{4}.$$

But $\quad \dfrac{x^2}{8} + \dfrac{y^2}{2} = 1 \quad \therefore \quad \dfrac{x^2}{8} + \dfrac{x^2}{8} = 1 \quad \therefore \quad 2x^2 = 8 \quad \therefore \quad x^2 = 4$

$$\therefore \quad x = \pm 2 \quad \therefore \quad y^2 = \frac{x^2}{4} = 1 \quad \therefore \quad y = \pm 1$$

Hence, the points of stationary values are (2, 1), (2, –1), (–2, 1), (– 2, –1).

Now, the stationary value of $f(x, y) = xy$ at (2, 1) = 2, and at (2, –1) = – 2

at (–2, 1) = – 2, and at (– 2, – 1) = 2

i.e. either 2 or – 2.

Hence, the maximum value of $f(x, y)$ is 2 and the minimum value is – 2.

Example 3 (b) : Find the value of $x^2 + y^2$ subjected to the condition $x + y = 2$ by Lagrange's method.

Sol. : Step I : We have to find the stationary value of

$$u \equiv f \equiv x^2 + y^2 \quad \text{.................... (1)}$$

with the condition $\quad \Phi \equiv x + y - 2 = 0$ .... (2)

Consider the Lagrange's function

$$F(x, y) = f(x, y) + \lambda\, \Phi(x, y) = x^2 + y^2 + \lambda(x + y - 2)$$

$$\therefore \quad dF = (2x + \lambda)\, dx + (2y + \lambda)\, dy$$

$\therefore \quad dF = 0$ gives $\quad 2x + \lambda = 0$ and $2y + \lambda = 0$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), we get $\quad x = -\dfrac{\lambda}{2}, \quad y = -\dfrac{\lambda}{2}.$ .... (4)

Putting these values in (2), we get $\quad -\dfrac{\lambda}{2} - \dfrac{\lambda}{2} - 2 = 0 \quad \therefore \quad \lambda = -2.$

Now, from (4), we get when $\lambda = -2, \quad x = 1, \quad y = 1.$

Hence, $f(x, y) = x^2 + y^2$ is maximum at $x = 1$, $y = 1$ and maximum value $= 1^2 + 1^2 = 2$.

Example 4 (b) : If $u \equiv x^2 + y^2 + xy$ with the condition $\Phi \equiv ax^2 + by^2 - ab = 0$, prove that the stationary values of u are given by $4(u - a)(u - b) = ab$.

Sol. : Step I : We have $\quad u \equiv f \equiv x^2 + y^2 + xy$ .... (1)

with the condition $\quad \Phi \equiv ax^2 + by^2 - ab = 0$ .... (2)

Consider the Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\, \Phi(x, y, z)$$

$$= (x^2 + y^2 + xy) + \lambda(ax^2 + by^2 - ab)$$

$$\therefore \quad dF = 2x\, dx + 2y\, dy + x\, dy + y\, dx + \lambda\, 2ax\, dx + \lambda\, 2by\, dy$$

$$= (2x + y + \lambda\, 2ax)\, dx + (2y + x + \lambda\, 2by)\, dy$$

$\therefore \quad dF = 0$ gives $\quad 2x + y + \lambda\, 2ax = 0, \;\; 2y + x + \lambda\, 2by = 0$ .... (3)

Step II : We have to eliminate x, y and λ from (1), (2) and (3).

Multiply the first equation in (3) by x and second by y and add.

$$\therefore \quad 2(x^2+y^2+xy)+2\lambda(ax^2+by^2)=0 \qquad \text{.................... (4)}$$

Using (1) and (2) in (4), we get, $2u+2\lambda ab=0 \quad \therefore \ \lambda=-\dfrac{u}{ab}$

Putting this value of λ in the two equations of (3), we get

$$2x+y-\frac{2ux}{b}=0, \ \ 2y+x-\frac{2uy}{a}=0$$

$$\therefore \quad 2\left(1-\frac{u}{b}\right)x=-y \ \text{ and } \ 2\left(1-\frac{u}{a}\right)y=-x$$

$$\therefore \quad \frac{x}{y}=-\frac{1}{2[1-(u/b)]} \ \text{ and } \ -2\left(1-\frac{u}{a}\right)=\frac{x}{y}$$

Equating the two values of x/y.

$$\frac{1}{2[1-(u/b)]}=2\left(1-\frac{u}{a}\right) \qquad \therefore \ 4\left(1-\frac{u}{a}\right)\left(1-\frac{u}{b}\right)=1$$

$$\therefore \quad 4(a-u)(b-u)=ab \quad i.e. \quad 4(u-a)(u-b)=ab$$

The roots of this equation are the stationary values.

Type II : Three Independent Variables : Class (b) : 6 Marks

Example 1 (b) : If $u=x^2+y^2+z^2$ where $\Phi=ax+by+cz-p=0$, find the stationary value of u.

Sol. : Step I : We have to find the stationary value of

$$u\equiv f\equiv x^2+y^2+z^2 \qquad \text{.................... (1)}$$

with the condition that $\quad \Phi\equiv ax+by+cz-p=0 \qquad$ .... (2)

Consider the Lagrange's function

$$F(x,y,z)=f(x,y,z)+\lambda\,\Phi(x,y,z)$$
$$=(x^2+y^2+z^2)+\lambda(ax+by+cz-p)$$

$$\therefore dF=2x\,dx+2y\,dy+2z\,dz+\lambda a\,dx+\lambda\,b\,dy+\lambda\,c\,dz$$
$$=(2x+\lambda\,a)\,dx+(2y+\lambda\,b)\,dy+(2z+\lambda\,c)\,dz$$

$$\therefore dF=0 \text{ gives, } 2x+\lambda\,a=0, \ \ 2y+\lambda\,b=0, \ \ 2z+\lambda\,c=0 \qquad \text{.................... (3)}$$

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), $\quad x=-\dfrac{a\lambda}{2}, \ \ y=-\dfrac{b\lambda}{2}, \ \ z=-\dfrac{c\lambda}{2}. \qquad$ .... (4)

Putting these values in (2),

$$-\frac{a^2\lambda}{2}-\frac{b^2\lambda}{2}-\frac{c^2\lambda}{2}-p=0 \qquad \therefore \ \lambda=-\frac{2p}{a^2+b^2+c^2}$$

Hence, from (4),

$$x=\frac{ap}{a^2+b^2+c^2}, \ y=\frac{bp}{a^2+b^2+c^2}, \ z=\frac{cp}{a^2+b^2+c^2}$$

Hence, the stationary value of u is

$$u = x^2 + y^2 + z^2 = \frac{(a^2+b^2+c^2)p^2}{(a^2+b^2+c^2)^2} = \frac{p^2}{a^2+b^2+c^2}.$$

Example 2 (b) : Prove that the stationary value of $x^m y^n z^p$ under the condition $x+y+z=a$ is

$$m^m n^n p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}.$$

Sol. : Step I : We have to find the stationary value of

$$u = f \equiv x^m y^n z^p \qquad \text{.................... (1)}$$

with the condition that $\quad \Phi \equiv x+y+z=a \qquad$ .... (2)

Consider the Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\,\Phi(x, y, z) = x^m y^n z^p + \lambda(x+y+z-a)$$

$$\therefore \quad dF = mx^{m-1}y^n z^p\,dx + n\,y^{n-1}x^m z^p\,dy + p\,z^{p-1}x^m y^n\,dz + \lambda\,dx + \lambda\,dy + \lambda\,dz$$

$$= \left(\frac{m}{x}x^m y^n z^p + \lambda\right)dx + \left(\frac{n}{y}x^m y^n z^p + \lambda\right)dy + \left(\frac{p}{z}x^m y^n z^p + \lambda\right)dz$$

$$\therefore \quad dF = \left(\frac{mu}{x} + \lambda\right)dx + \left(\frac{nu}{y} + \lambda\right)dy + \left(\frac{pu}{z} + \lambda\right)dz = 0$$

$$\therefore \quad dF = 0 \text{ gives } \quad \frac{mu}{x} + \lambda = 0,\quad \frac{nu}{y} + \lambda = 0,\quad \frac{pu}{z} + \lambda = 0 \qquad \text{.................... (3)}$$

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3) $\quad x = -mu/\lambda,\ y = -nu/\lambda,\ z = -pu/\lambda. \qquad$ .... (4)

Putting these values in (2),

$$-\frac{mu}{\lambda} - \frac{nu}{\lambda} - \frac{pu}{\lambda} = a \quad \therefore \ -\frac{(m+n+p)}{a}u = \lambda$$

Putting this value of λ in (4), we get,

$$x = -\frac{mu}{\lambda} = \frac{ma}{m+n+p},\quad y = \frac{na}{m+n+p},\quad z = \frac{pa}{m+n+p}$$

Hence, the stationary value of u is

$$u = x^m y^n z^p = \left(\frac{ma}{m+n+p}\right)^m \cdot \left(\frac{na}{m+n+p}\right)^n \cdot \left(\frac{pa}{m+n+p}\right)^p$$

$$= m^m n^n p^p \left(\frac{a}{m+n+p}\right)^{m+n+p}.$$

Example 3 (b) : If x, y, z are the angles of a triangle, show that $u = \cos x \cos y \cos z$ is maximum when the triangle is equilateral.

Sol. : Step I : We have $\quad u \equiv f \equiv \cos x \cos y \cos z \qquad$ .... (1)

with the condition $\quad \Phi \equiv x+y+z \equiv \pi \qquad$ .... (2)

Consider the Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\,\Phi(x, y, z)$$

$$= (\cos x \cos y \cos z) + \lambda(x+y+z-\pi)$$

$\therefore \quad dF = (-\sin x \cos y \cos z)\, dx + (-\cos x \sin y \cos z)\, dy$
$$+ (-\cos x \cos y \sin z)\, dz + \lambda\, dx + \lambda\, dy + \lambda\, dz$$
$$= (-\sin x \cos y \cos z + \lambda)\, dx + (-\cos x \sin y \cos z + \lambda)\, dy$$
$$+ (-\cos x \cos y \sin z + \lambda)\, dz$$

$\therefore \quad dF = 0$ gives $\quad -\sin x \cos y \cos z + \lambda = 0$;

$$-\cos x \sin y \cos z + \lambda = 0 ; \quad -\cos x \cos y \sin z + \lambda = 0 \quad \text{.......... (3)}$$

Step II : We have to eliminate x, y, z, λ from (1), (2) and (3).

From the equations in (3), we get

$$\sin x \cos y \cos z = \cos x \sin y \cos z = \cos x \cos y \sin z = \lambda$$

Dividing by $\cos x \cos y \cos z$, we get $\quad \tan x = \tan y = \tan z$

Hence, $x = y = z$.

This means the triangle is equilateral.

Example 4 (b) : Find the minimum distance of the origin from the plane $3x + 2y + z - 12 = 0$.

Sol. : If (x, y, z) is any point on the plane, its distance from the origin is $\sqrt{x^2 + y^2 + z^2}$.

Step I : We have to find the stationary value of

$$u = f = \sqrt{x^2 + y^2 + z^2} \quad \text{.......... (1)}$$

with the condition that $\quad \Phi = 3x + 2y + z - 12 = 0 \quad$ .... (2)

Consider the Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\, \Phi(x, y, z)$$
$$= \sqrt{x^2 + y^2 + z^2} + \lambda(3x + 2y + z - 12)$$

$$\therefore \quad dF = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\, dx + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\, dy$$
$$+ \frac{z}{\sqrt{x^2 + y^2 + z^2}}\, dz + 3\lambda\, dx + 2\lambda\, dy + \lambda\, dz$$
$$= \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} + 3\lambda\right) dx + \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} + 2\lambda\right) dy + \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} + \lambda\right) dz$$

$\therefore\ dF = 0$ gives

$$x + 3\lambda \sqrt{x^2 + y^2 + z^2} = 0, \quad y + 2\lambda \sqrt{x^2 + y^2 + z^2} = 0, \quad z + \lambda \sqrt{x^2 + y^2 + z^2} = 0$$

i.e. $\quad x + 3\lambda u = 0, \quad y + 2\lambda u = 0, \quad z + \lambda u = 0 \quad$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), we have $\quad x = -3\lambda u, \quad y = -2\lambda u, \quad z = -\lambda u \quad$ .... (4)

Putting these values in (2),

$$-9\lambda u - 4\lambda u - \lambda u - 12 = 0 \qquad \therefore \quad \lambda u = -\frac{12}{14} = -\frac{6}{7}$$

Hence, from (4) we get $\quad x = \frac{18}{7}, \quad y = \frac{12}{7}, \quad z = \frac{6}{7}$.

Hence, the stationary value of u from (1) is

$$u = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{18^2}{7^2}+\frac{12^2}{7^2}+\frac{6^2}{7^2}} = \sqrt{\frac{504}{49}} = \sqrt{\frac{72}{7}}.$$

Example 5 (b) : Find the minimum distance from the point (1, 2, 0) to the cone $z^2 = x^2 + y^2$.

Sol. : If (x, y, z) is any point on the cone, its distance from the point (1,2, 0) is

$$\sqrt{(x-1)^2+(y-2)^2+(z-0)^2}$$

When the distance is minimum, its square is also minimum.

Step I : We have to find the stationary value of

$$u \equiv f \equiv (x-1)^2+(y-2)^2+z^2 \qquad \text{.......... (1)}$$

with the condition that $\Phi = x^2+y^2-z^2$ .... (2)

Consider the Lagrange's function.

$$\begin{aligned} F(x, y, z) &= f(x, y, z) + \lambda\, \Phi(x, y, z) \\ &= (x-1)^2+(y-2)^2+z^2+\lambda(x^2+y^2-z^2) \end{aligned}$$

$\therefore$ $dF = 0$ gives

$$2(x-1)\,dx + 2(y-2)\,dy + 2z\,dz + \lambda(2x\,dx+2y\,dy-2z\,dz) = 0$$

$$\therefore \quad (x-1)+\lambda x = 0, \quad (y-2)+\lambda y = 0, \quad z - \lambda z = 0 \qquad \text{.......... (3)}$$

Step II : We have to eliminate x, y, z, λ from (1), (2) and (3).

From (3),

$\because$ $z-\lambda z = 0$ $\quad\therefore$ $z(\lambda-1)=0$ $\quad\therefore$ $\lambda = 1, z = 0$

Hence, $x-1+x=0$ $\quad\therefore$ $2x = 1$ $\quad\therefore$ $x = 1/2$

$y-2+y=0$ $\quad\therefore$ $2y=2$ $\quad\therefore$ $y = 1$

Hence, the required point is (1/2, 1, 0).

The distance between (1, 2, 0) and (1/2, 1, 0) is

$$= \sqrt{\left[\left(1-\frac{1}{2}\right)^2+(2-1)^2+(0-0)^2\right]} = \sqrt{\frac{1}{4}+1} = \frac{\sqrt{5}}{2}.$$

Example 6 (b) : Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2+y^2+z^2=1$.

Sol. : The distance of the point (3, 4, 12) from any point (x, y, z) on the sphere is

$$\sqrt{[(x-3)^2+(y-4)^2+(z-12)^2]} \text{ where } x^2+y^2+z^2=1.$$

If this distance is maximum or minimum, so will be the square of the distance.

Step I : Thus, we have to maximise or minimise.

$$u \equiv f \equiv (x-3)^2+(y-4)^2+(z-12)^2 \qquad \text{.......... (1)}$$

with the condition that $\Phi \equiv x^2+y^2+z^2-1=0$ .... (2)

Consider the Lagrange's function

$$\begin{aligned} F(x, y, z) &= f(x, y, z) + \lambda\, \Phi(x, y, z) \\ &= [(x-3)^2+(y-4)^2+(z-12)^2] + \lambda(x^2+y^2+z^2-1) \end{aligned}$$

$\therefore \quad dF = 2(x-3)\,dx + 2(y-4)\,dy + 2(z-12)\,dz + 2\lambda\, x\,dx + 2\lambda\, y\,dy + 2\lambda\, z\,dz$

$\quad = [2(x-3) + 2\lambda x]\,dx + [2(y-4) + 2\lambda y]\,dy + [2(z-12) + 2\lambda z]\,dz$

$\therefore \quad dF = 0$ gives, $2(x-3) + 2\lambda x = 0$, $2(y-4) + 2\lambda y = 0$ and $2(z-12) + 2\lambda z = 0$

$\therefore \quad (x-3) + \lambda x = 0,\ (y-4) + \lambda y = 0,\ (z-12) + \lambda z = 0$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), $x = \dfrac{3}{1+\lambda},\ y = \dfrac{4}{1+\lambda},\ z = \dfrac{12}{1+\lambda}$ .... (4)

Putting these values in (2), we get

$$\frac{9}{(1+\lambda)^2} + \frac{16}{(1+\lambda)^2} + \frac{144}{(1+\lambda)^2} = 1 \quad \therefore \quad (1+\lambda)^2 = 169 \quad \therefore \quad 1+\lambda = \pm 13$$

Hence, from (4),

$$x = \frac{3}{13},\ y = \frac{4}{13},\ z = \frac{12}{13} \qquad \text{or} \qquad x = -\frac{3}{13},\ y = -\frac{4}{13},\ z = -\frac{12}{13}$$

$\therefore$ The minimum distance from (1) is

$$\sqrt{\left[\left(\frac{3}{13}-3\right)^2 + \left(\frac{4}{13}-4\right)^2 + \left(\frac{12}{13}-12\right)^2\right]} = 12$$

The maximum distance from (1) is

$$= \sqrt{\left[\left(\frac{3}{13}+3\right)^2 + \left(\frac{4}{13}+4\right)^2 + \left(\frac{12}{13}+12\right)^2\right]} = 14.$$

Example 7 (b) : If $\dfrac{3}{x} + \dfrac{4}{y} + \dfrac{5}{z} = 6$, find the values of x, y, z such that $x + y + z$ is minimum.

Sol. : Step I : We have to find the stationary values of

$u \equiv f \equiv x + y + z$ .... (1)

with the condition that $\Phi \equiv \dfrac{3}{x} + \dfrac{4}{y} + \dfrac{5}{z} - 6 = 0$ .... (2)

Consider the Lagrange's function

$F(x, y, z) = f(x, y, z) + \lambda\, \Phi(x, y, z)$

$F(x, y, z) = (x + y + z) + \lambda\left(\dfrac{3}{x} + \dfrac{4}{y} + \dfrac{5}{z} - 6\right)$

$\therefore \quad dF \equiv dx + dy + dz - \dfrac{3\lambda}{x^2}dx - \dfrac{4\lambda}{y^2}dy - \dfrac{5\lambda}{z^2}dz$

$\quad = \left(1 - \dfrac{3\lambda}{x^2}\right)dx + \left(1 - \dfrac{4\lambda}{y^2}\right)dy + \left(1 - \dfrac{5\lambda}{z^2}\right)dz$

$\therefore \quad dF = 0$ gives, $1 - \dfrac{3\lambda}{x^2} = 0,\ 1 - \dfrac{4\lambda}{y^2} = 0,\ 1 - \dfrac{5\lambda}{z^2} = 0$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), $x = \sqrt{3\lambda},\ y = \sqrt{4\lambda},\ z = \sqrt{5\lambda}.$ .... (4)

Putting these values in (2),

$$\frac{3}{\sqrt{3\lambda}}+\frac{4}{\sqrt{4\lambda}}+\frac{5}{\sqrt{5\lambda}}=6 \quad \therefore\ \sqrt{3}+\sqrt{4}+\sqrt{5}=6\sqrt{\lambda}$$

$$\therefore\quad \sqrt{\lambda}=\frac{(\sqrt{3}+\sqrt{4}+\sqrt{5})}{6}.$$

Hence, from (4),

$$x=\frac{(\sqrt{3}+\sqrt{4}+\sqrt{5})}{6}\cdot\sqrt{3},\quad y=\frac{(\sqrt{3}+\sqrt{4}+\sqrt{5})}{6}\cdot\sqrt{4},\quad z=\frac{(\sqrt{3}+\sqrt{4}+\sqrt{5})}{6}\cdot\sqrt{6}.$$

Example 8 (b) : A space probe in the form of the ellipsoid $4x^2+y^2+4z^2=16$ enters the earth's surface when its surface begins to heat. After one hour the temperature at (x, y, z) on the surface of the probe is given by

$$T(x, y, z)=8x^2+4yz-16z+600.$$

Find the hottest point on the surface of the probe.

Sol. : Step I : We have $\quad u=T(x, y, z)=f=8x^2+4yz-16z+600 \quad$ .... (1)

with the condition $\quad \Phi=4x^2+y^2+4z^2=16 \quad$ .... (2)

Consider the Lagrange's function

$$\begin{aligned} F(x, y, z) &= f(x, y, z)+\lambda\,\Phi(x, y, z) \\ &= (8x^2+4yz-16z+600)+\lambda\,(4x^2+y^2+4z^2-60) \end{aligned}$$

$$\therefore\quad dF=16x\,dx+4z\,dy+4y\,dz-16dz+\lambda\,8x\,dx+\lambda\,2y\,dy+\lambda\,8z\,dz$$

$$=(16x+8\lambda\,x)\,dx+(4z+2\lambda\,y)\,dy+(4y-16+8\lambda\,z)\,dz$$

$\therefore\quad dF=0$ gives $\quad 16x+8\lambda\,x=0,\ 4z+2\lambda\,y=0,\ (4y-16)+8\lambda\,z=0 \quad$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

Now, from (3), we get $8x(\lambda+2)=0$, $\quad x=0$ (which is absurd) or $\lambda=-2$

$\therefore\quad 4z+2\lambda\,y=0$ gives $4z-4y=0 \quad \therefore\ z=y.$

And $4y-16+8\lambda\,z=0$ gives $4y-16-16z=0$

$\therefore\quad y-4-4z=0.$ But $z=y.$

$\therefore\quad 3y=-4 \qquad \therefore\ y=-4/3, \qquad \therefore\ z=-4/3.$

Putting these values in (2), we get

$$4x^2+\frac{16}{9}+4\cdot\frac{16}{9}=16 \quad \therefore\ 4x^2=4\cdot\frac{16}{9} \quad \therefore\ x^2=\frac{16}{9} \quad \therefore\ x=\pm\frac{4}{3}$$

Hence, the temperature is maximum at $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$.

Example 9 (b) : The temperature T at any point (x, y, z) is $T=400\,xyz^2$. Find the highest temperature on the surface of the sphere $x^2+y^2+z^2=1$.

Sol. : Step I : We have to maximize

$u=f=400\,xyz^2$ .... (1)

with the condition that $\quad \Phi=x^2+y^2+z^2-1=0 \quad$ .... (2)

Consider the Lagrange's function

$$F(x, y, z)=f(x, y, z)+\lambda\,\Phi(x, y, z)=400\,xyz^2+\lambda\,(x^2+y^2+z^2-1)$$

$\therefore \quad dF = 400yz^2\,dx + 400xz^2\,dy + 800xyz\,dz + 2\lambda\,x\,dx + 2\lambda\,y\,dy + 2\lambda\,z\,dz$

$\quad = (400\,yz^2 + 2\lambda\,x)\,dx + (400xz^2 + 2\lambda\,y)\,dy + (800\,xyz + 2\lambda\,z)\,dz$

$\therefore \quad dF = 0$ gives

$400yz^2 + 2\lambda\,x = 0; \quad 400xz^2 + 2\lambda\,y = 0; \quad 800xyz + 2\lambda\,z = 0$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

Multiply the equations in (3) by x, y, z respectively and add.

$\therefore \quad 1600xyz^2 + 2\lambda(x^2 + y^2 + z^2) = 0$

$\therefore \quad 800xyz^2 + \lambda(x^2 + y^2 + z^2) = 0$ .... (4)

By using (1) and (2), in (4) we get $\quad 2u + \lambda = 0 \quad \therefore \lambda = -2u.$

Putting this value of λ in (3), *i.e.* in $400yz^2 + 2\lambda\,x = 0$

i.e. in $\quad 400xyz^2 + 2\lambda\,x^2 = 0 \quad \therefore \quad u - 4u\,x^2 = 0 \quad \therefore \quad x = \pm\dfrac{1}{2}$

Similarly, from

$$400xyz^2 + 2\lambda\,y^2 = 0 \quad \therefore \quad u - 4uy^2 = 0 \quad \therefore \quad y = \pm\frac{1}{2}$$

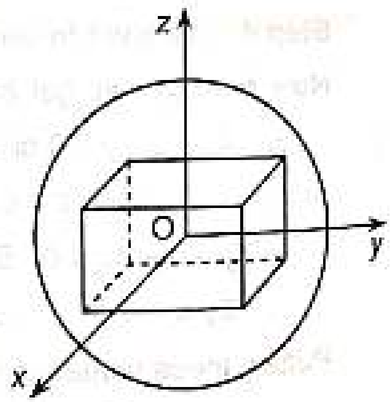
$$800xyz^2 + 2\lambda\,z^2 = 0 \quad \therefore \quad u - 2uz^2 = 0 \quad \therefore \quad z = \pm\frac{1}{\sqrt{2}}$$

Hence, maximum temperature $= 400\,xyz^2 = 400\dfrac{1}{2}\cdot\dfrac{1}{2}\cdot\dfrac{1}{2} = 50.$

Example 10 (b) : Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Sol. : Let the radius of the sphere be a and let the sides of the rectangular solid be $2x$, $2y$ and $2z$.

Hence, the volume V of the solid is given by $V = 8xyz$ with the condition that $x^2 + y^2 + z^2 = a^2$.



Step I : We have to find the stationary value of

$$u = f = 8xyz \qquad \text{.................... (1)}$$

with the condition that $\quad \Phi \equiv x^2 + y^2 + z^2 - a^2 = 0$ .... (2)

Consider the Lagrange's function,

$$F(x, y, z) = f(x, y, z) + \lambda\,\Phi(x, y, z)$$

$$= 8xyz + \lambda\,(x^2 + y^2 + z^2 - a^2)$$

$\therefore \quad dF = 8yz\,dx + 8zx\,dy + 8xy\,dz + \lambda\,2x\,dx + \lambda\,2y\,dy + \lambda\,2z\,dz$

$\quad = (8yz + 2\lambda\,x)\,dx + (8zx + 2\lambda\,y)\,dy + (8xy + 2\lambda\,z)\,dz$

$\therefore \quad dF = 0$ gives, $\quad \lambda\,x = -4yz \quad \therefore \quad \lambda\,x^2 = -4xyz$

$\quad \lambda\,y = -4zx \quad \therefore \quad \lambda\,y^2 = -4xyz$

$\quad \lambda\,z = -4xy \quad \therefore \quad \lambda\,z^2 = -4xyz$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), we get $\quad \lambda\,x^2 = \lambda\,y^2 = \lambda\,z^2 \quad \therefore \quad x^2 = y^2 = z^2 \quad \therefore \quad x = y = z.$

Now, from (2), $\quad 3x^2 = a^2 \quad \therefore \quad x = \dfrac{a}{\sqrt{3}} \quad \therefore \quad x = y = z = \dfrac{a}{\sqrt{3}}.$

Hence, the volume is maximum when the rectangular solid is a cube of side $2a/\sqrt{3}$.

The maximum volume $= 8\,xyz = \dfrac{8a^3}{3\sqrt{3}}$.

Example 11 (b) : If $xyz = 8$, find the values of x, y, z for which $u = \dfrac{5\,xyz}{x+2y+4z}$ is maximum.

Sol. : We have to maximise $\dfrac{5\,xyz}{x+2y+4z}$ with the condition that $xyz = 8$.

This means we have to maximise $v = \dfrac{40}{x+2y+4z}$

i.e. we have to minimise $\dfrac{x+2y+4z}{40}$ *i.e.* $x+2y+4z$.

Step I : We have to minimise $u = f = x+2y+4z$ .... (1)

with the condition that $\Phi = xyz - 8 = 0$ .... (2)

Consider the Lagrange's function.

$$F(x, y, z) = f(x, y, z) + \lambda\,\Phi(x, y, z) = x + 2y + 4z + \lambda\,(xyz - 8)$$

$$\therefore \quad dF = dx + 2dy + 4\,dz + \lambda\,yz\,dx + \lambda\,xz\,dy + \lambda\,xy\,dz$$

$$= (1 + \lambda\,yz)\,dx + (2 + \lambda\,xz)\,dy + (4 + \lambda\,xy)\,dz$$

$\therefore \quad dF = 0$ gives $\quad 1 + \lambda\,yz = 0,\ 2 + \lambda\,zx = 0,\ 4 + \lambda\,xy = 0$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), $\quad x + \lambda\,xyz = 0$

From (2), $\quad xyz = 8 \quad \therefore\ x + 8\lambda = 0 \quad \therefore\ x = -8\lambda$

Similarly, $\quad 2y + \lambda\,xyz = 0 \quad \therefore\ 2y + 8\lambda = 0 \quad \therefore\ y = -4\lambda$ and

$\quad 4z + \lambda\,xyz = 0 \quad \therefore\ 4z + 8\lambda = 0 \quad \therefore\ z = -2\lambda$.

Putting these values in (2), we get,

$$(-8\lambda)(-4\lambda)(-2\lambda) - 8 = 0 \quad \therefore\ \lambda = -1/2$$

But $\quad x = -8\lambda \quad \therefore\ x = 4 \ ; \quad y = -4\lambda \quad \therefore\ y = 2$

and $\quad z = -2\lambda \quad \therefore\ z = 1$.

Example 12 (b) : Using Lagrange's method, find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$.

Sol. : Step I : We have to minimise $u \equiv f = x^2 + y^2 + z^2$ .... (1)

with the condition that $\Phi = xyz - a^3$ .... (2)

Consider the Lagrange's function.

$$F(x, y, z) = f(x, y, z) + \lambda\,\Phi(x, y, z) = x^2 + y^2 + z^2 + \lambda\,(xyz - a^3)$$

$$\therefore \quad dF = 2x\,dx + 2y\,dy + 2z\,dz + \lambda\,yz\,dx + \lambda\,zx\,dy + \lambda\,xy\,dz$$

$\therefore \quad dF = 0$ gives $\quad 2x + \lambda\,yz = 0,\ \ 2y + \lambda\,zx = 0,\ \ 2z + \lambda\,xy = 0$ .... (3)

Step II : We have to eliminate x, y, z, λ from (1), (2) and (3).

Multiply the equations in (3) by x, y, z respectively

$$\therefore \quad 2x^2 + \lambda\,xyz = 0$$

From (2), $xyz = a^3$ $\quad \therefore \; 2x^2 + \lambda a^3 = 0 \quad \therefore \; 2x^2 = -a^3\lambda$

$2y^2 + \lambda\, xyz = 0 \quad \therefore \; 2y^2 + \lambda a^3 = 0 \quad \therefore \; 2y^2 = -a^3\lambda$

$2z^2 + \lambda\, xyz = 0 \quad \therefore \; 2z^2 + \lambda a^3 = 0 \quad \therefore \; 2z^2 = -a^3\lambda$

But from (2), $x^2 y^2 z^2 = a^6$

$$\therefore \left(-\frac{a^3\lambda}{2}\right)\left(-\frac{a^3\lambda}{2}\right)\left(-\frac{a^3\lambda}{2}\right) = a^6 \quad \therefore \; \lambda^3 = -\frac{8}{a^3} \quad \therefore \; \lambda = -\frac{2}{a}.$$

But $2x^2 = -a^3\lambda = 2a^2 \quad \therefore \; x^2 = a^2$

$2y^2 = -a^3\lambda = 2a^2 \quad \therefore \; y^2 = a^2$

$2z^2 = -a^3\lambda = 2a^2 \quad \therefore \; z^2 = a^2$

$\therefore \; u = x^2 + y^2 + z^2 = 3a^2$ is minimum value.

Example 13 (b) : The sum of three positive numbers is 1. Determine the maximum value of their product.

Sol. : Step I : We have to maximize $u = f \equiv xyz$ .... (1)

with the condition that $\Phi \equiv x + y + z = 1$ .... (2)

Consider the Lagrange's function.

$$F(x, y, z) = f(x, y, z) + \lambda\, \Phi(x, y, z) = xyz + \lambda(x + y + z - 1)$$

$$\therefore \; dF = yz\,dx + zx\,dy + xy\,dz + \lambda\,dx + \lambda\,dy + \lambda\,dz$$

$$= (yz + \lambda)\,dx + (zx + \lambda)\,dy + (xy + \lambda)\,dz$$

$\therefore \; dF = 0$ gives $yz + \lambda = 0, \; zx + \lambda = 0, \; xy + \lambda = 0$ .... (3)

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

Hence from (3), $xyz + \lambda x = 0, \; xyz + \lambda y = 0, \; xyz + \lambda z = 0$ .... (4)

But from (1), $xyz = u \quad \therefore \; u + \lambda x = 0, \; u + \lambda y = 0, \; u + \lambda z = 0$ .... (5)

$$\therefore \; 3u + \lambda(x + y + z) = 0$$

But $x + y + z = 1 \quad \therefore \; 3u + \lambda = 0 \quad \therefore \; \lambda = -3u.$

Hence, from (5), we get $u - 3ux = 0 \quad \therefore \; 1 - 3x = 0 \quad \therefore \; x = 1/3$

Similarly, we get $y = 1/3, \; z = 1/3$.

$\therefore$ The maximum value of $xyz = 1/27$.

Example 14 (b) : If $u = \dfrac{a^3}{x^2} + \dfrac{b^3}{y^2} + \dfrac{c^3}{z^2}$ where $x + y + z = 1$, prove that the stationary value of u is given by $x = \dfrac{a}{a+b+c}, \; y = \dfrac{b}{a+b+c}, \; z = \dfrac{c}{a+b+c}$.

Sol. : Step I : We have to find the stationary value of

$$u = f = \frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2} \qquad \text{..................... (1)}$$

with the condition that $\Phi \equiv x + y + z - 1 = 0$ .... (2)

Consider the Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda\, \Phi(x, y, z) = \left(\frac{a^3}{x^2} + \frac{b^3}{y^2} + \frac{c^3}{z^2}\right) + \lambda(x + y + z - 1)$$

$$\therefore \quad dF = -\frac{2a^3}{x^3}dx - \frac{2b^3}{y^3}dy - \frac{2c^3}{z^3}dz + \lambda\,dx + \lambda\,dy + \lambda\,dz$$

$$= \left(-\frac{2a^3}{x^3} + \lambda\right)dx + \left(-\frac{2b^3}{y^3} + \lambda\right)dy + \left(-\frac{2c^3}{z^3} + \lambda\right)dz$$

$$\therefore \quad dF = 0 \text{ gives} \quad x^3 = \frac{2a^3}{\lambda},\ y^3 = \frac{2b^3}{\lambda},\ z^3 = \frac{2c^3}{\lambda} \qquad \ldots\ldots (3)$$

Step II : We have to eliminate x, y, z, λ from (1), (2) and (3)

Putting the above values of x, y, z from (3), in (2), we get

$$\frac{\sqrt[3]{2}\cdot a}{\sqrt[3]{\lambda}} + \frac{\sqrt[3]{2}\cdot b}{\sqrt[3]{\lambda}} + \frac{\sqrt[3]{2}\cdot c}{\sqrt[3]{\lambda}} = 1$$

$$\therefore \quad \sqrt[3]{\lambda} = \sqrt[3]{2}\,(a+b+c) \quad \therefore \ \lambda = 2(a+b+c)^3 \qquad \ldots\ldots (4)$$

Hence, from (3), we get $\quad x^3 = \dfrac{a^3}{(a+b+c)^3}$ i.e. $x = \dfrac{a}{a+b+c}$.

Similarly, we get $\quad y = \dfrac{b}{a+b+c},\ z = \dfrac{c}{a+b+c}$.

Example 15 (b) : If $u = a^3x^2 + b^3y^2 + c^3z^2$ where $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$, show that the stationary value of u is given by $x = \dfrac{\Sigma a}{a}$, $y = \dfrac{\Sigma a}{b}$, $z = \dfrac{\Sigma a}{c}$ where $\Sigma a = a + b + c$.

Sol. : Step I : We have to find the stationary values of

$$u = f \equiv a^3x^2 + b^3y^2 + c^3z^2 \qquad \ldots\ldots (1)$$

with the condition that $\quad \Phi \equiv \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1 = 0 \qquad \ldots\ldots (2)$

Consider the Lagrange's function.

$$F(x, y, z) = f(x, y, z) + \lambda\,\Phi(x, y, z)$$

$$= a^3x^2 + b^3y^2 + c^3z^2 + \lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1\right)$$

$$\therefore \quad dF = 2a^3x\,dx + 2b^3y\,dy + 2c^3z\,dz - \frac{\lambda}{x^2}dx - \frac{\lambda}{y^2}dy - \frac{\lambda}{z^2}dz$$

$$= \left(2a^3x - \frac{\lambda}{x^2}\right)dx + \left(2b^3y - \frac{\lambda}{y^2}\right)dy + \left(2c^3z - \frac{\lambda}{z^2}\right)dz$$

$$\therefore \quad dF = 0 \text{ gives} \quad 2a^3x - \frac{\lambda}{x^2} = 0,\ 2b^3y - \frac{\lambda}{y^2} = 0,\ 2c^3z - \frac{\lambda}{z^2} = 0 \qquad \ldots\ldots (3)$$

Step II : We have to eliminate x, y, z and λ from (1), (2) and (3).

From (3), $\quad x^3 = \dfrac{\lambda}{2a^3},\ y^3 = \dfrac{\lambda}{2b^3},\ z^3 = \dfrac{\lambda}{2c^3} \qquad \ldots\ldots (4)$

Putting these values in (2), $\quad \dfrac{\sqrt[3]{2}\cdot a}{\sqrt[3]{\lambda}} + \dfrac{\sqrt[3]{2}\cdot b}{\sqrt[3]{\lambda}} + \dfrac{\sqrt[3]{2}\cdot c}{\sqrt[3]{\lambda}} = 1$

$$\therefore \quad \sqrt[3]{\lambda} = \sqrt[3]{2}\,(a+b+c) \quad \therefore \ \lambda = 2(a+b+c)^3$$

Hence, from (4), $x^3 = \dfrac{(a+b+c)^3}{a^3}$ $\therefore\ x = \dfrac{a+b+c}{a}$

Similarly, $y = \dfrac{a+b+c}{b}$, $z = \dfrac{a+b+c}{c}$.

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Type I : Two Independent Variables (Use Lagrange's Method) : Class (b) : 6 Marks

1. Find the points on the ellipse $x^2 + 2y^2 = 1$ at which $f(x, y) = xy$ takes extreme values.
[**Ans.** : $(\pm 1/\sqrt{2},\ 1/2), (\pm 1/\sqrt{2}, -1/2)$]

2. Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$. [**Ans.** : 39]

3. Find the points on the curve $xy^2 = 54$ which are nearest to the origin.
[**Ans.** : $(3, \pm 3\sqrt{2})$]

4. Find the minimum value of $x + y$ subject to the condition $xy = 16$, $x > 0$, $y > 0$. [**Ans.** : 8]

5. Find the maximum value of xy subject to the condition $x + y = 16$. [**Ans.** : 64]

6. Find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $9x^2 + 16y^2 = 144$. [**Ans.** : $4\sqrt{2}, 3\sqrt{2}$]

Type II : Three independent variable (Use Lagrange's Method) : Class (b) : 6 Marks

1. Find the point on the plane $x + 2y + 3z = 13$ which is nearest to the point (1, 1, 1).
(M.U. 2007) [**Ans.** : 3 / 2, 2, 5 / 2]

2. Find the minimum and maximum values of the function, $f(x, y, z) = x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$.
[**Ans.** : The points are (–1, 2, – 5) and (1, – 2, 5). The distances are – 30, 30.]

3. If $u = x^2 + y^2 + z^2$ where $x + y + z = 1$, show that u is stationary when $x = y = z = 1/3$.

4. Find the stationary value of $x^2 + y^2 + z^2$ when $2x + 3y + 4z = 5$. [**Ans.** : 25 / 29]

5. If 120 is divided into three numbers such that the sum of their product taken two at a time is maximum, find the numbers. [**Ans.** : 40, 40, 40]

6. Divide a into three parts such that their product is maximum. [**Ans.** : $a/3$, $a/3$, $a/3$]

7. A rectangular box open at the top is to have volume of 32 cu. units. Find its dimensions if the material required is least. [**Ans.** : 4, 4, 2]

8. Determine minimum distance from the origin to the plane $x + 2y + 3z = 14$.
[**Ans.** : $\sqrt{14}$]

9. If the distance of any point (x, y, z) on the plane $6x + 2y + 3z - 14 = 0$ is $r = \sqrt{x^2 + y^2 + z^2}$, find the minimum value of r. [**Ans.** : $r = 2$]

10. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ which are at maximum distance from (2, 1, 3). [**Ans.** : $\pm 2/\sqrt{14}, \pm 1/\sqrt{14}, \pm 3/\sqrt{14}$]

11. Find the points on the surface $z^2 = xy + 1$ nearest to the origin. [**Ans.** : 0, 0, ± 1]

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (a) : 3 Marks

State the conditions for $z = f(x, y)$ to have a maxima or minima.

Class (a) : 3 Marks

Verify whether (2/3, – 2/3) is a stationary value of $x^2 + 2xy - y^3$. **[Ans. : Yes]**

Summary

For maxima or minima we must have

$$rt - s^2 > 0$$

$\therefore$ $f(x, y)$ is maximum if r (or t) < 0

$f(x, y)$ is minimum if r (or t) > 0.

☆ ☆ ☆

CHAPTER 8

Successive Differentiation

1. Introduction

If $f(x)$ is differentiable we can obtain $f'(x)$. But if $f'(x)$ is also differentiable function of x we can differentiate it and get its derivative which is called the second derivative of $f(x)$ and is denoted by $f''(x)$. By continuing the procedure if possible, we get third and higher order derivatives and denote them by $f'''(x)$, $f^{iv}(x)$, $f^{v}(x)$, ... $f^{n}(x)$, ...

If we write $y = f(x)$, then the derivatives can also be denoted by

$$y_1,\ y_2,\ y_3,\ \ldots\ldots,\ y_n,\ \ldots\ldots$$

or by $\dfrac{dy}{dx},\ \dfrac{d^2y}{dx^2},\ \dfrac{d^3y}{dx^3},\ \ldots\ldots\ \dfrac{d^ny}{dx^n},\ \ldots\ldots$

The values of these derivatives at $x = a$ are denoted by

$$f^n(a),\ y_n(a)\ \text{ or }\ \left[\frac{d^ny}{dx^n}\right]_{x=a}$$

2. Derivatives of n^{th} Order

We now obtain n^{th} order derivatives of some standard functions.

(1) If $y = (ax+b)^m$ then,

$$\boxed{y_n = m(m-1)(m-2)\ldots(m-n+1)\,a^n(ax+b)^{m-n} \text{ if } n < m} \qquad \ldots\ldots\ldots\ldots (1)$$

Proof : We have

$$y_1 = m \cdot a \cdot (ax+b)^{m-1},\quad y_2 = m(m-1)\cdot a^2 \cdot (ax+b)^{m-2}$$

By generalisation (See Note 2 on page 8-5)

$$y_n = m(m-1)(m-2)\ldots(m-n+1)\cdot a^n \cdot (ax+b)^{m-n} \text{ if } n < m$$

If $n = m$, $\quad y_n = m(m-1)(m-2)\ldots(m-n+1)\ldots 3\cdot 2\cdot 1\cdot a^n$

$$= m!\,a^n \qquad \ldots\ldots\ldots\ldots (1\text{-A})$$

If $n > m$, $\quad y_n = 0 \qquad \ldots\ldots\ldots\ldots (1\text{-B})$

(2) If $y = (ax+b)^{-m}$ then,

$$\boxed{y_n = (-1)^n m(m+1)(m+2)\ldots(m+n-1)\,a^n(ax+b)^{-m-n}} \qquad \ldots\ldots\ldots\ldots (2)$$

Proof : Changing the sign of m in the above result,

$$y_n = (-m)(-m-1)(-m-2)\ldots(-m-n+1)\,a^n(ax+b)^{-m-n}$$

$$= (-1)^n m(m+1)(m+2)\ldots(m+n-1)\,a^n(ax+b)^{-m-n}$$

Writing $m(m+1) \ldots\ldots (m+n-1)$ in the reverse order and adjusting the terms,

$$\therefore \quad y_n = (-1)^n (m+n-1)(m+n-2)\ldots.(m+2)(m+1)\,m \cdot \frac{a^n}{(ax+b)^{m+n}}$$

$$= \frac{(-1)^n (m+n-1)(m+n-2)\ldots.m(m-1)(m-2)\ldots.3\cdot 2\cdot 1}{(m-1)(m-2)\ldots.3\cdot 2\cdot 1} \times \frac{a^n}{(ax+b)^{m+n}}$$

$\therefore$ If $y = \dfrac{1}{(ax+b)^m}$ then

$$\therefore \quad \boxed{y_n = (-1)^n \cdot \frac{(m+n-1)!}{(m-1)!} \cdot \frac{a^n}{(ax+b)^{m+n}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2-A)}$$

Cor. 1 : Putting $a = 1$, $b = a$, we get if $y = \dfrac{1}{(x+a)^m}$ then,

$$\boxed{y_n = (-1)^n \frac{(m+n-1)!}{(m-1)!} \cdot \frac{1}{(x+a)^{m+n}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2-B)}$$

Cor. 2 : If $y = \dfrac{1}{x^m}$ then, putting $a = 1$, $b = 0$, in 2 (A),

$$\boxed{y_n = (-1)^n \cdot \frac{(m+n-1)!}{(m-1)!} \cdot \frac{1}{x^{m+n}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2-C)}$$

e.g. if $y = \dfrac{1}{(x-2)^2}$, then putting $m = 2$, $a = 1$, $b = 2$ in (2-A)

$$y_n = (-1)^n \cdot \frac{(2+n-1)!}{(2-1)!} \cdot \frac{1}{(x-2)^{n+2}} = \frac{(-1)^n (n+1)!}{(x-2)^{n+2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2-D)}$$

(3) If $y = x^m$, then,

$$\boxed{y_n = m(m-1)(m-2) \ldots\ldots (m-n+1) \cdot x^{m-n} \text{ if } n < m} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

Proof : Proceeding as in the case (1) or by putting $a = 1$ and $b = 0$ there, we get,

$$y_n = m(m-1)(m-2) \ldots. (m-n+1)\, x^{m-n} \quad \text{if } n < m$$
$$= m! \qquad \text{if } n = m$$
$$= 0 \qquad \text{if } n > m$$

Example 1 : Find y_n if $y = \dfrac{x^n - 1}{x-1}$.

Sol. : $y = x^{n-1} + x^{n-2} + \ldots\ldots + 1$ [By division]

$\therefore \quad y_n = 0$. [By (1-B)]

Example 2 : If $y = (x^2-1)^n$, find y_{2n}.

Sol. : $y = (x^2)^n - {}^nC_1 (x^2)^{n-1} + \ldots\ldots + (-1)^n$ [By Binomial Theorem]

$\therefore \quad y_{2n} = 2n!$ [By (1-A)]

(4) **If** $y = \dfrac{1}{(ax+b)}$, **then** $\boxed{y_n = \dfrac{(-1)^n \cdot n!\,a^n}{(ax+b)^{n+1}}}$ .... **(4)**

Proof : Proceeding as in the case (1) or by putting $m = -1$ there, we get

$$y_n = (-1)(-2)(-3)\ldots(-n)\cdot a^n \cdot (ax+b)^{-1-n} = \frac{(-1)^n \cdot n!\,a^n}{(ax+b)^{n+1}}$$

Cor. : Putting $a = 1$, if $y = \dfrac{1}{x+b}$, then $\boxed{y_n = \dfrac{(-1)^n\, n!}{(x+b)^{n+1}}}$ .... **(4-A)**

(5) **If** $y = \log(ax+b)$, **then** $\boxed{y_n = \dfrac{(-1)^{n-1}\cdot(n-1)!\,a^n}{(ax+b)^n}}$ .... **(5)**

Proof : We have $y_1 = \dfrac{a}{(ax+b)}$.

Now, we can replace n by $n-1$, m by 1 in the case (2) above and multiply it by a to get [From 2-A]

$$y_n = \frac{(-1)^{n-1}\cdot(n-1)!\,a^n}{(ax+b)^n}$$

Cor. : Putting $a = 1$, if $y = \log(x+b)$, then $\boxed{y_n = \dfrac{(-1)^{n-1}(n-1)!}{(x+b)^n}}$ .... **(5-A)**

(6) **If** $y = a^{mx}$, **then** $\boxed{y_n = m^n a^{mx} (\log a)^n}$ .... **(6)**

Proof : We have $y_1 = ma^{mx}\cdot\log a$, $\quad y_2 = m^2\cdot a^{mx}\cdot(\log a)^2$

By generalisation, $\quad y_n = m^n \cdot a^{mx}\cdot(\log a)^n$.

Cor. : If $m = 1$, *i.e.* if $y = a^x$, $\boxed{y_n = a^x (\log a)^n}$ .... **(6-A)**

(7) **If** $y = e^{mx}$, **then** $\boxed{y_n = m^n e^{mx}}$ .... **(7)**

Proof : Proceeding as in the case (6) or by putting $a = e$ there, we get

$$y_n = m^n \cdot e^{mx}\cdot(\log e)^n = m^n \cdot e^{mx}$$

Cor. : If $m = 1$ *i.e.* if $y = e^x$, $\boxed{y_n = e^x}$ .... **(7-A)**

(8) **If** $y = \sin(ax+b)$ **then** $\boxed{y_n = a^n \sin\left(ax+b+\dfrac{n\pi}{2}\right)}$ .... **(8)**

Proof : We have $\quad y_1 = a\cos(ax+b) = a\sin\left(ax+b+\dfrac{\pi}{2}\right)$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{2\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax+b+\frac{2\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$

By generalisation, $y_n = a^n\sin\left(ax+b+\dfrac{n\pi}{2}\right)$.

Cor. : If $b = 0$, *i.e.* if $y = \sin ax$, $\boxed{y_n = a^n \sin\left(ax + \frac{n\pi}{2}\right)}$ .... **(8-A)**

Similarly, we can prove that.

(9) If $y = \cos(ax + b)$ then $\boxed{y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)}$ .... **(9)**

Cor. : If $b = 0$, *i.e.* if $y = \cos ax$, $\boxed{y_n = a^n \cos\left(ax + \frac{n\pi}{2}\right)}$ .... **(9-A)**

(10) If $y = e^{ax} \sin(bx + c)$ then $\boxed{y_n = r^n e^{ax} \sin(bx + c + n\Phi)}$ .... **(10)**

where, $r = \sqrt{a^2 + b^2}$ and $\Phi = \tan^{-1}\left(\frac{b}{a}\right)$.

Proof : We have

$$y_1 = e^{ax} \cdot a \cdot \sin(bx + c) + e^{ax} \cdot b \cdot \cos(bx + c)$$

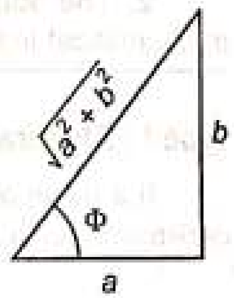
$$= e^{ax} [a \sin(bx + c) + b \cos(bx + c)]$$

We write this as

$$y_1 = e^{ax} \cdot \sqrt{a^2 + b^2}\left[\frac{a}{\sqrt{a^2 + b^2}} \sin(bx + c) + \frac{b}{\sqrt{a^2 + b^2}} \cos(bx + c)\right]$$

If we write $\frac{a}{\sqrt{a^2 + b^2}} = \cos\Phi$, $\frac{b}{\sqrt{a^2 + b^2}} = \sin\Phi$

i.e. if we put $r = \sqrt{a^2 + b^2}$ and $\frac{b}{a} = \tan\Phi$, then,



$$y_1 = r e^{ax} \sin(bx + c + \Phi).$$

Thus, y_1 is what y becomes when y is multiplied by r and the angle is increased by Φ. By the same reasoning.

$$y_2 = r^2 e^{ax} \sin(bx + c + 2\Phi);$$

$$y_3 = r^3 e^{ax} \sin(bx + c + 3\Phi).$$

By generalisation, $y_n = r^n e^{ax} \sin(bx + c + n\Phi)$

Cor. : If $c = 0$, *i.e.* if $y = e^{ax} \sin bx$, then

$\boxed{y_n = r^n e^{ax} \sin(bx + n\Phi)}$ .... **(10-A)**

Similarly, we can prove that.

(11) If $y = e^{ax} \cos(bx + c)$ then

$\boxed{y_n = r^n e^{ax} \cos(bx + c + n\Phi)}$ .... **(11)**

Cor. : If $c = 0$, *i.e.* if $y = e^{ax} \cos bx$

$\boxed{y_n = r^n e^{ax} \cos(bx + n\Phi)}$ **(M.U. 2005, 07)** .... **(11-A)**

(12) If $y = k^x \sin(bx + c)$ then $\boxed{y_n = r^n k^x \sin(bx + c + n\Phi)}$ ....(12)

where $r = \sqrt{(\log k)^2 + b^2}$ and $\Phi = \tan^{-1}\left(\dfrac{b}{\log k}\right)$.

Proof : Since $k^x = e^{x \log k} = e^{ax}$ where $a = \log k$

$\therefore \quad y = e^{ax} \cdot \sin(bx + c)$

$\therefore \quad y_n = r^n e^{ax} \sin(bx + c + n\Phi)$

$\therefore \quad y_n = r^n k^x \sin(bx + c + n\Phi) \qquad [\because e^{ax} = k^x]$

where, $r = \sqrt{a^2 + b^2} = \sqrt{(\log k)^2 + b^2}$ and $\Phi = \tan^{-1}\left(\dfrac{b}{a}\right) = \tan^{-1}\left(\dfrac{b}{\log k}\right)$.

(13) If $y = k^x \cos(bx + c)$ then $\boxed{y_n = r^n k^x \cos(bx + c + n\Phi)}$ ....(13)

where $r = \sqrt{(\log k)^2 + b^2}$ and $\Phi = \tan^{-1}\left(\dfrac{b}{\log k}\right)$.

You can prove it as above.

Note ....

1. If $y = u + v$ or $y = u - v$ we can obtain in certain cases $y_n = u_n + v_n$ or $y_n = u_n - v_n$ by using the above standard results. However, to put the given function as the sum or difference, sometimes we have to rearrange or simplify the given functions as illustrated below.

2. The above proofs are not rigorous. For rigorous proofs we have to use the method of mathematical induction.

Type I : Algebraic Functions

If a given algebraic function can be expressed in terms of **Linear Partial Fractions** its n^{th} derivative can be obtained by using the result (2).

Solved Examples : Class (a) : 4 Marks

Example 1 (a) : If $y = \dfrac{x}{(x+1)^4}$, find y_n.

Sol. : We have $y = \dfrac{x}{(x+1)^4} = \dfrac{(x+1)-1}{(x+1)^4} \quad \therefore \quad y = \dfrac{1}{(x+1)^3} - \dfrac{1}{(x+1)^4}$

Putting $m = 3, 4$ in the result (2-B)

$$\therefore \quad y_n = \frac{(-1)^n (n+2)!}{2! \cdot (x+1)^{n+3}} - \frac{(-1)^n (n+3)!}{3! \cdot (x+1)^{n+4}} = \frac{(-1)^n (n+2)!}{2(x+1)^{n+3}} - \frac{(-1)^n (n+3)!}{6(x+1)^{n+4}}$$

$$= \frac{(-1)^n (n+2)!}{6(x+1)^{n+3}}\left[3 - \frac{(n+3)}{(x+1)}\right] = \frac{(-1)^n (n+2)!}{6(x+1)^{n+4}}(3x - n).$$

Example 2 (a) : Find the n^{th} derivative of $\dfrac{x}{(x-1)(x-2)(x-3)}$. **(M.U. 2016)**

Sol. : Let $y = \dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{a}{(x-1)} + \dfrac{b}{(x-2)} + \dfrac{c}{(x-3)}$

$\therefore \quad x = a(x-2)(x-3) + b(x-1)(x-3) + c(x-1)(x-2)$

Putting $x = 1$, $1 = a(-1)(-2)$ $\therefore$ $a = 1/2$

Putting $x = 2$, $2 = b(1)(-1)$ $\therefore$ $b = -2$

Putting $x = 3$, $3 = c(2)(1)$ $\therefore$ $c = 3/2$

$$\therefore \quad y = \frac{1}{2}\cdot\frac{1}{x-1} - 2\cdot\frac{1}{x-2} + \frac{3}{2}\cdot\frac{1}{x-3}$$

By result (4-A), putting $b = -1, -2, -3$, we get

$$\therefore \quad y_n = \frac{1}{2}\cdot\frac{(-1)^n \cdot n!}{(x-1)^{n+1}} - 2\cdot\frac{(-1)^n \cdot n!}{(x-2)^{n+1}} + \frac{3}{2}\cdot\frac{(-1)^n \cdot n!}{(x-3)^{n+1}}$$

$$\therefore \quad y_n = (-1)^n \cdot n!\left[\frac{1}{2}\cdot\frac{1}{(x-1)^{n+1}} - \frac{2}{(x-2)^{n+1}} + \frac{3}{2}\cdot\frac{1}{(x-3)^{n+1}}\right]$$

Example 3 (a) : Find the n^{th} derivative of $\frac{2}{(x-1)(x-2)(x-3)}$. **(M.U. 1996)**

Sol. : Let $y = \frac{2}{(x-1)(x-2)(x-3)} = \frac{a}{(x-1)} + \frac{b}{(x-2)} + \frac{c}{(x-3)}$

$$\therefore \quad 2 = a(x-2)(x-3) + b(x-1)(x-3) + c(x-1)(x-2)$$

Putting $x = 1$, $2 = a(-1)(-2)$ $\therefore$ $a = 1$

Putting $x = 2$, $2 = b(1)(-1)$ $\therefore$ $b = -2$

Putting $x = 3$, $2 = c(2)(1)$ $\therefore$ $c = 1$

$$\therefore \quad y = \frac{1}{x-1} - \frac{2}{x-2} + \frac{1}{x-3}$$

By result (4-A), we get,

$$y_n = (-1)^n \cdot n!\left[\frac{1}{(x-1)^{n+1}} - \frac{2}{(x-2)^{n+1}} + \frac{1}{(x-3)^{n+1}}\right].$$

Example 4 (a) : Find the n^{th} derivative of $\frac{x^2}{(x+2)(2x+3)}$. **(M.U. 1993, 95, 2006)**

Sol. : We have to express the given expression in terms of **partial fractions.** Since the degree of the numerator is equal to the degree of denominator we first divide the numerator by the denominator and then obtain the partial fractions.

$$\therefore \quad y = \frac{x^2}{(x+2)(2x+3)} = \frac{1}{2} - \frac{4}{x+2} + \frac{9}{2(2x+3)}$$

Using the result (4-A), we get,

$$y_n = -\frac{4(-1)^n \cdot n!}{(x+2)^{n+1}} + \frac{9(-1)^n \cdot n!\, 2^n}{2(2x+3)^{n+1}}.$$

Example 5 (a) : If $y = \frac{x^3}{(x+1)(x-2)}$, find y_n. **(M.U. 2017)**

Sol. : Since the degree of the numerator is higher than the degree of the denominator, we divide the numerator x^3 by the denominator $x^2 - x - 2$.

$$\therefore \quad \frac{x^3}{(x+1)(x-2)} = (x+1) + \frac{3x+2}{(x+1)(x-2)}$$

Now, let $\dfrac{3x+2}{(x+1)(x-2)} = \dfrac{a}{x+1} + \dfrac{b}{x-2}$ $\quad \therefore \; 3x+2 = a(x-2) + b(x+1)$

Putting $x = -1$, $\quad -1 = -3a \quad \therefore \; a = \dfrac{1}{3}$

Putting $x = 2$, $\quad 8 = 3b \quad \therefore \; b = \dfrac{8}{3}$

$$\therefore \; y = (x+1) + \frac{1}{3} \cdot \frac{1}{x+1} + \frac{8}{3} \cdot \frac{1}{x-2}$$

By result (4-A), we get

$$y_n = 0 + \frac{1}{3} \cdot \frac{(-1)^n \cdot n!}{(x+1)^{n+1}} + \frac{8}{3} \cdot \frac{(-1)^n \cdot n!}{(x-2)^{n+1}} \qquad \text{If } n \geq 2$$

$$= \frac{(-1)^n \cdot n!}{3}\left[\frac{1}{(x+1)^{n+1}} + 8 \cdot \frac{(-1)^n \cdot n!}{(x-2)^{n+1}}\right]$$

Example 6 (a) : Find the n^{th} derivative of $\dfrac{x}{x^2+a^2}$.

Sol. : Let $\quad y = \dfrac{x}{x^2+a^2} = \dfrac{A}{(x+ai)} + \dfrac{B}{(x-ai)}$

$\therefore \; x = A(x-ai) + B(x+ai)$

Putting $x = ai$, $\quad ai = B \cdot 2ai \quad \therefore \; B = 1/2$

Putting $x = -ai$, $\quad -ai = A(-2ai) \quad \therefore \; A = 1/2$

$$\therefore \; y = \frac{1}{2}\left[\frac{1}{(x+ai)} + \frac{1}{(x-ai)}\right]$$

By result (4-A), we get,

$$y_n = \frac{1}{2}(-1)^n \cdot n!\left[\frac{1}{(x+ai)^{n+1}} + \frac{1}{(x-ai)^{n+1}}\right].$$

Example 7 (a) : If $y = \dfrac{1}{1+x+x^2+x^3}$, find y_n. **(M.U. 1997, 2009)**

Sol. : Let $\quad y = \dfrac{1}{(1+x)(1+x^2)} = \dfrac{a}{1+x} + \dfrac{b}{x+i} + \dfrac{c}{x-i}$

$\therefore \; 1 = a(x+i)(x-i) + b(1+x)(x-i) + c(1+x)(x+i)$

Putting $x = -1$, $\quad 1 = a(-1+i)(-1-i) = a(1-i^2) = 2a \quad \therefore \; a = 1/2$

Putting $x = i$, $\quad 1 = c(1+i) \cdot 2i \quad \therefore \; c = 1/[2(1+i)i]$

$$\therefore \; c = \frac{1}{2i(1+i)} = \frac{1}{2i} \cdot \frac{1}{i-1} \cdot \frac{i+1}{i+1} = \frac{1}{2i} \cdot \frac{i+1}{i^2-1} = -\frac{i+1}{4i}$$

Putting $x = -i$, $\quad 1 = b(1-i) \cdot (-2i)$

$\therefore \; b = 1/[(-2)(1-i)i]$

$$\therefore \; b = -\frac{1}{2i(1-i)} = -\frac{1}{2i} \cdot \frac{1}{i+1} \cdot \frac{i-1}{i-1} = -\frac{1}{2i} \cdot \frac{i-1}{i^2-1} = \frac{i-1}{4i}$$

$$\therefore \; y = \frac{1}{2} \cdot \frac{1}{1+x} + \frac{i-1}{4i} \cdot \frac{1}{x+i} - \frac{i+1}{4i} \cdot \frac{1}{x-i}$$

By result (4-A), we get

$$y_n = \frac{(-1)^n \cdot n!}{2}\left[\frac{1}{(1+x)^{n+1}} + \frac{i-1}{4i} \cdot \frac{1}{(x+i)^{n+1}} - \frac{i+1}{4i} \cdot \frac{1}{(x-i)^{n+1}}\right]$$

Example 8 (a) : If $y = \dfrac{x^2+4}{(x-1)^2(2x+3)}$, find y_n. **(M.U. 1992, 2012)**

Sol. : Let $\dfrac{x^2+4}{(x-1)^2(2x+3)} = \dfrac{a}{x-1} + \dfrac{b}{(x-1)^2} + \dfrac{c}{2x+3}$

$\therefore \quad x^2 + 4 = a(x-1)(2x+3) + b(2x+3) + c(x-1)^2$

Putting $x = 1$, $5 = 5b \quad \therefore \; b = 1$

Putting $x = -\dfrac{3}{2}$, $\dfrac{9}{4} + 4 = \left(-\dfrac{5}{2}\right)^2 c \quad \therefore \; \dfrac{25}{4} = \left(-\dfrac{5}{2}\right)^2 c \quad \therefore \; c = 1$

Comparing the coefficients of x^2 on both sides, we get,

$1 = 2a + c \quad \therefore \; 1 = 2a + 1 \quad \therefore \; a = 0$

$$\therefore \quad \frac{x^2+4}{(x-1)^2(2x+3)} = \frac{1}{(x-1)^2} + \frac{1}{(2x+3)}.$$

Alternatively the constant a, b, c can be obtained more quickly by a little adjustment of the terms as follows.

We can write the r.h.s. as

$$\frac{x^2+4}{(x-1)^2(2x+3)} = \frac{(x^2-2x+1)+(2x+3)}{(x-1)^2(2x+3)} = \frac{(x-1)^2+(2x+3)}{(x-1)^2(2x+3)}$$

$$= \frac{(x-1)^2}{(x-1)^2(2x+3)} + \frac{(2x+3)}{(x-1)^2(2x+3)} = \frac{1}{2x+3} + \frac{1}{(x-1)^2}$$

Hence, by (4) and (2B), we get

$$y_n = \frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}} + \frac{(-1)^n (n+1)!}{(x-1)^{n+2}} = (-1)^n\left[\frac{n!\,2^n}{(2x+3)^{n+1}} + \frac{(n+1)!}{(x-1)^{n+2}}\right]$$

Example 9 (a) : If $y = \dfrac{8x}{x^3 - 2x^2 - 4x + 8}$, find y_n. **(M.U. 2007, 09)**

Sol. : We have $y = \dfrac{8x}{x^3 - 2x^2 - 4x + 8} = \dfrac{8x}{(x-2)(x^2-4)}$

$\therefore \quad y = \dfrac{8x}{(x-2)^2(x+2)} = \dfrac{1}{x-2} + \dfrac{4}{(x-2)^2} - \dfrac{1}{x+2}$ [By partial fractions]

We know that

if $y = \dfrac{1}{ax+b}$, $\quad y_n = \dfrac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$ [By (4-A), page 8-3]

$\therefore$ If $u = \dfrac{1}{x-2}$, $\quad u_n = \dfrac{(-1)^n\, n!(1)^n}{(x-2)^{n+1}}$

If $v = \dfrac{1}{(x-2)^2}$, $\quad v_n = \dfrac{(-1)^n (n+1)!(1)^n}{(x-2)^{n+2}}$ [By (2-D), page 8-2]

$$\text{If } w=\frac{1}{x+2},\quad w_n=\frac{(-1)^n n!(1)^n}{(x+2)^n}$$

$$\therefore\quad y_n=\frac{(-1)^n n!}{(x-2)^{n+1}}+4\cdot\frac{(-1)^n(n+1)!}{(x-2)^{n+2}}-\frac{(-1)^n}{(x+2)^n}.$$

Example 10 (a) : Find the n^{th} derivative of $\dfrac{x^2}{(x-1)^2(x+2)}$.

Sol. : Let $\quad y=\dfrac{x^2}{(x-1)^2(x+2)}=\dfrac{a}{x-1}+\dfrac{b}{(x-1)^2}+\dfrac{c}{x+2}$

$$\therefore\quad x^2=a(x-1)(x+2)+b(x+2)+c(x-1)^2$$

Putting $x=1$, we get $\quad 1=b\cdot 3 \quad \therefore\ b=1/3$

Putting $x=-2$, we get $\quad 4=c\cdot 9 \quad \therefore\ c=4/9$

Putting $x=0$, we get $\quad 0=-2a+2b+c \quad \therefore\ a=5/9$

$$\therefore\quad y=\frac{5}{9}\cdot\frac{1}{x-1}+\frac{1}{3}\cdot\frac{1}{(x-1)^2}+\frac{4}{9}\cdot\frac{1}{x+2}$$

$\therefore$ If $u=\dfrac{1}{x-1},\quad u_n=\dfrac{(-1)^n\cdot n!\cdot(1)^n}{(x-1)^{n+1}}$ [By (4-A), page 8-3]

If $v=\dfrac{1}{(x-1)^2},\quad v_n=\dfrac{(-1)^n\cdot(n+1)!\cdot(1)^n}{(x-1)^{n+2}}$ [By (2B), page 8-2, putting $m=2$]

If $w=\dfrac{1}{x+2},\quad w_n=\dfrac{(-1)^n\cdot n!\cdot(1)^n}{(x+2)^n}$ [By (4-A)]

$$\therefore\quad y_n=\frac{5}{9}\cdot\frac{(-1)^n\cdot n!}{(x-1)^{n+1}}+\frac{1}{3}\cdot\frac{(-1)^n\cdot(n+1)!}{(x-1)^{n+2}}+\frac{4}{9}\cdot\frac{(-1)^n\cdot n!}{(x+2)^{n+1}}$$

$$\therefore\quad y_n=\frac{(-1)^n\cdot n!}{9}\left[\frac{5}{(x-1)^{n+1}}+\frac{3(n+1)}{(x-1)^{n+2}}+\frac{4}{(x+2)^{n+1}}\right].$$

Example 11 (a) : Prove that the value of n^{th} differential coefficient of $x^3/(x^2-1)$ for $x=0$ is 0 if n is even and $-n!$ if n is odd and greater than 1. **(M.U. 1995, 2002, 03)**

Sol. : We have $y=\dfrac{x^3}{x^2-1}=x+\dfrac{x}{x^2-1}$

$\therefore\quad y=x+\dfrac{1}{2}\left[\dfrac{1}{x-1}+\dfrac{1}{x+1}\right]$ [By partial fractions]

$\therefore\quad y_n=\dfrac{1}{2}\left[\dfrac{(-1)^n n!}{(x-1)^{n+1}}+\dfrac{(-1)^n n!}{(x+1)^{n+1}}\right]$ [By (4-A), page 8-3]

Putting $x=0$, $\quad y_n(0)=\dfrac{(-1)^n n!}{2}\left[\dfrac{1}{(-1)^{n+1}}+\dfrac{1}{(1)^{n+1}}\right]$

When n is even, $(n+1)$ is odd, $\quad\therefore\ y_n(0)=\dfrac{(-1)^n n!}{2}[-1+1]=0.$

When n is odd, $(n+1)$ is even, $\quad\therefore\ y_n(0)=\dfrac{(-1)^n n!}{2}[1+1]=-n!.$

Example 12 (a) : If $y = x\log\left(\frac{x-1}{x+1}\right)$, prove that $y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

(M.U. 1999)

Sol. : We have $\quad y = x\log(x-1) - x\log(x+1)$

$$\therefore\ y_1 = \frac{x}{x-1} + \log(x-1) - \frac{x}{x+1} - \log(x+1) = \log(x-1) - \log(x+1) + \frac{x}{x-1} - \frac{x}{x+1}$$

$$= \log(x-1) - \log(x+1) + \frac{(x-1)+1}{x-1} - \frac{(x+1)-1}{x+1} \qquad \textbf{[Note this]}$$

$$\therefore\ y_1 = \log(x-1) - \log(x+1) + 1 + \frac{1}{x-1} - 1 + \frac{1}{x+1}$$

By (5) and by (4-A), page 8-3 (Changing n to $n-1$), we get

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^{n-1}} - \frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} + \frac{(-1)^{n-1}(n-1)!}{(x-1)^n} + \frac{(-1)^n(n-1)!}{(x+1)^n}$$

$$= (-1)^{n-2}(n-2)!\left[\frac{1}{(x-1)^{n-1}} - \frac{1}{(x+1)^{n-1}} + \frac{(-1)(n-1)}{(x-1)^n} + \frac{(-1)(n-1)}{(x+1)^n}\right]$$

$$= (-1)^{n-2}(n-2)!\left[\frac{x-1}{(x-1)^n} - \frac{x+1}{(x+1)^n} + \frac{-n+1}{(x-1)^n} + \frac{-n+1}{(x+1)^n}\right]$$

$$\therefore\ y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$$

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Find the n^{th} derivatives of following : Class (a) : 4 Marks

1. $\frac{x}{x^2-a^2}$ 2. $\frac{1}{x^4-a^4}$ 3. $\frac{x^4}{(x-1)(x-2)}$ 4. $\frac{x^2}{1-x^4}$

5. $\frac{1}{6x^2-5x+1}$ 6. $\frac{x+1}{x^2-4}$ 7. $\frac{x}{x^3-6x^2+11x-6}$ 8. $\frac{x}{x^2+9}$

9. $\frac{x^2}{(x-1)(2x+3)}$ 10. $\frac{x}{1-4x^2}$ **(M.U. 1993)** 11. $\frac{x}{(x+1)^5}$ 12. $\frac{4x}{(x-1)^2(x+1)}$ **(M.U. 1983, 2002, 04)**

13. $\frac{1}{(3x-2)(x-3)^2}$ **(M.U. 1995)** 14. $\frac{x^2+4x+1}{x^2+2x^2-x-2}$ **(M.U. 1984)**

15. If $y = x\cot h^{-1}x$, prove that $y_n = \frac{(-1)^n(n-2)!}{2}\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$.

16. If $y = (x^2-1)^n$, prove that $y_{2n} = 2n!$.

[Ans. : (1) $\frac{(-1)^n n!}{2}\left[\frac{1}{(x+a)^{n+1}} + \frac{1}{(x-a)^{n+1}}\right]$

(2) $\frac{(-1)^n n!}{4a^3}\left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right] - \frac{(-1)^n n!}{4a^3 i}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$

(3) $y = x^2 + 3x + 7 + \dfrac{16}{(x-2)} - \dfrac{1}{(x-1)}$; $y_n = (-1)^n\, n!\left[\dfrac{16}{(x-2)^{n+1}} - \dfrac{1}{(x-1)^{n+1}}\right]$ for $n \geq 3$

(4) $\dfrac{(-1)^n\, n!}{4}\left[\dfrac{1}{(1+x)^{n+1}} + \dfrac{1}{(1-x)^{n+1}}\right] - \dfrac{(-1)^n\, n!}{4i}\left[\dfrac{1}{(x-i)^{n+1}} - \dfrac{1}{(x+i)^{n+1}}\right]$

(5) $(-1)^n\, n!\left[\dfrac{2^{n+1}}{(2x-1)^{n+1}} - \dfrac{3^{n+1}}{(3x-1)^{n+1}}\right]$ (6) $(-1)^n\, n!\left[\dfrac{3}{4}\cdot\dfrac{1}{(x-2)^{n+1}} + \dfrac{1}{4}\cdot\dfrac{1}{(x+2)^{n+1}}\right]$

(7) $(-1)^n\, n!\left[\dfrac{1}{2(x-1)^{n+1}} - \dfrac{2}{(x-2)^{n+1}} + \dfrac{3}{2(x-3)^{n+1}}\right]$

(8) $\dfrac{(-1)^n\, n!}{2}\left[\dfrac{1}{(x+3i)^{n+1}} + \dfrac{1}{(x-3i)^{n+1}}\right]$ (9) $\dfrac{(-1)^n\, n!}{10}\left[\dfrac{2}{(x-1)^{n+1}} - \dfrac{9\cdot 2^n}{(2x+3)^{n+1}}\right]$

(10) $\dfrac{(-1)^n\, n!}{4}\left[\dfrac{(-2)^n}{(1-2x)^{n+1}} - \dfrac{(2)^n}{(1+2x)^{n+1}}\right]$ (11) $\dfrac{(-1)^n\,(n+3)!}{4!(x+1)^5}(4x-n)$

(12) $(-1)^n\left[\dfrac{n!}{(x-1)^{n+1}} + \dfrac{2(n+1)!}{(x-1)^{n+2}} - \dfrac{n!}{(x+1)^{n+1}}\right]$

(13) $y = \dfrac{9}{49}\cdot\dfrac{1}{3x-2} - \dfrac{3}{49}\cdot\dfrac{1}{x-3} + \dfrac{1}{7}\cdot\dfrac{1}{(x-3)^2}$

$$y_n = \frac{9}{49}\cdot\frac{(-1)^n\cdot n!\cdot 3^n}{(3x-2)^{n+1}} - \frac{3}{49}\cdot\frac{(-1)^n\cdot n!}{(x-3)^{n+1}} + \frac{1}{7}\cdot\frac{(-1)^n\cdot(n+1)!}{(x-3)^{n+2}}$$

(14) $(-1)^n\cdot n!\left[\dfrac{1}{(x+1)^{n+1}} + \dfrac{1}{(x-1)^{n+1}} - \dfrac{1}{(x+2)^{n+1}}\right]$

(15) **Hint :** $y = x\tanh^{-1}\dfrac{1}{x} = \dfrac{x}{2}\log\left(\dfrac{x+1}{x-1}\right)$. Now proceed as in solved Ex. 12 above.]

Type II : Trigonometric Functions

If a given trigonometric function can be expressed as the sum or difference of $\sin(ax+b)$, $\cos(ax+b)$, $e^{ax}\sin(bx+c)$, $e^{ax}\cos(bx+c)$, $k^{ax}\sin(bx+c)$, $k^{ax}\cos(bx+c)$ then by using results (8), (9), (10), (11), (12), (13) we can obtain y_n. Note that in examples of this type we express product of trigonometric functions as the sum and also the powers of sine and cosine by increasing the angle.

Solved Examples : Class (a) : 4 Marks

Example 1 (a) : If $y = \sin rx + \cos rx$, prove that $y_n = r^n[1+(-1)^n \sin 2rx]^{1/2}$. Find $y_8(\pi)$ where $r = 1/4$. **(M.U. 2004, 07, 14, 15)**

Sol. : We have by (8) and (9), putting $a = r$, $b = 0$,

$$y_n = r^n\left[\sin\left(rx + \frac{n\pi}{2}\right) + \cos\left(rx + \frac{n\pi}{2}\right)\right]$$

$$= r^n\left[\left\{\sin\left(rx + \frac{n\pi}{2}\right) + \cos\left(rx + \frac{n\pi}{2}\right)\right\}^2\right]^{1/2}$$ **[Note this]**

$$\therefore \quad y_n = r^n \left[\sin^2\left(rx + \frac{n\pi}{2}\right) + \cos^2\left(rx + \frac{n\pi}{2}\right) + 2\sin\left(rx + \frac{n\pi}{2}\right)\cos\left(rx + \frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n \left[1 + 2\sin\left(rx + \frac{n\pi}{2}\right) \cdot \cos\left(rx + \frac{n\pi}{2}\right)\right]^{1/2}$$

$$= r^n \left[1 + \sin 2\left(rx + \frac{n\pi}{2}\right)\right]^{1/2} = r^n\left[1 + \sin(2rx + n\pi)\right]^{1/2}$$

$$= r^n\left[1 + (-1)^n \sin 2rx\right]^{1/2} \text{ since } \sin(n\pi + \theta) = (-1)^n \sin\theta.$$

To find, $y_8(\pi)$, we put $x = \pi$, $n = 8$ and $r = 1/4$ (given)

$$\therefore \quad y_8(\pi) = \left(\frac{1}{4}\right)^8 \left[1 + (-1)^8 \sin 2 \cdot \frac{1}{4} \cdot \pi\right]^{1/2}$$

$$= \frac{1}{2^{16}}\left[1 + 1 \cdot \sin\frac{\pi}{2}\right]^{1/2} = \frac{1}{2^{16}} \cdot 2^{1/2} = \left(\frac{1}{2}\right)^{31/2}.$$

Restatement : If $y = \sin\theta + \cos\theta$, then prove that $y_n = r^n\left[1 + (-1)^n \sin 2\theta\right]^{1/2}$ where $\theta = rx$. **(M.U. 2015)**

Sol. : Same as above.

Example 2 (a) : Find the n^{th} derivative of $y = \sin 2x \cdot \cos 6x$.

Sol. : We have $y = \sin 2x \cos 6x$

$$\therefore \quad y = \frac{1}{2} \cdot [2\sin 2x \cos 6x] = \frac{1}{2}[\sin 8x - \sin 4x]$$

Now by result (8), we get

$$\therefore \quad y_n = \frac{1}{2}\left[8^n \sin\left(8x + \frac{n\pi}{2}\right) - 4^n \sin\left(8x + \frac{n\pi}{2}\right)\right].$$

Example 3 (a) : Find the n^{th} derivative of $y = \cos x \cos 2x \cos 3x$. **(M.U. 2013)**

Sol. : We have $y = \cos x \cos 2x \cos 3x$.

$$\therefore \quad y = \frac{1}{2}\cos x \cdot 2\cos 2x \cos 3x = \frac{1}{2}\cos x\left[\cos 5x + \cos x\right]$$

$$= \frac{1}{2}\cos x \cos 5x + \frac{1}{2}\cos^2 x = \frac{1}{4}\left[\cos 6x + \cos 4x + 1 + \cos 2x\right]$$

Now, by result (9), page 8-4, we get

$$y_n = \frac{1}{4}\left[6^n \cos\left(6x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 2^n \cos\left(2x + \frac{n\pi}{2}\right)\right].$$

Example 4 (a) : Find the n^{th} derivative of $y = \sin x \sin 2x \sin 3x$. **(M.U. 2014)**

Sol. : We have

$$y = \frac{1}{2}\sin x \cdot 2\sin 2x \cdot \sin 3x = -\frac{1}{2}\sin x\left[\cos 5x - \cos x\right]$$

$$= -\frac{1}{2}\sin x \cos 5x + \frac{1}{2}\sin x \cos x$$

$$= -\frac{1}{4} \cdot 2\sin x \cos 5x + \frac{1}{4} \cdot 2\sin x \cos x$$

$$\therefore \quad y = -\frac{1}{4}\left[\sin 6x + \sin(-4x)\right] + \frac{1}{4}\sin 2x$$

$$= -\frac{1}{4}\sin 6x + \frac{1}{4}\sin 4x + \frac{1}{4}\sin 2x$$

$$= \frac{1}{4}\sin 2x + \frac{1}{4}\sin 4x - \frac{1}{4}\sin 6x$$

Now, by result (8), page 8-3, we get

$$y_n = \frac{1}{4}\left[2^n \sin\left(2x + \frac{n\pi}{2}\right) + 4^n \sin\left(4x + \frac{n\pi}{2}\right) - 6^n \sin\left(6x + \frac{n\pi}{2}\right)\right].$$

Example 5 (a) : If $y = \sin^2 x \cos^3 x$, find y_n. **(M.U. 2002, 19)**

Sol. : We have $\quad y = \sin^2 x \cos^3 x$

$$\therefore \quad y = \sin^2 x \cos^2 x \cdot \cos x = \frac{1}{4}(\sin 2x)^2 \cos x$$

$$= \frac{1}{8}(1 - \cos 4x)\cos x = \frac{1}{8}(\cos x - \cos 4x \cos x)$$

$$= \frac{1}{8}\cos x - \frac{1}{16}\left[\cos 5x + \cos 3x\right]$$

By using the result (9-A), we get

$$y_n = \frac{1}{8}\cos\left(x + \frac{n\pi}{2}\right) - \frac{1}{16}\cdot 5^n \cos\left(5x + \frac{n\pi}{2}\right) - \frac{1}{16}\cdot 3^n \cos\left(3x + \frac{n\pi}{2}\right)$$

Example 6 (a) : If $y = \sin^5 x \cos^3 x$, find y_n.

Sol. : We have

$$y = \sin^5 x \cos^3 x = \sin^2 x\,(\sin^3 x \cos^3 x)$$

$$= \sin^2 x \left(\frac{2\sin x \cos x}{2}\right)^3 = \frac{1 - \cos 2x}{2} \cdot \frac{1}{2^3} \cdot \sin^3 2x$$

$$= \frac{1}{16}(1 - \cos 2x) \cdot \left(\frac{3\sin 2x - \sin 6x}{4}\right) \qquad [\because \ \sin 3\theta = 3\sin\theta - 4\sin^3\theta]$$

$$= \frac{1}{64}\left[3\sin 2x - \sin 6x - 3\cos 2x \sin 2x + \sin 6x \cos 2x\right]$$

$$= \frac{1}{64}\left[3\sin 2x - \sin 6x - \frac{3}{2}\sin 4x + \frac{1}{2}(\sin 8x + \sin 4x)\right]$$

$$= \frac{1}{64}\left[3\sin 2x - \sin 6x - \sin 4x + \frac{1}{2}\sin 8x\right]$$

By using 8(A), page 8-4, we get

$$y_n = \frac{1}{64}\left[3 \cdot 2^n \sin\left(2x + \frac{n\pi}{2}\right) - 6^n \sin\left(6x + \frac{n\pi}{2}\right) - 4^n \sin\left(4x + \frac{n\pi}{2}\right) + \frac{1}{2}\cdot 8^n \sin\left(8x + \frac{n\pi}{2}\right)\right]$$

Example 7 (a) : Find the n^{th} derivative of $e^{2x}\sin\dfrac{x}{2}\cos\dfrac{x}{2}$.

Sol. : We have

$$y = \frac{1}{2}e^{2x} \cdot 2\sin\frac{x}{2} \cdot \cos\frac{x}{2} = \frac{1}{2}e^{2x}\sin x$$

By (10), page 8-4, if $y = e^{ax}\sin(bx+c)$, then

$$y_n = r^n e^{ax}\sin(bx+c+n\Phi).$$

Hence, put $a = 2$, $b = 1$, $c = 0$.

$$\therefore \quad y_n = \frac{1}{2}r^n e^{2x}\sin(x+n\Phi)$$

where, $r = \sqrt{a^2+b^2} = \sqrt{4+1} = \sqrt{5}$ and $\Phi = \tan^{-1}\dfrac{b}{a} = \tan^{-1}\dfrac{1}{2}$.

Example 8 (a) : If $y = e^x\sin x$, show that $y_n = 2^{n/2}e^x\sin\left(x+\dfrac{n\pi}{4}\right)$.

Sol. : We have $\quad y = e^x\sin x$

By (10), page 8-4

$$y_n = r^n e^{ax}\sin(x+n\Phi)$$

where $r = \sqrt{a^2+b^2} = \sqrt{1+1} = \sqrt{2}$ and $\Phi = \tan^{-1}\left(\dfrac{b}{a}\right) = \tan^{-1}(1) = \dfrac{\pi}{4}$.

Example 9 (a) : If $y = e^x(\sin x+\cos x)$, prove that $y_n = (\sqrt{2})^{n+1}e^x\sin\left[x+\dfrac{(n+1)\pi}{4}\right]$.

Sol. : We have, $\quad y = e^x\sin x + e^x\cos x$

Using results (10) and (11), [where $a = 1$, $b = 1$, $c = 0$]

$$y_n = (\sqrt{2})^n e^x\sin(x+n\Phi) + (\sqrt{2})^n e^x\cos(x+n\Phi)$$

where $\Phi = \tan^{-1}(1/1) = \pi/4$.

$$\therefore \quad y_n = (\sqrt{2})^n \cdot e^x\left[\sin\left(x+\frac{n\pi}{4}\right)+\cos\left(x+\frac{n\pi}{4}\right)\right]$$

$$= (\sqrt{2})^n \cdot e^x\left[\sin\left(x+\frac{n\pi}{4}\right)+\sin\left(\frac{\pi}{2}+x+\frac{n\pi}{4}\right)\right]$$

$$= (\sqrt{2})^n \cdot e^x 2\sin\left(x+\frac{n\pi}{4}+\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{4}\right)$$

$$\therefore \quad y_n = (\sqrt{2})^{n+1} \cdot e^x\sin\left[x+\frac{(n+1)\pi}{4}\right].$$

Aliter : We have,

$$y = e^x \cdot \sqrt{2}\left(\sin x\frac{1}{\sqrt{2}}+\cos x\frac{1}{\sqrt{2}}\right) = e^x \cdot \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$$

Using the result (10),

$$y_n = \sqrt{2}\cdot(\sqrt{2})^n e^x\sin\left(x+\frac{\pi}{4}+\frac{n\pi}{4}\right) = (\sqrt{2})^{n+1}e^x\sin\left[x+\frac{(n+1)\pi}{4}\right].$$

Example 10 (a) : If $y = e^{ax}\cos^2 x \sin x$, find y_n. **(M.U. 2018)**

Sol. : We have, $\cos^2 x \cdot \sin x = \cos x \sin x \cos x$

$$= \frac{1}{2}\cos x \sin 2x = \frac{1}{4}[\sin 3x + \sin x]$$

$$\therefore \quad y = \frac{1}{4} \cdot e^{ax}(\sin 3x + \sin x).$$

Using the result (10),

$$y_n = \left(\frac{1}{4}\right)[r_1^n\, e^{ax}\sin(3x + n\,\Phi_1) + r_2^n\, e^{ax}\sin(x + n\,\Phi_2)]$$

where, $r_1 = \sqrt{a^2+9}$ and $\Phi_1 = \tan^{-1}(3/a)$

$r_2 = \sqrt{a^2+1}$ and $\Phi_2 = \tan^{-1}(1/a)$.

Example 11 (a) : If $y = 2^x \cos 9x$, find y_n. **(M.U. 2009)**

Sol. : First we note that $2^x = e^{x\log 2} = e^{ax}$ where $a = \log 2$.

$$\therefore \quad y = e^{ax}\cos 9x \qquad \text{.......................... (1)}$$

$$\therefore \quad y_1 = e^{ax} \cdot a\cos 9x - e^{ax}\sin 9x$$

$$= e^{ax}(a\cos 9x - 9\sin 9x)$$

$$= e^{ax} \cdot \sqrt{a^2+9^2}\left[\frac{a}{\sqrt{a^2+9^2}}\cos 9x - \frac{9}{\sqrt{a^2+9^2}}\sin 9x\right]$$

If we put $\dfrac{a}{\sqrt{a^2+9^2}} = \cos\theta$ and $\dfrac{9}{\sqrt{a^2+9^2}} = \sin\theta$

i.e. if we put $r = \sqrt{a^2+9^2}$ and $\tan\theta = \dfrac{9}{a}$ then

$$y_1 = r\,e^{ax}(\cos\theta\cos 9x - \sin\theta\sin 9x) = r\,e^{ax}\cos(9x+\theta)$$

Thus, y_1 is what y in (1) becomes when y is multiplied by r and the angle $9x$ is increased by θ.

$$\therefore \quad y_2 = r^2 e^{ax}\cos(9x + 2\theta)$$

$$y_3 = r^3 e^{ax}\cos(9x + 3\theta)$$

....

$$\therefore \quad y_n = r^n\, e^{ax}\cos(9x + n\theta)$$

where, $r = \sqrt{a^2+9^2} = \sqrt{(\log 2)^2 + 9^2}$, $a = \log 2$ and $\theta = \tan^{-1}\left(\dfrac{9}{\log 2}\right)$.

Example 12 (a) : If $y = 2^n \sin^2 x \cos x$, find y_n. **(M.U. 2016)**

Sol. : First we note that $2^x = e^{x\log 2} = e^{ax}$, where, $a = \log 2$.

Now, $\sin^2 x\cos x = \sin x \cdot \sin x\cos x$

$$= \frac{1}{2}\sin x \cdot 2\sin x \cdot \cos x = \frac{1}{2}\sin x\sin 2x$$

$$= \frac{1}{4} \cdot 2\sin 2x\sin x = -\frac{1}{4}(\cos 3x - \cos x) = \frac{1}{4}(\cos x - \cos 3x)$$

$\therefore \quad y = \frac{1}{4} e^{ax} (\cos x - \cos 3x)$

By result (11), page 8-4

$$y_n = \frac{1}{4} [r_1^n e^{ax} (\cos x + n\Phi_1) - r_2^n e^{ax} (\cos 3x + n\Phi_2)]$$

where, $\quad r_1 = \sqrt{(\log 2)^2 + 1}, \quad a = \log 2, \quad \Phi_1 = \tan^{-1}\left(\frac{1}{\log 2}\right)$

and $\quad r_2 = \sqrt{(\log 2)^2 + 3^2}, \quad a = \log 2, \quad \Phi_2 = \tan^{-1}\left(\frac{3}{\log 2}\right).$

Example 13 (a) : If $y = 2^x \sin^2 x \cos^3 x$, find y_n. **(M.U. 1996, 2009, 12)**

Sol. : First note that $2^x = e^{x \log 2} = e^{ax}$ where $a = \log 2$

and $\quad \sin^2 x \cos^3 x = 4 \cdot \frac{\sin^2 x \cos^2 x \cdot \cos x}{4} = \frac{\sin^2 2x \cdot \cos x}{4}$

$$= \frac{(1 - \cos 4x)}{8} \cdot \cos x = \frac{1}{8} \cos x - \frac{1}{8} \cos 4x \cos x$$

$$= \frac{1}{8} \cos x - \frac{2}{16} \cos 4x \cos x$$

$\therefore \quad \sin^2 x \cos^3 x = \frac{1}{8} \cos x - \frac{1}{16} (\cos 5x + \cos 3x)$

$\therefore \quad y = \frac{1}{8} e^{ax} \cos x - \frac{1}{16} e^{ax} \cos 5x - \frac{1}{16} e^{ax} \cos 3x$

Using formula (10), we get,

$$y_n = \frac{1}{8} r_1^n e^{ax} \cos (x + n\Phi_1) - \frac{1}{16} r_2^n e^{ax} \cos (5x + n\Phi_2) - \frac{1}{16} r_3^n e^{ax} \cos (3x + n\Phi_3)$$

$$= \frac{1}{8} r_1^n 2^x \cos (x + n\Phi_1) - \frac{1}{16} r_2^n 2^x \cos (5x + n\Phi_2) - \frac{1}{16} r_3^n 2^x \cos (3x + n\Phi_3)$$

where, $\quad r_1 = \sqrt{(\log 2)^2 + 1^2}, \; \Phi_1 = \tan^{-1}\left(\frac{1}{\log 2}\right), \; r_2 = \sqrt{(\log 2)^2 + 5^2}$

$\Phi_2 = \tan^{-1}\left(\frac{5}{\log 2}\right), \; r_3 = \sqrt{(\log 2)^2 + 3^2}, \; \Phi_3 = \tan^{-1}\left(\frac{3}{\log 2}\right).$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Find n^{th} derivatives of the following : Class (a) : 4 Marks

1. $\sin x \cos 3x$
2. $\sin 2x \sin 3x \cos 4x$ **(M.U. 2004)**
3. $\sin x \sin 2x \sin 3x$ **(M.U. 2014)**
4. $\sin^3 3x$
5. $\sin^4 x$
6. $\cos^4 x$
7. $\cos^2 x \sin^3 x$
8. $e^{x \cos \alpha} \cos (x \sin \alpha)$
9. $e^x \cos 2x \cos x$ **(M.U. 1981, 2002, 16)**
10. $e^{5x} \cos x \cos 3x$ **(M.U. 1983, 2007)**
11. $e^x \sin^2 x \cos x$
12. $e^x \cos^2 x \cos x$
13. $e^x \cos^2 x \sin x$
14. $e^{ax} \cos 2x \sin x$
15. $2^x \sin^2 x \cos x$ **(M.U. 2013, 16)**

16. If $y = \cos h\, 2x$, prove that $y_n = 2^n \sin h\, 2x$ if n is odd and $y_n = 2^n \cos h\, 2x$ if n is even. **(M.U. 1989, 2002)**

[**Ans. :** (1) $\frac{1}{2}\left[4^n \sin\left(4x+\frac{n\pi}{2}\right) - 2^n \sin\left(2x+\frac{n\pi}{2}\right)\right]$

(2) $\frac{1}{4}\left[5^n \cos\left(5x+\frac{n\pi}{2}\right) + 3^n \cos\left(3x+\frac{n\pi}{2}\right) - 9^n \cos\left(9x+\frac{n\pi}{2}\right) - \cos\left(x+\frac{n\pi}{2}\right)\right]$

(3) $\frac{1}{4}\left[\sin\left(x+\frac{n\pi}{2}\right) + 4^n \sin\left(4x+\frac{n\pi}{2}\right) + 6^n \sin\left(6x+\frac{n\pi}{2}\right)\right]$

(4) $\frac{3}{4}\cdot 3^n \sin\left(3x+\frac{n\pi}{2}\right) - \frac{1}{4}\cdot 9^n \sin\left(9x+\frac{n\pi}{2}\right)$

(5) $y = \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x;\quad y_n = -\frac{1}{2}\cdot 2^n \cos\left(2x+\frac{n\pi}{2}\right) + \frac{1}{8}\cdot 4^n \cos\left(4x+\frac{n\pi}{2}\right)$

(6) $y = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x;\quad y_n = \frac{1}{2}\cdot 2^n \cos\left(2x+\frac{n\pi}{2}\right) + \frac{1}{8}\cdot 4^n \cos\left(4x+\frac{n\pi}{2}\right)$

(7) $y = \frac{1}{16}\left[2\sin\left(x+\frac{n\pi}{2}\right) + 3^n \sin\left(3x+\frac{n\pi}{2}\right) - 5^n \sin\left(5x+\frac{n\pi}{2}\right)\right]$

(8) $y = e^{x\cos\alpha}(x\sin\alpha + \alpha)$

(9) $\frac{1}{2}\cdot e^x\left[10^{n/2}\cos(3x + n\tan^{-1}3) + 2^{n/2}\cos\left(x+\frac{n\pi}{4}\right)\right]$

(10) $\frac{1}{2}\cdot e^{5x}\left\{41^{n/2}\cos\left[4x+n\tan^{-1}\left(\frac{4}{5}\right)\right] + 29^{n/2}\cos\left[2x+n\tan^{-1}\left(\frac{2}{5}\right)\right]\right\}$

(11) $-\frac{1}{4}(10)^{3/2} e^x \cos(3x + n\tan^{-1}3) + \frac{1}{4}2^{n/2} e^x \cos(x + n\tan^{-1}1)$

(12) $\frac{e^x}{4}\left[10^{n/2}\cos(3x + n\tan^{-1}3) + 2^{n/2}\cos\left(x+\frac{n\pi}{4}\right)\right]$

(13) $\frac{e^x}{4}\left[10^{n/2}\sin(3x + n\tan^{-1}3) + 2^{n/2}\sin\left(x+\frac{n\pi}{4}\right)\right]$

(14) $\frac{e^{ax}}{2}\left\{(a^2+9)^{n/2}\sin\left[3x+n\tan^{-1}\left(\frac{3}{a}\right)\right] + (a^2+1)^{n/2}\sin\left[x+n\tan^{-1}\left(\frac{1}{a}\right)\right]\right\}$

(15) $-\frac{1}{4}r_1^n 2^x \cos(3x + n\Phi_1) + \frac{1}{4}r_2^n 2^x \cos(x + n\Phi_2)$

where, $r_1 = \sqrt{(\log 2)^2 + 3^2},\ \Phi_1 = \tan^{-1}\left(\frac{3}{\log 2}\right),$

$r_2 = \sqrt{(\log 2)^2 + 1^2},\ \Phi_2 = \tan^{-1}\left(\frac{1}{\log 2}\right).$]

Type III : Based on De Moivre's Theorem

In some algebraic functions n^{th} derivative can be put in an elegant form by using De Moivre's Theorem. Also n^{th} derivatives of some inverse trigonometric functions can be put in a very compact form by using De Moivre's Theorem.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : If $y = \dfrac{1}{x^2+a^2}$, prove that

$$y_n = \frac{(-1)^n \cdot n!}{a^{n+2}} \sin^{n+1}\theta \sin(n+1)\theta \text{ where } \theta = \tan^{-1}\left(\frac{a}{x}\right).$$ **(M.U. 2018)**

Sol. : We have $y = \dfrac{1}{x^2+a^2} = \dfrac{1}{x^2-a^2 i^2} = \dfrac{1}{2ai}\left[\dfrac{1}{x-ai} - \dfrac{1}{x+ai}\right]$

By result (4-A), proved on page 8-3,

$$y_n = \frac{1}{2ai}\left[\frac{(-1)^n \cdot n!}{(x-ai)^{n+1}} - \frac{(-1)^n \cdot n!}{(x+ai)^{n+1}}\right]$$

$$= \frac{(-1)^n \cdot n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right] \qquad \text{.................... (A)}$$

Let $x = r\cos\theta$, $a = r\sin\theta$, so that $r^2 = x^2 + a^2$ and $\theta = \tan^{-1}(a/x)$

Now, $\dfrac{1}{(x-ai)^{n+1}} = \dfrac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}}$

$$= \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta} \qquad \text{[By De Moivre's Theorem]}$$

$$\therefore \quad \frac{1}{(x-ai)^{n+1}} = \frac{1}{r^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta]$$

And $\dfrac{1}{(x+ai)^{n+1}} = \dfrac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}$

$$= \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta} \qquad \text{[By De Moivre's Theorem]}$$

$$\therefore \quad \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}}[\cos(n+1)\theta - i\sin(n+1)\theta]$$

$$\therefore \quad \frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}} \cdot 2i\sin(n+1)\theta$$

Putting these values in (A), we get

$$\therefore \quad y_n = (-1)^n \cdot n! \cdot \frac{1}{a} \cdot \frac{1}{r^{n+1}} \sin(n+1)\theta$$

But $r = \dfrac{a}{\sin\theta}$, $\quad \left(\because \ a = r\sin\theta \ \therefore \ r^{n+1} = \dfrac{a^{n+1}}{\sin^{n+1}\theta}\right)$

$$\therefore \quad y_n = (-1)^n \cdot n! \cdot \frac{1}{a^{n+2}} \sin^{(n+1)}\theta \sin(n+1)\theta.$$

Example 2 (b) : If $y = \dfrac{x}{x^2+a^2}$, prove that

$$y_n = (-1)^n \cdot n!\, a^{-n-1} \sin^{n+1}\theta \cos(n+1)\theta \quad \text{where } \theta = \tan^{-1}(a/x).$$ **(M.U. 2007)**

Sol. : We have

$$y = \frac{x}{x^2+a^2} = \frac{x}{(x+ai)(x-ai)} = \frac{1}{2}\left[\frac{1}{x+ai} + \frac{1}{x-ai}\right] \qquad \text{[By partial fractions]}$$

By result (4-A), proved on page 8-3

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+ai)^{n+1}} + \frac{(-1)^n \cdot n!}{(x-ai)^{n+1}}\right]$$

$$= \frac{(-1)^n \cdot n!}{2}\left[\frac{1}{(x+ai)^{n+1}} + \frac{1}{(x-ai)^{n+1}}\right] \quad \text{.................... (A)}$$

Let $x = r\cos\theta$, $a = r\sin\theta$, so that $r^2 = (x^2 + a^2)$ and $\theta = \tan^{-1}(a/x)$

$$\therefore \quad \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta + i\sin\theta)^{n+1}}$$

$$= \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta + i\sin(n+1)\theta} \quad \text{[By De Moivre's Theorem]}$$

$$= \frac{1}{r^{n+1}} \cdot [\cos(n+1)\theta - i\sin(n+1)\theta] \quad \text{[By Corollary 1]}$$

And, $$\frac{1}{(x-ai)^{n+1}} = \frac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}}$$

$$= \frac{1}{r^{n+1}} \cdot \frac{1}{\cos(n+1)\theta - i\sin(n+1)\theta}$$

$$= \frac{1}{r^{n+1}} \cdot [\cos(n+1)\theta + i\sin(n+1)\theta]$$

Adding the two results,

$$\frac{1}{(x-ai)^{n+1}} + \frac{1}{(x+ai)^{n+1}} = \frac{1}{r^{n+1}} \cdot 2\cos(n+1)\theta$$

Putting these values in (A), we get

$$y_n = \frac{1}{2}(-1)^n \cdot n! \cdot \frac{1}{r^{n+1}} \cdot 2\cos(n+1)\theta$$

$$\therefore \quad y_n = (-1)^n \cdot n!\, r^{-n-1} \cos(n+1)\theta$$

But $r = \dfrac{a}{\sin\theta}$, $\quad (\because\ a = r\sin\theta)$

$$\therefore \quad y_n = (-1)^n \cdot n!\, a^{-n-1} \sin^{(n+1)}\theta \cos(n+1)\theta.$$

Example 3 (b) : If $y = \tan^{-1} x$, prove that $y_n = \dfrac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}} \sin\left(n\tan^{-1}\dfrac{1}{x}\right)$.

Sol. : Differentiating $y = \tan^{-1} x$, w.r.t. x, we get,

$$y_1 = \frac{1}{x^2+1} = \frac{1}{(x+i)(x-i)} = \frac{1}{2i}\left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]$$

Differentiating $(n-1)$ times, *i.e.* replacing n by $(n+1)$, by result (4-A), page 8-3

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+i)^n}\right]$$

$$= \frac{(-1)^{n-1} \cdot (n-1)!}{2i}\left[\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right] \quad \text{.................... (1)}$$

Putting $x = r\cos\theta$, $1 = r\sin\theta$

i.e. $r=\sqrt{1+x^2}$ and $\theta=\tan^{-1}(1/x)$, we get

$$\frac{1}{(x-i)^n}=\frac{1}{r^n(\cos\theta-i\sin\theta)^n}=\frac{1}{r^n}(\cos n\theta+i\sin n\theta)$$

Similarly, $\frac{1}{(x+i)^n}=\frac{1}{r^n(\cos\theta+i\sin\theta)^n}=\frac{1}{r^n}(\cos n\theta-i\sin n\theta)$

$$\therefore\quad \frac{1}{(x-i)^n}-\frac{1}{(x+i)^n}=\frac{2i}{r^n}\sin n\theta$$

Hence, from (1), we get,

$$y_n=\frac{(-1)^{n-1}(n-1)!}{2i}\cdot\frac{2i}{r^n}\sin n\theta=(-1)^{n-1}(n-1)!\cdot\frac{1}{r^n}\sin n\theta \qquad \text{.....................} (2)$$

Now, we put $r=\sqrt{1+x^2}$ and $\theta=\tan^{-1}\left(\frac{1}{x}\right)$ in (2).

$$\therefore\quad y_n=\frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}}\sin\left[n\tan^{-1}\left(\frac{1}{x}\right)\right]$$

Remark ....

You may stop at the result (1) above if asked to find y_n only.

Example 4 (b) : If $y=\tan^{-1}x$, prove that

$$y_n=(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta \qquad \text{where } \theta=\tan^{-1}(1/x).$$

Sol. : In the result (2), obtained above we put $r=\frac{1}{\sin\theta}$ *i.e.*, $\frac{1}{r}=\sin\theta$.

$$\therefore\quad y_n=(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$$

Example 5 (b) : Find the n^{th} derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Sol. : If we put $x=\tan\alpha$, then

$$y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{2\tan\alpha}{1-\tan^2\alpha}\right)=\tan^{-1}(\tan 2\alpha)=2\alpha=2\tan^{-1}x$$

Hence, as proved in Ex. 3 above

$$y_n=2\cdot\frac{(-1)^{n-1}(n-1)!}{(x^2+1)^{n/2}}\sin\left[n\tan^{-1}\left(\frac{1}{x}\right)\right].$$

Example 6 (b) : If $y=\tan^{-1}\left(\frac{1+x}{1-x}\right)$, prove that

$y_n=(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$ where $\theta=\tan^{-1}(1/x)$. **(M.U. 1995, 2001, 03, 04, 05)**

Sol. : Put $1=\tan(\pi/4)$ and $x=\tan\alpha$,

$$\therefore\quad y=\tan^{-1}\left[\frac{\tan(\pi/4)+\tan\alpha}{1-\tan(\pi/4)\cdot\tan\alpha}\right]=\tan^{-1}\tan\left(\frac{\pi}{4}+\alpha\right)$$

$$\therefore\quad y=\frac{\pi}{4}+\alpha=\frac{\pi}{4}+\tan^{-1}x \qquad \therefore\quad y_1=\frac{1}{1+x^2}=\frac{1}{2i}\left[\frac{1}{x-i}-\frac{1}{x+i}\right]$$

Now proceeding as in Ex. (3) and (4) above, we get,

$$y_n=(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta \text{ where } \theta=\tan^{-1}(1/x).$$

Example 7 (b) : If $y = \cos^{-1}\left(\dfrac{x - x^{-1}}{x + x^{-1}}\right)$, prove that

$y_n = 2\,(-1)^{n-1}\,(n-1)\,!\,\sin^n\theta\,\sin n\theta$ where $\theta = \tan^{-1}(1/x)$. **(M.U. 2004)**

Sol. : Putting $x = \tan\alpha$ $\therefore$ $\alpha = \tan^{-1} x$.

$$y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) = \cos^{-1}\left(\frac{\tan^2\alpha - 1}{\tan^2\alpha + 1}\right)$$

$$= \cos^{-1}\left[-\frac{(\cos^2\alpha - \sin^2\alpha)/\cos^2\alpha}{1/\cos^2\alpha}\right]$$

$$= \cos^{-1}[-(\cos^2\alpha - \sin^2\alpha)] = \cos^{-1}[-(\cos 2\alpha)]$$

$$= \cos^{-1}\cos(\pi + 2\alpha) = \pi + 2\alpha = \pi + 2\tan^{-1} x$$

$$\therefore\quad y_1 = 2\cdot\frac{1}{x^2+1}$$

Now, proceeding as in Ex. (3) and (4) above, we get

$y_n = 2\,(-1)^{n-1}\,(n-1)\,!\,\sin^n\theta\,\sin n\theta$ where $\theta = \tan^{-1}(1/x)$.

Example 8 (b) : If $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, prove that

$y_n = \dfrac{1}{2}(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$ where $\theta = \tan^{-1}(1/x)$ [or $\theta = \cot^{-1} x$]

Sol. : Putting $x = \tan\alpha$ $\therefore$ $\alpha = \tan^{-1} x$

$$y = \tan^{-1}\left(\frac{\sec\alpha - 1}{\tan\alpha}\right) = \tan^{-1}\left[\frac{(1-\cos\alpha)/\cos\alpha}{\sin\alpha/\cos\alpha}\right]$$

$$= \tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right) = \tan^{-1}\left[\frac{2\sin^2(\alpha/2)}{2\sin(\alpha/2)\cos(\alpha/2)}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\alpha}{2}\right)\right] = \frac{\alpha}{2} = \frac{1}{2}\tan^{-1} x$$

$$\therefore\quad y_1 = \frac{1}{2}\cdot\frac{1}{x^2+1}$$

$$\therefore\quad y_n = \frac{1}{2}\cdot(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$$

where $\theta = \tan^{-1}(1/x)$. [As in Ex. 3 and 4]

Example 9 (b) : Expand $y = \cos^9 x$ in cosines of multiples of x and then find y_n. **(M.U. 2008)**

Sol. : Let $x = \cos\theta + i\sin\theta$, $\dfrac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore\quad \left(x + \frac{1}{x}\right) = 2\cos\theta \qquad \therefore\quad 2^9\cos^9\theta = \left(x + \frac{1}{x}\right)^9$$

$$\therefore\quad 2^9\cos^9\theta = x^9 + 9x^8\frac{1}{x} + 36\cdot x^7\frac{1}{x^2} + 84x^6\cdot\frac{1}{x^3} + 126x^5\cdot\frac{1}{x^4}$$

$$+ 126x^4\cdot\frac{1}{x^5} + 84x^3\cdot\frac{1}{x^6} + 36x^2\cdot\frac{1}{x^7} + 9x\cdot\frac{1}{x^8} + \frac{1}{x^9}$$

$$\therefore \quad 2^9\cos^9\theta = \left(x^9 + \frac{1}{x^9}\right) + 9\left(x^7 + \frac{1}{x^7}\right) + 36\left(x^5 + \frac{1}{x^5}\right) + 84\left(x^3 + \frac{1}{x^3}\right) + 126\left(x + \frac{1}{x}\right)$$

But $\quad x^m + \dfrac{1}{x^m} = \cos m\theta + i\sin m\theta + \cos m\theta - i\sin m\theta = 2\cos m\theta$

Putting $m = 9, 7, 5, 3, 1$ successively, we get

$$\therefore \quad 2^9\cos^9\theta = 2\cos 9\theta + 9\cdot 2\cos 7\theta + 36\cdot 2\cos 5\theta + 84\cdot 2\cos 3\theta + 126\cdot 2\cos\theta$$

$$\therefore \quad \cos^9\theta = \frac{1}{256}[\cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta]$$

But if $y = \cos ax$, $\quad y_n = a^n\cos[ax + n(\pi/2)]$ [By Cor. of (9-A), page 8-4]

If $y = \cos^9 x$,

$$\therefore \quad y_n = \frac{1}{256}\left[9^n\cos\left(9\theta + \frac{n\pi}{2}\right) + 9\cdot 7^n\cos\left(7\theta + \frac{n\pi}{2}\right) + 36\cdot 5^n\cos\left(5\theta + \frac{n\pi}{2}\right)\right.$$
$$\left. + 84\cdot 3^n\cos\left(3\theta\frac{n\pi}{2}\right) + 126^n\cdot\cos\left(\theta + \frac{n\pi}{2}\right)\right]$$

Similarly, we can prove that if $y = \sin^9 x$, then

$$y_n = \frac{1}{256}\left[9^n\sin\left(9\theta + \frac{n\pi}{2}\right) - 9\cdot 7^n\sin\left(7\theta + \frac{n\pi}{2}\right) + 36\cdot 5^n\sin\left(5\theta + \frac{n\pi}{2}\right)\right.$$
$$\left. - 84\cdot 3^n\sin\left(3\theta + \frac{n\pi}{2}\right) + 126\cdot\sin\left(\theta + \frac{n\pi}{2}\right)\right]$$

Example 10 (b) : If $y = \dfrac{1}{x^2 + x + 1}$, prove that $\quad y_n = \dfrac{2(-1)^n}{\sqrt{3}}\cdot\dfrac{(n!)}{r^{n+1}}\sin(n+1)\theta$

where $\theta = \cot^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)$ and $r = \sqrt{x^3 + x + 1}$. **(M.U. 2004, 07)**

Sol. : We have $y = \dfrac{1}{x^2 + x + 1} = \dfrac{1}{[x + (1/2)]^2 + (3/4)}$

Let $x + \dfrac{1}{2} = X$ then $y = \dfrac{1}{X^2 - \left(\sqrt{3}\,i/2\right)^2}$

$$\therefore \quad y = \frac{1}{\sqrt{3}\,i}\cdot\left[\frac{1}{(X - \sqrt{3}\,i/2)} - \frac{1}{(X + \sqrt{3}\,i/2)}\right]$$

$$\therefore \quad y_n = \frac{(-1)^n\cdot n!}{\sqrt{3}\,i}\left[\frac{1}{(X - \sqrt{3}\,i/2)^{n+1}} - \frac{1}{(X + \sqrt{3}\,i/2)^{n+1}}\right]$$ [By (4-A), page 8-3]

Now, put $X = r\cos\theta$, $\sqrt{3}/2 = r\sin\theta$

$$\therefore \quad r = \sqrt{X^2 + (3/4)} = \sqrt{x^2 + x + 1}$$

and $\quad \cot\theta = \dfrac{2X}{\sqrt{3}} = \dfrac{2x+1}{\sqrt{3}} \quad \therefore \ \theta = \cot^{-1}\left(\dfrac{2x+1}{3}\right)$

Now, $\quad \dfrac{1}{(X - \sqrt{3}\cdot i/2)^{n+1}} = \dfrac{1}{(r\cos\theta - i\,r\sin\theta)^{n+1}} = \dfrac{1}{r^{n+1}(\cos\theta - i\sin\theta)^{n+1}}$

$$= \frac{1}{r^{n+1}}[\cos(n+1)\theta + i\sin(n+1)\theta]$$ [By Cor. (4), page 2-2]

Similarly, $\dfrac{1}{(X+\sqrt{3}\cdot i/2)^{n+1}}=\dfrac{1}{r^{n+1}}[\cos(n+1)\theta-i\sin(n+1)\theta]$

$$\therefore\quad y_n=\frac{(-1)^n\cdot n!}{\sqrt{3}\cdot i}\left[\frac{2i}{r^{n+1}}\cdot\sin(n+1)\theta\right]=2\cdot\frac{(-1)^n\cdot n!}{\sqrt{3}\cdot r^{n+1}}\cdot\sin(n+1)\theta.$$

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

1. If $y=\dfrac{1}{x^2+1}$, prove that $y_n=(-1)^n\cdot n!\sin^{n+1}\theta\sin(n+1)\theta$ where $\theta=\tan^{-1}\left(\dfrac{1}{x}\right)$.

2. If $y=\dfrac{x}{x^2+1}$, prove that $y_n=(-1)^n\cdot n!\sin^{n+1}\theta\cos(n+1)\theta$ where $\theta=\tan^{-1}\left(\dfrac{1}{x}\right)$.

3. If $y=\tan^{-1}\left(\dfrac{x}{a}\right)$, prove that $y_n=(-1)^{n-1}(n-1)!\,a^{-n}\sin^n\theta\sin n\theta$ where $\theta=\tan^{-1}\left(\dfrac{1}{x}\right)$.

4. If $y=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, prove that $y_n=2(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$ where $\theta=\tan^{-1}\left(\dfrac{1}{x}\right)$.

5. If $y=\sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right)$, prove that $y_n=2(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta$ where $\theta=\tan^{-1}\left(\dfrac{1}{x}\right)$.

(M.U. 1998)

6. If $y=x\tan^{-1}x$, find y_n.

   [**Ans. :** $(-1)^{n-1}(n-1)!\sin^n\theta\cos\theta+(-1)^{n-2}(n-2)!\sin^{n-1}\theta\sin(n-1)\theta$ where $\theta=\tan^{-1}(1/x)$.]

7. If $y=\sin^7 x$, find y_n.

   [**Ans. :** $y_n=-\dfrac{1}{64}\left[7^n\sin\left(7x+\dfrac{n\pi}{2}\right)-7\cdot5^n\sin\left(5x+\dfrac{n\pi}{2}\right)+21\cdot3^n\sin\left(3x+\dfrac{n\pi}{2}\right)-35\sin\left(x+\dfrac{n\pi}{2}\right)\right]$

8. If $y=\sin^4 x\cos^3 x$, find y_n. **(M.U. 1996)**

   [**Ans. :** $y_n=\dfrac{1}{64}\left[7^n\cos\left(7x+\dfrac{n\pi}{2}\right)-\cos\left(5x+\dfrac{n\pi}{2}\right)-3\cdot3^n\cos\left(3x+\dfrac{n\pi}{2}\right)+3\cos\left(x+\dfrac{n\pi}{2}\right)\right]$

9. If $y=\sin^5 x$, find y_n.

   [**Ans. :** $y_n=\dfrac{1}{16}\left[5^n\sin\left(5x+\dfrac{n\pi}{2}\right)-5\cdot3^n\sin\left(3x+\dfrac{n\pi}{2}\right)+10\sin\left(x+\dfrac{n\pi}{2}\right)\right]$

3. Leibnitz's Theorem

If $y = uv$ where u and v are functions of x possessing derivatives of n^{th} order, then

$$\boxed{y_n = {}^nC_0\, u_n\, v + {}^nC_1\, u_{n-1}\, v_1 + {}^nC_2\, u_{n-2}\, v_2 + \ldots + {}^nC_r\, u_{n-r}\, v_r + \ldots + {}^nC_n\, u v_n}$$

Proof : We prove the theorem by Mathematical Induction.

Baron Gottfried Wilhelm Von Leibnitz (1646-1716)

An outstanding German mathematician, philospher, physicist, diplomat and linguist developed calculus independently. He had encyclopedic knowledge. The notations of d/dx for the derivative and $\int$ for integral that we use are due to him. The word **'function' was first suggested by Leibnitz.** He entered the university of Leipzig at the age of 15 to study law and got his doctorate degree from the university of Aldorf. In 1672 he invented a calculating machine. He played a key role in founding Berlin Academy of Sciences in 1700. He was given the title of baron (noble man) by the Russian Emperor, Peter the Great. He was never married.

Step 1 : We first prove that the theorem is true for $n = 1, 2$ by direct differentiation :

$$y_1 = u_1 v + u v_1$$

$$y_2 = u_2 v + u_1 v_1 + u_1 v_1 + u v_2 = u_2 v + 2u_1 v_1 + u v_2$$

Thus, the theorem is true for $n = 1, 2$.

Step 2 : Let us assume that the theorem is true for a particular value of n, say m.

Hence, $y_m = {}^mC_0\, u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots$

$$+ {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r\, u_{m-r}\, v_r + \ldots + {}^mC_m\, u v_m$$

Now, differentiating both sides, we get,

$$y_{m+1} = {}^mC_0\, u_{m+1} v + {}^mC_0\, u_m v_1 + {}^mC_1\, u_m v_1 + {}^mC_1\, u_{m-1} v_2$$
$$+ {}^mC_2\, u_{m-1} v_2 + {}^mC_2\, u_{m-2} v_3 + \ldots + {}^mC_{r-1}\, u_{m-r+2}\, v_{r-1}$$
$$+ {}^mC_{r-1}\, u_{m-r+1} v_r + {}^mC_r\, u_{m-r+1} v_r + {}^mC_r\, u_{m-r}\, v_{r+1} + \ldots$$
$$\ldots + {}^mC_m\, u_1 v_m + {}^mC_m\, u v_{m+1}$$

$$y_{m+1} = {}^mC_0\, u_{m+1} v + ({}^mC_0 + {}^mC_1)\, u_m v_1 + ({}^mC_1 + {}^mC_2)\, u_{m-1} v_2 + \ldots$$
$$\ldots + ({}^mC_{r-1} + {}^mC_r)\, u_{m-r+1} v_r + \ldots + {}^mC_m\, u v_{m+1}$$

But we know that

$${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r, \quad {}^mC_0 = {}^{m+1}C_0, \quad {}^mC_m = {}^{m+1}C_{m+1}$$

$$\therefore \quad y_{m+1} = {}^{m+1}C_0\, u_{m+1} v + {}^{m+1}C_1\, u_m v_1 + {}^{m+1}C_2\, u_{m-1} v_2 + \ldots$$
$$\ldots + {}^{m+1}C_r\, u_{m+1-r}\, v_r + {}^{m+1}C_{m+1}\, u v_{m+1}$$

This proves that if the theorem is true for $n = m$ then it is true for $n = m + 1$.

Step 3 : But we have proved that in step 1, the theorem is true for $n = 1, 2$. Hence, by step 2, it is true for $n = 3$. Since it is true for $n = 3$ by step 2 again it is true for $n = 4$.

Hence, we can conclude that the theorem is true for every value of $n = 1, 2, 3, 4, ...$

To apply this theorem,

1. First determine the function whose n^{th} derivative is known and treat it as u.
2. The first term then is $u_n\, v$.
3. Then write the next binomial coefficient n. Multiply it by previous derivative of u and by the next derivative of v. Continue this process.
4. At some stage you will find that the derivative of v is zero and the above process stops.

The theorem can also be stated as

$$\boxed{y_n = u_n\, v + n\, u_{n-1}\, v_1 + \frac{n(n-1)}{2!}\, u_{n-2}\, v_2 + \frac{n(n-1)(n-2)}{3!}\, u_{n-3}\, v_3 + ...+ u\, v_n}$$

Type I : Class (b) : 6 Marks

Example 1 (b) : If $y = x^4 \cos 3x$, find y_n.

Sol. : Let $u = \cos 3x$, $v = x^4$ then $u_n = 3^n \cos\left(3n + \frac{n\pi}{2}\right)$. Now, $y = uv$.

By Leibnitz' Theorem

$$y_n = u_n\, v + n\, u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2}\, v_2 + ...$$

$$\therefore \quad y_n = 3^n \cos\left(3x + \frac{n\pi}{2}\right) \cdot x^4 + n \cdot 3^{n-1} \cos\left(3x + \overline{n-1} \cdot \frac{\pi}{2}\right) 4x^3$$

$$+ \frac{n(n-1)}{2!} \cdot 3^{n-2} \cos\left(3x + \overline{n-2} \cdot \frac{\pi}{2}\right) 12x^2$$

$$+ \frac{n(n-1)(n-2)}{3!} \cdot 3^{n-3} \cos\left(3x + \overline{n-3} \cdot \frac{\pi}{2}\right) 24x$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 3^{n-4} \cos\left(3x + \overline{n-4} \cdot \frac{\pi}{2}\right) 24$$

$$\therefore \quad y_n = 3^n \cos\left(3x + \frac{n\pi}{2}\right) \cdot x^4 + 4n \cdot 3^{n-1} \cos\left(3x + \overline{n-1} \cdot \frac{\pi}{2}\right) \cdot x^3$$

$$+ 6n(n-1) \cdot 3^{n-2} \cos\left(3x + \overline{n-2} \cdot \frac{\pi}{2}\right) \cdot x^2$$

$$+ 4n(n-1)(n-2) \cdot 3^{n-3} \cos\left(3x + \overline{n-3} \cdot \frac{\pi}{2}\right) \cdot x$$

$$+ n(n-1)(n-2)(n-3) \cdot 3^{n-4} \cos\left(3x + \overline{n-4} \cdot \frac{\pi}{2}\right)$$

Example 2 (b) : If $y = 7^x x^7$, find y_5.

Sol. : Let $u = 7^x$, $v = x^7$ then $u_n = 7^x (\log 7)^n$. Now, $y = uv$

By Leibnitz's Theorem,

$$y_5 = u_5\, v + 5\, u_4\, v_1 + \frac{5 \cdot 4}{2!} u_3\, v_2 + \frac{5 \cdot 4 \cdot 3}{3!} u_2\, v_3 + 5\, u_1\, v_4 + u\, v_5$$

$$= 7^x \cdot (\log 7)^5 \cdot x^7 + 5 \cdot 7^x \cdot (\log 7)^4 \cdot 7 \cdot x^6 + 10 \cdot 7^x \cdot (\log 7)^3 \cdot 42\, x^5$$
$$+ 10 \cdot 7^x \cdot (\log 7)^2 \cdot 210\, x^4 + 5 \cdot 7^x \cdot (\log 7) \cdot 840\, x^3 + 7^x \cdot 2520\, x^2$$

$$\therefore \quad y_5 = (\log 7)^5 \cdot 7^x x^7 + 35 \cdot (\log 7)^4 \cdot 7^x x^6 + 420 \cdot (\log 7)^3 \cdot 7^x x^5$$
$$+ 2100 \cdot (\log 7)^2 \cdot 7^x x^4 + 4200 \cdot (\log 7) \cdot 7^x x^3 + 2520 \cdot 7^x x^2.$$

Example 3 (b) : If $y = x^n \log x$, prove that $y_{n+1} = \dfrac{n!}{x}$. **(M.U. 2003, 04, 08)**

Sol. : We have $y = x^n \log x$.

Differentiating it w.r.t. x, $\quad y_1 = nx^{n-1} \log x + x^n \cdot \dfrac{1}{x}$

Multiplying by x throughout $\quad xy_1 = nx^n \log x + x^n = ny + x^n$ **[Note this]**

Applying Leibnitz's Theorem

$$xy_{n+1} + n(1) \cdot y_n = ny_n + n! \qquad \text{[By (1-A), page 8-1]}$$

$$\therefore \quad xy_{n+1} = n! \quad \therefore \quad y_{n+1} = \frac{n!}{x}. \qquad \text{[See also Ex. 7, page 8-39]}$$

Example 4 (b) : If $y = \dfrac{\log x}{x}$, prove that $y_5 = \dfrac{5!}{x^6}\left[1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} - \log x\right]$. **(M.U. 2010)**

Sol. : Let $u = \dfrac{1}{x}$ and $v = \log x$, so that $u_n = (-1)^n \cdot \dfrac{n!}{x^{n+1}}$ [By (4) page 8-3]

By Leibnitz's Theorem

$$y_5 = u_5 v + 5 u_4 v_1 + \frac{5 \cdot 4}{2!} u_3 v_2 + \frac{5 \cdot 4}{2!} u_2 v_3 + 5 u_1 v_4 + u v_5$$
$$= (-1)^5 \cdot \frac{5!}{x^6} \log x + 5 \cdot (-1)^4 \cdot \frac{4!}{x^5} \cdot \frac{1}{x} + 10(-1)^3 \cdot \frac{3!}{x^4} \cdot \left(-\frac{1}{x^2}\right)$$
$$+ 10(-1)^2 \cdot \frac{2!}{x^3}\left(\frac{2}{x^3}\right) + 5(-1) \cdot \frac{1}{x^2}\left(-\frac{2 \cdot 3}{x^4}\right) + \frac{1}{x}\left(\frac{2 \cdot 3 \cdot 4}{x^5}\right)$$
$$= -\frac{5!}{x^6} \log x + \frac{5!}{x^6} + \frac{5!}{2x^6} + \frac{5!}{3x^6} + \frac{5!}{4x^6} + \frac{5!}{5x^6}$$

$$\therefore \quad y_5 = \frac{5!}{x^6}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \log x\right]$$

[In general,

$$y_n = u_n v + n\, u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \ldots..$$
$$= \frac{(-1)^n n!}{x^{n+1}} \cdot \log x + \frac{n(-1)^{n-1}(n-1)!}{x^n} \cdot \frac{1}{x} + \frac{n(n-1)}{2!}(-1)^{n-2}\frac{(n-2)!}{x^{n-1}}\left(-\frac{1}{x^2}\right) + \ldots.$$
$$= \frac{(-1)^n n!}{x^{n+1}} \cdot \log x - \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n n!}{x^{n+1}}\left(\frac{1}{2}\right) + \ldots.$$
$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - \left(1 + \frac{1}{2} + \ldots. + \frac{1}{n}\right)\right]. \text{]}$$

Example 5 (b) : If $y = x^2 e^{2x}$, prove that at $x = 0$ $y_n = 2^{n-2} \cdot n(n-1)$. **(M.U. 2004)**

Sol. : Let $e^{2x} = u,\ x^2 = v \quad \therefore \quad u_n = 2^n e^{2x}$. Now $y = uv$.

By Leibnitz's Theorem,

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots..$$

$$= 2^n \cdot e^{2x} \cdot x^2 + n \cdot 2^{n-1} \cdot e^{2x}(2x) + \frac{n(n-1)}{2!} \cdot 2^{n-2} \cdot e^{2x}(2)$$

When $x = 0$, $y_n = \dfrac{n(n-1)}{2!} \cdot 2^{n-2}\, 2 = n(n-1) \cdot 2^{n-2}$.

Example 6 (b) : If $y = x \log(x+1)$, prove that $y_n = \dfrac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$. **(M.U. 1991)**

Sol. : Let $u = \log(x+1)$, $v = x$ then $u_n = \dfrac{(-1)^{n-1}(n-1)!}{(x+1)^n}$. [By (5-A), page 8-3]

By Leibnitz's Theorem,

$$y_n = u_n v + n u_{n-1} v_1 + \ldots.$$

$$= \frac{(-1)^{n-1}(n-1)!}{(x+1)^n} \cdot x + \frac{n(-1)^{n-2}(n-2)!}{(x+1)^{n-1}} \cdot (1)$$

$$= \frac{(-1)^{n-2}(-1)(n-2)!(n-1)x}{(x+1)^n} + \frac{(-1)^{n-2}(n-2)!\ n(x+1)}{(x+1)^n}$$

$$\therefore \quad y_n = \frac{(-1)^{n-2}(n-2)!}{(x+1)^n} \cdot [-nx + x + nx + n] = \frac{(-1)^{n-2}(n-2)!(x+n)}{(x+1)^n}$$

Alternatively : We have $y_1 = \dfrac{x}{1+x} + \log(1+x) \quad \therefore \quad y_1 = 1 - \dfrac{1}{1+x} + \log(1+x)$

Now using formulae (4) and (5), page 8-3, we get

$$y_n = -\frac{(-1)^{n-1}(n-1)!}{(1+x)^n} + \frac{(-1)^{n-2}(n-2)!}{(1+x)^{n-1}}$$

$$= \frac{(-1)^{n-2}(n-2)!}{(1+x)^n} \cdot [(n-1) + (1+x)]$$

$$= \frac{(-1)^{n-2}(n-2)!(n+x)}{(1+x)^n}.$$

Type II : Class (c) : 8 Marks

Example 1 (c) : If $x = \tan \log y$ (or $y = e^{\tan^{-1} x}$), prove that

$$(1 + x^2)\, y_{n+1} + (2nx - 1)\, y_n + n(n-1)\, y_{n-1} = 0. \quad \textbf{(M.U. 1983, 2001, 17)}$$

Sol. : From data $\tan^{-1} x = \log y$ i.e. $y = e^{\tan^{-1} x}$

Differentiating it w.r.t. x,

$$y_1 = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2} = \frac{y}{1+x^2}$$

$$\therefore \quad (1 + x^2)\, y_1 - y = 0.$$

Applying Leibnitz's Theorem to each term, we get

$$(1+x^2)\,y_{n+1}+n(2x)\,y_n+\frac{n(n-1)}{2!}\cdot 2\cdot y_{n-1}-y_n=0$$

i.e. $$(1+x^2)\,y_{n+1}+(2nx-1)\,y_n+n(n-1)\,y_{n-1}=0.$$

Example 2 (c) : If $y=\cos(m\sin^{-1}x)$ [or $y=\sin(m\sin^{-1}x)$], prove that

$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}+(m^2-n^2)\,y_n=0.$$

Hence, obtain $y_n(0)$. **(M.U. 2008, 09, 13, 17)**

Sol. : We have $y_1=-\sin(m\sin^{-1}x)\cdot m\cdot\dfrac{1}{\sqrt{1-x^2}}$

$$\therefore\quad \sqrt{1-x^2}\cdot y_1=-m\sin(m\sin^{-1}x).$$

Differentiating again, we get

$$\sqrt{1-x^2}\cdot y_2-\frac{xy_1}{\sqrt{1-x^2}}=-m\cos(m\sin^{-1}x)\cdot m\cdot\frac{1}{\sqrt{1-x^2}}$$

$$\therefore\quad (1-x^2)\,y_2-xy_1+m^2y=0 \quad \text{.................... (1)}$$

Applying Leibnitz's Theorem to each term, we get

$$(1-x^2)\,y_{n+2}+n(-2x)\,y_{n+1}+\frac{n(n-1)}{2!}(-2)\,y_n-[\,xy_{n+1}+n\,y_n\,]+m^2y_n=0$$

$$(1-x^2)\,y_{n+2}-2nxy_{n+1}-n(n-1)\,y_n-xy_{n+1}-n\,y_n+m^2y_n=0$$

$$\therefore\quad (1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}+(m^2-n^2)\,y_n=0 \quad \text{.................... (2)}$$

Putting $x=0$ in $y=\cos(m\sin^{-1}x)$, we get $y(0)=\cos(0)=1$.

Putting $x=0$ in $y_1=\sin(m\sin^{-1}x)\dfrac{1}{\sqrt{1-x^2}}$, we get $y_1(0)=\sin(0)=0$.

Putting $x=0$ in (1), we get $y_2(0)=-m^2\,y(0)=-m^2$.

Putting $x=0$ in (2), we get $y_{n+2}(0)=(n^2-m^2)\,y_n(0)$ .... (3)

Putting $n=1,3,5$ .... in (3), we get,

$$y_3(0)=(1^2-m^2)\,y_1(0)=0$$
$$y_5(0)=(3^2-m^2)\,y_3(0)=0$$

....

....

$$\therefore\quad y_n(0)=0 \quad \text{if } n \text{ is odd.}$$

Putting $n=2,4,6$ .... in (3), we get

$$y_4(0)=(2^2-m^2)\,y_2(0)=(2^2-m^2)(-m^2)$$
$$y_6(0)=(4^2-m^2)\,y_4(0)=(4^2-m^2)(2^2-m^2)(-m^2)$$

....

....

$$\therefore\quad y_n(0)=(n^2-m^2)\,.....\,(4^2-m^2)(2^2-m^2)(-m^2) \text{ if } n \text{ is even.}$$

Remark ....

In Example 1, we wanted a relation in y_{n+1} and hence we obtained y_1 and applied Leibnitz's Theorem. In Example 2, we wanted a relation in y_{n+2} and hence we obtained y_2 first and then applied Leibnitz's Theorem.

Example 3 (c) : If $y = a\cos(\log x) + b\sin(\log x)$, then show that

$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$ **(M.U. 1996, 2002, 04, 05, 19)**

Sol. : We have $y = a\cos(\log x) + b\sin(\log x)$

$$\therefore \quad y_1 = -a\sin(\log x)\cdot\frac{1}{x} + b\cos(\log x)\cdot\frac{1}{x}$$

$$\therefore \quad xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t. x,

$$xy_2 + y_1 = -a\cos(\log x)\cdot\frac{1}{x} - b\sin(\log x)\cdot\frac{1}{x}$$

$$\therefore \quad x^2 y_2 + xy_1 + y = 0$$

Applying Leibnitz's Theorem to each term, we get

$$x^2 y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + [xy_{n+1} + n(1)y_n] + y_n = 0$$

$$\therefore \quad x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 - n + n + 1)y_n = 0$$

$$\therefore \quad x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0.$$

Example 4 (c) : If $y = e^{m\sin^{-1}x}\left[\text{or } x = \sin\left(\frac{1}{m}\log y\right)\right]$, prove that

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$ **(M.U. 1982, 86, 94, 2004)**

Sol. : We have $y_1 = e^{m\sin^{-1}x}\cdot m\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{my}{\sqrt{1-x^2}}$

$$\therefore \quad \sqrt{1-x^2}\cdot y_1 = my \qquad \text{.................... (1)}$$

Differentiating again w.r.t. x,

$$\sqrt{1-x^2}\cdot y_2 - \frac{xy_1}{\sqrt{1-x^2}} = my_1 = \frac{m^2 y}{\sqrt{1-x^2}}$$

$$\therefore \quad (1-x^2)y_2 - xy_1 = m^2 y \qquad \text{.................... (2)}$$

Applying Leibnitz's Theorem to each term, we get

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - [xy_{n+1} + ny_n] = m^2 y_n$$

$$\therefore \quad (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$$

Example 5 (c) : If $y = e^{a\sin^{-1}x}$, prove that

(i) $(1-x^2)y_{n+2} - (2x+1)xy_{n+1} - (x^2+a^2)y_n = 0$ **(M.U. 2004)**

(ii) $y_{2m}(0) = a(a^2+2^2)(a^2+4^2)(a^2+6^2).....[a^2+(2m-2)^2]$

(iii) $y_{2m+1}(0) = a^2(a^2+1^2)(a^2+3^2)(a^2+5^2).....[a^2+(2m-1)^2]$ **(M.U. 1985)**

Sol. : We have $y = e^{a\sin^{-1}x}$, ... (1)

Replacing m by a in the above example, we get,

$$y_1 = \frac{ay}{\sqrt{1-x^2}} \qquad \text{.................... (2)}$$

$\therefore \quad (1-x^2)\,y_2 - xy_1 = a^2 y$ .... (3)

(i) $\quad (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+a^2)\,y_n = 0$ .... (4)

Putting $x = 0$, in (1), we get $\quad y(0) = e^0 = 1.$

Putting $x = 0$, in (2), we get $\quad y_1(0) = a\,y(0) = a.$

Putting $x = 0$, in (3), we get $\quad y_2(0) = a^2\,y(0) = a^2.$

Putting $x = 0$, in (4), we get $\quad y_{n+2}(0) = (n^2+a^2)\,y_n(0)$ .... (5)

Putting $n = 1$, in (5), we get $\quad y_3(0) = (a^2+1^2)\,y_1(0) = a\,(a^2+1^2)$

Putting $n = 2$, in (5), we get $\quad y_4(0) = (a^2+2^2)\,y_2(0) = a^2\,(a^2+2^2)$

Putting $n = 3$, in (5), we get $\quad y_5(0) = (a^2+3^2)\,y_3(0) = a\,(a^2+1^2)\,(a^2+3^2)$ .... (6)

Putting $n = 4$, in (5), we get $\quad y_6(0) = (a^2+4^2)\,y_4(0) = a^2\,(a^2+2^2)\,(a^2+4^2)$ .... (7)

Generalising from (7) and (6), we get

(ii) $\quad y_{2m}(0) = a^2\,(a^2+2^2)\,(a^2+4^2)\,(a^2+6^2)\,.....\,[a^2+(2m-2)^2]$

(iii) $\quad y_{2m+1}(0) = a\,(a^2+1^2)\,(a^2+3^2)\,(a^2+5^2)\,.....\,[a^2+(2m-1)^2].$

Example 6 (c) : If $\log y = \tan^{-1} x$, (or $y = e^{\tan^{-1} x}$), prove that

$$(1+x^2)\,y_{n+2} + [\,2\,(n+1)\,x - 1\,]\,y_{n+1} + n\,(n+1)\,y_n = 0. \qquad \textbf{(M.U. 1999, 2016)}$$

Sol. : Since $\log y = \tan^{-1} x, \quad y = e^{\tan^{-1} x}$

$$\therefore \quad y_1 = e^{\tan^{-1} x} \cdot \frac{1}{1+x^2} \qquad \therefore \quad (1+x^2)\,y_1 = y$$

Differentiating once again,

$$(1+x^2)\,y_2 + 2xy_1 = y_1$$

$$\therefore \quad (1+x^2)\,y_2 + (2x-1)\,y_1 = 0.$$

Applying Leibnitz's Theorem,

$$(1+x^2)\,y_{n+2} + n(2x)\,y_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n + (2x-1)\,y_{n+1} + n \cdot 2y_n = 0$$

$$\therefore \quad (1+x^2)\,y_{n+2} + [\,2\,(n+1)\,x - 1\,]\,y_{n+1} + n\,(n+1)\,y_n = 0.$$

Example 7 (c) : If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1-x^2)\,y_{n+1} - (2n+1)xy_n - n^2 y_{n-1} = 0.$

(M.U. 1994, 95, 2009, 14)

Sol. : We have $\sqrt{1-x^2} \cdot y = \sin^{-1} x.$

Differentiating it w.r.t. x,

$$\sqrt{1-x^2} \cdot y_1 - \frac{xy}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad (1-x^2)\,y_1 - xy - 1 = 0$$

Applying Leibnitz's Theorem,

$$(1-x^2)\,y_{n+1} + n(-2x)\,y_n + \frac{n(n-1)}{2!}(-2)\,y_{n-1} - [\,xy_n + n \cdot y_{n-1}\,] = 0$$

$$\therefore \quad (1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2 y_{n-1} = 0.$$

Example 8 (c) : If $\cos^{-1}\left(\frac{y}{b}\right)=\log\left(\frac{x}{n}\right)^n$ (or $y=b\cos[\,n(\log x-\log n)\,]$), prove that

$$x^2\cdot y_{n+2}+(2n+1)\,xy_{n+1}+2n^2\,y_n=0.$$ **(M.U. 1986, 90, 2011)**

Sol. : We have $y=b\cos[\,n(\log x-\log n)\,]$

$$\therefore\quad y_1=-b\sin[\,n(\log x-\log n)\,]\cdot n\cdot\frac{1}{x}\qquad\therefore\quad xy_1=-nb\sin[\,n(\log x-\log n)\,]$$

Differentiating again,

$$xy_2+y_1=-nb\cos[\,n(\log x-\log x)]\cdot n\cdot\frac{1}{x}$$

$$\therefore\quad x^2y_2+xy_1=-n^2\,b\cos[\,n(\log x-\log n)\,]=-n^2y$$

$$\therefore\quad x^2y_2+xy_1+n^2y=0.$$

Applying Leibnitz's Theorem,

$$x^2y_{n+2}+n(2x)\,y_{n+1}+\frac{n(n-1)}{2!}\cdot 2\cdot y_n+xy_{n+1}+ny_n+n^2y_n=0$$

$$\therefore\quad x^2y_{n+2}+(2n+1)\,xy_{n+1}+2n^2y_n=0.$$

Example 9 (c) : If $y=2x\sqrt{1-x^2}$, prove that

$$(1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2-4)\,y_n=0.$$

Sol. : We have $y_1=2\left[\frac{x(-x)}{\sqrt{1-x^2}}+\sqrt{1-x^2}\cdot 1\right]$

$$\therefore\quad y_1\sqrt{1-x^2}=2[-x^2+1-x^2]=-4x^2+2$$

Differentiating again,

$$y_1\cdot\frac{(-x)}{\sqrt{1-x^2}}+y_2\sqrt{1-x^2}=-8x$$

$$\therefore\quad -xy_1+y_2(1-x^2)=-8x\sqrt{1-x^2}\qquad\therefore\quad (1-x^2)\,y_2-xy_1+4y=0$$

Applying Leibnitz's Theorem,

$$(1-x^2)\,y_{n+2}+n(-2x)\,y_{n+1}+\frac{n(n-1)}{2!}(-2)\,y_n-[\,xy_{n+1}+n(1)\,y_n]+4y_n=0$$

$$\therefore\quad (1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-(n^2-4)=0.$$

Example 10 (c) : If $y=\left(x+\sqrt{x^2-1}\right)^m$, prove that

$$(x^2-1)\,y_{n+2}+(2n+1)\,xy_{n+1}+(n^2-m^2)\,y_n=0.$$ **(M.U. 2016)**

Sol. : We have

$$y_1=m\left(x+\sqrt{x^2-1}\right)^{m-1}\left[1+\frac{x}{\sqrt{x^2-1}}\right]$$

$$=m\left(x+\sqrt{x^2-1}\right)^{m-1}\cdot\frac{\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}=\frac{m\left(x+\sqrt{x^2-1}\right)^m}{\sqrt{x^2-1}}=\frac{my}{\sqrt{x^2-1}}$$

$\therefore \quad \sqrt{x^2-1} \cdot y_1 = my$

Differentiating this, again,

$$\sqrt{x^2-1} \cdot y_2 + y_1 \cdot \frac{x}{\sqrt{x^2-1}} = m\,y_1 = m \cdot \frac{my}{\sqrt{x^2-1}}$$

Multiplying throughout by $\sqrt{x^2-1}$,

$$(x^2-1)\,y_2 + x\,y_1 = m^2\,y.$$

Now, applying Leibnitz's theorem, we get

$$(x^2-1)\,y_{n+2} + n\,(2x)\,y_{n+1} + \frac{n(n-1)}{2!} \cdot 2 \cdot y_n + x \cdot y_{n+1} + n \cdot 1 \cdot y_n = m^2\,y_n$$

$$\therefore \quad (x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$$

Example 11 (c) : If $x = \cos h\left(\frac{1}{m}\log y\right)$, prove that

$(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2+m^2)\,y_n = 0.$ **(M.U. 2011)**

Sol. : We have $x = \cos h\left(\frac{1}{m}\log y\right)$

$\therefore \quad \cos^{-1} x = \frac{1}{m}\log y \qquad \therefore \quad m\cos h^{-1}\,x = \log y$ .... (1)

But $\quad \cos h^{-1}x = \log\left(x+\sqrt{x^2-1}\right)$ [By 2, page 3-34]

$\therefore \quad m\cos h^{-1}x = m\log\left(x+\sqrt{x^2-1}\right)$

$= \log\left(x+\sqrt{x^2-1}\right)^m$ [$\because$ $\log a^m = m\log a$] .... (2)

From (1) and (2), we get

$$\log y = \log\left(x+\sqrt{x^2-1}\right)^m \qquad \therefore \quad y = \left(x+\sqrt{x^2+1}\right)^m$$

Now, proceed as Ex. 10 above.

Note ....

In Ex. 1, page 8-27 and Ex. 11 above, we are given $x = f(y)$. Hence, we first express $y = f(x)$ and then apply Leibnitz's Theorem.

Example 12 (c) : If $y^{1/m} + y^{-1/m} = 2x$, prove that

$(x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$ **(M.U. 2007, 10, 13, 16)**

Sol. : We have $y^{1/m} + \frac{1}{y^{1/m}} = 2x \quad \therefore \quad y^{2/m} - 2x\,y^{1/m} + 1 = 0$

This is a quadratic in $y^{1/m}$

$$\therefore \quad y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1} \qquad \therefore \quad y = \left(x \pm \sqrt{x^2-1}\right)^m$$

Taking + sign before the radical

$$\therefore\quad y_1 = m\left(x+\sqrt{x^2-1}\right)^{m-1}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right)$$

$$= m\left(x+\sqrt{x^2-1}\right)^{m}\cdot\frac{1}{\sqrt{x^2-1}} = \frac{my}{\sqrt{x^2-1}}$$

$$\therefore\quad \sqrt{x^2-1}\cdot y_1 = my$$

Differentiating again w.r.t. x,

$$\sqrt{x^2-1}\cdot y_2 + \frac{x}{\sqrt{x^2-1}}\cdot y_1 = my_1$$

$$(x^2-1)\,y_2 + xy_1 = m\sqrt{x^2-1}\cdot y_1 = m\cdot my = m^2 y$$

$$\therefore\quad (x^2-1)\,y_2 + xy_1 - m^2 y = 0$$

Applying Leibnitz's Theorem,

$$(x^2-1)\,y_{n+2} + n(2x)\,y_{n+1} + \frac{n(n-1)}{2!}(2)\,y_n + xy_{n+1} + n(1)\,y_n - m^2 y_n = 0$$

$$\therefore\quad (x^2-1)\,y_{n+2} + (2n+1)\,xy_{n+1} + (n^2-m^2)\,y_n = 0.$$

Example 13 (c) : If $x = \sin\theta$, $y = \sin 2\theta$, prove that

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2-4)\,y_n = 0. \qquad \textbf{(M.U. 2003)}$$

Sol. : We first express the given parametric function as $y = f(x)$.

We have $y = \sin 2\theta = 2\sin\theta\cos\theta$. But $\sin\theta = x$

$$\therefore\quad \cos\theta = \sqrt{1-x^2} \qquad \therefore\quad y = 2x\sqrt{1-x^2}$$

Now proceed as in the above example.

Example 14 (c) : If $x = \sin\theta$ and $y = \cos m\theta$, prove that

$$(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} + (m^2-n^2)\,y_n = 0.$$

Sol. : We have $\theta = \sin^{-1} x$,

$$\therefore\ y = \cos m\theta = \cos(m\sin^{-1} x)$$

Now, proceed as in Ex. 2.

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Find the n^{th} derivative of y if : Class (b) : 6 Marks

1. $y = x^3 e^x$ 2. $y = x^2 a^x$ 3. $y = x^2 \sin x$ **(M.U. 1994)**
4. $y = x^2 e^{mx}$ 5. $y = x^3 \sin 2x$ 6. $y = (2x+3)^2 e^x$
7. $y = (x+3)^3 \sin 3x$.

[**Ans. :** (1) $y_n = e^x x^3 + 3ne^x\cdot x^2 + 3n(n-1)e^x x + n(n-1)(n-2)\,e^x$

(2) $y_n = a^x(\log a)^n x^2 + n\cdot a^x\cdot(\log a)^{n-1}\cdot 2x + n(n-1)\cdot a^x(\log a)^{n-2}$

(3) $y_n = \sin\left(x+\frac{n\pi}{2}\right)x^2 + 2n\sin\left(x+\overline{n-1}\cdot\frac{\pi}{2}\right)x + n(n-1)\sin\left(x+\overline{n-2}\cdot\frac{\pi}{2}\right)x$

(4) $y_n = m^n e^{mx} x^2 + n\cdot m^{n-1} e^{mx}\cdot 2x + n(n-1)\cdot m^{n-2} e^{mx}$

(5) $y_n = 2^n \sin\left(2x + \frac{n\pi}{2}\right) \cdot x^3 + n \cdot 3x^2\, 2^{n-1} \sin\left(2x + \overline{n-1} \cdot \frac{\pi}{2}\right)$

$+ n(n-1) \cdot 3x\, 2^{n-2} \sin\left(2x + \overline{n-2} \cdot \frac{\pi}{2}\right)$

$+ n(n-1)(n-2) \cdot 2^{n-3} \sin\left(2x + \overline{n-3} \cdot \frac{\pi}{2}\right)$

(6) $y_n = e^x [(2x+3)^2 + 4n(2x+3) + 4n(n-1)]$

(7) $y_n = 3^n \sin\left(3x + \frac{n\pi}{2}\right) \cdot (x+3)^3 + n \cdot 3^n \sin\left(3x + \overline{n-1} \cdot \frac{\pi}{2}\right) \cdot (x+3)^2$

$+ n(n-1) \cdot 3^{n-1} \sin\left(3x + \overline{n-2} \cdot \frac{\pi}{2}\right) \cdot (x+3)$

$+ n(n-1)(n-2) \cdot 3^{n-3} \sin\left(3x + \overline{n-3} \cdot \frac{\pi}{2}\right)$]

EXERCISE - V

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (c) : 8 Marks

1. If $y = \cos^{-1} x$, prove that $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0$. **(M.U. 2010)**
2. If $y = \sin^{-1} x$, prove that $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0$.
   Also find $y_9(0)$ and $y_{10}(0)$. **(M.U. 2003)** [**Ans. :** $y_9(0) = 1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$ and $y_{10}(0) = 0$]
3. If $y = (x + \sqrt{a^2 + x^2})^m$, prove that $a^2 y_{n+2} + (n^2 - m^2) y_n = 0$ at $x = 0$. **(M.U. 1992)**
4. If $y = \tan^{-1} x$, prove that $(1+x^2) y_{n+2} + 2(n+1) xy_{n+1} + n(n+1) y_n = 0$.
   Hence, deduce that $y_n(0) = 0$ if n is even and $y_n(0) = (n-1)!$ if n is odd. **(M.U. 2018)**
5. If $y = (x + \sqrt{a^2 + x^2})^2$, prove that $(a^2 + x^2) y_{n+2} + (2n+1) xy_{n+1} + (n^2 - 4) y_n = 0$.
6. If $y = \left(\frac{1+x}{1-x}\right)^r$, prove that $(1-x^2) y_{n+1} - 2(r + nx) y_n - n(n-1) y_{n-1} = 0$.
7. If $y = (x^2 - 1)^n$, prove that $(x^2 - 1) y_{n+2} + 2xy_{n+1} - n(n+1) y_n = 0$. **(M.U. 1987, 86)**
8. If $y = (1-x)^{-\alpha} \cdot e^{-\alpha x}$, prove that $(1-x) y_{n+1} - (n + \alpha x) y_n - n\alpha\, y_{n-1} = 0$.
9. If $y = \sin(m \sin^{-1} x)$ or if $m \sin^{-1} x = \sin^{-1} y$, prove that
   $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} + (m^2 - n^2) y_n = 0$. **(M.U. 2000, 02, 04, 05)**
   Hence, deduce that $y_n(0) = 0$ if n is even and
   $y_n(0) = (n^2 - m^2) \ldots\ldots (3^2 - m^2)(1^2 - m^2)\, m$ if n is odd.
10. If $y = e^{\alpha \sin^{-1} x}$, prove that $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 + \alpha^2) y_n = 0$.
11. If $y = (\sin^{-1} x)^2$, prove that $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - n^2 y_n = 0$. **(M.U. 1988, 97)**
12. If $y = (1-x^2)^{1/2} \sin^{-1} x$, prove that $(1-x^2) y_{n+1} - (2n-1) xy_n - n(n-2) y_{n-1} = 0$. **(M.U. 1995)**
13. If $y = e^{m \cos^{-1} x}$ or $x = \cos[\log(y^{1/m})]$, prove that
    $(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 + m^2) y_n = 0$. **(M.U. 1995, 2015)**

14. If $\sin^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2 y_{n+2} + (2n+1)\, x y_{n+1} + 2n^2\, y_n = 0$.

15. If $x = \cos h\left(\frac{1}{m}\log y\right)$, prove that $(x^2 - 1)\, y_{n+2} + (2n+1)\, x y_{n+1} + (n^2 - m^2)\, y_n = 0$.

(M.U. 1993)

16. If $y = \sin h^{-1} x$, prove that $(1 + x^2)\, y_{n+2} + (2n+1)\, x y_{n+1} + n^2\, y_n = 0$. (M.U. 1987)

17. If $y = \frac{\log x}{x}$, prove that $y_n = \frac{(-1)^n \cdot n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} \ldots\ldots - \frac{1}{n}\right]$. (M.U. 1991, 96)

18. If $y = \sqrt{\left(\frac{1+x}{1-x}\right)}$, prove that $y = (1 - x^2)\, y_1$ and hence, prove that

$$(1 - x^2)\, y_n - [2(n-1)\, x + 1]\, y_{n-1} - (n-1)(n-2)\, y_{n-2} = 0.$$ (M.U. 1983, 2004, 06)

19. If $y = x^n \log x$, prove that $x^2 y_2 - (2n-1)\, x y_1 + n^2 y = 0$.

And hence, prove that $x^2 y_{p+2} + (2p - 2n + 1)\, x y_{p+1} + (p - n)^2 y_p = 0$. (M.U. 1984, 88)

20. If $y = \sin \log (x^2 + 2x + 1)$, prove that

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)\, y_{n+1} + (n^2 + 4)\, y_n = 0.$$ (M.U. 1992, 2001, 12, 15, 18)

21. If $y = \cos [\, \log (x^2 - 2x + 1)\,]$, prove that

$$(x-1)^2\, y_{n+2}\, (2n+1)\, (x-1)\, y_{n+1} + (4 + n^2)\, y_n = 0.$$ (M.U. 1987, 92)

22. If $y = \cos (m \log x)$, prove that $x^2 y_{n+2} + (2n+1)\, x y_{n+1} + (m^2 + n^2)\, y_n = 0$.

23. If $y = \tan^{-1}\left[\frac{a+x}{a-x}\right]$, prove that $(a^2 + x^2)\, y_{n+2} + 2(n+1)\, x y_{n+1} + n(n+1)\, y_n = 0$.

(M.U. 1998, 2002)

24. If $y = \sec^{-1} x$, prove that

$$x(x^2 - 1)\, y_{n+2} + [(2 + 3n)\, x^2 - (n+1)]\, y_{n+1} + n(3n+1)\, x y_n + n^2 (n-1)\, y_{n-1} = 0.$$

25. If $y = \log\left(x + \sqrt{x^2 + a^2}\right)^2$, prove that $(x^2 + a^2)\, y_{n+2} + (2n+1)\, x y_{n+1} + n^2 y_n = 0$.

(M.U. 1998)

26. If $y = \left[\log\left(x + \sqrt{x^2 + 1}\right)\right]^2$, prove that $y_{n+2}(0) = -n^2 y_n(0)$. (M.U. 1997)

27. If $y = \frac{Lx + M}{x^2 - 2Bx + C}$, prove that $\frac{(x^2 - 2Bx + C)}{(n+1)(n+2)}\, y_{n+2} + 2\frac{(x - B)}{(n+1)}\, y_{n+1} + y_n = 0$.

28. If $y = \frac{\sin h^{-1} x}{\sqrt{1 + x^2}}$, prove that $(1 + x^2)\, y_{n+2} + (2n+3)\, x y_{n+1} + (n+1)^2\, y_n = 0$.

29. If $x = e^t$ and $y = \cos m\, t$, prove that $x^2 y_{n+2} + (2n+1)\, x y_{n+1} (m^2 + n^2)\, y_n = 0$.

30. If $x = \cos\theta$, $\theta = \frac{1}{m}\log y$, prove that $(1 - x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2 + m^2)\, y_n = 0.)$

EXERCISE - VI

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (c) : 8 Marks

1. If $y = x^2 \sin x$, prove that $y_n = (x^2 - n^2 + n) \sin\left(x + \frac{n\pi}{2}\right) - 2n\,x \cos\left(x + \frac{n\pi}{2}\right)$. **(M.U. 1998)**

2. If $y = e^x x^k$, prove that $(y_1 - y)\,x = ky$ and $xy_{n+2} + (n+1-k-x)\,y_{n+1} - (n+1)\,y_n = 0$.

   Further show that if $Y = \frac{d^n}{dx^n}(x^{n+1} e^x)$ then $x\,(Y_2 - Y_1) = (n+1)\,Y$. **(M.U. 1984)**

Miscellaneous Examples : Class (c) : 8 Marks

Example 1 (c) : If $y = (x-1)^n$, prove that

$$y + \frac{y_1}{1!} + \frac{y_2}{2!} + \frac{y_3}{3!} + \ldots\ldots + \frac{y_n}{n!} = x^n. \qquad \textbf{(M.U. 2002, 15)}$$

Sol. : We have $y = (x-1)^n$

$$\therefore \quad y_1 = n\,(x-1)^{n-1}$$

$$y_2 = n\,(n-1)\,(x-1)^{n-2}$$

$$y_3 = n\,(n-1)\,(n-2)\,(x-1)^{n-3}$$

$$y_4 = n\,(n-1)\,(n-2)\,(n-3)\,(x-1)^{n-4}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore \quad y_n = n\,(n-1)\,(n-2)\,\ldots..\,[\,n-(n-1)\,]\,(x-1)^{n-n} = n!\,(x-1)^0$$

Putting these values in l.h.s., we have,

$$\text{l.h.s.} = (x-1)^n + \frac{n}{1!}(x-1)^{n-1} + \frac{n(n-1)}{2!}(x-1)^{n-2} + \frac{n(n-1)(n-2)}{3!}(x-1)^{n-3} + \ldots.. + \frac{n!}{n!}(x-1)^0$$

$$\therefore \quad \text{l.h.s.} = [\,x-1+1]^n = x^n \qquad [\text{ By Binomial Theorem }]$$

Hence, the result.

Example 2 (c) : If $y = \tan x$, prove that

$$y_n(0) - {}^nC_2\,y_{n-2}(0) + {}^nC_4\,y_{n-4}(0) + \ldots. = \sin\left(\frac{n\pi}{2}\right). \qquad \textbf{(M.U. 1990, 2003)}$$

Sol. : We have $y \cos x = \sin x$.

Now, applying Leibnitz's Theorem to both sides,

$$y_n(\cos x) + {}^nC_1\,y_{n-1}(-\sin x) + {}^nC_2\,y_{n-2}(-\cos x) + {}^nC_3\,y_{n-3}(\sin x) + {}^nC_4\,y_{n-4}(\cos x) + \ldots. = \sin\left(x + \frac{n\pi}{2}\right)$$

Now put $x = 0$,

$$\therefore \quad y_n(0) - {}^nC_2\,y_{n-2}(0) + {}^nC_4\,y_{n-4}(0) - \ldots.. = \sin\frac{n\pi}{2}.$$

Example 3 (c) : If $y = \log\left(x+\sqrt{x^2+1}\right)$, $\left(\text{or if } e^y = x+\sqrt{x^2+1}\right)$, prove that

$$y_{2n}(0) = 0 \text{ and } y_{2n+1}(0) = (-1)^n \cdot 1^2 \cdot 3^2 \cdot 5^2 \ldots..(2n-1)^2.$$

(M.U. 2002)

Sol. : Since $y = \log\left(x+\sqrt{x^2+1}\right)$ .... (1)

$$y_1 = \frac{1}{x+\sqrt{x^2+1}}\left[1+\frac{x}{\sqrt{x^2+1}}\right] = \frac{1}{\sqrt{x^2+1}} \quad \ldots\ldots\ldots\ldots (2)$$

$$\therefore \quad \sqrt{x^2+1} \cdot y_1 = 1$$

Differentiating again,

$$\sqrt{x^2+1} \cdot y_2 + \frac{x}{\sqrt{x^2+1}} \cdot y_1 = 0 \qquad \therefore \quad (x^2+1) \cdot y_2 + xy_1 = 0 \quad \ldots\ldots\ldots\ldots (3)$$

Applying Leibnitz's Theorem

$$(x^2+1) \cdot y_{n+2} + n(2x) \cdot y_{n+1} + \frac{n(n-1)}{2!}(2) \cdot y_n + x \cdot y_{n+1} + n \cdot (1)\, y_n = 0$$

$$\therefore \quad (x^2+1)\, y_{n+2} + (2n+1)\, xy_{n+1} + n^2 y_n = 0 \quad \ldots\ldots\ldots\ldots (4)$$

Putting $x = 0$ in (1), (2), (3) we get

$$y(0) = \log(1) = 0, \quad y_1(0) = 1, \quad y_2(0) = 0$$

Putting $x = 0$ in (4), we get

$$y_{n+2}(0) = -n^2\, y_n(0) \quad \ldots\ldots\ldots\ldots (5)$$

Putting $n = 1, 2, 3, 4$ .... in (5), we get

$y_3(0) = -1^2 \cdot y_1(0) = -1^2,$

$y_4(0) = -2^2 \cdot y_2(0) = 0,$

$y_5(0) = -3^2 \cdot y_3(0) = 3^2 \cdot 1^2,$

$y_6(0) = -4^2 \cdot y_4(0) = 0,$

$y_7(0) = -5^2 \cdot y_5(0) = -5^2 \cdot 3^2 \cdot 1^2,$

$y_8(0) = -6^2 \cdot y_6(0) = 0,$

. . . .

. . . .

Hence, by generalisation

$$y_{2n}(0) = 0 \text{ and } y_{2n+1}(0) = (-1)^n\,[1^2 \cdot 3^2 \cdot 5^2 \ldots..(2n-1)^2]$$

Example 4 (c) : If $y = (\sin^{-1} x)^2$, prove that $y_n(0) = 0$ if n odd and $y_n(0) = 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 \ldots. (n-2)^2$ if $n \neq 2$ and n is even. **(M.U. 2012, 13)**

Sol. : We have $y = (\sin^{-1} x)^2$ .... (1)

$$\therefore \quad y_1 = 2(\sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}} \quad \ldots\ldots\ldots\ldots (2)$$

$$\therefore \quad \sqrt{1-x^2} \cdot y_1 = 2\sin^{-1} x$$

Differentiating again, $\sqrt{1-x^2} \cdot y_2 - y_1 \cdot \dfrac{x}{\sqrt{1-x^2}} = \dfrac{2}{\sqrt{1-x^2}}$

$$\therefore \quad (1-x^2)\, y_2 - x \cdot y_1 = 2 \quad \ldots\ldots\ldots\ldots (3)$$

By Leibnitz's theorem,

$$(1-x^2)\,y_{n+2}+n(-2x)\,y_{n+1}+\frac{n(n-1)}{2!}(-2)\,y_n-[\,xy_{n+1}+n(1)\,y_n\,]=0$$

$$\therefore\quad (1-x^2)\,y_{n+2}-(2n+1)\,xy_{n+1}-n^2\,y_n=0 \qquad \text{.......... (4)}$$

Putting $x=0$, in (4), $\qquad y_{n+2}(0)=n^2\,y_n(0) \qquad$ .... (5)

Putting $x=0$ in (1), (2) and (3), we get,

$$y(0)=(0),\quad y_1(0)=0,\quad y_2(0)=2.$$

Putting $n=1, 3, 5, \ldots\ldots$ in (5), we get,

$$y_3(0)=1^2\cdot y_1(0)=0,$$
$$y_5(0)=3^2\cdot y_3(0)=0,$$
$$y_7(0)=5^2\cdot y_5(0)=0,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$y_n(0)=0 \quad \text{if } n \text{ is odd.}$$

Putting $n=2, 4, 6, \ldots\ldots$ in (5), we get,

$$y_4(0)=2^2\cdot y_2(0)=2^2\cdot 2,\quad y_6(0)=4^2\cdot y_4(0)=4^2\cdot 2^2\cdot 2$$
$$y_n(0)=2\cdot 2^2\cdot 4^2\cdot 6^2\ \ldots\ (n-2)^2 \qquad \text{if } n \text{ is even and } n \neq 2.$$

Example 5 (c) : If $y=(\sin^{-1}x)^2$, prove that at $x=0$, $y_{n+2}(0)=n^2 y_n(0)$.

If, further, $y=(\sin^{-1}x)^2=a_0+a_1x+a_2x^2+\ldots\ldots+a_n x^n \ldots\ldots$,

prove that $\quad (n+1)(n+2)\,a_{n+2}=n^2\cdot a_n$.

Sol. : We have proved above in (5) that

$$y_{n+2}(0)=n^2\,y_n(0)$$

Now consider

$$y\ =\ a_0+a_1x+\ldots\ldots+a_nx^n+\ldots\ldots$$
$$\therefore\ y_1\ =\ a_1+2a_2x+\ldots\ldots+n\,a_nx^{n-1}+\ldots\ldots$$
$$y_2\ =\ 2a_2+\ldots\ldots+n(n-1)\,a_nx^{n-2}+\ldots\ldots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_n\ =\ n(n-1)(n-2)\ \ldots\ 2\cdot 1\cdot a_n+\ldots\ldots \text{ terms in } x.$$
$$=\ n!\,a_n+\ldots\ldots \text{ terms in } x.$$
$$y_n(0)=n!\,a_n$$

Similarly, $y_{n+2}(0)=(n+2)!\,a_{n+2}$.

But $\quad y_{n+2}(0)=n^2\,y_n(0)$

$$\therefore\quad (n+2)!\,a_{n+2}=n^2\,n!\,a_n$$
$$\therefore\quad (n+2)(n+1)\,n!\cdot a_{n+2}=n^2\cdot n!\cdot a_n$$
$$\therefore\quad (n+1)(n+2)\,a_{n+2}=n^2\,a_n.$$

Example 6 (c) : If $y=e^{m\tan^{-1}x}=a_0+a_1x+\ldots\ldots+a_nx^n+\ldots\ldots$, prove that at $x=0$,

$$y_{n+2}(0)-m\,y_{n+1}(0)-n(n+1)\,y_n(0)=0.$$

Hence, or otherwise deduce that

$$(n+1)\,a_{n+1}+(n-1)\,a_{n-1}=m\,a_n.$$

Sol. : Proceeding as in Ex. 6, page 8-30, we can prove that

$$(1+x^2)\,y_{n+2}+[\,2(n+1)\,x-m\,]\,y_{n+1}+n(n+1)\,y_n=0.$$

Putting $x=0$, we get

$$y_{n+2}(0)-m\,y_{n+1}(0)+n(n+1)\,y_n(0)=0 \qquad \text{.................... (1)}$$

Now, if

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$\therefore\ y_1 = a_1 + 2a_2 x + \ldots + n\,a_n x^{n-1} + \ldots$$

$$y_2 = 2a_2 + \ldots + n(n-1)\,a_n x^{n-1} + \ldots$$

. . . .

. . . .

$y_n = n!\,a_n + \ldots$ terms in x as above

Putting $x=0$, we get $\quad y_n(0) = n!\,a_n$ .... (2)

Changing n to $n-1$ in (1),

$$y_{n+1}(0)-m\,y_n(0)+(n-1)\,n\cdot y_{n-1}(0)=0 \qquad \text{.................... (3)}$$

Putting $n=n+1$ and $n-1$ in (2), we get,

$$y_{n+1}(0)=(n+1)!\,a_{n+1}, \qquad y_{n-1}(0)=(n-1)!\,a_{n-1}$$

Putting these values in (3),

$$(n+1)!\,a_{n+1}-m\cdot n!\,a_n+(n-1)\,n!\,a_{n-1}=0$$

$$\therefore\ (n+1)\,a_{n+1}-m\,a_n+(n-1)\,a_{n-1}=0$$

$$\therefore\ (n+1)\,a_{n+1}+(n-1)\,a_{n-1}=m\,a_n.$$

Example 7 (c) : If $I_n=\dfrac{d^n}{dx^n}(x^n\log x)$, prove that $I_n=n\,I_{n-1}+(n-1)!$.

Hence, show that $I_n=n!\left[\log x+1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}\right]$.

Sol. : We have $I_n=\dfrac{d^n}{dx^n}[\,x^n\log x\,]=\dfrac{d^{n-1}}{dx^{n-1}}\cdot\dfrac{d}{dx}(x^n\log x)$

$$=\frac{d^{n-1}}{dx^{n-1}}\left[n\,x^{n-1}\log x+x^n\cdot\frac{1}{x}\right]$$

$$=n\cdot\frac{d^{n-1}}{dx^{n-1}}[\,x^{n-1}\log x\,]+\frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$=n\,I_{n-1}+(n-1)! \qquad \text{[By (1-A), page 8-1]}$$

Dividing both sides by $n!$, we get,

$$\frac{I_n}{n!}=\frac{1}{(n-1)!}I_{n-1}+\frac{1}{n}$$

Putting $n=2,3,4,\ldots,n$,

$$\frac{I_2}{2!}=\frac{1}{1!}\cdot I_1+\frac{1}{2}; \qquad \frac{I_3}{3!}=\frac{1}{2!}\cdot I_2+\frac{1}{3}; \qquad \frac{I_4}{4!}=\frac{1}{3!}\cdot I_3+\frac{1}{4};$$

$$\ldots\ldots\ldots\ldots; \qquad \frac{I_n}{n!}=\frac{1}{(n-1)!}\cdot I_{n-1}+\frac{1}{n}.$$

Adding all these equalities ($I_2, I_3, \ldots$ are cancelled), we get

$$\frac{I_n}{n!} = I_1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}.$$

Now, $I_1 = \frac{d}{dx}(x \log x) = 1 + \log x$ $\quad \therefore \quad I_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right].$

Example 8 (c) : If $U_n = D^n(x^{n-1} \log x)$, prove that

$U_n = (n-1)\, U_{n-1}$ and hence deduce that $U_n = \dfrac{(n-1)!}{x}$.

Sol. : We have

$$U_n = \frac{d^n}{dx^n}(x^{n-1} \log x) = \frac{d^{n-1}}{dx^{n-1}} \cdot \frac{d}{dx}(x^{n-1} \log x)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[(n-1)\, x^{n-2} \cdot \log x + x^{n-1} \cdot \frac{1}{x}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[(n-1)\, x^{(n-1)-1} \cdot \log x + x^{n-2}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[(n-1)\, x^{(n-1)-1} \cdot \log x\right] + \frac{d^{n-1}}{dx^{n-1}}(x^{n-2})$$

$\therefore \quad U_n = (n-1) U_{n-1}$ [Second term is zero] [By (1-B), page 8-1]

Applying the result repeatedly,

$$U_n = (n-1)(n-2)\, U_{n-2}$$

$$= (n-1)(n-2)(n-3)\, U_{n-3}$$

$$= (n-1)(n-2)(n-3) \ldots 3 \cdot 2 \cdot 1 \cdot U_1$$

$\therefore \quad U_n = (n-1)!\, U_1$

But $U_1 = \frac{d}{dx}(x^0 \log x) = \frac{d}{dx}(\log x) = \frac{1}{x}$ $\quad \therefore \quad U_n = (n-1)! \frac{1}{x}.$

Example 9 (c) : If $y = (x^2 - 1)^n$, prove that $(x^2-1)\, y_{n+2} + 2x\, y_{n+1} - n(n+1)\, y_n = 0$.

Hence, deduce that if $I_n = \dfrac{d^n}{dx^n}(x^2-1)^n$, then $\left\{\dfrac{d}{dx}(1-x^2)\dfrac{dI_n}{d_x}\right\} = -n(n+1)\, I_n$.

Sol. : We have $y = (x^2-1)^n$. $\quad \therefore \quad y_1 = n(x^2-1)^{n-1} \cdot 2x$

$\therefore \quad (x^2-1)\, y_1 = n(x^2-1)^n \cdot 2x = 2n\, xy$ **[Note this]**

Differentiating again,

$$(x^2-1)\, y_2 + 2xy_1 = 2n\, xy_1 + 2n\, y$$

$$\therefore \quad (x^2-1)\, y_2 - 2(n-1)\, xy_1 - 2n\, y = 0$$

Now applying Leibnitz's Theorem

$$(x^2-1)\, y_{n+2} + n(2x)\, y_{n+1} + \frac{n(n-1)}{2!} \cdot 2y_n - 2(n-1)[x\, y_{n+1} + n\, y_n] - 2n\, y_n = 0$$

$$\therefore \quad (x^2-1)\, y_{n+2} + 2x\, y_{n+1} - n(n+1)\, y_n = 0 \qquad \ldots\ldots\ldots (1)$$

We first note that $I_n = \dfrac{d^n}{dx^n}(x^2-1)^n = \dfrac{d^n}{dx^n}(y) = y_n$

Now, $\dfrac{d}{dx}\left\{(1-x^2)\dfrac{dI_n}{dx}\right\} = \dfrac{d}{dx}\left\{(1-x^2)\dfrac{d}{dx}(y_n)\right\} = \dfrac{d}{dx}\{(1-x^2)y_{n+1}\}$

$$= (1-x^2)\,y_{n+2} - 2x\,y_{n+1} = -n(n+1)\,y_n \qquad \text{[From (1)]}$$

Example 10 (c) : If u is a function of x and $y = e^{ax}\,u$, prove that

$$D^n y = e^{ax}(D+a)^n u. \qquad \textbf{(M.U. 2006)}$$

Sol. : Let $v = e^{ax}$, then

$$v_1 = a\,e^{ax},\quad v_2 = a^2\,e^{ax},\quad v_3 = a^3\,e^{ax},\quad \ldots\ldots\ldots,\quad v_n = a^n\,e^{ax}.$$

By Leibnitz's Theorem,

$$D^n y = D^n(uv) = {}^nC_0\,u_n\,v + {}^nC_1\,u_{n-1}\,v_1 + {}^nC_2\,u_{n-2}\,v_2 + \ldots.. + {}^nC_n\,u\,v_n$$

Treating D as an operator, we write $D^n\,u = u_n$.

$$\therefore\quad D^n\,y = {}^nC_0\,D^n\,u\cdot v + {}^nC_1\,D^{n-1}\,u\cdot v_1 + {}^nC_2\,D^{n-2}\,u\cdot v_2 + {}^nC_3\,D^{n-3}\,u\cdot v_3 + \ldots.. + {}^nC_n\,u\cdot v_n$$

$$= {}^nC_0\,D^n\,u\cdot e^{ax} + {}^nC_1 D^{n-1}\,u\cdot ae^{ax} + {}^nC_2\,D^{n-2}\,u\cdot a^2e^{ax} + {}^nC_3\,D^{n-3}\,u\cdot a^3e^{ax} + \ldots.. + {}^nC_n\,u\cdot a^ne^{ax}$$

$$= e^{ax}\,[\,{}^nC_0\,D^n\,u + {}^nC_1 D^{n-1}\,u\cdot a + {}^nC_2\,D^{n-2}\,u\cdot a^2 + {}^nC_3\,D^{n-3}\,u\cdot a^3 + \ldots.. + {}^nC_n\,u\cdot a^n\,]$$

$$= e^{ax}\,[\,{}^nC_0\,D^n + {}^nC_1\,D^{n-1}\,a + {}^nC_2\,D^{n-2}\,a^2 + {}^nC_3\,D^{n-3}a^3 + \ldots.. + {}^nC_n\,a^n\,]\cdot u$$

$$\therefore\quad D^n\,y = e^{ax}(D+a)^n\cdot u.$$

Example 11 (c) : If $y = [\log(x+\sqrt{1+x^2})]^2$, prove that

$$(1+x^2)\,y_{n+2} + (2n+1)\,xy_{n+1} + n^2y_n = 0.$$

Hence, deduce that $y_{n+2}(0) = -n^2y_n(0)$. **(M.U. 2002)**

Sol. : We have $y = [\log(x+\sqrt{1+x^2})]^2$ .... (1)

Differentiating it w.r.t. x, we get

$$y_1 = 2\left[\log\left(x+\sqrt{1+x^2}\right)\right]\cdot\frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

$$\therefore\quad y_1 = 2\left[\log\left(x+\sqrt{1+x^2}\right)\right]\cdot\frac{1}{\sqrt{1+x^2}}$$

$$\therefore\quad \sqrt{1+x^2}\cdot y_1 = 2\cdot\log\left(x+\sqrt{1+x^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Differentiating again,

$$\sqrt{1+x^2}\cdot y_2 + \frac{x}{\sqrt{1+x^2}}\cdot y_1 = 2\cdot\frac{1}{x+\sqrt{1+x^2}}\left[1+\frac{x}{\sqrt{1+x^2}}\right] = \frac{2}{\sqrt{1+x^2}}$$

$$\therefore\quad (1+x^2)\,y_2 + xy_1 = 2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

Applying Leibnitz's Theorem, we get

$$(1+x^2)\,y_{n+2} + n\cdot(2x)\,y_{n+1} + \frac{n(n-1)}{2!}(2)\,y_n + xy_{n+1} + n\cdot(1)\,y_n = 0$$

$$\therefore \quad (1+x^2)\,y_{n+2}+(2n+1)\,xy_{n+1}+n^2 y_n=0 \qquad \text{.....................} \ (4)$$

Putting $x=0$ in (4), we get $y_{n+2}(0)=-n^2 y_n(0)$.

Example 12 (c) : Using Leibnitz's Theorem for x^{2n}, prove that

$$1+\frac{n^2}{1!}+\frac{n^2(n-1)^2}{1^2\cdot 2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2}+\ldots\ldots=\frac{(2n)!}{(n!)^2}.$$

Sol. : We obtain n^{th} derivative of $y=x^{2n}$ in two different ways.

By applying the formula (1), page 8-1 to $y=x^{2n}$.

$$y_n=2n(2n-1)(2n-2)\ldots..(2n-n+1)\,x^{2n-n}$$

$$=\frac{2n(2n-1)(2n-2)\ldots(n+1)(n)\ldots.3\cdot 2\cdot 1}{n(n-1)\ldots.3\cdot 2\cdot 1}x^n=\frac{(2n)!}{n!}x^n.$$

By applying Leibnitz's rule to $y=x^n\cdot x^n$, we have

$$y_n=D^n(x^n)\cdot x^n+n\,D^{n-1}(x^n)\cdot D(x^n)$$

$$+\frac{n(n-1)}{2!}D^{n-2}(x^n)\,D^2(x^n)+\frac{n(n-1)(n-2)}{3!}D^{n-3}(x^n)\,D^3(x^n)+\ldots..$$

$$=n!\,x^n+n\cdot\frac{(n!\,x)}{1!}(n\,x^{n-1})+\frac{n(n-1)}{2!}\left(\frac{n!}{2!}x^2\right)(n)(n-1)\,x^{n-2}$$

$$+\frac{n(n-1)(n-2)}{3!}\left(\frac{n!}{3!}x^3\right)(n)(n-1)(n-2)\,x^{n-3}+\ldots.$$

$$\therefore \quad y_n=n!\,x^n\left[1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2\cdot 2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2}+\ldots..\right]$$

Equating the two results, we get,

$$n!\left[1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2\cdot 2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2}+\ldots\right]=\frac{(2n)!}{n!}$$

$$\therefore \quad 1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2\cdot 2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2\cdot 2^2\cdot 3^2}+\ldots=\frac{(2n)!}{(n!)^2}.$$

Example 13 (c) : If $y=x^{n-1}\cdot e^{1/x}$, prove that $y_n=(-1)^n\,x^{-(n+1)}\cdot e^{1/x}$. **(M.U. 1994)**

Sol. : We have

$$y=x^{n-1}\left[1+\frac{1}{x}+\frac{1}{2!}\left(\frac{1}{x}\right)^2+\frac{1}{3!}\left(\frac{1}{x}\right)^3+\ldots..+\frac{1}{(n-1)!}\left(\frac{1}{x}\right)^{n-1}\right.$$

$$\left.+\frac{1}{n!}\left(\frac{1}{x}\right)^n+\frac{1}{(n+1)!}\left(\frac{1}{x}\right)^{n+1}+\ldots\right] \qquad \text{.....................} \ (A)$$

[For expansion of $e^{1/x}$. See 4(a), page 12-2]

$$\therefore \quad y=\left[x^{n-1}+x^{n-2}+\frac{x^{n-3}}{2!}+\frac{x^{n-4}}{3!}+\ldots..+\frac{1}{(n-1)!}\right]$$

$$+\frac{1}{n!}\cdot\frac{1}{x}+\frac{1}{(n+1)!}\cdot\frac{1}{x^2}+\frac{1}{(n+2)!}\cdot\frac{1}{x^3}+\ldots..$$

If we take the nth derivative of y then the derivatives of the bracketed terms on the r.h.s. are zero by (1), page 8-1.

Now, we use Cor. (2), page 8-2 for the remaining terms on the r.h.s.

If $y = \frac{1}{x^m}$, then $y_n = (-1)^n \frac{(m+n-1)!}{(m-1)!} \cdot \frac{1}{x^{m+n}}$.

We put $m = 1, 2, 3, \ldots$ successively.

$$\therefore \quad y_n = \frac{1}{n!} \cdot (-1)^n \cdot \frac{n!}{0!} \cdot \frac{1}{x^{n+1}} + \frac{(-1)^n}{(n+1)!} \cdot \frac{(n+1)!}{(1)!\, x^{n+2}} + \frac{(-1)^n}{(n+2)!} \cdot \frac{(n+2)!}{2!} \cdot \frac{1}{x^{n+3}} + \ldots$$

$$= \frac{(-1)^n}{x^{n+1}} \left[1 + \frac{1}{1!\, x} + \frac{1}{2!\, x^2} + \ldots\right] = \frac{(-1)^n}{x^{n+1}} \cdot e^{1/x}. \quad \text{[By (A) above]}$$

Example 14 (c) : If $(x + y) = 1$, prove that

$$\frac{d^n}{dx^n}(x^n y^n) = n!\,[\, y^n - ({}^nC_1)^2 y^{n-1} \cdot x + ({}^nC_2)^2 y^{n-2} x^2 - \ldots + (-1)^n x^n \,] \qquad \text{(M.U. 2006)}$$

Sol. : Since $x + y = 1 \quad \therefore \quad y = 1 - x$

$$\frac{d^n}{dx^n}(x^n y^n) = \frac{d^n}{dx^n}[\, x^n (1-x)^n \,]$$

Let $u = x^n$, $v = (1-x)^n = y^n$. Then $\frac{d^n}{dx^n}(x^n y^n) = \frac{d^n}{dx^n}[u \cdot v]$

$$\begin{aligned}
\text{Now} \quad u_1 &= n x^{n-1} \\
u_2 &= n(n-1)\, x^{n-2} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
u_{n-2} &= n(n-1) \ldots (n-n+3) \cdot x^{n-n+2} = \frac{n!}{2} x^2 \\
u_{n-1} &= n(n-1) \ldots (3)(2)\, x = n!\, x \\
u_n &= n(n-1) \ldots 3 \cdot 2 \cdot 1 = n! \\
v_1 &= n(1-x)^{n-1}(-1) = (-1)\, n\, y^{n-1} \\
v_2 &= n(n-1)(1-x)^{n-2}(-1)^2 \\
&= (-1)^2\, n(n-1)\, y^{n-2} \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
v_{n-1} &= n(n-1) \ldots 3 \cdot 2\, (1-x)(-1)^{n-1} \\
&= n!\, y\, (-1)^{n-1} \\
v_n &= n!\, (-1)^n
\end{aligned}$$

$$\therefore \quad \frac{d^n}{dx^n}(x^n y^n) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \ldots + u v^n$$

$$= n!(y)^n + nc_1 (n!\, x)(-1)\, n\, y^{n-1} + nc_2 \cdot \frac{n!}{2} x^2 \cdot n(n-1)\, y^{n-2}(-1)^2 + \ldots$$

$$= n!\,[\, y^n - (nc_1)^2 y^{n-1} x + (nc_2)^2 y^{n-2} x^2 + \ldots + (-1)^n x^n \,]$$

EXERCISE - VII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (c) : 8 Marks

1. If $y = \cot x$, prove that

$${}^nC_1\, y_{n-1}(0) - {}^nC_3\, y_{n-3}(0) + {}^nC_5\, y_{n-5}(0) - \ldots\ldots = \cos\frac{n\pi}{2}.$$ **(M.U. 1996)**

2. If $y = (\sin^{-1} x)^2$, then prove that $y_{2n+1}(0) = 0$ and

$y_{2n}(0) = 2^{2n-1}[(n-1)!]^2$. **(M.U. 1993)**

EXERCISE - VIII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (a) : 4 Marks

1. State and prove Leibnitz's Theorem.

2. If $y = \dfrac{1}{ax+b}$, prove that $y_n = \dfrac{(-1)^n(m+n-1)!}{(m-1)!} \cdot \dfrac{a^n}{(ax+b)^{m+n}}$. **(M.U. 1998)**

3. If $y = e^{ax}\sin bx + c$, prove that $y_n = r^n e^{ax}\sin(bx + c + n\Phi)$

where $r = \sqrt{a^2+b^2}$, $\Phi = \tan^{-1}\left(\dfrac{b}{a}\right)$.

4. If $y = k^x \sin(bx+c)$, prove that $y_n = r^n k^n \sin(bx + c + n\Phi)$

where $r = \sqrt{(\log k)^2 + b^2}$, $\Phi = \tan^{-1}\left(\dfrac{b}{\log k}\right)$.

EXERCISE - IX

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 4 Marks

1. If $y = \cos^2 x$, find y_n. $\left[\textbf{Ans. :}\ 2^{n-1}\cos\left(2x + \dfrac{n\pi}{2}\right)\right]$

2. If $y = \sin^2 x$, find y_n. $\left[\textbf{Ans. :}\ -2^{n-1}\cos\left(2x + \dfrac{n\pi}{2}\right)\right]$

3. If $y = 2\sin x\cos x$, find y_n. $\left[\textbf{Ans. :}\ 2^n\sin\left(2x + \dfrac{n\pi}{2}\right)\right]$

4. If $y = (2x+4)^{30}$, find y_{30}. $[\textbf{Ans. :}\ 2^{30}\cdot 30!]$

5. If $y = \log(2x-3)$, find y_n. $\left[\textbf{Ans. :}\ (-1)^{n-1}\cdot\dfrac{(n-1)!\,2^n}{(2x-3)^n}\right]$

6. If $y = \dfrac{1}{(x-3)^2}$, find y_n. $\left[\textbf{Ans. :}\ \dfrac{(-1)^n(n+1)!}{(x-3)^{n+2}}\right]$

7. If $y = \dfrac{1}{(x+1)^3}$, find y_n. $\left[\textbf{Ans. :}\ \dfrac{(-1)^n(n+2)!}{(x+1)^{n+3}}\right]$

8. If $y = \sin 3x$, find y_n. $\left[\textbf{Ans. :}\ 3^n\sin\left(3x + \dfrac{n\pi}{2}\right)\right]$

9. If $y = \cos 4x$, find y_n. $\left[\textbf{Ans. :}\ 4^n \cos\left(4x + \frac{n\pi}{2}\right)\right]$

10. If $y = e^x \sin x$, find y_n. $\left[\textbf{Ans. :}\ 2^{n/2} \cdot e^x \sin\left(x + \frac{n\pi}{4}\right)\right]$

11. If $y = e^{3x} \sin 3x$, find y_n. $\left[\textbf{Ans. :}\ 18^{n/2} \cdot e^{3x} \sin\left(3x + \frac{n\pi}{4}\right)\right]$

12. If $y = x\, e^{2x}$, find y_n. $\left[\textbf{Ans. :}\ 2^n x e^{2x} + n 2^{n-1} e^{2x}\right]$

13. If $y = x \sin x$, find y_n. $\left[\textbf{Ans. :}\ x \sin\left(x + \frac{n\pi}{2}\right) + n \sin\left(x + \frac{(n-1)\pi}{2}\right)\right]$

14. If $y = x^2 e^x$, find y_n. $\left[\textbf{Ans. :}\ e^x (x^2 + 2nx + n^2 - n)\right]$

15. If $f(x) = \sum_{n=0} 2^n x^n$ then find the value of $f^{42}(0)$. [**Ans. :** $42!\, 2^{42}$]

Summary

1. If $y = (ax + b)^m$ then,
$y_n = m(m-1)(m-2) \ldots (m-n+1)\, a^n (ax+b)^{m-n}, \quad n < m$

2. If $y = x^m$, then $y_n = m(m-1)(m-2) \ldots (m-n+1) \cdot x^{m-n}$ if $n < m$.

3. If $y = \frac{1}{(ax+b)^m}$, then $y_n = (-1)^n \cdot \frac{(m+n-1)!}{(m-1)!} \cdot \frac{a^n}{(ax+b)^{m+n}}$

4. If $y = \frac{1}{ax+b}$, then $y_n = \frac{(-1)^n \cdot n!\, a^n}{(ax+b)^{n+1}}$

5. If $y = \log(ax+b)$, then $y_n = (-1)^{n-1} \cdot \frac{(n-1)!\, a^n}{(ax+b)^n}$.

6. If $y = a^{mx}$, then $y_n = m^n a^{mx} (\log a)^n$.

7. If $y = e^{mx}$, then $y_n = m^h e^{mx}$.

8. If $y = \sin(ax+b)$ then $y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$.

9. If $y = \cos(ax+b)$ then $y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$.

10. If $y = e^{ax} \sin(bx+c)$ then $y_n = r^n e^{ax} \sin(bx + c + n\Phi)$.
where, $r = \sqrt{a^2 + b^2}$ and $\Phi = \tan^{-1}\left(\frac{b}{a}\right)$.

11. If $y = e^{ax} \cos(bx+c)$ then $y_n = r^n e^{ax} \cos(bx + c + n\Phi)$
where r and Φ are as above.

12. If $y = a^x \sin(bx+c)$, then
$y_n = r^n a^x \sin(bx + c + n\Phi)$ where $r = \sqrt{(\log a)^2 + b^2}$ and $\Phi = \tan^{-1}\left(\frac{b}{\log a}\right)$.

13. If $y = a^x \cos(bx + c)$, then

$$y_n = r^n a^x \cos(bx + c + n\Phi) \text{ where } r = \sqrt{(\log a)^2 + b^2} \text{ and } \Phi = \tan^{-1}\left(\frac{b}{\log a}\right).$$

Leibnitz's Theorem : If $y = uv$, then

$$y_n = {}^nC_0\, u_n v + {}^nC_1\, u_{n-1} v_1 + {}^nC_2\, u_{n-2} v_2 + \ldots + {}^nC_r\, u_{n-r} v_r + \ldots + {}^nC_n\, u v_n$$

☆ ☆ ☆

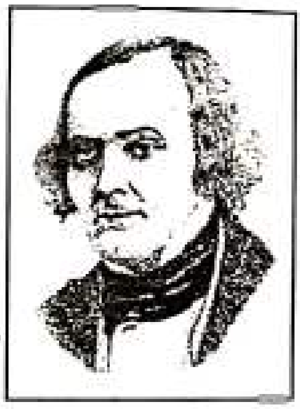
CHAPTER

9 Rank of A Matrix

1. Introduction

You have studied matrices in XIIth standard. We shall briefly review what you have studied in the XIIth standard and then we shall study some new topics in matrices. Matrices were discovered by two great English Mathematicians, Arthur Cayley and James Jospth Sylvester.

Arthor Cayley (1821 - 1895)



A great British mathematician. He had shown his talent at the age of 17 when he was recognised by his teachers as "above the first". He had published his first paper in mathematics at the age of 20 and in the next five years he published 25 papers, when he was at Cambridge. In 1846 he left Cambridge to study law. He worked as a lawyer for the next 14 years but in the same period he published more than 200 papers. But in 1863 he left law and again joined the faculty at Cambridge University. He pursued his mathematical interest till his death. Cayley knew French, German, Italian, Greek and Latin besides English.

James Jospth Sylvester (1814 - 1897)

He got his degree from Cambridge. At the age of 24 he become professor of natural philosophy at the university of London. He taught at the university of Virginia for a year and then returned to England. From 1879-1883 he was professor of mathematics at Johns Hophins University. He founded the American Journal of Mathematics in 1878.

2. Definition

You are already familiar with matrices and with some operations on it. To repeat, a **matrix** is a system of mn numbers arranged in m rows and n columns. It is called an $m \times n$ matrix. Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

We denote this matrix by $A = [a_{ij}]_{m \times n}$

3. Types of Matrices

1. Row and Column Matrix : A matrix having only one row is called a **row matrix** and a matrix having only one column is called a **column matrix**.

For example, [3], [3, 2], [3, 2, 1] are row matrices and [3], $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ are column matrices.

2. A Square Matrix : If the number of rows of a matrix is equal to the number of columns *i.e.* if $m = n$, then the matrix is called a **square matrix** and the number of rows (or the number of columns) is called the **order** of the square matrix.

For example $[3]$, $\begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & -2 & 1 \\ 4 & 0 & 6 \\ -1 & -2 & 6 \end{bmatrix}$ are square matrices of order 1, 2, 3 respectively.

3. Diagonal Elements : In a square matrix the elements lying along the diagonal of the matrix *i.e.* the elements a_{ii} (*i.e.* the order of the row = the order of the column) are called **diagonal elements** of the matrix.

In the above matrices 3; 4, 0; 2, 0, 6 are diagonal elements.

4. Diagonal Matrix : A square matrix whose all non-diagonal elements are zero is called a **diagonal matrix**. Thus, a matrix $A = [a_{ij}]$ is a diagonal matrix if (i) A is a square matrix and (ii) $a_{ij} = 0$ if $i \neq j$ for all i (and j).

For example, $[3]$, $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ are diagonal matrices.

5. Trace of A Matrix : The sum of all diagonal elements of a square matrix is called the **trace** of a matrix.

Thus, if $A = [a_{ij}]_{n \times n}$ then $\sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots.. + a_{nn}$ is called the trace of A.

In the case of matrices given in (2), the traces of the matrices are 3, 4 + 0 = 4, 2 + 0 + 6 = 8, respectively.

6. Determinant of A Square Matrix : If we evaluate the determinant of a square matrix, it is called the determinant of the matrix and is denoted by $|A|$.

7. Singular Matrix : A square matrix A whose **determinant is zero** is called a **singular matrix**.

For example, $[0]$, $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 4 & 3 \\ 3 & 5 & 3 \\ 1 & -7 & 1 \end{bmatrix}$ are singular matrices.

8. Non-singular Matrix : A square matrix which is not singular *i.e.* whose determinant is not zero is called a **non-singular matrix**.

For example, $[3]$, $\begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 & 3 \\ 3 & 4 & 5 \\ 6 & -7 & 8 \end{bmatrix}$ are non-singular matrix.

9. Zero or Null Matrix : A matrix square or rectangular whose **all elements are zero** is called a zero **matrix or a null matrix** and is denoted by O.

For example, $[0]$, $\begin{bmatrix} 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices.

10. Unit Matrix or Identity Matrix : A diagonal matrix whose all diagonal elements are equal to one is called a **unit matrix** or an **identity matrix** and is denoted by I.

For example, $[1]$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices.

In a unit matrix all **diagonal elements are one** and **all non-diagonal elements are zero.**

11. Scalar Matrix : A diagonal matrix whose all diagonal elements are equal is called a **scalar matrix**.

For example, $[4]$, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$ are scalar matrices.

In a scalar matrix, **all diagonal elements are equal** and **all non-diagonal elements are zero.**

12. Transpose of a Matrix : *A matrix obtained from a given matrix A by interchanging rows and columns is called the* ***transpose*** *of a given matrix and is denoted by* A' *or by* A^T. Clearly $(A')' = A$.

Clearly if A is an $m \times n$ matrix then the transpose A' is an $n \times m$ matrix.

e.g. if $A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 4 & 0 \\ -1 & 3 & -1 \end{bmatrix}$ then $A' = \begin{bmatrix} 3 & 1 & -1 \\ -1 & 4 & 3 \\ 2 & 0 & -1 \end{bmatrix}$

Note that $|A| = |A'|$

13. Triangular Matrices : A matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for $i > j$ and is said to be **lower triangular** if $a_{ij} = 0$ for $i < j$.

e.g. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ are respectively upper and lower triangular matrices.

If A is a square matrix then clearly it is upper triangular if all its elements **below the leading diagonal are zero** and is lower triangular if all its elements **above its leading diagonal are zero.**

e.g. $\begin{bmatrix} 2 & 4 & 5 & 6 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -4 & -4 & 3 & 0 \\ -3 & -3 & 1 & -1 \end{bmatrix}$ are upper and lower triangular matrices.

14. Conjugate of a Matrix : The matrix obtained from a given matrix by replacing each element by its complex conjugate is called the **conjugate** of the given matrix and is denoted by $\overline{A}$.

e.g. If $A = \begin{bmatrix} 2 & 1+i & i \\ -i & 3i+2 & 0 \\ 2-3i & 5i & 4 \end{bmatrix}$ then $\overline{A} = \begin{bmatrix} 2 & 1-i & -i \\ i & -3i+2 & 0 \\ 2+3i & -5i & 4 \end{bmatrix}$

Note ....

The conjugate of $z = a + ib$ is $\overline{z} = a - ib$. In other words the conjugate of a complex number is obtained by changing the sign of i.

15. Transposed Conjugate of a Matrix : The transpose of a conjugate complex of a given matrix *i.e.* $\overline{A}'$ is called the **transposed conjugate** of the given matrix A and is denoted by A^{θ} or by A^{*}. Clearly $(A^{\theta})^{\theta} = A$.

e.g. If $A = \begin{bmatrix} 1+2i & 3-i & 4 \\ 5i & 3+2i & 0 \\ -2i & 1-i & 2-i \end{bmatrix}$, then

$$\overline{A} = \begin{bmatrix} 1-2i & 3+i & 4 \\ -5i & 3-2i & 0 \\ 2i & 1+i & 2+i \end{bmatrix} \text{ and } A^{\theta} = \begin{bmatrix} 1-2i & -5i & 2i \\ 3+i & 3-2i & 1+i \\ 4 & 0 & 2+i \end{bmatrix}$$

16. Symmetric And Skew-symmetric Matrix : A square matrix $A = [a_{ij}]$ is said to be **symmetric** if its (i, j)th element is the same as its (j, i)th element, *i.e.* if $a_{ij} = a_{ji}$ for all i, j.

In other words, if in a square matrix elements symmetrical with respect to the leading diagonal are equal then the matrix is called symmetric matrix.

$$\begin{bmatrix} a & h & g & k \\ h & b & l & m \\ g & l & c & n \\ k & m & n & d \end{bmatrix}$$

For example, the following matrices are symmetric matrices.

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 2 & 3+i & 2 \\ 3+i & 1 & 3i \\ 2 & 3i & 0 \end{bmatrix}$$

Class (a) : 4 Marks

(a) Theorem : The necessary and sufficient condition for a square matrix to be symmetric is that $A = A'$.

Proof : Let $A = [a_{ij}]$ be an n-rowed square symmetric matrix. This means $a_{ij} = a_{ji}$ and A' *i.e.* the transpose of A is also an n-rowed square matrix.

Now, (i, j)th element of A' = (j, i)th element of $A = a_{ji} = a_{ij}$

= (i, j)th element of A

$\therefore \quad A' = A.$

Conversely if $A' = A$. A must be an n-rowed square matrix.

Also, (i, j)th element of A = (i, j)th element of A' $\quad [\because A = A']$

= (j, i)th element A

$\therefore$ A is a symmetric matrix.

(b) Skew-symmetric Matrix : A square matrix $A = [a_{ij}]$ is said to be **skew-symmetric** if the (i, j)th element is equal to the negative of (j, i)th element,

i.e. $\quad a_{ij} = -a_{ji}$ for all i, j.

In other words if in a square matrix elements placed symmetrically with respect to the leading diagonal are equal in magnitude but **opposite** in sign and the diagonal elements are zero then the matrix is called skew-symmetric.

$$\begin{bmatrix} 0 & h & -g & k \\ -h & 0 & -l & m \\ g & l & 0 & -n \\ -k & -m & n & 0 \end{bmatrix}$$

If A is a skew-symmetric matrix then,

$a_{ii} = -a_{ii}$ *i.e.* $2a_{ii} = 0$ *i.e.* $a_{ii} = 0$

i.e. **its diagonal elements are zero.**

For example, the following matrices are skew-symmetric matrices.

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \begin{bmatrix} 0 & b & g \\ -b & 0 & h \\ -g & -h & 0 \end{bmatrix}, \begin{bmatrix} 0 & i & -2 \\ -i & 0 & -3i \\ -2 & 3i & 0 \end{bmatrix}$$

Class (a) : 4 Marks

Theorem : The necessary and sufficient condition for a square matrix to be skew-symmetric is that $A' = -A$.

Proof : Let $A = [a_{ij}]$ be an n-rowed square skew-symmetric matrix. Then $a_{ij} = -a_{ji}$. Since, A is n-rowed square matrix, A', $-A$ are also n-rowed square matrices.

Now, (i, j)th element of $A' = (j, i)$th element of $A = a_{ji} = -a_{ij}$

$= (i, j)$th element of $(-A)$

$\therefore \quad A' = -A.$

Conversely if $A' = -A$ then A must be a square matrix.

Also, (i, j)th element of A = the negative of the (i, j)th element of A' $\quad [\because A = -A']$

= the negative of the (j, i)th element of A.

Hence, A is a skew-symmetric matrix.

Example 1 (a) : If A is a skew-symmetric matrix and X is a column matrix then show that $X'AX$ is a null matrix. **(M.U. 2002)**

Sol. : Since A is a skew-symmetric matrix $A' = -A$.

Let A be a square matrix of order n and X be a column matrix of order $n \times 1$. Now X' is a row matrix of order $1 \times n$. Hence, $X'AX$ is a matrix of order 1×1.

Let $X'AX = B$. Since B is or order 1×1, $B' = B$ and hence, B is symmetric.

Now, consider, $\quad (X'AX)' = B' \quad \therefore \; X'A(X')' = B' \quad \therefore \; X'A'X'' = B'$.

But $A' = -A$ and $X'' = X$ and $B' = B$.

$\therefore \; X'(-A)X'' = B' \quad \therefore \; -(X'AX) = B$

$\therefore \; -B = B \quad \therefore \; 2B = O \quad \therefore \; B = O.$

Example 2 (a) : Let a skew-symmetric matrix be

$$A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \text{ and a column matrix be } X = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

We shall now verify the above result that $X'AX = O$.

$$\text{Now,} \quad X'AX = [4,\ 5]\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = [4,\ 5]\begin{bmatrix} 15 \\ -12 \end{bmatrix}$$

$$= [\,60 - 60\,] = [\,O\,].$$

Example 3 (a) : Show by an example that a skew-symmetric matrix of third order is singular.

Sol. : Consider the skew-symmetric matrix

$$A = \begin{vmatrix} 0 & b & g \\ -b & 0 & h \\ -g & -h & 0 \end{vmatrix}$$

$$\therefore \; A = 0\begin{vmatrix} 0 & h \\ -h & 0 \end{vmatrix} - b\begin{vmatrix} -b & h \\ -g & 0 \end{vmatrix} + g\begin{vmatrix} -b & 0 \\ -g & -h \end{vmatrix} = -bgh + bgh = O$$

17. Hermitian And Skew Hermitian Matrices :

Definition - Hermitian : A square matrix $A = [a_{ij}]$ is said to **Hermitian** if (i, j)th element of A is equal to the conjugate complex of (j, i)th element of A. *i.e.* $a_{ij} = \bar{a}_{ji}$ for all i and j.

If A is a Hermitian matrix than $a_{ii} = \bar{a}_{ii} \;\; \therefore a_{ii}$ is real. In other words **in a Hermitian matrix every diagonal element is real** and elements symmetrically placed with respect to the principal diagonal are complex conjugates of each other.

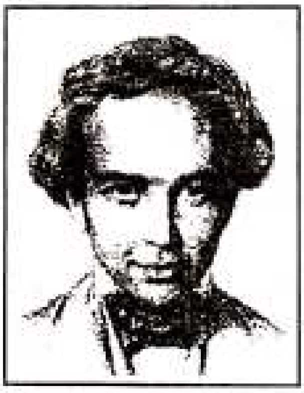
In other words in a Hermitian matrix elements symmetrically placed with respect to the leading diagonal are complex conjugates of each other *i.e.*, of the form $a + ib$ and $a - ib$ *i.e.* the signs of imaginary parts differ and the diagonal elements are real.

$$\begin{bmatrix} a & \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \\ z_1 & b & \bar{z}_4 & \bar{z}_5 \\ z_2 & z_4 & c & \bar{z}_6 \\ z_3 & z_5 & z_6 & d \end{bmatrix}$$

The following are Hermitian matrices.

$$\begin{bmatrix} a & c+id \\ c-id & b \end{bmatrix}, \begin{bmatrix} 2 & 3+2i & -i \\ 3-2i & 3 & 5 \\ i & 5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 2+3i & 4-i \\ 2-3i & 6 & 6+2i \\ 4+i & 6-2i & 7 \end{bmatrix}$$

Charles Hermite (1822 - 1901)



A French mathematician who made fundamental contributions to algebra, matrix and analysis. He was the first mathematician to prove that the **number** ***e*** (the base of natural logarithm) **is a transcendental number** (a number different from a rational or an irrational number) *i.e.* it is not the root of any polynomial equation with rational coefficients.

Example : Show that the above matrices are Hermitian. **(M.U. 2004)**

Sol. : We shall show that the last matrix is Hermitian.

The diagonal elements 2, 6, 7 are real.

The elements symmetrically placed with respect to the principal diagonal are $2+3i$, $2-3i$, $4-i$, $4+i$ and $6+2i$, $6-2i$.

They are conjugates of each other.

Hence, the matrix is Hermitian.

Theorem 1 : The necessary and sufficient condition for a square matrix A to be Hermitian is that $A = A^{\theta}$.

Proof : Let $A = [a_{ij}]$ be an n-rowed square Hermitian matrix. This means $a_{ij} = \bar{a}_{ji}$ and A^{θ} *i.e.* the transposed conjugate is also an n-rowed square matrix.

Now, (i, j)th element of A^{θ} = Conjugate of (j, i)th elements of $A = \bar{a}_{ji}$

= a_{ij} = (i, j)th element of A

$\therefore \quad A^{\theta} = A.$

Conversely if $A^{\theta} = A$, A must be an n-rowed square matrix.

Also, (i, j)th element of A = (i, j)th element of A^{θ}

= Conjugate of (j, i)th element of A

$\therefore$ A is a Hermitian matrix.

Definition - Skew-Hermitian : A square matrix $A = [a_{ij}]$ is said to be **Skew-Hermitian** if (i, j)th element of A is equal to the negative of the complex conjugate of (j, i)th element of A *i.e.* $a_{ij} = -\bar{a}_{ji}$ for all i and j.

If A is a Skew-Hermitian matrix then $a_{ii} = -\bar{a}_{ii}$ $\therefore a_{ii} + \bar{a}_{ii} = 0$. Hence, a_{ii} must be either purely imaginary number or zero. Thus, **the diagonal elements of a Skew-Hermitian matrix must**

be purely imaginary numbers or zero and elements symmetrically placed with respect to the principal diagonal are negative complex conjugates of each other *i.e.*, these elements differ in the sign of the real part.

In other words in a Skew-Hermitian matrix if the element is z then the element symmetrically placed is the negative complement *i.e.* $-\overline{z}$ and all diagonal elements are either zero or purely imaginary numbers.

$$\begin{bmatrix} 0 & -\overline{z}_1 & -\overline{z}_2 & -\overline{z}_3 \\ z_1 & k_i & -\overline{z}_4 & -\overline{z}_5 \\ z_2 & z_4 & 0 & -\overline{z}_6 \\ z_3 & z_5 & z_6 & m_j \end{bmatrix}$$

(k_i, m_j are purely imaginary)

The following are Skew-Hermitian matrices.

$$\begin{bmatrix} 0 & 2+3i \\ -2+3i & i \end{bmatrix}, \begin{bmatrix} 2i & 1+i & -3+2i \\ -1+i & 0 & 2-i \\ 3+2i & -2-i & -3i \end{bmatrix}, \begin{bmatrix} 3i & 2+i & -1+i \\ -2+i & i & 3-i \\ 1+i & -3-i & 0 \end{bmatrix}$$

Example : Show that the above matrices are Skew-Hermitian.

Sol. : We shall prove that the last matrix is Skew-Hermitian.

The diagonal elements $3i$, i, 0 are either purely imaginary or zero.

The elements symmetrically placed with respect to the principal diagonal are $2+i$, $-2+i$, $-1+i$, $1+i$ and $3-i$ and $-3-i$.

The conjugate of $2+i$ is $2-i$ and its negative is $-(2-i) = -2+i$. Similarly, $3-i$ and $-3-i$ and $-1+i$, $1+i$ negative conjugates of each other.

Hence, the matrix is Skew-Symmetric.

Theorem 2 : The necessary and sufficient condition for a square matrix A to be Skew-Hermitian is that $A^{\theta} = -A$.

Proof : Let $A = [a_{ij}]$ be an n-rowed square Skew-Hermitian matrix. Then $a_{ij} = -\overline{a}_{ji}$. Since, A is n-rowed square matrix, A^{θ} and $-A$ are also n-rowed square matrices.

Now, (i, j)th element of A^{θ} = conjugate of (j, i)th element of $A = \overline{a}_{ji}$

$= -a_{ij} = (i, j)$th element of $-A$.

$\therefore \quad A^{\theta} = -A$.

Conversely, if $A^{\theta} = -A$, A must be an n-rowed square matrix.

Also, (i, j)th element of A = $-(i, j)$th element of A^{θ}

$= -$ (conjugate of (j, i)th element of A)

$\therefore$ A is a Skew-Hermitian matrix.

4. Operations on Matrices

1. Equality of Two Matrices : Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be **equal** if (i) they are of the same type *i.e.* they have the same number of rows and the same number of columns and (ii) all the elements in the corresponding places are equal.

For example, $\begin{bmatrix} 2 & 3 & 1 \\ -2 & 4 & 8 \end{bmatrix}$ and $\begin{bmatrix} \sqrt{4} & 3 & \sqrt[3]{1} \\ -2 & 2^2 & 2^3 \end{bmatrix}$ are equal matrices.

But $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ are not equal matrices.

2. Multiplication of A Matrix By A Scalar : The matrix obtained by multiplying each element of a matrix A by a constant k is called a **scalar multiple** of A and is denoted by kA.

For example, if $A = \begin{bmatrix} 2 & 3 & 1 \\ -2 & 4 & 8 \end{bmatrix}$, then $5A = \begin{bmatrix} 10 & 15 & 5 \\ -10 & 20 & 40 \end{bmatrix}$.

3. Addition Or Subtraction of Two Matrices : If $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same type then the matrix $C = [c_{ij}]$ where $c_{ij} = a_{ij} + b_{ij}$ is called the **sum** of the matrices A and B and the matrix $D = [d_{ij}]$ where $d_{ij} = a_{ij} - b_{ij}$ is called the **difference** of the matrices A and B. The sum is denoted by $A + B$ and the difference is denoted by $A - B$.

4. Product of Two Matrices : If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix then the matrix $C = [c_{ij}]$ where $c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$ is called the **product** of the two matrices A and B.

More easily if $R_1, R_2, \ldots R_m$ are the rows of A and $C_1, C_2, \ldots, C_p$ are the columns of B, then the product AB is given by

$$AB = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} [C_1, C_2, \ldots, C_p] = \begin{bmatrix} R_1C_1 & R_1C_2 & \ldots & R_1C_p \\ R_2C_1 & R_2C_2 & \ldots & R_2C_p \\ \ldots & \ldots & \ldots & \ldots \\ R_mC_1 & R_mC_2 & \ldots & R_mC_p \end{bmatrix}$$

5. Powers of Diagonal Matrices : If D is a diagonal matrix then we have a very easy rule to find $D^2, D^3, \ldots$, etc. The n^{th} power of a diagonal matrix is a diagonal matrix whose diagonal elements are the n^{th} powers of the corresponding diagonal elements of D.

Thus, if $D = \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$, then $D^n = \begin{bmatrix} d_1^n & 0 & 0 & 0 \\ 0 & d_2^n & 0 & 0 \\ 0 & 0 & d_3^n & 0 \\ 0 & 0 & 0 & d_4^n \end{bmatrix}$.

For example, if $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, then $A^5 = \begin{bmatrix} 32 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 1024 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \text{ and } A^{-5} = \begin{bmatrix} 1/32 & 0 & 0 \\ 0 & -1/243 & 0 \\ 0 & 0 & 1/1024 \end{bmatrix}$$

Notes ....

1. The above multiplication of two matrices is better remembered as **row by column** multiplication.
2. If AB exists then BA does not necessarily exist. If A and B are both square matrices of the same order then only AB and BA are both defined.
3. If AB and BA are both defined it is **not** necessarily true that $AB = BA$.
4. If $AB = AC$ it is **not** necessarily true that $B = C$. Similarly, if $BA = CA$, it is not necessarily true that $B = C$.
5. If $AB = 0$ it is **not** necessarily true that $A = 0$ or $B = 0$.

5. Properties of Matrices [Class (a) : 4 Marks]

Property 1 (a) : If A is Hermitian matrix then iA is Skew-Hermitian. **(M.U. 2003, 07)**

Proof : Let A be a Hermitian matrix $\quad \therefore \quad A^{\theta} = A$.

Now, $\quad (iA)^{\theta} = \bar{i}A^{\theta} = -iA^{\theta} = -iA \qquad [\because A^{\theta} = A]$

$\therefore \; iA$ is a Skew-Hermitian matrix.

Example 1 (a) : Show that the matrix A given below is Hermitian and iA is Skew-Hermitian.

$$A = \begin{bmatrix} 3 & 1+i & 2 \\ 1-i & -4 & 3-i \\ 2 & 3+i & 0 \end{bmatrix}$$

Sol. : Since all diagonal elements are real, and non-diagonal elements symmetrically placed with respect to the principal diagonal are complex conjugates, A is Hermitian.

$$\text{Now,} \quad iA = \begin{bmatrix} 3i & -1+i & 2i \\ 1+i & -4i & 1+3i \\ 2i & -1+3i & 0 \end{bmatrix}$$

Since all diagonal elements are imaginary or zero and non-diagonal elements symmetrically placed with respect to the principal diagonal are negative complex conjugates, iA is Skew-Hermitian.

Property 2 (a) : If A is a Skew-Hermitian matrix, prove that iA is Hermitian.

Proof : Let A be a Skew-Hermitian matrix $\quad \therefore \quad A^{\theta} = -A$

Now, $\quad (iA)^{\theta} = \bar{i}A^{\theta} = -iA^{\theta} = -i(-A) = iA \qquad [\because A^{\theta} = -A]$

$\therefore \; iA$ is Hermitian.

Example 2 (a) : Show that the matrix A given below is Skew-Hermitian and iA is Hermitian.

$$A = \begin{bmatrix} 2i & 1+i & -1+3i \\ -1+i & 0 & 5i \\ 1+3i & 5i & -i \end{bmatrix}$$

Sol. : Since all diagonal elements are imaginary or zero and non-diagonal elements symmetrically placed with respect to the principal diagonal are negative complex conjugates, A is Skew-Hermitian.

$$\text{Now,} \quad iA = \begin{bmatrix} -2 & -1+i & -3-i \\ -1-i & 0 & -5 \\ -3+i & -5 & 1 \end{bmatrix}$$

Property 7 (a) : Show that every square matrix can be uniquely expressed as the sum of a Hermitian matrix and a Skew-Hermitian matrix. **(M.U. 1999, 2003, 05, 07, 08, 14)**

Proof : Let A be a given square matrix.

Now we can write,

$$A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta}) = P + Q, \text{ say} \quad \text{where } P = \frac{1}{2}(A + A^{\theta}) \text{ and } Q = \frac{1}{2}(A - A^{\theta})$$

Now, $P^{\theta} = \frac{1}{2}(A + A^{\theta})^{\theta} = \frac{1}{2}\left[(A^{\theta} + (A^{\theta})^{\theta}\right] = \frac{1}{2}(A^{\theta} + A) = \frac{1}{2}(A + A^{\theta}) = P$

$\therefore$ P is Hermitian

Also, $Q^{\theta} = \frac{1}{2}(A - A^{\theta})^{\theta} = \frac{1}{2}\left[(A^{\theta} - (A^{\theta})^{\theta}\right] = \frac{1}{2}(A^{\theta} - A) = -\frac{1}{2}(A - A^{\theta}) = -Q$

$\therefore$ Q is Skew-Hermitian.

Thus, we have expressed A as sum of a Hermitian and a Skew-Hermitian matrices.

To prove uniqueness let $A = R + S$, where R is Hermitian and S is Skew-Hermitian, be another representation of A.

Now, $A^{\theta} = (R + S)^{\theta} = R^{\theta} + S^{\theta} = R - S \qquad [\because R^{\theta} = R, S^{\theta} - S]$

$\therefore$ $\frac{1}{2}(A + A^{\theta}) = \frac{1}{2}[(R + S) + (R - S)] = R \quad \therefore R = P$

And $\frac{1}{2}(A - A^{\theta}) = \frac{1}{2}[(R + S) - (R - S)] = S \quad \therefore S = Q$

Hence, the representation $A = P + Q$ is unique.

(We shall verify this property below by an example.)

Example 4 (a) : Express the matrix $A = \begin{bmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{bmatrix}$ as the sum of a Hermitian and a Skew-Hermitian matrix. **(M.U. 2005, 19)**

Sol. : We have $A' = \begin{bmatrix} 2+3i & -2i & 4 \\ 2 & 0 & 2+5i \\ 3i & 1+2i & -i \end{bmatrix} \quad \therefore A^{\theta} = (\bar{A}') = \begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}$

$\therefore$ $A + A^{\theta} = \begin{bmatrix} 4 & 2+2i & 4+3i \\ 2-2i & 0 & 3-3i \\ 4-3i & 3+3i & 0 \end{bmatrix}; \quad A - A^{\theta} = \begin{bmatrix} 6i & 2-2i & -4+3i \\ -2-2i & 0 & -1+7i \\ 4+3i & 1+7i & -2i \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^{\theta}),\ Q = \frac{1}{2}(A - A^{\theta})$

But, we know that P is Hermitian and Q is Skew-Hermitian and $A = P + Q$

$$\therefore A = P + Q = \begin{bmatrix} 2 & 1+i & (4+3i)/2 \\ 1-i & 0 & (3-3i)/2 \\ (4-3i)/2 & (3+3i)/2 & 0 \end{bmatrix} + \begin{bmatrix} 3i & 1-i & (-4+3i)/2 \\ -1-i & 0 & (-1+7i)/2 \\ (4+3i)/2 & (1+7i)/2 & -i \end{bmatrix}$$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Express the following matrices as the sum of a Hermitian and a Skew-Hermitian matrices and check your results : Class (a) : 4 Marks

1. $\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$ **(M.U. 2015)**

2. $\begin{bmatrix} 2 & 2+i & 3 \\ -2+i & 0 & 4i \\ -i & 3-i & 1-i \end{bmatrix}$ **(M.U. 2003)**

3. $\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}$ **(M.U. 2003)**

4. $\begin{bmatrix} 2-i & 3+i & 2i \\ 3 & 0 & 4-i \\ -5 & 2-i & 3+i \end{bmatrix}$

5. $\begin{bmatrix} 2 & -3i & 1-i \\ 2-3i & 4i & -5 \\ 4+i & 1+i & -3i \end{bmatrix}$

6. $\begin{bmatrix} 1 & 1+i & 2+3i \\ 1-i & 2 & -i \\ 2-3i & i & 0 \end{bmatrix}$ **(M.U. 2004)**

7. $\begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$ **(M.U. 2002)**

8. $\begin{bmatrix} 2+3i & 0 & 4i \\ 5 & i & 8 \\ 1-i & -3+i & 6 \end{bmatrix}$ **(M.U. 2002, 03)**

9. $\begin{bmatrix} 2 & 4+i & 6i \\ 6 & 5-i & 4 \\ 0 & 1-i & 8i \end{bmatrix}$ **(M.U. 2003)**

10. $\begin{bmatrix} 2 & 4+i & 4i \\ 3i & 6-i & 2 \\ 6 & 4-2i & 1-i \end{bmatrix}$ **(M.U. 2004)**

11. $\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$ **(M.U. 2015)**

Propery 8 (a) : Show that every square matrix A can be uniquely expressed as $P+iQ$ when P and Q are Hermitian matrices. **(M.U. 2006, 07, 08, 09, 2012)**

Proof : Let $P=\frac{1}{2}(A+A^{\theta})$ and $Q=\frac{1}{2i}(A-A^{\theta})$ then

$$P+iQ=\frac{1}{2}(A+A^{\theta})+\frac{1}{2}(A-A^{\theta})=A$$

Now, $P^{\theta}=\frac{1}{2}(A+A^{\theta})^{\theta}=\frac{1}{2}\left[(A^{\theta})+(A^{\theta})^{\theta}\right]=\frac{1}{2}[A^{\theta}+A]=\frac{1}{2}(A+A^{\theta})=P$

$\therefore$ P is Hermitian.

Also, $Q^{\theta}=\left[\frac{1}{2i}(A-A^{\theta})\right]^{\theta}=\overline{\left(\frac{1}{2i}\right)}(A-A^{\theta})^{\theta}=-\frac{1}{2i}\left[(A^{\theta})-(A^{\theta})^{\theta}\right]$

$$=-\frac{1}{2i}(A^{\theta}-A)=\frac{1}{2i}(A-A^{\theta})=Q$$

$\therefore$ Q also is Hermitian.

To prove uniqueness let $A=R+iS$ be another representation of A, where R and S are Hermitian matrices.

Now $A^{\theta}=(R+iS)^{\theta}=R^{\theta}+(iS)^{\theta}=R^{\theta}-iS^{\theta}=R-iS$ (because R and S are Hermitian)

$\therefore$ $A+A^{\theta}=R+iS+R-iS=2R$

$\therefore$ $R=\frac{1}{2}(A+A^{\theta})=P$

Also, $A - A^{\theta} = (R + iS) - (R - iS) = 2iS \quad \therefore \quad S = \dfrac{1}{2i}(A - A^{\theta}) = Q$

Hence, the representation $A = P + iQ$ is unique.

(We shall verify this property below by an example.)

Example 5 (a) : Express the following matrix A as $P + iQ$, where P, Q are both Hermitian.

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}$$ **(M.U. 2003, 11)**

Sol. : From A, we get $A' = \begin{bmatrix} 2 & i & 1+2i \\ 3-i & 0 & 1 \\ 2+i & 1-i & 3i \end{bmatrix}$

$$A^{\theta} = (\overline{A}') = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}$$

Now let $P = \dfrac{1}{2}(A + A^{\theta}) = \dfrac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$

and let $Q = \dfrac{1}{2i}(A - A^{\theta}) = \dfrac{1}{2i}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$

If can be verified that both P and Q are Hermitian and that $P + iQ = A$.

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Express the following matrices as $P + iQ$, where P and Q are both Hermitian : Class (a) : 4 Marks

1. $\begin{bmatrix} 1+2i & 2 & 3-i \\ 2+3i & 2i & 1-2i \\ 1+i & 0 & 3+2i \end{bmatrix}$ **2.** $\begin{bmatrix} 1-i & 2+3i & 3-i \\ 2 & 2-i & 1+2i \\ 3i & 0 & 1+i \end{bmatrix}$

3. $\begin{bmatrix} 2 & 3-i & 1-i \\ 2-i & 3+i & 2+i \\ 1+i & 0 & -3i \end{bmatrix}$ **4.** $\begin{bmatrix} 2i & -3 & 1-i \\ 0 & 2+3i & 1+i \\ -3i & 3+2i & 2-5i \end{bmatrix}$ **(M.U. 2006)**

Propery 9 (a) : Prove that every Hermitian matrix A can be written as $B + iC$, where B is real symmetric and C is real skew-symmetric matrix. **(M.U. 1997, 2005, 09, 11)**

Proof : Let A be a Hermitian matrix. Then $A^{\theta} = A$.

Let $B = \dfrac{1}{2}(A + \overline{A})$ and $C = \dfrac{1}{2i}(A - \overline{A})$

We know that if $z = x + iy$ is a complex number then $\bar{z} = x - iy$ and hence $\frac{1}{2}(z + \bar{z}) = x$ is real and also $\frac{1}{2i}(z - \bar{z}) = \frac{1}{2i} \cdot 2iy = y$ is real. Hence, B and C are real matrices.

Now, we can write

$$A = \frac{1}{2}(A + \bar{A}) + i \cdot \left[\frac{1}{2i}(A - \bar{A})\right] = B + iC$$

But $B' = \frac{1}{2}(A + \bar{A})' = \frac{1}{2}\left[A' + (\bar{A})'\right]$

$$= \frac{1}{2}\left[A' + A^{\theta}\right] = \frac{1}{2}\left[(A^{\theta})' + (A^{\theta})^{\theta}\right] \qquad [\because A^{\theta} = A]$$

$\therefore$ $B' = \frac{1}{2}\left[((\bar{A})')' + A\right] = \frac{1}{2}(\bar{A} + A) = B$ $\qquad \therefore$ B is symmetric

And $C' = \left[\frac{1}{2i}(A - \bar{A})\right]' = \frac{1}{2i}(A - \bar{A})' = \frac{1}{2i}(A' - \bar{A}')$

$$= \frac{1}{2i}(A' - A^{\theta}) = \frac{1}{2i}\left[(A^{\theta})' - A\right] \qquad [\because A^{\theta} = A]$$

$\therefore$ $C' = \frac{1}{2i}\left[((\bar{A})')' - A\right] = \frac{1}{2i}(\bar{A} - A) = -\frac{1}{2i}(A - \bar{A}) = -C$

$\therefore$ C is Skew-symmetric.

(We shall verify this property below by an example.)

Example 6 (a) : Express the following Hermitian matrix A as $P + iQ$, where P is real symmetric and Q is real skew-symmetric matrix.

$$A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$$

Sol. : We first note that the given matrix A is Hermitian.

Now $\bar{A} = \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ -i & -3-i & -1 \end{bmatrix}$

Let $P = \frac{1}{2}(A + \bar{A}) = \frac{1}{2}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 0 & -6 \\ 0 & -6 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$

and $Q = \frac{1}{2i}(A - \bar{A}) = \frac{1}{2i}\begin{bmatrix} 0 & 2i & -2i \\ -2i & 0 & -2i \\ 2i & 2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

It can be easily seen that P is real symmetric and Q is real skew- symmetric matrix and $A = P + iQ$.

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Express the following Hermitian matrices as $P + iQ$, where P is real symmetric and Q is real skew-symmetric : Class (a) : 4 Marks

1. $\begin{bmatrix} 1 & 2+i & -1+i \\ 2-i & 1 & 2i \\ -1-i & -2i & 0 \end{bmatrix}$ **(M.U. 2003)**

2. $\begin{bmatrix} 2 & 2+i & -2i \\ 2-i & 3 & i \\ 2i & -i & 1 \end{bmatrix}$ **(M.U. 2005)**

3. $\begin{bmatrix} 3 & 2-i & 1+2i \\ 2+i & 2 & 3-2i \\ 1-2i & 3+2i & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$

Property 10 (a) : Prove that every Skew-Hermitian matrix A can be written as $B + iC$ where, B is real skew-symmetric and C is real symmetric matrix.

Proof : Let A be a Skew-Hermitian matrix. Then $A^\theta = -A$.

Let $B = \dfrac{1}{2}(A + \overline{A})$ and $C = \dfrac{1}{2i}(A - \overline{A})$

We know that if $z = x + iy$ is a complex number then $\overline{z} = x - iy$ and hence $\dfrac{1}{2}(z + \overline{z}) = x$ is real and also $\dfrac{1}{2i}(z - \overline{z}) = \dfrac{1}{2i}\cdot 2iy = y$ is real. Hence, B and C are real matrices.

Now, we can write,

$$A = \frac{1}{2}(A + \overline{A}) + i\left[\frac{1}{2i}(A - \overline{A})\right] = B + iC$$

But $\quad B' = \dfrac{1}{2}(A + \overline{A})' = \dfrac{1}{2}\left[A' + (\overline{A})'\right] = \dfrac{1}{2}\left[A' + A^\theta\right] = \dfrac{1}{2}\left[(-A^\theta)' + (-A^\theta)^\theta\right] \qquad \left[\because\ A^\theta = -A\right]$

$$\therefore\ B' = \frac{1}{2}\left[\{(-\overline{A})'\}' - A\right] = \frac{1}{2}(-\overline{A} - A) = -\frac{1}{2}(A + \overline{A}) = -B$$

$\therefore\ B$ is skew-symmetric.

And $\quad C' = \left[\dfrac{1}{2i}(A - \overline{A})\right]' = \dfrac{1}{2i}(A - \overline{A})' = \dfrac{1}{2i}(A' - \overline{A}') = \dfrac{1}{2i}(A' - A^\theta)$

$$= \frac{1}{2i}\left[(-A^\theta)' - (-A^\theta)^\theta\right] \qquad \left[\because\ A^\theta = -A\right]$$

$$\therefore\ C' = \frac{1}{2i}\left[\{(-\overline{A})'\}' + A\right] = \frac{1}{2i}\left[-\overline{A} + A\right] = \frac{1}{2i}(A - \overline{A}) = C$$

$\therefore\ C$ is symmetric.

(We shall verify this property below by an example.)

Example 7 (a) : Express the following skew-Hermitian matrix A as $P + iQ$, where P is real skew-symmetric and Q is real symmetric matrix.

$$A = \begin{bmatrix} 2i & 2+i & 1-i \\ -2+i & -i & 3i \\ -1-i & 3i & 0 \end{bmatrix}$$ **(M.U. 2002, 06)**

Sol. : We first note that A is a Skew-Hermitian matrix.

Now $\bar{A} = \begin{bmatrix} -2i & 2-i & 1+i \\ -2-i & i & -3i \\ -1+i & -3i & 0 \end{bmatrix}$

Let $P = \frac{1}{2}(A+\bar{A}) = \frac{1}{2}\begin{bmatrix} 0 & 4 & 2 \\ -4 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

and $Q = \frac{1}{2i}(A-\bar{A}) = \frac{1}{2i}\begin{bmatrix} 4i & 2i & -2i \\ 2i & -2i & 6i \\ -2i & 6i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 3 \\ -1 & 3 & 0 \end{bmatrix}$

It can be seen that P is real skew-symmetric and Q is real symmetric matrix and $A = P + iQ$.

EXERCISE - V

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Express the following Skew-Hermitian matrices as $P + iQ$, where P is real and skew-symmetric and Q is real and symmetric : Class (a) : 4 Marks

1. $\begin{bmatrix} i & 1-i & 2+3i \\ -1-i & 2i & -3i \\ -2+3i & -3i & -i \end{bmatrix}$

2. $\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 2-3i & 1+i \\ -2-3i & 2i & 2-i \\ -1+i & -2-i & -i \end{bmatrix}$

4. $\begin{bmatrix} i & 2i & -1+3i \\ 2i & 2i & 2-i \\ 1+3i & -2-i & 3i \end{bmatrix}$

6. Inverse of A Matrix (Review)

If A is a square matrix and B is another matrix if it exists such that $AB = BA = I$, then A is called **invertible** matrix and B is called the **inverse** of A and is denoted by A^{-1}.

You have studied in Std. XII, how to find the inverse of a given matrix by adjoint method. In order to maintain continuity in the development of the subject we shall briefly take the **review** of this part below.

(a) Minors and Cofactors (Review)

The **Minor** of the element a_{ij} of a square matrix A is the determinant obtained from $|A|$ by **deleting i^{th} row and j^{th} column.** Thus, if

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ then,}$$

Minor of $a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, Minor of $c_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, Minor of $b_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

The **cofactor** of the element a_{ij} is denoted by A_{ij} and is defined as the minor of a_{ij} with the sign given by $(-1)^{i+j}$.

Thus, $A_{ij} = (-1)^{i+j}$ **(the minor of a_{ij}).**

For the above determinant, the cofactors of a_1, c_2, b_3 respectively are

$$A_1 = (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},\ C_2 = (-1)^{2+3}\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix},\ B_3 = (-1)^{3+2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$\therefore\quad A_1 = (b_2c_3 - b_3c_2),\ C_2 = -(a_1b_3 - a_3b_1),\ B_3 = -(a_1c_2 - a_2c_1)$$

(b) Determinant of a square matrix (Review)

Consider a 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Corresponding to the matrix there is a determinant.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{32} \end{vmatrix}$$

The value of this determinant is obtained in usual manner. **The sum of the products of the elements of any row or column with the cofactors of corresponding elements of that row or column is equal to the value of the determinant.**

Thus, in the above determinant.

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

or $\quad \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \quad$ and so on.

(c) The Adjoint of A Square Matrix (Review)

Let $A = [a_{ij}]$ be a given square matrix and A_{ij} denote the cofactor of a_{ij}. Then the transpose of the matrix $[A_{ij}]$ is called the **adjoint matrix** or simply **adjoint** of the matrix A.

Let us denote the matrix A by $\quad A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

and the cofactor of the element a_{ij} by A_i. Then the matrix of the cofactors is

$$\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

The transpose of this matrix is called the adjoint of A denoted by adj. A.

$$\therefore\quad \text{adj } A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

In other words, **the transpose of the matrix of the cofactors is called the adjoint of the matrix A.**

If $A=\begin{bmatrix}2 & 3 & 1\\ 0 & -1 & 2\\ 1 & -2 & 3\end{bmatrix}$ then the matrix of cofactors is $\begin{bmatrix}1 & 2 & 1\\ -11 & 5 & 7\\ 7 & -4 & -2\end{bmatrix}$.

The transpose of this matrix is the adjoint. $\therefore$ adj. $A=\begin{bmatrix}1 & -11 & 7\\ 2 & 5 & -4\\ 1 & 7 & -2\end{bmatrix}$

Example : If $A=\begin{bmatrix}-1 & -2 & -2\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$, prove that adj. $A=3\overline{A}$.

Sol. : $A_1=\begin{vmatrix}1 & -2\\ -2 & 1\end{vmatrix}=1-4=-3;$ $\quad A_2=-\begin{vmatrix}2 & -2\\ 2 & 1\end{vmatrix}=-(2+4)=-6;$

$A_3=\begin{vmatrix}2 & 1\\ 2 & -2\end{vmatrix}=-4-2=-6.$

$B_1=-\begin{vmatrix}-2 & -2\\ -2 & 1\end{vmatrix}=-(-2-4)=6;$ $\quad B_2=\begin{vmatrix}-1 & -2\\ 2 & 1\end{vmatrix}=-1+4=3;$

$B_3=-\begin{vmatrix}-1 & -2\\ 2 & -2\end{vmatrix}=-(2+4)=-6.$

$C_1=\begin{vmatrix}-2 & -2\\ 1 & -2\end{vmatrix}=4+2=6;$ $\quad C_2=-\begin{vmatrix}-1 & -2\\ 2 & -2\end{vmatrix}=-(2+4)=-6;$

$C_3=\begin{vmatrix}-1 & -2\\ 2 & 1\end{vmatrix}=-1+4=3.$

Matrix of cofactors $=\begin{bmatrix}-3 & -6 & -6\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix}$

Adjoint of A = transpose of the above matrix $=\begin{bmatrix}-3 & 6 & 6\\ -6 & 3 & -6\\ -6 & -6 & 3\end{bmatrix}=3\begin{bmatrix}-1 & 2 & 2\\ -2 & 1 & -2\\ -2 & -2 & 1\end{bmatrix}$

$\overline{A}$ = transpose of the given matrix $=\begin{bmatrix}-1 & 2 & 2\\ -2 & 1 & -2\\ -2 & -2 & 1\end{bmatrix}$

$\therefore$ adj. $A=3\overline{A}$.

(d) Inverse by Adjoint method (Review)

To refresh your memory, we shall find inverse of one matrix by the adjoing method.

Example : Use the adjoint method to find the inverse of

$$A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 3\\ 3 & 1 & 2\end{bmatrix}$$

Sol. : There are various methods of computing inverse of a matrix. We shall revise here to calculate the inverse from its adjoint. Other methods are explained at proper places.

Instead of taking cofactors of elements in the rows and then taking the transpose we may take the cofactors of the elements in columns.

We have $|A| = 2(4-3) - 3(2-9) + 1(1-6)$

$= 2 + 21 - 5 = 18$

Cofactors of the elements of the first column are

$$\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1, \quad -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = -(6-1) = -5, \quad \begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix} = 9 - 2 = 7$$

Cofactors of the elements of the second column are

$$-\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -(2-9) = 7, \quad \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1, \quad -\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = -(6-1) = -5$$

Cofactors of the elements of the third column are

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 1 - 6 = -5, \quad -\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -(2-9) = 7, \quad \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$\therefore \quad \text{adj. } A = \begin{vmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{vmatrix} \qquad \therefore \quad A^{-1} = \frac{1}{|A|}\text{adj. } A = \frac{1}{18}\begin{vmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{vmatrix}$$

Remark ....

Note that the signs are to be taken alternatively positive and negative.

7. Orthogonal Matrices

Definition : A real square matrix A is called orthogonal if $AA' = A'A = I$. Clearly for an othrogonal matrix $A^{-1} = A'$.

Properties of orthogonal matrices : Class (a) : 4 Marks

(a) If A is an orthogonal matrix then $|A| = \pm 1$ **(M.U. 1997, 2006)**

Since a determinant is unchanged by interchange or rows and columns $|A'| = |A|$. Further, by definition if A is orthogonal $AA' = I$.

$$\therefore \quad |AA'| = |I| \qquad \therefore \quad |A| \cdot |A'| = |I| \, |A^2| = 1 \qquad \therefore \quad |A| = \pm 1.$$

Note ....

If A is an orthogonal square matrix of order n, since $|A| = \pm 1$, the rank of $A = n$. [For rank see § 11].

(b) If A is an orthogonal matrix then A' is also orthogonal.

Proof : Let A be an orthogonal matrix so that $AA' = I = A'A$.

Now, taking transpose of both sides of $AA' = I$, we get

$$(AA')' = I' \qquad \therefore \qquad A'(A')' = A'A = I$$

Since the product of A' and the transpose of A' is I, A' is orthogonal.

(c) If A is orthogonal then A^{-1} is also orthogonal.

Proof : Let A be an orthogonal matrix, so that

$$AA' = I = A'A$$

Now, taking inverse of both sides of $AA' = I$, we get

$$(AA')^{-1} = I^{-1} \qquad \therefore \ (A')^{-1} A^{-1} = I \qquad \therefore \ (A^{-1})' A^{-1} = I.$$

Since the product of A^{-1} and its transpose $(A^{-1})'$ is I, A^{-1} is orthogonal.

(d) If A is an orthogonal matrix then A^{-1} exists and is equal to A'

We know that if A and B are two matrices such that $AB = BA = 1$, then B is called the inverse of A and the necessary and sufficient condition for A to have inverse is $|A| \neq 0$.

As seen above, if A is orthogonal $|A| = \pm 1 \neq 0$. $\quad \therefore \ A^{-1}$ exists.

Further, $\quad AA' = I \qquad \therefore \ A^{-1}(AA') = A^{-1} \cdot I$

$\therefore \ (A^{-1}A) A' = A^{-1} \qquad \therefore \ IA' = A^{-1} \qquad \therefore \ A' = A^{-1}$.

Example 1 : If $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ is orthogonal, what is the value of

$$abc + 2fgh - af^2 - bg^2 - ch^2 \ ?$$ **[Ans. : $|A| = \pm 1$]**

Example 2 : If A as given above is orthogonal, what is A^{-1} ? **[Ans. : A]**

(e) If A and B are two orthogonal square matrices or order n then AB and BA are also orthogonal. **(M.U. 2003)**

Proof : Since A and B are square matrices of order n, AB and BA are defined and are square matrices of order n.

Since A, B are orthogonal $|A| \neq 0$, $|B| \neq 0 \ (= \pm 1)$ and A^{-1}, B^{-1} exist.

Further, $\quad |AB| = |A| \cdot |B| \neq 0 \qquad \therefore \ (AB)^{-1}$ exists.

Now, $\quad (AB)' = B'A'$

Further, $\quad (AB)'(AB) = B'A'AB = B'(A'A)B$

$$= B'IB = B'B = I$$

Hence, $(AB)'$ is the inverse of AB. $\qquad \therefore \ AB$ is orthogonal.

Solved Examples : Class (a) : 4 Marks

Example 1 (a) : Prove that the following matrix is orthogonal and hence find A^{-1}.

$$A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$ **(M.U. 2005, 18)**

Sol. : $\quad A' = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix} \qquad \therefore \ AA' = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$

A is orthogonal and A' is the inverse of A.

Example 2 (a) : If $A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$ is orthogonal find a, b, c.

Also find A^{-1}. (State the rank of A^2.) **(M.U. 2003, 04, 06, 07, 08)**

Sol. : Since for orthogonality $AA' = I$.

$$\begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \frac{1}{9} + \frac{4}{9} + a^2 = 1 \qquad \therefore \; a^2 = \frac{4}{9} \qquad \therefore \; a = \pm\frac{2}{3}$$

$$\frac{4}{9} + \frac{1}{9} + b^2 = 1 \qquad \therefore \; b^2 = \frac{4}{9} \qquad \therefore \; b = \pm\frac{2}{3}$$

$$\frac{4}{9} + \frac{4}{9} + c^2 = 1 \qquad \therefore \; c^2 = \frac{1}{9} \qquad \therefore \; c = \pm\frac{1}{3}.$$

$$A^{-1} = A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & \pm 2 \\ 2 & 1 & \pm 2 \\ 2 & -2 & \pm 1 \end{bmatrix}.$$

[Since for orthogonal matrix A, $|A| = \pm 1$, the rank of A is 3 by note on page 9-21 and hence the rank of $A^2 = 3$. $\quad (\because \; |A^2| = |A|\,|A| = \pm 1)$]

Example 3 (a) : Is the following matrix orthogonal? If not, can it be converted into an orthogonal matrix? If yes, how ?

$$A = \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$ **(M.U. 2003, 04)**

Sol. : For orthogonality $AA' = I$.

$$\text{Now,} \quad AA' = \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} = \begin{bmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{bmatrix} = 81 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $AA' \neq I$ but $81I$, A is not orthogonal.

But the matrix A can be converted into an orthogonal matrix.

$$\text{Since,} \quad AA' = 81I, \quad \frac{1}{81}AA' = 1 \quad \therefore \quad \frac{1}{9}A \cdot \frac{1}{9}A' = I$$

$$\therefore \quad \frac{1}{9}A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} = \begin{bmatrix} -8/9 & 1/9 & 4/9 \\ 4/9 & 4/9 & 7/9 \\ 1/9 & -8/9 & 4/9 \end{bmatrix}$$ is the required orthogonal matrix.

Example 4 (a) : Is the following matrix orthogonal ? If not, can it be converted into an orthogonal matrix ? If yes, how ?

$$A = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

Sol. : For orthogonal $AA' = I$.

But $$AA' = \begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $AA' \neq I$, but $9A$, A is not orthogonal. But matrix A can be converted into an orthogonal matrix.

Since, $AA' = 9I$, $\frac{1}{9}AA' = I$ $\therefore$ $\frac{1}{3}A \cdot \frac{1}{3}A' = I$

$$\therefore \quad \frac{1}{3}A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$ is the required orthogonal matrix.

EXERCISE - VI

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 4 Marks

(A) Prove that the following matrices are orthogonal and hence find A^{-1}.

1. $\frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix}$ **(M.U. 2000, 02, 06)**

2. $\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ **(M.U. 2002)**

3. $\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$ **(M.U. 2005, 06, 09)**

5. $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6. $\frac{1}{\sqrt{6}}\begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ 2 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$

7. $\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8. $\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ **(M.U. 1999)**

9. $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & -2 \end{bmatrix}$

10. $\frac{1}{7}\begin{bmatrix} 2 & 3 & -6 \\ 6 & 2 & 3 \\ -3 & 6 & 2 \end{bmatrix}$

11. $\frac{1}{13}\begin{bmatrix} -3 & 4 & 12 \\ 12 & -3 & 4 \\ 4 & 12 & -3 \end{bmatrix}$

12. $\frac{1}{11}\begin{bmatrix} 2 & 6 & -9 \\ 6 & 7 & 6 \\ 9 & -6 & -2 \end{bmatrix}$

13. $\frac{1}{21}\begin{bmatrix} 4 & 5 & 20 \\ 20 & 4 & -5 \\ 5 & -20 & 4 \end{bmatrix}$

14. $\frac{1}{9}\begin{bmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{bmatrix}$ **(M.U. 2004)**

[Answers are not given.]

Class (a) : 4 Marks

(B) **1.** If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal, find a, b, c. **(M.U. 2003)**

(**Hint :** $AA' = \frac{1}{9}\begin{bmatrix} 5+a^2 & 4+ab & ac-2 \\ 4+ab & 5+b^2 & bc+2 \\ -2+ac & bc+2 & 8+c^2 \end{bmatrix} = I$

$\therefore \quad \frac{5+a^2}{9} = 1, \quad \frac{5+b^2}{9} = 1, \quad \frac{8+c^2}{9} = 1.$) [**Ans. :** (i) 2, −2, 1 ; (ii) −2, 2, −1.]

2. If $3A = \begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ and if A is orthogonal, find a, b, c.

(**Hint :** $AA' = \frac{1}{9}\begin{bmatrix} a^2+b^2+c^2 & -2a+b+2c & a-2b+2c \\ -2a+b+2c & 9 & 0 \\ a-2b+2c & 0 & a \end{bmatrix} = I$

$\therefore \quad a^2+b^2+c^2 = 9, \quad -2a+b+2c = 0$

$a-2b+2c = 0.$) **(M.U. 2004)** [**Ans. :** $a = 2, b = 2, c = 1$]

3. Find a, b, c if $\frac{1}{9}\begin{bmatrix} a & 1 & b \\ c & b & 7 \\ 1 & a & c \end{bmatrix}$ is orthogonal. [**Ans. :** $a = \pm 8, b = \pm 4, c = \pm 4$]

4. If $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is orthogonal, find the conditions.

[**Ans. :** $a_1^2 + b_1^2 = 1, \ a_1 a_2 + b_1 b_2 = 0$ etc.]

5. Find a, b, c, if A is orthogonal where $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$. **(M.U. 2004, 08, 15)**

[**Ans. :** $a = 1, b = -8, c = 4$]

6. Show that the matrix $\begin{bmatrix} \cos\phi\cos\theta & \sin\phi & \cos\phi\sin\theta \\ -\sin\phi\cos\theta & \cos\phi & -\sin\phi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

is orthogonal and find its inverse. **(M.U. 1995, 2007)** [**Ans. :** $A^{-1} = A'$]

(C) **1.** If $A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}$, find A^{-1}. **(M.U. 2002, 04, 10)**

[**Ans. :** A is orthogonal. $A^{-1} = A'$.]

2. If $A = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2/3} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$, find A^{-1}. [**Ans. :** A is orthogonal, $A^{-1} = A'$.]

8. Unitary Matrices

Definition : A square matrix A is said to be unitary if the product of A and its transpose of conjugate complex *i.e.* $\overline{A}'$ *i.e.* A^θ is a unit matrix.

In symbols a square matrix A is unitary if

$$\boxed{A^\theta A = A A^\theta = I}$$

Clarly for a unitary matrix $A^{-1} = A^\theta$.

Properties of Unitary Matrices : Class (a) : 4 Marks

(a) Determinant of a unitary matrix is of modulus unity.

Proof : Let A be a unitary matrix. Then $AA^\theta = I$

Taking determinants of both sides, we get

$$|AA^\theta| = |I| \quad \therefore\ |A||A^\theta| = 1$$

$$\therefore\ |A|\cdot|\overline{A}'| = 1 \quad \therefore\ |A|\cdot|\overline{A}| = 1 \quad \therefore\ |A|^2 = 1 \qquad [\because |\overline{A}| = |A|]$$

Hence, the result.

Remark ....

Note that the determinant of the unitary matrix given in Ex. 2 below.

$$= \frac{1}{2}(1+i)\cdot\frac{1}{2}(1-i) - \frac{1}{2}(-1+i)\frac{1}{2}(1+i)$$

$$= \frac{1}{4}(1-i^2) - \frac{1}{4}(i^2-1) = \frac{1}{4}(2) + \frac{1}{2}(2) = 1$$

Similarly, you can prove that the 'modulus' of the determinant of the matrix given in Ex. 1 also is unity.

(b) If A unitary, show that A^{-1} is also unitary.

Proof : Since A is unitary

$$AA^\theta = I \quad \therefore\ (AA^\theta)^{-1} = I^{-1} \quad \therefore\ (A^\theta)^{-1}\cdot A^{-1} = I^{-1}$$

$$\therefore\ (A^\theta)^{-1}\cdot A^{-1} = I \quad \therefore\ (A^{-1})^\theta\cdot A^{-1} = I$$

Hence, A^{-1} is unitary.

(c) If A unitary, prove that A' is also unitary.

Proof : Since A is unitary

$$AA^\theta = I \qquad \therefore\ (AA^\theta)' = I' = I$$

$$\therefore\ (A^\theta)' A' = I \qquad \therefore\ (A')^\theta A' = I.$$

Hence, A' is also unitary.

(d) If A, B are two unitary matrices, prove that AB is also unitary.

Proof : We have

$$\begin{aligned}(AB)(AB)^\theta &= (AB)(B^\theta A^\theta) = A(BB^\theta)A^\theta \\ &= AIA^\theta && [\because B \text{ is unitary}] \\ &= AA^\theta = I && [\because A \text{ is unitary}]\end{aligned}$$

Similarly, we can prove that $(AB)^\theta(AB) = I$.

Hence, AB is also unitary.

Solvex Examples : Class (a) : 4 Marks

Example 1 (a) : Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary. **(M.U. 2017)**

Sol. : Let us denote the given matrix by A.

$$\therefore \quad A' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \qquad \therefore \quad A^{\theta} = (\bar{A}') = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

Now, $$A^{\theta}A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore \quad A^{\theta}A = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is unitary.

Example 2 (a) : Prove that the matrix $\begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$ is unitary. **(M.U. 2003, 06, 10, 11)**

Sol. : Let us denote the given matrix by A.

$$\therefore \quad A' = \frac{1}{2}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} \qquad \therefore \quad A^{\theta} = (\bar{A}') = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$\therefore \quad A^{\theta}A = \frac{1}{4}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} (1-i^2)+(1-i^2) & -(1-i)^2+(1-i)^2 \\ -(1+i)^2+(1+i^2) & (1-i)^2+(1-i^2) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is unitary.

Example 3 (a) : Show that the matrix $A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is unitary and hence, find A^{-1}.

(M.U. 1995, 98, 2006, 18)

Sol. : We have

$$A' = \frac{1}{2}\begin{bmatrix} \sqrt{2} & i\sqrt{2} & 0 \\ -i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \therefore \quad A^{\theta} = (\bar{A}') = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \quad A^{\theta}A = \frac{1}{4}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2+2 & 0 & 0 \\ 0 & 2+2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A \text{ is unitary and } A^{-1} = A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example 4 (a) : If $N = \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$ then show that $(I - N)(I + N)^{-1}$ is a unitary matrix.

(M.U. 2005, 07, 09, 11)

Sol. : We have

$$I - N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$I + N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1+2i \\ -1+2i & 1 \end{bmatrix}$$

$$\text{adj.}(I + N) = \begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix} \qquad \left[\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then adj. } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right]$$

$$|I + N| = 1 + (1 - 4i^2) = 6$$

$$\therefore \quad (I + N)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

Now, $$(I - N)(I + N)^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1-2i \\ 1-2i & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

Let $$A = (I - N)(I + N)^{-1} = \frac{1}{6}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}$$

$$\therefore \quad A' = \frac{1}{6}\begin{bmatrix} -4 & 2-4i \\ -2-4i & -4 \end{bmatrix} \; ; \; A^{\theta} = \frac{1}{6}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

Hence, $$AA^{\theta} = \frac{1}{36}\begin{bmatrix} -4 & -2-4i \\ 2-4i & -4 \end{bmatrix}\begin{bmatrix} -4 & 2+4i \\ -2+4i & -4 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, $(I - N)(I + N)^{-1}$ is a unitary matrix.

Example 5 (a) : Show that the matrix $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.

(M.U. 2013, 17)

Sol. : We have $$A^{\theta} = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

$$\therefore \quad A^{\theta}A = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$$

But by data A is unitary. $\quad \therefore \quad A^{\theta}A = I$

$$\therefore \quad \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & 0 \\ 0 & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

EXERCISE - VII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (a) : 4 Marks

Show that the following matrix is unitary.

$$\begin{bmatrix} \dfrac{2+i}{3} & \dfrac{2i}{3} \\ \dfrac{2i}{3} & \dfrac{2-i}{3} \end{bmatrix}$$ **(M.U. 2003, 08)** (**Hint :** Show that $AA^{\theta} = I$.)

9. Elementary Transformations

The following operations on a matrix are **called** elementary transfor-mations.

(a) Elementary Row Transformations

The following three operations on a matrix are called elementary row transformations.

(i) Interchange of i^{th} and j^{th} rows, denoted by R_{ij}.

(ii) Multiplying the elements of i^{th} row by any non-zero number k, denoted by $k\,R_i$ $(k \neq 0)$.

(iii) Adding to the element of i^{th} row the corresponding elements of j^{th} row multiplied by any number k, denoted by $R_i + kR_j$.

(b) Elementary Column Transformations

The following three operations on a matrix are called **elementary column transformations.**

(i) Interchange of i^{th} and j^{th} columns, denoted by C_{ij}.

(ii) Multiplying the elements of the i^{th} column by any non-zero number k, denoted by $k\,C_i$ $(k \neq 0)$.

(iii) Adding to the elements of i^{th} column the corresponding elements of the j^{th} column multiplied by any number k, denoted by $C_i + kC_j$.

10. Elementary Matrices

Definition : A matrix obtained from a unit matrix by performing on it a *single* elementary row or column transformation is called an **elementary matrix.**

$$e.g. \text{ if } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{by } R_{12} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{by } 3\,C_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{by } C_3 + 2C_1 \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Theorem : Every elementary row (column) transformation can be brought about by pre-multiplication (post multiplication) of corresponding elementary matrix.

We accept this theorem without proof. This theorem states that if we write $A = IA$ and perform elementary row transformations on A on the l.h.s. and the same row transformations on the prefactor I on the r.h.s., the equality is not changed. Similarly if we write $A = AI$ and perform elementary column transformations on A on the l.h.s. and the same column transformations on the post factor I on the r.h.s., the equality is not changed.

11. Rank of A Matrix

Let A be a given matrix rectangular or square. From this matrix select any r rows and from these r rows select only r columns thus getting a *square matrix of order* $r \times r$. The determinant of this matrix of order $r \times r$ is called the **minor of order** r.

$$e.g. \text{ if } A = \begin{bmatrix} 1 & 2 & -3 & 0 & 2 \\ 2 & 3 & 4 & -1 & 1 \\ 5 & -2 & 3 & 2 & 2 \\ -1 & 0 & 2 & 3 & 4 \end{bmatrix}$$

By selecting two rows *viz.* 3^{rd} and 4^{th} *i.e.* deleting first two rows, we get,

$$\begin{bmatrix} 5 & -2 & 3 & 2 & 2 \\ -1 & 0 & 2 & 3 & 4 \end{bmatrix}$$

From this by selecting two columns *viz.* 3^{rd} and 4^{th} *i.e.* deleting 1^{st}, 2^{nd} and 5^{th} columns we get a matrix of order 2×2.

$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ The determinant of this matrix $\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$ is a minor of order 2.

Again selecting only 3 rows, say, by deleting 1^{st} row only from A, we get

$$\begin{bmatrix} 2 & 3 & 4 & -1 & 1 \\ 5 & -2 & 3 & 2 & 2 \\ -1 & 0 & 2 & 3 & 4 \end{bmatrix}$$

By selecting only 3 columns, say, deleting 2^{nd} and 3^{rd} columns, we get the matrix $\begin{bmatrix} 2 & -1 & 1 \\ 5 & 2 & 2 \\ -1 & 3 & 4 \end{bmatrix}$ which is of order 3×3.

The determinant of this matrix $\begin{vmatrix} 2 & -1 & 1 \\ 5 & 2 & 2 \\ -1 & 3 & 4 \end{vmatrix}$ is a minor or order 3.

Clearly from a given matrix we can obtain a number of minors.

(a) Rank

Definition : Let A be a non-zero matrix. Then the integer r is called the *rank* of the matrix A if,

(i) there exists at least one minor of order r of A which is non-zero and

(ii) every minor of order greater than r is zero.

Remarks ....

1. The rank of a null matrix *i.e.* zero matrix is zero.
2. The rank of a non-singular square matrix of order r is r.

Class (a) : 4 Marks

Example : Find the rank of the matrix $A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{bmatrix}$.

Now, $\quad |A| = 2\,(18 - 16) - 4\,(9 - 8) + 1\,(24 - 24) = 0$

Further, $\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0$, $\begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} = 0$, $\begin{vmatrix} 3 & 6 \\ 4 & 8 \end{vmatrix} = 0$.

But $\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} = 8 - 6 = 2 \neq 0$ $\therefore$ Rank = 2.

(b) Invariance of Rank through Elementary transformations

We assume the following theorem.

***Theorem :* The rank of a matrix remains unchanged by elementary transformations *i.e.* if from a given matrix *A* we get another matrix *B* by performing elementary transformations on *A* then**

$$\boxed{\text{rank of } A = \text{rank of } B}$$

From the above theorem the following corollary can be obtained.

***Corollary :* Pre-multiplication and/or post multiplication by a finite sequence of elementary row and/or column matrices does not alter the rank of a matrix.**

Note ....

While the value of the minors of a matrix may get changed by elementary transformations their zero or non-zero character remains unaffected.

(c) *Theorem :* The rank of the transpose of a matrix *A* is same as the matrix *A*.

Proof : Let A be an $m \times n$ matrix and let its rank be r. Then there is at least one square sub-matrix R of A whose determinant is not equal to zero *i.e.* $|R| \neq 0$. .... (i)

Let R' be the transpose of R and A' be the transpose of A. Then R' is a sub-matrix of A'. Since, the value of the determinant is not altered by interchange of rows and columns, $|R'| = |R| \neq 0$.

If S is a submatrix of order $(r+1)$ of A and S' is the transpose of S then S' is a sub-matrix of A' of order $(r+1)$. Since, $|S| = 0$, $|S'| = 0$. .... (ii)

From (i) and (ii) it follows that the rank of $A' = r$.

(d) *Theorem :* The rank of the product of two matrices cannot exceed the rank of either matrix.

Proof : We shall accept this theorem without proof. The theorem can be restated as follows. "If r_1 is the rank of A, r_2 is the rank of B, if A and B are conformable *i.e.*, if we can obtain the product AB and if r is the rank of AB, then $r \leq r_1$, $r \leq r_2$."

Cor. : If r is the rank of he matrix A then the rank of A^n is less than or equal to A.

Example : Find the ranks of A, B, AB and verify the above theorem if

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix}$$

Sol. : We have seen above that the rank of $A = 2$.

Now, $|B| = 1(5-6) - 2(15-8) + 3(9-4) = 0$

But $\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 10 - 6 = 4 \neq 0$

Hence, the rank of $B = 2$.

Further, $AB = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 4 & 8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 18 & 11 & 19 \\ 29 & 18 & 31 \\ 40 & 25 & 43 \end{bmatrix}$

Now, by $\begin{matrix} R_2 - R_1 \\ R_3 - R_2 \end{matrix}$, $AB = \begin{bmatrix} 18 & 11 & 19 \\ 11 & 7 & 21 \\ 11 & 7 & 21 \end{bmatrix}$ By $R_3 - R_2$, $AB = \begin{bmatrix} 18 & 11 & 19 \\ 11 & 7 & 21 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore \ |AB| = 0$

But $\begin{vmatrix} 11 & 19 \\ 7 & 21 \end{vmatrix} = 11 \times 21 - 19 \times 7 \neq 0$ $\therefore$ The rank of $AB = 2$.

Thus, the theorem is verified.

(e) Observations

From the definition of the rank of a matrix, we observe the following properties of the rank.

(i) The rank of a square matrix of order n with $|A| \neq 0$ is n.

(ii) The rank of a unit matrix of order n is n.

(iii) The rank of a null matrix of order n is zero.

(iv) The rank of a diagonal matrix of order n whose all diagonal elements are non-zero is n.

(v) The rank of a scalar matrix of order n is n.

12. Echelon Form

Before defining the term echelon form let us have a look at such a form of a matrix. Look at the following matrix

$$\begin{bmatrix} 1 & 2 & -4 & 6 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What do you observe ? The two rows in the bottom have all elements zero. The number of zeros before the first non-zero element in a row goes on increasing as you go down the matrix. In every column all the elements below the first non-zero element are zero.

A matrix in such a form is said to be in **echelon form.**

We show below a 5 × 8 matrix in echelon form.

$$\begin{bmatrix} a & * & * & * & * & * & * & * \\ 0 & b & * & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The starred entries may take any value, zero, positive or negative. Since the non-zero entries show a "Stairecase Pattern" (steps), the form is called **echelon (meaning steps) form**.

Definition : A matrix is said to be in **echelon form** if it has the following two properties.

1. If any row has all elements zero then such a row appears at the bottom of the matrix. If there are more such rows having all elements zero then they are grouped at the bottom.
2. If there are some rows which do not have all elements zero then they are arranged in such a way that the number of zeros before the first non-zero element go an increasing as we move down the matrix.

Such a matrix is said to be in **echelon form.**

Replacing the * by any real number in the following matrices we an obtain an **echelon form.**

$$\begin{bmatrix} 1 & * & * \\ 0 & 3 & * \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 0 & 2 & * & * \\ 0 & 0 & 0 & 4 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & * & * & * & * \\ 0 & 4 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 5 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrices are in **echelon form,**

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & -2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & -5 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

EXERCISE - VIII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

1. State whether the following matrices are in echelon form.

(1) $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 1 & 3 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 5 & 4 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (5) $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 5 & -1 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(All of these matrices are in echelon form.)

2. State which of the following matrices are in echelon form.

(a) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ [**Ans. :** (b), (c), (e), (f)]

(a) Echelon Form and Elementary Row Transformations

By a series of finite number of **row transformations** a matrix can be transformed into an echelon form.

It may be noted that, an echelon form of a given matrix **is not unique**. It depends upon the sequence of transformations.

(b) To Reduce a Matrix to Echelon Form by Row Transformations

(i) If a_{11} is zero, use the row operation R_{ij} and bring the first element in the first row non-zero.

(ii) If now the first element in the first row is not unity divide the first row by suitable number, so that the first element is unity.

(iii) Using suitable values of k and by subtracting k-th times the first row from the remaining rows, reduce all the first elements in the remaining rows to zero.

(iv) Repeat this procedure for other rows, so that the matrix A is reduced to echelon form.

Note ....

1. Unfortunately there is no unanimity in the definition of echelon form, so far as the first non-zero element is concerned.

2. Some authors insist that the first non-zero element in every row must be unity while others allow it to be any number.

3. Some authors call the above echelon form as row-echelon form. They define in a similar manner column echelon form to be obtained by column transformations only.

Example 1 : Reduce the matrix to echelon form : $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & -3 & -1 \\ -1 & 0 & 5 & 7 \end{bmatrix}$

Sol. : We have $A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & -3 & -1 \\ -1 & 0 & 5 & 7 \end{bmatrix}$

By $\begin{matrix} R_2 - 3R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -2 & -6 & -10 \\ 0 & 2 & 6 & 10 \end{bmatrix}$; By $\begin{matrix} R_3 + R_2 \\ -R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is the required echelon form.

Example 2 : Reduce the matrix to echelon form : $A = \begin{bmatrix} 0 & 2 & -6 & -2 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Sol. : We have $A = \begin{bmatrix} 0 & 2 & -6 & -2 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

By R_{12} $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -6 & -2 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$; By $\begin{matrix} R_3 - 3R_1 \\ R_4 - R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -6 & -2 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$;

$$\text{By } \begin{array}{l} R_3 - \frac{1}{2}R_2 \\ R_4 - \frac{1}{2}R_2 \end{array} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & -6 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ This is the required echelon form.

EXERCISE - IX

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Reduce the following matrices to echelon from.

1. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 0 \end{bmatrix}$, 2. $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 4 \\ 2 & 5 & 11 & 6 \end{bmatrix}$, 3. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$, 4. $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$.

13. Rank in Echelon Form

From the definition of rank given on page 9-30 and from the example solved on the same page you might have noted that to find the rank of a matrix, we shall be required to check a large of number of determinants of various minors. But the work is considerably reduced by reducing the matrix to echelon form.

The rank of a matrix in echelon form is equal to the number of rows containing non-zero elements.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Reduce the following matrix to echelon form and find its ranks

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Sol. : We have $A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ By $\xrightarrow[R_3 - R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

By $\xrightarrow[R_3 + 2R_4]{R_2 - 3R_4} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ By $\xrightarrow{R_{24}} \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in echelon form. The number of non-zero rows is 2. $\therefore$ The rank = 2.

Example 2 (b) : Reduce the following matrix to echelon form and find its ranks

$$\begin{bmatrix} 1 & 3 & 2 & 2 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 8 \end{bmatrix}$$

Sol. : We have $A=\begin{bmatrix}1&3&2&2\\1&2&1&3\\2&4&3&4\\3&7&4&8\end{bmatrix}$

By $\xrightarrow{\substack{R_2-R_1\\R_3-2R_1\\R_4-3R_1}}\begin{bmatrix}1&3&2&2\\0&-1&-1&1\\0&-2&-1&0\\0&-2&-2&2\end{bmatrix}$ By $\xrightarrow{\substack{R_4-2R_2\\R_3-2R_2}}\begin{bmatrix}1&3&2&2\\0&-1&-1&1\\0&0&1&-2\\0&0&0&0\end{bmatrix}$

This is in echelon form. The number of non-zero rows is 3.

$\therefore$ The rank = 3.

Example 3 (b) : Reduce the following matrix to echelon form and find its rank.

$$\begin{bmatrix}8&1&3&6\\0&3&2&2\\-8&-1&-3&-6\end{bmatrix}$$

Sol. : We have $A=\begin{bmatrix}8&1&3&6\\0&3&2&2\\-8&-1&-3&-6\end{bmatrix}$. By R_3-R_1 $\begin{bmatrix}8&1&3&6\\0&3&2&2\\0&0&0&0\end{bmatrix}$.

This is echelon form. The number of non-zero rows is 2.

$\therefore$ The rank = 2.

Example 4 (b) : Reduce the following matrix to echelon form and hence find its rank.

$$\begin{bmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&8&7&5\end{bmatrix}$$

Sol. : We have $A=\begin{bmatrix}1&2&3&0\\2&4&3&2\\3&2&1&3\\6&8&7&5\end{bmatrix}$

By $\substack{R_2-2R_1\\R_3-3R_1\\R_4-(R_1+R_2+R_3)}\begin{bmatrix}1&2&3&0\\0&0&-3&2\\0&-4&-8&3\\0&0&0&0\end{bmatrix}$. By R_{32} $\begin{bmatrix}1&2&3&0\\0&-4&-8&3\\0&0&-3&2\\0&0&0&0\end{bmatrix}$.

This is echelon form. The number of non-zero rows is 3.

$\therefore$ The rank = 3.

Example 5 (b) : Reduce the following matrix to echelon form and hence find its rank.

$$\begin{bmatrix}1&0&2&1\\0&4&-2&1\\1&-1&4&0\\-2&2&-8&0\end{bmatrix}$$

Sol. : We have $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 1 & -1 & 4 & 0 \\ -2 & 2 & -8 & 0 \end{bmatrix}$. By $\begin{matrix} R_3 - R_1 \\ R_4 + 2R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

By $4R_3 \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & -4 & 8 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, By $R_3 + R_2 \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

This is echelon form. The number of non-zero rows is 3.

$\therefore$ The rank = 3.

EXERCISE - X

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Determine the ranks of the following matrices by reducing them to echelon form : Class (b) ; 6 Marks

1. $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 9 \\ 1 & 3 & 4 & 0 \end{bmatrix}$ **2.** $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 4 \\ 2 & 5 & 11 & 6 \end{bmatrix}$ **3.** $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$ **5.** $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 4 & 7 \\ 4 & -1 & 7 \\ 2 & 1 & 5 \end{bmatrix}$ **6.** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 8 & 5 & 14 & 17 \\ 1 & 5 & 5 & 7 \end{bmatrix}$

[**Ans. : (1)** $r = 2$, **(2)** $r = 3$, **(3)** $r = 3$, **(4)** $r = 2$, **(5)** $r = 2$, **(6)** $r = 2$.]

14. Normal Form or (First) Canonical Form

Every $m \times n$ matrix of rank r can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r denotes unit matrix of order r by a finite sequence of elementary transformations. This form is called the **normal form or the first canonical form** of the matrix A.

For example, $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in normal form.

Note ....

The above theorem enables us to determine the rank of a given matrix. We reduce the given matrix to normal form through elementary (row and column) transformations. From this we get the rank directly.

Working Rule

(i) If $a_{11} \neq 1$ interchange suitably, two rows or two columns such that first element is 1. If this is not possible divide the first row or the first column by a_{11} so that the first element is 1.

(ii) By performing operations of the type $R_2 + kR_1$, $C_2 + kC_1$ etc. bring zero everywhere in the first row and first column.

(iii) Repeat the above two operations now on each of the remaining rows by using the first row.

Solved Example : Class (a) : 4 Marks

Example 1 (a) : For the matrix $A = [a_{ij}]_{3\times 3}$ where $a_{ij} = i \times j$, find the rank.

Sol. : By definition, when $i = 1, j = 1, a_{11} = 1 \times 1 = 1$,

when $i = 1, j = 2, a_{12} = 1 \times 2 = 2$, etc.

$$\therefore \quad A = \begin{bmatrix} 1\times1 & 1\times2 & 1\times3 \\ 2\times1 & 2\times2 & 2\times3 \\ 3\times1 & 3\times2 & 3\times3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Since the first element in the first row is 1 and the first element in the second row is 2, we perform $R_2 - 2R_1$. Since the first element in the third element is 3, we perform $R_3 - 3R_1$.

$$\text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{By } \begin{matrix} C_2 - 2C_1 \\ C_3 - 3C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In the process of the above operations, we get all elements in the second and third row zero. So also the elements in the second and third columns are zero.

$\therefore$ Rank = 1.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Find the rank of the matrix by reducing it to normal form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$ **(M.U. 2016)**

Sol. : We have $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

$$\text{By } \xrightarrow[\;]{\begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} \qquad \text{By } \xrightarrow[\;]{\begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\text{By } \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{By } \xrightarrow{C_3 - C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{By } \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ which is in normal form. $\therefore$ Rank = 2.

Example 2 (b) : Reduce the following matrix to normal form and find its rank.

$$\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$$

(M.U. 1996, 2003, 09, 14)

Sol. : We have $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

Since the first element in the first row is 1 and the first element in the second row is 1, we perform $R_2 - R_1$. Since the first element in the third row is 5, we perform $R_3 - 5R_1$.

By $\xrightarrow[R_3 - 5R_1]{R_2 - R_1} \begin{bmatrix} 1 & -1 & 3 & 6 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$

Since the first element in the first column is 1 and the first element in the second column is -1, we perform $C_2 - (-C_1)$ *i.e.,* $C_2 + C_1$. Since the first element in the second column is 3, we perform $C_3 - 3C_1$. Since the first element in the fourth column is 6, we perform $C_4 - 6C_1$.

By $\xrightarrow[\substack{C_3 - 3C_1 \\ C_4 - 6C_1}]{C_2 + C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 8 & -12 & -19 \end{bmatrix}$

Since a_{22} is 4 and a_{32} is 8, we perform $R_3 - 2R_2$.

By $\xrightarrow{R_3 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Since a_{22} is 4, we perform $\left(\frac{1}{4}\right)R_2$ to bring 1 there.

By $\xrightarrow{\left(\frac{1}{4}\right)R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3/2 & -5/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Since a_{23} is $-\frac{3}{2}$, we perform $C_3 + \frac{2}{3}C_2$ and since a_{24} is $-\frac{5}{2}$, we perform $C_5 + \frac{2}{5}C_2$.

By $\xrightarrow[C_4 + (2/5)C_2]{C_3 + (2/3)C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore \xrightarrow{C_{34}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [I_3 \quad 0]$ which is in normal form.

$\therefore$ Rank of $A = 3$.

Example 3 (b) : Find the rank of the matrix by reducing it to normal form.

$$\begin{bmatrix} 1 & -1 & -2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

(M.U. 2005, 06)

Sol. : We have $A = \begin{bmatrix} 1 & -1 & -2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ By $\xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & -1 & -2 & -3 \\ 0 & 5 & 8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

By $\xrightarrow{\substack{C_2 + C_1 \\ C_3 + 2C_1 \\ C_4 + 3C_1}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 8 & 14 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$ By $\xrightarrow{R_{24}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 5 & 8 & 14 \end{bmatrix}$ By $\xrightarrow{\substack{R_3 - 3R_2 \\ R_4 - 5R_2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 4 \end{bmatrix}$

By $\xrightarrow{C_4 - 2C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 8 & 4 \end{bmatrix}$ By $\xrightarrow{R_4 - 8R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 12 \end{bmatrix}$ By $\xrightarrow{C_4 + 2C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 12 \end{bmatrix}$

By $\xrightarrow{\left(\frac{1}{12}\right)R_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$ which is in normal form. $\therefore$ Rank of $A = 4$.

Example 4 (b) : Reduce the following matrix to normal form and find its rank.

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

(M.U. 1997, 2004)

Sol. : We have $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ By $\xrightarrow{R_{12}} \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

By $\xrightarrow{\substack{C_2 + C_1 \\ C_3 + 2C_1 \\ C_4 + 4C_1}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 3 & 7 \\ 3 & 4 & 9 & 10 \\ 6 & 9 & 12 & 17 \end{bmatrix}$ By $\xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 + 6R_1}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$

By $\xrightarrow{R_2 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$ By $\xrightarrow{\substack{C_3 + 6C_2 \\ C_4 + 3C_2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 33 & 22 \\ 0 & 9 & 66 & 44 \end{bmatrix}$

$$\xrightarrow[\text{By } R_4-9R_2]{R_3-4R_2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&33&22\\0&0&66&44\end{bmatrix}\quad \xrightarrow{\text{By } R_4-2R_3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&33&22\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow{\text{By } \frac{1}{11}R_3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&3&2\\0&0&0&0\end{bmatrix}\quad \xrightarrow{\text{By } C_3-C_4}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&2\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow{\text{By } C_4-2C_3}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}I_3&0\\0&0\end{bmatrix}$$ which is in normal form. $\therefore$ Rank of $A=3$.

Example 5 (b) : Reduce the matrix to normal form and find its rank.

$$\begin{bmatrix}2&3&4&5\\3&4&5&6\\4&5&6&7\\9&10&11&12\end{bmatrix}$$ **(M.U. 1993, 2003)**

Sol. : We have $$A=\begin{bmatrix}2&3&4&5\\3&4&5&6\\4&5&6&7\\9&10&11&12\end{bmatrix}$$

$$\xrightarrow[\substack{R_3-R_2\\R_4-R_3}]{\text{By } R_2-R_1}\begin{bmatrix}2&3&4&5\\1&1&1&1\\1&1&1&1\\5&5&5&5\end{bmatrix}\quad \xrightarrow[\substack{R_3-R_2\\R_4-5R_2}]{\text{By } R_1-2R_2}\begin{bmatrix}0&1&2&3\\1&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\quad \xrightarrow{\text{By } R_{12}}\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow[\substack{C_3-C_2\\C_4-C_3}]{\text{By } C_2-C_1}\begin{bmatrix}1&0&0&0\\0&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\quad \xrightarrow[C_4-C_3]{\text{By } C_3-C_2}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}I_2&0\\0&0\end{bmatrix}$$ which is in normal form.

$\therefore$ Rank of $A=2$.

Example 6 (b) : Reduce the following matrix to normal form and hence find its rank.

$$\begin{bmatrix}2&-4&3&1&0\\1&-2&1&-4&2\\0&1&-1&3&1\\4&-7&4&-4&5\end{bmatrix}$$ **(M.U. 2004)**

Sol. : We have $$A=\begin{bmatrix}2&-4&3&1&0\\1&-2&1&-4&2\\0&1&-1&3&1\\4&-7&4&-4&5\end{bmatrix}$$

$$\xrightarrow{\text{By } R_{21}} \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix} \quad \xrightarrow[R_4 - 4R_1]{\text{By } R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 1 & -4 & 2 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & -3 \end{bmatrix}$$

$$\xrightarrow[\substack{C_3 - C_1 \\ C_4 + 4C_1 \\ C_5 - 2C_1}]{\text{By } C_2 + 2C_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & 0 & 12 & 3 \end{bmatrix} \quad \xrightarrow{\text{By } R_4 - (R_2 + R_3)} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{By } R_{23}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[\substack{C_4 - 3C_2 \\ C_5 - C_2}]{\text{By } C_3 + C_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[C_5 + 4C_3]{\text{By } C_4 - 9C_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ which is in normal form. $\therefore$ Rank of $A = 3$.

Example 7 (b) : Reduce the following matrix to normal form and find its rank.

$$\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$$ **(M.U. 2003)**

Sol. : We have $$A = \begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}.$$

$$\xrightarrow{\text{By } R_{12}} \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 7 & 12 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix} \quad \xrightarrow{\text{By } \frac{1}{3}R_3} \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \end{bmatrix}$$

$$\xrightarrow{\text{By } R_3 - R_1} \begin{bmatrix} 1 & 1 & 2 & 3 & 5 \\ 3 & 2 & 5 & 7 & 12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[\substack{C_3 - 2C_2 \\ C_4 - 3C_1 \\ C_5 - 5C_1}]{\text{By } C_2 - C_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{By } R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[\substack{C_4 - 2C_1 \\ C_5 - 3C_1}]{\text{By } C_3 - C_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{By } (-1)R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$ which is in normal form. $\therefore$ Rank of $A = 2$.

Example 8 (b) : Reduce the following matrix to normal form and hence find its rank.

$$\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$$

Sol. : By $\begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ By $\begin{matrix} C_2 - 2C_1 \\ C_3 + 2C_1 \\ C_4 - 3C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

By $R_3 + R_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$ By $C_{34} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

By $\frac{1}{3}C_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$ which is in normal form. $\therefore$ Rank of $A = 4$.

Example 9 (b) : Find the rank of the matrix, $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 3 & 4 & -1 & 3 \\ 6 & 10 & 4 & 6 \end{bmatrix}$.

Sol. : By $\xrightarrow[R_4 - (R_1 + R_2 + R_3)]{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}}$ $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 0 & -3 & -1 & 0 \\ 0 & -5 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now, $\begin{vmatrix} 1 & 3 & 2 \\ 0 & -3 & -1 \\ 0 & -5 & -7 \end{vmatrix} = 21 - 5 = 16 \neq 0.$ $\therefore$ Rank of $A = 3$.

Example 10 (b) : Reduce the following matrix to normal form and hence find its rank.

$$A = \begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$$ **(M.U. 2012)**

Sol. : By $\xrightarrow{R_{12}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 1 \\ 3 & 3 & 3 & 1 \\ 1 & 4 & 2 & 0 \\ 0 & -4 & -1 & 2 \end{bmatrix}$ By $\xrightarrow[R_4 - R_1]{\substack{R_2 - 2R_1 \\ R_3 - 3R_1}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & -4 & -1 & 2 \end{bmatrix}$

$$\xrightarrow{\text{By } R_5 + R_4} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[C_4 - 2C_1]{\text{By } C_3 - C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 3 & 0 & -5 \\ 0 & 4 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 + 4R_2]{\text{By } R_3 + 3R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[C_4 - 3C_2]{\text{By } C_3 - C_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{By } R_4 - R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -3 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[C_4 - \frac{14}{3}C_3]{\text{By } -R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{By } -\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$ which is in normal form. $\therefore$ Rank of $A = 3$.

Example 11 (b) : Reduce A to normal form and find its rank where

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$$ **(M.U. 2002)**

Sol. : $\xrightarrow[\substack{R_3 - 15R_1 \\ R_4 - 6R_1}]{\text{By } R_2 - 4R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -6 & -12 & -18 \\ 0 & -18 & -36 & -54 \\ 0 & -6 & -12 & -18 \end{bmatrix} \quad \xrightarrow[\substack{R_4 - R_2 \\ (-1/6)R_2}]{\text{By } R_3 - 3R_2} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\xrightarrow[\substack{C_3 - 5C_1 \\ C_4 - 7C_1}]{\text{By } C_2 - 3C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \xrightarrow[C_4 - 3C_1]{\text{By } C_3 - 2C_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

which is in normal form. $\therefore$ Rank of $A = 2$.

Example 12 (b) : Find the rank of the matrix.

$$A = \begin{bmatrix} 3 & 2 & 1 & 4 \\ -1 & 3 & 2 & 2 \\ 2 & 5 & 3 & 6 \\ 5 & 7 & 4 & 10 \end{bmatrix}$$

5. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 4 & 3 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(M.U. 2015)**

6. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ **(M.U. 2005)**

7. $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

8. $\begin{bmatrix} 3 & -2 & 0 & 1 \\ 0 & 2 & 2 & 7 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$ **(M.U. 2016)**

9. $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix}$ **(M.U. 2005)**

10. $\begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}$ **(M.U. 2007, 09)**

11. $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$

12. $\begin{bmatrix} 3 & 1 & 2 & 1 \\ 2 & 0 & 3 & 2 \\ 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 1 \\ 2 & 4 & 6 & 8 \end{bmatrix}$

[**Ans. :** **(1)** $r = 2$, **(2)** $r = 2$, **(3)** $r = 3$, **(4)** $r = 3$, **(5)** $r = 3$, **(6)** $r = 3$, **(7)** $r = 2$, **(8)** $r = 4$, **(9)** $r = 2$, **(10)** $r = 4$, **(11)** $r = 3$, **(12)** $r = 4$.]

Class (b) : 6 Marks

(B) Find the ranks of the following matrices.

1. $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 6 \end{bmatrix}$ **(M.U. 1994)**

2. $\begin{bmatrix} 6 & 1 & 3 & 6 \\ 4 & 2 & 6 & 1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 13 \end{bmatrix}$ **(M.U. 2005)**

3. $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$ **(M.U. 2005)**

4. $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 2 & 3 & 0 \\ 9 & 8 & 0 & 8 \end{bmatrix}$ **(M.U. 2003)**

5. $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

6. $\begin{bmatrix} 3 & 4 & -2 & 1 \\ 5 & 8 & 4 & 2 \\ 8 & 12 & 2 & 3 \\ 13 & 20 & 6 & 5 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 6 & -2 & 6 & 10 \\ -3 & 3 & -3 & -3 & -3 \\ 1 & -2 & 4 & 3 & 5 \\ 2 & 0 & 4 & 6 & 10 \\ 1 & 0 & 2 & 3 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 3 & 2 & -4 & 3 & 6 \\ 1 & -2 & 3 & 4 & -3 \\ 2 & -4 & 6 & 8 & -6 \\ 3 & -6 & 9 & 12 & -9 \\ 5 & -2 & 2 & 11 & 0 \end{bmatrix}$

9. $\begin{bmatrix} 25 & 31 & 17 & 43 \\ 75 & 94 & 53 & 132 \\ 75 & 94 & 54 & 134 \\ 25 & 32 & 20 & 48 \end{bmatrix}$ **(M.U. 2003)**

[**Ans. : (1)** $r = 3$, **(2)** $r = 2$, **(3)** $r = 2$, **(4)** $r = 3$, **(5)** $r = 3$, **(6)** $r = 2$, **(7)** $r = 3$, **(8)** $r = 2$, **(9)** $r = 3$.]

Class (b) : 6 Marks

(C) **1.** For the following matrix A, find the value of k for which the rank of A is 3. Also find the ranks of A for the remaining values of k.

$$A = \begin{bmatrix} 1 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2k+2 & 3k+4 \end{bmatrix}$$ **(M.U. 2015)**

[**Ans. : (i)** For $k \neq \pm 2$, rank of A is 3. **(ii)** For $k = \pm 2$, rank $= 2$]

2. Find the rank of the matrix given below by using only row transformations.

$$\begin{bmatrix} 0 & b-a & c-a & b+c \\ a-b & 0 & c-b & c+a \\ a-c & b-c & 0 & a+b \\ b+c & c+a & a+b & 0 \end{bmatrix}$$ **(M.U. 2003)**

(**Hint :** Take $R_1 + R_4$, $R_2 + R_4$, $R_3 + R_4$. Take $R_1 / (a+b)$, $R_2 / (a+c)$, $R_3 / (a+b)$. Then $R_1 - R_3$, $R_2 - R_3$. Consider any minor of order 2. **Rank = 2.**)

3. Find the possible values of k for which the rank of A is 1, 2, 3, where

$$A = \begin{bmatrix} k & 4 & 4 \\ 4 & k & 4 \\ 4 & 4 & k \end{bmatrix}$$ **(M.U. 2004, 11)**

[**Ans. :** **(i)** If $k^3 - 48k + 128 \neq 0$, the rank is 3,
(ii) If $k^3 - 48k + 128 = 0$, but $k \neq 4$, the rank is 2,
(iii) If $k = 4$, the rank is 1.]

Class (a) : 4 Marks

(D) 1. Find the rank of the matrix given below.

$$\begin{bmatrix} 6 & 1 & 3 & 8 \\ 3 & 2 & 6 & -1 \\ 9 & 3 & 9 & 7 \\ 15 & 4 & 12 & 15 \end{bmatrix}$$ **(M.U. 2002) [Ans. : 2]**

2. If x is a rational number, find the rank of $A - xI$ where I is the identity matrix of order 3, and

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$ **(M.U. 2002) [Ans. : $r = 3$ if $x \neq 1$]**

3. Find the rank of the matrix given below.

$$\begin{bmatrix} 25 & 31 & 17 & 43 \\ 75 & 94 & 53 & 132 \\ 75 & 94 & 54 & 134 \\ 25 & 32 & 20 & 48 \end{bmatrix}$$ **(M.U. 2003, 10)**

(**Hint :** Take $R_2 - 3R_1$, $R_3 - 3R_1$, $R_4 - R_1$. Then take $R_4 - R_3$.) **[Ans. : Rank = 3]**

4. Find the rank of the matrix given below.

$$\begin{bmatrix} 37 & 41 & 53 & 65 \\ 74 & 83 & 108 & 133 \\ 111 & 124 & 161 & 198 \\ 37 & 42 & 55 & 68 \end{bmatrix}$$

(**Hint :** Take $R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - R_1$. Then $R_3 - R_2$, $R_4 - R_2$.) **[Ans. : Rank = 2]**

5. If $A = [a_{ij}]$ is a square matrix of order 3 where $a_{ij} = i + j$, find the rank of A. **(M.U. 2003)**

(**Hint :** $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$\because \quad a_{ij} = i + j, \; a_{11} = 1 + 1, \; a_{12} = 1 + 2$, etc.) [**Ans. :** Rank = 2]

6. Find the rank of $A = \begin{bmatrix} x-1 & x+1 & x \\ -1 & x & 0 \\ 0 & 1 & 1 \end{bmatrix}$ where x is real. **(M.U. 2003)** [**Ans. :** Rank = 3]

7. If A is the matrix given in Ex. 11, page 9-44 and if $f(x) = x^2 - 5x + 6$, check whether $f(A)$ is singular. **(M.U. 2002)**

(**Hint :** $f(x) = (x-3)(x-2)$

$\therefore \quad |f(A)| = |A - 3| \cdot |A - 2|$

Show that $|A - 3|$ has two rows identical and hence, it is zero.

$\therefore \quad |f(A)| = 0$. Hence, $f(A)$ is singular.)

8. Find the ranks of the following matrices.

(1) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 \end{bmatrix}$ **(2)** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 8 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$ **(3)** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 7 & 9 & 11 & 13 \\ 2 & 4 & 6 & 8 \end{bmatrix}$

[**Ans. :** **(1)** 4, **(2)** 1, **(3)** 2]

9. Find the rank of

(i) $A = [a_{ij}]_{3 \times 3}$ where $a_{ij} = \dfrac{i}{j}$. **(M.U. 2006)**

(ii) $A = [a_{ij}]_{3 \times 3}$ where $a_{ij} = i - j$.

(iii) $A = [a_{ij}]_{3 \times 3}$ where $a_{ij} = i + j$. [**Ans. :** (i) 1, **(ii)** 2, **(iii)** 2.]

15. Reduction of a Matrix A to Normal Form PAQ

Theorem : If A is a matrix of rank r, then there exist non-singular matrices P and Q such that PAQ is in the normal form

i.e. $$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

To obtain the matrices P and Q use the following procedure.

Working Rule

1. If A is an $m \times n$ matrix, write $\boxed{A = I_m A I_n}$

Scanned by CamScanner

2. Now, apply only row operations on A on l.h.s. and the same row operations on the prefactor I_m in such a way that $a_{11} = 1$ and all other elements in the first column of I_m are zero. For this if $a_{11} = 1$, we perform operations of the type $R_2 - k_1 R_1$, $R_3 - k_2 R_2$, etc.
3. Then, apply only column operations on A on l.h.s. and the same column operations on the post factor I_n in such a way that all elements in the first row of I_n except a_{11} (= 1) are zero. For this we perform operations of the type $C_2 - k_1' C_1$, $C_3 - k_2' C_2$, etc.
4. Repeat the same type of operations on the remaining rows and columns of A and also on I_m and I_n, so that A is reduced to $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$, find two matrices P and Q such that PAQ is in normal form. **(M.U. 1999, 2011)**

(Find also the rank of A).

Sol. : We first write $A = I_3 A I_3$

i.e. $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - 3R_1$ $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_2 - C_1$, $C_3 - C_1$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $-\frac{1}{2} R_2$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + 2R_2$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_3 - C_2$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ -2 & -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

(The rank of the matrix is 2).

Example 2 (b) : For the following matrix A find non-singular matrices P and Q such that PAQ is in normal form and hence or otherwise find the rank of A.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

Sol. : Let $A = I_3 \; A I_3$ i.e., $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} C_2 - C_1 \\ C_3 - 2C_1 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 - R_1$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $C_3 - C_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_3 + R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore \quad \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$

$\therefore \quad P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Rank of $A = 2$.

Example 3 (b) : Find non-singular matrices P and Q such that $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$ is reduced to normal form. Also find its rank. **(M.U. 2006, 07, 09, 16, 18)**

Sol. : We first write $A = I_3 A I_4$

i.e. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

By $R_2 - 2R_1$ and $R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $C_2 - 2C_1$, $C_3 - 3C_1$, $C_4 - 4C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & 0 & 0 & -12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By $-\frac{1}{3}C_2, -\frac{1}{2}C_3, -\frac{1}{5}C_5$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 3/2 & 4/5 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & -1/5 \end{bmatrix}$$

By $C_3 - C_2$, $C_4 - C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 12/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 5/6 & -7/10 \\ 0 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & -1/5 \end{bmatrix}$$

By $(5/12)\ C_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & 5/6 & -7/24 \\ 0 & -1/3 & 1/3 & 0 \\ 0 & 0 & 1/2 & -5/24 \\ 0 & 0 & 0 & -1/12 \end{bmatrix}$$

By C_{34}

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2/3 & -7/24 & 5/6 \\ 0 & -1/3 & 0 & 1/3 \\ 0 & 0 & -5/24 & 1/2 \\ 0 & 0 & -1/12 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = 3$.

Example 4 (b) : Find non-singular matrices P and Q, such that $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ is reduced to normal form. Also find its rank. **(M.U. 2008, 18, 19)**

Sol. : We first write $A = I_3\, A\, I_4$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } \begin{matrix} C_2 - 2C_1 \\ C_3 - 3C_1 \\ C_4 - 2C_1 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } R_2 + R_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } \begin{matrix} C_3 - C_2 \\ C_4 - 3C_2 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } -R_2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ \qquad \therefore$ The rank of $A = 2$.

Example 5 (b) : Find non-singular matrices P and Q such that PAQ is in normal form. Also find the rank of A, and A^{-1}.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$ **(M.U. 2002, 08, 09, 10, 11, 15)**

Sol. : $A = I_3 A I_3$ i.e. $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\text{By } \xrightarrow{R_2 + R_1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } \xrightarrow{C_2 - 2C_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{By } \xrightarrow{C_3 + 2C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $C_2 + 2C_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

By $R_2 + 2R_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ which is in normal form.

$\therefore$ Rank of $A = 3$.

To find A^{-1} we see that, $I = PAQ \quad \therefore \; AQ = P^{-1} \quad \therefore \; A^{-1}AQ = A^{-1}P^{-1} \quad \therefore \; Q = A^{-1}P^{-1}$.

$\therefore \; QP = A^{-1}P^{-1}P = A^{-1}$

Now, $QP = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \quad \therefore \; A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

EXERCISE - XII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

(A) Find non-singular matrices P and Q such that PAQ is in normal form where A is

1. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ **(M.U. 2016)**

2. $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ **(M.U. 2000, 02, 03, 17)**

4. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 3 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ **(M.U. 2013)**

6. $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ **(M.U. 1995)**

7. $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 0 & 2 & 2 \\ 4 & 1 & 9 & 7 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 3 & 4 & 7 \\ -3 & 4 & 7 & -9 \\ 5 & 4 & 6 & -5 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 4 & 8 & 0 & 12 \\ 5 & 4 & 9 & 9 \end{bmatrix}$ **(M.U. 2002)**

Class (b) : 6 Marks

(B) Find non-singular matrices P and Q such that PAQ is in normal form. Also find their ranks.

1. $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$ **(M.U. 1998, 2005)**

2. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 8 & 5 & 8 \end{bmatrix}$ **(M.U. 2004, 08, 10)**

3. $\begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$ **(M.U. 2003, 13)**

4. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 1 & 0 & 6 \end{bmatrix}$ **(M.U. 2003)**

5. $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ **(M.U. 2004, 07)**

6. $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 2 & 3 & 6 & 4 \\ 6 & 5 & 15 & 10 \end{bmatrix}$ **(M.U. 2004)**

7. $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(M.U. 1997, 2005)**

8. $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$ **(M.U. 2002, 06)**

9. $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 0 & 2 & 2 \\ 4 & 1 & 9 & 7 \end{bmatrix}$ **(M.U. 2003)**
10. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 1 & 0 & 6 \end{bmatrix}$ **(M.U. 2003)**
11. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ **(M.U. 2002)**
12. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 3 \\ 5 & 6 & 10 & 2 \end{bmatrix}$ **(M.U. 2002)**

[**Ans. :** **(1)** $r = 2$, **(2)** $r = 2$, **(3)** $r = 3$, **(4)** $r = 2$, **(5)** $r = 3$, **(6)** $r = 3$, **(7)** $r = 2$, **(8)** $r = 2$, **(9)** $r = 3$, **(10)** $r = 2$, **(11)** $r = 3$, **(12)** $r = 3$.]

(C) Find non-singular matrices P and Q such that PAQ is in normal form. Hence, find A^{-1}.

1. $\begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ **(M.U. 1998, 2012)**

[**Ans. :** **(1)** $\frac{1}{6}\begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$ **(2)** $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ **(3)** $\begin{bmatrix} 3/2 & -1/4 & -9/4 \\ -1 & 1/2 & 3/2 \\ 1/2 & -1/4 & -1/4 \end{bmatrix}$

Class (b) : 6 Marks

(D) **1.** If A and B are as given below, find the rank of A by reducing it to normal form. Find $3A - B$. Hence, or otherwise, show that $3A^2 - AB = 2A$. Also find the rank of $3A^2 - AB$.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 1 & 1 \\ 2 & 6 & 3 & 5 \\ 2 & 4 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 & 3 & 6 \\ 0 & 4 & 3 & 3 \\ 6 & 18 & 7 & 15 \\ 6 & 12 & 6 & 10 \end{bmatrix}$$ **(M.U. 2005)**

[**Ans. :** Rank of $A = 2 =$ rank of $3A^2 - AB$.]

2. If A is an orthogonal matrix, find a, b, c, where $A = \frac{1}{3}\begin{bmatrix} a & b & c \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$.

Hence, find the inverse of A. Also find the ranks of A^2 and $3A$. **(M.U. 2004)**

[**Ans. :** $a = \pm 1$, $b = \pm 2$, $c = -1$, $A^{-1} = A$. And $|A| = \pm 1 \neq 0$.
$|A^2| = |A| \cdot |A| \neq 0$. $\therefore$ Rank of $A = 3 =$ Rank of $A^2 =$ Rank of $3A$.]

EXERCISE - XIII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (a) : 4 Marks

1. Define the following terms

(i) Transpose of a matrix. **(ii)** Triangular matrix. **(iii)** Conjugate of a matrix. **(iv)** Transposed conjugate of a matrix. **(v)** Symmetric and Skew-symmetric matrix. **(vi)** Hermitian and Skew-Hermitian matrix. **(vii)** Adjoint of a square matrix. **(viii)** Inverse of a matrix. **(ix)** Singular matrix. **(x)** Rank of a matrix. **(xi)** Normal form or Canonical form. **(xii)** Elementary transformation of a matrix.

2. Prove that, the necessary and sufficient condition for a square matrix to be symmetric is $A = A'$.

3. Prove that, the necessary and sufficient condition for a square matrix to be skew-symmetric is $A' = -A$.

4. Prove that the necessary and sufficient condition for a square matrix A to be Hermitian is $A^{\theta} = A$.

5. Prove that, the necessary and sufficient condition for a square matrix A to be Skew-Hermitian is $A^{\theta} = -A$.

6. Prove that the diagonal elements of a skew-symmetric matrix are zero. **(M.U. 2000)**

7. Prove that the diagonal elements of a Hermitian matrix are real.

8. Prove that the diagonal elements of a Skew-Hermitian matrix are imaginary numbers or zero.

9. Prove that if A is Hermitian (Skew-Hermitian) then iA is Skew-Hermitian (Hermitian). **(M.U. 2003, 06)**

10. Prove that every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix. **(M.U. 1998)**

11. Prove that, every square matrix can be uniquely expressed as the sum of a Hermitian and a Skew-Hermitian matrix. **(M.U. 1996, 97, 99)**

12. Prove that, every square matrix A can be uniquely expressed as $P + iQ$, where P and Q are Hermitian matrices.

13. Prove that, every Hermitian matrix A can be expressed as $B + iC$, where B and C are real symmetric and skew-symmetric matrices respectively. **(M.U. 1997)**

14. Prove that, every Skew-Hermitian matrix A can be written as $B + iC$, where B is real skew-symmetric and C is real symmetric matrix.

15. If A, B are orthogonal square matrices of order n then prove that AB is also orthogonal. **(M.U. 2003)**

EXERCISE - XIV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Short Answer Questions

(A) 1. State true or false (2 Marks)

(i) Every diagonal matrix is a square matrix.

(ii) In a diagonal matrix all non-diagonal elements are zero.

(iii) In a diagonal matrix no diagonal element is zero.

(iv) In a diagonal matrix all diagonal elements are equal.

(v) The trace of a zero matrix is zero.

(vi) The trace of unit matrix of order n is n.

(vii) The determinant of a zero square matrix is zero.

(viii) The determinant of every unit matrix is 1.

(ix) The determinant of a singular matrix is zero.

(x) All diagonal elements of a scalar matrix are equal.

(xi) All diagonal elements of a diagonal matrix are equal.

(xii) Every diagonal matrix is a scalar matrix. **(M.U. 2002)**

(xiii) Every scalar matrix is a diagonal matrix.

(xiv) The determinants of A and A' are equal.

(xv) The determinant of a square triangular matrix is equal to the product of its diagonal elements.

(xvi) Every square triangular matrix having one diagonal element zero is singular.

[**Ans. :** (i) True, (ii) True, (iii) False, (iv) False, (v) True, (vi) True, (vii) True, (viii) True, (ix) True, (x) True, (xi) False, (xii) False, (xiii) True, (xiv) True, (xv) True, (xvi) True.]

2. Prove that the diagonal elements of a skew-symmetric matrix are all zero.

[**Ans. :** § 3 (16) (b) page 9-5]

3. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ and $f(x) = x^2 - 5x + 6$, find $f(A)$. Is $f(A)$ non-singular ? **(M.U. 2002)**

[**Ans. :** $f(A) = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$, $|f(A)| = 9 \neq 0 \quad \therefore f(A)$ is non-singular]

4. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, show that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ where n is a positive integer. **(M.U. 2002)**

(**Hint :** Find A^2 and then use mathematical induction.)

(B) State whether the following statements are true or false. Give an example in support of your answer. (2 Marks)

1. All diagonal elements of a triangular matrix are zero.

2. All diagonal element of a triangular matrix are unity.

3. All elements above the leading diagonal of a lower triangular matrix are zero.

4. All elements below the leading diagonal of an upper triangular matrix are zero.

5. For a symmetric matrix $A = \overline{A}$.

6. All diagonal elements of a symmetric matrix are zero.

7. All diagonal elements of a skew-symmetric matrix are zero. **(M.U. 2000, 04)**

8. For a skew-symmetric matrix $\overline{A} = -A$.

9. All diagonal elements of Hermitian matrix are real. **(M.U. 2000, 04)**

10. For a Hermitian matrix $A = A^\theta$.

11. The diagonal elements of a Skew-Hermitian matrix are either imaginary or zero.

12. For a Skew-Hermitian matrix $A^\theta = -A$.

13. If A is orthogonal then $|A| = \pm 1$.

14. If A is orthogonal then A^{-1} does not exist.

15. If A is orthogonal then $A^{-1} = A'$.

16. If A and B are two orthogonal square matrices of order n then AB also orthogonal.

17. Modulus of the determinant of unitary matrix is unity.

18. If A is unitary then A^{-1} is not unitary.

19. If A is unitary then A' is not unitary.

20. If A is a non-singular matrix and $AB = AC$, then $B = C$. **(M.U. 2000)**

21. If $AB = AC$ and $B = C$, then $|A| \neq 0$.

22. If A is a skew-symmetric matrix and X is a column matrix then $X'AX = 0$. **(M.U. 2002)**

23. If A is skew-Hermitician then iA is Hermitian. **(M.U. 2002)**

24. If $A = [a_{ij}]$ is a square matrix of order 3 where $a_{ij} = i/j$, then the rank of A is 3. **(M.U. 2004)**

25. If $A = [a_{ij}]$ is a square matrix or order 3 where $a_{ij} = i + j$, then the rank of $A = 2$. **(M.U. 2003)**

26. Every orthogonal matrix is non-singular. **(M.U. 2004)**

27. If A is an orthogonal matrix of order 3, then the rank of A is 3.

28. The rank of unit matrix or order 10 is 10.

29. The rank of a null matrix of any order is zero.

30. If A is a row matrix with one element unity then the rank of A is 1. **(M.U. 2003)**

31. If A is a column matrix with one element unity then the rank of A is 1.

32. If each element of a square matrix of order n (> 1) is n, then the rank of A is n. **(M.U. 2003)**

33. If each element of a square matrix of order n (> 1) is n then the rank of A is 1.

34. If A is a symmetric matrix then adj. A is also symmetric.

35. If A is a non-singular matrix then the rank of A = rank of A^2. **(M.U. 2003)**

36. If A is a square matrix then adj. adj. $A = A$. **(M.U. 2003)**

37. If A is a square matrix of order 3 having each element 2, then the rank of A is 2.

[**Ans. : (1)** False, **(2)** False, **(3)** True, **(4)** True, **(5)** True, **(6)** False, **(7)** True, **(8)** True, **(9)** True, **(10)** True, **(11)** True, **(12)** True, **(13)** True, **(14)** False, **(15)** True, **(16)** True, **(17)** True, **(18)** False, **(19)** False, **(20)** True, **(21)** True, **(22)** True, **(23)** True, **(24)** False, **(25)** True, **(26)** True, **(27)** True, **(28)** True, **(29)** True, **(30)** True, **(31)** True, **(32)** False, **(33)** True, **(34)** True, **(35)** True, **(36)** True, **(37)** False.]

2. Give one example each of a 3×3 real symmetric matrix B and real skew-symmetric matrix C and verify that $B + iC$ is a Hermitian matrix. **(M.U. 2004)**

3. Give one example each of a 3×3 real skew-symmetric matrix B, and real symmetric matrix C and verify that $B + iC$ is skew-Hermitian.

4. Do as directed.

(i) Find the value of x for which the following matrix is singular.

$$\begin{bmatrix} 6 & x & 0 \\ 3 & 0 & 2 \\ 18 & 6 & 0 \end{bmatrix}$$

[**Ans. :** 2]

(ii) Find the condition that the following matrix is unitary.

$$\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

[**Ans. :** $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$]

Summary

1. **Orthogonal Matrix :** $AA' = A'A = I.$ Hence, $A^{-1} = A'$
2. **Unitary Matrix :** $A^{\theta} A = AA^{\theta} = I.$ Hence, $A^{-1} = A^{\theta}$
3. **Echelon Form :**
$$\begin{bmatrix} a & * & * & * & * & * & * \\ 0 & b & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * \\ 0 & 0 & 0 & 0 & 0 & d & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
4. **Normal Form :**
$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad e.g., \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

☆ ☆ ☆

CHAPTER 10

Linear Equations

1. Introduction

In this chapter we shall learn what is meant by linear dependence and linear independence of vectors. This concept will be required while solving linear equations. We shall then learn the methods of solving homogeneous and non-homogeneous linear equations using matrix methods.

2. Vectors

We know that an ordered pair (x_1, y_1) represents a vector in two dimensional geometry, an ordered triple $(x_1\ y_1, z_1)$ represents a vector in three dimensional geometry. Similarly, an ordered quadruple (x_1, x_2, x_3, x_4) represents a vector in four dimensional geometry. In this way, we can think of an ordered n tuple of numbers $(x_1, x_2, x_3, \ldots\ldots, x_n)$. An ordered n-tuple of numbers is called an ***n*-vector**. In other words, an ordered set of n numbers is an n-vector. Thus, the n numbers $x_1, x_2, x_3, \ldots\ldots x_n$ taken in order denote a vector X. The numbers $x_1, x_2, \ldots\ldots x_n$ themselves are called **components** of the vector X. A vector may be written as a *row vector* or as a *column vector*. If A is an $m \times n$ matrix then each row will be an n-vector and each column will be an m-vector.

(a) Linear Dependence

A set of r vectors $X_1, X_2, X_3, \ldots\ldots X_r$ is said to be **linearly dependent** if there exist r numbers $k_1, k_2, k_3, \ldots\ldots k_r$ **not all zero** such that

$$k_1X_1 + k_2X_2 + k_3X_3 + \ldots\ldots + k_rX_r = O$$

Here, O denotes zero vector whose all components are zero. Such vectors are called **linearly dependent** because then at least one vector can be expressed as a linear combination of the others. For example, if $k_1 \neq 0$ then we can express X_1 as

$$X_1 = -\frac{1}{k_1}(k_2X_2 + k_3X_3 + \ldots\ldots + k_rX_r) \qquad \text{[See Examples, 1, 2, 3, 6, 7 below.]}$$

(b) Linear Independence

A set of r vectors $X_1, X_2, X_3, \ldots\ldots X_r$ is said to be linearly independent if every relation of the type

$$k_1X_1 + k_2X_2 + k_3X_3 + \ldots\ldots + k_rX_r = O$$

implies that $k_1 = k_2 = k_3 = \ldots\ldots = k_r = O$.

Here, as above O denotes zero vector whose all r components are zero. [See Examples 4 and 5 below.]

(c) Linear Combination of Vectors

A vector X which can be expressed in the form

$$X = k_1X_1 + k_2X_2 + \ldots\ldots + k_rX_r$$

where $k_1, k_2, k_3, \ldots\ldots k_r$ are any numbers is said to be a **linear combination** of the vectors $X_1, X_2, \ldots\ldots X_r$.

The following two results are clear :

1. If a set of vectors is linearly dependent then at least one member can be expressed as a linear combination of the remaining vectors.

2. If a set of vectors is linearly independent then no member of the set can be expressed as a linear combination of the other members.

Now, we shall consider problems in which we shall be required to check whether the given vectors are linearly dependent or independent. For this, we write the given set of vectors in matrix form, consider a matrix equation, perform row operations only and transform the matrix in the following form

$$\begin{bmatrix} a_{11} & * & * & * \\ 0 & a_{22} & * & * \\ 0 & 0 & a_{33} & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then equate the two sides of the equation and get the required result. This is easily understood through the following equations.

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Are the vectors $X_1 = [1, 3, 4, 2]$, $X_2 = [3, -5, 2, 6]$, $X_3 = [2, -1, 3, 4]$ linearly dependent ? If so, express X_1 as a linear combination of the others. **(M.U. 2002, 10, 15)**

Sol. : Consider the matrix equation

$$k_1X_1 + k_2X_2 + k_3X_3 = O \qquad \text{.....................} \text{ (I)}$$

$$k_1[1, 3, 4, 2] + k_2[3, -5, 2, 6] + k_3[2, -1, 3, 4] = [0, 0, 0, 0]$$

$$\therefore \quad k_1 + 3k_2 + 2k_3 = 0, \quad 3k_1 - 5k_2 - k_3 = 0,$$

$$4k_1 + 2k_2 + 3k_3 = 0, \quad 2k_1 + 6k_2 + 4k_3 = 0.$$

This can be written in matrix form as

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 6 & 4 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - 3R_1 \\ R_3 - 4R_1 \\ R_4 - 2R_1 \end{matrix}$ $\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} (-1/7)R_2 \\ (-1/5)R_3 \end{matrix}$ $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_3 - R_2$ $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$k_1 + 3k_2 + 2k_3 = 0, \quad 2k_2 + k_3 = 0.$$

If we put $k_3 = -2t$, $k_2 = t$, $k_1 = t$.

Now, form (i), we get $tX_1 + tX_2 - 2tX_3 = 0 \qquad \therefore \; X_1 + X_2 - 2X_3 = 0$

Since k_1, k_2, k_3 are not all zero, the vectors are linearly dependent and $X_1 = -X_2 + 2X_3$ is the relation between them.

Note ....

If the rank of the matrix of the coefficients is r, it contains r linearly independent vectors and the remaining vectors (if any) can be expressed as a linear combinations of these vectors.

In the above example $r = 2$ hence, the remaining one vector can be expressed in terms of these two independent victors.

Example 2 (b) : Determine the linear dependence or independence of vectors (2, –1, 3, 2), (1, 3, 4, 2) and (3, – 5, 2, 2). Find the relation between them if dependent.

(M.U. 1995, 97, 98, 2011)

Sol. : Consider the matrix equation

$$k_1X_1 + k_2X_2 + k_3X_3 = O \qquad \text{.....................(i)}$$

$$\therefore \quad k_1[2, -1, 3, 2] + k_2[1, 3, 4, 2] + k_3[3, -5, 2, 2] = [0, 0, 0, 0]$$

$$\therefore \quad 2k_1 + k_2 + 3k_3 = 0, \qquad -k_1 + 3k_2 - 5k_3 = 0,$$

$$3k_1 + 4k_2 + 2k_3 = 0, \qquad 2k_1 + 2k_2 + 2k_3 = 0.$$

This can be written as

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \xrightarrow{R_{12}} \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad (-1)R_1 \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1}} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 7 & -7 \\ 0 & 13 & -13 \\ 0 & 8 & -8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} (1/7)R_2 \\ (1/13)R_3 \\ (1/8)R_4 \end{matrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_3 - R_2 \\ R_4 - R_3 \end{matrix} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 - 3k_2 + 5k_3 = 0, \qquad k_2 - k_3 = 0$$

If we put $k_3 = t$, then $k_2 = t$ and $k_1 = 3k_2 - 5k_3 = 3t - 5t = -2t$.

Now from (i), we get $-2tX_1 + tX_2 + tX_3 = 0$

$$\therefore \quad 2tX_1 - tX_2 - tX_3 = 0$$

$$\therefore \quad 2X_1 - X_2 - X_3 = 0.$$

Since k_1, k_2, k_3 are not all zero, the vectors are linearly dependent and $X_1 = \frac{1}{2}X_2 + \frac{1}{2}X_3$ is the relation between them.

Example 3 (b) : Show that the vectors X_1, X_2, X_3 are linearly independent and vector X_4 depends upon them, where $X_1 = (1, 2, 4)$, $X_2 = (2, -1, 3)$, $X_3 = (0, 1, 2)$, $X_4 = (-3, 7, 2)$

(M.U. 1995, 96, 98)

Sol. : Consider the matrix equation

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = O$$

$$\therefore \quad k_1 [1, 2, 4] + k_2 [2, -1, 3] + k_3 [0, 1, 2] = [0, 0, 0]$$

$$\therefore \quad k_1 + 2k_2 + 0k_3 = 0, \quad 2k_1 - k_2 + k_3 = 0, \quad 4k_1 + 3k_2 + 2k_3 = 0.$$

This can be written as,

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 + 2k_2 = 0, \quad -5k_2 + k_3 = 0, \quad k_3 = 0.$$

$$\therefore \quad k_3 = 0, \quad k_2 = 0, \quad k_1 = 0.$$

Since, all k's are zero. $\therefore$ X_1, X_2, X_3 are linearly independent.

Now, consider the matrix equation

$$k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 = O \qquad \text{.................... (i)}$$

$$\therefore \quad k_1 [1, 2, 4] + k_2 [2, -1, 3] + k_3 [0, 1, 2] + k_4 [-3, 7, 2] = 0$$

$$\therefore \quad k_1 + 2k_2 + 0k_3 - 3k_4 = 0, \quad 2k_1 - k_2 + k_3 + 7k_4 = 0, \quad 4k_1 + 3k_2 + 2k_3 + 2k_4 = 0.$$

This can be written as

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \xrightarrow[R_3 - 4R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 + 2k_2 - 3k_4 = 0, \quad -5k_2 + k_3 + 13k_4 = 0, \quad k_3 + k_4 = 0.$$

If we put $k_4 = t$ $\qquad \therefore \quad k_3 = -t$

$$\therefore \quad -5k_2 - t + 13t = 0 \qquad \therefore \quad k_2 = \frac{12}{5} t$$

$$\therefore \quad k_1 + \frac{24}{5} t - 3t = 0 \qquad \therefore \quad k_1 = -\frac{9}{5} t$$

Putting the values of k_1, k_2, k_3, k_4 in (i), we get,

$$-\frac{9}{5} t X_1 + \frac{12}{5} t X_2 - t X_3 + t X_4 = 0$$

$\therefore \quad \frac{9}{5} X_1 - \frac{12}{5} X_2 + X_3 - X_4 = 0 \qquad \therefore \quad 9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$

Since k_1, k_2, k_3, k_4 are not zero, the vectors are linearly dependent.

$\therefore \quad X_4 = \frac{9}{5} X_1 - \frac{12}{5} X_2 + X_3$ is the relation between them.

Example 4 (b) : Examine whether the vectors $X_1 = [3, 1, 1]$, $X_2 = [2, 0, -1]$, $X_3 = [4, 2, 1]$ are linearly independent. **(M.U. 2003, 04, 05, 14, 16)**

Sol. : Consider the matrix equation

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = O$$

$$\therefore \quad k_1 [3, 1, 1] + k_2 [2, 0, -1] + k_3 [4, 2, 1] = [0, 0, 0]$$

$$\therefore \quad 3k_1 + 2k_2 + 4k_3 = 0, \quad k_1 + 0k_2 + 2k_3 = 0, \quad k_1 - k_2 + k_3 = 0.$$

This can be written as

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By} \quad R_{13} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{By} \quad R_3 - R_2 \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 - k_2 + k_3 = 0, \quad k_2 + k_3 = 0, \quad 4k_2 = 0$$

$$\therefore \quad k_2 = 0, \ k_3 = 0, \quad \therefore \ k_1 = 0.$$

Since, all k_1, k_2, k_3 are zero the vectors are linearly independent.

Example 5 (b) : Show that the following vectors are linearly independent

$X_1 = [1, 2, -1, 0]$, $X_2 = [1, 3, 1, 2]$, $X_3 = [4, 2, 1, 0]$, $X_4 = [6, 1, 0, 1]$. **(M.U. 2012)**

Sol. : Consider the matrix equation

$$k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 = O$$

$$\therefore \quad k_1 [1, 2, -1, 0] + k_2 [1, 3, 1, 2] + k_3 [4, 2, 1, 0] + k_4 [6, 1, 0, 1] = [0, 0, 0, 0]$$

$$\therefore \quad \begin{aligned} k_1 + k_2 + 4k_3 + 6k_4 &= 0 \\ 2k_1 + 3k_2 + 2k_3 + k_4 &= 0 \\ -k_1 + k_2 + k_3 &= 0 \\ 2k_2 + k_4 &= 0 \end{aligned}$$

This can be written as

$$\begin{bmatrix} 1 & 1 & 4 & 6 \\ 2 & 3 & 2 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 2 & 5 & 6 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 - 2R_2 \\ R_4 - R_3 \end{matrix} \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -11 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_4 + \frac{5}{17} R_3 \begin{bmatrix} 1 & 1 & 4 & 6 \\ 0 & 1 & -6 & -1 \\ 0 & 0 & 17 & 28 \\ 0 & 0 & 0 & 55/17 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad k_1 + k_2 + 4k_3 + 6k_4 = 0$$
$$k_2 - 6k_3 + k_4 = 0$$
$$17k_3 + 28k_4 = 0$$
$$\frac{55}{17} k_4 = 0$$
$$\therefore \quad k_4 = 0, \; k_3 = 0, \; k_2 = 0, \; k_1 = 0$$

Since all k's are zero, the vectors are independent.

Example 6 (b) : Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

Discuss and find the relation of linear dependence amongst its row vectors. **(M.U. 2006, 12)**

Sol. : Let the row vectors (1, 1, –1, 1) (1, –1, 2, –1) and (3, 1, 0, 1) be denoted by X_1, X_2, X_3 respectively. Now, consider the matrix equation,

$$k_1 X_1 + k_2 X_2 + k_3 X_3 = O \quad \text{.......................} (1)$$

$$\therefore \quad k_1 [1, 1, -1, 1] + k_2 [1, -1, 2, -1] + k_3 [3, 1, 0, 1] = [0, 0, 0, 0]$$

$$\therefore \quad k_1 + k_2 + 3k_3 = 0, \qquad k_1 - k_2 + k_3 = 0,$$
$$-k_1 + 2k_2 + 0k_3 = 0, \qquad k_1 - k_2 + k_3 = 0.$$

This can be written as

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{.......................} (2)$$

$$\text{By} \quad \xrightarrow{\begin{matrix} R_2 - R_1 \\ R_3 + R_2 \\ R_4 - R_2 \end{matrix}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \xrightarrow{\left(-\frac{1}{2}\right) R_2} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\quad R_3 - R_2 \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The matrix on the r.h.s. is in echelon form. There are two non-zero rows.

$\therefore$ The rank of the matrix = 2.

The given matrix A is the transpose of the matrix on the l.h.s. of (2). Hence, the rank of $A = 2$.

Now, $\quad k_1 + k_2 + 3k_3 = 0$ and $k_2 + k_3 = 0$.

If we put $k_3 = t$, $\quad k_2 = -t$ and $k_1 = t - 3t = -2t$.

Since k_1, k_2, k_3 are not zero, the vectors are linearly dependent.

Now, from (1), we get $\quad -2t\,X_1 - t\,X_2 + t\,X_3 = 0$

$\therefore \quad 2X_1 + X_2 - X_3 = 0$. This is the required relation.

Example 7 (b) : Show that the rows of the matrix $\begin{bmatrix} 1 & 0 & -5 & 6 \\ 3 & -2 & 1 & 2 \\ 5 & -2 & -9 & 14 \\ 4 & -2 & -4 & 8 \end{bmatrix}$ are linearly dependent and express any row as a linear combination of other rows. **(M.U. 1996)**

Sol. : Let the row vectors of the given matrix be denoted by X_1, X_2, X_3, X_4. Now, consider the matrix equation

$$k_1 X_1 + k_2 X_2 + k_3 X_3 + k_4 X_4 = O \qquad \text{.....................} (1)$$

$\therefore \quad k_1 [1, 0, -5, 6] + k_2 [3, -2, 1, 2] + k_3 [5, -2, -9, 14] + k_4 [4, -2, -4, 8] = [0, 0, 0, 0]$

$\therefore \quad k_1 + 3k_2 + 5k_3 + 4k_4 = 0, \qquad 0 - 2k_2 - 2k_3 - 2k_4 = 0,$

$\quad -5k_1 + k_2 - 9k_3 - 4k_4 = 0, \quad 6k_1 + 2k_2 + 14k_3 + 8k_4 = 0$

This can be written as,

$$\begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & -2 & -2 & -2 \\ -5 & 1 & -9 & -4 \\ 6 & 2 & 14 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\quad \xrightarrow{\substack{R_3 + 5R_1 \\ R_4 - 6R_1}} \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & -2 & -2 & -2 \\ 0 & 16 & 16 & 16 \\ 0 & -16 & -16 & -16 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\quad \xrightarrow{\substack{R_3 + 8R_2 \\ R_4 - 8R_3 \\ -\frac{1}{2}R_2}} \begin{bmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad k_1 + 3k_2 + 5k_3 + 4k_4 = 0$ and $\quad k_2 + k_3 + k_4 = 0$.

Let $k_4 = t$, $k_3 = s$, $k_2 = -s - t$.

$\therefore \quad k_1 = -3k_2 - 5k_3 - 4k_4 = 3s + 3t - 5s - 4t = -2s - t$.

Since k_1, k_2, k_3, k_4 are not zero the vectors are linearly dependent.

Now, from (1), we get

$$(-2s-t)X_1 + (-s-t)X_2 + sX_3 + tX_4 = 0$$

$$\therefore \quad sX_3 = (2s+t)X_1 + (s+t)X_2 + tX_4 \qquad \text{.................... (2)}$$

This expresses X_3 in terms of X_1, X_2, X_4.

Note ....

You can verify the validity of (ii) by putting the values of X_1, X_2, X_3, X_4.

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Class (b) : 6 Marks

(A) Examine whether the following vectors are linearly independent or dependent.

1. [1, –1, 1], [2, 1, 1], [3, 0, 2] **(M.U. 2015)**

2. [1, 1, 1, 3], [1, 2, 3, 4], [2, 3, 4, 7]

3. [3, 1, – 4], [2, 2, – 3], [0, – 4, 1]

4. [1, 1, 1], [1, 2, 3], [2, 3, 8]

5. [2, 1, 1], [1, 3, 1], [1, 2, –1]

6. [1, 1, –1], [2, –3, 5], [2, –1, 4] **(M.U. 2003)**

[**Ans. :** **(1)** Dependent $X_1 + X_2 - X_3 = 0$, **(2)** Dependent $X_1 + X_2 - X_3 = 0$, **(3)** Dependent $2X_1 = 3X_2 + X_3$, **(4)** Independent, **(5)** Independent, **(6)** Independent.]

Class (b) : 6 Marks

(B) Are the following vectors linearly dependent? If so, find the relation between them.

1. [1, 2, 1], [2, 1, 4], [4, 5, 6], [1, 8, –3] [**Ans. :** $(-5t-2s)X_1 + (2t-s)X_2 + sX_3 + tX_4 = 0$.]

2. [2, –1, 4], [0, 1, 2], [6, –1, 14], [4, 0, 12] [**Ans. :** $X_4 = 2(X_1 + X_2)$]

3. [1, 4, 5], [5, 2, 1], [2, –1, 3], [3, –6, 11]. Also find a smaller linearly independent set.

[**Ans. :** $X_2 - 4X_3 + X_4 = 0$, Smaller independent set is [1, 0, 0], [0, 1, 0] [0, 0, 1]]

4. [2, 3, 4, –2], [–1, –2, –2, 1], [1, 1, 2, –1] [**Ans. :** $X_1 + X_2 - X_3 = 0$]

5. [1, 0, 2, 1], [3, 1, 2, 1], [4, 6, 2, – 4], [– 6, 0, –3, – 4] **(M.U. 2004)**

[**Ans. :** $2X_1 - 6X_2 + X_3 - 2X_4 = 0$]

6. [2, 1, –3], [1, –1, 2], [5, 1, –4] [**Ans. :** $2X_1 - X_2 + X_3 = 0$]

Class (b) : 6 Marks

(C) Show that the rows of the following matrix are linearly dependent and find the relationship between them.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 1 \\ 4 & 6 & 2 & -4 \\ -6 & 0 & -3 & -4 \end{bmatrix}$$

(M.U. 2004) [**Ans. :** $2X_1 - 6X_2 + X_3 - 2X_4 = 0$]

3. Non-Homogeneous Linear Equations

The equations of the type $2x + 3y = 4$, $x - y + 2z = 3$, $x + 2y - 3z + u = 7$ are called linear equations. In these equations only the first degree terms like kx, ly, mz, nu only appear and the terms like $\sqrt{xy}$, x^2, $y^{2/3}$, ..., $\sin x$, $\log y$, e^z, ..., etc. do not appear.

The most general form of a set of linear equations in n unknowns is

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots\dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots\dots + a_{2n} x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_{m1} x_1 + a_{m2} x_2 + \dots\dots + a_{mn} x_n &= b_m \end{aligned} \qquad \dots\dots (1)$$

Such equations are called **non-homogeneous** equations. If on the other hand all $b_1, b_2, \dots, b_n$ are zero, the equations are called **homogeneous** equations.

If the matrix of the coefficients is denoted by A, the matrix of unknowns by X and the matrix of the constants on the r.h.s. by B then the equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_m \end{bmatrix} \qquad i.e. \quad AX = B.$$

The matrix $[\,A,\ B\,]$ *i.e.* the matrix formed by the coefficients and the constants is called the **augmented matrix.**

Thus, the matrix of the coefficient and the augmented matrix are :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \quad [A,\ B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ \dots & \dots & \dots & \dots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \vdots & b_m \end{bmatrix}$$

The set of n numbers $s_1, s_2, \dots, s_n$ such that if we put $x_1 = s_1, x_2 = s_2, \dots, s_n = s_n$ in l.h.s. of (1), and if we get r.h.s. then the set $(s_1, s_2, \dots, s_n)$ is called the solution set.

The set $(s_1, s_2, \dots, s_n)$ satisfying the above relations is called a **solution set.**

A finite set of linear equations is called a **system of linear equations.**

If the system has a solution set, *i.e.*, if we are able to solve the equations and get $s_1, s_2, \dots, s_n$, then the set is called **consistent** otherwise it is called **inconsistent.**

4. Solutions of a System of Linear Equations

(a) System of equation in one variable

The simplest 'system' of linear equations is one equation in one unknown *viz.* $ax = b$ where a, b are constants and x is the unknown. There are the following three possibilities depending upon the values of a and b.

(i) If $a \neq 0$ then there is **unique solution** $x = b / a$ for any b. The coefficient matrix is $[\,a\,]$ and its inverse is $[\,1 / a\,]$. For example, if $4x = 12$ then $x = 12 / 4 = 3$.

(ii) If $a = 0$, $b = 0$ then the equations reduces to $0x = 0$. It has **infinity of solutions**. (Any value of x satisfies the equation $0x = 0$.)

(iii) If $a = 0$ but $b \neq 0$, the equation reduces to $0x = b$ *i.e.* $0 = b$ which is absurd. In this case the equation is **inconsistent** *i.e.* the system has no solution.

The three cases are shown below geometrically on the number line or on the x-axis.

Unique solution **Infinity of solution** **No solution**

Fig. 10.1

We shall now consider the system of 2 and 3 equations.

(b) System of Equations in Two Variables

(i) Unique Solution : Consider the following pairs of equations

$$2x + 3y = 5 \qquad \text{and} \qquad 4x + 3y = 7$$

By subtracting the first equation from the second, we get $2x = 2$ *i.e.* $x = 1$ and putting $x = 1$ in any one of them we get $y = 1$.

Hence, the solution is (1, 1). Geometrically this means the two lines intersect in one unique point.

(ii) Infinity of solutions : Consider the following pairs of equations

$$2x + 3y = 5 \qquad \text{and} \qquad 4x + 6y = 10$$

If we divide the second equation by 2, we get the first equation. The two equations are thus the same. By putting $x = 1$, in any one of them we get $y = 1$, putting $x = 2$, we get $y = 1/3$. Thus, we get infinite solutions. Geometrically, this means the two lines are coincident and any point on the line satisfies the two equations.

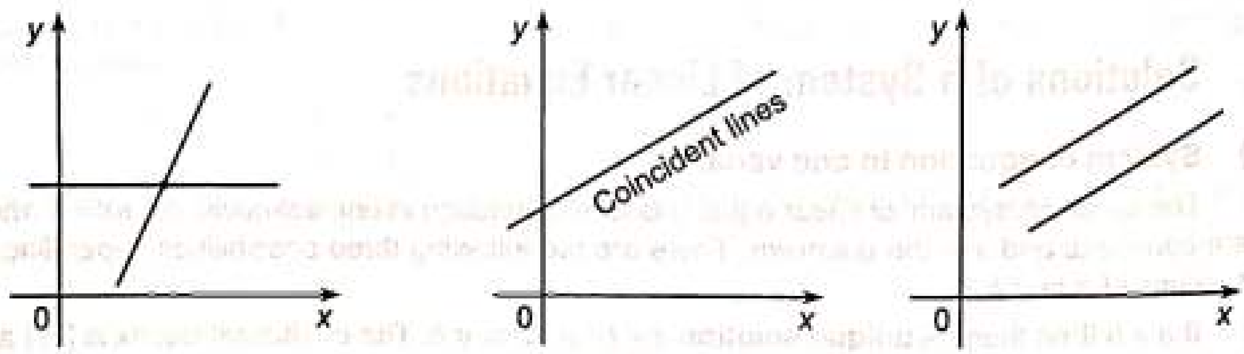
(iii) No solution : Consider the following pairs of equations

$$2x + 3y = 4 \qquad \text{and} \qquad 2x + 3y = 6$$

If we subtract the first equation from the second, we get an absurd result $2 = 0$. This is so because the two lines have the same slope $-2/3$ and hence they are parallel. They do not intersect and the system has no solution.

The three cases are shown geometrically in Fig. 10.2.

Unique solution **Infinite solutions** **No solution**



The two lines intersect in one point. The two lines are coincident The two lines are parallel

Fig. 10.2

(c) System of Equations in Three Variables

(i) Unique solution : Consider the following three equations

$$2x + 3y + 5z = 10$$
$$3x + 6y + 2z = 11$$
$$x + y + 4z = 6$$

The system has unique solution (1, 1, 1). Geometrically this means the three planes intersect in one point.

(ii) Infinite number of solutions : Consider the following three equations

$$2x + 35y - 39z + 12 = 0$$
$$6x + 6y - 7z - 8 = 0$$
$$12x - 15y + 16z - 28 = 0$$

The system has infinite number of solutions. Geometrically this means the three planes intersect in a line.

Again consider the following system of equations.

$$x + 2y + 3z + 4 = 0$$
$$3x + 6y + 9z + 12 = 0$$
$$2x + 3y + 4z + 5 = 0$$

This system also has infinite number of solutions. Geometrically this means the first two planes which are coincident intersect the third plane in a line.

Consider again the following system of equations.

$$2x + 3y + 4z = 7$$
$$4x + 6y + 8z = 14$$
$$6x + 9y + 12z = 21$$

This system has doubly infinite solutions. Geometrically this means the three planes are coincident.

(iii) No solution : Consider the following system of equations

$$x + 2y + 3z = 4$$
$$2x + 4y + 6z = 8$$
$$3x + 6y + 9z = 12$$

This system has no solution. Geometrically this means the three planes are parallel.

Consider the following system

$$2x + 3y + 4z = 6$$
$$2x + 3y + 4z = 8$$
$$4x + 5y + 3z = 10$$

This system also has no solution. Geometrically this means the first two planes are parallel and the third plane intersects them in two parallel lines.

Consider the following system.

$$x + 4y - 6z = 1$$
$$2x - 3y + 5z = 1$$
$$3x + y - z = 2$$

The system also has no solution. Geometrically this means the three planes form a triangular prism; the line of intersection of any two planes is parallel to the third plane.

Consider the following system.

$$2x + 3y + 4z = 5$$
$$4x + 6y + 8z = 10$$
$$2x + 3y + 4z = 7$$

This system also has no solution. The first two planes are coincident and the third plane is parallel to them.

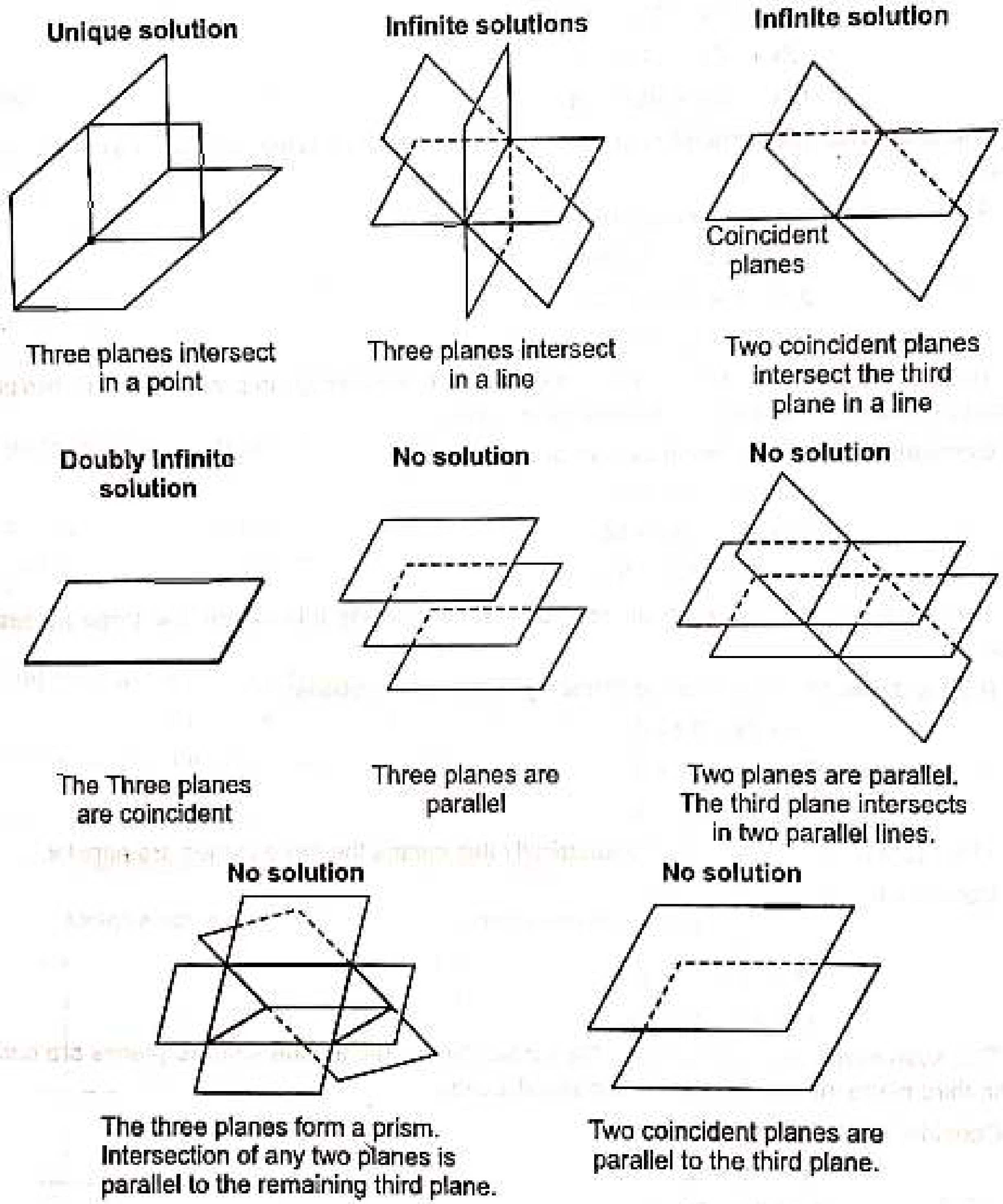


Fig. 10.3

5. Condition of Consistency

If in a system there n unknowns and n equations *i.e.* if the number of equations is equal to the number of unknowns then the matrix of the coefficients is a square matrix. If the determinant of this matrix is zero *i.e.* if $|A| = 0$ **the equations are inconsistent.**

However, if the number of equations is less than the number of unknowns this criterian cannot be used. In this case the system of equations $AX = B$ **is consistent** *i.e.* possesses a solution if and only if the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A, B]$.

6. Consistency in Echelon Form

We now consider how to solve a system of non-homogeneous linear equations by reducing the matrix of the coefficients to echelon form. [For Echelon Form, see § 12, page 9-32.]

(i) Write the given system of m equations in n unknowns in the matrix form $AX = B$.

(ii) Apply elementary row transformations on A as well as on the column matrix B till you get an echelon form.

(iii) Then rewrite the equations as a set of linear equations.

(iv) We know that the rank of a matrix in echelon form is equal to the number of rows containing non-zero elements.

This enables us to determine the rank of A and the rank of the augmented matrix $[A, B]$.

Case I : Rank $A <$ Rank $[A, B]$.

In this case the equations are **inconsistent** *i.e.* they have no solution. (See Ex. 4, 5).

Case II : Rank $A =$ Rank $[A, B]$.

In this case the equations are **consistent** *i.e.* they possess a solution. Further,

(a) If $r = n$, *i.e.* if the rank of A is equal to the number of unknowns, the system has **unique solution**. (See Ex. 1, 2). (Also note that the system has unique solution if the coefficient matrix is non-singular.)

(b) If $r < n$, if the rank of A is less than the number of unknowns the system has **infinite solutions**. In this case $n - r$ unknowns called parameters can be assigned arbitrary values. The remaining unknowns then can be expressed in terms of these parameters. (See Ex. 3)

Type I : To Solve The Equations : Class (b) : 6 Marks

Example 1 (b) : Show that the equations $x + y = 1$, $2x + 3y = 1$, $5x - y = 11$ are consistent and solve them.

Sol. : For consistency of the equations, we know that we should have the ranks of the coefficient matrix A and of the augmented matrix $[A, B]$ equal.

The equations can be written as

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 11 \end{bmatrix} \quad \text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 6 \end{bmatrix}$$

$$\text{By} \quad R_3 + 6R_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Now, $A \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $[A, B] \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ The rank of A is **equal** to the rank of $[A, B] = 2$.

$\therefore$ The equations are **consistent**. Further since the **rank** of the coefficient matrix A is **equal** to the number of **unknowns**, the system has **unique** solution. Now, the equations can be written as

$x + y = 1$; $y = -1$ $\therefore$ $x = 2$.

Note ....

Geometrically this means the given three lines intersect in one point $(2, -1)$, which can be verified by substitution.

Example 2 (b) : Solve the equations

$x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$. **(M.U. 2003, 05)**

Sol. : We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \qquad \text{By } \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{By } R_3 - 3R_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Now, $A \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $[A, B] \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

$\therefore$ The rank of A is **equal** to the rank of $[A, B] = 3$.

$\therefore$ The equations are **consistent**. Further, the **rank** of A is **equal** to the number of **unknowns**. Hence, the system has **unique** solution.

Now, the equations can be written as,

$x + y + z = 3$, $y + 2z = 1$, $2z = 0$.

$\therefore$ $z = 0$, $y = 1$ and $x = 2$.

Note ....

Geometrically this means the given three planes intersect in one point $(2, 1, 0)$ which can be verified by substitution.

Example 3 (b) : Test for consistency and solve

$5x_1 + 3x_2 + 7x_3 = 4$, $3x_1 + 26x_2 + 2x_3 = 9$, $7x_1 + 2x_2 + 10x_3 = 5$.

Sol. : We have $\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

By $\begin{matrix} 7R_1 \\ 5R_3 \end{matrix}$ $\begin{bmatrix} 35 & 21 & 49 \\ 3 & 26 & 2 \\ 35 & 10 & 50 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 9 \\ 25 \end{bmatrix}$

By $\begin{matrix} R_3 - R_1 \\ 5R_2 \end{matrix}$ $\begin{bmatrix} 35 & 21 & 49 \\ 15 & 130 & 10 \\ 0 & -11 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 45 \\ -3 \end{bmatrix}$

By $R_2 - \left(\frac{3}{7}\right)R_1$ $\begin{bmatrix} 35 & 21 & 49 \\ 0 & 121 & -11 \\ 0 & -11 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 33 \\ -3 \end{bmatrix}$

By $R_3 + \left(\frac{1}{11}\right)R_2$ $\begin{bmatrix} 35 & 21 & 49 \\ 0 & 121 & -11 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 28 \\ 33 \\ 0 \end{bmatrix}$

By $\left(\frac{1}{7}\right)R_1, \left(\frac{1}{11}\right)R_2$ $\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$

The **ranks** of the coefficient matrix and of the augmented matrix **are equal** (to 2). Hence, the equations are **consistent**. But $r = 2 < 3$, the number of unknowns. Hence, the equations have **infinite** solutions. The equations now can be written as

$$5x_1 + 3x_2 + 7x_3 = 4, \quad 11x_2 - x_3 = 3.$$

Putting $x_3 = t$, $x_2 = \frac{3}{11} + \frac{t}{11}$ $\therefore$ $x_1 = \frac{7}{11} - \frac{16}{11}t$, where t is a parameter.

$\therefore$ The system has infinite solutions.

Note ....

Geometrically this means the given three planes intersect in a line.

Example 4 (b) : Test for consistency and if possible solve

$2x - 3y + 7z = 5, \quad 3x + y - 3z = 13, \quad 2x + 19y - 47z = 32.$ **(M.U. 2002)**

Sol. : We have $\begin{bmatrix} 2 & -3 & 7 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 13 \\ 32 \end{bmatrix}$

By $R_1 - R_2$ $\begin{bmatrix} -1 & -4 & 10 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ 13 \\ 32 \end{bmatrix}$ By $-R_1$ $\begin{bmatrix} 1 & 4 & -10 \\ 3 & 1 & -3 \\ 2 & 19 & -47 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \\ 32 \end{bmatrix}$

By $\begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$ $\begin{bmatrix} 1 & 4 & -10 \\ 0 & -11 & 27 \\ 0 & 11 & -27 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ 16 \end{bmatrix}$ By $R_3 + R_2$ $\begin{bmatrix} 1 & 4 & -10 \\ 0 & -11 & 27 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ 5 \end{bmatrix}$

The rank of the coefficient matrix A is 2 and the rank of the augmented matrix $[A, B] = 3$. Since, the **ranks are not equal**, the system of equations is **inconsistent**. This is clear even otherwise because the last row gives,

$$0x + 0y + 0z = 5 \text{ which is absurd.}$$

$\therefore$ The equations are inconsistent.

Note ....

Geometrically this means the line of intersection of any two planes is parallel to the third plane *i.e.* the given three planes form a triangular prism. Further, by eliminating y from 1st and 2nd we get $11x - 2z = 44$ and from 1st and 3rd, we get $11x - 2z = 43$ thus, verifying the inconsistency.

Example 5 (b) : Test the consistency of the following equations and solve them if they are consistent

$$2x - y + z = 9, \quad 3x - y + z = 6, \quad 4x - y + 2z = 7; \quad -x + y - z = 4. \qquad \textbf{(M.U. 2004, 14)}$$

Sol. : We have

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 7 \\ 4 \end{bmatrix} \qquad \text{By } R_{14} \quad \begin{bmatrix} -1 & 1 & -1 \\ 3 & -1 & 1 \\ 4 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 7 \\ 9 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 + 3R_1 \\ R_3 + 4R_1 \\ R_4 + 2R_1 \end{matrix} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 3 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 18 \\ 23 \\ 17 \end{bmatrix} \qquad \text{By } R_2 - 2R_4 \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -16 \\ 23 \\ 17 \end{bmatrix}$$

$$\text{By } \quad R_{24} \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \\ 23 \\ -16 \end{bmatrix} \qquad \text{.................... (1)}$$

$$\text{By } \quad R_3 - 3R_2 \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 17 \\ -28 \\ -16 \end{bmatrix}$$

The rank of the matrix A = the rank of the matrix $\begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 3.$

The rank of the augmented matrix $[A, B]$ =

the rank of the matrix $\begin{bmatrix} -1 & 1 & -1 & 4 \\ 0 & 1 & -1 & 17 \\ 0 & 0 & 1 & -28 \\ 0 & 0 & 0 & -16 \end{bmatrix} = 4.$

$\therefore$ The equations are **inconsistent**. This is clear even otherwise, because the last equation from (1), is $0x + 0y + 0z = -16$ which is absurd.

$\therefore$ The equations are inconsistent.

(By adding 1st and 4th equations we get $x = 13$ and by adding 2nd and 4th equations we get $x = 5$. This verifies the inconsistency.)

Example 6 (b) : Establish the consistency of the following equations by considering the ranks of suitable matrices and solve them if possible.

$$x+y+z=6, \quad x-y+2z=5, \quad 3x+y+z=8, \quad 2x-2y+3z=7.$$ **(M.U. 2005)**

Sol. : We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix} \quad \text{By} \xrightarrow[R_4-2R_1]{R_2-R_1,\ R_3-3R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & -2 \\ 0 & -4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -10 \\ -5 \end{bmatrix}$$

$$\text{By} \xrightarrow[R_4-2R_3]{R_3-R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -9 \\ 15 \end{bmatrix} \quad \text{By} \xrightarrow{R_4+\frac{5}{3}(R_3)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \\ -9 \\ 0 \end{bmatrix}$$

$\therefore$ The rank of A is **equal** to the rank of $[A, B] = 3$.

$\therefore$ The equations are **consistent**. Further, the rank of A is **equal** to the number of unknowns $= 3$. Hence, the system has **unique** solution.

Now, the equations can be written as

$$x+y+z=6, \quad -2y+z=-1, \quad -3z=-9. \qquad \therefore \ z=3, \ y=2 \text{ and } x=1.$$

Example 7 (b) : Test for consistency the following equations and solve them if consistent

$$x-2y+3t=2, \quad 2x+y+z+t=-4, \quad 4x-3y+z+7t=8.$$ **(M.U. 2003, 07, 08, 09)**

Sol. : We have

$$\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 1 & 1 \\ 4 & -3 & 1 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} \quad \text{By} \ \begin{matrix} R_2-2R_1 \\ R_3-4R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 5 & 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_3-R_3 \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 5 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix is **equal** to the rank of the augmented matrix $= 2$. Hence, the equations are **consistent**. But the rank of A $(= 2)$ is **less** than the number of unknowns $(= 4)$.

$\therefore$ The number of parameters $= 4-2 = 2$.

$\therefore$ The equations have **doubly infinite** solutions.

Now, $\quad x-2y+0z+3t=2 \quad$ and $\quad 5y+z-5t=0$.

Putting $t=t_1$, $z=t_2$, we get $\quad 5y=-t_2+5t_1 \qquad \therefore \ y=-\dfrac{1}{5}t_2+t_1$.

and $\quad x=2+2y-3t=2-\dfrac{2}{5}t_2+2t_1-3t_1=2-\dfrac{2}{5}t_2-t_1$.

Example 8 (b) : Test for consistency and solve if consistent.

$$\begin{aligned} x_1-2x_2+x_3-\ \ x_4 &= 2 \\ x_1+2x_2+\qquad 2x_4 &= 1 \\ 4x_2-x_3+3x_4 &= -1 \end{aligned}$$

(M.U. 2012)

Sol. : We have
$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 1 & 2 & 0 & 2 \\ 0 & 4 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

By $R_2 - R_1$
$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 4 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

By $R_3 - R_2$
$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix is equal to the rank of the augmented matrix = 2. Hence, the equations are **consistent**. But the rank of A (= 2) is **less** than the number of unknowns (= 4).

$\therefore$ The number of parameters = 4 − 2.

$\therefore$ The equations have **doubly infinite** solutions.

Now, $x_1 - 2x_2 + x_3 - x_4 = 2$ and $4x_2 - x_3 + 3x_4 = -1$.

Putting $x_3 = t_1$, $x_4 = t_2$, we get

$$4x_2 = -1 + x_3 - 3x_4 \qquad \therefore \quad x_2 = -\frac{1}{4} + \frac{t_1}{4} - \frac{3t_2}{4}$$

and $x_1 = 2 + 2x_2 - x_3 + x_4$

$$= 2 + 2\left(-\frac{1}{4} + \frac{t_1}{4} - \frac{3}{4}t_2\right) - t_1 + t_2 = \frac{3}{2} - \frac{t_1}{2} - \frac{t_2}{2}.$$

Example 9 (b) : Test for consistency the following equations and solve them if consistent.

$$x_1 - x_2 + x_3 - x_4 + x_5 = 1, \qquad 2x_1 - x_2 + 3x_3 + 4x_5 = 2,$$
$$3x_1 - 2x_2 + 2x_3 + x_4 + x_5 = 1, \qquad x_1 + x_3 + 2x_4 + x_5 = 0.$$

Sol. : We have
$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 2 & -1 & 3 & 0 & 4 \\ 3 & -2 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

By $\xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1}}$
$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 4 & -2 \\ 0 & 1 & 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix}$$

By $\xrightarrow{R_{34}}$
$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

$$\text{By } \xrightarrow{\substack{R_3 - R_2 \\ R_4 - R_3}} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\text{By } \xrightarrow{R_4 - R_3} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$\therefore$ The rank of the coefficient matrix is **equal** to the rank of the augmented matrix = 3. Hence, the equations are **consistent**. But the rank of A (= 3) is less than the number of unknowns (= 5).

$\therefore$ The number of parameters = 5 − 3 = 2.

$\therefore$ Hence, the equations have **doubly infinite** solutions.

Now, $x_1 - x_2 + x_3 - x_4 + x_5 = 1$

$x_2 + x_3 + 2x_4 + 2x_5 = 0$

$-x_3 + x_4 - 2x_5 = -1$

If $x_4 = t_1,\ x_5 = t_2$

$-x_3 + t_1 - 2t_2 = -1 \qquad \therefore\ x_3 = 1 + t_1 - 2t_2$

Now, $x_2 + x_3 + 2x_4 + 2x_5 = 0$ gives $x_2 + 1 + t_1 - 2t_2 + 2t_1 + 2t_2 = 0$

$\therefore\ x_2 = -1 - 3t_1$

And $x_1 + 1 + 3t_1 + 1 + t_1 - 2t_2 - t_1 + t_2 = 1$

$\therefore\ x_1 = -1 - 3t_1 + t_2.$

Example 10 (b) : Solve by matrices

$x + x_2 - 2x_3 + x_4 + 3x_5 = 1,\ \ 2x_1 - x_2 + 2x_3 + 2x_4 + 6x_5 = 2,\ \ 3x_1 + 2x_2 - 4x_3 - 3x_4 - 9x_5 = 3.$

Sol. : We have
$$\begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 2 & -1 & 2 & 2 & 6 \\ 3 & 2 & -4 & -3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & -2 & 1 & 3 \\ 0 & -3 & 6 & 0 & 0 \\ 0 & -1 & 2 & -6 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ The rank of the coefficient matrix is **equal** to the rank of the augmented matrix = 3. Hence, the equations are **consistent**. But the rank of A (= 3) is less than the number of unknowns (= 5).

$\therefore$ The numbers of parameters = 5 − 3 = 2.

$\therefore$ Hence, equations have **doubly infinite** solutions.

Now, $x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1$, $\quad -3x_2 + 6x_3 = 0 \quad \therefore\ x_2 = 2x_3$

And $- x_2 + 2x_3 - 6x_4 - 18x_5 = 0$

$\therefore$ If $x_3 = t_1$, $x_2 = 2t_1 \quad \therefore\ -2t_1 + 2t_1 - 6x_4 - 18x_5 = 0 \quad \therefore\ x_4 + 3x_5 = 0$

$\therefore$ If $x_5 = t_2$, $x_4 = -3t_2$

Now, $x_1 + x_2 - 2x_3 + x_4 + 3x_5 = 1$

$\therefore\ x_1 + 2t_1 - 2t_1 - 3t_2 + 3t_2 = 1 \quad \therefore\ x_1 = 1$

$\therefore\ x_1 = 1,\ x_2 = 2t_1,\ x_3 = t_1,\ x_4 = -3t_2,\ x_5 = t_2.$

Type II : On Conditions For Solutions : Class (b) : 6 Marks

Example 1 (b) : For what value of λ the equations $3x - 2y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$, will have no unique solution ? Will the equations have any solution for this value of λ ? **(M.U. 2004)**

Sol. : We have, (taking the equations in reverse order)

$$\begin{bmatrix} 1 & 2 & -\lambda \\ 2 & 1 & 1 \\ 3 & -2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

By $\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$
$$\begin{bmatrix} 1 & 2 & -\lambda \\ 0 & -3 & 1+2\lambda \\ 0 & -8 & 4\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix} \quad \text{.....................} \ (1)$$

The equations have unique solutions if the coefficient matrix is non-singular.

$\therefore\ -12\lambda + 8 + 16\lambda \neq 0,\ 4\lambda \neq -8 \quad \therefore\ \lambda \neq -2.$

$\therefore$ The equations have unique solutions if $\lambda \neq -2$ and the equations have no unique solutions if $\lambda = -2$.

Further, if $\lambda = -2$, we have from (1)

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -8 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$$

By $R_3 - \dfrac{8}{3}R_2$
$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -20/3 \end{bmatrix}$$

$0x + 0y + 0z = -20/3$ which is absured. (Also the rank of $A = 2 <$ the rank of $[A, B] = 3$)

$\therefore$ The equations are **inconsistent**.

For $\lambda = -2$ there is no solution.

Example 2 (b) : Show that if $\lambda \neq -5$ the system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$, $6x + 5y + \lambda z = -3$ has a unique solution. If $\lambda = -5$, show that the equations are consistent. Determine the solutions in each case.

Sol. : We have
$$\begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

By R_{12} $\begin{bmatrix} 1 & 2 & -3 \\ 3 & -1 & 4 \\ 6 & 5 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$ By $\begin{matrix} R_2 - 3R_1 \\ R_3 - 6R_1 \end{matrix}$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 13 \\ 0 & -7 & \lambda+18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 9 \end{bmatrix}$

By $R_3 - R_2$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -7 & 13 \\ 0 & 0 & \lambda+5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \\ 0 \end{bmatrix}$

(a) If $\lambda \neq -5$ determinant of the coefficient matrix is not equal zero. *i.e.* the coefficient matrix is not singular and hence, the system has **unique** solutions. Now, we have,

$x + 2y - 3z = -2,\ -7y + 13z = 9$ and $(\lambda + 5)\, z = 0.$

But $\lambda + 5 \neq 0 \quad \therefore\ z = 0 \quad \therefore\ y = -9/7,\ x = 4/7.$

(b) If $\lambda = -5$, *i.e.* $\lambda + 5 = 0$, the rank of the coefficient matrix is **equal** to the rank of augmented matrix = 2. Hence, the equations are **consistent**.

But the rank of A (= 2) is less than the number of unknowns (= 3).

$\therefore$ The number of parameters = 3 − 2 = 1.

$\therefore$ The equations have **infinite** solutions.

The equations then become $x + 2y - 3z = -2$, $-7y + 13z = 9$. Putting $z = t$ the solutions are

$\therefore\quad z = t,\ y = (13t - 9)/7,\ x = (4 - 5t)/7.$

Example 3 (b) : For what value of λ the equations $x + y + z = 1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z = \lambda^2$ have a solution and solve them completely in each case. **(M.U. 2006, 08, 09, 13)**

Sol. : We have $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix}$

By $\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 1 \end{bmatrix}$

By $R_3 - 3R_2$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda - 1 \\ \lambda^2 - 3\lambda + 2 \end{bmatrix}$ .... (1)

The given equations will be consistent if the rank of A = rank of $[A, B]$.

This requires $\lambda^2 - 3\lambda + 2 = 0$ *i.e.* $(\lambda - 2)(\lambda - 1) = 0. \quad \therefore\ \lambda = 2$ or $\lambda = 1$.

Hence, the equations are **consistent** if $\lambda = 2$ or $\lambda = 1$.

(a) If $\lambda = 2$ the equation (1) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Now, the rank of A (= 2) is less than the number of unknowns (= 3).

$\therefore$ The number of parameters = 3 − 2 = 1.

$\therefore$ The equations have **infinite** solutions. $\quad \therefore\ x + y + z = 1,\ y + 3z = 1.$

Putting $z = t$, $y = 1 - 3t$, we get $x = 2t$ which is the general solution.

(b) If $\lambda = 1$, the equation (1) becomes

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x + y + z = 1, \; y + 3z = 0.$

Putting $z = t$, $y = -3t$, $x = 1 + 2t$ which is the general solution.

Example 4 (b) : Investigate for what values of λ and μ the equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have **(i)** no solution, **(ii)** a unique solution, **(iii)** infinite number of solutions.

(M.U. 2002, 04, 07, 16, 18)

Sol. : We have
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

By
$$\begin{matrix} R_2 - R_1 \\ R_3 - R_2 \end{matrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & \lambda - 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ \mu - 10 \end{bmatrix}$$

(i) The system has unique solution if the coefficient matrix is non-singular (or the rank A, $r =$ the number of unknowns, $n = 3$).

This requires $\lambda - 3 \neq 0 \quad \therefore \lambda \neq 3$.

$\therefore$ If $\lambda \neq 3$, (μ may have any value) the system has unique solution.

(ii) If $\lambda = 3$ the coefficient matrix and the augmented matrix become

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & \mu - 10 \end{bmatrix}$$

The rank of $A = 2$, and the rank of $[A, B]$ will be also 2 if $\mu = 10$.

Thus, if $\lambda = 3$ and $\mu = 10$, the system is consistent. But the rank of A (= 2) is less than the number of unknowns (= 3). Hence the equations will posses **infinite** solutions.

(iii) If $\lambda = 3$, and $\mu \neq 10$, the rank of $A = 2$, and the rank of $[A, B] = 3$. They are not equal and the equations will be **inconsistent** and will **not** posses any solution.

Example 5 (b) : If the augmented matrix of a system of equations is

$$\left[\begin{array}{ccc:c} 1 & -2 & 1 & 3 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & \lambda + 1 & \mu - 3 \end{array}\right]$$

find the values of λ and μ such that the system has (i) unique solution, (ii) no solution, (iii) infinitely many solutions.

(M.U. 2002)

Sol. : (i) The system has a unique solution if the coefficient matrix is non-singular.

$\therefore \quad \lambda + 1 \neq 0 \quad \therefore \lambda \neq -1.$

(ii) If $\lambda = -1$, the coefficient matrix and the augmented matrix become

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \left[\begin{array}{ccc:c} 1 & -2 & 1 & 2 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & \mu - 3 \end{array}\right]$$

The rank of A is 2 and the rank of [A, B] will be also 2 if $\mu - 3 = 0$ *i.e.* $\mu = 3$.

$\therefore$ If $\lambda = 1$ and $\mu = 3$, the system is consistent. But the rank of A (= 2) is less than the number of unknowns (= 3). Hence, will have **infinite** solutions.

(iii) If $\lambda = -1$ and $\mu \neq 3$, the rank of A is 2 and the rank of [A, B] is 3. The system then the equations will be **inconsistent** and will have no solution.

Example 6 (b) : Determine the values of k for which the following equations are consistent. Also solve the system for these values of k.

$$x + 2y + z = 3, \qquad x + y + z = k, \qquad 3x + y + 3z = k^2.$$ **(M.U. 1995, 2014)**

Sol. : We have
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ k \\ k^2 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}$
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ k-3 \\ k^2-9 \end{bmatrix}$$

By $R_3 - 5R_2$
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ k-3 \\ k^2-5k+6 \end{bmatrix}$$

The equation will be consistent if the rank of A = rank of [A, B].

This requires $k^2 - 5k + 6 = 0 \quad \therefore \ (k-3)(k-2) = 0 \quad \therefore \ k = 3$ or $k = 2$.

Hence, the equations are **consistent** if $k = 3$ or $k = 2$.

(a) When $k = 3$,
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}$
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - 5R_2$
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Now, the rank of A (= 2) is less than the number of unknowns (= 3). Hence, the number of parameters = 3 − 2 = 1. The equations will have **infinite** solutions.

$\therefore \quad x + 2y + z = 3, \ y = 0, \quad 0z = 0$ *i.e.*, $z = t$.

$\therefore \quad x = 3 - t, \ y = 0, \ z = t$.

(b) When $k = 2$,
$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}$
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -5 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}$$

$$\text{By} \quad R_3 - 5R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Now, the rank of A (= 2) is less than the number of unknowns (= 3). Hence, the number of parameters = 3 − 2 = 1. The equations will have **infinite** solutions.

$\therefore \quad x + 2y + z = 3, \; y = 1, \qquad 0z = 0 \;$ *i.e.,* $\; z = t$

$\therefore \quad x = 1 - t, \; y = 1, \; z = t.$

Example 7 (b) : Show that if $\lambda \neq 0$, the system of equations

$$2x_1 + x_2 = a, \qquad x + \lambda x_2 - x_3 = b, \qquad x_2 + 2x_3 = c.$$

has a unique solution for every value of a, b, c. If $\lambda = 0$, determine the relation satisfied by a, b, c such that the system is consistent. Find the general solution by taking $\lambda = 0$, $a = 1$, $b = 1$, $c = -1$.

Sol. : We have
$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \text{.....................} (1)$$

i.e. $\qquad AX = B$

The system has a unique solution if $|A| \neq 0$.

$$\therefore \quad \begin{vmatrix} 2 & 1 & 0 \\ 1 & \lambda & -1 \\ 0 & 1 & 2 \end{vmatrix} \neq 0$$

$\therefore \quad 2(2\lambda + 1) - 1(2 + 0) \neq 0 \qquad \therefore \; 4\lambda \neq 0 \qquad \therefore \; \lambda \neq 0.$

$\therefore$ The system has a unique solution if $\lambda \neq 0$, a, b, c may have any value. If $\lambda = 0$, the system is either inconsistent or has infinite number of solutions. Putting $\lambda = 0$ we have from (1)

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \text{By} \quad R_{12} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ a \\ c \end{bmatrix}$$

$$\text{By} \; R_2 - 2R_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ a - 2b \\ c \end{bmatrix} \qquad \text{By} \quad R_3 - R_2 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b \\ a - 2b \\ c - a + 2b \end{bmatrix}$$

$\therefore \quad x_1 - x_3 = b \;$ and $\; x_2 + 2x_3 = a - 2b \;$ and $\; 0 = c - a + 2b.$

$\therefore$ If $c - a + 2b = 0$ *i.e.* if $a = 2b + c$, the rank of A is **equal** to the rank of the augmented matrix. Hence, the equations will be **consistent**. But the number of parameters = the number of unknowns − the rank = 3 − 2 = 1. Hence, the system will have **infinite** solutions.

If we put $x_3 = t$, $\; x_1 = b + t$, and $x_2 = a - 2b - 2t$.

In particular if $a = 1$, $\; b = 1$, $c = -1$, $x_3 = t$, $\; x_1 = 1 + t$, $\; x_2 = -1 - 2t$.

Example 8 (b) : Show that the equations $-2x + y + z = a$, $\; x - 2y + z = b$, $\; x + y - 2z = c$ have no solutions unless $a + b + c = 0$ in which case they have infinitely many solutions. Find these solutions when $a = 1$, $b = 1$, $c = -2$.

(M.U. 1997, 2017)

Sol. : We have
$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

By $$R_{21}\begin{bmatrix}1 & -2 & 1\\ -2 & 1 & 1\\ 1 & 1 & -2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}b\\ a\\ c\end{bmatrix}$$

By $$\begin{matrix}R_2+2R_1\\ R_3-R_1\end{matrix}\begin{bmatrix}1 & -2 & 1\\ 0 & -3 & 3\\ 0 & 3 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}b\\ a+2b\\ c-b\end{bmatrix}$$

By $$R_3+R_2\begin{bmatrix}1 & -2 & 1\\ 0 & -3 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}b\\ a+2b\\ a+b+c\end{bmatrix} \qquad \text{.................... (1)}$$

The given equations will be consistent if the rank of A = rank of $[A, B]$.

This requires $a+b+c=0$.

Since, when $a+b+c=0$, rank of $A=2$ is less than the number of unknowns 3, the system has **infinite** solutions.

Now, when $a=1$, $b=1$, $c=-2$ from (1), we get

$$\begin{bmatrix}1 & -2 & 1\\ 0 & -3 & 3\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 3\\ 0\end{bmatrix}$$

Now, the rank of A (= 2) is less than the number of unknowns (= 3). The number of parameters $=3-2=1$.

$\therefore \quad x-2y+z=1, \; -3y+3z=3$

Let $z=t$, $\quad \therefore \; y=-1+t$

$\therefore \quad x=1+2y-z=1-2+2t-t=-1+t.$

$\therefore \quad x=-1+t, \; y=-1+t, \; z=t.$

Example 9 (b) : Show that the following system of equations is consistent if a, b, c are in A.P.

$$3x+4y+5z=a, \qquad 4x+5y+6z=b, \qquad 5x+6y+7z=c.$$

Find the solution when $a=1$, $b=2$, $c=3$. **(M.U. 1997, 99, 2002, 03, 10, 11)**

Sol. : We have $$\begin{bmatrix}3 & 4 & 5\\ 4 & 5 & 6\\ 5 & 6 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}$$

By $$\begin{matrix}R_2-R_1\\ R_3-R_2\end{matrix}\begin{bmatrix}3 & 4 & 5\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-a\\ c-b\end{bmatrix}$$

By $$R_3-R_2\begin{bmatrix}3 & 4 & 5\\ 1 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}a\\ b-a\\ a+c-2b\end{bmatrix} \qquad \text{.................... (1)}$$

The given system will be consistent if the rank of A = the rank of $[A, B]$.

This requires $a+c-2b=0 \quad \therefore \quad \dfrac{a+c}{2}=b \quad \therefore \; a, b, c$ are in A.P.

Putting $a=1$, $b=2$, $c=3$ in (1), we get

$$\begin{bmatrix} 3 & 4 & 5 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \therefore \quad R_{1,2}\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

By $$R_2 - 3R_1\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Now, the rank of A (= 2) is less than the number of unknowns (= 3). The number of parameters $= 3 - 2 = 1$.

$\therefore \quad x + y + z = 1, \quad y + 2z = -2.$

Putting $z = t, \; y = -2 - 2t.$

$\therefore \quad x = 1 - y - z = 1 + 2 + 2t - t = 3 + t.$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

(A) Test for consistency the following equations and if possible solve them :
Class (b) : 6 Marks

1. $2x_1 - 3x_2 + 5x_3 = 1, \; 3x_1 + x_2 - x_3 = 2, \; x_1 + 4x_2 - 6x_3 = 1$

[**Ans. :** $x_1 = \frac{7}{11} - \frac{2}{11}t, \; x_2 = \frac{1}{11} + \frac{17}{11}t, \; x_3 = t$]

2. $x + 2y - z = 1, \; x + y + 2z = 9, \; 2x + y - z = 2.$ **(M.U. 1999)** [**Ans. :** $x = 2, \; y = 1, \; z = 3$]

3. $x_1 - 3x_2 - 8x_3 = -10, \; 3x_1 + x_2 - 4x_3 = 0, \; 2x_1 + 5x_2 + 6x_3 = 13.$

[**Ans. :** $x_1 = -1 + 2t, \; x_2 = 3 - 2t, \; x_3 = t$]

4. $2x_1 + x_3 = 4, \; x_1 - 2x_2 + 2x_3 = 7, \; 3x_1 + 2x_2 = 1.$

[**Ans. :** $x_1 = 2 - \frac{t}{2}, \; x_2 = -\frac{5}{2} + \frac{3}{4}t, \; x_3 = t$]

5. $x + 2y + 3z = 14, \; 3x + y + 2z = 11, \; 2x + 3y + z = 11.$ **(M.U. 1996)**

[**Ans. :** $x = 1, \; y = 2, \; z = 3$]

6. $x_1 - x_2 + 2x_3 + x_4 = 2, \; 3x_1 + 2x_2 + x_4 = 1, \; 4x_1 + x_2 + 2x_3 + 2x_4 = 3.$

[**Ans. :** $x_3 = t_1, \; x_4 = t_2, \; x_2 = (-5 + 6t_1 + 2t_2)/5, \; x_1 = (5 - 4t_1 - 3t_2)/5$]

7. $x_1 + 3x_2 - x_3 = 4, \; 2x_1 + x_2 + x_3 = 7, \; 2x_1 - 4x_2 + 4x_3 = 6, \; 3x_1 + 4x_2 = 11.$

[**Ans. :** $x_1 = \frac{17 - 4t}{5}, \; x_2 = \frac{1 + 3t}{5}, \; x_3 = t$]

8. $2x - y - z = 2, \; x + 2y + z = 2, \; 4x - 7y - 5z = 2.$ [**Ans. :** $x = \frac{6 + t}{5}, \; y = \frac{2 - 3t}{5}, \; z = t$]

9. $4x - 2y + 6z = 8, \; x + y - 3z = -1, \; 15x - 3y + 9z = 21.$ **(M.U. 1998, 2003)**

[**Ans. :** $x = 1, \; y = -2 + 3t, \; z = t$]

10. $2x_1 + x_2 + x_3 = 4, \; x_1 - x_2 + 3x_3 = 3, \; 4x_1 - x_2 - x_3 = 2.$ **(M.U. 1999)**

[**Ans. :** $x_1 = 1, \; x_2 = 1 \; x_3 = 1$]

11. $2x_1 + x_2 - x_3 + 3x_4 = 11, \quad x_1 - 2x_2 + x_3 + x_4 = 8,$
$4x_1 + 7x_2 + 2x_3 - x_4 = 0, \quad 3x_1 + 5x_2 + 4x_3 + 4x_4 = 17.$

[**Ans. :** $x_1 = 2, \; x_2 = -1, \; x_3 = 1, \; x_4 = 3$]

12. $4x - 2y + 6z = 8,\ x + y - 3z = -1,\ 15x - 3y + 9z = 21.$ **(M.U. 2002)**

[**Ans. :** $x = 1, y = 3t - 2, z = t$]

13. $6x + y + z = -4,\ 2x - 3y - z = 0,\ -x - 7y - 2z = 7.$ **(M.U. 2007)**

[**Ans. :** $x = -1, y = -2, z = -4$]

14. $x + 2y - z = 1,\ x + y + 2z = 9,\ 2x + y - z = 2.$ **(M.U. 2004)** [**Ans. :** $x = 2, y = 1, z = 3$]

15. $2x_1 + x_2 - x_3 + 3x_4 = 8,\quad x_1 + x_2 + x_3 - x_4 = -2,$
$3x_1 + 2x_2 - x_3 = 6,\quad 4x_2 + 3x_3 + 2x_4 = -8.$

[**Ans. :** $x_1 = 2,\ x_2 = -1,\ x_3 = -2,\ x_4 = 1$]

(B) Class (b) : 6 Marks

1. Investigate for what value of λ and μ the equations $2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ have **(i)** no solution, **(ii)** a unique solution, **(iii)** an infinite number of solutions.

(M.U. 2005, 07, 11, 16, 17, 18)

[**Ans. :** **(i)** No solution if $\lambda = 5, \mu \neq 9$
(ii) Unique solution if $\lambda \neq 5$, μ may have any value
(iii) Infinite solutions if $\lambda = 5, \mu = 9$.]

2. For what value of λ the equations $x + 2y + z = 3,\ x + y + z = \lambda,\ 3x + y + 3z = \lambda^2$ have a solution and solve them completely in each case? **(M.U. 2014)**

[**Ans. :** **(i)** $\lambda = 2,\ x = 1 - t,\ y = 1,\ z = t$; **(ii)** $\lambda = 3,\ x = 3 - t,\ y = 0, z = t$.]

3. For what value of λ the equations $x + y + z = 1,\ 2x + y - 4z = \lambda,\ 4x + 5y + 10z = \lambda^2$ have a solution and solve them completely in each case?

[**Ans. :** **(i)** $\lambda = 2,\ x = 1 + 5t,\ y = -6t,\ z = t,$
(ii) $\lambda = -3,\ x = -4 + 5t,\ y = 5 - 6t,\ z = t$]

4. For what value of λ the equations $x + y + 4z = 1,\ 2x + 2y + 3z = 5,\ \lambda x + 3y + 6z = 4$ will have no unique solution? Will the equations have any solution for this value of λ ?

[**Ans. :** $\lambda = 3$, Inconsistent]

5. For what value of λ the equations $\lambda x + 2y - 2z - 1 = 0,\ 4x + 2\lambda y - z = 2, 6x + 6y + \lambda z = 3$ have **(i)** a unique solution **(ii)** infinity of solutions ? Find the solutions in the second case.

[**Ans. : (i)** $\lambda \neq 2$, **(ii)** when $\lambda = 2,\ x = \frac{1}{2} - t,\ y = t,\ z = 0$]

6. Find the values of k for which the equations

$$x + y + z = 1,\quad x + 2y + 3z = k \quad \text{and} \quad x + 5y + 9z = k^2$$

have a solution. Solve them for these values of k. **(M.U. 2004, 08)**

[**Ans. : (i)** $k = 1, k = 3$, **(ii)** $1 + t, -2t, t$; $t - 1, 2(1 - t), t$]

7. Determine λ and μ if the system

$$3x - 2y + z = \mu,\quad 5x - 8y + 9z = 3,\quad 2x + y + \lambda z = -1$$

will have a unique solution. **(M.U. 2015)** [**Ans. :** $\lambda \neq 3$, μ may have any value.]

8. For what value of k the set of equations

$$2x - 3y + 6z - 5t = 3,\quad y - 4z + t = 1,\quad 4x - 5y + 8z - 9t = k$$

has **(i)** no solution, **(ii)** an infinite number of solutions ?

[**Ans. :** $k = 7,\ x = 3r + s + 3,\ y = 4r - s + 1,\ z = r,\ t = s$]

9. Investigate for what values of λ and μ the equations

$x+2y+3z=4, \quad x+3y+4z=5, \quad x+3y+\lambda z=\mu$

have (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions. **(M.U. 2003, 06)**

[**Ans. :** (i) No solution if $\lambda = 4, \ \lambda \neq 5$,

(ii) a unique solution if $\lambda \neq 4$, μ may have any value,

(iii) an infinite number of solutions if $\lambda = 4, \mu = 5$.]

10. Find the values of k for which the following system of equations has (i) no solution, (ii) a unique solution, (iii) an infinite number of solutions.

$kx+y+z=1, \quad x+ky+z=1, \quad x+y+kz=1$ **(M.U. 2003)**

[**Ans. :** (i) $k=-2$, (ii) $k \neq 1, k \neq -2$, (iii) $k=1$]

11. Determine the values of k for which the following equations have a solution. Also solve these equations for these values of k.

$2x+y+z=1, \quad x+3y+z=k, \quad 3x+4y+2z=2k^2.$ **(M.U. 2004)**

[**Ans. :** (i) $k=-1/2, k=1$; (ii) $0{\cdot}7-2t/5-(2+t)/5$; (iii) $2(1-t)/5, (1-t)/5, t$]

12. Find the values of λ for which the system of equations

$x+y+4z=1, \quad x+2y-2z=1, \quad \lambda x+y+z=1$

will have (i) unique solution, (ii) no solution ?

Solve the equations by elementary row transformations for $\lambda = 1/2$.

[**Ans. :** The system will have unique solution if the coefficient matrix is not singular. $\lambda \neq 7/10$. The system has no solution if $\lambda = 7/10$. For $\lambda = 1/2$, $x=-3/2, \ y=3/2, \ z=1/4$.]

---

7. Homogeneous Linear Equations

Consider the following two pairs of equations

$$x+2y=0, \ 2x-y=0 \quad \text{.....................} \ (i)$$

$$\text{and} \quad 3x-y=0, \ 6x-2y=0. \quad \text{.....................} \ (ii)$$

We see that $x=0, y=0$ is the only solution of the pair (1). Further $x=0, y=0$ is also a solution of the pair (2). But in addition $x=1, y=3$ is also a solution of (2). In fact $x=C, y=3C$ where C is any real number is also a solution (2). Thus, the pair (2) has infinitely many solutions. The solution $x=0, y=0$ is called **zero-solution**. All other solutions are called **non-zero solutions.**

We know that, a set of non-homogeneous linear equations can be written in matrix form as $AX=B$. Similarly a set of homogeneous equations of the above type can be written as

$$AX=O$$

A is called the **coefficient matrix**, X is called the **matrix of the unknowns,** and O is the **zero matrix.**

(a) Solution of $AX=0$

As seen above the null column matrix is obviously a solution of $AX=0$. The solution $X=0$ is called the *trivial solution* or the zero-solution. A solution $X \neq 0$ is called non-trivial solution or non-zero solution.

(b) Important Results

Suppose we have a system of m equations in n unknowns. Then, the coefficient matrix A will be $m \times n$. Let r be the rank of the matrix A.

Case I : **If $r = n$** *i.e.* the rank of the matrix **A is equal** to the number of unknowns then $x_1 = x_2 = ... = x_n = 0$ *i.e.* only possible solution is **zero-solution** or **trivial solution.** (See Ex. 1, 2)

Case II : **If $r < n$** *i.e.* the rank of the matrix **A is less than** the number of unknowns then the system has **non-trivial solutions**. The number of indepen-dent solutions *i.e.* the number of parameters is equal to $n - r$. (See Ex. 3, 4 and 5.)

(c) Working Rule to solve Homogeneous Equations

(i) Write the given system of m equations in n unknown in the matrix form $AX = 0$.

(ii) **Apply elementary row transformations only** and reduce the coefficient matrix A to **echelon form.**

(iii) Then rewrite the matrix equation as a set of linear equations.

(iv) If $r = n$ then only trivial solution is the possible solution.

If $r < n$ then non-trivial solution is possible.

The number of independent solutions *i.e.* the number of parameters $= n - r$.

Type I : To Solve The Equations : Class (b) : 6 Marks

Example 1 (b) : Solve $x_1 + 2x_2 + 3x_3 = 0,\quad 2x_1 + 3x_2 + x_3 = 0,$
$4x_1 + 5x_2 + 4x_3 = 0,\quad x_1 + 2x_2 - 2x_3 = 0.$

Sol. : We have
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 4 & 5 & 4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - 2R_1 \\ R_3 - 4R_1 \\ R_4 - R_1 \end{matrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -3 & -8 \\ 0 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_3 - 3R_2 \\ R_4 - R_2 \end{matrix}$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

A is now in echelon form.

The rank of the matrix $A = 3 =$ the number of unknowns. Hence, the system has *trivial solutions.* This is also clear otherwise. The matrix equation ultimately gives

$x_1 + 2x_2 + 3x_3 = 0,\ x_2 + 5x_3 = 0,\ 7x_3 = 0$

$\therefore\ x_3 = 0,\ x_2 = 0$ and $x_1 = 0$

$\therefore$ The system has only *trivial solutions.*

Example 2 (b) : Solve $3x + y - 5z = 0,\quad 5x + 3y - 6z = 0,$
$x + y - 2z = 0,\quad x - 5y + z = 0.$

Sol. : Taking the third and fourth equations as first and second, we have

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -5 & 1 \\ 3 & 1 & -5 \\ 5 & 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 - 5R_1 \end{matrix}$ $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -6 & 3 \\ 0 & -2 & 1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} \frac{1}{3}R_2 \\ \frac{1}{2}R_4 \end{matrix}$ $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ By $R_3 - R_2$ $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_{2,4} \\ R_{34} \end{matrix}$ $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ By $R_3 - 2R_2$ $\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

A is now in echelon form.

The rank = 3 and the number of unknowns = 3.

Hence, only possible solution is the trivial solution $x = 0$, $y = 0$, $z = 0$.

This is also clear otherwise. We get from the above matrix equality,

$$-3z = 0 \quad \therefore \quad z = 0.$$
$$-y + 2z = 0 \quad \therefore \quad y = 0,$$
$$x + y - 2z = 0 \quad \therefore \quad x = 0.$$

Example 3 (b): Solve $x_1 - 2x_2 + 3x_3 = 0$, $2x_1 + 5x_2 + 6x_3 = 0$.

Sol. : We have $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ By $R_2 - 2R_1$ $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 9 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore \quad x_1 - 2x_2 - 3x_3 = 0, \; 9x_2 = 0.$

The rank of the matrix 2 < 3, the number of unknowns. The system has therefore, **non-trivial solution** also. The number of independent solutions is $3 - 2 = 1$.

Now form the above equations, we get $x_1 - 2x_2 + 3x_3 = 0$ and $x_2 = 0$; putting $x_3 = t$, we get $x_1 = -3t$.

$\therefore$ The solutions are $x_1 = -3t$, $x_2 = 0$, $x_3 = t$.

(You can verify the answer by directly putting the values in the given equations.)

Example 4 (b) : Solve by matrix method,

$$x + y + 2z = 0, \quad x + 2y + 3z = 0, \quad x + 3y + 4z = 0, \quad 3x + 4y + 7z = 0.$$

Sol. : We have $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 3R_1 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_3 - 2R_2 \\ R_4 - R_2 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The rank of the matrix 2 < 3 the number of unknowns.

The system has **non-trivial** solution also. The number of independent solutions is $3 - 2 = 1$.

From the above equations, we get, $x + y + 2z = 0$, $y + z = 0$.

Putting $z = -t$, we get $y = t$.

$\therefore \quad x = -y - 2z = -t + 2t = t \qquad \therefore \quad x = t,\ y = t,\ z = -t.$

Example 5 (b) : Solve the equations $x_1 + x_2 - x_3 + x_4 = 0,$
$x_1 - x_2 + 2x_3 - x_4 = 0,$
$3x_1 + x_2 + x_4 = 0.$ **(M.U. 2002, 03)**

Sol. : We have
$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 - 3R_1 \end{matrix}$
$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - R_2$
$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 + x_2 - x_3 + x_4 = 0,\ -2x_2 + 3x_3 - 2x_4 = 0$

The rank of the matrix $A = 2 < 4$, the number of unknowns. The system has **non-trivial solutions** and the number of independent solutions is $4 - 2 = 2$.

$\therefore$ Putting $x_4 = t_1$, $x_3 = t_2$ we get $x_2 = \frac{3}{2} t_2 - t_2$ and $x_1 = -\frac{1}{2} t_2$.

(You can verify the answer by directly substituting the values in the given equations.)

Type II : To Find The Conditions : Class (b) : 6 Marks

Example 1 (b) : Show that the system of equations

$$2x - 2y + z = \lambda x, \qquad 2x - 3y + 2z = \lambda y, \qquad -x + 2y = \lambda z$$

can posses a non-trivial solution only if $\lambda = 1$, $\lambda = -3$. Obtain the general solution in each case. **(M.U. 2003, 19)**

Sol. : We have
$$\begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has non-trivial solution if the rank of A is less than the number of unknowns, 3. (See Case II, page 10-29). The rank of A will be less than three if $|A| = 0$.

Now, $\begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\therefore \quad (2-\lambda)(\lambda^2 + 3\lambda - 4) + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$

$\therefore \quad (2-\lambda)(\lambda + 4)(\lambda - 1) - 4(\lambda - 1) - (\lambda - 1) = 0$

$\therefore \quad (\lambda-1)[2\lambda+8-\lambda^2-4\lambda-4-1]=0 \qquad \therefore \quad (\lambda-1)(-\lambda^2-2\lambda+3)=0$

$\therefore \quad (\lambda-1)(\lambda-1)(\lambda+3)=0 \qquad \therefore \quad \lambda=1,\ \lambda=-3.$

(i) If $\lambda=1$, we have,

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_2-2R_1 \\ R_3+R_1 \end{matrix}\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x-2y+z=0$. Putting $z=t_1,\ y=t_2$.

$\therefore \quad$ The solution is $x=2t_2-t_1,\ y=t_2,\ z=t_1$.

(ii) If $\lambda=-3$, we have,

$$\begin{bmatrix} 5 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_{13}\begin{bmatrix} -1 & 2 & 3 \\ 2 & 0 & 2 \\ 5 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2+2R_1 \\ R_3+5R_1 \end{matrix}\begin{bmatrix} -1 & 2 & 3 \\ 0 & 4 & 8 \\ 0 & 8 & 16 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} -R_1 \\ (1/4)R_2 \\ R_3-2R_2 \end{matrix}\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x-2y+3z=0, \quad y+2z=0.$

Putting $z=t, \quad y=-2t$.

$\therefore \quad$ The solution is $x=-t,\ y=-2t,\ z=t$.

Example 2 (b) : For what value of λ, the following system of equations possesses a non-trivial solution ? Obtain the solution for real values of λ.

$3x_1+x_2-\lambda x_3=0,\ 4x_1-2x_2-3x_3=0,\ 2\lambda x_1+4x_2-\lambda x_3=0.$ **(M.U. 2003, 12, 13)**

Sol. : We have $\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The system will have nontrivial solution if the rank of A is less than the number of unknowns 3. (Case II, page 10-29). The rak of A will be less than 3 if $|A|=0$.

Now, $\begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix}=0$

$\therefore \quad 3[-2\lambda+12]-1[4\lambda+6\lambda]-\lambda[16+4\lambda]=0$

$\therefore \quad -4\lambda^2-32\lambda+36=0 \qquad \therefore \quad \lambda^2+8\lambda-9=0$

$\therefore \quad (\lambda+9)(\lambda-1)=0 \qquad \therefore \quad \lambda=1,-9.$

(i) When $\lambda=1$, we have,

$$\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_1-R_3\begin{bmatrix} 1 & -3 & -2 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2-4R_1 \\ R_3-2R_1 \end{matrix}\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 10 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_3-R_2 \\ \frac{1}{5}R_2 \end{matrix}\begin{bmatrix} 1 & -3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 - 3x_2 + 2x_3 = 0, \ \ 2x_2 + x_3 = 0$

Let $x_2 = t \quad \therefore \ x_3 = -2t \qquad \therefore \ x_1 = 3x_2 - 2x_3 = -t.$

Hence, when $\lambda = 1$, the solution is $x_1 = -t, \ \ x_2 = t, \ \ x_3 = -2t.$

(ii) When $\lambda = -9$, we have,

$$\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_{12}\begin{bmatrix} 4 & -2 & -3 \\ 3 & 1 & 9 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_2 - R_1\begin{bmatrix} 1 & -3 & -12 \\ 3 & 1 & 9 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 + 18R_1 \end{matrix}\begin{bmatrix} 1 & -3 & -12 \\ 0 & 10 & 45 \\ 0 & -50 & -225 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} (1/5)R_2 \\ -(1/25)R_3 \end{matrix}\begin{bmatrix} 1 & -3 & -12 \\ 0 & 2 & 9 \\ 0 & 2 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By} \quad R_3 - R_2\begin{bmatrix} 1 & -3 & -12 \\ 0 & 2 & 9 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x_1 - 3x_2 - 12x_3 = 0 \qquad \therefore \ 2x_2 + 9x_3 = 0$

Let $x_3 = 2t \quad \therefore \ x_2 = -9t \qquad \therefore \ x_1 = 3x_2 + 12x_3 = -27t + 24t = -3t.$

Hence, when $\lambda = -9$, the solution is $x_1 = -3t, \ \ x_2 = -9t, \ \ x_3 = 2t.$

Example 3 (b) : Discuss for all values of k the system of equations

$2x + 3ky + (3k+4)z = 0, \quad x + (k+4)y + (4k+2)z = 0,$

$x + 2(k+1)y + (3k+4)z = 0.$ **(M.U. 1995)**

Sol. : Taking the second equation first, consider the determinant of the coefficients.

$$\begin{vmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2k+2 & 3k+4 \end{vmatrix} = 0$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \longrightarrow \begin{vmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{vmatrix} = 0 \qquad \text{.................... (1)}$$

$\therefore \quad (k-8)(-k+2) + 5k(k-2) = 0$

$\therefore \quad (k-2)(-k+8+5k) = 0 \quad \therefore \ 4(k-2)(k+2) = 0 \qquad \therefore \ k = 2, \ -2.$

(i) **When $k \neq \pm 2$**

The rank of the coefficient matrix is equal to the number of unknowns (= 3). Hence, the only possible solution is zero solution

$\therefore \quad x = 0, \ \ y = 0, \ \ z = 0$

[Even otherwise from the last row of (1) we get $(k-2)(y-z) = 0$

$\therefore \quad y = 0, \ \ z = 0$. Then from the first row we get $x = 0$.]

(ii) **When $k = 2$:** From (1), we get,

$$\begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad x + 6y + 10z = 0, \ \ 6y + 10z = 0$

If we put $z = t$, $y = (-5/3)\,t$, $x = 0$

$\therefore\ x = 0,\ y = (-5/3)\,t,\ z = t.$

(iii) When $k = -2$: From (1), we get,

$$\begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{By } \xrightarrow[(-1/10)R_2]{R_3-(2/5)R_2} \begin{bmatrix} 1 & 2 & -6 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ x + 2y - 6z = 0,\ y - z = 0$

Putting $z = t$, $y = t$ $\therefore\ x = 4t$ $\therefore\ x = 4t,\ y = t,\ z = t.$

Example 4 (b) : Find the value of k for which the system of equations has a non-trivial solution

$$\begin{aligned} (3k-8)\,x + 3y + 3z &= 0 \\ 3x + (3k-8)\,y + 3z &= 0 \\ 3x + 3y + (3k-8)\,z &= 0 \end{aligned}$$

Sol. : We consider the determinant of the coefficient matrix.

$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

The system has non-trivial solution if the rank of A is less than the number of unknowns three. This, requires $|A| = 0$.

$$\text{By } R_3 - R_2 \quad \begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 0 & -3k+11 & 3k-11 \end{vmatrix} = 0$$

$$\text{By } C_2 + C_3 \quad \begin{vmatrix} 3k-8 & 6 & 3 \\ 3 & 3k-5 & 3 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0$$

$\therefore\ [\,(3k-8)(3k-5) - 6\times 3\,]\,(3k-11) = 0$

$\therefore\ (3k-11)(9k^2 - 15k - 24k + 40 - 18) = 0$

$\therefore\ (3k-11)(9k^2 - 39k + 22) = 0$

$\therefore\ (3k-11)(3k-11)(3k-2) = 0$

$\therefore\ k = 11/3,\ 2/3.$

The system has non-trivial solution if $k = 2/3,\ 11/3$.

[Or by direct simplification of the determinant, we get

$27k^3 - 216k^2 + 495k - 242 = 0$

$\therefore\ (3k-2)(9k^2 - 66k + 121) = 0$

$\therefore\ (3k-2)(3k-11)^2 = 0$]

Example 5 (b) : Show that the system of equations

$$ax + by + cz = 0, \qquad bx + cy + az = 0, \qquad cx + ay + bz = 0$$

has a non-trivial solution if $a + b + c = 0$ or if $a = b = c$. Find the non-trivial solution when the condition is satisfied.

(M.U. 2003, 06, 09)

Sol. : We have $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ *i.e.* $AX=0$ .... (1)

The system has non-trivial solution if $|A|=0$ *i.e.* if

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$$

$\therefore \quad a(bc-a^2)-b(b^2-ac)+c(ab-c^2)=0$

$\therefore \quad a^3+b^3+c^3-3abc=0$

$\therefore \quad (a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$ **[Note this]**

$\therefore \quad a+b+c=0$ or $a^2+b^2+c^2-ab-bc-ca=0$

$\therefore \quad \frac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]=0$ **[Note this]**

$\therefore \quad a=b,\ b=c,\ c=a.$

$\therefore$ The system has non-trivial solution if $a+b+c=0$ or if $a=b=c$.

Now, the system (1) can be written as (by $R_3+R_1+R_2$)

$$\begin{bmatrix} a & b & c \\ b & c & a \\ a+b+c & a+b+c & a+b+c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad ax+by+cz=0\ ; \qquad bx+cy+az=0\ ;$

$(a+b+c)(x+y+z)=0.$

If $a+b+c=0$, the above system reduces to only two distinct equations.

$ax+by+cz=0\ ; \qquad bx+cy+az=0.$

$$\therefore \quad \frac{x}{ab-c^2}=-\frac{y}{a^2-bc}=\frac{z}{ac-b^2}=t, \text{ say}$$

$\therefore$ The solution is $x=(ab-c^2)\,t,\ \ y=(bc-a^2)\,t,\ \ z=(ac-b^2)\,t.$

If $a=b=c$, the system is equivalent to a single equation

$x+y+z=0.$

$\therefore$ If $y=t_1,\quad z=t_2$ then $x=-(t_1+t_2)$ is the required solution.

Aliter : The system has non-trivial solution if the rank of the coefficient matrix on the l.h.s. of A is less than the number of unknowns *i.e.* if $\rho(A)=2$.

The rank will be less than 3, if the determinant of A is zero. The determinant is zero if $a+b+c=0$ or the first and the second rows are identical *i.e.* if $a=b,\ b=c,\ c=a$ *i.e.* $a=b=c$.

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

(A) Solve the following equations : Class (b) : 6 Marks

1. $x_1-2x_2+3x_3=0,\ 2x_1+5x_2+6x_3=0.$ **[Ans. :** $x_1=-3t,\ x_2=0,\ x_3=t$ **]**
2. $2x_1-x_2+3x_3=0,\ 3x_1+2x_2+x_3=0,\ x_1-4x_2+5x_3=0.$ **[Ans. :** $x_1=-t,\ x_2=t,\ x_3=t$ **]**
3. $x_1-x_2+x_3=0,\ x_1+2x_2+x_3=0,\ 2x_1+x_2+3x_3=0$ **[Ans. :** $x_1=x_2=x_3=0$ **]**

4. $x_1 + x_2 + x_3 + x_4 = 0$, $2x_1 + x_2 - x_4 = 0$, $x_1 + 3x_2 + 2x_3 + 4x_4 = 0$.
[**Ans. :** $x_1 = t$, $x_2 = -t$, $x_3 = -t$, $x_4 = t$]

5. $7x_1 + x_2 - 2x_3 = 0$, $x_1 + 5x_2 - 4x_3 = 0$, $3x_1 - 2x_2 + x_3 = 0$, $2x_1 - 7x_2 + 5x_3 = 0$.
[**Ans. :** $x_1 = \frac{3}{17}t$, $x_2 = \frac{13}{17}t$, $x_3 = t$]

6. $4x - y + 2z + t = 0$, $2x + 3y - z - 2t = 0$, $7y - 4z - 5t = 0$, $2x - 11y + 7z + 8t = 0$.
[**Ans. :** $x = -\frac{5r+s}{14}$, $y = \frac{4r+5s}{7}$, $z = r$, $t = s$]

7. $3x_1 + 4x_2 - x_3 - 9x_4 = 0$, $2x_1 + 3x_2 + 2x_3 - 3x_4 = 0$,
$2x_1 + x_2 - 14x_3 - 12x_4 = 0$, $x_1 + 3x_2 + 13x_3 + 3x_4 = 0$.
(M.U. 2004, 16) [**Ans. :** $x_1 = 11t$, $x_2 = -8t$, $x_3 = t$, $x_4 = 0$]

(B) Class (b) : 6 Marks

1. Find the value of λ for which the following equations have non-zero solutions. Also solve the equations.
$x + 2y + 3z = \lambda x$, $3x + y + 2z = \lambda y$, $2x + 3y + z = \lambda z$. **(M.U. 2003, 05, 06)**
[**Ans. :** $\lambda = 6$, $x = y = z = t$]

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (a) : 4 Marks

State the conditions when a system of non-homogeneous linear equations has
(i) a unique solution ; (ii) infinite solutions ; (iii) no solutions.

Summary

1. **Vectors are independent if**
$$k_1 X_1 + k_2 X_2 + \ldots\ldots + k_r X_r = O$$
implies $k_1 = 0$, $k_2 = 0$, ..., $k_r = 0$.

Vectors are dependent if
$$k_1 X_1 + k_2 X_2 + \ldots\ldots + k_r X_r = O \quad \text{and} \quad k_1, k_2, \ldots\ldots, k_r \text{ not all are zero.}$$

2. **Consistency for non-homogeneous equations**
(1) Inconsistent if rank $A <$ Rank [A, B]
(2) Consistent if rank A = rank B
(a) If $r = n$, unique solution ; (b) If $r < n$, infinite solutions.

3. **Homogeneous equations**
(1) If $r = n$, zero solution, trivial solutions.
(2) If $r < n$, non-trivial solutions.

☆ ☆ ☆

CHAPTER 11

Numerical Solutions of Transcendental and Linear Equations

1. Introduction

In this chapter, we are going to study methods of solving equations - algebraic and transcendental by methods, known as numerical methods. You might have come across some algebraic equations, or transcendental equations where usual analytical methods fail. In such cases numerical methods are used. However, while analytical methods give exact values, numerical methods give approximate values. The difference between the two, as you know, we call error.

2. Algebraic and Transcendental Equations

An expression of the form

$$P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$$

where $a_0, a_1, a_2, \ldots\ldots a_n$ are constants and n is a positive integer, is called an **algebraic polynomial of degree n** if $a_0 \neq 0$. The values of x which satisfy the equation $P_n(x) = 0$ are called the **zeroes** of the polynomial and every polynomial of n^{th} degree has n zeroes. Geometrically speaking a zero of polynomial equation $P_n(x) = 0$ is the value of x where the curve $y = P_n(x)$ intersects the x-axis.

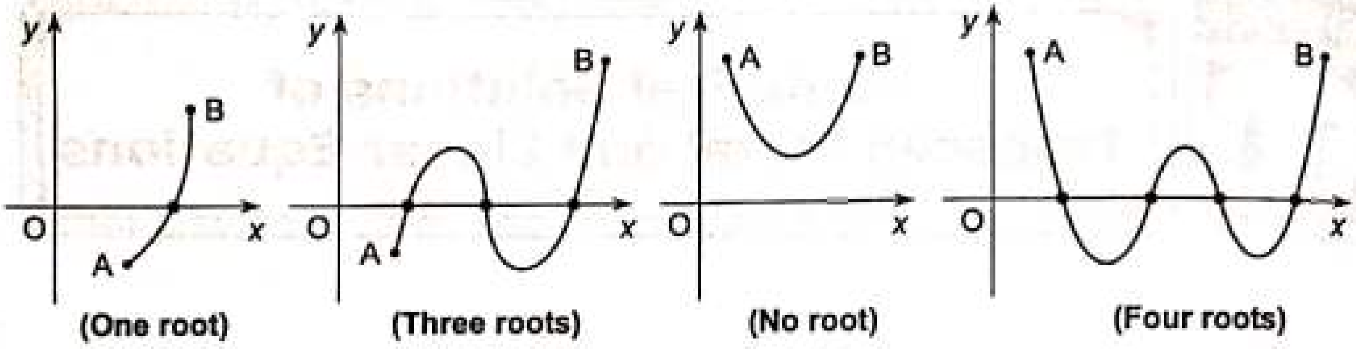
The equation $P_n(x) = 0$ is called an **algebraic equation**, if $P_n(x)$ is an algebraic polynomial. On the other hand, if $P_n(x)$ contains other functions, such as trigonometric, logarithmic, exponential also, then $P_n(x) = 0$ is called a **transcendental equation** *e.g.* $x^5 - 4x^3 + 2x - 1 = 0$ is an algebraic equation and $2x^3 - \log(x+2) \cdot \tan x + e^x = 0$ is a transcendental equation.

You know how to solve the equation of the type $ax^2 + bx + c = 0$. Either by factorisation or by using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, we can find the roots of this equation. In general, an equation is solved by factorisation. But in many cases the method of factorisation fails, because the polynomial is not factorisable. In such cases numerical methods studied in the following articles are used. These methods are in a way tedious but with the help of calculating machines or computers, we can, in a short time obtain the solution of the given equation to desired accuracy.

3. Theorems on the Roots of $P_n(x) = 0$

We state below some theorems on the real roots of a polynomial equation $P_n(x) = 0$.

(i) **If $P_n(x)$ is a real and continuous function between $x = a$ and $x = b$ and if $P_n(a)$ and $P_n(b)$ are of opposite signs, then there are one or odd number of real roots of $P_n(x) = 0$ between $x = a$ and $x = b$.**



If $P_n(a)$ and $P_n(b)$ are of the same sign then there are no or even number of real roots of $P_n(x) = 0$ between $x = a$ and $x = b$.

(ii) If $P_n(x)$ is an algebraic polynomial of odd degree with real coefficients, then it has at least one real root of sign opposite to the sign of (a_0 / a_n).

If $P_n(x) = 0$ is of even degree, then it has at least two real roots, one positive and one negative provided $a_n < 0$.

4. Methods of Solving the Equations

We shall now learn two methods of solving the equation $P_n(x) = 0$. We first find an approximate root of the equation and then successively improve upon it. This is the general approach of all these methods. We consider below :

1. Regula-Falsi Method.
2. Newton-Raphson Method.

5. Regula Falsi Method (False Position Method)

Let $y = f(x)$ be the given equation and the curve intersect the x-axis in R. Then the x-coordinate *i.e.* abscissa of R is the exact root of the given equation. Suppose further, we know that x lies between two values x_1 and x_2. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Let P and Q be the points (x_1, y_1) and (x_2, y_2).

Since x lies between x_1 and x_2 the points P and Q must lie on opposite sides of the x-axis *i.e.* $y_1 \times y_2 = f(x_1) \cdot f(x_2) < 0$. Now, consider the line PQ intersecting the x-axis in R_1. Suppose the abscissa of R_1 is x_3. Then x_3 is closer to x than x_1 and x_2 and this is the first approximation of the root.

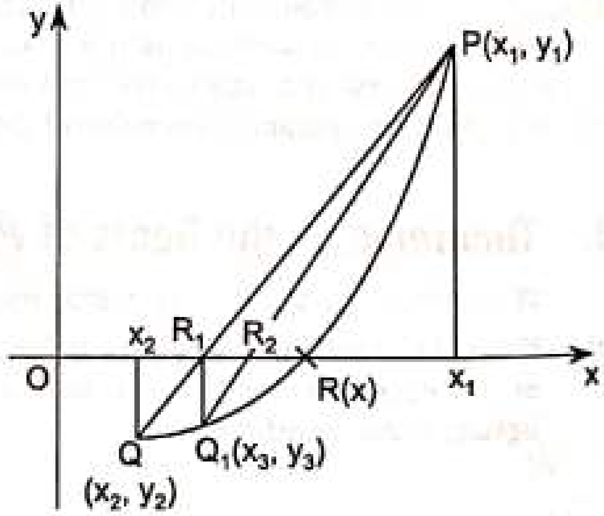
To find x_3 algebraically, consider the equation of the line PQ through $P(x_1, y_1)$ and $Q(x_2, y_2)$. Its equation by two-point form is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

i.e. $\quad y(x_1 - x_2) + x(y_2 - y_1) = x_1 y_2 - x_2 y_1.$

To find the point of inter-section R_1 of the line PQ with the x-axis, we put $y = 0$ and get,

$$x = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$$



Thus, $\boxed{x_3 = \dfrac{x_1 y_2 - x_2 y_1}{y_2 - y_1}}$

Suppose, $Q_1(x_3, y_3)$ is the point on the curve whose x-coordinate is equal to x_3 *i.e.* the x-coordinate of R_1. We now find the point where the secant PQ_1 intersects the x-axis. If this point is $R_2(x_4, 0)$, then x_4 is a still better approximation of x.

In this way we can find better and better approximations of the root of the equation $f(x) = 0$.

This method is known as Regula Falsi or False Position Method.

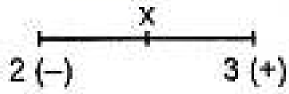
Example 1 (b) : Find the approximate value of the root of the equation $x^3 - 2x^2 - 5 = 0$ lying between 2 and 3 by false position method. (Take three iterations)

Sol. : Let $y = f(x) = x^3 - 2x^2 - 5, \quad x_1 = 2, \; x_2 = 3$

$\therefore \quad y_1 = f(x_1) = f(2) = 8 - 8 - 5 = -5 < 0$

$y_2 = f(x_2) = f(3) = 27 - 18 - 5 = 4 > 0$

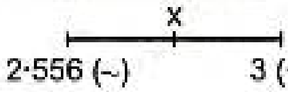
Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.



$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(4) - 3(-5)}{4 - (-5)} = \frac{23}{9} = 2{\cdot}556 \qquad \ldots\ldots\ldots\ldots (1)$$

$\therefore \quad y_3 = (2{\cdot}556)^3 - 2\,(2{\cdot}556)^2 - 5 = -1{\cdot}368 < 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·556 to 3, there is a root between 2·556 and 3.

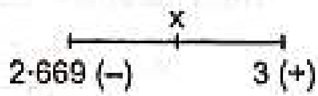


First Iteration : Let $x_1 = 3$, $y_1 = 4$, $x_2 = 2{\cdot}556$, $y_2 = -1{\cdot}368$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(-1{\cdot}368) - 2{\cdot}566\,(4)}{-1{\cdot}368 - 4} = 2{\cdot}669 \qquad \ldots\ldots\ldots\ldots (2)$$

Now, $\quad y_3 = (2{\cdot}669)^3 - 2\,(2{\cdot}669)^2 - 5 = -0{\cdot}234 < 0.$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·669 to 3, there is a root between 2·669 and 3.

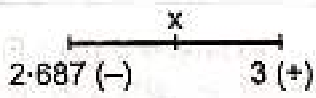


Second Iteration : Let $x_1 = 3$, $y_1 = 4$, $x_2 = 2{\cdot}669$, $y_2 = -0{\cdot}234$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(-0{\cdot}234) - 2{\cdot}669\,(4)}{-0{\cdot}234 - 4} = 2{\cdot}687 \qquad \ldots\ldots\ldots\ldots (3)$$

Now, $\quad y_3 = (2{\cdot}687)^3 - 2\,(2{\cdot}687)^2 - 5 = -0{\cdot}0399 < 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·687 to 3, there is a root between 2·687 and 3.



Third Iteration : Let $x_1 = 3$, $y_1 = 4$, $x_2 = 2{\cdot}687$, $y_2 = -0{\cdot}0399$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(-0{\cdot}0399) - 2{\cdot}687\,(4)}{-0{\cdot}0399 - 4} = 2{\cdot}690 \qquad \ldots\ldots\ldots\ldots (4)$$

$\therefore \quad x = 2{\cdot}690.$

Example 2 (b) : Find the root of the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by the method of false position. (Take three iterations)

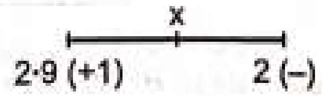
Sol. : Let $y = f(x) = x^3 - 5x - 7, \quad x_1 = 2, \; x_2 = 3$

$\therefore \quad y_1 = f(x_1) = f(2) = 2^3 - 5\,(2) - 7 = -9 < 0$

$y_2 = f(x_2) = f(3) = 3^3 - 5\,(3) - 7 = 5 > 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

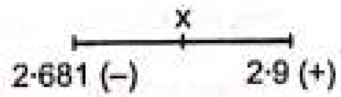
Since the difference between y_1 and y_2 is large, we try $x_2 = 2{\cdot}9$.



$$\therefore \quad y_2 = (2{\cdot}9)^3 - 5\,(2{\cdot}9) - 7 = 2{\cdot}889 > 0$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 2·9, there is a root between 2·9 and 2.

Let $x_1 = 2{\cdot}9,\ x_2 = 2,\ y_1 = 2{\cdot}889,\ y_2 = -9$.



$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}9(-9) - 2(2{\cdot}889)}{-9 - 2{\cdot}889} = 2{\cdot}681$$

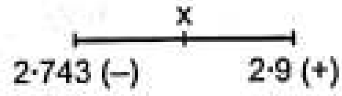
$$\therefore \quad y_3 = (2{\cdot}681)^3 - 5\,(2{\cdot}681) - 7 = -1{\cdot}135$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·681 to 2·9, there is a root between 2·681 and 2·9.

First Iteration : Let $x_1 = 2{\cdot}681,\ x_2 = 2{\cdot}9,\ y_1 = -1{\cdot}135,\ y_2 = 2{\cdot}889$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}681(2{\cdot}889) - 2{\cdot}9(-1{\cdot}135)}{2{\cdot}889 - (-1{\cdot}135)} = 2{\cdot}743$$

Now, $\quad y_3 = (2{\cdot}743)^3 - 5\,(2{\cdot}743) - 7 = -0{\cdot}076 < 0$.

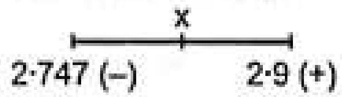


Since $f(x)$ changes its sign from negative to positive as x goes from 2·743 to 2·9, there is a root between 2·743 and 2·9.

Second Iteration : Let $x_1 = 2{\cdot}743,\ x_2 = 2{\cdot}9,\ y_1 = -0{\cdot}076,\ y_2 = 2{\cdot}889$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}743\,(2{\cdot}889) - 2{\cdot}9\,(-0{\cdot}076)}{2{\cdot}889 - (-0{\cdot}076)} = 2{\cdot}747$$

Now, $\quad y_3 = (2{\cdot}747)^3 - 2\,(2{\cdot}747) - 7 = -0{\cdot}006 < 0$



Since $f(x)$ changes its sign from negative to positive as x goes from 2·747 to 2·9, there is a root between 2·9 and 2·747.

Third Iteration : Let $x_1 = 2{\cdot}747,\ x_2 = 2{\cdot}9,\ y_1 = -0{\cdot}006,\ y_2 = 2{\cdot}889$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}747\,(2{\cdot}889) - 2{\cdot}9\,(-0{\cdot}006)}{2{\cdot}889 - (-0{\cdot}006)} = 2{\cdot}747$$

Hence, the root of the equation correct to three places of decimals = 2.747.

Note ....

Note that by starting with a narrow interval we arrive at a more accurate result.

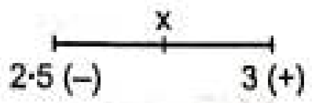
Example 3 (b) : Find approximate value of the root of the equation $x^3 - 9x + 1 = 0$ lying between 2·5 and 3 by false position method.

Sol. : Let $\quad y = f(x) = x^3 - 9x + 1, \quad x_1 = 2{\cdot}5,\ x_2 = 3$

$$\therefore \quad y_1 = f(x_1) = f(2{\cdot}5) = (2{\cdot}5)^3 - 9\,(2{\cdot}5) + 1 = -5{\cdot}875 < 0$$

$$y_2 = f(x_2) = f(3) = 3^3 - 9\,(3) + 1 = 1 > 0$$

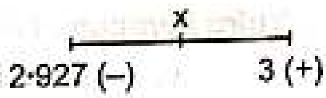
Since $f(x)$ changes its sign from negative to positive as x goes from 2·5 to 3, there is a root between 2·5 and 3.



$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}5\,(1) - 3\,(-5{\cdot}875)}{1 - (-5{\cdot}875)} = 2{\cdot}927$$

$$\therefore \quad y_3 = (2{\cdot}927)^3 - 9\,(2{\cdot}927) + 1 = -0{\cdot}266 < 0$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·927 to 3, there is a root between 2·927 and 3.

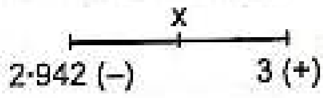


First Iteration : Let $x_1 = 3$, $x_2 = 2{\cdot}927$, $y_1 = 1$, $y_2 = -0{\cdot}266$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(-0{\cdot}266) - 2{\cdot}927(1)}{-0{\cdot}266 - 1} = 2{\cdot}942$$

Now, $y_3 = (2{\cdot}942)^3 - 9\,(2{\cdot}942) + 1 = -0{\cdot}014 < 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 2·942 to 3, there is a root between 2·942 and 3.

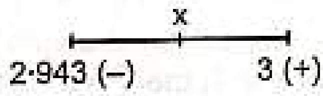


Second Iteration : Let $x_1 = 3$, $x_2 = 2{\cdot}942$, $y_1 = 1$, $y_2 = -0{\cdot}014$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(-0{\cdot}014) - 2{\cdot}942(1)}{-0{\cdot}014 - 1} = 2{\cdot}943$$

Now, $y_3 = (2{\cdot}943)^3 - 9\,(2{\cdot}943) + 1 = -0{\cdot}003 < 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 2·943 to 3, there is a root between 2·943 and 3.



Third Iteration : Let $x_1 = 2{\cdot}943$, $x_2 = 3$, $y_1 = 0{\cdot}003$, $y_2 = 1$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}943(1) - 3(-0{\cdot}003)}{1 - (-0{\cdot}003)} = 2{\cdot}943$$

Hence, the root of the equation nearest to three places of decimals = 2·943.

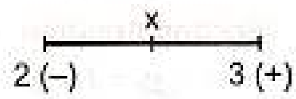
Example 4 (b) : By using Regula falsi method solve $2x - 3\sin x - 5 = 0$. **(M.U. 2018)**

Sol. : Let $\quad y = f(x) = 2x - \sin x - 5$

We first find that $\quad y_0 = f(0) < 0, \qquad y_1 = f(1) < 0,$

$\qquad y_2 = f(2) < 0 \quad$ and $\quad y_3 = f(3) > 0.$

Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and x.



First iteration : Let $x_1 = 2$, $x_2 = 3$.

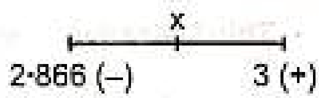
$$\therefore \quad y_1 = f(x_1) = -3{\cdot}7279 \quad \text{and} \quad y_2 = f(x_2) = 0{\cdot}5766.$$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(0{\cdot}5766) - 3(-3{\cdot}7279)}{0{\cdot}5766 - (-3{\cdot}7279)}$$

$$= \frac{12{\cdot}336}{4{\cdot}3045} = 2{\cdot}8660$$

$$\therefore \quad y_3 = 2\,(2{\cdot}8660) - 3\sin(2{\cdot}8660) - 5 = -0{\cdot}0845 < 0$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·8660 to 3, there is a root between 2·8660 and 3.



Second Iteration : Let $x_1 = 2{\cdot}8660$, $x_2 = 3$.

$$\therefore \quad y_1 = -0{\cdot}0845 \quad \text{and} \quad y_2 = 0{\cdot}5766.$$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(2{\cdot}8660)(0{\cdot}5766) - 3(-0{\cdot}0845)}{0{\cdot}5766 - (-0{\cdot}0845)}$$

$$= \frac{1{\cdot}9060}{0{\cdot}6611} = 2{\cdot}8830$$

$$\therefore \quad y_3 = 2\,(2{\cdot}8830) - 3\sin(2{\cdot}8830) - 5 = -0{\cdot}001161 < 0$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·8830 to 3, there is a root between 2·8830 and 3.

Third Iteration : Let $x_1 = 2{\cdot}8830$, $x_2 = 3$.

$\therefore \quad y_1 = -0{\cdot}001161$ and $y_2 = 0{\cdot}5766$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(2{\cdot}883)(0{\cdot}5766) - 3(-0{\cdot}001161)}{0{\cdot}5766 - (-0{\cdot}001161)} = \frac{1{\cdot}6658}{0{\cdot}5778} = 2{\cdot}8830$$

Hence, the root of the equation nearest to three places of decimals = 2·8830.

Example 5 (b) : Find the root of $x^2 - x - 1 = 0$ by regula falsi method upto third approximation.

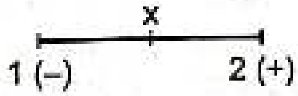
Sol. : Let $\quad y = f(x) = x^2 - x - 1$

We first find the interval in which a root lies as it is not given.

Let $\quad x = 0, \quad f(x) = 0 - 0 - 1 = -1 < 0$

Let $\quad x = 1, \quad f(x) = 1 - 1 - 1 = -1 < 0$

Let $\quad x = 2, \quad f(x) = 4 - 2 - 1 = 1 > 0$

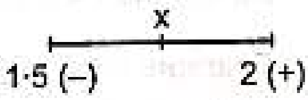


Since $f(x)$ changes its sign from negative to positive as x goes from 1 to 2, there is a root between 1 and 2.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{1(1) - (2)(-1)}{1 - (-1)} = \frac{3}{2} = 1{\cdot}5$$

First Iteration : Let $x_1 = 1{\cdot}5$, $x_2 = 2$.

$\therefore \quad y_1 = f(x_1) = f(1{\cdot}5) = (1{\cdot}5)^2 - 1{\cdot}5 - 1 = -0{\cdot}25 < 0$

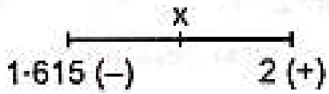


Since $f(x)$ changes its sign from negative to positive as x goes from 1·5 to 2, there is a root between 1·5 and 2.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(1{\cdot}5)(1) - 2(-0{\cdot}25)}{1 - (-0{\cdot}25)} = \frac{1{\cdot}5 + 0{\cdot}5}{1{\cdot}25} = 1{\cdot}6$$

Second Iteration : Let $x_1 = 1{\cdot}6$, $x_2 = 2$.

$\therefore \quad y_1 = f(x_1) = f(1{\cdot}6) = (1{\cdot}6)^2 - 1{\cdot}6 - 1 = -0{\cdot}04 < 0$



Since $f(x)$ changes its sign from negative to positive as x goes from 1·6 to 2, there is a root between 1·6 and 2.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(1{\cdot}6)(1) - 2(-0{\cdot}04)}{1 - (-0{\cdot}04)} = \frac{1{\cdot}6 + 0{\cdot}08}{1{\cdot}04} = \frac{1{\cdot}68}{1{\cdot}04} = 1{\cdot}615$$

$\therefore \quad y_3 = (1{\cdot}615)^2 - 1{\cdot}615 - 1 = -0{\cdot}007 < 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 1·615 to 2, there is a root between 2 and 1·615.

Third Iteration : Let $x_1 = 2$, $x_2 = 1{\cdot}615$ and $y_1 = 1$, $y_2 = -0{\cdot}007$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(-0{\cdot}007) - 1{\cdot}615(1)}{-0{\cdot}007 - 1} = 1{\cdot}615$$

The root of the equation correct to three places of decimals = 1·615.

Example 6 (b) : Find the root of the equation $x^3 - 4x - 9 = 0$ by false position method lying between 2 and 3. **(M.U. 2019)**

Sol. : Let $\quad y = f(x) = x^3 - 4x - 9$. $\quad$ Here, $x_1 = 2$ and $x_2 = 3$.

$\therefore \quad y_1 = f(x_1) = f(2) = 2^3 - 4(2) - 9 = 8 - 8 - 9 = -9 < 0$

$\quad\quad y_2 = f(x_2) = f(3) = 3^3 - 4(3) - 9 = 27 - 12 - 9 = 6 > 0$

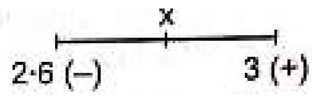
Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

The root is given by

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(6) - 3(-9)}{6-(-9)} = \frac{39}{15} = 2 \cdot 6$$

Now, $y_3 = f(x_3) = f(2 \cdot 6) = (2 \cdot 6)^3 - 4(2 \cdot 6) - 9 = -1 \cdot 82 > 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·6 to 3, there is a root between 2·6 and 3.

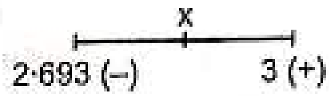


First Iteration : Let $x_1 = 2 \cdot 6$, $x_2 = 3$, $y_1 = -1 \cdot 82$, $y_2 = 6$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2 \cdot 6(6) - 3(-1 \cdot 82)}{6 - (-1 \cdot 82)} = 2 \cdot 693$$

$$y_3 = f(x_3) = (2 \cdot 693)^3 - 4(2 \cdot 693) - 9 = -0 \cdot 242 > 0$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·693 to 3, there is a root between 2·693 and 3.

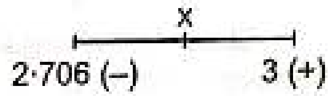


Second Iteration : Let $x_1 = 2 \cdot 693$, $x_2 = 3$, $y_1 = -0 \cdot 262$, $y_2 = 6$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2 \cdot 693(6) - 3(-0 \cdot 262)}{6 - (-0 \cdot 262)} = 2 \cdot 7058 = 2 \cdot 706$$

$$y_3 = f(x_3) = (2 \cdot 706)^3 - 4(2 \cdot 706) - 9 = -0 \cdot 009 > 0$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·706 to 3, there is a root between 2·706 and 3.



Third Iteration : Let $x_1 = 3$, $x_2 = 2 \cdot 706$, $y_1 = 6$, $y_2 = -0 \cdot 009$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(-0 \cdot 03) - 2 \cdot 706(6)}{-0 \cdot 009 - 6} = 2 \cdot 706$$

Hence, the root correct to three places of decimals = 2·706.

Example 7 (b) : Find the real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position. (Take two iterations)

Sol. : We have $\quad f(x) = x^3 - 2x - 5$

We first find the interval in which a root lies as it is not given.

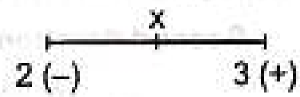
Let $\quad x = 0, \quad f(x) = 0 - 0 - 5 = -5 < 0$

Let $\quad x = 1, \quad f(x) = 1 - 2 - 5 = -6 < 0$

Let $\quad x = 2, \quad f(x) = 8 - 4 - 5 = -1 < 0$

Let $\quad x = 3, \quad f(x) = 27 - 6 - 5 = 16 > 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.



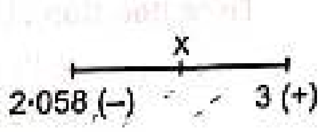
Let $x_1 = 2$, $x_2 = 3$, $y_1 = -1$, $y_2 = 16$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{32 + 3}{17} = 2 \cdot 058$$

Now, $\quad y_3 = (2 \cdot 058)^3 - 2(2 \cdot 058) - 5$

$$= 8 \cdot 7164 - 4 \cdot 116 - 5 = -0 \cdot 3996 < 0.$$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·058 to 3, there is a root between 2·058 and 3.

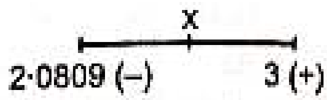


First Iteration : Let $x_1 = 2{\cdot}058,\ x_2 = 3,\ y_1 = -0{\cdot}3996,\ y_2 = 16.$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}058\,(16) - 3\,(-0{\cdot}3996)}{16 - (-0{\cdot}3996)}$$

$$= \frac{32{\cdot}928 + 1{\cdot}1988}{16{\cdot}3996} = 2{\cdot}0809$$

Now, $y_3 = (2{\cdot}0809)^3 - 2\,(2{\cdot}0809) - 5 = -0{\cdot}1512 < 0.$



Since $f(x)$ changes its sign from negative to positive as x goes from 2·0809 to 3, there is a root between 2·0809 and 3.

Second Iteration : Let $x_1 = 2{\cdot}0809,\ x_2 = 3,\ y_1 = -0{\cdot}1512,\ y_2 = 16.$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2{\cdot}0809\,(16) - 3\,(-0{\cdot}1512)}{16 - (-0{\cdot}1512)}$$

$$= \frac{33{\cdot}2944 + 0{\cdot}4536}{16{\cdot}1512} = 2{\cdot}0895$$

$$\therefore \quad x = 2{\cdot}089$$

Example 8 (b) : Using regula falsi method find the root of the equation $x^2 - 2x - 1 = 0$.

Sol. : We have $\quad f(x) = x^2 - 2x - 1$

$$\therefore \quad f(0) = 0 - 0 - 1 = -1 < 0; \qquad f(1) = 1 - 2 - 1 = -2 < 0;$$

$$f(-1) = 1 + 2 - 1 = 2 > 0.$$

Since $f(x)$ changes its sign from positive to negative as x goes from -1 to 0, there is a root between -1 and 0.

Let $\quad x_1 = -1,\ x_2 = 0,\ y_1 = 2,\ y_2 = -1.$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(-1)(-1) - (0)(2)}{-1 - 2} = -\frac{1}{3} = -0{\cdot}3333$$

Now, $y_3 = (-0{\cdot}3333)^2 - 2\,(-0{\cdot}3333) - 1 = -0{\cdot}2223 < 0$

Since $f(x)$ changes its sign from positive to negative as x goes from -1 to $-0{\cdot}333$, there is a root between -1 and $-0{\cdot}333$.

First Iteration : Let $x_1 = -1,\ x_2 = -0{\cdot}3333,\ y_1 = 2,\ y_2 = -0{\cdot}2223.$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(-1)(-0{\cdot}2223) - (-0{\cdot}3333)(2)}{-0{\cdot}2223 - 2} = -0{\cdot}3999$$

Now, $y_3 = (-0{\cdot}3999)^2 - 2\,(-0{\cdot}3999) - 1 = -0{\cdot}0402 < 0.$

Since $f(x)$ changes its sign from positive to negative as x goes from -1 to $-0{\cdot}3999$, there is a root between -1 and $-0{\cdot}3999$.

Second Iteration : Let $x_1 = -1,\ x_2 = -0{\cdot}3999,\ y_1 = 2,\ y_2 = -0{\cdot}0402$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(-1)(-0{\cdot}0402) - (-0{\cdot}3999)(2)}{-0{\cdot}0402 - 2} = -0{\cdot}4117$$

Now, $y_3 = (-0{\cdot}4117)^2 - 2\,(-0{\cdot}4117) - 1 = -0{\cdot}0071 < 0.$

Since $f(x)$ changes its sign from positive to negative as x goes from -1 to $-0{\cdot}4117$, there is a root between -1 and $-0{\cdot}4117$.

Third Iteration : Let $x_1 = -1,\ x_2 = -0{\cdot}04117,\ y_1 = 2,\ y_2 = -0{\cdot}0071$

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(-1)(-0{\cdot}0071) - (-0{\cdot}4117)(2)}{-0{\cdot}0071 - 2} = -0{\cdot}4138$$

$$\therefore \quad x = -0{\cdot}4138.$$

Example 9 (b) : Find the root of the equation $x^3 - 9x + 1$ between 2 and 3 using regula-falsi method.

Sol. : We are given $x_1 = 2$, $x_2 = 3$.

Now, $f(x_1) = 2^3 - 18 + 1 = -9 < 0$ and $f(x_2) = 3^3 - 27 + 1 = 1 > 0$.

As $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

Let $x_1 = 2$, $x_2 = 3$ $\quad \therefore \quad y_1 = -9$, $y_2 = 1$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(2)(1) - (3)(-9)}{1-(-9)} = \frac{29}{10} = 2{\cdot}9$$

Now, $y_3 = (2{\cdot}9)^3 - 9\,(2{\cdot}9) + 1 = -0{\cdot}711 < 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 2·9 to 3, there is a root between 2·9 and 3.

First Iteration : Let $x_1 = 2{\cdot}9$, $x_2 = 3$, $y_1 = -0{\cdot}711$, $y_2 = 1$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(2{\cdot}9)(1) - (3)(-0{\cdot}711)}{1-(-0{\cdot}711)} = 2{\cdot}9415$$

Now, $y_3 = (2{\cdot}9415)^2 - 9\,(2{\cdot}9415) + 1 = -0{\cdot}0224 < 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 2.9415 to 3, there is a root between 2·9415 and 3.

Second Iteration : Let $x_1 = 2{\cdot}9415$, $x_2 = 3$, $y_1 = -0{\cdot}0224$, $y_2 = 1$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(2{\cdot}9415)(1) - (3)(-0{\cdot}0224)}{1-(-0{\cdot}0224)} = 2{\cdot}9428$$

Now, $y_3 = (2{\cdot}9428)^2 - 9\,(2{\cdot}9428) + 1 = -0{\cdot}0003 < 0$.

Since y_3 is very close to zero, we do not take next iteration.

$\therefore \quad x = 2{\cdot}9428$.

Example 10 (b) : Using regula-falsi method solve $x^2 + x - 3 = 0$.

Sol. : We have $\quad f(x) = x^2 + x - 3$

$\therefore \quad f(0) = 0 + 0 - 3 = -3 < 0; \qquad f(1) = 1 + 1 - 3 = -1 < 0;$

$\quad f(-1) = 1 - 1 - 3 = -3 < 0; \qquad f(2) = 4 + 2 - 3 = 3 > 0.$

Since $f(x)$ changes its sign from negative to positive as x goes from 1 to 2, there is a root between 1 and 2.

Let $x_1 = 1$, $x_2 = 2$, $y_1 = -1$, $y_2 = 3$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(1)(3) - (2)(-1)}{3-(-1)} = \frac{5}{4} = 1{\cdot}25$$

Now, $y_3 = (1{\cdot}25)^2 + 1{\cdot}25 - 3 = -0{\cdot}1875 < 0$

And $\quad f(2) = 3 > 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 1.2515 to 2, there is a root between 1·25 and 2.

First Iteration : Let $x_1 = 1{\cdot}25$, $x_2 = 2$, $y_1 = -0{\cdot}1875$, $y_2 = 3$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(1{\cdot}25)(3) - (2)(-0{\cdot}1875)}{3-(-0{\cdot}1875)} = 1{\cdot}2941$$

Now, $y_3 = (1{\cdot}2941)^2 + 1{\cdot}2941 - 3 = -0{\cdot}0312 < 0$.

And $\quad f(2) = 3 > 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 1.2941 to 3, there is a root between 1·2941 and 3.

Second Iteration : Let $x_1 = 1{\cdot}2941$, $x_2 = 2$, $y_1 = -0{\cdot}2312$, $y_2 = 3$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(1{\cdot}2941)(3) - (2)(-0{\cdot}0312)}{3-(-0{\cdot}0312)} = 1{\cdot}3014$$

Now, $y_3 = (1{\cdot}3014)^2 + 1{\cdot}3014 - 3 = -0{\cdot}004 < 0$.

Since y_3 is sufficiently close to zero we do not take the next iteration.

$\therefore \quad x = 1{\cdot}3014$.

Example 11 (b) : Obtain the root of $x^3 - x - 1 = 0$ by regula falsi method. (Take three iterations).

(M.U. 2018)

Sol. : We have $\quad f(x) = x^3 - x - 1$

$\therefore \quad f(0) = 0 - 0 - 1 = -1 < 0; \qquad f(1) = 1 - 1 - 1 = -1 < 0;$

$f(2) = 8 - 2 - 1 = 5 > 0.$

Since $f(x)$ changes its sign from negative to positive as x goes from 1 and 2, there is a root between 1 and 2.

Let $x_1 = 1$, $x_2 = 2$, $y_1 = -1$, $y_2 = 5$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{1(5) - 2(-1)}{5-(-1)} = \frac{7}{6} = 1{\cdot}667$$

Now, $y_3 = (1{\cdot}667)^3 - 1{\cdot}667 - 1 = 1{\cdot}9654 > 0$

Since $f(x)$ changes its sign from negative to positive as x goes from 1 to 1·667, there is a root between 1 and 1·9654.

First Iteration : Let $x_1 = 1$, $x_2 = 1{\cdot}667$, $y_1 = -1$, $y_2 = 1{\cdot}9654$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{1(1{\cdot}9654) - 1{\cdot}667(-1)}{1{\cdot}9654 - (-1)} = 1{\cdot}2249$$

Now, $y_3 = (1{\cdot}2249)^3 - 1{\cdot}2249 - 1 = -0{\cdot}3871 < 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 1·2249 to 1·667, there is a root between 1·2249 and 1·667.

Second Iteration : Let $x_1 = 1{\cdot}2249$, $x_2 = 1{\cdot}667$, $y_1 = -0{\cdot}3871$, $y_2 = 1{\cdot}9654$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(1{\cdot}2249)(1{\cdot}9654) - (1{\cdot}667)(-0{\cdot}3871)}{1{\cdot}9654 - (-0{\cdot}3871)}$$

$$= \frac{3{\cdot}0527}{2{\cdot}3525} = 1{\cdot}2976$$

Now, $y_3 = (1{\cdot}2976)^3 - 1{\cdot}2976 - 1 = -0{\cdot}1127 < 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 1·2976 to 1·667, there is a root between 1·2976 and 1·667.

Third Iteration : Let $x_1 = 1{\cdot}2976$, $x_2 = 1{\cdot}667$, $y_1 = 0{\cdot}1127$, $y_2 = 1{\cdot}9654$.

$$\therefore \quad x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{(1{\cdot}2976)(1{\cdot}9654) - (1{\cdot}667)(-0{\cdot}1127)}{1{\cdot}9654 - (-0{\cdot}1127)}$$

$$= \frac{2{\cdot}7382}{2{\cdot}0781} = 1{\cdot}3176$$

$\therefore \quad x = 1{\cdot}3176$.

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Using false position method, find the approximate value of the root of the equation.

1. $x^3 + x - 1 = 0$ lying between 0·5 and 1.
2. $x^3 - 2x + 0{\cdot}5 = 0$ lying between 0 and 1.
3. $x^3 + 2x - 20 = 0$ lying between 2 and 3.
4. $3x - \cos x - 1 = 0$ lying between 0 and 1.
5. $x^3 - 3x - 5 = 0$ lying between 2 and 2·5.
6. $x \log_{10} x - 1{\cdot}2 = 0$ lying between 2 and 3.
7. $x^3 - 4x + 1 = 0$
8. $x^3 + 2x^2 - 8 = 0$
9. $x^3 - 9x + 1 = 0$
10. $x^3 - x - 4 = 0$

[**Ans. :** **(1)** 0·672, **(2)** 0·5, **(3)** 2·38, **(4)** 0·6071, **(5)** 2·279,
(6) 2·7406, **(7)** 0·2541, **(8)** 1·5038, **(9)** 0·1109, **(10)** 1·7945]

6. Newton-Raphson Method

Newton-Raphson method uses Taylor's series to find an approximate value of a root.

Let x_0 be an approximate value of a root of the equation $f(x) = 0$. Let $x_0 + h$ be the exact value of he corresponding root where h is very small. Since, $x_0 + h$ is an exact root of $f(x) = 0$, we get $f(x_0 + h) = 0$.

But by Taylor's theorem, we have

$$f(x_0 + h) = f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$$

Since, h is very small, neglecting second and higher order terms in h, we get,

$$f(x_0) + h f'(x_0) = 0 \quad i.e. \quad h = -\frac{f(x_0)}{f'(x_0)}$$

$\therefore$ The first approximation of the root gives

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Considering x_1 as the approximation of the root and $x_1 + h$ as the exact value of the root by the above reasoning, we get, the second approximation as,

$$x_2 = x_1 + h = x_1 - \frac{f(x_1)}{f'(x_1)}$$

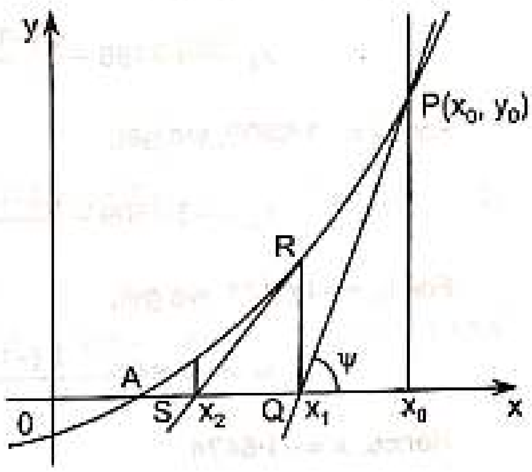
Continuing the process, we have

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

This is known as Newton-Raphson Method.

Geometrical Explanation

Let the equation be $f(x) = 0$ and let $y = f(x)$ be the curve as shown in the figure. Then x-coordinate of A where the curve intersects the x-axis is the root of the equation. Newton-Raphson method consists of replacing the part of the curve between the starting point $P[x_0, f(x_0)]$ and the point $A[a, f(a)]$ by the tangent at P.



Suppose, we start with $P(x_0, y_0)$. If the tangent at P intersects the x-axis in Q and if the x-coordinate of Q is x_1, then as seen above

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

If the ordinate at Q intersects the curve in R, then R is $(x_1, f(x_1))$. By drawing a tangent at R, we get a point S whose x-coordinate x_2 is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Thus, we get closer and closer to the point A.

Example 1 (b) : Using Newton-Raphson method find the root of the equation $2x^3 - 3x + 4 = 0$ lying between -2 and -1 correct to four places of decimals.

Sol. : Since $f(-2) = 2(-2)^3 - 3(-2) + 4 = -6$

and $f(-1) = 2(-1)^3 - 3(-1) + 4 = 5$

Since $f(x)$ changes its sign from negative to positive as x goes from -2 to -1, there is a root between -2 and -1.

We start with $x_0 = -1$.

Since, $f(x) = 2x^3 - 3x + 4$, $f'(x) = 6x^2 - 3$.

$$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \ldots..$$

$$\therefore \quad x_{n+1} = x_n - \frac{2x_n^3 - 3x_n + 4}{6x_n^2 - 3}$$

For $x_0 = -1$, we get, $\quad x_1 = -1 - \dfrac{2(-1)^3 - 3(-1) + 4}{6(-1)^2 - 3} = -2.6667$

For $x_1 = -2.6667$, we get,

$$x_2 = -2.6667 - \frac{2(-2.6667)^3 - 3(-2.6667) + 4}{6(-2.6667)^3 - 3} = -2.0131$$

For $x_2 = -2.0131$, we get,

$$x_3 = -2.0131 - \frac{2(-2.0131)^3 - 3(-2.0131) + 4}{6(-2.0131)^2 - 3} = -1.7186$$

For $x_3 = -1.7186$, we get,

$$x_4 = -1.7186 - \frac{2(-1.7186)^3 - 3(-1.7186) + 4}{6(-1.7186)^2 - 3} = -1.6509$$

For $x_4 = -1.6509$, we get,

$$x_5 = -1.6509 - \frac{2(-1.6509)^3 - 3(-1.6509) + 4}{6(-1.6509)^2 - 3} = -1.6474$$

For $x_5 = -1.6474$, we get,

$$x_6 = -1.6474 - \frac{2(-1.6474)^3 - 3(-1.6474) + 4}{6(-1.6474)^2 - 3} = -1.6474$$

Hence, $x = -1.6474$.

Example 2 (b) : Compute the real root of $x \log_{10} x - 1{\cdot}2 = 0$ correct to three places of decimals using Newton-Raphson method. **(M.U. 2019)**

Sol. : We first note that $f(x) = x \log_{10} x - 1{\cdot}2$.

$\therefore \quad f(1) = 1 \log_{10} 1 - 1{\cdot}2 = -1{\cdot}2, \quad f(2) = 2 \log_{10} 2 - 1{\cdot}2 = -0{\cdot}5979,$
$f(3) = 3 \log_{10} 3 - 1{\cdot}2 = 0{\cdot}2313.$

Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

Now, $f'(x) = x \cdot \dfrac{1}{x \log_e 10} + \log_{10} x = (\log_e 10)^{-1} + \log_{10} x = 0.4343 + \log_{10} x$

Hence, by Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots..$$
$$= x_n - \frac{x \log_{10} x - 1.2}{0{\cdot}4343 + \log_{10} x}$$

For $x_0 = 3$, $\quad x_1 = 3 - \dfrac{3 \log_{10} 3 - 1.2}{0{\cdot}4343 + \log_{10} 3} = 2{\cdot}74615$

For $x_1 = 2{\cdot}74615$, $\quad x_2 = 2{\cdot}74615 - \dfrac{(2{\cdot}74615) \cdot \log(2{\cdot}74615) - 1{\cdot}2}{0{\cdot}4343 + \log 2{\cdot}74615} = 2{\cdot}7406$

For $x_2 = 2{\cdot}7406$, $\quad x_3 = 2{\cdot}7406$. $\therefore$ Hence, $x = 2{\cdot}7406$.

Example 3 (b) : Using Newton-Raphson method for the equation $x^3 - 5x + 3 = 0$, find the root starting with $x_0 = 1$ as initial value with an accuracy of 0·0001.

Sol. : Since $f(x) = x^3 - 5x + 3$, $f'(x) = 3x^2 - 5$.

$$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 5x_n + 3}{3x_n^2 - 5}$$

When $x_0 = 1$, $\quad x_1 = 1 - \dfrac{(1)^3 - 5(1) + 3}{3(1)^2 - 5} = 0{\cdot}5$

When $x_1 = 0{\cdot}5$, $\quad x_2 = 0{\cdot}5 - \dfrac{(0{\cdot}5)^3 - 5(0{\cdot}3) + 3}{3(0{\cdot}5)^2 - 5} = 0{\cdot}6471$

When $x_2 = 0{\cdot}6471$, $\quad x_3 = 0{\cdot}6471 - \dfrac{(0{\cdot}6471)^3 - 5(0{\cdot}6471) + 3}{3(0{\cdot}6471)^2 - 5} = 0{\cdot}6566$

When $x_3 = 0{\cdot}6566$, $\quad x_4 = 0{\cdot}6566 - \dfrac{(0{\cdot}6566)^3 - 5(0{\cdot}6566) + 3}{3(0{\cdot}6566)^2 - 5} = 0{\cdot}6566$

Hence, $x = 0{\cdot}6566$.

Example 4 (b) : Find a positive root of $x^3 + x - 1 = 0$ by Newton-Raphson method, correct to three places of decimals.

Sol. : Since $f(x) = x^3 + x - 1$, $f(0) = -1 < 0$ and $f(1) = 1 > 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 0 to 1, there is root between 0 and 1.

By Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1} = \frac{2x_n^3 + 1}{3x_n^2 + 1}$$

We start with $x_0 = 1$.

For $x_0 = 1$, $\quad x_1 = \frac{2 \cdot 1^3 + 1}{3 \cdot 1^2 + 1} = \frac{3}{4} = 0 \cdot 75$

For $x_1 = 0 \cdot 75$, $\quad x_2 = \frac{2(0 \cdot 75)^3 + 1}{3(0 \cdot 75)^2 + 1} = \frac{1 \cdot 84375}{2 \cdot 6875} = 0 \cdot 6860$

For $x_2 = 0 \cdot 6860$, $\quad x_3 = \frac{2(0 \cdot 6860)^3 + 1}{3(0 \cdot 6860)^2 + 1} = \frac{1 \cdot 6457}{2 \cdot 4118} = 0 \cdot 6823$

For $x_3 = 0 \cdot 6823$, $\quad x_4 = \frac{2(0 \cdot 6823)^3 + 1}{3(0 \cdot 6823)^2 + 1} = \frac{1 \cdot 6353}{2 \cdot 3966} = 0 \cdot 6823$

Hence, $x = 0 \cdot 6823$.

Example 5 (b) : Find a root of $x^3 - 9x^2 - 18 = 0$ by Newton-Raphson method.

Sol. : Since $f(x) = x^3 - 9x^2 + 18$, $f(0) = 18 > 0$, $f(1) = 10 > 0$, $f(2) = -10 < 0$.

Since $f(x)$ changes its sign from positive to negative as x goes from 1 to 2, there is a root between 1 and 2.

By Newton-Raphson Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 9x_n + 18}{3x_n^2 - 18x_n} = \frac{2x_n^3 - 9x_n^2 - 18}{3x_n^2 - 18x_n}$$

We start with $x_0 = 2$.

For $x_0 = 2$, $\quad x_1 = \frac{2(8) - 9(4) - 18}{3(4) - 18(2)} = \frac{38}{24} = 1 \cdot 5833$

For $x_1 = 1 \cdot 5833$, $\quad x_2 = \frac{2(1 \cdot 5833)^3 - 9(1 \cdot 5833)^2 - 18}{3(1 \cdot 5833)^2 - 18(1 \cdot 5833)} = \frac{32 \cdot 6233}{20 \cdot 9787} = 1 \cdot 5551$

For $x_2 = 1 \cdot 5551$, $\quad x_3 = \frac{2(1 \cdot 5551)^3 - 9(1 \cdot 5551)^2 - 18}{3(1 \cdot 5551)^2 - 18(1 \cdot 5551)} = \frac{32 \cdot 2435}{20 \cdot 7368} = 1 \cdot 5549$

For $x_3 = 1 \cdot 5549$, $\quad x_4 = \frac{2(1 \cdot 5549)^3 - 9(1 \cdot 5549)^2 - 18}{3(1 \cdot 5549)^2 - 18(1 \cdot 5549)} = 1 \cdot 5549$

Hence, $x = 1 \cdot 5549$.

Example 6 (b) : Find the root of $x^4 - x - 10 = 0$ which is near to 2 correct to two places of decimals using Newton-Raphson method.

Sol. : Since $f(x) = x^4 - x - 10$; $\quad f(1) = -10 < 0$ and $f(2) = 4 > 0$.

Since $f(x)$ changes its sign from negative to positive as x goes from 1 to 2, there is a root between 1 and 2. Since we are required to find the root near to 2.

By Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 + x_n - 10}{4x_n^3 - 1} = \frac{3x_n^4 + 10}{3x_n^3 - 1}$$

We start with 2.

For $x_0 = 2$, $\quad x_1 = \dfrac{3(2)^4 + 10}{4(2)^3 - 1} = 1{\cdot}8710$

For $x_1 = 1{\cdot}8710$, $\quad x_2 = \dfrac{3(1{\cdot}8710)^4 + 10}{4(1{\cdot}8710)^3 - 1} = \dfrac{46{\cdot}7635}{25{\cdot}1988} = 1{\cdot}8588$

For $x_2 = 1{\cdot}8588$, $\quad x_3 = \dfrac{3(1{\cdot}8588)^4 + 10}{4(1{\cdot}8588)^3 - 1} = \dfrac{45{\cdot}8139}{24{\cdot}6896} = 1{\cdot}8556$

Hence, $x = 1{\cdot}8556$.

Example 7 (b) : Find the real root of $x^3 - 2x - 5 = 0$ correct to three decimal places using Newton-Raphson method. **(M.U. 2017)**

Sol. : Since $f(x) = x^3 - 2x - 5$; $\quad f(0) = -5 < 0,\ f(1) = -6 < 0,\ f(2) = -1 < 0,\ f(3) = 16 > 0.$

Since $f(x)$ changes its sign from negative to positive as x goes from 2 to 3, there is a root between 2 and 3.

By Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

Since $f(2)$ is nearer to zero than $f(3)$, we start with $x_0 = 2$.

For $x_0 = 2$, $\quad x_1 = \dfrac{2(2)^3 + 5}{3(2)^2 - 2} = 2{\cdot}1$

For $x_1 = 2{\cdot}1$, $\quad x_2 = \dfrac{2(2{\cdot}1)^3 + 5}{3(2{\cdot}1)^2 - 2} = \dfrac{23{\cdot}522}{11{\cdot}23} = 2{\cdot}0946$

For $x_2 = 2{\cdot}0946$, $\quad x_3 = \dfrac{2(2{\cdot}0946)^3 + 5}{3(2{\cdot}0946)^2 - 2} = \dfrac{23{\cdot}3795}{11{\cdot}1620} = 2{\cdot}0946$

Hence, $x = 2{\cdot}0946$.

Example 8 (b) : Using Newton-Rapson method solve $3x - \cos x - 1 = 0$ correct to three decimal places. **(M.U. 2017)**

Sol. : Since $f(0) = -2 < 0$ and $f(1) = 1{\cdot}45 > 0$.

$f(x)$ changes its sign from negative to positive as x goes from 0 to 1. There is a root between 0 and 1.

Since $f(1)$ is nearer to zero than $f(0)$, we start with $x_0 = 1$.

By Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

We start with $x_0 = 1$.

For $x_0 = 1$, $\quad x_1 = 1 - \dfrac{3 - \cos 1 - 1}{3 + \sin 1} = 1 - \dfrac{1{\cdot}4596}{3{\cdot}8415} = 1 - 0{\cdot}3799 = 0{\cdot}6200$

For $x_1 = 0{\cdot}6200$, $\quad x_2 = 0{\cdot}62 - \dfrac{3(0{\cdot}62) - \cos(0{\cdot}62) - 1}{3 + \sin(0{\cdot}62)} = 0{\cdot}62 - \dfrac{0{\cdot}04612}{3{\cdot}5810}$

$$= 0{\cdot}62 - 0{\cdot}0129 = 0{\cdot}6071$$

For $x_2 = 0{\cdot}6071$, $\quad x_3 = 0{\cdot}6071 - \dfrac{3(0{\cdot}6071) - \cos(0{\cdot}6071) - 1}{3 + \sin(0{\cdot}6071)}$

$$= 0{\cdot}6071 - 0{\cdot}0000 = 0{\cdot}6071$$

Hence, the required root upto 4 places of decimals is 0·6071.

Example 9 (b) : Derive a formula to compute $\sqrt[3]{N}$ where N is a positive number and hence, estimate $\sqrt[3]{11}$ upto three places of decimals.

Sol. : Let $x = \sqrt[3]{N}$ $\quad \therefore \quad x^3 - N = 0$

Now, by Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $f(x_n) = x_n^3 - N$ $\quad \therefore \quad f'(x_n) = 3x_n^2$

Iterative Formula : $\quad x_{n+1} = x_n - \dfrac{x_n^3 - N}{3x_n^2} = \dfrac{2x_n^3 + N}{3x_n^2}$

This is the required iterative formula, to find $\sqrt[3]{N}$.

Now, to find $\sqrt[3]{11}$ we start with $x_0 = 2$ and $N = 11$.

Since $\sqrt[3]{8} = 2$ and $\sqrt[3]{27} = 3$, it is easy to see that $\sqrt[3]{11}$ is nearer to 2 than to 3. Hence, we start with $x_0 = 2$.

For $x_0 = 2$, $\quad x_1 = \dfrac{2(2)^3 + 11}{3(2)^2} = 2{\cdot}25$

For $x_1 = 2{\cdot}25$, $\quad x_2 = \dfrac{2(2{\cdot}25)^3 + 11}{3(2{\cdot}25)^2} = \dfrac{33{\cdot}78125}{15{\cdot}1875} = 2{\cdot}2243$

For $x_2 = 2{\cdot}2243$, $\quad x_3 = \dfrac{2(2{\cdot}2243)^3 + 11}{3(2{\cdot}2243)^2} = \dfrac{33{\cdot}0095}{14{\cdot}8425} = 2{\cdot}2240$

Hence, $x = 2{\cdot}2240$.

Example 10 (b) : Find an iterative formula for $\sqrt{N}$ where N is a positive number and hence, find $\sqrt{10}$.

Sol. : Let $x = \sqrt{n}$ $\quad \therefore \quad x^2 - N = 0$

To solve this equation by Newton-Raphson iterative formula, we have $f(x) = x^2 - N$. Hence, Newton-Raphson formula gives,

Iterative Formula : $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^2 - N}{2x_n} = \dfrac{x_n^2 + N}{2x_n}$

This is the required iterative formula.

To find $\sqrt{10}$, we start with $x_0 = 3$ and $N = 10$.

Now, $\quad x_{n+1} = \dfrac{x_n^2 + 10}{2\,x_n}$

For $x_0 = 3$, $\quad x_1 = \dfrac{3^2 + 10}{2(3)} = \dfrac{19}{6} = 3{\cdot}166$

For $x_1 = 3{\cdot}166$, $\quad x_2 = \dfrac{(3{\cdot}166)^2 + 10}{2(3{\cdot}166)} = \dfrac{20{\cdot}0273}{6{\cdot}3332} = 3{\cdot}1622$

For $x_2 = 3{\cdot}1622$, $\quad x_3 = \dfrac{(3{\cdot}1622)^2 + 10}{2(3{\cdot}1622)} = \dfrac{19{\cdot}9995}{6{\cdot}3244} = 3{\cdot}1622$

Hence, $\sqrt{10} = 3{\cdot}1622$.

Example 11 (b) : Using Newton-Raphson method find $\sqrt[5]{35}$ correct to four decimal places.

Sol. : Let $x = \sqrt[5]{35}$ $\quad \therefore \; x^5 = 35 \quad \therefore \; x^5 - 35 = 0$

By Newton-Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{Let } f(x) = x^5 - 35.$$

Iterative Formula : $\quad x_{n+1} = x_n - \dfrac{x_n^{\,5} - 35}{5x_n^{\,4}} = \dfrac{4x_n^{\,5} + 35}{5x_n^{\,4}}$

We start with $x_0 = 2$.

For $x_0 = 2$, $\quad x_1 = \dfrac{4(2)^5 + 35}{5(2)^4} = \dfrac{163}{80} = 2{\cdot}0375$

For $x_1 = 2{\cdot}0375$, $\quad x_2 = \dfrac{4(2{\cdot}0375)^5 + 35}{5(2{\cdot}0375)^4} = \dfrac{175{\cdot}4585}{86{\cdot}1709} = 2{\cdot}0362$

For $x_2 = 2{\cdot}0362$, $\quad x_3 = \dfrac{4(2{\cdot}0362)^5 + 35}{5(2{\cdot}0362)^4} = \dfrac{175{\cdot}0101}{85{\cdot}9512} = 2{\cdot}0362$

$\therefore \quad \sqrt[5]{35} = 2{\cdot}0362$

Example 12 (b) : Using Newton-Raphson method find approximate value of $\sqrt[3]{150}$. (Take two iterations)

Sol. : Let $x = \sqrt[3]{150}$ $\quad \therefore \; x^3 = 150 \quad \therefore \; x^3 - 150 = 0$

By Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{Let } f(x) = x^3 - 150.$$

Iterative Formula : $\quad x_{n+1} = x_n - \dfrac{x_n^{\,3} - 150}{3x_n^{\,2}} = \dfrac{2x_n^{\,3} + 150}{3x_n^{\,2}}$

We start with $x_0 = 5$.

For $x_0 = 5$, $\quad x_1 = \dfrac{2(5)^3 + 150}{3(5)^2} = \dfrac{400}{75} = 5{\cdot}3333$

For $x_1 = 5{\cdot}3333$, $\quad x_2 = \dfrac{2(5{\cdot}3333)^3 + 150}{3(5{\cdot}3333)^2} = \dfrac{453{\cdot}4017}{85{\cdot}3323} = 5{\cdot}3134$

For $x_2 = 5{\cdot}3134$, $\quad x_3 = \dfrac{2(5{\cdot}3134)^3 + 150}{3(5{\cdot}3134)^2} = \dfrac{450{\cdot}0181}{84{\cdot}6966} = 5{\cdot}3133$

$\therefore \quad \sqrt[3]{150} = 5{\cdot}3133$

Example 13 (b) : Using Newton-Raphson Method evaluate $\frac{1}{\sqrt{12}}$, taking three iterations.

Sol. : Let $x = \frac{1}{\sqrt{12}}$ $\quad \therefore \quad x^2 - \frac{1}{12} = 0$

By Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \qquad \text{Let } f(x) = x^2 - \frac{1}{12}.$$

Iterative Formula :

$$x_{n+1} = x_n - \frac{x_n^3 - (1/12)}{2x_n} = \frac{x_n^2 + (1/12)}{x_n} = \frac{x_n^2 + 0{\cdot}0833}{2x_n}$$

Since $f(0) = -\frac{1}{12}$ and $f(1) = \frac{11}{12}$.

The root lies between 0 and 1.

We start with $x_0 = 1$.

For $x_0 = 1$, $\quad x_1 = \frac{1{\cdot}0833}{2} = 0{\cdot}54165$

For $x_1 = 0{\cdot}54165$, $\quad x_2 = \frac{(0{\cdot}54165)^2 + 0{\cdot}0833}{2(0{\cdot}54165)} = 0{\cdot}3477$

For $x_2 = 0{\cdot}3477$, $\quad x_3 = \frac{(0{\cdot}3477)^2 + 0{\cdot}0833}{2(0{\cdot}3477)} = 0{\cdot}2936$

For $x_3 = 0{\cdot}2936$, $\quad x_4 = \frac{(0{\cdot}2936)^2 + 0{\cdot}0833}{2(0{\cdot}2936)} = \frac{0{\cdot}1692}{0{\cdot}5872} = 0{\cdot}2887$

$$\therefore \quad \frac{1}{\sqrt{12}} = 0{\cdot}2887.$$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Using Newton-Raphson method, find the root of the following equations upto four places of decimals.

1. $x^3 + x - 1 = 0$. [**Ans. :** 0·6823]
2. $xe^x - 2 = 0$. [**Ans. :** 0·8526]
3. $x^3 - 2x + 0{\cdot}5 = 0$. [**Ans. :** 0·2586]
4. $x - \cos x = 0$. [**Ans. :** 0·7391]
5. $e^{-x} - \sin x = 0$. [**Ans. :** 0·5885]
6. $x \sin x + \cos x = 0$. [**Ans. :** 2·7984]
7. $x^3 - 9x + 1 = 0$. [**Ans. :** 0·1113]
8. $3x - \cos x - 1 = 0$. [**Ans. :** 0·6071]
9. $2 \tan x = 5e^{-x}$. [**Ans. :** 0·8290]
10. $xe^x - 1 = 0$. [**Ans. :** 0·5671]
11. $x^2 - 5x + 2 = 0$. [**Ans. :** 0·4384]
12. $x^3 + 2x - 20 = 0$. [**Ans. :** 2·4695]
13. $x \sin x + \cos x = 0$. [**Ans. :** 2·7984]
14. $\sin x = 1 - x$. [**Ans. :** 0·5110]
15. $x^3 - 5x + 3 = 0$, start with $x_0 = 2$. [**Ans. :** 1·8342]
16. Find by Newton-Raphson method upto four places of decimals.

    (i) $\sqrt{30}$, **(ii)** $\sqrt[3]{15}$, **(iii)** $\sqrt{38}$, **(iv)** $\sqrt[3]{79}$.

    [**Ans. :** **(i)** 5·4772, **(ii)** 2·4662, **(iii)** 6·1644, **(iv)** 4·2908.]

7. Solutions of Linear Algebraic Equations

We have learnt how to solve linear equations in three or four unknowns by matrix inversion method *i.e.* by using the inverse of the coefficient matrix. Now, we shall learn four more methods of solving linear algebraic equations. They are :-

1. **Gauss-Jacobi's Iterative Method**
2. **Gauss-Seidel Iterative Method**

8. Iterative Methods

The above method of solving simultaneous linear equations is called **direct methods.** In all this method the solutions of the equations are arrived at after a certain fixed amount of computations. There is another class of methods of solving simultaneous equations called **iterative methods.** In these methods we start with certain assumptions as to the values of the variables. By applying a method of this type we get a better approximation. We repeat (iterate) this procedure as many times as we want till we arrive at a desired accuracy.

9. Gauss-Jacobi's or Jacobi's Method

Consider the following system of equations,

$$\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned} \qquad \text{.....................} \ \textbf{(1)}$$

When a_1, b_2, c_3 are **large** as compared to remaining coefficients, we write the equations as

$$\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned} \qquad \text{.....................} \ \textbf{(2)}$$

We now start with the assumption that the roots of these equations are $x = x_0$, $y = y_0$, $z = z_0$. Putting these values in (2) the first approximation is given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

We now assume that the roots of the equations are $x = x_1$, $y = y_1$, $z = z_1$. Putting these values in (2), we get a better approximation given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

We repeat (iterate) the procedure as many times as we want till we arrive at a desired accuracy.

Carl Gustav Jacob Jacobi (1804 - 1851)

He was educated at the university of Berlin. He was initially an unsalaried lecturer paid from the fees of the students. But was made professor of mathematics at Königsbergh two years later. He was prolific contributor to various fields of mathematics. He made significant contributions to Elliptic Functions, Analysis, Differential Equations, Calculus of Variations and Infinite Series.

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Solve the following equations by Gauss-Jacobi's (Jcacobi's) iteration method.

$$15x + 2y + z = 18, \quad 2x + 20y - 3z = 19, \quad 3x - 6y + 25z = 22.$$

Sol. : We first write the equations as

$$x = \frac{1}{15}(18 - 2y - z)$$

$$y = \frac{1}{20}(19 - 2x + 3z) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ (1)$$

$$z = \frac{1}{25}(22 - 3x + 6y)$$

(i) First Iteration : We start with the approximation $x = 0, \ y = 0, \ z = 0$.

$$\therefore \ x_1 = \frac{18}{15} = 1{\cdot}2, \quad y_1 = \frac{19}{20} = 0{\cdot}95, \quad z_1 = \frac{22}{25} = 0{\cdot}88$$

(ii) Second Iteration : Putting these values on the right hand side of (1), we get,

$$x_2 = \frac{1}{15}[18 - 2(0{\cdot}95) - 0{\cdot}88] = 1{\cdot}0147$$

$$y_2 = \frac{1}{20}[19 - 2(1{\cdot}2) + 3(0{\cdot}88)] = 0{\cdot}962$$

$$z_2 = \frac{1}{25}[22 - 3(1{\cdot}2) + 6(0{\cdot}95)] = 0{\cdot}964$$

(iii) Third Iteration : Putting these values on the right hand side of (1), we get,

$$x_3 = \frac{1}{15}[18 - 2(0{\cdot}962) - 0{\cdot}964] = 1{\cdot}0075$$

$$y_3 = \frac{1}{20}[19 - 2(1{\cdot}0147) + 3(0{\cdot}964)] = 0{\cdot}9931$$

$$z_3 = \frac{1}{25}[22 - 3(1{\cdot}0147) + 6(0{\cdot}962)] = 0{\cdot}9891$$

(iv) Fourth Iteration : Putting these values on the right hand side of (1) again, we get,

$$x_4 = \frac{1}{15}[18 - 2(0{\cdot}9931) - 0{\cdot}9891] = 1{\cdot}0016$$

$$y_4 = \frac{1}{20}[19 - 2(1{\cdot}0075) + 3(0{\cdot}9891)] = 0{\cdot}9976$$

$$z_4 = \frac{1}{25}[22 - 3(1{\cdot}0075) + 6(0{\cdot}9931)] = 0{\cdot}9974$$

(v) Fifth Iteration : Putting these values on the right hand side of (1) again, we get

$$x_5 = \frac{1}{15}[18 - 2(0{\cdot}9976) - 0{\cdot}9974] = 1{\cdot}0005$$

$$y_5 = \frac{1}{20}[19 - 2(1{\cdot}0016) + 3(0{\cdot}9974)] = 0{\cdot}9994$$

$$z_5 = \frac{1}{25}[22 - 3(1{\cdot}0016) + 6(0{\cdot}9976)] = 0{\cdot}9980$$

(vi) Sixth Iteration : Putting these values on the right hand side of (1) again, we get

$$x_6 = \frac{1}{15}[18 - 2(0{\cdot}9994) - 0{\cdot}9980] = 1{\cdot}0002$$

$$y_6 = \frac{1}{20}[19 - 2(1{\cdot}0005) + 3(0{\cdot}9980)] = 0{\cdot}9996$$

$$z_6 = \frac{1}{25}[22 - 3(1{\cdot}0005) + 6(0{\cdot}9994)] = 0{\cdot}9998$$

Comparing the values obtained in the 5th and 6th iterations, we see that

$$x = 1, \ \ y = 1, \ \ z = 1.$$

Example 2 (c) : Solve by Gauss-Jacobi's (Jacobi's) method

$$4x + y + 3z = 17, \qquad x + 5y + z = 14, \qquad 2x - y + 8z = 12.$$

Sol. : We first write the equations as

$$x = \frac{1}{4}(17 - y - 3z)$$

$$y = \frac{1}{5}(14 - x - z) \qquad \text{.....................} \ (1)$$

$$z = \frac{1}{8}(12 - 2x + y)$$

(i) First Iteration : We start with the approximation $x = 0, \ y = 0, \ z = 0.$

$$\therefore \ \ x_1 = \frac{17}{4} = 4{\cdot}25, \quad y_1 = \frac{14}{5} = 2{\cdot}8, \quad z_1 = \frac{12}{8} = 1{\cdot}5.$$

(ii) Second Iteration : Putting these values on r.h.s. of (1), we get,

$$x_2 = \frac{1}{4}[17 - 2{\cdot}8 - 3(1{\cdot}5)] = 2{\cdot}425$$

$$y_2 = \frac{1}{5}(14 - 4{\cdot}25 - 1{\cdot}5) = 1{\cdot}65$$

$$z_2 = \frac{1}{8}[12 - 2(4{\cdot}25) + 2{\cdot}8] = 0{\cdot}7875$$

(iii) Third Iteration : Putting these values on r.h.s. of (1), we get,

$$x_3 = \frac{1}{4}\left[17 - 1\cdot65 - 3(0\cdot7875)\right] = 3\cdot2469$$

$$y_3 = \frac{1}{5}(14 - 2\cdot425 - 0\cdot7875) = 2\cdot1575$$

$$z_3 = \frac{1}{8}(12 - 2\cdot2425 + 1\cdot65) = 1\cdot1$$

(iv) Fourth Iteration : Putting these values on r.h.s. of (1) again, we get,

$$x_4 = \frac{1}{4}\left[17 - 2\cdot1575 - 3(1\cdot1)\right] = 2\cdot8856$$

$$y_4 = \frac{1}{5}(14 - 3\cdot2469 - 1\cdot1) = 1\cdot9306$$

$$z_4 = \frac{1}{8}\left[12 - 2(3\cdot2469) + 2\cdot1575\right] = 0\cdot9580$$

(v) Fifth Iteration : Putting these values on r.h.s. of (1) again, we get,

$$x_5 = \frac{1}{4}\left[17 - 1\cdot9306 - 3(0\cdot9580)\right] = 3\cdot0488$$

$$y_5 = \frac{1}{5}(14 - 2\cdot8856 - 0\cdot9580) = 2\cdot0313$$

$$z_5 = \frac{1}{8}\left[12 - 2(2\cdot8856) + 1\cdot9306\right] = 1\cdot0199$$

$\therefore \quad x = 3, \ y = 2, \ z = 1.$

Example 3 (c) : Solve by Gauss-Jacobi's (Jacobi's) method (Take three iterations).

$$10x + y + z = 12, \qquad x + 10y + z = 12, \qquad x + y + 10z = 12.$$

Sol. : We first write the equations as

$$x = \frac{1}{10}[12 - y - z]$$

$$y = \frac{1}{10}[12 - x - z] \qquad \text{......................} \ (1)$$

$$z = \frac{1}{10}[12 - x - y]$$

(i) First Iteration : We start with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$. Putting these values in (1), we get

$$x_1 = \frac{12}{10} = 1\cdot2, \quad y_1 = \frac{12}{10} = 1\cdot2, \quad z_1 = \frac{12}{10} = 1\cdot2.$$

(ii) Second Iteration : Putting these values in (1), we get

$$x_2 = \frac{1}{10}[12 - 1\cdot2 - 1\cdot2] = \frac{9\cdot6}{10} = 0\cdot96$$

$$y_2 = \frac{1}{10}[12 - 1\cdot2 - 1\cdot2] = \frac{9\cdot6}{10} = 0\cdot96$$

$$z_2 = \frac{1}{10}[12 - 1\cdot2 - 1\cdot2] = \frac{9\cdot6}{10} = 0\cdot96$$

(iii) Third Iteration : Putting these values in (1), we get

$$x_3 = \frac{1}{10}[12 - 0{\cdot}96 - 0{\cdot}96] = \frac{10{\cdot}08}{10} = 1{\cdot}008$$

$$y_3 = \frac{1}{10}[12 - 0{\cdot}96 - 0{\cdot}96] = \frac{10{\cdot}08}{10} = 1{\cdot}008$$

$$z_3 = \frac{1}{10}[12 - 0{\cdot}96 - 0{\cdot}96] = \frac{10{\cdot}08}{10} = 1{\cdot}008$$

$\therefore \quad x = 1{\cdot}008, \quad y = 1{\cdot}008, \quad z = 1{\cdot}008.$

Example 4 (c) : Solve by Gauss-Jacobi's (Jacobi's) method (Take three iterations).

$$20x + y - 2z = 17, \qquad 3x + 20y - z = -18, \qquad 2x - 3y + 20z = 25.$$

Sol. : We first write the equations as

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z] \qquad \text{.....................} \ (1)$$

$$z = \frac{1}{20}[25 - 2x + 3y]$$

(i) First Iteration : We start with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$. Putting these values in (1), we get

$$x_1 = \frac{17}{20} = 0{\cdot}85, \quad y_1 = -\frac{18}{20} = -0{\cdot}9, \quad z_1 = \frac{25}{20} = 1{\cdot}25$$

(ii) Second Iteration : Putting these values in (1), we get

$$x_2 = \frac{1}{20}[17 + 0{\cdot}9 + 2(1{\cdot}25)] = \frac{20{\cdot}4}{20} = 1{\cdot}02$$

$$y_2 = \frac{1}{20}[-18 - 3(0{\cdot}85) + 1{\cdot}25] = -\frac{19{\cdot}3}{20} = -0{\cdot}965$$

$$z_2 = \frac{1}{20}[25 - 2(0{\cdot}85) + 3(-0{\cdot}9)] = \frac{20{\cdot}6}{20} = 1{\cdot}03$$

(iii) Third Iteration : Putting these values in (1), we get

$$x_3 = \frac{1}{20}[17 - (-0{\cdot}965) + 2(1{\cdot}03)] = \frac{20{\cdot}025}{20} = 1{\cdot}0013$$

$$y_3 = \frac{1}{20}[-18 - 3(1{\cdot}02) + 1{\cdot}03] = -\frac{20{\cdot}03}{20} = -1{\cdot}0015$$

$$z_3 = \frac{1}{20}[25 - 2(1{\cdot}02) + 3(-0{\cdot}965)] = \frac{20{\cdot}065}{20} = 1{\cdot}0033$$

$\therefore \quad x = 1{\cdot}0013, \quad y = -1{\cdot}0015, \quad z = 1{\cdot}0033.$

Example 5 (c) : Solve by Gauss-Jacobi's (Jacobi's) method (Take three iterations)

$$5x - y + z = 10, \qquad x + 2y = 6, \qquad x + y + 5z = -1.$$

Sol. : We first write the equations as

$$x = \frac{1}{5}[10 + y - z]$$

$$y = \frac{1}{2}[6 - x - 0] \qquad \text{.....................} \ (1)$$

$$z = \frac{1}{5}[-1 - x - y]$$

(i) First Iteration : We start with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$. Putting these values in (1), we get

$$x_1 = \frac{10}{5} = 2, \quad y_1 = \frac{6}{2} = 3, \quad z_1 = -\frac{1}{5} = -0{\cdot}2$$

(ii) Second Iteration : Putting these values in (1), we get

$$x_2 = \frac{1}{5}[10 + 3 - (-0{\cdot}2)] = \frac{13{\cdot}2}{5} = 2{\cdot}64$$

$$y_2 = \frac{1}{2}[6 - 2 - 0] = \frac{4}{2} = 2$$

$$z_2 = \frac{1}{5}[-1 - 2 - 3] = -\frac{6}{5} = -1{\cdot}2$$

(iii) Third Iteration : Putting these values in (1), we get

$$x_3 = \frac{1}{5}[10 + 2 - (-1{\cdot}2)] = \frac{13{\cdot}2}{5} = 2{\cdot}64$$

$$y_3 = \frac{1}{2}[6 - 2{\cdot}64 - 0] = \frac{3{\cdot}36}{2} = 1{\cdot}68$$

$$z_3 = \frac{1}{5}[-1 - 2{\cdot}64 - 2] = -\frac{5{\cdot}64}{5} = -1{\cdot}128$$

$\therefore \quad x = 2{\cdot}64, \quad y = 1{\cdot}68, \quad z = -1{\cdot}128.$

Example 6 : Solve the equations $5x - y + z = 10$, $2x + 4y = 12$, $x + y + 5z = -1$ by Jacobi's method. (Take three iterations only) **(M.U. 2017)**

Sol. : We first write the equations as

$$x = \frac{1}{5}[10 + y - z]$$

$$y = \frac{1}{4}[12 - 2x + 0] \qquad \text{.....................} \ (1)$$

$$z = \frac{1}{5}[-1 - x - y]$$

(i) First Iteration : We start with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$. Putting these values in (1), we get

$$x_1 = \frac{1}{5}[10 + 0 - 0] = 2$$

$$y_1 = \frac{1}{4}[12 - 0 + 0] = 3$$

$$z_1 = \frac{1}{5}[-1 - 0 - 0] = -0{\cdot}2$$

(ii) Second Iteration : Putting these values in (1), we get

$$x_2 = \frac{1}{5}[10 + 3 - 0{\cdot}2] = 2{\cdot}64$$

$$y_2 = \frac{1}{4}[12 - 4 + 0] = 2$$

$$z_2 = \frac{1}{5}[-1 - 2 - 3] = -1{\cdot}2$$

(iii) Third Iteration : Putting these values in (1), we get

$$x_3 = \frac{1}{5}[10 + 2 + 1{\cdot}2] = 2{\cdot}64$$

$$y_3 = \frac{1}{4}[12 - 2 \times 2{\cdot}64] = 1{\cdot}68$$

$$z_3 = \frac{1}{5}[-1 - 2{\cdot}64 - 2] = -1{\cdot}28$$

Hence, after three iterations, $x = 2{\cdot}64$, $y = 1{\cdot}68$, $z = -1{\cdot}28$.

EXERCISE - III

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Solve the following equations by Gauss-Jacobi's (Jacobi's) method : Class (c) : 8 Marks

1. $10x + y + 2z = 13$, $2x + 10y + 3z = 15$, $x + 3y + 10z = 14$.
2. $15x + y - z = 14$, $x + 20y + z = 23$, $2x - 3y + 18z = 35$.
3. $12x + 2y + z = 27$, $2x + 15y - 3z = 16$, $2x - 3y + 25z = 26$.
4. $8x - y + 2z = 13$, $x - 10y + 3z = 17$, $3x + 2y + 12z = 25$.
5. $14x - y + 3z = 18$, $2x - 14y + 3z = 19$, $x - 3y + 16z = 20$.
6. $5x + 2y + z = 12$, $x + 4y + 2z = 15$, $x + 2y + 5z = 20$.
7. $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$.

(**Ans.** : Actual values are : (1) $x = 1, y = 1, z = 1$; (2) $x = 1, y = 1, z = 2$; (3) $x = 2, y = 1, z = 1$; (4) $x = 1, y = -1, z = 2$; (5) $x = 1, y = -1, z = 1$; (6) $x = 1, y = 2, z = 3$; (7) $x = 1, y = -1, z = 1$.

10. Gauss-Seidel Method

This is a modification of Gauss-Jacobi's method in which as soon as a new approximation of an unknown is obtained, it is used immediately in the next calculation.

Consider as before the system of equations,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2 \qquad \text{.....................} \ (1)$$
$$a_3 x + b_3 y + c_3 z = d_3$$

We write the equations as

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \qquad \text{.....................} \ (2)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

We now start, as before with the assumption that the roots of the equations are $x = x_0$, $y = y_0$, $z = z_0$. Putting these values in the first equation of (2), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

Now, we make use of this value of x instead of x_0 to calculate y_1 from second equation of (2). Thus, we put $x = x_1$, $z = z_0$ in the second equation of (2) and get,

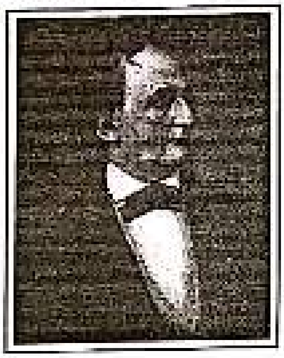
$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

We use this value of y, instead of y_0 and the new value of x *i.e.* x_1 to calculate z from the 3rd equation of 2 *i.e.* we put $x = x_1$, $y = y_1$ and get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

The process is continued till we get desired degree of accuracy. As soon as we obtain a new approximation, it is immediately used in the next calculation.

Philipp Ludwig von Seidel (1821 - 1896)



Philipp Ludwig von Seidel was a German mathematician. Some mathematicians give the credit of developing the concept of uniform convergence to Seidel. The lunar crater Seidel is named after him. He is known for Gauss-Seidel numerical method of solving linear equations.

Solved Examples : Class (c) : 8 Marks

Example 1 (c) : Solve the following equations by Gauss-Seidel method

$27x + 6y - z = 85, \quad 6x + 15y + 2z = 72, \quad x + y + 54z = 110.$ **(M.U. 2005, 14, 19)**

Sol. : We first write the equations as

$$x = \frac{1}{27}(85 - 6y + z) \qquad \text{.....................} (1)$$

$$y = \frac{1}{15}(72 - 6x - 2z) \qquad \text{.....................} (2)$$

$$z = \frac{1}{54}(110 - x - y) \qquad \text{.....................} (3)$$

(i) First Iteration : We start with the approximation $y = 0$, $z = 0$ and we get from (1)

$$\therefore \quad x_1 = \frac{85}{27} = 3{\cdot}15$$

We use this approximation to find y_1 *i.e.* we put $x = 3{\cdot}15$, $z = 0$ in (2).

$$\therefore \quad y_1 = \frac{1}{15}[72 - 6(3{\cdot}15)] = 3{\cdot}54$$

We use these values of x_1 and y_1 to find z_1 *i.e.* we put $x = 3{\cdot}15$ and $y = 3{\cdot}54$ in (3).

$$\therefore \quad z_1 = \frac{1}{54}(110 - 3{\cdot}15 - 3{\cdot}54) = 1{\cdot}91$$

(ii) Second Iteration : We use the latest values of y and z to find x *i.e.* we put $y = 3{\cdot}54$, $z = 1{\cdot}91$ in (1) to get

$$x_2 = \frac{1}{27}[85 - 6(3{\cdot}54) + 1{\cdot}91] = 2{\cdot}43$$

We put $x = 2{\cdot}43$, $z = 1{\cdot}91$ to find y from (2). Thus,

$$y_2 = \frac{1}{15}[72 - 6(2{\cdot}43) - 2(1{\cdot}91)] = 3{\cdot}57$$

We put $x = 2{\cdot}43$, $y = 3{\cdot}57$ in (3) to find z. Thus,

$$z_2 = \frac{1}{54}(110 - 2{\cdot}43 - 3{\cdot}57) = 1{\cdot}93$$

(iii) Third Iteration : Putting $y = 3{\cdot}57$, $z = 1{\cdot}93$ in (1), we get,

$$x_3 = \frac{1}{27}(85 - 6(3{\cdot}57) + 1{\cdot}93) = 2{\cdot}43$$

Putting $x = 2{\cdot}43$, $z = 1{\cdot}93$ in (2), we get,

$$y_3 = \frac{1}{15}[72 - 6(2{\cdot}43) - 2(1{\cdot}93)] = 3{\cdot}57$$

Putting $x = 2{\cdot}43$, $y = 3{\cdot}57$ in (3), we get

$$z_3 = \frac{1}{54}(110 - 2{\cdot}43 - 3{\cdot}57) = 1{\cdot}93$$

Since, the second and third iterations give the same values

$x = 2{\cdot}43$, $y = 3{\cdot}57$; $z = 1{\cdot}93$.

Example 2 (c) : Solve the following equations by Gauss-Seidel method

$10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$, $2x_1 + 2x_2 + 10x_3 = 14$. **(M.U. 2008, 13)**

Sol. : We first write the equations as

$$x_1 = \frac{1}{10}(12 - x_2 - x_3) \qquad \text{.....................} (1)$$

$$x_2 = \frac{1}{10}(13 - 2x_1 - x_3) \qquad \text{.....................} (2)$$

$$x_3 = \frac{1}{10}(14 - 2x_1 - 2x_2) \qquad \text{.....................} (3)$$

(i) First Iteration : We start with the approximation $x_2 = 0$, $x_3 = 0$ and then we get from (1)

$$\therefore \quad x_{1,1} = \frac{12}{10} = 1{\cdot}2$$

We use this approximation to find x_2 *i.e.* we put $x_1 = 1{\cdot}2$, $x_3 = 0$ in (2)

$$\therefore \quad x_{2,1} = \frac{1}{10}[13 - 2(1{\cdot}2)] = 1{\cdot}06$$

We use these values of x_1 and x_2 to find x_3.

$$\therefore \quad x_{3,1} = \frac{1}{10}[14 - 2(1{\cdot}2) - 2(1{\cdot}06)] = 0{\cdot}948$$

(ii) Second Iteration : We use latest values of x_2 and x_3 to find x_1 *i.e.* we put $x_2 = 1{\cdot}06$ and $x_3 = 0{\cdot}948$ in (1)

$$\therefore \quad x_{1,2} = \frac{1}{10}(12 - 1{\cdot}06 - 0{\cdot}948) = 0{\cdot}9992$$

We use $x_1 = 0{\cdot}9992$, $x_3 = 0{\cdot}948$ to find x_2 from (2)

$$\therefore \quad x_{2,2} = \frac{1}{10}[13 - 2(0{\cdot}9992) - 0{\cdot}948] = \frac{1}{10}(10{\cdot}0536) = 1{\cdot}00536$$

We put $x_1 = 0{\cdot}9992$, $x_2 = 1{\cdot}00536$ in (3) to find x_3.

$$\therefore \quad x_{3,2} = \frac{1}{10}\left[14 - 2(0{\cdot}9992) - 2(1{\cdot}00536)\right] = \frac{1}{10}(9{\cdot}99088) = 0{\cdot}999088$$

(iii) Third Iteration : Now, we put $x_2 = 1{\cdot}00536$, $x_3 = 0{\cdot}99908$ in (1) to find x_1.

$$\therefore \quad x_{1,3} = \frac{1}{10}(12 - 1{\cdot}00536 - 0{\cdot}999088) = \frac{1}{10}(9{\cdot}995532) = 0{\cdot}9995552$$

We put $x_1 = 0{\cdot}999552$, $x_3 = 0{\cdot}999088$ in (2) to find x_2.

$$\therefore \quad x_{2,3} = \frac{1}{10}\left[13 - 2(0{\cdot}9995552) - 0{\cdot}999088\right] = \frac{1}{10}(10{\cdot}0018) = 1{\cdot}00018$$

We put $x_1 = 0{\cdot}999552$, $x_2 = 1{\cdot}00018$ in (3) to find x_3.

$$\therefore \quad x_{3,3} = \frac{1}{10}\left[14 - 2(0{\cdot}999552) - 2(1{\cdot}00018)\right] = \frac{1}{10}(10{\cdot}00052) = 1{\cdot}00052$$

Since, the second and third iterations give the same values

$$x_1 = 1, \; x_2 = 1, \; x_3 = 1.$$

Example 3 (c) : Solve the following equations by Gauss-Seidel method.

$28x + 4y - z = 32, \quad 2x + 17y + 4z = 35, \quad x + 3y + 10\,z = 24.$ **(M.U. 2009)**

Sol. : We first write the equations as

$$x = \frac{1}{28}[32 - 4y + z] \quad \text{.....................} (1)$$

$$y = \frac{1}{17}[35 - 2x - 4z] \quad \text{.....................} (2)$$

$$z = \frac{1}{10}[24 - x - 3y] \quad \text{.....................} (3)$$

(i) First Iteration : We start with the approximation $y = 0$, $z = 0$ and then we get from (1),

$$\therefore \quad x_1 = \frac{32}{28} = 1{\cdot}1429$$

We use this approximation to find y *i.e.* we put $x = 1{\cdot}1429$, $z = 0$ in (2)

$$\therefore \quad y_1 = \frac{1}{17}[35 - 2(1{\cdot}1429)] = 1{\cdot}9244$$

We use these values of x_1 and y_1 to find z_1 *i.e.* we put $x = 1{\cdot}1429$, $y = 1{\cdot}9244$ in (3),

$$\therefore \quad z_1 = \frac{1}{10}[24 - 1{\cdot}1429 - 3(1{\cdot}92244)] = 1{\cdot}8084$$

(ii) Second Iteration : We use latest values of y and z to find x *i.e.* we put $y = 1{\cdot}9244$, $z = 1{\cdot}8084$ in (1).

$$\therefore \quad x_2 = \frac{1}{28}[32 - 4(1{\cdot}9244) + 1{\cdot}8084] = 0{\cdot}9325$$

We put $x = 0{\cdot}9325$, $z = 1{\cdot}8084$ in (2).

$$\therefore \quad y_2 = \frac{1}{17}[35 - 2(0{\cdot}9325) - 4(1{\cdot}8044)] = 1{\cdot}5236$$

We put $x = 0{\cdot}9325$, $y = 1{\cdot}5236$ in (3).

$$\therefore \quad z_2 = \frac{1}{10}[24 - 0{\cdot}9325 - 3(1{\cdot}15236)] = 1{\cdot}8488$$

(iii) Third Iteration : We use the latest values of y and z to find x *i.e.* we put $y = 1{\cdot}5236$, $z = 1{\cdot}8488$ in (1).

$$\therefore \quad x_3 = \frac{1}{28}\left[32 - 4\,(1{\cdot}5236) + 1{\cdot}8488\right] = 0{\cdot}9913$$

We put $x = 0{\cdot}9913$, $z = 1{\cdot}8488$ in (2).

$$\therefore \quad y_3 = \frac{1}{17}\left[35 - 2\,(0{\cdot}9913) - 4\,(1{\cdot}8488)\right] = 1{\cdot}5070$$

We put $x = 0.9913$, $y = 1{\cdot}5070$ in (3).

$$\therefore \quad z_3 = \frac{1}{10}\left[24 - 0{\cdot}9913 - 3\,(1{\cdot}5070)\right] = 1{\cdot}8488$$

(iv) Fourth Iteration : We use the latest values of y and z to find x *i.e.* we put $y = 1{\cdot}5070$, $z = 1{\cdot}8488$ in (1).

$$\therefore \quad x_4 = \frac{1}{28}\left[32 - 4\,(1{\cdot}5070) - 4\,(1{\cdot}8488)\right] = 0{\cdot}9936$$

We put $x = 0{\cdot}9936$, $z = 1{\cdot}8488$ in (2).

$$\therefore \quad y_4 = \frac{1}{17}\left[35 - 2\,(0{\cdot}9936) - 4\,(1{\cdot}8488)\right] = 1{\cdot}5069$$

We put $x = 0{\cdot}9936$, $y = 1{\cdot}5069$ in (3)

$$\therefore \quad z_4 = \frac{1}{10}\left[24 - (0{\cdot}9936) - 3\,(1{\cdot}5069)\right] = 1{\cdot}8486$$

Since the third and fourth iterations give the same value upto two places of decimals, we get after rounding, $x = 0{\cdot}99$, $y = 1{\cdot}51$, $z = 1{\cdot}85$.

Example 4 (c) : Solve the following equations by Gauss-Seidel method, upto four iterations.

$4x - 2y - z = 40, \quad x - 6y + 2z = -28, \quad x - 2y + 12z = -86.$ **(M.U. 2016)**

Sol. : We first write the equations as

$$x = \frac{1}{4}\left[40 + 2y + z\right] \qquad \text{.....................} (1)$$

$$y = \frac{1}{6}\left[28 + x + 2z\right] \qquad \text{.....................} (2)$$

$$z = \frac{1}{12}\left[-86 - x + 2y\right] \qquad \text{.....................} (3)$$

(i) First iteration : We start with the approximation $y = 0$, $z = 0$ and then we get from (1),

$$\therefore \quad x_1 = \frac{1}{4}(40) = 10$$

We use this approximation to find y *i.e.* we put $x = 10$, $z = 0$ in (2).

$$\therefore \quad y_1 = \frac{1}{6}\left[28 + 10 + 2\,(0)\right] = 6{\cdot}3333$$

We use these values of x_1 and y_1 to find z_1 *i.e.* we put $x = 4$, $y = 6{\cdot}3333$ in (3).

$$\therefore \quad z_1 = \frac{1}{12}\left[-86 - 10 + 2\,(6{\cdot}3333)\right] = -6{\cdot}944$$

(ii) Second Iteration : We use latest values of y and z to find x *i.e.* we put $y = 6{\cdot}3333$, $z = -6{\cdot}9444$ in (1)

$$\therefore \quad x_2 = \frac{1}{4}\left[40 + 2\,(6{\cdot}3333) - 6{\cdot}9444\right] = 11{\cdot}4306$$

We put $x = 11{\cdot}4306$, $z = -6{\cdot}9444$ in (2).

$$\therefore \quad y_2 = \frac{1}{6}\left[28 + 11{\cdot}4306 + 2\,(-6{\cdot}9444)\right] = 4{\cdot}2569$$

We put $x = 11{\cdot}4306$, $y = 4{\cdot}2569$ in (3).

$$\therefore \quad z_2 = \frac{1}{12}\left[-86 - 11{\cdot}4306 + 2\,(4{\cdot}2569)\right] = -7{\cdot}4097$$

(iii) Third Iteration : We use the latest values of y and z to find x *i.e.* we put $y = 4{\cdot}2569$, $z = -7{\cdot}4097$ in (1).

$$\therefore \quad x_3 = \frac{1}{4}\left[40 + 2\,(4{\cdot}2569) - 7{\cdot}4097\right] = 10{\cdot}2760$$

We put $x = 10{\cdot}2760$, $z = -7{\cdot}4097$ in (2).

$$\therefore \quad y_3 = \frac{1}{6}\left[28 + 10{\cdot}2760 + 2\,(-7{\cdot}4097)\right] = 3{\cdot}9094$$

We put $x = 10{\cdot}2760$, $y = 3{\cdot}9094$ in (3).

$$\therefore \quad z_3 = \frac{1}{12}\left[-86 - 10{\cdot}2760 + 2\,(3{\cdot}9094)\right] = -7{\cdot}3714$$

(iv) Fourth Iteration : We use the latest values of y and z to find x *i.e.* we put $y = 3{\cdot}9094$, $z = -7{\cdot}3714$ in (1).

$$\therefore \quad x_4 = \frac{1}{4}\left[40 + 2\,(3{\cdot}9094) - 7{\cdot}3714\right] = 10{\cdot}1118$$

We put $x = 10{\cdot}1118$, $z = -7{\cdot}3714$ in (2).

$$\therefore \quad y_4 = \frac{1}{6}\left[28 + 10{\cdot}1118 + 2\,(-7{\cdot}3714)\right] = 3{\cdot}8948$$

We put $x = 10{\cdot}1118$, $y = 3{\cdot}8448$ in (3).

$$\therefore \quad z_4 = \frac{1}{12}\left[-86 - 10{\cdot}1118 + 2\,(3{\cdot}8948)\right] = -7{\cdot}3602$$

Hence, upto two places of decimals

$x = 10{\cdot}11$, $y = 3{\cdot}89$, $z = -7{\cdot}36$.

Example 5 (c) : Use Gauss-Seidel method to obtain the solution of the system

$3x_1 - 0{\cdot}1x_2 - 0.2x_3 = 7{\cdot}85$, $\quad 0{\cdot}1x_1 + 7x_2 - 0{\cdot}3x_3 = -19{\cdot}3$,

$0{\cdot}3x_1 - 0{\cdot}2x_2 + 10x_3 = 71{\cdot}4$. **(M.U. 1998, 2000, 03, 04)**

Sol. : We first write the equations as

$$x_1 = \frac{1}{3}\left[7{\cdot}85 + 0{\cdot}1\,x_2 + 0{\cdot}2\,x_3\right] \qquad \text{.....................} \ (1)$$

$$x_2 = \frac{1}{7}\left[-19{\cdot}3 - 0{\cdot}1\,x_1 + 0{\cdot}3\,x_3\right] \qquad \text{.....................} \ (2)$$

$$x_3 = \frac{1}{10}\left[71{\cdot}3 - 0{\cdot}3\,x_1 + 0{\cdot}2\,x_2\right] \qquad \text{.....................} \ (3)$$

(i) First Iteration : We start with the approximation $x_2 = 0$, $x_3 = 0$ and then get from (1),

$$\therefore \quad x_{1,1} = \frac{7.85}{3} = 2.6167$$

We use these approximations to find x_2 *i.e.* we put $x_1 = 2.6167$ and $x_3 = 0$, in (2).

$$\therefore \quad x_{2,1} = \frac{1}{7}[-19.3 - 0.1(2.6167) + 0.3(0)] = -2.7945$$

We use these values of x_1 and x_2 to find x_3 *i.e.* we put $x_1 = 2.6167$, $x_2 = -2.7945$ in (3).

$$\therefore \quad x_{3,1} = \frac{1}{10}[71.4 - 0.3(2.6167) + 0.2(-2.7945)] = 7.0056$$

(ii) Second Iteration : We use latest values of x_2 and x_3 to find x_1 *i.e.* we put $x_2 = -2.7945$, $x_3 = 7.0056$ in (1).

$$\therefore \quad x_{1,2} = \frac{1}{3}[7.85 + 0.1(-2.7945) + 0.2(7.0056)] = 2.9906$$

We use these approximations to find x_2 *i.e.* we put $x_1 = 2.9906$, $x_3 = 7.0056$ in (2)

$$\therefore \quad x_{2,2} = \frac{1}{7}[-19.3 - 0.1(2.9906) + 0.3(7.0056)] = -2.4996$$

We use these values of x_1 and x_2 to find x_3 *i.e.* we put $x_1 = 2.9906$, $x_2 = -2.4996$ in (3).

$$\therefore \quad x_{3,2} = \frac{1}{10}[71.4 - 0.3(2.9906) + 0.2(-2.4946)] = 7.0003$$

(iii) Third Iteration : We use latest values of x_2 and x_3 to find x_1 *i.e.* we put $x_2 = -2.4996$, $x_3 = 7.0003$ in (1)

$$\therefore \quad x_{1,3} = \frac{1}{3}[7.85 + 0.1(-2.4996) + 0.2(7.0003)] = 3.000$$

We use these approximations to find x_2 *i.e.* we put $x_1 = 3.000$, $x_3 = 7.0003$ in (2).

$$\therefore \quad x_{2,3} = \frac{1}{7}[-19.3 - 0.1(3.000) + 0.3(7.0003)] = -2.500$$

We use these approximations to find x_3 *i.e.* we put $x_1 = 3.000$, $x_2 = -2.500$ in (3).

$$\therefore \quad x_{3,3} = \frac{1}{10}[71.4 - 0.3(3.000) + 0.2(-2.500)] = 7.000$$

Hence, the values are $x_1 = 3$, $x_2 = -2.5$, $x_3 = 7$.

Example 6 (c) : Solve the following equations by Gauss-Seidel method upto four iterations.

$$25x + 2y + z = 69, \qquad 2x + 10y + z = 63, \qquad x + y + z = 43.$$

Sol. : We first write the equation as

$$x = \frac{1}{25}[69 - 2y - z] \quad \text{....................} (1)$$

$$y = \frac{1}{10}[63 - 2x - z] \quad \text{....................} (2)$$

$$z = 43 - x - y \quad \text{....................} (3)$$

(i) First Iteration : We start with the approximation $y = 0$, $z = 0$ and then we get from (1),

$$\therefore \quad x_1 = \frac{69}{25} = 2.76$$

We use this approximation to find y *i.e.* we put $x = 2{\cdot}76$ and $z = 0$ in (2).

$$\therefore \quad y_1 = \frac{1}{10}\left[63 - 2(2{\cdot}76) - 0\right] = 5{\cdot}748$$

We use these values of x_1 and y_1 to find z_1 *i.e.* we put $x = 2{\cdot}76$, $y = 5{\cdot}748$ in (3).

$$\therefore \quad z_1 = 43 - 2{\cdot}76 - 5{\cdot}748 = 34{\cdot}492$$

(ii) Second Iteration : We use latest values of y and z to find x *i.e.* we put $y = 5{\cdot}748$, $z = 34{\cdot}492$ in (1).

$$\therefore \quad x_2 = \frac{1}{25}\left[69 - 2(5{\cdot}748) - (34{\cdot}492)\right] = 0{\cdot}9205$$

We put $x = 0{\cdot}9205$, $z = 34{\cdot}492$ in (2).

$$\therefore \quad y_2 = \frac{1}{10}\left[63 - 2(0{\cdot}9205) - 34{\cdot}492\right] = 2{\cdot}6667$$

We put $x = 0{\cdot}9205$, $y = 2{\cdot}6667$ in (3).

$$\therefore \quad z_2 = 43 - 0{\cdot}9205 - 2{\cdot}6667 = 39{\cdot}4128$$

(iii) Third Iteration : We use latest values of y and z to find x *i.e.* we put $y = 2{\cdot}6667$, $z = 39{\cdot}4128$ in (1).

$$\therefore \quad x_3 = \frac{1}{25}\left[69 - 2(2{\cdot}6667) - 39{\cdot}4128\right] = 0{\cdot}9701$$

We put $x = 0{\cdot}9701$, $z = 39{\cdot}4128$ in (2).

$$\therefore \quad y_3 = \frac{1}{10}\left[63 - 2(0{\cdot}9701) - 39{\cdot}4128\right] = 2{\cdot}1647$$

We put $x = 0{\cdot}9701$, $y = 2{\cdot}1647$ in (3).

$$\therefore \quad z_3 = 43 - 0{\cdot}9701 - 2{\cdot}1647 = 39{\cdot}8652$$

(iv) Fourth iteration : We use latest values of y and z to find x.

$$\therefore \quad x_4 = \frac{1}{25}\left[69 - 2(2{\cdot}1647) - 39{\cdot}8652\right] = 0{\cdot}9922$$

We put $x = 0{\cdot}9952$, $z = 39{\cdot}8652$ in (2).

$$\therefore \quad y_4 = \frac{1}{10}\left[63 - 2(0{\cdot}9952) - 39{\cdot}8652\right] = 2{\cdot}1144$$

We put $x = 0{\cdot}9922$, $y = 2{\cdot}1144$ in (3).

$$\therefore \quad z_4 = 43 - 0{\cdot}9922 - 2{\cdot}1144 = 39{\cdot}8934$$

Hence, we get $x = 0{\cdot}99$, $y = 2{\cdot}11$, $z = 39{\cdot}89$.

Example 7 (c) : Apply Gauss-Seidel iteration method to solve the equations

$20x + y - 2z = 17, \quad 3x + 20y - z = -18, \quad 2x - 3y + 20z = 25.$ **(M.U. 2003, 11, 17)**

Sol. : We first write the equations as

$$x = \frac{1}{20}\left[17 - y + 2z\right] \quad \text{.....................} \ (1)$$

$$y = \frac{1}{20}\left[-18 - 3x + z\right] \quad \text{.....................} \ (2)$$

$$z = \frac{1}{20}\left[25 - 2x + 3y\right] \quad \text{.....................} \ (3)$$

(i) First Iteration : We start with the approximation $y = 0$, $z = 0$ and then get form (1).

$$x_1 = \frac{17}{20} = 0{\cdot}85$$

We use this approximation to find y *i.e.* we put $x = 0{\cdot}85$, $z = 0$ in (2)

$$\therefore \quad y_1 = \frac{1}{20}\left[-18 - 3\,(0{\cdot}85) - 0\right] = -1{\cdot}0275$$

We use these values of x_1 and y_1 to find z_1 *i.e.* we put $x = 0{\cdot}85$, $y = -1{\cdot}0275$ in (3).

$$\therefore \quad z_1 = \frac{1}{20}\left[25 - 2\,(0{\cdot}85) + 3\,(-1{\cdot}0275)\right] = 1{\cdot}0109$$

(ii) Second Iteration : We use latest values of y and z to find x *i.e.* we put $y = -1{\cdot}0275$, $z = 1{\cdot}0109$ in (1).

$$\therefore \quad x_2 = \frac{1}{20}\left[17 - (-1{\cdot}0275) + 2\,(1{\cdot}0109)\right] = 1{\cdot}0025$$

We put $x = 1{\cdot}0025$, $z = 1{\cdot}0109$ in (2).

$$\therefore \quad y_2 = \frac{1}{20}\left[-18 - 3\,(1{\cdot}0025) + 1{\cdot}0109\right] = -0{\cdot}9998$$

We put $x = 1{\cdot}0025$, $y = -0{\cdot}9998$ in (3).

$$\therefore \quad z_2 = \frac{1}{20}\left[25 - 2\,(1{\cdot}0025) + 3\,(-0{\cdot}9998)\right] = 0{\cdot}9998$$

(iii) Third Iteration : We use latest values of y and z to find x *i.e.* we put $y = -0{\cdot}9998$, $z = 0{\cdot}9998$ in (1).

$$\therefore \quad x_3 = \frac{1}{20}\left[17 - (-0{\cdot}9998) + 2\,(0{\cdot}9998)\right] = 1{\cdot}000$$

We put $x = 1{\cdot}000$, $z = 0{\cdot}9998$ in (2).

$$\therefore \quad y_3 = \frac{1}{20}\left[-18 - 3\,(1{\cdot}000) + 0{\cdot}9998\right] = -1{\cdot}000$$

We put $x = 1{\cdot}000$, $y = -1{\cdot}000$ in (3).

$$\therefore \quad z_3 = \frac{1}{20}\left[25 - 2\,(1{\cdot}000) + 3\,(-1{\cdot}000)\right] = 1$$

Hence, we get $x = 1$, $y = -1$, $z = 1$.

Example 8 : Solve the following equations by Gauss-Seidel method.

$$15x + 2y + z = 18, \quad 2x + 20y - 3z = 19, \quad 3x - 6y + 25z = 22.$$

Sol. : We first write the equations as

$$x = \frac{1}{15}[18 - 2y - z] \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$y = \frac{1}{20}[19 - 2x + 3z] \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$z = \frac{1}{25}[22 - 3x + 6y] \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

(i) First Iteration : We start with the approximation $y_0 = 0$, $z_0 = 0$ and then from (1), we get

$$x_1 = \frac{18}{15} = 1{\cdot}2$$

We use this approximation to find y from (2), *i.e.*, we put $x = 1\cdot2$ and $z = 0$ in (2) and get

$$y_1 = \frac{1}{20}[19 - 2(1\cdot2) - 0] = \frac{16\cdot6}{20} = 0\cdot83$$

We use these values of x and y to find z from (3), *i.e.*, we put $x = 1\cdot2$, $y = 0\cdot83$ in (3) and get

$$z_1 = \frac{1}{25}[22 - 3(1\cdot2) + 6(0\cdot83)] = \frac{23\cdot38}{25} = 0\cdot9352$$

(ii) Second Iteration : We use the latest values of y and z in (1) to find x, *i.e.*, we put $y = 0\cdot83$ and $z = 0\cdot9352$in (1) and get

$$x_2 = \frac{1}{15}[18 - 2(0\cdot83) - 0\cdot9352] = \frac{15\cdot4048}{15} = 1\cdot0270$$

We now put $x = 1\cdot027$ and $z = 0\cdot9352$ in (2) and get

$$y_2 = \frac{1}{20}[19 - 2(1\cdot027) + 3(0\cdot9352)] = \frac{19\cdot7516}{20} = 0\cdot9876$$

We put $x = 1\cdot0270$ and $y = 0\cdot9876$ in (3) and get

$$z_2 = \frac{1}{25}[22 - 3(1\cdot0270) + 6(0\cdot9876)] = \frac{28\cdot8446}{25} \approx 0\cdot9938$$

(iii) Third Iteration : We use the latest values of y and z to find x, *i.e.*, we put $y = 0\cdot9876$ and $z = 0\cdot9938$ in (1) and get

$$x_3 = \frac{1}{15}[18 - 2(0\cdot9876) - 0\cdot9938] = \frac{15\cdot031}{15} = 1\cdot0021$$

We put $x = 1\cdot0021$ and $z = 0\cdot9938$ in (2) and get

$$y_3 = \frac{1}{20}[19 - 2(1\cdot0021) + 3(0\cdot9938)] = \frac{19\cdot9772}{20} = 0\cdot9989$$

We put $x = 1\cdot0021$ and $y = 0\cdot9989$ in (3) and get

$$z_3 = \frac{1}{25}[22 - 3(1\cdot0021) + 6(0\cdot9989)] = \frac{24\cdot9871}{25} = 0\cdot9995$$

Hence, we get $x = 1\cdot0021$, $y = 0\cdot9989$, $z = 0\cdot9995$.

(Thus, $x = 1$, $y = 1$, $z = 1$. Check these values.)

Example 9 (c) : Using Gauss-Seidel method solve the following the equations.

$$5x - y = 9, \quad x - 5y + z = -4, \quad y - 5z = 6$$

with initial approximations $x_0 = 1\cdot5$, $y_0 = 0\cdot5$, $z = -0\cdot5$. (Take two iterations) **(M.U. 2014)**

Sol. : We rewrite equations as follows.

$$5x - y + 0 = 9,$$
$$x - 5y + z = -4,$$
$$0 + y - 5z = 6$$

We write these equations as

$$x = \frac{1}{5}[9 + y] \quad \text{.....................} (1)$$

$$y = \frac{1}{5}[4 + x + z] \quad \text{.....................} (2)$$

$$z = \frac{1}{5}[-6 + y] \quad \text{.....................} (3)$$

(i) First Iteration : We start with $y_0 = 0{\cdot}5$ and $z_0 = -0{\cdot}5$ and putting these values in (1), we get

$$x_1 = \frac{1}{5}[9 + 0{\cdot}5] = \frac{9{\cdot}5}{5} = 1{\cdot}9$$

We put $x = 1{\cdot}9$ and $z = -0{\cdot}5$ in (2) and get

$$y_1 = \frac{1}{5}[4 + 1{\cdot}9 - 0{\cdot}5] = \frac{5{\cdot}4}{5} = 1{\cdot}08$$

We use these values of x and y in (3) to find z, *i.e.*, we put $x = 1{\cdot}9$ and $y = 1{\cdot}08$ in (3) and get

$$z_1 = \frac{1}{5}[-6 + 1{\cdot}08] = -0{\cdot}984$$

(ii) Second Iteration : We use the latest values of y and z to find x, *i.e.*, we put $y = 1{\cdot}08$ and $z = -0{\cdot}984$ in (1) and get

$$x_2 = \frac{1}{5}[9 + 1{\cdot}08] = \frac{10{\cdot}08}{5} = 2{\cdot}016$$

We put $x = 2{\cdot}016$ and $z = -0{\cdot}984$ in (2) and get

$$y_2 = \frac{1}{5}[4 + 2{\cdot}016 - 0{\cdot}984] = \frac{5{\cdot}032}{5} = 1{\cdot}0064$$

We put these values of x and y in (3), *i.e.*, we put $x = 2{\cdot}016$ and $y = 1{\cdot}0064$ in (3) and get

$$z_2 = \frac{1}{5}[-6 + 1{\cdot}0064] = \frac{-4{\cdot}9936}{5} = -0{\cdot}9972$$

Hence, $x = 2{\cdot}016$, $y = 1{\cdot}0064$, $z = -0{\cdot}9972$.

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

(A) Solve the following equations by Gauss-Seidel method. Class (c) : 8 Marks

1. $15x + y + z = 17$, $2x + 15y + z = 18$, $x + 2y + 15z = 18$.
2. $10x + 2y + z = 9$, $2x + 20y - 2z = -44$, $-2x + 3y + 10z = 22$.
3. $82x - 3y + z = 75$, $x + 75y - 2z = 153$, $3x - 2y + 85z = -86$.
4. $20x + y - z = 40$, $2x + 18y + z = 21$, $x + 2y + 25z = -21$.
5. $43x + 2y + 3z = 91$, $3x + 53y + z = 60$, $2x - 4y + 49z = 49$. **(May 2016)**
6. $25x + 2y - 3z = 48$, $3x + 27y - 2z = 56$, $x + 2y + 23z = 52$, start with (1, 1, 0).
7. $14x_1 + 2{\cdot}3x_2 + 3{\cdot}7x_3 = 6{\cdot}5$, $3{\cdot}3x_1 + 16x_2 + 4{\cdot}3x_3 = 10{\cdot}3$, $2{\cdot}5x_1 + 1{\cdot}9x_2 + 41x_3 = 8{\cdot}8$.
8. $15x + 2y + z = 22$, $x + 14y + 2z = 35$, $x + 2y + 15z = 50$.
9. $83x_1 + 11x_2 - 4x_3 = 45$, $7x_1 + 52x_2 - 13x_3 = 104$, $3x_1 + 8x_2 + 29x_3 = 71$.
10. $5x + y - z = 10$, $2x + 4y + z = 14$, $x + y + 8z = 20$. **(M.U. 2004)**

[**Ans. :** Actual values are :

(1) $x = 1, y = 1, z = 1$; (2) $x = 1, y = -2, z = 3$; (3) $x = 1, y = 2, z = -1$;
(4) $x = 2, y = 1, z = -1$; (5) $x = 2, y = 1, z = 1$; (6) $x = 2, y = 2, z = 2$;
(7) $x_1 = 0{\cdot}33, x_2 = 0{\cdot}53, x_3 = 0{\cdot}17$; (8) $x = 1, y = 2, z = 3$;
(9) $x_1 = 0{\cdot}45, x_2 = 1{\cdot}44, x_3 = 2$. (10) $x = 2, y = 2, z = 2.$]

(B) **Solve the following systems of linear equations by Gauss-Seidel Method by taking three iterations only : Class (c) : 8 Marks**

1. $4x + y + z = 5$, $x + 6y + 2z = 19$, $x + 2y + 5z = -10$. **(M.U. 1997, 98, 2000)**
2. $x_1 + 10x_2 + 4x_3 = 6$, $2x_1 - 4x_2 + 10x_3 = -15$, $9x_1 + 2x_2 + 4x_3 = 20$. **(M.U. 1997, 99)**
3. $10x_1 - 5x_2 - 2x_3 = 3$, $4x_1 - 10x_2 + 3x_3 = -3$, $x_1 + 6x_2 - 10x_3 = -3$. **(M.U. 1999, 2013)**

[**Ans. : (1)** $x = 1{\cdot}2060$, $y = 4{\cdot}2060$, $z = -3{\cdot}9484$
(2) $x_1 = 2{\cdot}7403$, $x_2 = 0{\cdot}9970$, $x_3 = -1{\cdot}6493$
(3) $x_1 = 0{\cdot}9147$, $x_2 = 0{\cdot}9368$, $x_3 = -0{\cdot}9536$]

Summary

1. Regular Falsi Method

If $y_1 < 0$, $y_2 > 0$, then there is a root between $x_1 > y_1$ and $x_3 = \dfrac{x_1 y_2 - x_2 y_1}{y_2 - y_1}$.

2. Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. Gauss-Jacobi's or Jacobi's Method

If
$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

then $x = \dfrac{1}{a_1}(d_1 - b_1 y - c_1 z)$, $y = \dfrac{1}{b_2}(d_2 - a_2 x - c_2 z)$, $z = \dfrac{1}{c_3}(d_3 - a_3 x - b_3 y)$

Start with $x_0 = 0$, $y_0 = 0$, $z_0 = 0$.

$$\therefore \quad x_1 = \frac{1}{a_1} \cdot d_1, \quad x_2 = \frac{1}{a_2} \cdot d_2, \quad x_3 = \frac{1}{a_3} \cdot d_3.$$

Use these values to find x_2, y_2, z_2 and so on.

4. Gauss-Seidel Method

If
$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

then, $x = \dfrac{1}{a_1}(d_1 - b_1 y - c_1 z)$, $y = \dfrac{1}{b_2}(d_2 - a_2 x - c_2 z)$, $z = \dfrac{1}{c_3}(d_3 - a_3 x - b_3 y)$

Use the value of x_2 obtained above to find y_2 and use x_2, y_2 obtained above to find z_2.

☆ ☆ ☆

CHAPTER 12
Expansions of Functions

1. Introduction

In this chapter we are going to study two very important theorems *viz.* Maclaurin's Theorem and Taylor's Theorem. These theorems enable us to expand a function satisfying certain conditions about the origin and also about a given point. If a function $f(x)$ has derivatives for any value of n and if we arrange the terms in ascending powers of x, we get a Maclaurin's series.

2. Maclaurin's Series

Assuming that a function $f(x)$ can be expanded in ascending powers of x, it is expressed as a power series as

$$\boxed{f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots \infty} \quad \ldots\ldots\ldots\ (A)$$

The above series is known as **Maclaurin's Series.**

Colin Maclaurin (1698 - 1746)

A well-known English mathematician best known for Maclaurin's Series. At the age of 15 he published his thesis on the power of gravity. He was appointed professor at the Marischal College, Aberhdeen at the age of 19. In 1775 he was appointed to a Chair of Mathematics at the University of Edinburgh. He is best remembered for **Geometric Organica (1720), Treatise On Fluxions (1742) and Treatise on Algebra.**

3. Some Standard Expansions : Class (a) : 3 Marks

In this article we shall obtain expansions of some standard functions using the Maclaurin's series given above.

(1a) Expansion of sin x

Let $f(x) = \sin x \quad \therefore \quad f'(x) = \cos x, \quad f''(x) = -\sin x,$

$f'''(x) = -\cos x, \quad f^{iv}(x) = \sin x$ and so on.

$\therefore \quad f(0) = 0, \ f'(0) = 1, \ f''(0) = 0, \ f'''(0) = -1, \ f^{iv}(0) = 0 \ \ldots\ldots$

Putting these values in (A), we get

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \infty}$$

Cor. : Putting $x = \frac{\pi}{2}$, we get $\sin\frac{\pi}{2} = 1$.

Hence, limit of $x - \frac{x^3}{3!} + \dots$ as $x \to \frac{\pi}{2}$ is 1.

Note ....

This is not rigorous derivation of the series. We need to prove that the remainder after *n* terms tends to zero as *n* tends to infinity.

(2a) Expansion of cos *x*

Let $f(x) = \cos x$ $\therefore f'(x) = -\sin x$, $f''(x) = -\cos x$

$f'''(x) = \sin x$, $f^{iv}(x) = \cos x$ and so on.

$\therefore f(0) = 1,\ f'(0) = 0,\ f''(0) = -1,\ f'''(0) = 0,\ f^{iv}(0) = 1 \dots$

Putting these values in (A), we get

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \infty}$$

(3a) Expansion of tan *x*

Let $y = \tan x$ $\therefore (y)_0 = 0$

$\therefore y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2$ $\therefore y_1(0) = 1 + y(0) = 1$

$y_2 = 2yy_1$ $\therefore y_2(0) = 2y(0)\, y_1(0) = 2(0)(1) = 0$

$y_3 = 2{y_1}^2 + 2yy_2$ $\therefore y_3(0) = 2(1)^2 = 2$

$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3$ $\therefore y_4(0) = 6(1)(0) + 2(0)(2) = 0$

$y_5 = 6{y_2}^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 = 6{y_2}^2 + 8y_1y_3 + 2yy_4$

$\therefore y_5(0) = 0 + 8(1)(2) + 0 = 16$

Putting these values in (A), we get

$$\tan x = x + \frac{x^3}{3!}\cdot 2 + \frac{x^5}{5!}\cdot 16 + \dots \quad \therefore \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \infty$$

Alternatively : $\tan x = \frac{\sin x}{\cos x} = \frac{x - (x^3/3!) + (x^5/5!) - \dots}{1 - (x^2/2!) + (x^4/4!) - \dots}$

By actual division, we get

$$\boxed{\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots}$$

(4a) Expansion of e^x

Let $f(x) = e^x$, $f'(x) = e^x$, $f''(x) = e^x$,

$f'''(x) = e^x$, $f^{iv}(x) = e^x$, and so on.

$\therefore f(0) = 1,\ f'(0) = 1,\ f''(0) = 1,\ f'''(0) = 1, \dots$

Putting these values in (A), we get,

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\infty}$$

Cor. 1 : Putting $a^x = e^{x \log a}$, we get,

$$\boxed{a^x = 1 + x\,(\log a) + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots\ldots}$$

Cor. 2 : Putting $x = 1$, we get the limit of

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \text{ as } x \to 1 \text{ equal to } e.$$

Cor. 3 : Putting $x = -1$, we get

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots\ldots = \frac{1}{e}.$$

(5a) Expansion of sin $h\,x$ **(M.U. 2015)**

Let $f(x) = \sin h\,x$, $f'(x) = \cos h\,x$, $f''(x) = \sin h\,x$

$f'''(x) = \cos h\,x$, $f^{iv}(x) = \sin h\,x$, and so on.

$\therefore$ $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f'''(0) = 1$, ...

Putting these values in (A), we get,

$$\boxed{\sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots\ldots}$$

(6a) Expansion of cos hx

Let $f(x) = \cos h\,x$, $f'(x) = \sin h\,x$, $f''(x) = \cos hx$

$f'''(x) = \sin h\,x$, $f^{iv}(x) = \cos h\,x$, and so on.

$\therefore$ $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$, ...

Putting these values in (A), we get,

$$\boxed{\cos hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots\infty}$$

Alternatively : Expansions of sin x, cos x, sin $h\,x$, cos $h\,x$ can also be obtained by using the definitions of these functions and the expansion of e^x as shown below.

1. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$$= \frac{1}{2i}\left[\left(1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \ldots\ldots\right) - \left(1 - ix + \frac{i^2x^2}{2!} - \frac{i^3x^3}{3!} + \ldots\ldots\right)\right]$$

$$= \frac{1}{2i}\left[2ix - \frac{2ix^3}{3!} + \ldots\ldots\right] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$$

2. $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$$= \frac{1}{2}\left[\left(1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \ldots\right) + \left(1 - ix + \frac{i^2x^2}{2!} - \frac{i^3x^3}{3!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[2 - \frac{2x^2}{2!} + \ldots\right] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

3. $\sin hx = \dfrac{e^{x} - e^{-x}}{2}$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3!} + \ldots\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

4. $\cos hx = \dfrac{e^{x} + e^{-x}}{2}$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right)\right]$$

$$= \frac{1}{2}\left[2 - \frac{2x^2}{2!} + \ldots\right] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

(7a) Expansion of tan hx

We have $\tan h\,x = \dfrac{\sin h\,x}{\cos h\,x} = \dfrac{x + (x^3/3!) + (x^5/5!) + \ldots}{1 + (x^2/2!) + (x^4/4!) + \ldots}$

By actual division, we get

$$\boxed{\tan h\,x = x - \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\infty}$$

(8a) Expansion of log $(1 + x)$ **(M.U. 2014)**

Let $f(x) = \log(1 + x)$, $f'(x) = \dfrac{1}{1+x}$, $f''(x) = -\dfrac{1}{(1+x)^2}$

$f'''(x) = \dfrac{2 \cdot 1}{(1+x)^3}$, $f^{iv}(x) = -\dfrac{3 \cdot 2 \cdot 1}{(1+x)^4}$, and so on.

$\therefore$ $f(0) = 0$, $f'(0) = 1$, $f''(0) = -1$,

$f'''(0) = 2!$, $f^{iv}(0) = -3!$, $f^{v}(0) = 4!$

Putting these values in (A), we get,

$$\log(1 + x) = x + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}2! + \frac{x^4}{4!}(-3!) + \ldots$$

$$\therefore \quad \boxed{\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\infty}$$

Cor. 1 : Putting $x = 1$, we get

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Cor. 2 : Putting $x = x - 1$, we get

$$\log x = \log[1 + (x-1)] = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \ldots$$ **(M.U. 2014)**

(9a) Expansion of log (1 – x)

Changing x to $-x$ in the above expansion, we get,

$$\boxed{\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\ldots\infty}$$

(10a) Expansion of tan h^{-1} x

$$\tan h^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$ [See § 10 (3), page 3-34]

$$= \frac{1}{2}[\log(1+x) - \log(1-x)]$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right]$$

$$\therefore \quad \boxed{\tan h^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\infty}$$

(11a) Expansion of $(1 + x)^m$

Since $f(x) = (1+x)^m$, $\quad f(0) = 1$

$f'(x) = m(1+x)^{m-1}$, $\quad f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2}$, $\quad f''(0) = m(m-1)$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$,

$f'''(0) = m(m-1)(m-2)$

If $|x| < 1$, then by (A), we get

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots$$

Note ....

If m is a positive integer, we get a finite number of terms on the r.h.s. of the above expansion.

Cor. 1 : If $m = -1$, we get from the above expression

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\infty$$

Cor. 2 : By changing x to $-x$, we get

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\infty$$

The expansions of the standard functions obtained above, are listed below for ready reference. Students are advised to memories these results.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots\infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\infty$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

$$\sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots$$

$$\cos hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\infty$$

$$\tan h\, x = x - \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\infty$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\infty$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\infty$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots\infty$$

$$\tan h^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots$$

4. Expansion's of Functions in Power Series

We have seen above how to use Maclaurin's Series to expand a given function in the form of a series. We shall apply this method to some more functions. However, in some cases the use of Maclaurin's Series is not advantageous. This is illustrated below. There are other methods of expanding a given function. When Maclaurin's method is not convenient, we shall use these methods.

5. Expansion Using Maclaurin's Series : Class (a) : 3 Marks

Example 1 (a) : By Maclaurin's Series expand $\log(1+e^x)$ in powers of x upto x^4.

(M.U. 1995, 97, 2001, 02, 06)

Sol. : Let $f(x) = \log(1+e^x) \quad \therefore \quad f'(x) = \dfrac{e^x}{1+e^x}$

$$f''(x) = \frac{(1+e^x)\cdot e^x - e^x \cdot e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$f'''(x) = \frac{(1+e^x)^2\cdot e^x - e^x\cdot 2(1+e^x)\cdot e^x}{(1+e^x)^4} = \frac{(1+e^x)e^x - 2e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$f^{iv}(x)=\frac{(1+e^x)^3(e^x-2e^{2x})-(e^x-e^{2x})\cdot 3(1+e^x)^2e^x}{(1+e^x)^6}$$

$$\therefore \quad f^{iv}(x)=\frac{(1+e^x)(e^x-2e^{2x})-3e^x(e^x-e^{2x})}{(1+e^x)^4}$$

$$\therefore \quad f(0)=\log 2,\ f'(0)=\frac{1}{2},\ f''(0)=\frac{1}{4},\ f'''(0)=0,\ f^{iv}(0)=-\frac{1}{8}$$

Hence, by the Maclaurin's Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots..,\text{ we get}$$

$$\log(1+e^x)=\log 2+\frac{1}{2}\cdot x+\frac{1}{8}\cdot x^2-\frac{1}{192}x^4+\ldots..$$

Example 2 (a) : Prove that, $\log\sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots..$

(M.U. 1999, 2003, 09, 11, 14, 15, 18)

Sol. : Let $y=\log\sec x\ \therefore\ y(0)=\log\sec 0=\log 1=0$

$$y_1=\frac{1}{\sec x}\cdot\sec x\tan x=\tan x \qquad \therefore\ y_1(0)=0$$

$$y_2=\sec^2 x=1+\tan^2 x=1+y_1^2 \qquad \therefore\ y_2(0)=1+y_1^2(0)=1$$

$$y_3=2y_1y_2 \qquad \therefore\ y_3(0)=2y_1(0)y_2(0)=0$$

$$y_4=2y_2^2+2y_1y_3 \qquad \therefore\ y_4(0)=2(1)^2=2$$

$$y_5=4y_2y_3+2y_2y_3+2y_1y_4=6y_2y_3+2y_1y_4 \qquad \therefore\ y_5(0)=0$$

$$y_6=6y_3^2+6y_2y_4+2y_2y_4+2y_1y_5=6y_3^2+8y_2y_4+2y_1y_5 \qquad \therefore\ y_6(0)=8\cdot 2=16$$

Hence, by the Maclaurin's Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots..,\text{ we get}$$

$$\log\sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots..$$

(For another method, see solved Ex. 1, page 12-24.)

Note ....

Note that sometimes it is convenient to express the derivatives in terms of the derivatives of the previous order as in the above example.

Example 3 (a) : Expand in powers of x, $e^x\sec x$.

Sol. : Let $y=e^x\sec x \qquad \therefore\ y(0)=1$

$$y_1=e^x\sec x+e^x\sec x\tan x=y+y\tan x \qquad \therefore\ y_1(0)=1$$

$$y_2=y_1+y_1\tan x+y\sec^2 x \qquad \therefore\ y_2(0)=1+0+1=2$$

Hence, by the Maclaurin's Series

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\frac{x^4}{4!}f^{iv}(0)+\ldots..,\text{ we get}$$

$$e^x\sec x=1+x+\frac{2x^2}{2!}+\ldots..$$

Example 4 (a) : Expand in powers of x, $e^{x \cos x}$.

Sol. : Let $y = e^{x \cos x}$ $\quad \therefore \; y(0) = 1$

$\therefore \; y_1 = e^{x \cos x}(\cos x - x \sin x) = y(\cos x - x \sin x)$

$\therefore \; y_1(0) = 1\,(1 - 0) = 1$

$\therefore \; y_2 = y_1(\cos x - x \sin x) + y(-\sin x - \sin x - x \cos x)$

$\therefore \; y_2(0) = 1\,(1 - 0) + 1\,(0) = 1$

Hence, by the Maclaurin's Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \ldots\ldots, \text{ we get}$$

$$e^{x \cos x} = 1 + x + \frac{x^2}{2} + \ldots\ldots$$

Note ....

It may be noted that some of the problems given in this section can also be solved by using standard series as shown below. We shall obtain the expansions by using Maclaurin's Series and by using expansions of the known functions in the following examples.

Example 5 (a) : Show that $\quad e^x \log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots$

Sol. : Let $\quad f(x) = e^x \log(1+x), \quad f'(x) = e^x \log(1+x) + \frac{e^x}{1+x}$

$$f''(x) = e^x \log(1+x) + \frac{e^x}{1+x} + \frac{e^x}{1+x} - \frac{e^x}{(1+x)^2}$$

$$f'''(x) = e^x \log(1+x) + \frac{e^x}{1+x} + \frac{e^x}{1+x} - \frac{e^x}{(1+x)^2} + \frac{e^x}{1+x} - \frac{e^x}{(1+x)^2} - \frac{e^x}{(1+x)^2} + \frac{2e^x}{(1+x)^3}$$

$\therefore \quad f(0) = 0, \; f'(0) = 0 + 1 = 1, \; f''(0) = 1, \; f'''(0) = 2$

Hence, by Maclaurin's Series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$\therefore \quad e^x \log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots$$

Allter : We have,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots \quad \text{and} \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots$$

$$\therefore \quad e^x \log(1+x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots\right)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} \ldots\ldots + x^2 - \frac{x^3}{2} + \ldots\ldots + \frac{x^3}{2!} \ldots\ldots = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots$$

$$\therefore \quad e^x \log(1+x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \ldots\ldots$$

Example 6 (a) : Prove that $\log(1+\sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} + ...$

(M.U. 1990, 92, 2001, 12)

Sol. : Let $f(x) = \log(1+\sin x)$ $\therefore$ $f'(x) = \dfrac{\cos x}{1+\sin x}$

$$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x \cos x}{(1+\sin x)^2} = -\frac{\sin x + 1}{(1+\sin x)^2} = -\frac{1}{1+\sin x}$$

$$\therefore \quad f'''(x) = \frac{\cos x}{(1+\sin x)^2}$$

$$\therefore \quad f(0) = 0,\ f'(0) = 1,\ f''(0) = -1,\ f'''(0) = 1$$

Hence, by Maclaurin's Series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ...$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - .....$$

Aliter : We have

$$\log(1+\sin x) = \sin x - \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{3} - \frac{(\sin x)^4}{4} + ...$$

$$= \left(x - \frac{x^3}{6} + ...\right) - \frac{1}{2}\left(x - \frac{x^3}{6} + ...\right)^2 + \frac{1}{3}\left(x - \frac{x^3}{6} + ...\right)^3 - \frac{1}{4}(x ...)^4 + ...$$

$$= x - \frac{x^3}{6} + ... - \frac{1}{2}\left(x^2 - \frac{2x^4}{6} + ...\right) + \frac{1}{3}(x^3 + ...) - \frac{1}{4}(x^4 + ...) + ...$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + ...$$

Example 7 (a) : Prove that $\sec^2 x = 1 + x^2 + \dfrac{2x^4}{3} + ...$ **(M.U. 2008, 16)**

Sol. : Let $f(x) = \sec^2 x$ $\therefore$ $f'(x) = 2\sec^2 x \tan x$

$$f''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

$$f'''(x) = 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x + 8\sec^4 x \tan x$$

$$= 8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$$

$$f^{iv}(x) = 16\sec^2 x \tan^4 x + 24\sec^4 x \tan^2 x + 64\sec^4 x \tan^2 x + 16\sec^6 x$$

$$\therefore \quad f(0) = 1,\ f'(0) = 0,\ f''(0) = 2,\ f'''(0) = 0,\ f^{iv}(0) = 16.$$

Hence, by Maclaurin's Series,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + ...$$

$$\therefore \quad \sec^2 x = 1 + x^2 + \frac{2}{3}x^4 + ...$$

Aliter : We have $\sec^2 x = \dfrac{1}{\cos^2 x} = \dfrac{2}{1+\cos 2x}$

$$\therefore \quad \sec^2 x = \frac{2}{\left[1+1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-.....\right]} = \frac{2}{\left[2-2x^2+\frac{2}{3}x^4-.....\right]}$$

$$= \frac{1}{\left[1-x^2+\frac{x^4}{3}-.....\right]} = \left[1-\left(x^2-\frac{x^4}{3}+.....\right)\right]^{-1}$$

$$= 1+\left(x^2-\frac{x^4}{3}+.....\right)+\left(x^2-\frac{x^4}{3}+.....\right)^2+..... \quad \text{[By Cor. 2 of (11a), page 12-5]}$$

$$= 1+x^2-\frac{x^4}{3}+x^4+..... = 1+x^2+\frac{2}{3}x^4+.....$$

6. Expansion of Implicit Functions by Maclaurin's Series

Class (a) : 3 Marks

Example 1 (a) : If $x=(1-y)(1-2y)$, prove that $y=1+x-2x^2+.....$

Sol. : Since $x=1-3y+2y^2$ when $x=0$, $y=1$ or $\frac{1}{2}$ $\quad\therefore\quad y(0)=1$

By differentiating the given function, we get

$$4yy_1-3y_1=1 \qquad \text{.....................} (1)$$

$$\therefore \quad 4y(0)\,y_1(0)-3y_1(0)=1$$

$$\therefore \quad 4y_1(0)-3y_1(0)=1 \qquad \therefore \quad y_1(0)=1$$

Again differentiating (1), we get

$$4y_1^2+4yy_2-3y_2=0$$

$$\therefore \quad 4y_1^2(0)+4y(0)\,y_2(0)-3y_2(0)=0 \qquad \therefore \quad y_2(0)=-4$$

Hence, by Maclaurin's Series

$$y=y(0)+xy_1(0)+\frac{x^2}{2!}y_2(0)+.... = 1+x-2x^2+.....$$

Example 2 (a) : If $x^3+2xy^2-y^3+x=1$, prove that $y=-1+x-\frac{x^2}{3}+.....$

Sol. : When $x=0$, $-y^3(0)=1$ $\quad\therefore\quad y(0)=-1$

By differentiating the given function, we get

$$3x^2+4\,xyy_1+2\,y^2-3\,y^2\,y_1+1=0 \qquad \text{.....................} (1)$$

$\therefore$ When $x=0$, $\quad 2y^2(0)-3y^2(0)\,y_1(0)+1=0$

$$\therefore \quad 2-3y_1(0)+1=0 \qquad \therefore \quad y_1(0)=1$$

Again differentiating (1), we get

$$6x+4yy_1+4x[\,y_1^2+yy_2]+4\,yy_1-6\,yy_1^2-3y^2\,y_2=0$$

$\therefore$ When $x=0$,

$$8y(0)\,y_1(0)-6y(0)\,y_1^2(0)-3y^2(0)\,y_2(0)=-8+6-3y_2(0)=0$$

$$\therefore \quad y_2(0) = -\frac{2}{3}$$

Hence, by Maclaurin's Series,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots = -1 + x - \frac{x^2}{3} + \ldots$$

EXERCISE - I

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Using Maclaurin's Series prove the following : Class (a) : 3 Marks

1. $a^x = 1 + x\log a + \dfrac{x^2}{2!}(\log a)^2 + \ldots$

2. $5^x = 1 + x\log 5 + \dfrac{x^2}{2!}(\log 5)^2 + \ldots$ **(M.U. 2004)**

3. $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \ldots$ **(M.U. 2018)**

4. $\log\cos x = -\dfrac{x^2}{2} - \dfrac{x^4}{12} - \dfrac{x^6}{45} - \ldots$

5. $\log(\sec x + \tan x) = x + \dfrac{x^3}{6} + \dfrac{x^5}{24} + \ldots$

6. $e^{\sin x} = 1 + x + \dfrac{x^2}{2} - \dfrac{x^4}{8} - \dfrac{x^5}{15} + \ldots$

7. $\dfrac{e^x}{1+e^x} = \dfrac{1}{2} + \dfrac{x}{4} - \dfrac{x^3}{48} + \ldots$

8. $\log(1+\cos x) = \log 2 - \dfrac{x^2}{4} - \dfrac{x^4}{96} - \ldots$

9. $\log(1+\tan x) = x - \dfrac{x^2}{2} + \dfrac{2x^3}{3} - \ldots$ **(M.U. 1995)**

10. $e^x \sin x = x + x^2 + \dfrac{2x^3}{3!} - \ldots$ (For another method see Ex. 3, page 12-22.)

11. $\log\sec\left(\dfrac{\pi}{4} + x\right) = \dfrac{1}{2}\log 2 + x + x^2 + \dfrac{2x^3}{3} + \ldots$ **(M.U. 1994)**

12. If $x^3 + y^3 + xy - 1 = 0$, prove that $y = 1 - \dfrac{x}{3} - \dfrac{26x^3}{81} - \ldots$ **(M.U. 1995)**

13. If $x = y(1 + y^2)$, prove that $y = x - x^3 + 3x^5 - \ldots$

14. If $y^3 + y - 2x = 0$, prove that $y = 2x - 8x^3 + 96x^5 - \ldots$

7. Method of using Standard Expansions : Class (a) : 3 Marks

Example 1 (a) : Expand in powers of x, $e^{x\sin x}$. **(M.U. 1995, 97, 2002, 04)**

Sol. : Let $x\sin x = y$

$$\therefore \quad e^{x\sin x} = e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots$$

$$\therefore \quad e^{x\sin x} = 1 + x\sin x + \frac{1}{2!}(x\sin x)^2 + \frac{1}{3!}(x\sin x)^3 + \ldots$$

$$= 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) + \frac{x^2}{2!}\left(x - \frac{x^3}{3!} + \ldots\right)^2 + \frac{x^3}{3!}\left(x - \ldots\right)^3 + \ldots$$

$$= 1 + x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) + \frac{x^2}{2!}\left(x^2 - \frac{2x^4}{3!} + \frac{x^6}{(3!)^2} + \ldots\right) + \frac{x^3}{3!}\left(x - \ldots\right)^3 + \ldots$$

$$\therefore \quad e^{x\sin x} = 1 + x^2 - \frac{x^4}{3!} + \frac{x^6}{120} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^6}{3!} + \ldots$$

$$= 1 + x^2 + \frac{1}{3}x^4 + \frac{1}{120}x^6 + \ldots$$

Example 2 (a) : Expand $\left[\frac{1+e^x}{2e^x}\right]^{1/2}$. **(M.U. 2004)**

Sol. : We have

$$\left[\frac{1+e^x}{2e^x}\right]^{1/2} = \left[\frac{1}{2}e^{-x} + \frac{1}{2}\right]^{1/2} = \left[\frac{1}{2}\left\{1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\right\} + \frac{1}{2}\right]^{1/2}$$

$$= \left[1 - \left(\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} + \ldots\right)\right]^{1/2}$$

$$= 1 - \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} - \ldots\right) - \frac{1}{8}\left(\frac{x}{2} - \frac{x^2}{4} + \ldots\right)^2 + \ldots$$

[By Binomial Theorem]

$$= 1 - \frac{x}{4} + \frac{3x^2}{32} - \ldots$$

Example 3 (a) : Expand $\log(1 + x + x^2 + x^3)$ upto x^8. **(M.U. 1997, 2002, 13)**

Sol. : We have $\log(1 + x + x^2 + x^3) = \log[(1 + x)(1 + x^2)]$

$$= \log(1 + x) + \log(1 + x^2)$$

$$= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} + \ldots\right]$$

$$+ \left[x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3}{8}x^8 + \ldots$$

Example 4 (a) : Prove that $\log(1 - x + x^2 - x^3) = -x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots$

Sol. : We have

$$\log(1 - x + x^2 - x^3) = \log[(1 - x)(1 + x^2)]$$

$$= \log(1 - x) + \log(1 + x^2)$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots + x^2 - \frac{x^4}{2} + \ldots$$

$$= -x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots$$

Example 5 (a) : Prove that $x \operatorname{cosec} x = 1 + \frac{x^2}{6} + \frac{7x^4}{360} + \ldots$ **(M.U. 2007, 17, 18)**

Sol. : We have $x \operatorname{cosec} x = \frac{x}{\sin x} = \frac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots} = \frac{1}{1 - \frac{x^2}{6} + \frac{x^4}{120} - \ldots}$

$$= \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120}\right) + \ldots\right]^{-1}$$

$$\therefore \quad x \operatorname{cosec} x = 1 + \left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right) + \left(\frac{x^2}{6} + \ldots\right)^2 \qquad \text{[By Cor. 2 of (11a), page 12-5]}$$

$$= 1 + \frac{x^2}{6} + \left(\frac{1}{36} - \frac{1}{120}\right)x^4 + \ldots$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \ldots$$

Example 6 (a) : Show that $\sin x \sin hx = x^2 - 8 \cdot \frac{x^6}{6!} + \ldots$

Sol. : Using the expansions of $\sin x$, $\sin hx$

$$\sin x \cdot \sin hx = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \ldots + \frac{x^4}{3!} - \frac{x^6}{(3!)^2} + \ldots + \frac{x^6}{5!} + \ldots$$

$$= x^2 + \left[\frac{2}{5!} - \frac{1}{(3!)^2}\right]x^6 + \ldots = x^2 - \frac{8}{6!}x^6 + \ldots$$

Example 7 (a) : Show that $e^{x \cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 + \ldots$ **(M.U. 2003, 10)**

Sol. : We have,

$$e^{x \cos x} = 1 + x \cos x + \frac{(x \cos x)^2}{2!} + \frac{(x \cos x)^3}{3!} + \frac{(x \cos x)^4}{4!} + \ldots$$

$$= 1 + x\left(1 - \frac{x^2}{2} + \ldots\right) + \frac{x^2}{2}\left(1 - \frac{x^2}{2} + \ldots\right)^2 + \frac{x^3}{6}\left(1 - \ldots\right)^3 + \frac{x^4}{24}\left(1 - \ldots\right)^4 + \ldots$$

$$= 1 + x - \frac{x^3}{2} + \ldots + \frac{x^2}{2} - \frac{x^4}{2} + \ldots + \frac{x^3}{6} + \ldots + \frac{x^4}{24} + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11}{24}x^4 + \ldots$$

Example 8 (a) : Show that $\sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \ldots$ **(M.U. 1984, 88, 98, 2002, 16)**

Sol. : We have $\sin(e^x - 1) = \sin\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots - 1\right)$

$$\therefore \quad \sin(e^x - 1) = \sin\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)$$

But $\quad \sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \ldots$

$$\therefore \quad \sin(e^x - 1) = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots - \frac{1}{6}\left(x + \frac{x^2}{2} + \ldots\right)^3 + \ldots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots - \frac{x^3}{6} - \frac{1}{4}x^4 + \ldots$$

$$= x + \frac{x^2}{2} - \frac{5}{24}x^4 + \ldots$$

Example 9 (a) : Prove that $\sqrt{1 + \sin x} = 1 + \dfrac{x}{2} - \dfrac{x^2}{8} - \dfrac{x^3}{48} + \dfrac{x^4}{384} - \ldots$ **(M.U. 2004, 05)**

Sol. : We have

$$\sqrt{1 + \sin x} = \sqrt{\sin^2(x/2) + \cos^2(x/2) + 2\sin(x/2)\cos(x/2)} \qquad \textbf{[Note this]}$$

$$= \sqrt{[\sin(x/2) + \cos(x/2)]^2} = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

$$= \left(\frac{x}{2}\right) - \frac{1}{6}\left(\frac{x}{2}\right)^3 + \ldots + 1 - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{1}{24}\left(\frac{x}{2}\right)^4 - \ldots$$

$$= \frac{x}{2} - \frac{x^3}{48} + \ldots + 1 - \frac{x^2}{8} + \frac{x^4}{384} - \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \ldots$$

Example 10 (a) : Prove that $e^{e^x} = e\left[1 + x + x^2 + \dfrac{5}{6}x^3 + \dfrac{5}{8}x^4 + \ldots\right]$ **(M.U. 2002)**

Sol. : We have $\quad e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

Let $\quad y = x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots \qquad \therefore \quad e^x = 1 + y$

$$e^{e^x} = e^{1+y} = e \cdot e^y = e\left[1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \ldots\right]$$

$$e^{e^x} = e \cdot \left[1 + \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\right) + \frac{1}{2!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^2\right.$$

$$\left. + \frac{1}{3!}\left(x + \frac{x^2}{2!} + \ldots\right)^3 + \frac{1}{4!}\left(x + \ldots\right)^4 + \ldots\right]$$

$$\therefore \quad e^{e^x} = e \cdot \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{1}{2!}\left(x^2 + \frac{x^4}{(2!)^2} + x^3 + \frac{x^4}{3} + \ldots\right)\right.$$

$$\left. + \frac{1}{3!}\left(x^3 + \frac{3}{2!}x^4 + \ldots\right) + \frac{1}{4!}\left(x^4 + \ldots\right)\right]$$

$$= e \cdot \left[1 + x + x^2 + \frac{5x^3}{6} + \frac{5}{8}x^4 + \ldots\right]$$

Example 11 (a) : Prove that $\log(x \cot x) = -\dfrac{x^2}{3} - \dfrac{7}{90}x^4 - \ldots$

Sol. : We have

$$\log(x \cot x) = \log\left(\frac{x \cos x}{\sin x}\right) = \log x + \log \cos x - \log \sin x$$

$$= \log x + \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) - \log\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$$

$$= \log x + \log\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right) - \log x - \log\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right)$$

$$= \log\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right) - \log\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \ldots\right)$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{24} - \ldots\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} - \ldots\right)^2 + \ldots$$

$$- \left(-\frac{x^2}{6} + \frac{x^4}{120} - \ldots\right) + \frac{1}{2}\left(-\frac{x^2}{6} + \frac{x^4}{120} - \ldots\right)^2 + \ldots$$

$$= -\frac{x^2}{3} - \frac{7}{90}x^4 - \ldots$$

Example 12 (a) : Prove that $(1+x)^x = 1 + x^2 - \dfrac{x^3}{2} + \dfrac{5x^4}{6} - \ldots$ **(M.U. 1995, 98, 2002, 04)**

Sol. : Let $\quad y = (1+x)^x \quad \therefore \quad \log y = x \log(1+x)$

$$\therefore \quad \log y = x\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) = \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \ldots\right) = z, \text{ say}$$

$$\therefore \quad y = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

$$= 1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \ldots\right) + \frac{1}{2!}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \ldots\right)^2$$

$$\therefore \quad y = 1 + x^2 - \frac{x^3}{2} + \frac{5x^4}{6} - \ldots$$

Example 13 (a) : Expand $(1+x)^{1/x}$ upto the term x^2. **(M.U. 2002)**

Sol. : Let $y=(1+x)^{1/x}$ $\therefore$ $\log y=\dfrac{1}{x}\cdot\log(1+x)$

$$\therefore\ \log y=\frac{1}{x}\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right]=1-\frac{x}{2}+\frac{x^2}{3}-\ldots=1+\left[-\frac{x}{2}+\frac{x^2}{3}-\ldots\right]$$

$$=1+z \text{ where } z=-\frac{x}{2}+\frac{x^2}{3}+\ldots$$

$$\therefore\ y=e^{1+z}=e\cdot e^z=e\cdot\left[1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots\right]$$

$$\therefore\ y=e\cdot\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}+\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\ldots\right)^2+\ldots\right]$$

$$=e\cdot\left[1-\frac{x}{2}+\frac{x^2}{3}+\frac{x^2}{8}+\ldots\right]=e\cdot\left[1-\frac{x}{2}+\frac{11x^2}{24}-\ldots\right]$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Find the expansion of $\log\tan\left(\dfrac{\pi}{4}+x\right)$ upto x^5. **(M.U. 1999, 2001)**

Sol. : We have,

$$\log\tan\left(\frac{\pi}{4}+x\right)=\log\left[\frac{\tan(\pi/4)+\tan x}{1-\tan(\pi/4)\tan x}\right]=\log\left(\frac{1+\tan x}{1-\tan x}\right)$$

$$=\log(1+\tan x)-\log(1-\tan x)$$

Now,

$$\log(1+\tan x)=\tan x-\frac{(\tan x)^2}{2}+\frac{(\tan x)^3}{3}-\frac{(\tan x)^4}{4}+\frac{(\tan x)^5}{5}-\ldots$$

$$=\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right)-\frac{1}{2}\left(x+\frac{x^3}{3}+\ldots\right)^2+\frac{1}{3}\left(x+\frac{x^3}{3}+\ldots\right)^3$$
$$-\frac{1}{4}\left(x+\ldots\right)^4+\frac{1}{5}\left(x+\ldots\right)^5+\ldots$$

$$=x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots-\frac{x^2}{2}-\frac{x^4}{3}-\ldots+\frac{1}{3}x^3+\frac{1}{3}x^5$$
$$+\ldots-\frac{1}{4}x^4+\ldots+\frac{1}{5}x^5+\ldots$$

$$=x-\frac{x^2}{2}+\frac{2}{3}x^3-\frac{7}{12}x^4+\frac{2}{3}x^5+\ldots$$

Changing the sign of x,

$$\log(1-\tan x)=-x-\frac{x^2}{2}-\frac{2}{3}x^3-\frac{7}{12}x^4-\frac{2}{3}x^5-\ldots$$

By subtraction, we get,

$$\log\tan\left(\frac{\pi}{4}+x\right)=\log(1+\tan x)-\log(1-\tan x)=2x+\frac{4}{3}x^3+\frac{4}{3}x^5+\ldots$$

Example 2 (b) : Prove that $(1+x)^{(1+x)}=1+x+x^2+\frac{x^3}{3}+\ldots$

Hence, find the approximate value of $(1\cdot 01)^{(1\cdot 01)}$.

Sol. : Let $z=(1+x)^{(1+x)}$

$$\therefore\quad \log z=(1+x)\log(1+x)=(1+x)\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\right]$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots+x^2-\frac{x^3}{2}+\ldots=x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots$$

$$\therefore\quad z=e^{x+(x^2/2)-(x^3/6)+\ldots}$$

$$\therefore\quad (1+x)^{(1+x)}=1+\left(x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots\right)+\frac{1}{2}\left(x+\frac{x^2}{2}+\ldots\right)^2+\frac{1}{6}\left(x+\ldots\right)^3+\ldots$$

$$=1+x+\frac{x^2}{2}-\frac{x^3}{6}+\ldots+\frac{1}{2}(x^2+x^3+\ldots)+\frac{1}{6}(x^3-\ldots)+\ldots$$

$$=1+x+x^2+\frac{x^3}{2}+\ldots$$

Now put $x=0\cdot 01$,

$$\therefore\quad (1\cdot 01)^{(1\cdot 01)}=1+0\cdot 01+(0\cdot 01)^3+\frac{(0\cdot 01)^3}{2}+\ldots=1\cdot 0101005$$

Example 3 (b) : Show that

$$e^{ax}\cos bx=1+ax+\frac{(a^2-b^2)}{2!}x^2+\frac{a(a^2-3b^2)}{3!}x^3+\ldots\qquad\textbf{(M.U. 1992, 2004)}$$

Hence, deduce that

1. $e^{x\cos\alpha}\cos(x\sin\alpha)=1+\cos\alpha+\frac{x^2}{2!}\cos 2\alpha+\frac{x^3}{3!}\cos 3\alpha+\ldots$
2. $e^{x}\cos x=1+2^{1/2}x\cos\frac{\pi}{4}+2^{2/2}\frac{x^2}{2!}\cos\left(\frac{2\pi}{4}\right)+2^{3/2}\frac{x^3}{3!}\cos\left(\frac{3\pi}{4}\right)+\ldots$

Sol. : Using the expansions of e^{ax} and $\cos bx$, we get

$$e^{ax}\cos bx=\left(1+ax+\frac{a^2x^2}{2!}+\frac{a^3x^3}{3!}+\ldots\right)\times\left(1-\frac{b^2x^2}{2!}+\frac{b^4x^4}{4!}-\ldots\right)$$

$$=1+ax+\frac{a^2x^2}{2!}+\frac{a^3x^3}{3!}+\ldots-\frac{b^2x^2}{2!}-\frac{ab^2x^3}{2!}+\ldots$$

$$=1+ax+\frac{(a^2-b^2)}{2!}x^2+\frac{a(a^2-3b^2)}{3!}x^3+\ldots\qquad\ldots\ldots(1)$$

For first deduction, compare $e^{ax}\cos bx$ with $e^{x\cos\alpha}\cos(x\sin\alpha)$.

It is clear that we have to put $a=\cos\alpha$ and $b=\sin\alpha$.

Putting these values of a and b in (1), we get,

$$e^{x\cos\alpha}\cos(x\sin\alpha)=1+\cos\alpha\cdot x+\frac{(\cos^2\alpha-\sin^2\alpha)}{2!}x^2+\cos\alpha\frac{(\cos^2\alpha-3\sin^2\alpha)}{3!}x^3+\ldots$$

But $\cos^2\alpha-\sin^2\alpha=\cos 2\alpha$ and

$$\begin{aligned}\cos\alpha(\cos^2\alpha-3\sin^2\alpha)&=\cos\alpha(\cos^2\alpha-3+3\cos^2\alpha)\\&=\cos\alpha(4\cos^2\alpha-3)\\&=4\cos^3\alpha-3\cos\alpha\\&=\cos 3\alpha\end{aligned}$$

$$\therefore\quad e^{x\cos\alpha}\cos x\sin\alpha=1+x\cos\alpha+\frac{x^2}{2!}\cos 2\alpha+\frac{x^3}{3!}\cos 3\alpha+\ldots \qquad \ldots\ldots\ldots (2)$$

For second deduction compare $e^{ax}\cos bx$ with $e^x\cos x$.

It is clear that, we have to put $a=1$ and $b=1$.

Putting these values of a and b in (1), we get,

$$e^x\cos x=1+x+0\frac{x^2}{2!}+(1-3)\frac{x^3}{3!}+\ldots \qquad \ldots\ldots\ldots (3)$$

But we can put the coefficient of x as $1=\sqrt{2}\cdot\cos\left(\frac{\pi}{4}\right)$

We can put the coefficient of x^2 as $0=(\sqrt{2})^2\cos\left(\frac{2\pi}{4}\right)$

We can put the coefficient of x^3 as $-2=2\cdot\sqrt{2}\cos\left(\frac{3\pi}{4}\right)=(2)^{3/2}\cos\left(\frac{3\pi}{4}\right)$

Hence, from (3) we get,

$$e^x\cos x=1+2^{1/2}x\cos\left(\frac{\pi}{4}\right)+2^{2/2}\frac{x^2}{2!}\cos\left(\frac{2\pi}{4}\right)+2^{3/2}\frac{x^3}{3!}\cos\left(\frac{3\pi}{4}\right)+\ldots$$

Example 4 (b) : Prove that $\log(1+e^x)=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+\ldots$

Sol. : We have

$$\log(1+e^x)=\log\left[1+1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right]=\log\left[2+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right]$$

$$=\log\left[2\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)\right]$$

$$=\log 2+\log\left[1+\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)\right]$$

$$=\log 2+\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\frac{x^4}{48}+\ldots\right)-\frac{1}{2}\left(\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{12}+\ldots\right)^2$$

$$+\frac{1}{3}\left(\frac{x}{2}+\frac{x^2}{4}+\ldots\right)^3-\frac{1}{4}\left(\frac{x}{2}+\ldots\right)^4+\ldots$$

$$\therefore \quad \log(1+e^x) = \log 2 + \frac{x}{2} + \left(\frac{1}{4} - \frac{1}{8}\right)x^2 + \left(\frac{1}{12} - \frac{1}{8} + \frac{1}{24}\right)x^3 + \left(\frac{1}{48} - \frac{1}{32} - \frac{1}{24} + \frac{1}{16} - \frac{1}{64}\right)x^4 + \ldots$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} + 0\,x^3 - \frac{1}{192}x^4 + \ldots$$

$$= \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{1}{192}x^4 + \ldots$$

Example 5 (b) : Expand in powers of x, $\dfrac{x}{e^x - 1}$.

Hence, prove that $\dfrac{x}{2}\left(\dfrac{e^x+1}{e^x-1}\right) = 1 + \dfrac{1}{12}x^2 - \dfrac{1}{720}x^4 + \ldots\ldots$ **(M.U. 1988, 89, 2001, 07, 14)**

Sol. : We have

$$\frac{x}{e^x - 1} = \frac{x}{1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots - 1} = \frac{x}{x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \ldots}$$

$$= \frac{1}{\left[1 + \left(\dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \dfrac{x^4}{5!} + \ldots\right)\right]}$$

$$= \left[1 + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \ldots\right)\right]^{-1} \qquad \text{[By Cor. 1 of (11a), page 12-5]}$$

$$\therefore \quad \frac{x}{e^x - 1} = 1 - \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \frac{x^4}{5!} + \ldots\right) + \left(\frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots\right)^2 - \left(\frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)^3 + \left(\frac{x}{2!} + \ldots\right)^4 - \ldots$$

$$= 1 - \frac{x}{2} - \frac{x^2}{6} - \frac{x^3}{24} - \frac{x^4}{120} - \ldots + \frac{x^2}{4} + \frac{x^3}{6} + \frac{x^4}{36} + \frac{x^4}{24} + \ldots - \frac{x^3}{8} - \frac{x^4}{8} - \ldots + \frac{x^4}{16} + \ldots$$

$$= 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \ldots$$

Now, $\dfrac{x}{2}\left(\dfrac{e^x+1}{e^x-1}\right) = \dfrac{x}{2}\left[\dfrac{e^x - 1 + 2}{e^x - 1}\right] = \dfrac{x}{2}\left[1 + \dfrac{2}{e^x - 1}\right] = \dfrac{x}{2} + \dfrac{x}{e^x - 1}$

$$\therefore \quad \frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right) = \frac{x}{2} + 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \ldots = 1 + \frac{x^2}{12} - \frac{x^4}{720}\ldots$$

Example 6 (b) : Find the values of a and b such that the expansion of

$$\log(1+x) - \frac{x(1+ax)}{1+bx}$$

In ascending powers of x will begin with the term x^4 and show that this term is $-x^4/36$.

Sol. : Let us expand the given function in powers of x.

$$\therefore \quad f(x) = \log(1+x) - x(1+ax)(1+bx)^{-1}$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - (x + ax^2)(1 - bx + b^2x^2 - b^3x^3 + \ldots)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - x + bx^2 - b^2x^3 + b^3x^4 + \ldots$$

$$- ax^2 + abx^3 - ab^2x^4 + \ldots$$

$$= \left(-\frac{1}{2} + b - a\right)x^2 + \left(\frac{1}{3} - b^2 + ab\right)x^3 + \left(-\frac{1}{4} + b^3 - ab^2\right)x^4 + \ldots$$

Since this is to begin with x^4 the coefficients of x^2 and x^3 must be zero.

$$\therefore \quad b - a = \frac{1}{2} \quad \text{and} \quad b^2 - ab - \frac{1}{3} = 0 \qquad \therefore \quad b = a + \frac{1}{2}.$$

$$\text{and} \quad \left(a + \frac{1}{2}\right)^2 - a\left(a + \frac{1}{2}\right) - \frac{1}{3} = 0$$

$$\therefore \quad a^2 + a + \frac{1}{4} - a^2 - \frac{a}{2} - \frac{1}{3} = 0 \qquad \therefore \quad \frac{a}{2} = \frac{1}{12} \qquad \therefore \quad a = \frac{1}{6}$$

$$\therefore \quad b = a + \frac{1}{2} = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$$

Now, the coefficient of x^4,

$$= -\frac{1}{4} + b^3 - ab^2 = -\frac{1}{4} + \frac{8}{27} - \frac{1}{6} \cdot \frac{4}{9} = -\frac{1}{36} \qquad \therefore \quad f(x) = -\frac{1}{36}x^4 + \ldots$$

Example 7 (b) : Obtain the expansion of $\frac{1+x^2}{1+x^4}$ and integrating it term by term deduce that

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots$$

Sol. : We have

$$\frac{1+x^2}{1+x^4} = (1+x^2)(1+x^4)^{-1} \qquad \text{[By Cor. 1 of (11a), page 12-5]}$$

$$= (1+x^2)(1 - x^4 + x^8 - x^{12} + x^{16} - \ldots)$$

$$= 1 - x^4 + x^8 - x^{12} + \ldots + x^2 - x^6 + x^{10} - \ldots$$

$$= 1 + x^2 - x^4 - x^6 + x^8 + x^{10} - x^{12} - \ldots$$

Now integrate both sides from 0 to 1.

$$\therefore \quad \int_0^1 \left(\frac{1+x^2}{1+x^4}\right) dx = \int_0^1 dx + \int_0^1 x^2\, dx - \int_0^1 x^4\, dx - \int_0^1 x^6\, dx + \int_0^1 x^8\, dx + \int_0^1 x^{10}\, dx - \ldots$$

$$= [x]_0^1 + \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^5}{5}\right]_0^1 - \left[\frac{x^7}{7}\right]_0^1 + \left[\frac{x^9}{9}\right]_0^1 + \left[\frac{x^{11}}{11}\right]_0^1 + \ldots$$

$$\therefore \quad \int_0^1 \left(\frac{1+x^2}{1+x^4}\right) dx = 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\ldots$$

Now to find the integral on l.h.s. put $x = \tan\theta \quad \therefore \; dx = \sec^2\theta \, d\theta$.

When $x = 0$, $\theta = 0$; when $x = 1$, $\theta = \pi/4$.

$$\therefore \quad \int_0^1 \frac{1+x^2}{1+x^4} dx = \int_0^{\pi/4} \frac{\sec^2\theta}{1+\tan^4\theta} \cdot \sec^2\theta \, d\theta = \int_0^{\pi/4} \frac{\sec^4\theta}{1+\tan^4\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{\sin^4\theta+\cos^4\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{(\sin^2\theta+\cos^2\theta)^2 - 2\sin^2\theta \cdot \cos^2\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{1}{1-(\sin^2 2\theta)/2} d\theta = \int_0^{\pi/4} \frac{2\, d\theta}{2-\sin^2 2\theta}$$

$$= \int_0^{\pi/4} \frac{2\, d\theta}{2\sin^2 2\theta + 2\cos^2 2\theta - \sin^2 2\theta}$$

$$= \int_0^{\pi/4} \frac{2\, d\theta}{\sin^2 2\theta + 2\cos^2 2\theta} = \int_0^{\pi/4} \frac{2\sec^2 2\theta}{\tan^2 2\theta + 2} d\theta$$

Now, put $\tan 2\theta = t \quad \therefore \; 2\sec^2 2\theta \, d\theta = dt$

When $\theta = 0$, $t = 0$; when $\theta = \pi/4$, $t = \infty$.

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_0^{\infty} = \frac{1}{\sqrt{2}}\left[\frac{\pi}{2}-0\right] = \frac{\pi}{2\sqrt{2}}.$$

Hence, $\dfrac{\pi}{2\sqrt{2}} = 1+\dfrac{1}{3}-\dfrac{1}{5}-\dfrac{1}{7}+\dfrac{1}{9}+\dfrac{1}{11}-\ldots$

Example 8 (b) : Show that

$$\tan^{-1}\left(\frac{x\sin\theta}{1-x\cos\theta}\right) = x\sin\theta + \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta + \ldots$$

Sol. : Let $y = \tan^{-1}\left(\dfrac{x\sin\theta}{1-x\cos\theta}\right) \quad \therefore \; \tan y = \dfrac{x\sin\theta}{1-x\cos\theta}$

$$\therefore \quad \frac{\sin y}{\cos y} = \frac{x\sin\theta}{1-x\cos\theta}$$

But $\sin t = \dfrac{e^{it}-e^{-it}}{2i}$ and $\cos t = \dfrac{e^{it}+e^{-it}}{2}$

$$\therefore \quad \frac{(e^{iy}-e^{-iy})/2i}{(e^{iy}+e^{-iy})/2} = \frac{x\sin\theta}{1-x\cos\theta}$$

$$\therefore \quad \frac{e^{iy}-e^{-iy}}{i(e^{iy}+e^{-iy})} = \frac{x\sin\theta}{1-x\cos\theta} \qquad \therefore \quad \frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}} = \frac{ix\sin\theta}{1-x\cos\theta}$$

By componendo and dividendo $\left[\text{i.e. using if } \dfrac{a}{b} = \dfrac{c}{d} \text{ then } \dfrac{a+b}{b-a} = \dfrac{c+d}{d-c}\right]$

$$\therefore \quad \frac{2e^{iy}}{2e^{-iy}} = \frac{1-x(\cos\theta - i\sin\theta)}{1-x(\cos\theta + i\sin\theta)}$$

$$\therefore \quad e^{2iy} = \frac{1-xe^{-i\theta}}{1-xe^{i\theta}} \qquad \therefore \quad 2iy = \log\left(\frac{1-xe^{-i\theta}}{1-xe^{i\theta}}\right)$$

$$\therefore \quad 2iy = \log(1-xe^{-i\theta}) - \log(1-xe^{i\theta})$$

$$= \left[-xe^{-i\theta} - \frac{x^2 e^{-2i\theta}}{2} - .....\right] - \left[-xe^{i\theta} - \frac{x^2 e^{2i\theta}}{2} - .....\right]$$

$$= x(e^{i\theta} - e^{-i\theta}) + \frac{x^2}{2}(e^{2i\theta} - e^{-2i\theta}) + ...$$

$$\therefore \quad 2iy = 2ix\sin\theta + \frac{x^2}{2}\cdot 2i\sin 2\theta + \frac{x^3}{3}\cdot 2i\sin 3\theta + ...$$

$$\therefore \quad y = x\sin\theta + \frac{x^2}{2}\cdot\sin 2\theta + \frac{x^3}{3}\cdot\sin 3\theta + ...$$

EXERCISE - II

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Prove the following : Class (a) : 3 Marks

1. $\cos^n x = 1 - n\cdot\frac{x^2}{2!} + n(3n-2)\cdot\frac{x^4}{4!} - ....$

2. $[\log(1+x)]^2 = x^2 - x^3 + \frac{11}{12}x^4 - ....$

3. $e^x \sin x = x + x^2 + \frac{x^3}{3} ....$

4. $\cos^3 x = 1 - \frac{3x^2}{2} + \frac{7x^4}{8} - ....$

5. $\frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} - ....$

6. $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + ...$

7. $\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right]$ and hence find $\log_e\left(\frac{11}{9}\right)$.

(M.U. 1993, 2002, 03) **[Ans. : 2·20067]**

8. $\log(1-x+x^2) = -x + \frac{x^2}{2} + \frac{2x^3}{3} - ....$ **(M.U. 2013)**

9. $\cos x \cos hx = 1 - \frac{2^2 x^4}{4!} + \frac{2^4 x^8}{8!} - ....$

10. $\log[\log(1+x)^{1/x}] = -\frac{x}{2} + \frac{5x^2}{24} + \frac{x^3}{8} + ...$

11. $\log\left(\frac{\sin hx}{x}\right) = \frac{x^2}{6} - \frac{x^4}{180} + \frac{x^6}{2835} - ....$

12. $\log(1+x+x^2) = x + \frac{x^2}{2} - \frac{2x^3}{3} + ...$

13. $e^x \cos x = 1 + x - \frac{x^3}{3} + ...$

14. $e^x \log(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ...$

15. $\cos^2 x = 1 - x^2 + \frac{1}{6}x^4 - \frac{2}{45}x^6 + ...$ **(M.U. 1991)**

16. $\log\left(\frac{\tan x}{x}\right) = \frac{x^2}{3} + \frac{7}{90}x^4 + ...$ **(M.U. 1999)**

17. $\log(1+x+x^2+x^3+x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...$ **(M.U. 1997)**

18. $\cos h^3 x = \frac{1}{4}\sum_{n=0}^{\infty} \frac{3^{2n}+3}{(2n)!} x^{2n}$

19. $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + ...$

$$= \left(\frac{x-1}{x}\right) + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + ...$$ **(M.U. 2014)**

20. $\log x = \log 2 + \left(\frac{x}{2}-1\right) - \frac{1}{2}\left(\frac{x}{2}-1\right)^2 + \frac{1}{3}\left(\frac{x}{2}-1\right)^3 - ....$

21. $\log\left(\frac{\sin x}{x}\right) = -\left(\frac{x^2}{6} + \frac{x^4}{180} + ...\right)$

22. $\log\cos h\, x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - ....$

23. $\log\left(\frac{1+e^{2x}}{x}\right) = \log 2 + \frac{x^2}{2} - \frac{x^4}{12} + ...$

Class (b) : 6 Marks

1. $\log\left(\frac{xe^x}{e^x-1}\right) = \frac{x}{2} - \frac{x^2}{24} + \frac{x^4}{2880} - ....$

2. $\tan^{-1}\left(\frac{x\cos\alpha}{1-x\sin\alpha}\right) = x\cos\alpha - \frac{x^2}{2}\sin 2\alpha - \frac{x^3}{3}\cos 3\alpha + ...$

3. $\sin h^3 x = \sum_{n=1} \frac{(3^n-3)\cdot[1-(-1)^n]\,x^n}{8\cdot n!}$

4. $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + ...$ **(M.U. 2001)**

8. Method of Inversion : Class (a) : 3 Marks

Example 1 (a) : If $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + ...\infty$, prove that $y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

Sol. : It is obvious that, we are given, $x = \log(1+y)$ $\therefore$ $1 + y = e^x$

$$\therefore \quad 1+y = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... \qquad \therefore \quad y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

Example 2 (a) : If $y = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$, prove that $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + ...$ **(M.U. 2005)**

Sol. : We have $1 + y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

$$\therefore \quad 1+y = e^x \qquad \therefore \quad x = \log(1+y) \qquad \therefore \quad x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + ...$$

Example 3 (a) : If $y = \sin^{-1} x$, prove that $x = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + ...$

Sol. : We have $x = \sin y$ $\therefore$ $x = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + ...$

Example 4 (a) : If $y = \tan^{-1} x$, prove that $x = y + \frac{y^3}{3} + \frac{2y^5}{15} + ...$

Sol. : We have $x = \tan y$ $\therefore$ $x = y + \frac{y^3}{3} + \frac{2y^5}{15} + ...$

EXERCISE - III

For solutions of this Exercise see Companion to Applied Mathematics - I

Class (a) : 3 Marks

(M.U. 2001, 02)

1. If $x = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \ldots$, find y as a series of x.

 [**Ans. :** $x = \cos y \quad \therefore \quad y = \cos^{-1} x$

 $\therefore \quad y = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left(x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{6} + \ldots\right)$

2. If $x = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots$, find y as a series of x.

 [**Ans. :** $x = \sin y \quad \therefore \quad y = \sin^{-1} x \quad \therefore \quad y = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \ldots$]

3. If $x = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \ldots$, find y as a series of x.

 [**Ans. :** $x = \tan^{-1} y \quad \therefore \quad y = \tan x \quad \therefore \quad y = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$]

4. If $x = y + \frac{y^3}{3!} + \frac{y^5}{5!} + \ldots$, find y as a series of x.

 [**Ans. :** $x = \sin h\, y \quad \therefore \quad y = \sin h^{-1} x \quad \therefore \quad y = x - \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \ldots$]

---

9. Method of Differentiation or Integration of Known Series

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Prove that $\log(\sec x) = \frac{1}{2} \cdot x^2 + \frac{1}{3} \cdot \frac{x^4}{4} + \frac{2}{15} \cdot \frac{x^6}{6} + \ldots$

(M.U. 2003, 09, 11, 14, 17)

Sol. : If $y = \log \sec x$, then $\frac{dy}{dx} = \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

$\therefore$ Integrating, $\quad y = a_0 + \frac{x^2}{2} + \frac{1}{3} \cdot \frac{x^4}{4} + \frac{2}{15} \cdot \frac{x^6}{6} + \ldots$

When $x = 0$, $y = 0 \quad \therefore \quad a_0 = 0$

$$\therefore \quad y = \frac{x^2}{2} + \frac{1}{3} \cdot \frac{x^4}{4} + \frac{2}{15} \cdot \frac{x^6}{6} + \ldots$$

(See also solved Ex. 2, page 12-7.)

Example 2 (a) : If $\log\sec x = \frac{1}{2} x^2 + A x^4 + B x^6 + \ldots$, find A and B.

Sol. : As proved above

$$\log\sec x = \frac{1}{2} x^2 + \frac{1}{12} x^4 + \frac{1}{45} x^6 + \ldots \quad \therefore \quad A = \frac{1}{12}, \quad B = \frac{1}{45}.$$

Example 3 (a) : Prove that $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ **(M.U. 2017)**

Hence, expand $\log(1 + x^2)$ in powers of x. **(M.U. 1999, 2001, 02, 08, 09, 10)**

Sol. : If $y = \tan^{-1} x$, then $\frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \ldots$

$\therefore$ Integrating, $y = a_0 + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$

When $x = 0$, $y = 0$ $\therefore a_0 = 0$

$$\therefore \quad y = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \qquad \ldots\ldots (1)$$

Differentiating (1) w.r.t. x, $\frac{dy}{dx} = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$

Multiply both sides by $2x$, $\therefore \frac{2x}{1+x^2} = 2x - 2x^3 + 2x^5 - 2x^7 + \ldots$

Now, integrate both sides

$$\therefore \quad \int \frac{2x}{1+x^2}\, dx = \int (2x - 2x^3 + 2x^5 - 2x^7 + \ldots)\, dx$$

$$\therefore \quad \log(1+x^2) = a_0 + x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$$

To find a_0, put $x = 0$ $\therefore a_0 = 0$

$$\therefore \quad \log(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$$

Example 4 (a) : If $y = \tan^{-1} x$, find $y_{101}'(0)$ and $y_{102}'(0)$.

Sol. : We have as proved above

$$y = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1} \cdot \frac{x^{2n-1}}{2n-1} + \ldots$$

When $n = 51$, the last term is $\frac{x^{101}}{101}$.

By differentiation $y_{101}' = (101)! \cdot \frac{x^0}{101} = 100!$

The next derivative $y_{102}' = 0$.

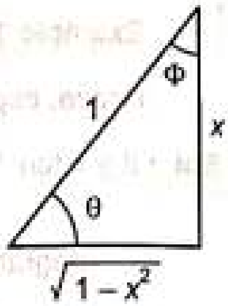
Example 5 (a) : Prove that

$$\sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{x^7}{7} + \ldots \qquad \textbf{(M.U. 2007)}$$

Sol. : If $y = \sin^{-1} x$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

Expanding by Binomial Theorem,

$$\frac{dy}{dx} = 1 - \left(-\frac{1}{2}\right) x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)}{2!} x^4 - \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(-\frac{1}{2} - 2\right)}{3!} x^6 + \ldots$$



$$\therefore \quad \frac{dy}{dx} = 1 + \frac{1}{2} \cdot x^2 + \frac{1 \cdot 3}{2 \cdot 4} x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^6 + \ldots$$

By integration,

$$y = a_0 + x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots$$

When $x = 0$, $y = 0$ $\quad \therefore \; a_0 = 0$

$$\therefore \quad y = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \ldots$$

Similarly, we can prove that

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left[x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \ldots \right]$$

[In the above triangle, $\sin \theta = x$,

$$\therefore \quad \theta = \sin^{-1} x \quad \text{and} \quad \Phi = \cos^{-1} x = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1} x. \;]$$

Example 6 (a) : Prove that $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \dfrac{2}{3} x^3 + \dfrac{8}{15} x^5 + \ldots$ **(M.U. 1985, 89, 95, 96)**

Sol. : Using the expansions of $\sin^{-1} x$ and of $1 / \sqrt{1-x^2}$ obtained above.

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = \left(x + \frac{1}{6} \cdot x^3 + \frac{3}{40} \cdot x^5 + \ldots \right) (1-x^2)^{-1/2}$$

$$= \left(x + \frac{1}{6} \cdot x^3 + \frac{3}{40} \cdot x^5 + \ldots \right) \left(1 + \frac{1}{2} x^2 + \frac{3}{8} x^4 + \ldots \right)$$

$$\therefore \quad \frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots + \frac{x^3}{2} + \frac{x^5}{12} + \ldots + \frac{3}{8} x^5 + \ldots$$

$$= x + \frac{2}{3} x^3 + \frac{8}{15} x^5 + \ldots$$

(For another method, see Ex. 2, page 12-31.)

Example 7 (a) : Prove that $\sinh^{-1} x = x - \dfrac{x^3}{6} + \dfrac{3x^5}{40} - \ldots$

Sol. : See solved Ex. 1, page 12-28 of the next method *i.e.* (10) Method of Substitution.

Example 8 (a) : Prove that $\log \left(x + \sqrt{1+x^2} \right) = x - \dfrac{x^3}{6} + \dfrac{3x^5}{40} - \ldots$

Sol. : Do as above.

Example 9 (a) : Prove that $\log \cos x = -\dfrac{x^2}{2} - \dfrac{x^4}{12} - \dfrac{x^6}{45} - \ldots$

Sol. : Let $y = \log \cos x$

$$\therefore \quad \frac{dy}{dx} = -\frac{\sin x}{\cos x} = -\tan x = -x - \frac{x^3}{3} - \frac{2x^5}{15} \ldots$$

Integrating both sides

$$y = \int\left(-x - \frac{x^3}{3} - \frac{2x^5}{15} - \ldots\right)dx = a_0 - \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \ldots$$

Putting $x = 0$, $y(0) = \log\cos 0 = \log 1 = 0$ $\quad\therefore\; a_0 = 0$

$$\therefore\; \log\cos x = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45}.$$

Example 10 (a) : Prove that $\tan h^{-1} x = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$

Sol. : Let $y = \tan h^{-1} x$.

We know that $\tan h^{-1} x = \dfrac{1}{2}\log\left(\dfrac{1+x}{1-x}\right)$ [See (3) of § 10, page 3-34]

$$\therefore\; y = \frac{1}{2}[\log(1+x) - \log(1-x)]$$

$$= \frac{1}{2}\left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)\right]$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

Additional Standard Results

1. $\sin^{-1} x = x + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{x^5}{5} + \ldots$
2. $\cos^{-1} x = \dfrac{\pi}{2} - \left[x + \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{x^5}{5} + \ldots\right]$
3. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$
4. $\sin h^{-1} x = x - \dfrac{1}{2}\cdot\dfrac{x^3}{3} + \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{x^5}{5} - \ldots$
5. $\tan h^{-1} x = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \ldots$

10. Method of Substitution

By a suitable substitution a given function can be reduced to a standard form as illustrated below.

Solved Examples : Class (a) : 3 Marks

Example 1 (a) : Expand $\sec^{-1}\left(\dfrac{1}{1-2x^2}\right)$. **(M.U. 1995)**

Sol. : Put $x = \sin\theta$ $\quad\therefore\; 1 - 2x^2 = 1 - 2\sin^2\theta = \cos 2\theta$

$$\therefore \quad \sec^{-1}\left(\frac{1}{1-2x^2}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right)$$

$$= \sec^{-1}\sec 2\theta = 2\theta = 2\sin^{-1} x$$

$$= 2\left[x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\right]$$

(Using the expansion of $\sin^{-1} x$ obtained above.)

Example 2 (a) : Expand $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$. **(M.U. 2002, 10)**

Sol. : Put $x = \tan\theta \quad \therefore \quad \sqrt{1+x^2} = \sqrt{\sec^2\theta} = \sec\theta$

$$\therefore \quad \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right) = \tan^{-1}\tan\left(\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\theta = \frac{1}{2}\tan^{-1} x = \frac{1}{2}\left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\right]$$

(Using the expansion of $\tan^{-1} x$ obtained above.)

Example 3 (a) : Expand $\tan^{-1}\left(\dfrac{p-qx}{q+px}\right)$.

Sol. : We have $\tan^{-1}\left(\dfrac{p-qx}{q+px}\right) = \tan^{-1}\left(\dfrac{(p/q)-x}{1+(p/q)\,x}\right)$

Now put $\dfrac{p}{q} = \tan\alpha$ and $x = \tan\theta$

$$\therefore \quad \tan^{-1}\left(\frac{p-qx}{q+px}\right) = \tan^{-1}\left(\frac{\tan\alpha - \tan\theta}{1+\tan\alpha\tan\theta}\right) = \tan^{-1}\tan(\alpha-\theta)$$

$$= \alpha - \theta = \tan^{-1}\left(\frac{p}{q}\right) - \tan^{-1} x$$

$$= \tan^{-1}\frac{p}{q} - \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right]$$

Solved Examples : Class (b) : 6 Marks

Example 1 (b) : Expand $\sin h^{-1}(3x+4x^3)$. **(M.U. 1989, 98)**

Sol. : Put $x = \sin h\,\theta$

$$\therefore \quad 3x + 4x^3 = 3\sin h\,\theta + 4\sin h^3\,\theta = \sin h\,3\theta$$

$$\therefore \quad \sin h^{-1}(3x+4x^3) = \sin h^{-1}(\sin h\,3\theta) = 3\theta = 3\sin h^{-1} x$$

Now, to obtain the expansion of $\sin h^{-1} x$, let

$$y = \sin h^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$ [See (1) of § 10, page 3-34]

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}=(1+x^2)^{-1/2}$$

Expanding by Binomial Theorem,

$$\frac{dy}{dx}=1-\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}\cdot x^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot x^6+\ldots$$ [See also Ex. 5, page 12-25]

Integrating, we get,

$$y=a_0+x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\ldots\ldots$$

But when $x=0$, $y=0$ $\quad\therefore\ a_0=0$

$$\therefore\quad \sin h^{-1}x=x-\frac{x^3}{6}+\frac{3x^5}{40}-\ldots\ldots$$

$$\therefore\quad \sin h^{-1}(3x+4x^3)=3\sin h^{-1}x=3\left[x-\frac{x^3}{6}+\frac{3x^5}{40}-\ldots\ldots\right]$$

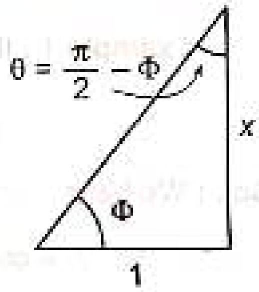
Example 2 (b) : Show that $\cos^{-1}[\tan h\log x]=\pi-2\left[x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots\right]$.

(M.U. 1994, 96, 2003, 06, 11, 17)

Sol. : Let $\cos^{-1}[\tan h\log x]=y$

$$\therefore\quad \cos y=\tan h(\log x)=\frac{e^{\log x}-e^{-\log x}}{e^{\log x}+e^{-\log x}}=\frac{x-x^{-1}}{x+x^{-1}}=\frac{x^2-1}{x^2+1}$$

$$\therefore\quad y=\cos^{-1}\left[\frac{x^2-1}{x^2+1}\right]$$



Now put $x=\cot\theta$,

$$\therefore\quad y=\cos^{-1}\left[\frac{\cot^2\theta-1}{\cot^2\theta+1}\right]=\cos^{-1}\left(\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$$

$$\therefore\quad y=\cos^{-1}\cos 2\theta=2\theta=2\cot^{-1}x$$

$$=2\left[\frac{\pi}{2}-\tan^{-1}x\right]=\pi-2\tan^{-1}x=\pi-2\left[x-\frac{x^3}{3}+\frac{x^5}{5}\ldots\right]$$

$$\therefore\quad \cos^{-1}[\tan h\log x]=\pi-2\left[x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\right]$$

EXERCISE - IV

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Prove the following : Class (a) : 3 Marks

1. $\sin^{-1}(3x-4x^3)=3\left[x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}+\ldots\right]$

2. $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)=2\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots\right)$ **(M.U. 2001)**

3. $\tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)=3\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-....\right)$

4. $\cot^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)=\dfrac{\pi}{2}-3\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-....\right)$

5. $\tan^{-1}\sqrt{\left\{\dfrac{1-x}{1+x}\right\}}=\dfrac{\pi}{4}-\dfrac{1}{2}\left(x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}+....\right)$

6. $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=2\left(x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+....\right)$

7. $\cos^{-1}\left(\dfrac{x-x^{-1}}{x-x^{-1}}\right)=\pi-2\left[x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+....\right]$

8. $\cos^{-1}(4x^3-3x)=3\left[\dfrac{\pi}{2}-\left(x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}+....\right)\right]$

9. $\cot^{-1}\sqrt{\left\{\dfrac{1-x}{1+x}\right\}}=\dfrac{\pi}{4}+\left(x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}+....\right)$

---

11. Use of Leinitz's Theorem

Example 1 : If $y=\sin(m\sin^{-1}x)$, prove that

$$y=mx+\frac{m(1^2-m^2)}{3!}x^3+\frac{m(1^2-m^2)(3^2-m^2)}{5!}x^5+....$$

Sol. : We have $\quad y=\sin(m\sin^{-1}x)$ .... (1)

$$\therefore\quad y_1=\cos(m\sin^{-1}x)\cdot m\cdot\frac{1}{\sqrt{1-x^2}}$$

$\therefore\quad \sqrt{1-x^2}\cdot y_1=m\cos(m\sin^{-1}x)$ .... (2)

Differentiating again, w.r.t. x,

$$\sqrt{1-x^2}\cdot y_2-y_1\frac{x}{\sqrt{1-x^2}}=-m^2\sin(m\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}$$

$\therefore\quad (1-x^2)y_2-xy_1+m^2y=0$ .... (3)

Now, by applying Leibnitz's theorem, we get

$$(1-x^2)y_{n+2}+ny_{n+1}(-2x)+\frac{n(n-1)}{2!}y_n(-2)-\left[xy_{n+1}+n\cdot1\cdot y_n\right]+m^2y_n=0$$

$\therefore\quad (1-x^2)y_{n+2}-x(2n+1)y_{n+1}+(m^2-n^2)y_n=0$ .... (4)

Putting $x=0$ in (1), (2), (3), (4), we get

$y(0)=0,\quad y_1(0)=m,\quad y_2(0)=0,\quad y_{n+2}(0)=(-m^2+n^2)\,y_n(0)$ .... (5)

Putting $n=1,2,3,4,....$ in (5), we get

$y_3(0)=(-m^2+1)\,m,\ \ y_4(0)=0,\ \ y_5(0)=(-m^2+3^2)(-m^2+1)\,m$ and so on.

But by Maclurin's series, we get

$$y = y(0) + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \ldots.$$

$$\therefore \quad y = 0 + x\,m + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}(-m^2+1) + \frac{x^4}{4!}\cdot 0 + \frac{x^5}{5!}\,m(-m^2+1)(-m^2+3^2) + \ldots.$$

$$\therefore \quad y = m\,x + m(-m^2+1)\cdot\frac{x^3}{3!} + m(-m^2+1)(-m^2+3^2)\cdot\frac{x^5}{5!} + \ldots.$$

Example 2 (c) : If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $y = x + \dfrac{2}{3}x^3 + \dfrac{2\cdot 4}{3\cdot 5}x^5 + \ldots.$

Sol. : In Ex. 7, page 8-30, we have obtained if $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ .... (1)

then $(1-x^2)\,y_1 - xy - 1 = 0$ .... (2)

and $(1-x^2)\,y_{n+1} - (2n+1)\,xy_n - n^2\,y_{n-1} = 0$ .... (3)

Putting $x = 0$ in (1), (2), (3), we get

$$y(0) = 0,\ \ y_1(0) = 1,\ \ y_{n+1}(0) = n^2\,y_{n-1}(0) \qquad \ldots\ldots\ldots\ldots (4)$$

Putting $n = 1$, in (4), $y_2(0) = 1^2\cdot y(0) = 0$

Putting $n = 2$, in (4), $y_3(0) = 2^2\cdot y_1(0) = 2^2$

Putting $n = 3$, in (4), $y_4(0) = 3^2\cdot y_2(0) = 0$

Putting $n = 4$, in (4), $y_5(0) = 4^2\cdot y_3(0) = 4^2\cdot 2^2$ and so on.

Now, by Maclaurin's series,

$$y = y + x\,y_1(0) + \frac{x^2}{2!}\,y_2(0) + \frac{x^3}{3!}\,y_3(0) + \ldots.$$

$$= 0 + x + 0 + \frac{x^3}{3!}\cdot 2^2 + 0 + \frac{x^5}{5!}\,4^2\cdot 2^2 + \ldots.$$

$$= x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \ldots.$$

(For another method, see Ex. 6, page 12-26.)

Example 3 (c) : If $y = e^{a\sin^{-1} x}$, prove that $y = 1 + ax + \dfrac{a^2x^2}{2!} + \dfrac{a(a^2+1)}{3!}x^3 + \ldots.$ and hence deduce that $e^{\theta} = 1 + \sin\theta + \dfrac{1}{2!}\sin^2\theta + \dfrac{2}{3!}\sin^3\theta + \ldots.$

Sol. : In Ex. 4, page 8-29, we have obtained if $y = e^{a\sin^{-1} x}$ .... (1)

then $\left(\sqrt{1-x^2}\right) y_1 = ay$, ... (2)

and $(1-x^2)\,y_2 - xy_1 = a^2 y$, ... (3)

$(1-x^2)\,y_{n+2} - (2x+1)\,xy_{n+1} = (n^2+a^2)\,y_n$ .... (4)

Putting $x = 0$ in (1), (2), (3), (4), we get

$$y(0) = 1,\ \ y_1(0) = a,\ \ y_2(0) = a^2,\ \ y_{n+2}(0) = (n^2+a^2)\,y_n \qquad \ldots\ldots\ldots\ldots (5)$$

Now, putting $n = 1, 2, 3, \ldots$ in (5), we get

$y_3(0) = (1^2 + a^2)\, y_1(0) = (a^2 + 1^2) \cdot a$

$y_4(0) = (2^2 + a^2)\, y_2(0) = (a^2 + 2^2)\, a^2$ and so on.

Now, by Maclaurin's series,

$$y = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots$$

$$= 1 + ax + \frac{a^2}{2!} x^2 + \frac{a(a^2+1^2)}{3!} x^3 + \ldots$$

For deduction put $a = 1$ and $x = \sin\theta$.

$$\therefore \quad y = e^{\theta} \quad \text{and} \quad e^{\theta} = 1 + \sin\theta + \frac{1}{2!}\sin^2\theta + \frac{2}{3!}\sin^3\theta + \ldots$$

Example 4 (c) : If $y = e^{m\tan^{-1}x} = a_0 + a_1 x + a_2 x^2 + \ldots$, prove that

(i) $y = 1 + mx + \dfrac{m^2}{2!} x^2 + \dfrac{m^2(m^2-2)}{3!} x^3 + \ldots$

(ii) $(n+1)\, a_{n+1} + (n-1)\, a_{n-1} = m\, a_n$.

Sol. : We have $\quad y = e^{m\tan^{-1}x}$ .... (1)

$$\therefore \quad y_1 = e^{m\tan^{-1}x}\, \frac{m}{1+x^2} \qquad \text{..................... (2)}$$

$\therefore \quad (1+x^2)\, y_1 = my$

Differentiating this again w.r.t. x,

$(1+x^2)\, y_2 + 2xy_1 = my_1$ .... (3)

Now, by Leibnitz's theorem,

$$(1+x^2)\, y_{n+2} + n(2x) \cdot y_{n+1} + \frac{n(n-1)}{2!} \cdot 2\, y_n + 2\, xy_{n+1} + 2 \cdot n \cdot 1 \cdot y_n = m\, y_{n+1}$$

$$\therefore \quad (1+x^2)\, y_{n+2} + 2(n+1)\, xy_{n+1} - m\, y_{n+1} + (n^2+n)\, y_n = 0 \qquad \text{..................... (4)}$$

Putting $x = 0$ in (1), (2), (3), (4), we get

$y(0) = 1, \;\; y_1(0) = m, \;\; y_2(0) = m\, y_1(0) = m^2,$

$y_{n+2}(0) = m\, y_{n+1}(0) - (n^2+n)\, y_n(0)$ .... (5)

Putting $n = 1, 2, 3, \ldots$ in (5), we get

$y_3(0) = m\, y_2(0) - (1^2+1)\, y_1 = 0$

$= m^3 - (1+1)\, m = m\,(m^2-2)$ and so on.

But by Maclaurin's series,

$$y = y(0) + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots$$

$$= 1 + mx + \frac{m^2}{2!} x^2 + \frac{m(m^2-2)}{3!} x^3 + \ldots$$

Now, for the second part compare the following two series.

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$$

$$y = y_0 + x\, y_1(0) + \frac{x^2}{2!} y_2(0) + \ldots + \frac{x^n}{n!} y_n(0) + \ldots$$

$\therefore$ Comparing the coefficients of x^n, we get $a_n = \dfrac{y_n(0)}{n!}$.

$$\left.\begin{aligned} \therefore \quad & y_n(0) = n! \cdot a_n \\ & y_{n+1}(0) = (n+1)! \cdot a_{n+1} \\ & y_{n-1}(0) = (n-1)! \cdot a_{n-1} \end{aligned}\right\} \qquad \text{.................... (6)}$$

Replacing n by $n-1$ in (5), we get

$$y_{n+1}(0) = m\, y_n(0) - n\,(n-1)\, y_{n-1}(0) \qquad \text{.................... (7)}$$

Putting the values of $y_n(0)$, $y_{n+1}(0)$ and $y_{n-1}(0)$ from (6), (7), we get

$$(n+1)! \cdot a_{n+1} = m\,(n!) \cdot a_n - n\,(n-1)\,(n-1)! \cdot a_{n-1}$$

$$= n!\,[\, m\, a_n - (n-1)\, a_{n-1}\,]$$

$$\therefore \quad (n+1)\, a_{n+1} + (n-1)\, a_{n-1} = m\, a_n.$$

Example 5 (c) : Prove that $\quad e^{\cos^{-1} x} = e^{\pi/2}\left[1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \ldots\right]$ **(M.U. 2008)**

Sol. : Let $y = e^{\cos^{-1} x}$

$$\therefore \quad y_1 = e^{\cos^{-1} x}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\therefore \quad \sqrt{1-x^2} \cdot y_1 = -y$$

$$\therefore \quad \sqrt{1-x^2} \cdot y_2 - \frac{x y_1}{\sqrt{1-x^2}} = -y_1 = \frac{y}{\sqrt{1-x^2}}$$

$$\therefore \quad (1-x^2)\, y_2 - x y_1 - y = 0$$

Applying Leibnitz's Theorem,

$$(1-x^2)\, y_{n+2} + n y_{n+1}(-2x) + \frac{n(n-1)}{2!} y_n(-2) - x y_{n+1} - n y_n - y_n = 0$$

$$\therefore \quad (1-x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2+1)\, y_n = 0 \qquad \text{.................... (1)}$$

Putting $x = 0$, in y, y_1 and y_2, we get

$$y(0) = e^{\pi/2}, \quad y_1(0) = -e^{\pi/2}, \quad y_2(0) = e^{\pi/2}$$

Putting $x = 0$, and $n = 1$ in (1), we get,

$$y_3(0) = 2\, y_1(0) = -2e^{\pi/2},$$

Putting $x = 0$ and $n = 2$ in (1), we get

$$y_4(0) = 5\, y_2(0) = 5e^{\pi/2},$$

$$\therefore \quad y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots$$

$$\therefore \quad e^{\cos^{-1} x} = e^{\pi/2} - x \cdot e^{\pi/2} + \frac{x^2}{2!} e^{\pi/2} - \frac{x^3}{3!} (2e^{\pi/2}) + \ldots$$

$$= e^{\pi/2}\left[1 - x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots\right]$$

Example 6 (c) : If $y^{1/m} - y^{-1/m} = 2x$, prove that

$$y = 1 + mx + \frac{m}{2!} \cdot x^2 + \frac{m(m^2 - 1^2)}{3!} x^3 + \frac{m^2(m^2 - 2^2)}{4!} x^4 + \ldots.$$

Sol. : We have $\quad y^{2/m} - 2xy^{1/m} - 1 = 0$

This is a quadratic in $y^{1/m}$.

$$y^{1/m} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

$$\therefore \quad y = \left(x \pm \sqrt{x^2 + 1}\right)^m \qquad \text{.................... (1)}$$

Taking only positive sign for convenience

$$y_1 = m\left(x + \sqrt{x^2 + 1}\right)^{m-1}\left[1 + \frac{x}{\sqrt{x^2 + 1}}\right]$$

$$= m\left(x + \sqrt{x^2 + 1}\right)^{m-1}\left(x + \sqrt{x^2 + 1}\right)\frac{1}{\sqrt{x^2 + 1}} = \frac{my}{\sqrt{x^2 + 1}}$$

$$\therefore \quad y_1\sqrt{x^2 + 1} = my \qquad \text{.................... (2)}$$

Differentiating again,

$$y_2 \cdot \sqrt{x^2 + 1} + y_1 \cdot \frac{x}{\sqrt{x^2 + 1}} = my_1$$

$$\therefore \quad (x^2 + 1)\, y_2 + xy_1 = my_1\sqrt{x^2 + 1} = m(my)$$

$$\therefore \quad (x^2 + 1)\, y_2 + xy_1 - m^2 y = 0 \qquad \text{.................... (3)}$$

Applying Leibnitz's Theorem

$$(x^2 + 1)\, y_{n+2} + n(2x)\, y_{n+1} + \frac{n(n-1)}{2!} 2 \cdot y_n + [\, xy_{n+1} + ny_n\,] - m^2 y_n = 0$$

$$(x^2 + 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} - (m^2 - n^2)\, y_n = 0 \qquad \text{.................... (4)}$$

Putting $x = 0$ in (1), $\quad y(0) = 1$

Putting $x = 0$ in (2), $\quad y_1(0) = my(0) = m$

Putting $x = 0$ in (3), $\quad y_2(0) = m^2\, y(0) = m^2$

Putting $x = 0$ in (4), $\quad y_{n+2}(0) = (m^2 - n^2)\, y_n(0) \qquad$ .... (5)

Putting $n = 1, 2, \ldots..$ in (5),

$$y_3(0) = (m^2 - 1^2) \cdot m, \quad y_4(0) = (m^2 - 2^2)\, m^2, \ldots..$$

By Maclaurin's Theorem,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!} y_2(0) + \ldots.$$

$$= 1 + mx + \frac{m^2}{2!} x^2 + \frac{m(m^2 - 1^2)}{3!} x^3 + \frac{m^2(m^2 - 2^2)}{4!} x^4 + \ldots.$$

EXERCISE - V

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Prove that

1. $e^{\tan^{-1}x} = 1 + x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \ldots.$

2. $e^{\sin^{-1}x} = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \ldots.$

3. $\log\left(x + \sqrt{1 + x^2}\right) = x - \dfrac{x^3}{3!} + (3^2 \cdot 1^2)\dfrac{x^5}{5!} + \ldots.$

12. Taylor's Series

Assuming that $f(x + h)$ can be expanded in **ascending powers of *h***, it is expressed as

$$\boxed{f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots + \frac{h^n}{n!} f^n(x) + \ldots\infty} \quad \ldots\ldots\ldots (1)$$

The above series is known as **Taylor's Series.**

Interchanging x and h we can express $f(x + h)$ in **ascending powers of *x*** as

$$\boxed{f(x + h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \ldots + \frac{x^n}{n!} f^n(h) + \ldots\infty} \quad \ldots\ldots\ldots (2)$$

Changing h to a, we have

$$\boxed{f(x + a) = f(a) + x f'(a) + \frac{x^2}{2} f''(a) + \ldots} \quad \ldots\ldots\ldots (3)$$

Replacing x by a and h by $x - a$ and noting that $a + h = a + (x - a) = x$, in (1), we get $f(x)$ as a power series of $(x - a)$.

$$\boxed{f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \ldots\infty} \quad \ldots\ldots\ldots (4)$$

Brook Taylor (1685 - 1731)

He was an English mathematician best known for Taylor's Theorem and Taylor's Series. Initially he had interest in law and got doctorate in law in 1714. But he had also keen interest in mathematics. His publication **'Methodus Incrementorum Directa et Inversa'** is considered as the beginning of new branch of mathematics called **"Calculus of finite differences"**. The famous Taylor's theorem remained unrecognised untill 1712 when Lagrange realised its powers. He was elected to Royal Society in the same year. From 1715 he took interest in the studies of religion and philosophy.

Type I : Expansion in powers of h (or x) : Class (a) : 3 Marks

Example 1 (a) : Show that $\log(x+h) = \log h + \dfrac{x}{h} - \dfrac{x^2}{2h^2} + \dfrac{x^3}{3h^3} - \ldots$

Sol. : Comparing with the form (2) of Taylor's Series, we see that $f(x) = \log x$.

$$\therefore \quad f'(h) = \frac{1}{h}, \quad f''(h) = \frac{-1}{h^2}, \quad f'''(h) = \frac{2}{h^3} \text{ and so on.}$$

Putting these values in

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} + f''(h) + \frac{x^3}{3!} f'''(h) + \ldots$$

$$\log(x+h) = \log h + \frac{x}{h} + \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \ldots$$

Example 2 (a) : Prove that

$$\log[\sin(x+h)] = \log \sin x + h \cot x - \frac{h^2}{2} \operatorname{cosec}^2 x + \frac{h^3}{3} \cdot \frac{\cos x}{\sin^3 x} - \ldots$$

Sol. : Let $f(x) = \log \sin x \quad \therefore f(x+h) = \log \sin(x+h)$

$$\therefore \quad f(x+h) = f(h) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots \qquad \ldots\ldots (1)$$

Now, $f'(x) = \dfrac{1}{\sin x} \cdot \cos x = \cot x,$

$f''(x) = -\operatorname{cosec}^2 x,$

$f'''(x) = 2 \operatorname{cosec} x \operatorname{cosec} x \cot x = 2 \cdot \dfrac{1}{\sin^2 x} \cdot \dfrac{\cos x}{\sin x}$

Putting these values in (1),

$$\log[\sin(x+h)] = \log \sin x + h \cot x - \frac{h^2}{2!} \operatorname{cosec}^2 x + \frac{h^3}{3} \cdot \frac{\cos x}{\sin^3 x} - \ldots$$

Example 3 (a) : Prove that $f\left(\dfrac{x^2}{1+x}\right) = f(x) - \dfrac{x}{1+x} \cdot f'(x) + \dfrac{x^2}{(1+x)^2} \cdot \dfrac{f''(x)}{2!} + \ldots$

Sol. : We first note that $\dfrac{x^2}{1+x} = x - \dfrac{x}{1+x} \quad \therefore$ Let $h = -\dfrac{x}{1+x}$

Then by Taylor's Theorem, we get

$$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} \cdot f'(x) + \frac{x^2}{(1+x)^2} \cdot \frac{f''(x)}{2!} + \ldots$$

Example 4 (a) : Show that $\sin h(x+a) = \sin ha + x \cos ha + \dfrac{x^2}{2!} \sin ha + \ldots$

Given that $\sin h\, 1{\cdot}5 = 2{\cdot}1293$, $\cos h\, 1{\cdot}5 = 2{\cdot}3524$, calculate $\sin h\, 1{\cdot}505$.

Sol. : By (3), we have $f(x+a) = f(a) + x f'(a) + \dfrac{x^2}{2!} f''(a) + \dfrac{x^3}{3!} f'''(a) + \ldots$

Let $f(x) = \sin hx \quad \therefore f'(a) = \cos h a, \; f''(a) = \sin ha$

$$\therefore \quad \sin h\,(x+a) = \sin ha + x\cos ha + \frac{x^2}{2!}\sin ha + \ldots..$$

Now put $a = 1{\cdot}5$ and $x = 0{\cdot}005$.

$$\therefore \quad \sin h\, 1{\cdot}505 = \sin h\,(1{\cdot}5) + 0{\cdot}005 \cos h\,(1{\cdot}5) + \frac{(0{\cdot}005)^2}{2!}\sin h\,(1{\cdot}5) + \ldots$$

$$= 2{\cdot}1293 + 0{\cdot}005\,(2{\cdot}3524) + (0{\cdot}000025)\,(2{\cdot}1293)$$

$$= 2{\cdot}1411$$

Example 5 (a) : Expand $\tan^{-1}(x+h)$ in powers of h and find the value of $\tan^{-1}(1{\cdot}003)$ upto five places of decimals.

Sol. : Let $f(x) = \tan^{-1} x$.

$$f'(x) = \frac{1}{1+x^2}, \quad f''(x) = -\frac{2x}{(1+x^2)^2},$$

$$f'''(x) = -2\left[\frac{(1+x^2)^2 \cdot 1 - x \cdot 2(1+x^2)\,2x}{(1+x^2)^4}\right]$$

$$= -2\left[\frac{1+x^2-4x^2}{(1+x^2)^3}\right] = \frac{2(3x^2-1)}{(1+x^2)^3}$$

By Taylor's Theorem,

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.$$

$$\therefore \quad \tan^{-1}(x+h) = \tan^{-1}x + h \cdot \frac{1}{1+x^2} + \frac{h^2}{2!}\left[-\frac{2x}{(1+x^2)^2}\right] + \frac{h^3}{3!}\left[\frac{2(3x^2-1)}{(1+x^2)^3}\right]$$

Now, put $x = 1$, $h = 0{\cdot}0003$

$$\therefore \quad \tan^{-1}(1{\cdot}0003) = \tan^{-1}1 + \frac{0{\cdot}0003}{2} + \frac{(0{\cdot}0003)^2}{2}\left[-\frac{2\cdot 1}{4}\right] + \frac{(0{\cdot}0003)^3}{3!}\left[\frac{2\cdot 2}{8}\right] + \ldots.$$

$$= \frac{\pi}{4} + 0{\cdot}000015 = 0{\cdot}785398 + 0{\cdot}000015$$

$$= 0{\cdot}78541 \text{ upto 5 places of decimals.}$$

Example 6 (a) : Show that $\sqrt{1+x+2x^2} = 1 + \frac{x}{2} + \frac{7}{8}x^2 - \frac{7}{16}x^3 + \ldots.$ by using Taylor's Theorem. **(M.U. 1996, 98)**

Sol. : Let $f(x) = \sqrt{x}$ then $f(x+h) = \sqrt{x+h}$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \ldots.$$

Now put $x = 1$ and $h = x + 2x^2$.

$$\therefore \quad f(x) = \sqrt{x} = x^{1/2}, \quad f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2},$$

$$f'''(x) = \frac{3}{8}x^{-5/2}.$$

When $x = 1$, $f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{4}$, $f'''(1) = \frac{3}{8}$

$$\therefore \quad \sqrt{1+x+2x^2} = 1+h\frac{1}{2}+\frac{h^2}{2!}\cdot\left(-\frac{1}{4}\right)+\frac{h^3}{3!}\left(\frac{3}{8}\right)+....$$

Putting $h = x+2x^2$,

$$\therefore \quad \sqrt{1+x+2x^2} = 1+\frac{1}{2}(x+2x^2)-\frac{1}{8}(x+2x^2)^2+\frac{1}{16}(x+2x^2)^3+....$$

$$= 1+\frac{1}{2}x+x^2-\frac{1}{8}(x^2+4x^3+....)+\frac{1}{16}(x^3+....)+....$$

$$= 1+\frac{1}{2}x+\frac{7x^2}{8}-\frac{7}{16}x^3+....$$

Type II : Expansion in powers of (x – a) : Class (a) : 3 Marks

Example 1 (a) : Expand $f(x) = \frac{1}{x}$ in powers of $(x-1)$.

Sol. : Let $f(x) = \frac{1}{x}$ and $a = 1$.

$$\because \quad f(x) = \frac{1}{x}, \qquad f(1) = 1$$

$$f'(x) = -\frac{1}{x^2}, \qquad f'(1) = -1$$

$$f''(x) = \frac{2\cdot 1}{x^3}, \qquad f''(1) = 2\cdot 1$$

$$f'''(x) = -\frac{3\cdot 2\cdot 1}{x^4}, \qquad f^{iv}(1) = -3\cdot 2\cdot 1$$

Now, $f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+....$

$$\therefore \quad f(x) = 1-(x-1)+(x-1)^2-(x-1)^3+.....$$

Example 2 (a) : Expand $f(x) = \frac{1}{1-x}$ in powers of $(x+2)$.

Sol. : Let $f(x) = \frac{1}{1-x}$ and $a = -2$.

$$\therefore \quad f(x) = \frac{1}{1-x}, \qquad f(-2) = \frac{1}{3}$$

$$f'(x) = \frac{1}{(1-x)^2}, \qquad f'(-2) = \frac{1}{3^2}$$

$$f''(x) = \frac{2}{(1-x)^3}, \qquad f''(-2) = \frac{2}{3^3}$$

$$f'''(x) = \frac{2\cdot 3}{(1-x)^4}, \qquad f'''(-2) = \frac{3!}{3^4} \quad \text{and so on.}$$

By Taylor's series,

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+....$$

Putting the above values, we get

$$f(x)=\frac{1}{3}+\frac{x+2}{3^2}+\frac{(x+2)^2}{3^3}+\frac{(x+2)^3}{3^4}+\ldots$$

Example 3 (a) : Expand $x^5-x^4+x^3-x^2+x-1$ in powers of $(x-1)$ and hence find (i) $f(11/10)$, **(ii)** $f(0\cdot 99)$.

Sol. : Let $f(x)=x^5-x^4+x^3-x^2+x-1$ and $a=1$ $\quad\therefore\ f(1)=0$

$\therefore\ f'(x)=5x^4-4x^3+3x^2-2x+1$ $\quad\therefore\ f'(1)=3$

$f''(x)=20x^3-12x^2+6x-2,$ $\quad\therefore\ f''(1)=12$

$f'''(x)=60x^2-24x+6,$ $\quad\therefore\ f'''(1)=42$

$f^{iv}(x)=120x-24,$ $\quad\therefore\ f^{iv}(1)=96$

$f^{v}(x)=120$ $\quad\therefore\ f^{v}(1)=120$

Now, $f(x)=f(a)+(x-a)f'(a)+\dfrac{(x-a)^2}{2!}f''(a)+\dfrac{(x-a)^3}{3!}f'''(a)+\ldots$

$$\therefore\ f(x)=0+(x-1)\cdot 3+\frac{(x-1)^2}{2!}\cdot 12+\frac{(x-1)^3}{3!}\cdot 42+\frac{(x-1)^4}{4!}\cdot 96+\frac{(x-1)^5}{5!}\cdot 120$$

$$=3(x-1)+6(x-1)^2+7(x-1)^3+4(x-1)^4+(x-1)^5$$

(i) To find $f\left(\dfrac{11}{10}\right)$, we put $x=\dfrac{11}{10}=1\cdot 1$ and $x-1=0\cdot 1$.

$$\therefore\ f(1\cdot 1)=3(0\cdot 1)+6(0\cdot 1)^2+7(0\cdot 1)^3+4(0\cdot 1)^4+(0\cdot 1)^5$$

$$=0\cdot 3+0\cdot 06+0\cdot 007+0\cdot 0004+0\cdot 00001$$

$$=0\cdot 36741.$$

(ii) To find $f(0\cdot 99)$, we put $x=0\cdot 99$ and $(x-1)=0\cdot 99-1=-0\cdot 01$.

$$\therefore\ f(0\cdot 99)=3(-0\cdot 01)+6(-0\cdot 01)^2+7(-0\cdot 01)^3+4(-0\cdot 01)^4+(-0\cdot 01)^5$$

$$=-0\cdot 02939.$$

Example 4 (a) : Expand $\tan^{-1}x$ in powers of $\left(x-\dfrac{\pi}{4}\right)$. **(M.U. 2019)**

Sol. : Let $f(x)=\tan^{-1}x$ and $a=\dfrac{\pi}{4}$.

$$\therefore\ f(x)=\tan^{-1}x,\quad f'(x)=\frac{1}{1+x^2},\quad f''(x)=-\frac{2x}{(1+x^2)^2}$$

$$\therefore\ f\left(\frac{\pi}{4}\right)=\tan^{-1}\left(\frac{\pi}{4}\right),\quad f'\left(\frac{\pi}{4}\right)=\frac{1}{1+(\pi/4)^2},\quad f''\left(\frac{\pi}{4}\right)=-\frac{\pi/2}{[1+(\pi^2/16)]^2},\ \text{etc.}$$

Now, $f(x)=f(a)+(x-a)f'(a)+\dfrac{(x-a)^2}{2!}f''(a)+\ldots$

$$\therefore\ \tan^{-1}x=\tan^{-1}\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\cdot\frac{1}{[1+(\pi^2/16)]}-\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right)^2\cdot\frac{1}{[1+(\pi^2/16)]^2}+\ldots$$

Example 5 (a) : Expand $\log\cos x$ about $\pi/3$.

Sol. : Let $f(x)=\log\cos x$ and $a=\pi/3$

$$\therefore\ f'(x)=\frac{1}{\cos x}(-\sin x)=-\tan x,\qquad f''(x)=-\sec^2 x,$$

$$f'''(x) = -2\sec x \cdot \sec x \tan x = -2\sec^2 x \tan x.$$

Now, by Taylor's Series,

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots.$$

$$\therefore \quad \log\cos x = \log\cos\left(\frac{\pi}{3}\right) + \left(x - \frac{\pi}{3}\right)\left(-\tan\frac{\pi}{3}\right) + \frac{[x-(\pi/3)]^2}{2!}\cdot\left(-\sec^2\frac{\pi}{3}\right)$$

$$+ \frac{[x-(\pi/3)]^3}{3!}\cdot\left(-2\sec^2\frac{\pi}{3}\tan\frac{\pi}{3}\right) + \ldots.$$

$$= \log\left(-\frac{1}{2}\right) + \left(x - \frac{\pi}{3}\right)(-\sqrt{3}) + \frac{[x-(\pi/3)]^2}{2!}\cdot(-4) + \frac{[x-(\pi/3)]^3}{3!}\cdot\left(-2\cdot 4\sqrt{3}\right) + \ldots.$$

$$= -\log 2 - \sqrt{3}\left(x - \frac{\pi}{3}\right) - 2\left(x - \frac{\pi}{3}\right)^2 - \frac{4}{\sqrt{3}}\left(x - \frac{\pi}{3}\right)^3 + \ldots.$$

Example 6 (a) : Expand e^x in powers of $(x-1)$. **(M.U. 1997, 2015)**

Sol. : Let $f(x) = e^x$ and $a = 1$.

$$\therefore \quad f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x \text{ etc.}$$

$$f(1) = e, \quad f'(1) = e, \quad f''(1) = e, \quad f'''(1) = e \text{ etc.}$$

Now, $f(x) = f(a) + (x-a) f'(a) + \dfrac{(x-a)^2}{2!} f''(a) + \ldots.$

$$\therefore \quad e^x = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \ldots..$$

$$\therefore \quad e^x = e + (x-1)\cdot e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \ldots..$$

$$= e\left[1 + (x-1) + \frac{1}{2!}\cdot(x-1)^2 + \frac{1}{3!}\cdot(x-1)^3 + \ldots..\right]$$

Example 7 (a) : Expand $2x^3 + 7x^2 + x - 1$ in powers of $x-2$. **(M.U. 2018)**

Sol. : Let $f(x) = 2x^3 + 7x^2 + x - 1$ and $a = 2$

$$\therefore \quad f'(x) = 6x^2 + 14x + 1, \quad f''(x) = 12x + 14, \quad f'''(x) = 12$$

$$\therefore \quad f(2) = 45, \quad f'(2) = 53, \quad f''(2) = 38, \quad f'''(2) = 12$$

Now, $f(x) = f(a) + (x-a) f'(a) + \dfrac{(x-a)^2}{2!} f''(a) + \ldots.$

$$\therefore \quad f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2) + \ldots..$$

$$2x^3 + 7x^2 + x - 1 = 45 + (x-2)\,53 + (x-2)^2\cdot 19 + (x-2)^3\cdot 2$$

Example 8 (a) : Expand $f(x) = (x+2)^4 + 5(x+2)^3 + 6(x+2)^2 + 7(x+2) + 8$ in ascending powers of $(x+1)$.

Sol. : By successive differentiation, we get,

$$f'(x) = 4(x+2)^3 + 15(x+2)^2 + 12(x+2) + 7,$$

$$f''(x) = 12(x+2)^2 + 30(x+2) + 12,$$

$$f'''(x) = 24(x+2) + 30, \qquad f^{iv}(x) = 24.$$

Now, $f(x) = f(a) + (x-a)f'(a) + \dfrac{(x-a)^2}{2!}f''(a) + \ldots$

Putting $x = -1$, $f(-1) = 27$, $f'(-1) = 38$, $f''(-1) = 54$, $f'''(-1) = 54$.

$$\therefore \quad f(x) = f(-1) + (x+1)f'(-1) + \frac{(x+1)^2}{2!}f''(-1) + \frac{(x+1)^3}{3!}f'''(-1) + \frac{(x+1)^4}{4!}f^{iv}(-1)$$

$$= 27 + 38(x+1) + 27(x+1)^2 + 9(x+1)^3 + (x+1)^4.$$

Example 9 (a) : By using Taylor's Theorem, arrange in powers of x

$7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$. **(M.U. 1992, 94, 2003, 13)**

Sol. : We know that

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ldots \qquad \text{.................... (1)}$$

Taking $h = 2$, we get,

$$f(x+2) = f(2) + x f'(2) + \frac{x^2}{2!}f''(2) + \frac{x^3}{3!}f'''(2) + \ldots$$

Here, $f(x+2) = 7 + (x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$

$\therefore \quad f(x) = 7 + x + 3x^3 + x^4 - x^5 \qquad \therefore \quad f'(x) = 1 + 9x^2 + 4x^3 - 5x^4$

$\therefore \quad f''(x) = 18x + 12x^2 - 20x^3 \qquad \therefore \quad f'''(x) = 18 + 24x - 60x^2$

$\therefore \quad f^{iv}(x) = 24 - 120x, f^{v}(x) = -120.$

Hence we get, $f(2) = 17$, $f'(2) = -11$, $f''(2) = -76$

$f'''(2) = -174$, $f^{iv}(2) = -216$, $f^{v}(2) = -120$.

Putting these values in (1), we get

Given Expression $= 17 - 11x - 38x - 29x^3 - 9x^4 - x^5$.

Type III : To Find Approximate Value : Class (a) : 3 Marks

Example 1 (a) : Using Taylor's Theorem, find $\sqrt{25{\cdot}15}$.

Sol. : Let $f(x) = \sqrt{x}$ and $h = 0{\cdot}15$, $x = 25$.

Then, $f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!}f''(x) + \ldots$

$$f(x) = \sqrt{x} \qquad \therefore \quad f'(x) = \frac{1}{2\sqrt{x}},\ f''(x) = -\frac{1}{4}x^{-3/2},\ f'''(x) = \frac{3}{8}x^{-5/2}$$

$$\therefore \quad \sqrt{x+h} = x^{1/2} + h\frac{1}{2}x^{-1/2} + \frac{h^2}{2!}\left(-\frac{1}{4}\right)x^{-3/2} + \frac{h^3}{3!}\left(\frac{3}{8}\right)x^{-5/2} + \ldots$$

$$\therefore \quad \sqrt{25{\cdot}15} = 5 + \frac{(0{\cdot}15)}{2}(5)^{-1} + \frac{(0{\cdot}15)^2}{2}\cdot\left(-\frac{1}{4}\right)(5)^{-3} + \frac{(0{\cdot}15)^3}{3!}\cdot\frac{3}{8}(5)^{-5}$$

$$= 5 + 0{\cdot}015 - 0{\cdot}0000225 = 5{\cdot}01498$$

Example 2 (a) : Calculate the value of $\sqrt{10}$ to four places of decimals by using Taylor's Theorem. **(M.U. 1995)**

Sol. : Let $f(x) = \sqrt{x}$, then by Taylor's Theorem

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

Now $\quad f(x) = x^{1/2}, \quad f'(x) = \frac{1}{2} x^{-1/2}, \quad f''(x) = -\frac{1}{4} x^{-3/2}$

$$f'''(x) = \frac{3}{8} x^{-5/2}, \quad f^{iv}(x) = -\frac{15}{16} x^{-7/2}, \ldots$$

$$\therefore \quad f(x+h) = x^{1/2} + h \cdot \frac{1}{2} x^{-1/2} + \frac{h^2}{2!}\left(-\frac{1}{4}\right) x^{-3/2} + \frac{h^3}{3!} \cdot \frac{3}{8} x^{-5/2} + \frac{h^4}{4!}\left(-\frac{15}{16}\right) x^{-7/2} + \ldots$$

Now put $x = 9$ and $h = 1$,

$$f(10) = \sqrt{10}$$

$$= 3 + \frac{1}{2} \cdot \frac{1}{3} - \frac{1}{8} \cdot \frac{1}{27} + \frac{1}{16} \cdot \frac{1}{243} - \frac{5}{128} \cdot \frac{1}{2187}$$

$$= 3 \cdot 16227.$$

Example 3 (a) : Apply Taylor's Theorem to find approximately the value of $f(11/10)$ where $f(x) = x^3 + 3x^2 + 15x - 10$.

Sol. : By Taylor's Theorem, we have,

$$f(x+h) = f(x) + hf'(x) \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

To find $f\left(\frac{11}{10}\right)$, we write $f\left(1 + \frac{1}{10}\right)$ *i.e.*, $x = 1$, $h = 0 \cdot 1$.

We put $x = 1$, $h = 0 \cdot 1$

$f(x) = x^3 + 3x^2 + 15x - 10, \qquad f(1) = 9$

$f'(x) = 3x^2 + 6x + 15, \qquad f'(1) = 24$

$f''(x) = 6x + 6 \qquad \therefore \; f''(1) = 12, \quad f'''(1) = 6.$

$$\therefore \quad f\left(\frac{11}{10}\right) = 9 + \frac{(0.1)}{1} \cdot 24 + \frac{(0.1)^2}{2!} 12 + \frac{(0.1)^3}{3!} 6 = 11 \cdot 461$$

Example 4 (a) : Find the approximate value of $\cos h\,(1 \cdot 505)$ given that $\sin h\,(1 \cdot 5) = 2 \cdot 1293$ and $\cos h\,(1 \cdot 5) = 2 \cdot 3524$.

Sol. : Let $\quad f(x) = \cos hx$.

Now, by Taylor's series,

$$f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

To find $\cos h\,(1 \cdot 505)$, we put $f(x) = \cos hx$, $x = 1 \cdot 5$ and $h = 0 \cdot 005$.

$$\therefore \quad \cos h\,(1 \cdot 505) = \cos h\,(1 \cdot 5 + 0 \cdot 005) = f(x) + h\,f'(x) + \frac{h^2}{2!} f''(x) + \ldots \quad \ldots\ldots\ldots\ldots (1)$$

$\therefore \quad f(x) = f(1 \cdot 5) = \cos h\,(1 \cdot 5) = 2 \cdot 3524$

$f'(x) = \sin hx \quad \therefore \; f'(1 \cdot 5) = \sin h\,(1 \cdot 5) = 2 \cdot 1293$

$f''(x) = \cos hx \quad \therefore \; f''(1 \cdot 5) = \cos h\,(1 \cdot 5) = 2 \cdot 3524$

Putting these values in (1), we get

$$\cos h\,(1{\cdot}505) = \cos h\,(1{\cdot}5) + 0{\cdot}005 \sin h\,(1{\cdot}5) + \frac{(1{\cdot}005)^2}{2!} \cos h\,(1{\cdot}5) + \ldots$$

$$= 2{\cdot}3524 + 0{\cdot}005\,(2{\cdot}1293) + \ldots$$

$$= 2{\cdot}3524 + 0{\cdot}01064$$

$$= 2{\cdot}3631$$

Miscellaneous Examples : Class (b) : 6 Marks

Example 1 (b) : Prove that

$$e^x = 1 + \tan x + \frac{1}{2!}\tan^2 x - \frac{1}{3!}\tan^3 x - \frac{7}{4!}\tan^4 x + \ldots \qquad \textbf{(M.U. 2002)}$$

Sol. : We assume $e^x = a_0 + a_1 \tan x + a_2 \tan^2 x + a_3 \tan^3 x + a_4 \tan^4 x + \ldots$ .... (1)

and putting the series of $\tan x$ determine the constants $a_0, a_1, \ldots$

But $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$

Putting this in the above series (1)

$$e^x = a_0 + a_1\left(x + \frac{1}{3}x^3 + \ldots\right) + a_2\left(x + \frac{1}{3}x^3 + \ldots\right)^2 + a_3\left(x + \frac{1}{3}x^3 + \ldots\right)^3 + a_4\left(x + \ldots\right)^4 + \ldots$$

$$= a_0 + a_1 x + a_2 x^2 + \left(\frac{a_1}{3} + a_3\right)x^3 + \left(\frac{2}{3}a_2 + a_4\right)x^4 + \ldots \qquad \text{.......... (2)}$$

But $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ .... (3)

Comparing the coefficients of powers of x in (2) and (3),

$$a_0 = 1,\ a_1 = 1,\ a_2 = \frac{1}{2!},\ \frac{a_1}{3} + a_3 = \frac{1}{3!} \quad \therefore\ a_3 = -\frac{1}{3!}$$

$$\frac{2}{3}a_2 + a_4 = \frac{1}{4!} \quad \therefore\ a_4 = -\frac{7}{4!}.$$

Hence, the result.

Alter : Let $y = e^x$ and $\tan x = t$ i.e. $x = \tan^{-1} t$.

$$\therefore\ y = e^{\tan^{-1} t} \qquad \therefore\ y_1 = e^{\tan^{-1} t} \cdot \frac{1}{1 + t^2}$$

$$\therefore\ (1 + t^2)\,y_1 = y \qquad \therefore\ (1 + t^2)\,y_2 + 2t\,y_1 = y_1$$

$$\therefore\ (1 + t^2)\,y_2 + (2t - 1)\,y_1 = 0$$

Applying Leibnitz's Theorem,

$$(1 + t^2)\,y_{n+2} + n(2t)\,y_{n+1} + \frac{n(n-1)}{2!}\cdot(2)\,y_n + (2t - 1)\,y_{n+1} + n(2)\cdot y_n = 0$$

$$\therefore\ (1 + t^2)\,y_{n+2} + (2nt + 2t - 1)\,y_{n+1} + (n^2 + n)\,y_n = 0 \qquad \text{..................... (1)}$$

Now, when $t = 0$, $y(0) = 1$, $y_1(0) = 1$, $y_2(0) = 1$.

Putting $t = 0$ in (1), $y_{n+2} = y_{n+1} - (n^2 + n)\, y_n$

Put $n = 1$, $y_3 = y_2 - 2y_1 = -1$

$n = 2$, $y_4 = y_3 - 6y_2 = -7$

$\therefore$ By Maclaurin's Series

$$y = y(0) + t\, y_1(0) + \frac{t^2}{2!} y_2(0) + \ldots$$

$$\therefore \quad y = e^x = 1 + t + \frac{t^2}{2!} - \frac{t^3}{3!} - \frac{7t^4}{4!} + \ldots$$

$$= 1 + (\tan x) + \frac{1}{2}(\tan x)^2 - \frac{1}{3!}(\tan x)^3 - \frac{7}{4!}(\tan x)^4 + \ldots$$

Example 2 (b) : Prove that if $y = e^{x\cos\alpha}\cos(x\sin\alpha)$, then $y_n = e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha)$. **(M.U. 1997)**

Hence, show that $\quad e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \ldots \infty$

Also deduce that $\quad \cos x = 1 - \frac{x^2}{2!} + \ldots$

Sol. : Let $y = e^{ax}\cos bx$

$$\therefore \quad y_1 = e^{ax} a\cos bx + e^{ax}\cdot(-b\sin bx)$$

$$= \sqrt{a^2+b^2}\cdot e^{ax}\left[\frac{a}{\sqrt{a^2+b^2}}\cos bx - \frac{b}{\sqrt{a^2+b^2}}\sin bx\right]$$

Put $\quad \frac{a}{\sqrt{a^2+b^2}} = \cos\Phi; \quad \frac{b}{\sqrt{a^2+b^2}} = \sin\Phi$

$$y_1 = \sqrt{a^2+b^2}\cdot e^{ax}\,[\cos bx\cos\Phi - \sin bx\sin\Phi]$$

$$= \sqrt{a^2+b^2}\cdot e^{ax}\cos(bx + \Phi)$$

This means y_1 is obtained by multiplying y by $\sqrt{a^2+b^2}$ and by increasing the angle by Φ.

$$\therefore \quad y_2 = \left(\sqrt{a^2+b^2}\right)^2 e^{ax}\cos(bx + 2\Phi)$$

. . . .

. . . .

$$y_n = \left(\sqrt{a^2+b^2}\right)^n e^{ax}\cos(bx + n\Phi)$$

Now, comparing with the given example, put $a = \cos\alpha$, $b = \sin\alpha$.

$$\therefore \quad a^2 + b^2 = \cos^2\alpha + \sin^2\alpha = 1$$

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}} = \cos\Phi \text{ gives } a = \cos\Phi. \text{ But } a = \cos\alpha$$

$$\therefore \quad \alpha = \Phi \quad \therefore \quad y_n = e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha) \quad \ldots\ldots\ldots\ldots (1)$$

Putting $x = 0$ in $y = e^{x\cos\alpha}\cos(x\sin\alpha)$, we get $\quad y(0) = 1$.

From (1), putting $n = 1$, $\quad y_1 = e^{x\cos\alpha}\cos(x\sin\alpha + \alpha)$

Putting $x = 0$, in y_1, we get $\quad y_1(0) = \cos\alpha$.

From (1), putting $n = 2$, $\quad y_2 = e^{x\cos\alpha}\cos(x\sin\alpha + 2\alpha)$

Putting $x = 0$ in y_2, we get $\quad y_2(0) = \cos(2\alpha)$ and so on.

Now, $\quad f(x) = f(0) + x f'(0) + \dfrac{x^2}{2!} f''(0) + \ldots$

$$\therefore \quad e^{x\cos\alpha}\cos(x\sin\alpha) = 1 + x\cos\alpha + \frac{x^2}{2!}\cos 2\alpha + \ldots$$

Now, putting $\alpha = \dfrac{\pi}{2}$, we get, $\cos x = 1 - \dfrac{x^2}{2!} + \ldots$

Example 3 (b) : Use Taylor's Theorem to prove that

$$\tan^{-1}(x+h) = \tan^{-1}x + h\sin z \cdot \frac{\sin z}{1} - (h\sin z)^2 \cdot \frac{\sin 2z}{2} + (h\sin z)^3 \cdot \frac{\sin 3z}{3} + \ldots \text{ where } z = \cot^{-1}x.$$

Sol. : Let $\quad f(x) = \tan^{-1}x \quad \therefore \quad f(x+h) = \tan^{-1}(x+h)$

$$\therefore \quad f'(x) = \frac{1}{1+x^2} = \frac{1}{1+\cot^2 z} = \sin^2 z \qquad [\text{ By data } z = \cot^{-1}x \text{ i.e., } \cot z = x]$$

$$f''(x) = 2\sin z\cos z \cdot \frac{dz}{dx} = \sin 2z \cdot \left(-\frac{1}{1+x^2}\right)$$

$$= -\sin 2z \cdot \frac{1}{1+\cot^2 z} = -\sin 2z \cdot \sin^2 z$$

$$f'''(x) = -[\sin 2z \cdot 2\sin z\cos z + 2\cos 2z\sin^2 z]\frac{dz}{dx}$$

$$= -2\sin z(\sin 2z\cos z + \cos 2z\sin z) \cdot \frac{dz}{dx}$$

$$= -2\sin z \cdot \sin 3z \cdot (-\sin^2 z)$$

$$= 2\sin 3z \cdot \sin^3 z \quad \text{and so on.}$$

Now, $\quad \tan^{-1}(x+h) = f(x+h) = f(x) + h f'(x) + \ldots$

$$= \tan^{-1}x + h\sin z\frac{\sin z}{1!} + \frac{h^2}{2!}(-\sin 2z \cdot \sin^2 z) + \frac{h^3}{3!}(2\sin 3z) \cdot \sin^3 z + \ldots$$

$$= \tan^{-1}x + h\sin z \cdot \frac{\sin z}{1!} - (h\sin z)^2 \cdot \frac{\sin 2z}{2} + (h\sin z)^3 \cdot \frac{\sin 3z}{3} + \ldots$$

EXERCISE - VI

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Type I : Expansion in powers of x [Class (a) : 3 Marks]

1. Expand $\log\tan\left(\dfrac{\pi}{4} + x\right)$ in powers of x. **(M.U. 1999)** [**Ans. :** $2x + \dfrac{4}{3}x^3 + \dfrac{4}{3}x^5 + \ldots$]

2. Show that $\quad \cos h(x+h) = \cos hx + h\sin hx + \dfrac{h^2}{2!}\cos hx + \dfrac{h^3}{3!}\sin hx + \ldots$

   Given that $\sin h(1\cdot5) = 2\cdot1293$, $\cos h(1\cdot5) = 2\cdot3524$, calculate $\cos h(1\cdot505)$.

   [**Ans. : 2·3631**]

3. Expand $\tan^{-1}(x+h)$ in powers of h and hence find the value of $\tan^{-1}(1\cdot003)$ upto 5 places of decimals. Given $\pi = 3\cdot141593$. **[Ans. :** $0\cdot78690$ **]**

4. Expand $\tan\left(\frac{\pi}{4}+x\right)$ and hence, find the value of tan (46°, 36') upto four places of decimals. **[Ans. :** $1\cdot0574$ **]**

5. Expand $\sin\left(\frac{\pi}{6}+x\right)$ upto x^4 and find sin (30°, 30')

[Ans. : $\sin\left(\frac{\pi}{6}+x\right) = \frac{1}{2}+\frac{\sqrt{3}}{2}x-\frac{1}{2}\cdot\frac{x^2}{2!}-\frac{\sqrt{3}}{2}\cdot\frac{x^3}{3!}+\frac{1}{2}\cdot\frac{x^4}{4!}+\ldots$; sin (30°, 30') = 0.5075 **]**

Type II : Expansion in powers of $(x-a)$ [Class (a) : 3 Marks]

1. Expand $7x^6-3x^5+x^2+2x$ in powers of $(x-1)$.
[Ans. : $7+29(x-1)+76(x-1)^2+110(x-1)^3+90(x-1)^4+39(x-1)^5+7(x-1)^6$ **]**

2. Expand $f(x) = x^3+3x^2+15x-10$ in powers of $(x-1)$ and hence deduce $f(11/10)$. **(M.U. 1999)**
[Ans. : $f(x) = 9+24(x-1)+6(x-1)^2+(x-1)^3$. Now, put $x = 1\cdot1$; $11\cdot461$ **]**

3. Expand x^3-3x^2+4x+3 in powers of $(x-2)$.
[Ans. : $x^3-3x^2+4x+3 = 7+4(x-2)+3(x-2)^2+(x-2)^3$ **]**

4. Expand log sin x in powers of $(x-2)$.
[Ans. : $\log\sin x = \log\sin 2+(x-2)\cot 2-\frac{1}{2}(x-2)^2\operatorname{cosec}^2 2+\ldots$ **]**

5. Expand $3x^3-2x^2+x-4$ in powers of $(x+2)$.
[Ans. : $-38+45(x+2)-20(x+2)^2+3(x+2)^3$ **]**

6. Expand $f(x) = x^5-x^4+x^3-x^2+x-1$ in powers of $(x-1)$ and hence find $f(0\cdot99)$. **(M.U. 2001)**
[Ans. : $f(x) = 3(x-1)+6(x-1)^2+7(x-1)^3+4(x-1)^4+7(x-1)^5$; $-0\cdot0294$ **]**

7. Express $x^5-5x^4+6x^3-7x^2+8x-9$ in powers of $(x-1)$. **(M.U. 2001)**
[Ans. : $f(x) = -6-3(x-1)-9(x-1)^2-4(x-1)^3+(x-1)^5$ **]**

8. Expand $f(x) = x^4-3x^3+2x^2-x+1$ in powers of $(x-3)$. **(M.U. 2005, 11)**
[Ans. : $f(x) = 16+38(x-3)+29(x-3)^2+9(x-3)^3+(x-3)^4$ **]**

9. Expand x^3-2x^2+3x+5 in powers of $(x-2)$. **(M.U. 1992)**
[Ans. : $11+7(x-2)+4(x-2)^2+(x-2)^3$ **]**

10. Expand $2x^3+3x^2-8x+7$ in terms of $(x-2)$. **(M.U. 2002, 04, 17)**
[Ans. : $19+28(x-2)+15(x-2)^2+2(x-2)^3$ **]**

11. Expand in $f(x) = \sqrt{1+x+2x^2}$ powers of $(x-1)$ using Taylor's series. **(M.U. 1996)**
[Ans. : $f(x) = 2+\frac{5}{4}(x-1)+\frac{7}{64}(x-1)^2+\ldots$ **]**

12. Expand $\tan^{-1}x$ in powers of $(x-1)$. **(M.U. 1996, 2015)**
[Ans. : $\tan^{-1}x = \frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\ldots$ **]**

13. Expand log x in powers of **(i)** $(x-1)$, **(ii)** $(x-2)$. **(M.U. 1991)**

Hence, show that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots$

[**Ans. :** (i) $\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$

(ii) $\log x = \log 2 + \frac{1}{2}(x-2) - \frac{1}{2!}\cdot\frac{1}{4}(x-2)^2 + \frac{1}{3!}\cdot\frac{1}{4}(x-2)^3 + \ldots$]

14. Expand $\sqrt{x}$ in powers of $(x-a)$. [**Ans. :** $\sqrt{x} = \sqrt{a} + \frac{1}{2\sqrt{a}}(x-a) - \frac{1}{8}\cdot\frac{(x-a)^3}{\sqrt{a^3}} - \ldots$]

15. Arrange in powers of x, by Taylor's Theorem, $7 + (x+2) + 3(x+2)^3 + (x+2)^4$.

(M.U. 1990, 2003) [**Ans. :** $49 + 69x + 42x^2 + 11x^3 + x^4$]

16. Arrange in powers of x, by Taylor's Theorem, $17 + 6(x+2) + 3(x+2)^3 + (x+2)^4 - (x+2)^5$.

[**Ans. :** $37 - 6x - 38x^2 - 29x^3 - 9x^4 - x^5$]

17. Expand $\sin x$ in powers of $\left(x - \frac{\pi}{2}\right)$. [**Ans. :** $\sin x = 1 - \frac{1}{2!}\cdot\left(x - \frac{\pi}{2}\right)^2 + \ldots$]

18. Expand $\cos x$ in powers of (i) $\left(x - \frac{\pi}{2}\right)$ (ii) $\left(x - \frac{\pi}{4}\right)$.

[**Ans. :** (i) $\cos x = -\left(x - \frac{\pi}{2}\right) + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x - \frac{\pi}{2}\right)^5 + \ldots$

(ii) $\cos x = \frac{1}{\sqrt{2}}\left[1 + \left(x - \frac{\pi}{4}\right) + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \ldots\right]$

19. Expand $\tan x$ in powers of $\left(x - \frac{\pi}{4}\right)$. [**Ans. :** $\tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \ldots$]

20. Expand $\sin x$ in powers of $(x-a)$.

[**Ans. :** $\sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{2!}\sin a - \frac{(x-a)^3}{3!}\cos a + \ldots$]

21. Expand $\cos\left(x + \frac{\pi}{6}\right)$ and find approximate value of cos 64°. [**Ans. :** 0·4384]

Type III : To find approximate value [Class (a) : 3 Marks]

1. Using Taylor's Theorem, evaluate $\sqrt{1{\cdot}02}$ upto 4 places of decimals. [**Ans. :** 1·0099]
2. Using Taylor's Theorem, find approximate value of sin (30°, 30'). [**Ans. :** 0·5075]
3. Using Taylor's series find $\sqrt{9{\cdot}12}$ correct to five places of decimals. [**Ans. :** 3·0199]

EXERCISE - VII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Theory : Class (a) : 3 Marks

1. Show that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$ and hence show that

$\cos hx = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$ and $\sin hx = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$ **(M.U. 1988)**

2. Expand $\tan x$ by Maclaurin's series. **(M.U. 1997)**

3. State Maclaurin's theorem and use it to find first four terms of $\tan\left(\frac{\pi}{4}+x\right)$ in powers of x and hence find the values of tan (46° 30') to four places of decimals. **(M.U. 1989, 98)**

[**Ans. :** 1.05378]

4. Expand $\sin x$ upto x^5.
5. Expand $\tan x$ upto x^5.
6. Expand $\tanh^{-1} x$ upto x^5.
7. Obtain the series for $\log(1+x)$ and hence find the series for $\log\left(\frac{1+x}{1-x}\right)$.

   Hence, find the value of $\log\left(\frac{11}{9}\right)$. **(M.U. 1993)** [**Ans. :** 0·20067]

EXERCISE - VIII

For solutions of this Exercise see **Companion to Applied Mathematics - I**

Short Answer Questions : Class (a) : 3 Marks

1. State Maclaurin's Series.
2. State Taylor's Series.
3. Write $\sin x$ in powers of x.
4. Write $\tan x$ in powers of x.
5. Write $\sinh^{-1} x$ in powers of x.
6. Write $\tanh^{-1} x$ in powers of x.
7. If $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots$, then find y. [**Ans. :** $y = e^x - 1$]
8. If $y = \sin^{-1} x$, then show that $x = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots$
9. If $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, prove that $y = 2\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$.
10. If $y = \sin^{-1}(3x - 4x^3)$, prove that $y = 3\left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots\right)$.
11. Find Taylor's series expansion of $y = \frac{1}{x}$ about $x = 1$.

    [**Ans. :** $1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots$]
12. Find Taylor's series expansion of $y = e^x$ about $x = 1$.

    [**Ans. :** $e\left[1 + (x-1) + \frac{1}{2!}(x-1)^2 + \frac{1}{3!}(x-1)^3 + \ldots\right]$]
13. Expand $f(x) = 4x^2 + 5x + 12$ in powers of $x - 1$. [**Ans. :** $21 + 13(x-1) + 8(x-1)^2$]
14. Expand $e^x \tan x$ upto first three terms. [**Ans. :** $x + x^2 + \frac{5}{6}x^3 + \ldots$]
15. Expand $e^x \sin x$ upto first three terms. [**Ans. :** $x + x^2 + \frac{x^3}{3} + \ldots$]

16. Expand tan hx in powers of x. [**Ans. :** $x-\frac{x^3}{3}+\frac{2}{15}x^5-\ldots$]

17. Find the limit of $f(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$ as $x\to\frac{\pi}{2}$. [**Ans. :** 1]

18. Find the limit of $f(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$ as $x\to 1$. [**Ans. :** e]

19. Find the limit of $f(x)=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$ as $x\to 1$. [**Ans. :** $\frac{1}{e}$]

20. Find the limit of $f(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$ as $x\to 1$. [**Ans. :** log 2]

21. If $f(x)=\tan^{-1}x$, find $f^{101}(0)$. [**Ans. :** 100]

Summary

1. $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots+\frac{x^n}{n!}f^n(0)+\ldots\infty$

2. $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\ldots\infty$

3. $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\ldots\infty$

4. $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots$

5. $\sin hx=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\frac{x^9}{9!}+\ldots$

6. $\cos hx=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots\infty$

7. $\tan h\,x=x-\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\infty$

8. $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\infty$

9. $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\ldots\infty$

10. $\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}-\ldots\infty$

11. $\sin^{-1}x=x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\ldots$

12. $\cos^{-1}x=\frac{\pi}{2}-\left[x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\ldots\right]$

13. $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$

14. $\sin h^{-1}x=x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}-\ldots$

15. $\tan h^{-1}x=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$

16. $(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$

17. $(1-x)^n=1-nx+\frac{n(n-1)}{2!}x^3-\ldots$

18. $(1+x)^{-1}=1-x+x^2-x^3+\ldots$

19. $(1-x)^{-1}=1+x+x^2+x^3+\ldots$

20. $(1+x)^{-2}=1-2x+3x^2-4x^3+\ldots$

21. $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+\ldots$

22. $f(x+a)=f(a)+xf'(a)+\frac{x^2}{2}f''(a)+\frac{x^3}{3}f'''(a)+\ldots$

☆☆☆

CHAPTER 13

Scilab

1. Introduction

Scilab (**Scie**nce **Lab**oratory) is an interactive software system developed for numerical computations and for drawing graphs. It can be used for matrix manipulations, for evaluating single, double and triple integrals, and for solving differential equations of the first order and of the first degree. It can also be used to solve certain types of problems in Fourier Series, Laplace Transforms, Vector Algebra, Vector Calculus, Complex Variables, Partial Differential Equations, Probability and Statistics etc.

2. Symbols

Scilab uses particular symbols for particular operations. These are explained below :

	Operation	Symbol		Example
(i)	addition	+	a+ b	2 + 3
(ii)	subtraction	–	a – b	5 – 2
(iii)	multiplication	*	a * b	4 * 9
(iv)	division	/	a / b	27 / 5
(v)	power	^	a^b	5^4
(vi)	exponential i.e. power of e	exp( )	exp(x)	exp(3)
(vii)	comments	//		//PARABOLA
(viii)	array	:	a:b:c	x = –1:0.1:1

(This gives values to x starting from –1 upto 1 in steps of 0.1 *i.e.* $x = -1, -0.9, -0.8, -0.7, \ldots\ldots, 0.7, 0.8, 0.9, 1.$)

(π is denoted by %pi) x : –%p : %pi / 16 : %pi

(This gives values to x starting from $-\pi$ upto π in steps of $\pi/16$

$$i.e.\ -\pi,\ -\frac{15\pi}{16},\ -\frac{14\pi}{16},\ \ldots\ldots,\ 0,\ \frac{\pi}{16},\ \frac{2\pi}{16},\ \ldots\ldots,\ \pi)$$

x : –2*%pi : %pi / 16 : 2*%pi

(This gives values to x starting from -2π upto 2π in steps of $\pi/16$)

('e' is denoted by %e)

Square root sqrt

Trigonometric function sin(x), – ; cos(x), –

Hyperbolic functions sinh(x), – ; cosh(x), –

3. Numerical Methods for Solving Transcendental and Linear Equations

(1) Regula Falsi Method

Programe :

```
deff('d = f(x)', 'd = ^3-100')
a = input ("Enter the value of a:")
b = input ("Enter the value of b:")
n = input ("Enter the number of iterations n:")
for i = 1 : n
   c = (a * f(b) – b * f(a)) / (f(b) – f(a))
   disp ( [i, c] )
if f(a) * f(c) < 0 then
   b = c
end
if f(b) * f(c) < 0 then
   a = c
end
c1 = (a * f(b) – b * f(a)) / (f(b) – f(a))
if abs (c1 – c) < 0.00001 then
   disp ("we get accurate roots")
   break;
   end
end
```

Example : Find approximate value of the root of the equation $y = x^3 - 2x^2 - 5$.

Sol. : startup execution:

```
   loading initial environment
--> deff ('y = f(x)', 'y = x^3 – 2*x^2 – 5')
--> a = 2
   a =
      2.
--> b = 3
   b =
      3.
--> n = 3
   n =
      3.
--> for i = 1:n
--> c = (a * f(b) – b * f(a)) / (f(b) – f(a))
--> disp ( [i, c] )
```

```
--> if f(a) * f(c) < 0 then
--> b = c
--> end
--> if f(b) * f(c) < 0 then
--> a = c
--> end
--> c1 = (a * f(b) – b * f(a)) / (f(b) – f(a))
--> if abs (c1–c) < 0.00001 then
--> disp ("we get accurate roots")
--> break;
--> end
--> end
c =
    2.5555556
    1.   2.5555556
a =
    2.5555556
c1 =
    2.6690501
c =
    2.6690501
    2.   2.6690501
a =
    2.6690501
c1 =
    2.687326
c =
    2.687326
    3.   2.687326
a =
    2.687326
c1 =
    2.6901398
```

EXERCISE - I

Solve the Examples given in Exercise - I, page 14-10 by Regula-Falsi method.

(2) Newton-Raphson Method

Program :

```
*Define function and its derivative
--> y = f(x)
--> fp(x) = y'(x)
N = 100; eps = 1.e^ – 5;
maxval=10000.0;
xx = x0;
while(N > 0)
xn = xx – f(xx) / fp(xx);
if (abs(f(xn)) < eps then
x = xn
disp (100 – N);
return (x);
end;
if (abs(f(xx)) > maxval) then
disp (100 – N);
error('solution diverges');
abort;
end;
N = N – 1;
xx = xn;
end;
error;
error('No convergence');
abort;
```

Example 1 : Using Newton-Raphson method find the root of the equation $y = 2x^3 - 3x + 4$.

Sol. : startup execution;

loading initial environment

```
--> deff ('[y] = f(x)', 'y = 2*x^3 – 3*x + 4');
--> deff ('[y] = fp(x)', 'y = 6*x^2 – 3');
--> x0 = – 1;
--> N = 3;
--> eps = 1.e^ – 5;
--> xx = x0;
--> while (N > 0)
--> xn = xx – (f(xx) / fp(xx));
--> if (abs (f(xn)) < eps) then
--> x = xn
```

```
--> disp (3 – N);
--> return(x);
--> end;
--> if (abs(f(xx)) > 1000) then
--> disp (3 – N);
--> error('solution diverges');
--> abort;
--> end;
--> N = N – 1;
--> xx = xn;
--> end;
x =
   – 1.7186014
```

Example 2 : Using Newton-Raphson method find the root of the equation $y = x^3 - 100$.

Sol. : startup execution;

loading initial environment

```
--> deff ('[y] = f(x)', 'y = x^3 – 100');
--> deff ('[y] = fp(x)', 'y = 3*x^2');
--> x0 = 4;
--> N = 4;
    N =
        4.
--> eps = 1.e^ – 5;
--> maxval = 10000.0;
--> xx = x0;
--> while (N > 0)
--> xn = xx – (f(xx) / fp(xx));
--> if (abs (f(xn)) < eps) then
--> x = xn
--> disp (4 – N);
--> return(x);
--> end;
--> if (abs(f(xx)) > maxval) then
--> disp (4 – N);
--> error('solution diverges');
--> abort;
--> end;
--> N = N – 1;
```

```
--> xx = xn;
--> end;
x =
   4.6440443
   1.
x =
   4.6415901
   2.
x = 4.6415888
   3.
```

EXERCISE - II

Solve the Examples given in Exercise - II, page 14-17 by Newton-Raphson method.

(3) Gauss Elimination Method

Programe : Define matrix A, b.

```
→ a = [ A, b ]                  // Augmented matrix
→ [ nA, mA ] = size (A)
→ n = nA
→ for k = 1 : n – 1, for i = k + 1 : n,  for j = k + 1 : n + 1,
  a (i, j ) = a (i, j ) – a (k, j ) * a (i, k ) / a (k, k); end;
  for j = 1 : k, a (i, j ) = 0; end; end; end;
→ a
→ x (n) = a (n, n + 1) / a (n, n);
→ for i = n – 1 : – 1 : 1, sum k = 0; for k = i + 1 : n,
  sum k = sum k + a (i, k) * x (k); end;
      x (i) = (a (i, n + 1) – sum k) / a (i, i ); end;
→ x
```

Example 1 : Solve the following system of linear equations by Gauss Elimination Method.

$$x + 2y + 3z - t = 10$$
$$2x + 3y - 3z - t = 1$$
$$3x + 2y - 4z + 3t = 2$$
$$2x - y + 2z + 3t = 7.$$

Sol. : Startup execution:

```
    loading initial environment
→ A = [ 1  2  3  – 1;  2  3  – 3  – 1;  3  2  – 4  3  2;  2  –1  2  3 ]
  A =
        1.    2.    3.   –1.
        2.    3.   –3.   –1.
        3.    2.   –4.    3.
        2.   –1.    2.    3.
```

Scanned by CamScanner

```
→ b = [ 10; 1; 2; 7 ]
  b =  10.
       1.
       2.
       7.

→ a = [ A, b ]
  a =
        1.   2.   3.  -1.  10.
        2.   3.  -3.  -1.   1.
        3.   2.  -4.   3.   2.
        2.  -1.   2.   3.   7.
→ [ nA, mA ] = size(A)
  mA = 4.
  nA = 4.
→ n = nA
  n = 4.
→ for k = 1 : n - 1, for i = k + 1 : n, for j = k + 1 : n + 1
→ a(i, j) = a(i, j) a(k, j) * a(i, k)/a(k, k); end;
→ for j = 1:k, a(i, j) = 0; end; end; end;
→ a
  a =
        1.   2.   3.          -1.          10.
        0.  -1.  -9.           1.         -19.
        0.   0.  23.           2.          48.
        0.   0.   0.  -3.5652174   -3.5652174
→ x(n) = a(n, n+1) / a(n, n);
→ for i = n - 1 : - 1 : 1, sum k = 0; for k = i + 1 : n,
   sum k + a (i, k) * x(k); end; x(i) = (a(i, n + 1) - sum k)/a(i, i); end
→ x
  x =  1.
       2.
       2.
       1.
```

Here, we get $x = 1$, $y = 2$, $z = 2$ and $t = 1$.

EXERCISE - III

Solve the equations of Exercise - III by Gauss elimination method given on page 14-25 of the text.

(4) Gauss-Jacobi's Method

Example 1 : Solve the following system of equations by Gauss-Jacobi method.

$$15x + 2y + z = 18$$
$$2x + 20y - 3z = 19$$
$$3x - 6y + 25z = 22$$

Sol. : startup execution :

loading initial environment

```
→ A = [ 15 2 1; 2 20 -3; 3 -6 25 ]
  A =  15.   2.    1.
        2.  20.   -3.
        3.  -6.   25.

→ b = [18; 19; 22]
  b =  18.
       19.
       22.

→ [n, m] = size(A)
  m = 3.
  n = 3.

→ x = zeros(n,1);

→ itmax = 7; eps = 0.001;        // Maximum iterations
```

Download Scilab book function Execute xbar.

```
→ iter = 1;  R = b - A * x;

→ Rave = xbar(R);
```

Execute function NextX

```
→ while((Rave>eps) & (iter<itmax))

→ x = Nextx(x, R, A)

→ R = b - A * x;

→ Rave = xbar(R);

→ disp(iter,'iteration');disp(Rave,'Average residual')

→ end
  x =
       1.2
       0.95
       0.88
```

```
iteration
    1.
Average residual
    1.7066667
x =
    1.0146667
    0.962
    0.964
iteration
    2.
Average residual
    0.4528889
x =
    1.0074667
    0.9931333
    0.98912
iteration
    3.
Average residual
    0.1285156
x =
    1.0016409
    0.9976213
    0.997456
iteration
    4.
Average residual
    0.0327923
x =
    1.0004868
    0.9994543
    0.9992322
iteration
    5.
Average residual
    0.0091798
```

```
x =
    1.0001239
    0.9998362
    0.9998106
iteration
    6.
Average residual
    0.0023942
x =
    1.0000345
    0.9999592
    0.9999458
iteration
    7.
Average residual
    0.0006575
```

EXERCISE - IV

Solve the equations of Exercise - IV by Gauss-Jacobi's method given on page 14-31 of the text.

(5) Gauss-Seidel Method

Programme :

```
a = input ('matrix a');
b = input ('matrix b');
i = input ('enter initial values');
for j = 1; 5
  x1 = (b(1) – (a(1, 2) * i(2)) – (a(1, 3) * i(3)))/a(1,1);
  i(1) = x1;
  x2 = (b(2) – (a(2,1) * i(1)) – (a(2,3) * i(3)))/a(2,2);
  i(2) = x2;
  x3 = (b(3) – (a(3,1) * i(1)) – (a(3, 2) * i(2)))/a(3,3);
  i(3) = x3;
  end
disp(i);
```

Example 1 : Solve the following system of equations by Gauss-Seidel method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

We start with the first iteration $x_1 = 3.15$, $y_1 = 3.54$, $z_1 = 1.91$.

Sol. : startup execution :

```
    loading initial environment
→  A = [ 27 6 -1; 6 15 2; 1 1 54 ]
   A =
        27.   6.  -1.
         6.  15.   2.
         1.   1.  54.
→  b = [ 85; 72; 110 ]
   b =  85.
        72.
        110.
→  i = [ 3.15; 3.54; 1.91 ]
   i =  3.15
        3.54
        1.91
→  for j = 1:5
→  x1 = (b(1) - A(1,2) * i(2)) - (A(1,3) * i(3)))/A(1,1);
→  i(1) = x1;
→  x2 = (b(2) - A(2,1) * i(1)) - (A(2,3) * i(3)))/A(2,2);
→  i(2) = x2;
→  x3 = (b(3) - A(3,1) * i(1)) - (A(3,2) * i(2)))/A(3,3);
→  i(3) = x3;
→  end
→  disp(i);
   2.4254763
   3.5730156
   1.9259539
```

Example 2 : Solve the following system of equations by Gauss-Seidel method.

$$10x_1 + x_2 + x_3 = 12$$
$$2x_1 + 10x_2 + x_3 = 13$$
$$2x_1 + 2x_2 + 10x_3 = 14$$

We start with the first iteration $x_1 = 1.2$, $y_1 = 1.06$, $z_1 = 0.948$.

Sol. :startup execution :

```
        loading initial environment
  → a = [ 10 1 1; 2 10 1; 2 2 10 ]
     a =
           10.    1.    1.
            2.   10.    1.
            2.    2.   10.
  → b = [ 12; 13; 14 ]
     b =   12.
           13.
           14.
  → i = [ 1.2; 1.06; 0.948 ]
     i =   1.2
           1.06
           0.948
  → for j = 1:5
  → x1 = (b(1) – a(1,2) * i(2)) – (a(1,3) * i(3)))/a(1,1);
  → i(1) = x1;
  → x2 = (b(2) – a(2,1) * i(1)) – (a(2,3) * i(3)))/a(2,2);
  → i(2) = x2;
  → x3 = (b(3) – a(3,1) * i(1)) – (a(3,2) * i(2)))/a(3,3);
  → i(3) = x3;
  → end
  → disp(i);
     1.
     1.0000000
     1.
```

EXERCISE - V

Solve the equations of Exercise - V by Gauss-Seidel method given on page 14-41 of the text.

☆ ☆ ☆