CS7015 (Deep Learning) : Lecture 23

Generative Adversarial Networks (GANs)

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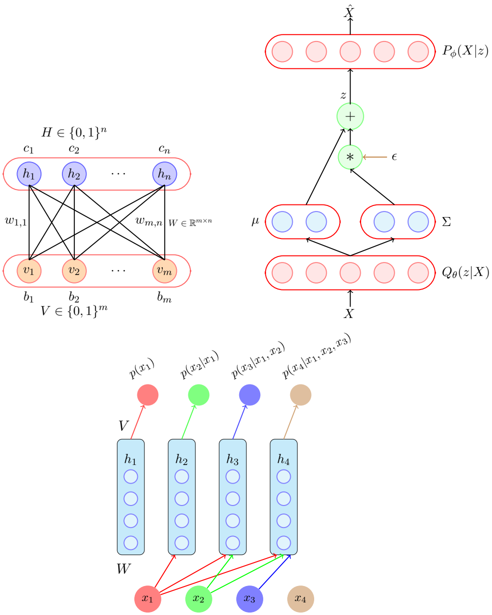
Module 23.1: Generative Adversarial Networks - The intuition

- So far we have looked at generative models which explicitly model the joint probability distribution or conditional probability distribution



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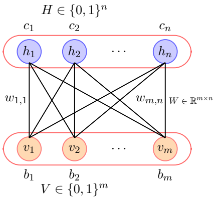

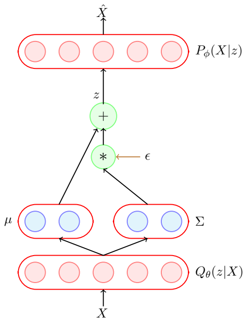

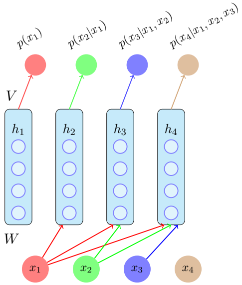

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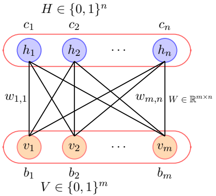

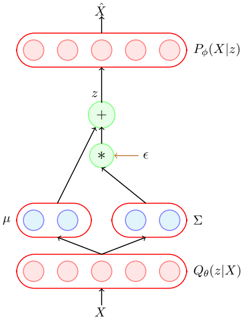

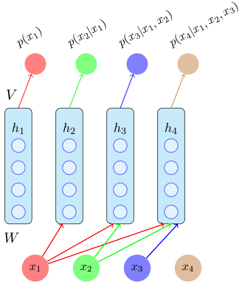

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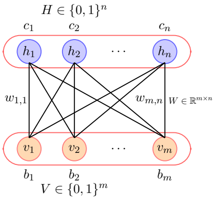

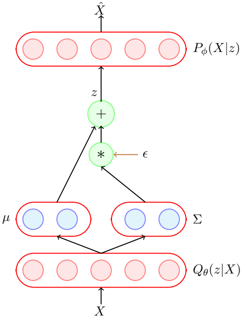

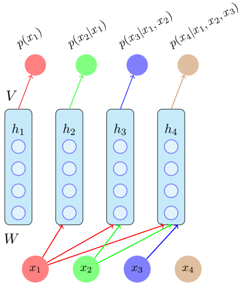

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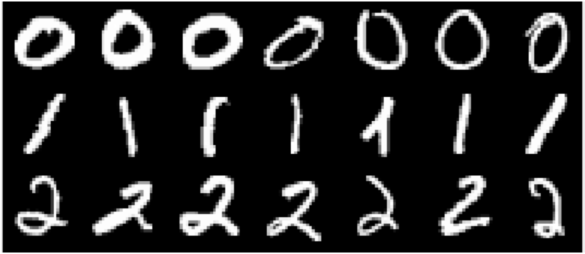
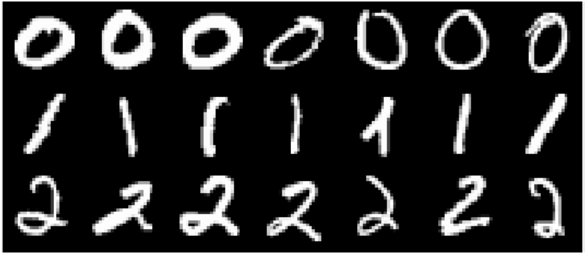


- As usual we are given some training data (say, MNIST images) which obviously comes from some underlying distribution



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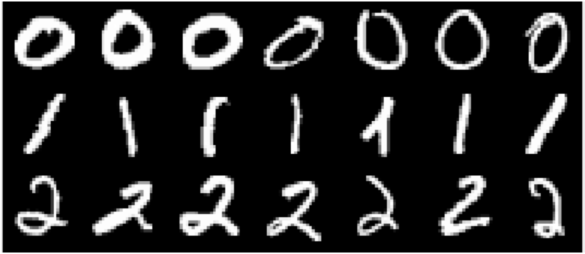

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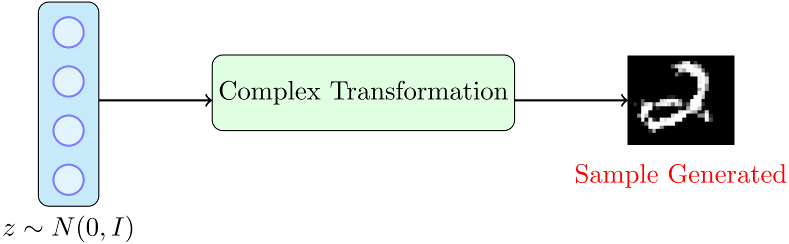
- Our goal is to generate more images from this distribution (*i.e.*, create images which look similar to the images from the training data)
- In other words, we want to sample from a complex high dimensional distribution which is intractable (recall RBMs, VAEs and AR models deal with this intractability in their own way)



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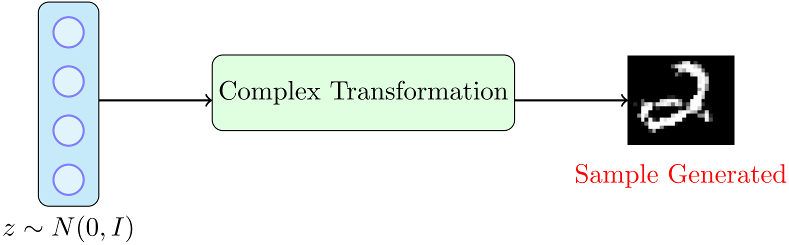

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- In other words, we will take a $z \sim N(0, I)$, learn to make a series of complex transformations on it so that the output looks as if it came from our training distribution

- What can we use for such a complex transformation?

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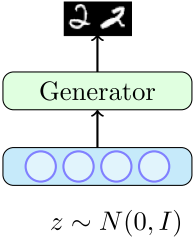

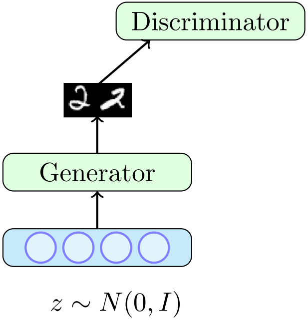

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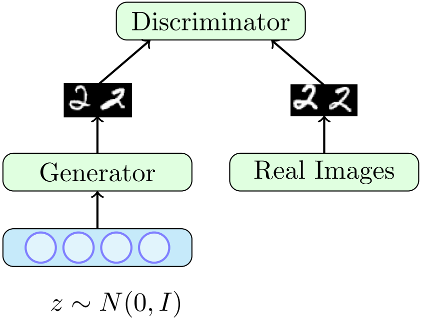

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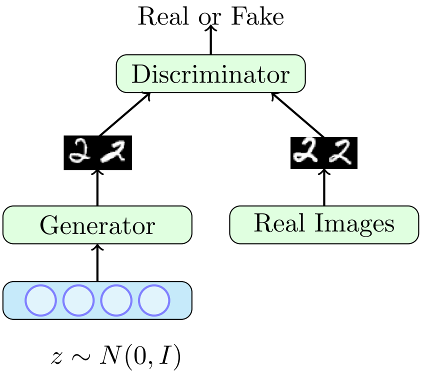

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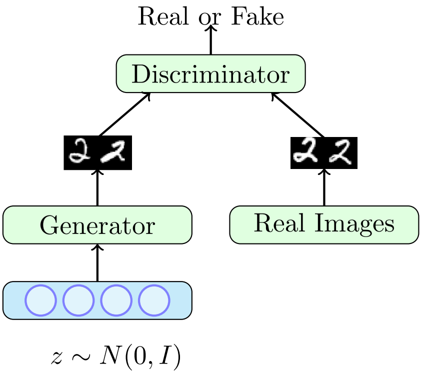
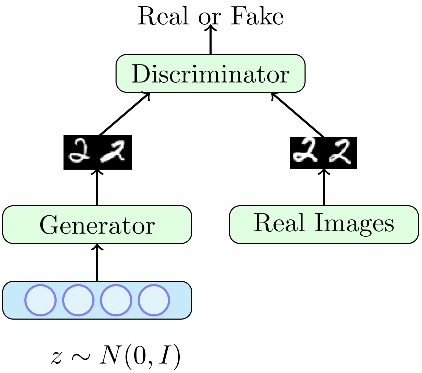
- The job of the discriminator is to get better and better at distinguishing between true images and generated (fake) images

- So let's look at the full picture





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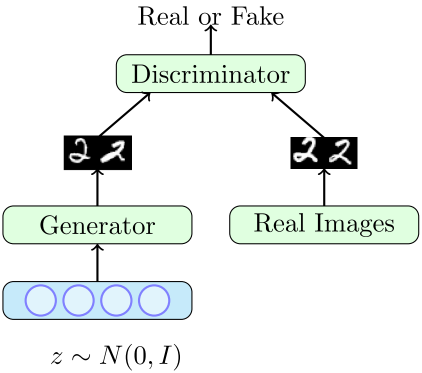
- Let G_ϕ be the generator and D_θ be the discriminator (ϕ and θ are the parameters of G and D, respectively)



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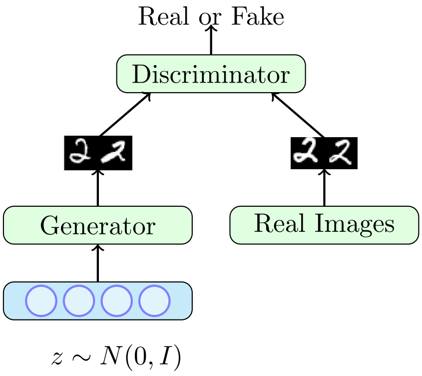

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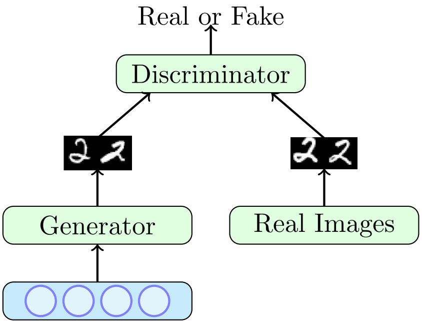
- Let G_ϕ be the generator and D_θ be the discriminator (ϕ and θ are the parameters of G and D, respectively)
- We have a neural network based generator which takes as input a noise vector $z \sim N(0, I)$ and produces $G_\phi(z) = X$
- We have a neural network based discriminator which could take as input a real X or a generated $X = G_\phi(z)$ and classify the input as real/fake

- What should be the objective function of the overall network?



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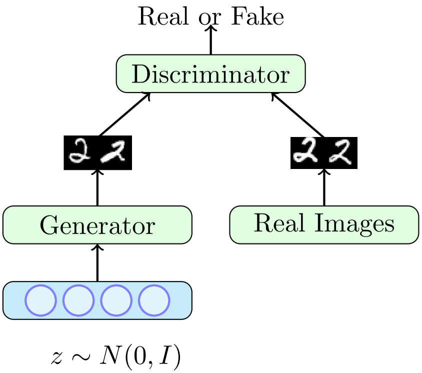

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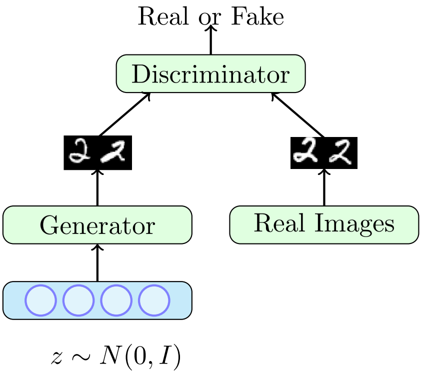

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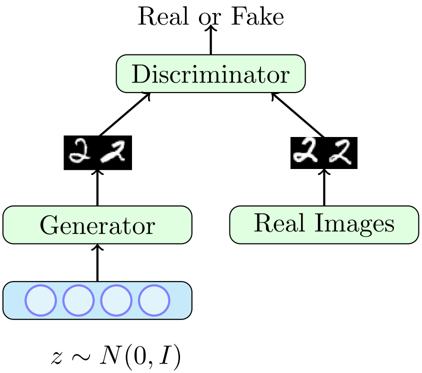

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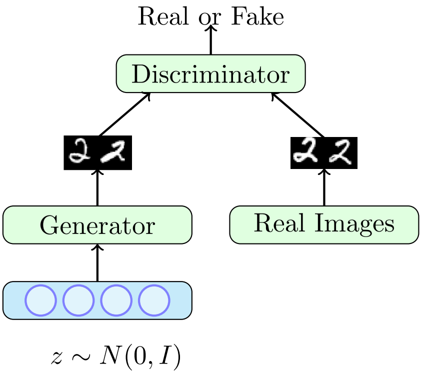

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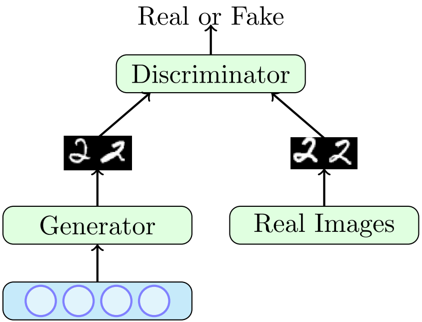
- This score will be between 0 and 1 and will tell us the probability of the image being real or fake
- For a given z, the generator would want to maximize $\log D_\theta(G_\phi(z))$ (log likelihood) or minimize $\log(1 - D_\theta(G_\phi(z)))$

- This is just for a single z and the generator would like to do this for all possible values of z,



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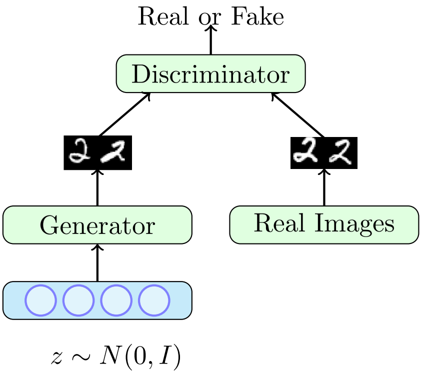

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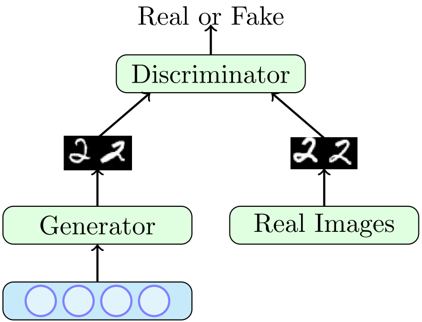
- For example, if z was discrete and drawn from a uniform distribution (*i.e.*, $p(z) = \frac{1}{N}\ \forall z$) then the generator's objective function would be

$$\min_{\phi} \sum_{i=1}^{N} \frac{1}{N} \log(1 - D_\theta(G_\phi(z)))$$



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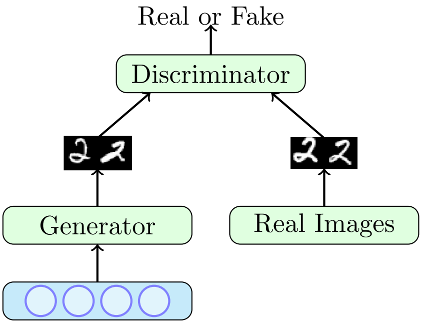
- However, in our case, z is continuous and not uniform ($z \sim N(0, I)$) so the equivalent objective function would be

$$\min_{\phi} \int p(z) \log(1 - D_\theta(G_\phi(z)))$$

$$\min_{\phi} E_{z\sim p(z)}[\log(1 - D_\theta(G_\phi(z)))]$$

- Now let's look at the discriminator



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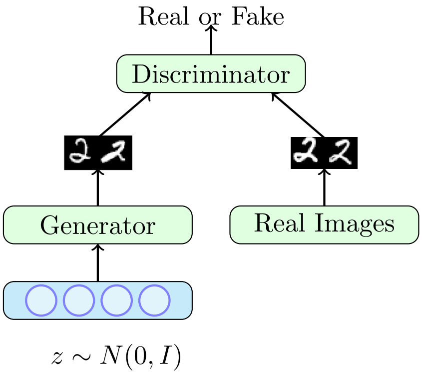

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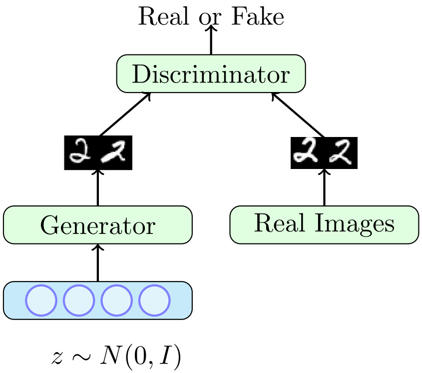

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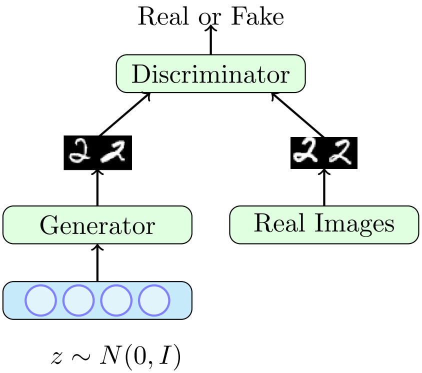

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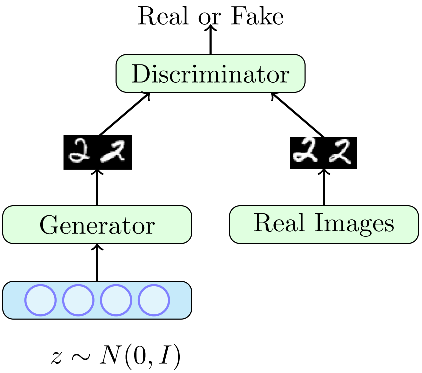
- And it should do this for all possible real images and all possible fake images
- In other words, it should try to maximize the following objective function

$$\max_{\theta} E_{x \sim p_{data}} [\log D_{\theta}(x)] + E_{z \sim p(z)} [\log(1 - D_{\theta}(G_{\phi}(z)))]$$



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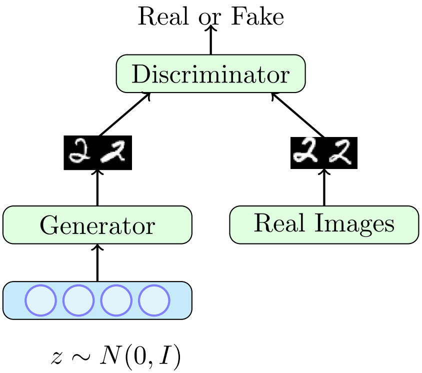

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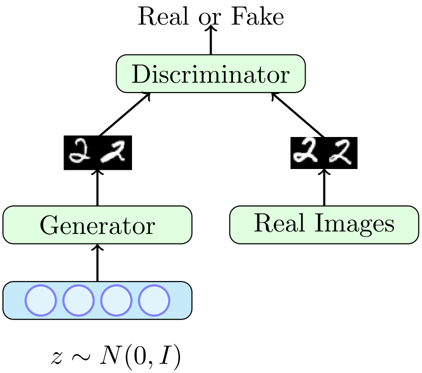

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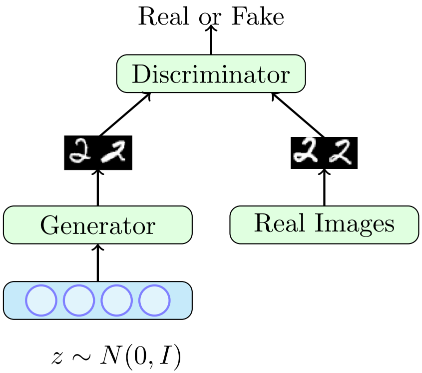

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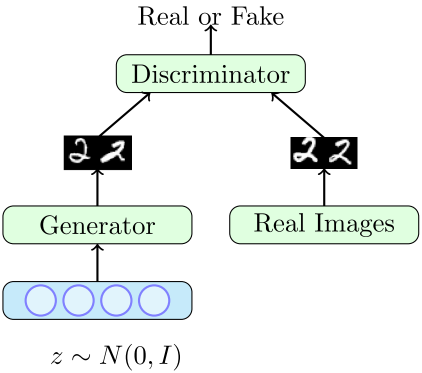
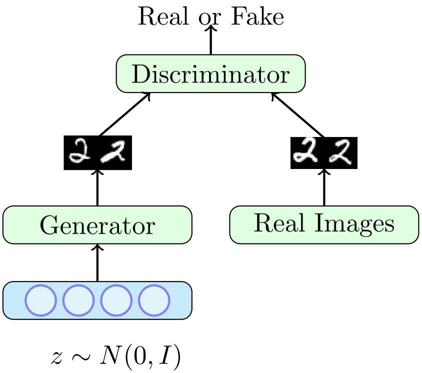
- The second term in the objective is w.r.t. the parameters of the generator (ϕ) as well as the discriminator (θ)
- The discriminator wants to maximize the second term whereas the generator wants to minimize it (hence it is a two-player game)

- So the overall training proceeds by alternating between these two step





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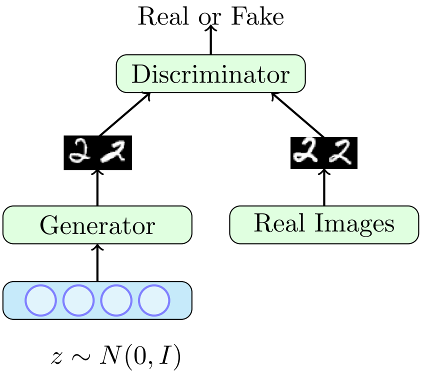

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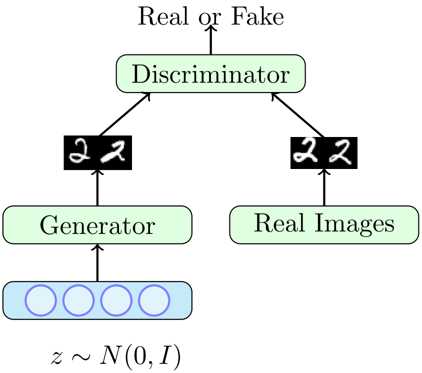

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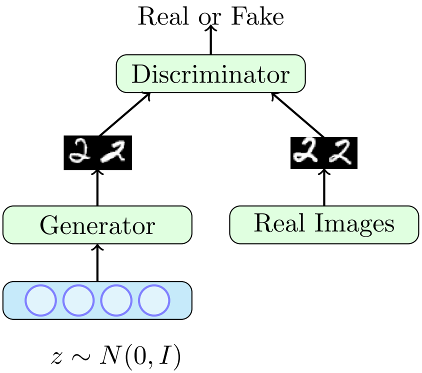

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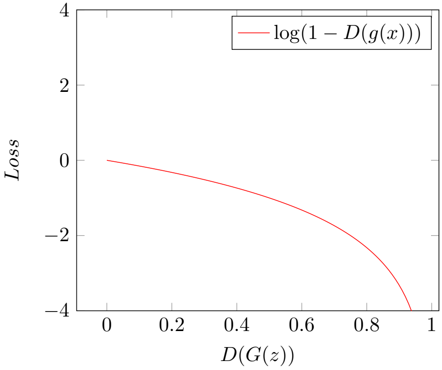
- Let us see why



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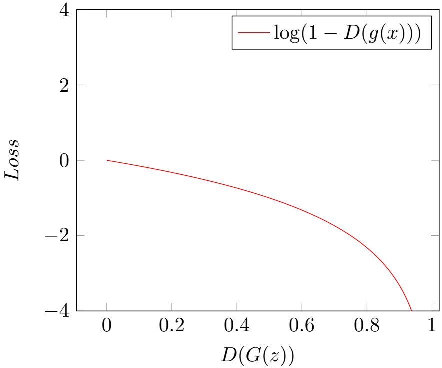

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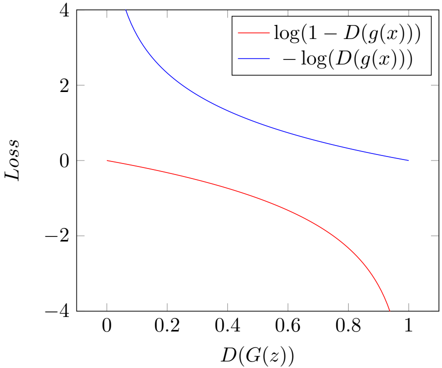

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- However, in this region where $D(G(z))$ is close to 0, the curve of the loss function is very flat and the gradient would be close to 0
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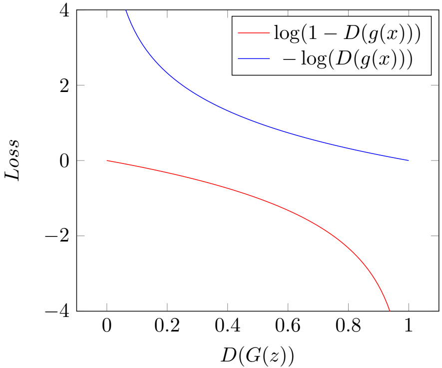

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- Trick: Instead of minimizing the likelihood of the discriminator being correct, maximize the likelihood of the discriminator being wrong
- In effect, the objective remains the same but the gradient signal becomes better

With that we are now ready to see the full algorithm for training GANs

1: **procedure** GAN TRAINING

11: **end procedure**

With that we are now ready to see the full algorithm for training GANs

procedure GAN TRAINING
    for number of training iterations **do**

    end for
end procedure

With that we are now ready to see the full algorithm for training GANs

```
procedure GAN Training
    for number of training iterations do
        for k steps do

        end for

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end procedure
```

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procedure GAN TRAINING
    for number of training iterations **do**
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            • Sample minibatch of m noise samples $\{\mathbf{z}^{(1)}, .., \mathbf{z}^{(m)}\}$ from noise prior $p_g(\mathbf{z})$

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With that we are now ready to see the full algorithm for training GANs

```
procedure GAN Training
    for number of training iterations do
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            • Sample minibatch of m noise samples {z^(1), .., z^(m)} from noise prior p_g(z)
            • Sample minibatch of m examples {x^(1), .., x^(m)} from data generating distribution p_data(x)
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```

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$$\nabla_\theta \frac{1}{m} \sum_{i=1}^{m} \left[\log D_\theta\left(x^{(i)}\right) + \log\left(1 - D_\theta\left(G_\phi\left(z^{(i)}\right)\right)\right)\right]$$

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$$\nabla_\phi \frac{1}{m} \sum_{i=1}^{m} \left[\log \left(D_\theta \left(G_\phi \left(z^{(i)} \right) \right) \right) \right]$$

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end procedure

Module 23.2: Generative Adversarial Networks - Architecture

- We will now look at one of the popular neural networks used for the generator and discriminator (Deep Convolutional GANs)

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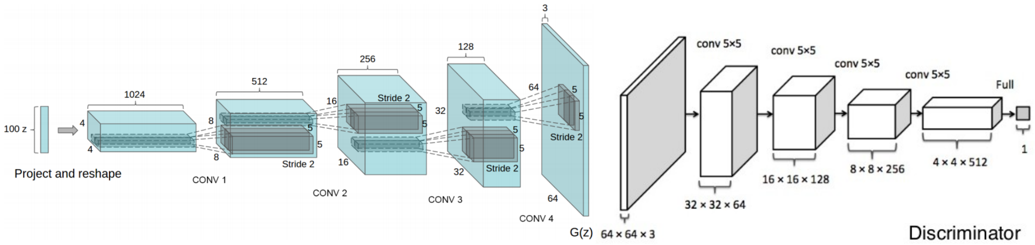

Figure: Generator (Redford et al 2015) (left) and discriminator (Yeh et al 2016) (right) used in DCGAN

Architecture guidelines for stable Deep Convolutional GANs

- Replace any pooling layers with strided convolutions (discriminator) and fractional-strided convolutions (generator).

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- Use LeakyReLU activation in the discriminator for all layers

Module 23.3: Generative Adversarial Networks - The Math Behind it

- We will now delve a bit deeper into the objective function used by GANs and see what it implies

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- Can we prove this formally even though the model is not explicitly computing this density?
- We will try to prove this over the next few slides

Theorem

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2. The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **only if** $p_G = p_{data}$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_{D} V(G, D)$ is achieved **if** $p_G = p_{data}$

The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_{D} V(G, D)$ is achieved **only if** $p_G = p_{data}$

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The 'if' part: The global minimum of the virtual training criterion $C(G) = \max\limits_{D} V(G,D)$ is achieved **if** $p_G = p_{data}$

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(c) Show that $a < b \; \forall \; p_G \neq p_{data}$(and hence the minimum $V(D, G)$ is achieved when $p_G = p_{data}$)

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- First let us look at the objective function again

$$\min_{\phi} \max_{\theta} \quad [\mathbb{E}_{x \sim p_{data}} \log D_\theta(x) + \mathbb{E}_{z \sim p(z)} \log(1 - D_\theta(G_\phi(z)))]$$

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- The above replacement follows from the *law of the unconscious statistician* (click to link of wikipedia page)

- Okay, so our revised objective is given by

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- This means for any given generator

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$$V(G, D_G^*) = \int_x p_{data}(x) \log D(x) + p_G(x) \log (1 - D(x))\, dx$$

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$$\begin{aligned} V(G, D_G^*) &= \int_x p_{data}(x) \log D(x) + p_G(x) \log (1 - D(x))\, dx \\ &= \int_x p_{data}(x) \log \frac{1}{2} + p_G(x) \log \left(1 - \frac{1}{2}\right) dx \\ &= \log 2 \int_x p_G(x) dx - \log 2 \int_x p_{data}(x) dx \end{aligned}$$

- This means for any given generator

$$D_G^*(G(x)) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$$

- Now the if part of the theorem says "if $p_G = p_{data}$ ...."
- So let us substitute $p_G = p_{data}$ into $D_G^*(G(x))$ and see what happens to the loss functions

$$D_G^* = \frac{p_{data}}{p_{data} + p_G} = \frac{1}{2}$$

$$\begin{aligned} V(G, D_G^*) &= \int_x p_{data}(x) \log D(x) + p_G(x) \log (1 - D(x))\, dx \\ &= \int_x p_{data}(x) \log \frac{1}{2} + p_G(x) \log \left(1 - \frac{1}{2}\right) dx \\ &= \log 2 \int_x p_G(x) dx - \log 2 \int_x p_{data}(x) dx \\ &= -2 \log 2 \qquad = -\log 4 \end{aligned}$$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G,D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D,G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

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The 'only if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G,D)$ is achieved **only if** $p_G = p_{data}$

- Show that when $V(D,G)$ is minimum then $p_G = p_{data}$

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- So what we have proved so far is that if the generator is optimal ($p_G = p_{data}$) the discriminator's loss value is $-\log 4$
- We still haven't proved that this is the minima
- For example, it is possible that for some $p_G \neq p_{data}$, the discriminator's loss value is lower than $-\log 4$
- To show that the discriminator achieves its lowest value "if $p_G = p_{data}$", we need to show that for all other values of p_G the discriminator's loss value is greater than $-\log 4$

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$$C(G) = \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log \left(1 - \frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right] dx$$

- To show this we will get rid of the assumption that $p_G = p_{data}$

$$
\begin{aligned}
C(G) &= \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log \left(1 - \frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right] dx \\
&= \int_x \left[p_{data}(x) \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) + p_G(x) \log \left(\frac{p_G(x)}{p_G(x) + p_{data}(x)} \right) + (\log 2 - \log 2)(p_{data} + p_G) \right] dx
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&= -\log 2 \int_x (p_G(x) + p_{data}(x))\, dx \\
&\quad + \int_x \left[p_{data}(x) \left(\log 2 + \log \left(\frac{p_{data}(x)}{p_G(x) + p_{data}(x)} \right) \right) + p_G(x) \left(\log 2 + \log \left(\frac{p_G(x)}{P p_G(x) + p_{data}(x)} \right) \right) \right] dx
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&= -\log 2(1+1) \\
&\quad + \int_x \left[p_{data}(x) \log\left(\frac{p_{data}(x)}{\frac{p_G(x)+p_{data}(x)}{2}}\right) + p_G(x) \log\left(\frac{p_G(x)}{\frac{p_G(x)+p_{data}(x)}{2}}\right)\right] dx
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C(G) &= \int_x \left[p_{data}(x) \log\left(\frac{p_{data}(x)}{p_G(x)+p_{data}(x)}\right) + p_G(x)\log\left(1-\frac{p_{data}(x)}{p_G(x)+p_{data}(x)}\right)\right] dx \\
&= \int_x \left[p_{data}(x) \log\left(\frac{p_{data}(x)}{p_G(x)+p_{data}(x)}\right) + p_G(x)\log\left(\frac{p_G(x)}{p_G(x)+p_{data}(x)}\right) + (\log 2 - \log 2)(p_{data}+p_G)\right] dx \\
&= -\log 2 \int_x (p_G(x)+p_{data}(x))\, dx \\
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&= -\log 2(1+1) \\
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&= -\log 4 + KL\left(p_{data} \| \frac{p_G(x)+p_{data}(x)}{2}\right) + KL\left(p_G \| \frac{p_G(x)+p_{data}(x)}{2}\right)
\end{aligned}
$$

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_{D} V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

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- Okay, so we have

$$C(G) = -\log 4 + KL\left(p_{data}||\frac{p_{data} + p_g}{2}\right) + KL\left(p_G||\frac{p_{data} + p_G}{2}\right)$$

- Okay, so we have

$$C(G) = -\log 4 + KL\left(p_{data} || \frac{p_{data} + p_g}{2}\right) + KL\left(p_G || \frac{p_{data} + p_G}{2}\right)$$

- We know that KL divergence is always ≥ 0

$$\therefore C(G) \geq -\log 4$$

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- Hence the minimum possible value of $C(G)$ is $-\log 4$
- But this is the value that $C(G)$ achieves when $p_G = p_{data}$ (and this is exactly what we wanted to prove)
- We have, thus, proved the **if part** of the theorem

Outline of the Proof

The 'if' part: The global minimum of the virtual training criterion $C(G) = \max_D V(G, D)$ is achieved **if** $p_G = p_{data}$

(a) Find the value of $V(D, G)$ when the generator is optimal *i.e.*, when $p_G = p_{data}$

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- In fact $KL\left(p_{data}\|\frac{p_{data}+p_g}{2}\right) + KL\left(p_G\|\frac{p_{data}+p_G}{2}\right)$ is the Jenson-Shannon divergence between p_G and p_{data}

$$KL\left(p_{data}\|\frac{p_{data}+p_g}{2}\right) + KL\left(p_G\|\frac{p_{data}+p_G}{2}\right) = JSD(p_{data}\|p_G)$$

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which is minimum only when $p_G = p_{data}$

Module 23.4: Generative Adversarial Networks - Some Cool Stuff and Applications

- In each row the first image was generated by the network by taking a vector z_1 as the input and the last images was generated by a vector z_2 as the input

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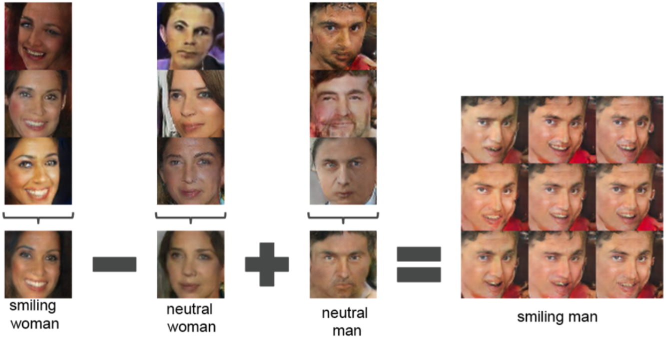
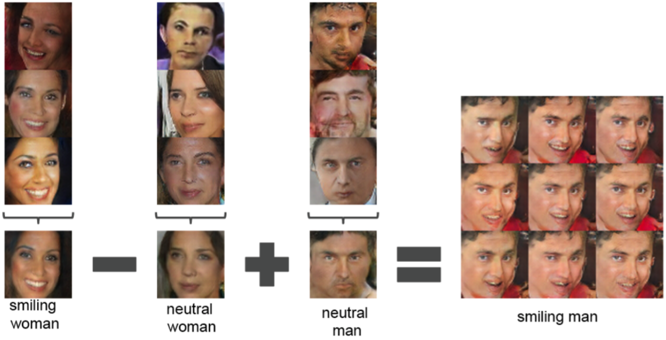
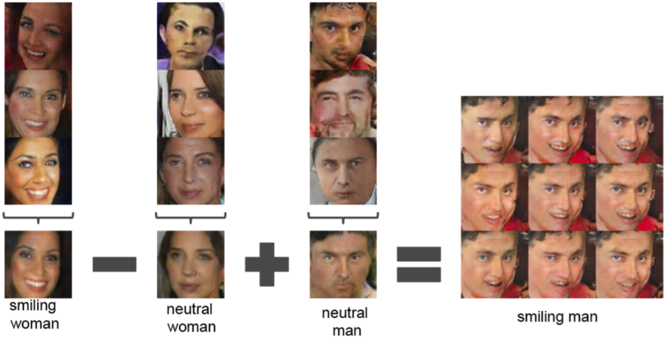
- In each row the first image was generated by the network by taking a vector z_1 as the input and the last images was generated by a vector z_2 as the input
- All intermediate images were generated by feeding z's which were obtained by interpolating z_1 and z_2 ($z = \lambda z_1 + (1 - \lambda) z_2$)
- As we transition from z_1 to z_2 in the input space there is a corresponding smooth transition in the image space also



- The first 3 images in the first column were generated by feeding some z_{11}, z_{12}, z_{13} respectively as the input to the generator



- The first 3 images in the first column were generated by feeding some z_{11}, z_{12}, z_{13} respectively as the input to the generator
- The fourth image was generated by taking an average of $z_1 = z_{11}, z_{12}, z_{13}$ and feeding it to the generator



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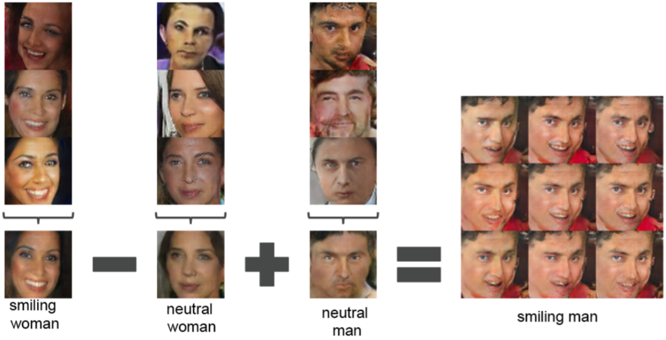
- Similarly we obtain the average vectors z_2 and z_3 for the 2nd and 3rd columns



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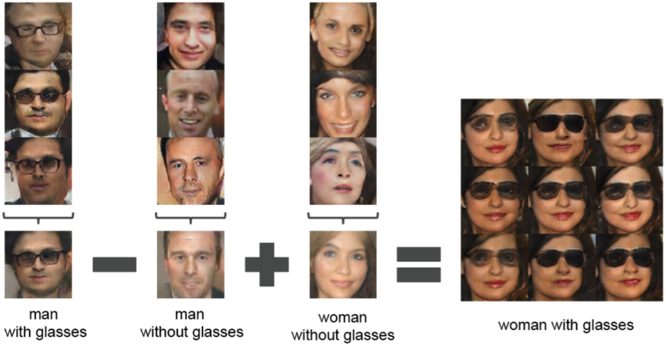
- Similarly we obtain the average vectors z_2 and z_3 for the 2nd and 3rd columns
- If we do a simple vector arithmetic on these averaged vectors then we see the corresponding effect in the generated images



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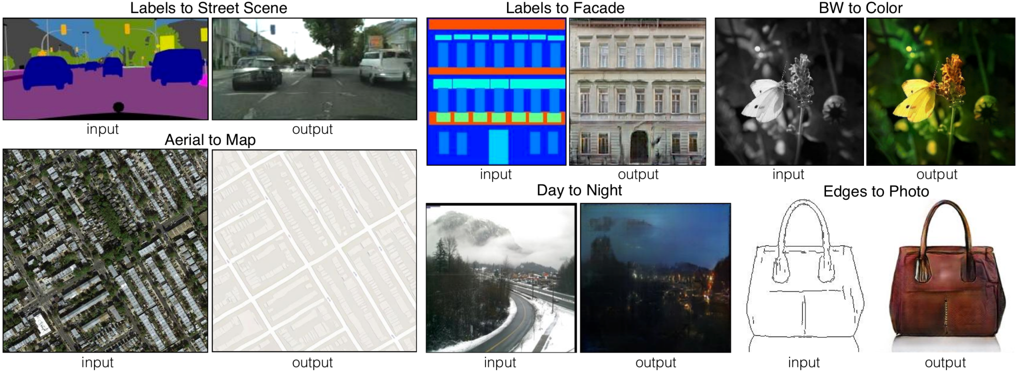

Phillip Isola, Jun-Yan Zhu, Tinghui Zhou, Alexei A. Efros, Image-to-Image Translation with Conditional Adversarial Networks, CVPR, 2017.

Module 23.4: Bringing it all together (the deep generative summary)

	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No

Table: Comparison of Generative Models

	RBMs	VAEs	AR models	GANs
Abstraction	Yes	Yes	No	No
Generation	Yes	Yes	Yes	Yes

Table: Comparison of Generative Models

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Compute P(X)	Intractable	Intractable	Tractable	No

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Type of GM	Undirected	Directed	Directed	Directed

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Generation	Yes	Yes	Yes	Yes
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Loss	KL-divergence	KL-divergence	KL-divergence	JSD

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Type of GM	Undirected	Directed	Directed	Directed
Loss	KL-divergence	KL-divergence	KL-divergence	JSD
Assumptions	X independent given z	X independent given z	None	N.A.

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Samples	Bad	Ok	Good	Good (best)

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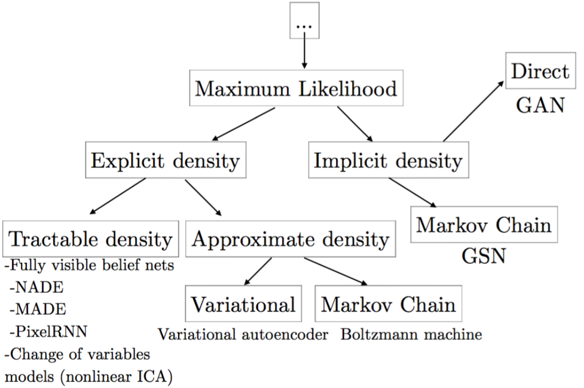
Assumptions	X independent given z	X independent given z	None	N.A.
Samples	Bad	Ok	Good	Good (best)

Table: Comparison of Generative Models

Recent works look at combining these methods: e.g. Adversarial Autoencoders (Makhzani 2015), PixelVAE (Gulrajani 2016) and PixelGAN Autoencoders (Makhzani 2017)



Source: Ian Goodfellow, NIPS 2016 Tutorial: Generative Adversarial Networks