GRAPHS

Graph: A graph $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its end points. An edge is said to connect its end points.

Infinite Graph: A graph with an infinite vertex set is called an infinite graph.

Finite Graph: A graph with a finite vertex set is called a finite graph.

Simple Graph: A graph in which each edge connects two different vertices and when no two edges connect the same pair of vertices is called a simple graph.

Multigraphs: Graphs that may have multiple edges connecting the same vertices are called multigraphs.

- ~~Graphs that may have multi~~
- When there are m different edges associated to the same unordered pair of vertices $\{u, v\}$, we also say that $\{u, v\}$ is an edge of multiplicity m.

- Edges that connect a vertex to itself are called loops.

- <u>Pseudographs</u>: Graphs that may include loops, and possibly multiple edges connecting the same pair of vertices, are called pseudographs.

- <u>Directed Graphs</u>: A directed graph (or digraph) (V,E) consists of a non empty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edges associated with the ordered pair (u,v) is said to start at u and end at v.

<u>Simple Directed Graph</u>: When a directed graph has no loops and has no multiple directed edges.

<u>Multiple Directed Edge</u>: When a directed graph has ~~multiple~~ loops and multiple directed edges.

<u>Mixed Graphs</u>: When there are m directed edges, each associated to an ordered pair of vertices (u,v), we say that, (u,v) is an edge of multiplicity m.

<u>Mixed Graph</u>: A graph with both directed and undirected edges is called a mixed graph.

TYPE	Edges	Multiple Edges?	Loops?
Simple Graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	~~No~~ Yes
Mixed graph	Directed and Undirected	Yes	Yes.

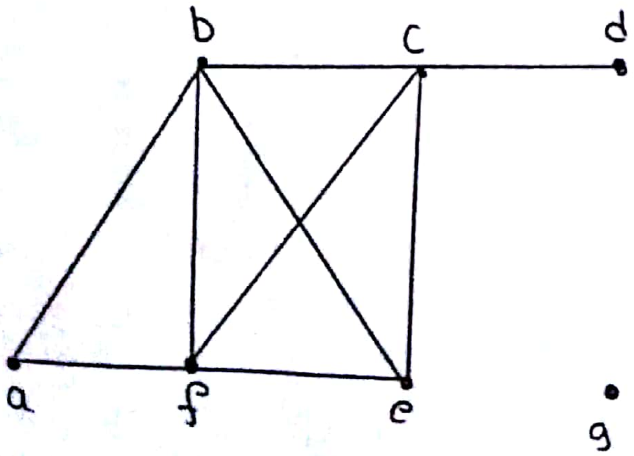
Figures for each type

GRAPH TERMINOLOGY

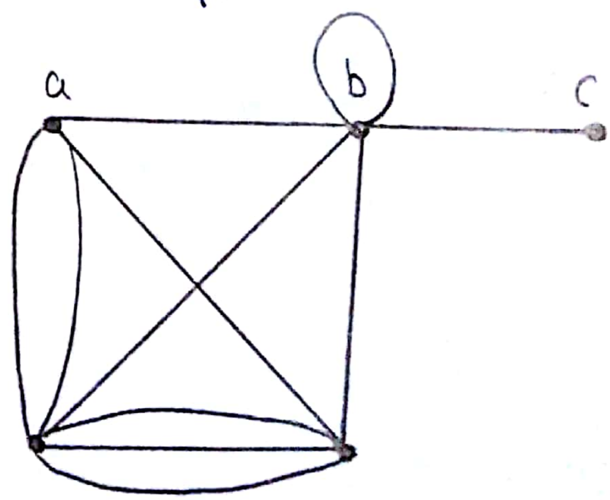
- Two vertices u and v in an undirected graph G are called <u>adjacent</u> (or neighbors) in G if u and v are endpoints of an edge of G. If e is associated with {u,v}, the edge e is called <u>incident</u> with the vertices u and v. The edge e is used to <u>connect</u> u and v. The vertices u and v are called <u>endpoints</u> of an edge associated with {u,v}.

- The <u>degree of a vertex</u> in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The ~~deg~~ degree of the vertex is denoted by deg( ).

- Example: What are the degree of the vertices in the graphs G and H? ~~Graph~~.



[G]



[H]

Solution: In G.

deg(a) = 2
deg(b) = 4
deg(c) = 4
deg(d) = 1
deg(e) = 3
deg(f) = 4
deg(g) = 0

In H

deg(a) = 4
deg(b) = 6
deg(e) = 6
deg(d) = 5
deg(c) = 1

- The vertex of degree zero is called isolated
- A vertex is pendant if and only if it has degree one.

- The Handshaking Theorem

Let, $G = (V, E)$ be an undirected graph with e edges. Then

$$2e = \sum_{u \in V} \deg(u)$$

(This applies even if multiple edges and loops are present).

- **Theorem:**

An undirected graph has an even number of vertices with odd degrees.

<u>Proof:</u> Let V_1 and V_2 be set of vertices with even degree and the set of vertices with odd degree, respectively in an undirected graph $G = (V, E)$. Then.

$$2e = \sum_{u \in V} \deg(u) = \sum_{u \in V_1} \deg(u) + \sum_{u \in V_2} \deg(u)$$

- Because deg(u) is even for $\in V_1$, the first term in the right-hand side of the last equality is even. Furthermore, the sum of two terms on the right-hand side of the last equality is even, because this sum is 2e. Hence, the second term in the sum is also even.
  Because all the terms in this sum are odd, there must be an even number of such terms.

- When (u,v) is an edge of the graph G with <u>directed edges</u>, u is said to be <u>adjacent</u> to v and v is said to be adjacent to u. The vertex u is called the <u>initial vertex</u> of (u,v) and v is called the <u>terminal</u> or end of (u,v).
  The initial vertex and the terminal vertex of the loop are same.

- In a graph with directed edges the in-degree of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex.

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- The out-degree of v, denoted by $deg^+(v)$, is the number of edges with v as their initial vertex.

Note: A loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

- Example: Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in the following figure:

Solution:

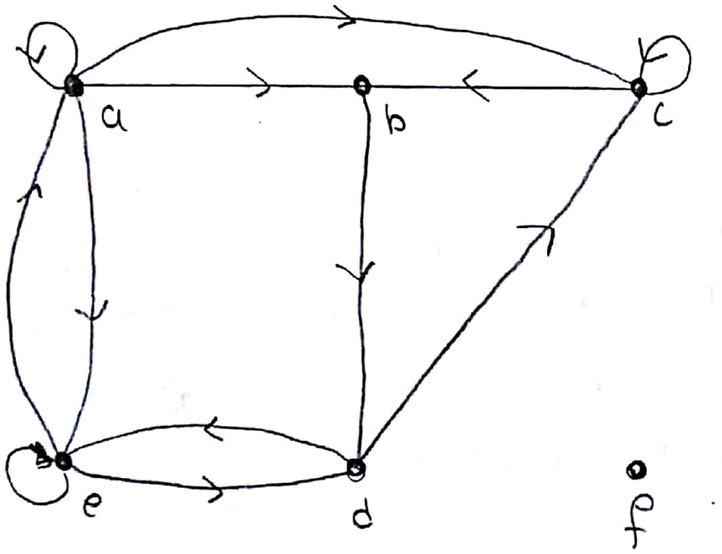


Fig: The directed graph G.

The in degrees in G are

$deg^-(a) = 2$
$deg^-(b) = 2$
$deg^-(c) = 3$
$deg^-(d) = 2$
$deg^-(e) = 3$
$deg^-(f) = 0$

The out degrees in G are

$deg^+(a) = 4$
$deg^+(b) = 1$
$deg^+(c) = 2$
$deg^+(d) = 2$
$deg^+(e) = 3$
$deg^+(f) = 0$

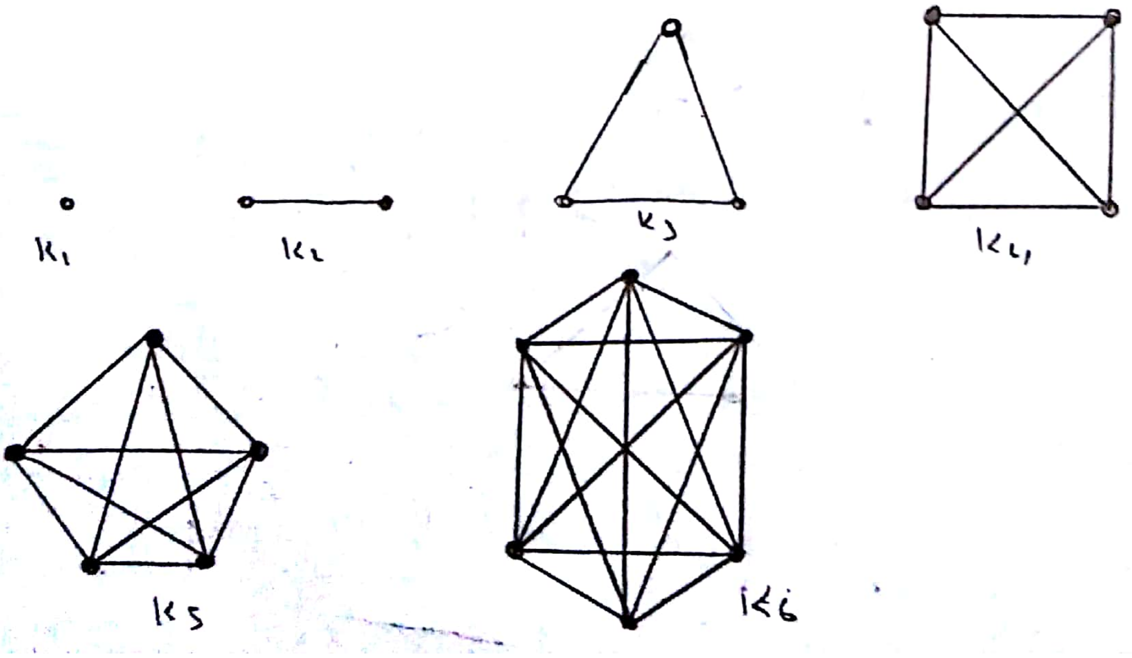
- Let $G=(V,E)$ be a graph with directed edges. Then

$$\sum_{v\in V} \deg^-(v) = \sum_{v\in V} \deg^+(v) = |E|$$

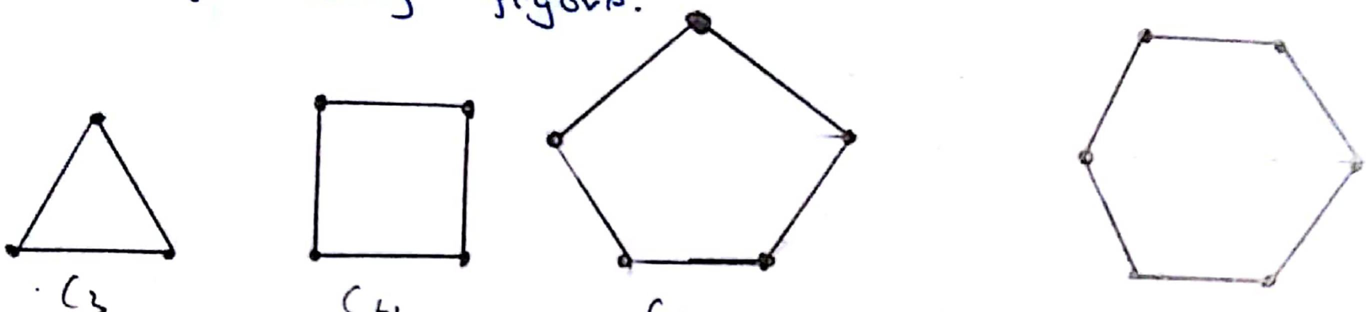
- The undirected graph that results from ignoring directions of edges is called the underlying undirected graph.
  A graph with directed edges and its underlying undirected graph have the same number of edges.

SOME SPECIAL GRAPHS:

- <u>Complete Graphs</u>: The complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices. The graph K_n, for $n=1,2,3,4,5,6$ is displayed in the following figure.



CYCLES: The cycles C_n, $n \geq 3$, consists of n vertices $1, 2, \cdots n$ and edges $\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}$ and $\{n, 1\}$. The cycles C_3, C_4, C_5 and C_6 are displayed in the following figure.

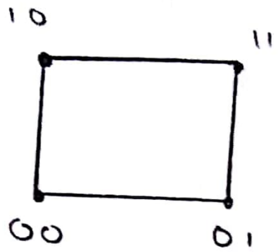


C_3 C_4 C_5 C_6

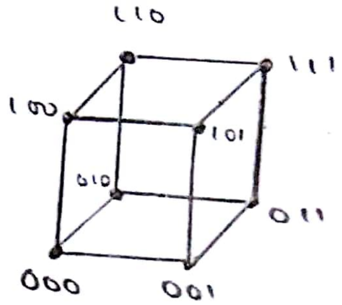
WHEELS: We obtain the wheel W_n when we add an additional vertex to the cycle C_n, for $n \geq 3$ and connect this new vertex to each of the n vertices in C_n, by new ~~e~~edges. The wheels W_3, W_4, W_5 and W_6 are displayed in the following figure.

n-cubes: The n-dimensional hypercube, or n-cube, denoted by Q_n; is the graph that has vertices representing the 2^n bit string of length n. Two vertices are adjacent if and only if the bit strings that they represent differ in exactly one bit position. The graphs Q_1, Q_2 and Q_3 are displayed in the following figure.

Q_1



Q_2



Q_3

- One can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefering the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit.

<u>Bipartite Graphs</u>: A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we calle the pair (V_1, V_2) a bipartition of the vertex set V of G.

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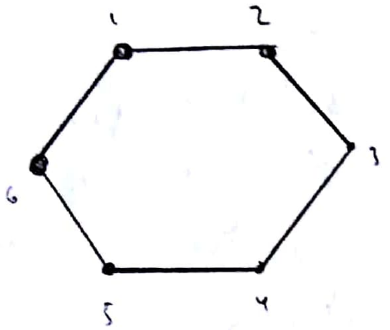
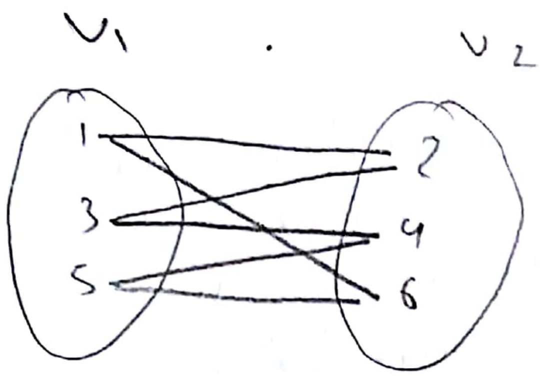
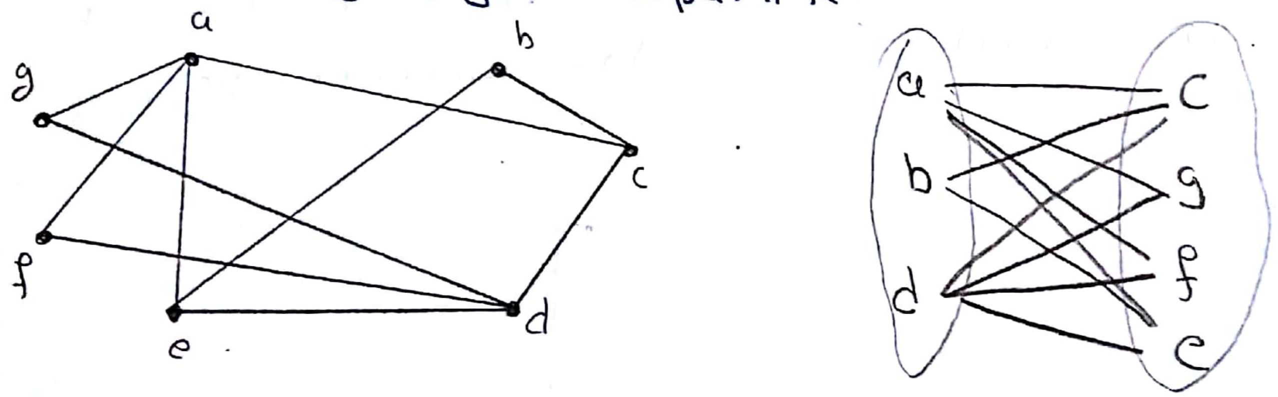
Ex C6 is bipartite, because its vertex set can be partitioned into the two sets $V_1 = \{1, 3, 5\}$ and $V_2 = \{2, 4, 6\}$ and every edge of C_6 connects a vertex in V_1 and a vertex in V_2.



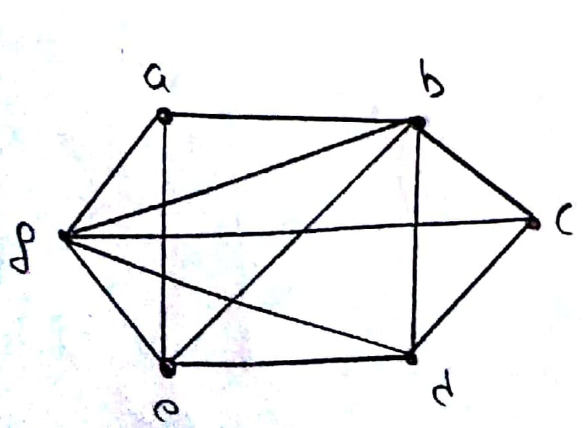


Ex: Are the graphs G and H displayed in the following figure bipartite.



G

$V_1 = \{a, b, d\}$ $V_2 = \{c, g, f, e\}$

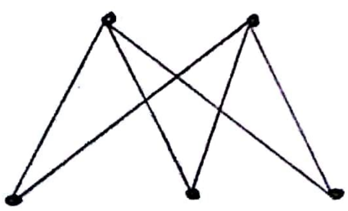


(H)

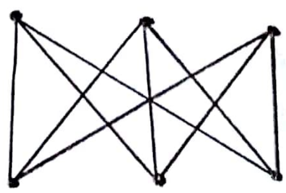
• Graph H is not bipartite because its vertex set cannot be partioned into two subsets so that the edges do not connect two vertices from the same subset.

- A simple graph is bipartite if and only if it is possible to assign one of two different colours to each vertex of the graph so that no two adjacent vertices are assigned the same color.

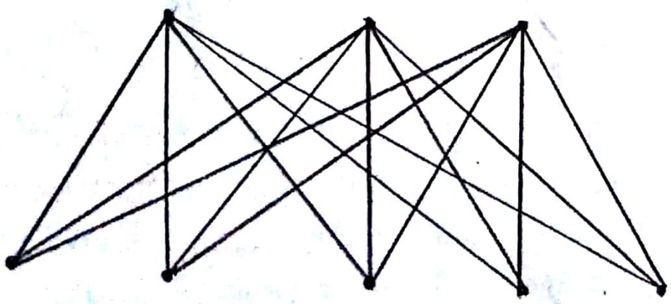
- The complete bipartite graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset. The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$ and $K_{2,6}$ are displayed in the following figure.



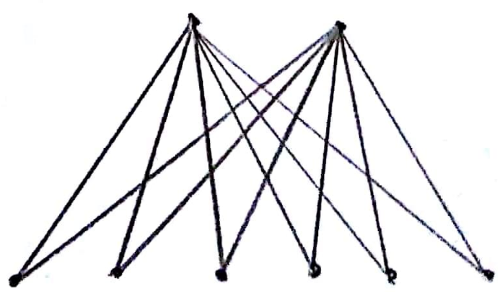
$K_{2,3}$



$K_{3,3}$



$K_{3,5}$



$K_{2,6}$

- A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$ where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

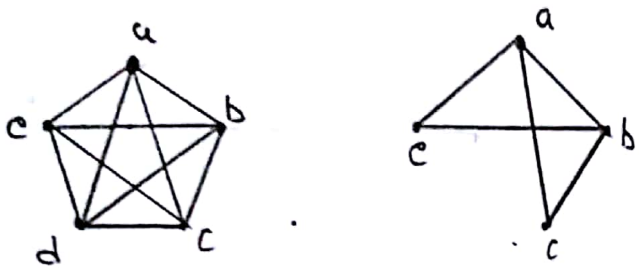
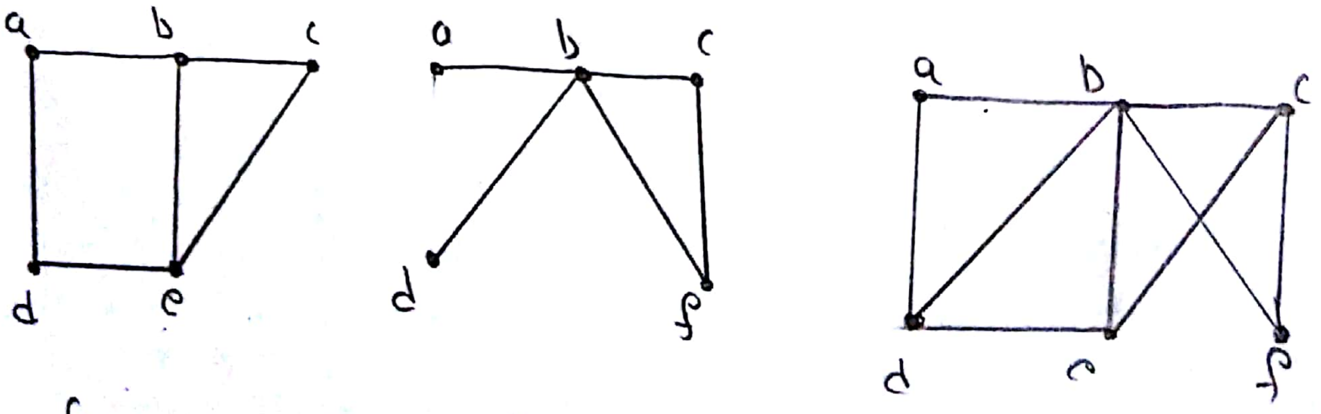


Fig: A subgraph of K_5.

- The union of two simple graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Ex.



G_1 G_2 $G_1 \cup G_2$

REPRESENTING GRAPHS:

- **Adjacency Lists:**

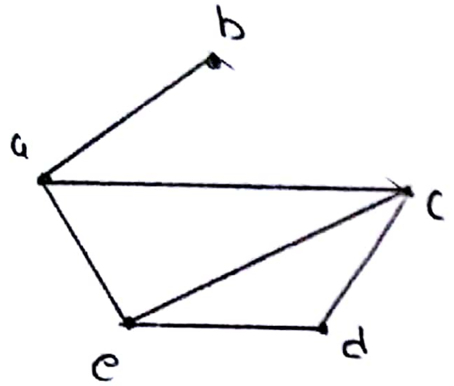


Figure: A simple graph

Vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Table: An adjacency list for a simple graph

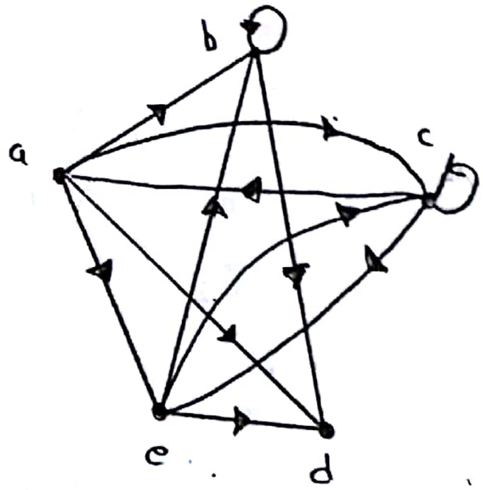


Fig: A directed graph

Initial vertex	Terminal vertices
a	b, c, d, e
b	b, d
c	c, a, e
d	
e	b, c, d

Table: An adjacency list for a directed graph h

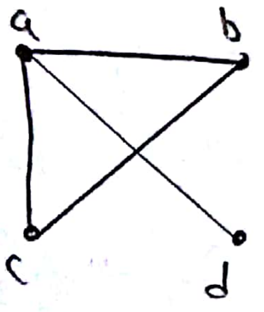
Adjacency Matrices:

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $1, 2, \ldots, n$. The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its ~~entry~~ (i, j)th entry when i and j are adjacent, and 0 as its $(i, j)^{th}$ entry when are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \text{ is an edge of } G \\ 0 & \text{otherwise.} \end{cases}$$

Example: Use an adjacency matrix to represent the graph shown in the following figure.

Solution:

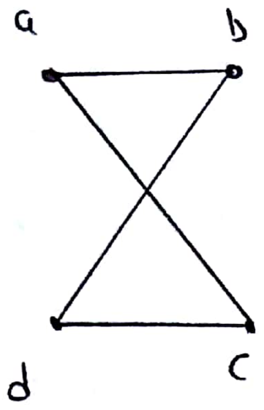


$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Simple Graphs

Example: Draw the graph with adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

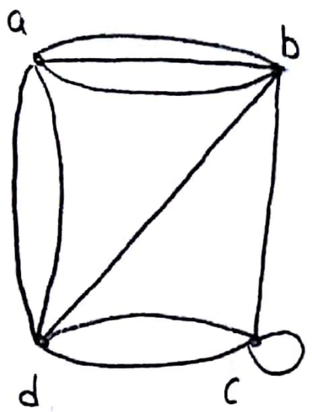


- The adjacency matrix of a simple graph is symmetric, that is, $a_{ij} = a_{ji}$, because both of these entries are 1, when i and j are adjacent and both are 0 otherwise. Furthermore, because a simple graph has no loops, each entry a_{ii}, $i = 1, 2, 3 \cdots n$, is 0.

- Adjacency matrices can also be used to represent undirected graphs with loops and with multiple edges. A loop at the vertex a_i is represented by a 1 at the (i, i)th position of the adjacency matrix. When multiple edges are present, the adjacency matrix is no longer a zero-one matrix, because the (i, j)th entry of this matrix equals the number of edges that are associated to $\{a_i, a_j\}$.
All undirected graphs, including multigraphs and pseudo graphs, have symmetric adjacency matrices.

Example: Use an adjacency matrix to represent the Pseudograph shown in following figure.

Solution:



$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

- If $A = [a_{ij}]$ is the adjacency matrix for the directed graph with respect to ~~the~~ its listing of the vertices, then

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases}$$

The adjacency matrix for a directed graph does not have to be symmetric, because there may not be an edge from a_i to a_j when there is an edge from a_i to a_j.

- In Adjacency matrix for a directed multigraph, a_{ij} equals the number of edges that are associated to $\{i,j\}$.

Such matrices are not zero-one matrices.

- When a simple graph contains relatively few edges, ~~a~~ adjacency lists are preferred to adjacency matrix to represent the graph.

But when simple graph contains many edges i.e graph is dense, adjacency matrix is preferred.

Incidence Matrices:

Let $G = (V, E)$ be an undirected graph. ~~&~~ Suppose that $1, 2, \ldots, n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } i, \\ 0 & \text{otherwise.} \end{cases}$$

Example: Represent the graph shown in the following figure with an incidence matrix.

Solution:

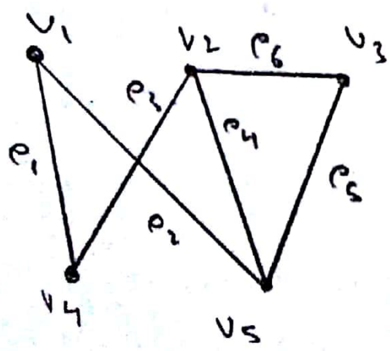
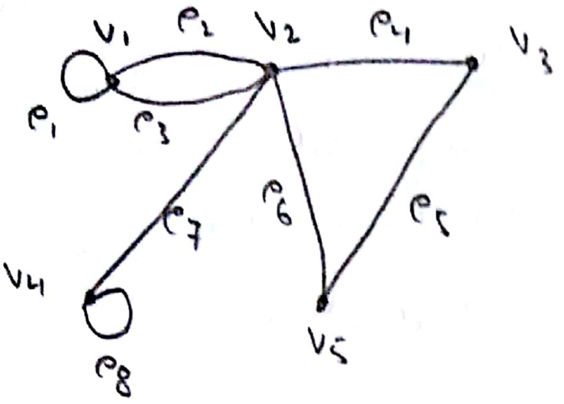


Fig: An undirected graph

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

- **Example:** Represent the pseudograph shown in the following figure using an incidence matrix.



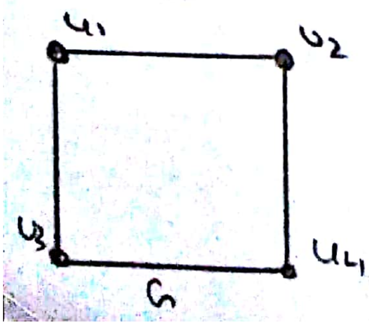
$$
\begin{array}{c|cccccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{array}
$$

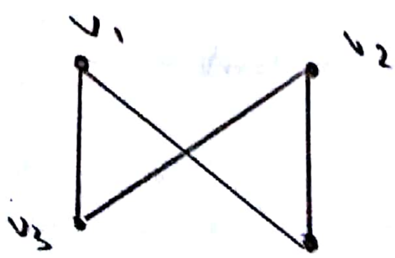
Isomorphism of Graphs:

- The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called isomorphism.

- **Ex** Show that the graphs $G = (V, E)$ and $H = (W, F)$ displayed in following figure is isomorphic.

Solution





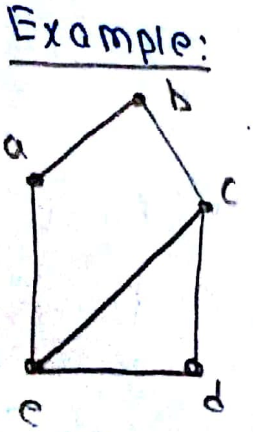
The function f with $f(u_1)=v_1$, $f(u_2)=v_4$, $f(u_3)=v_3$ and ~~f(u4)~~ $f(u_4)=v_2$ is one-to-one correspondence between V and W. The adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4 and each of the pairs $f(u_1)=v_1$ and $f(u_2)=v_4$, $f(u_1)=v_1$ and $f(u_3)=v_3$, $f(u_2)=v_4$ and $f(u_4)=v_2$, and $f(u_3)=v_3$ and $f(u_4)=v_2$ are adjacent in H.

- The
  - ↳ number of vertices
  - ↳ number of edges
  - ↳ the number of vertices of each degree

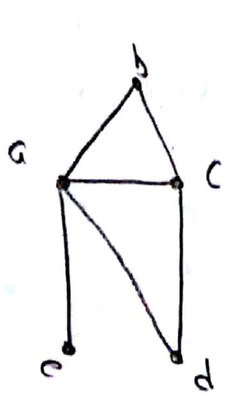
are all invariants under isomorphism.

If any of these quantities differ in two simple graphs, these graphs cannot be isomorphic. However, when these invariants are the same, it does not necessarily mean that the two graphs are isomorphic.

Example:



[G]



H

No. of vertex in G = 5
No. of vertex in H = 5

No. of edge in G = 6
" " " " H = 6

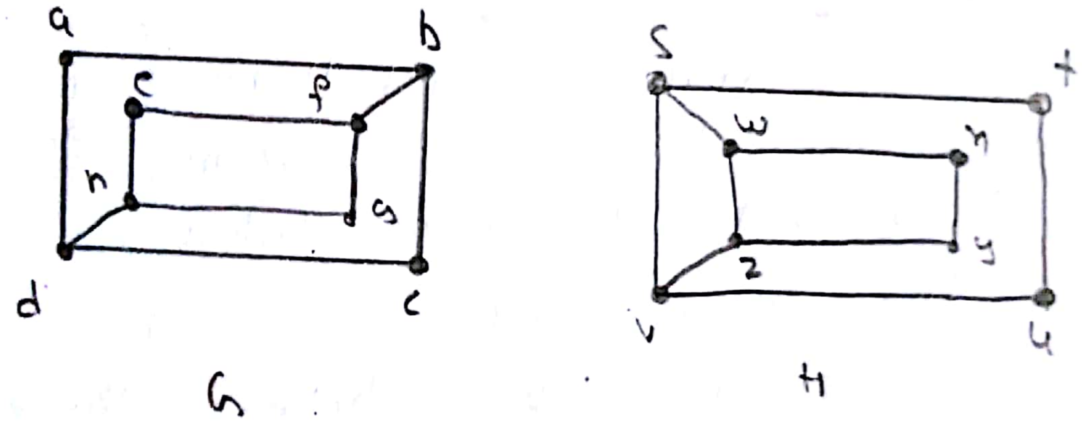
~~But~~

~~deg~~

G	H
deg(a) = 2	deg(a) = 4
deg(b) = 2	deg(b) = 2
deg(c) = 3	deg(c) = 3
deg(d) = 2	deg(d) = 2
deg(e) = 3	deg(e) = 1

Since, there is not any edge in ~~H~~ G with ~~deg~~ degree 1 or 4, G and H are not isomorphic.

Ex Are these graphs isomorphic?



Vertex = 8

No. of edges = 10

deg(a) = 2
deg(b) = 3
deg(c) = 2
deg(d) = 3
deg(e) = 2
deg(f) = 3
deg(g) = 2
deg(h) = 3

No. of vertex = 8

No. of vertex = 10

~~deg(a) =~~ deg(s) = 3
deg(t) = 2
deg(u) = 2
deg(v) = 3
deg(w) = 3
deg(x) = 2
deg(y) = 2
deg(z) = 3

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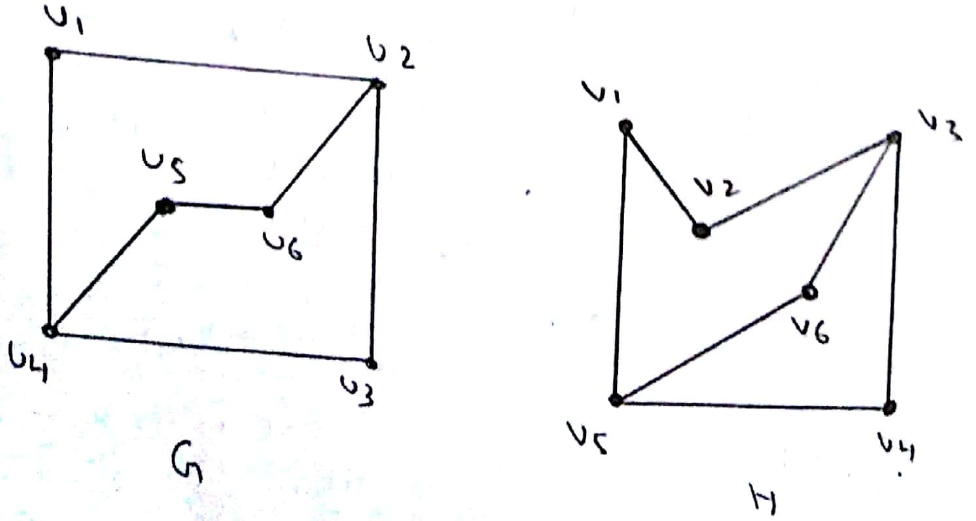
The G and H both have eight vertices and 10 edges. They also have four vertices of degree two and four of degree two. All these invariants agree but G and H are not isomorphic.

- $deg(a) = 2$ in G, 'a' must correspond to either t, u, x or y in H, because these are the vertices of degree two in H. However, each of these four vertices in H is adjacent to another vertex of degree two in H which is not true for a in G.

- To show that two graphs G and H are isomorphic, we can show that the adjacency matrix of H, when rows and columns are labeled to correspond to the images under f of the vertices in G that are the labels of these rows and columns in the adjacency matrix of G.

<u>Ex</u>



For G:

Vertices = 6

Edges = 7

$deg(u_1) = 2$

$deg(u_2) = 3$

$deg(u_3) = 2$

$deg(u_4) = 3$

$deg(u_5) = 2$

$deg(u_6) = 2$

For H

Vertices = 6

Edges = 7

$deg(v_1) = 2$

$deg(v_2) = 2$

$deg(v_3) = 3$

$deg(v_4) = 2$

$deg(v_5) = 3$

$deg(v_6) = 2$

- Both have two vertices of degree 3 and four vertices of degree 2.

- Because $deg(u_1) = 2$ and because u_1 is not adjacent to any other vertex of degree two, the image of u_1 must be either v_4 or v_6 the only vertices of degree two in H not adjacent to a vertex of degree two.

Let $f(u_1) = v_6$.

Because, u_2 is adjacent to u_1, the possible images of u_2 are v_3 and v_5.

Let $f(u_2) = v_3$, Continuing this wee get,

$f(u_1) = v_6$ $\quad$ $f(u_4) = v_5$

$f(u_2) = v_3$ $\quad$ $f(u_5) = v_1$

$f(u_3) = v_4$ $\quad$ $f(u_6) = v_2$.

To see whether f preserves edges, we examine the adjacency matrix of G.

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of H with the rows and columns labeled by the images of the correspon-ding vertices in G.

$$A_H = \begin{array}{c} \\ 6 \\ 3 \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccccc} 6 & 3 & 4 & 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Because, $A_G = A_H$, it follows that f preserve edges. We conclude that f is an isomorphism, so G and H are isomorphic.

m

GRAPH CONNECTIVITY

WALK: A walk in a graph G is a finite ordered set $W = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k)$ whose elements are alternately vertices and edges such that $1 \le i \le k$, the edge e_i has ends v_{i-1} and v_i.

The walk W is a v_0-v_k walk or walk from v_0 to v_k.

The number of edges appearing in the sequence of the path is called its length.

If the length of the walk is zero i.e. the walk has no edges, it contains only a single vertex and is called a trivial walk.

A walk is closed if it starts and ends at the same point, otherwise the walk is open.

Trail: A trail is a walk with no repeated edges.

A closed trail is a circuit.

Path: A path is a walk with no repeated vertices.

A closed path is a cycle.

- A circuit which does not repeat any vertices (except the initial and final vertex) is called a cycle.

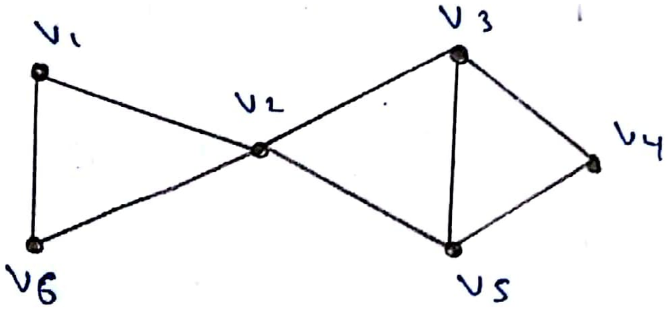
Thus, a cycle is a non-intersecting circuit and must have a length of three or more.

A cycle of length k is called k-cycle.

- While every cycle is a circuit, the converse is not always true.

Term	Repeated Edge	Repeated Vertex
WALK	✓	✓
PATH	✗	✗
TRAIL	✗	✓
CIRCUIT	✗	✓
CYCLE	✗	First & last only

Example:



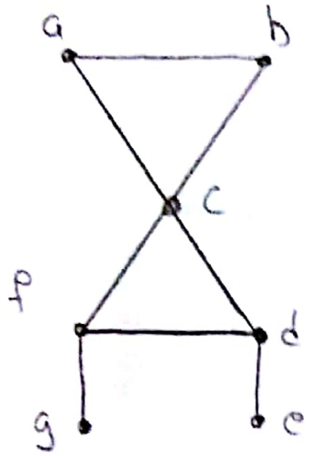
- The closed trail: $(v_1, v_2, v_3, v_4, v_5, v_2, v_6, v_1)$
  it is a circuit but not a cycle.

- The closed trail: $(v_2, v_3, v_4, v_5, v_2)$
  it is a circuit as well as cycle.

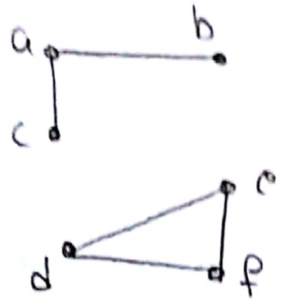
Connectedness In Undirected Graphs

- An undirected graph is called connected if there is a ~~path~~ walk between every pair of distinct vertices of the graph.

- **Example:**



G_1



G_2

G_1 = Connected

G_2 = not-connected.

- **Theorem**

- There is a trail between every pair of distinct vertices of a connected undirected graph.

Proof: Let u and v be two distinct vertices of the connected undirected graph $G = (V, E)$. Because G is connected, there is at least one ~~trail~~ walk between u and v. Let $x_0, x_1, \dots x_n$, where $x_0 = u$ and $x_n = v$, be the vertex sequence of a ~~path~~ walk of least length. This walk of least length is a trail.. To see this, suppose it is not a trail. Then $x_i = x_j$ for some i and j with $0 \leq i < j$.

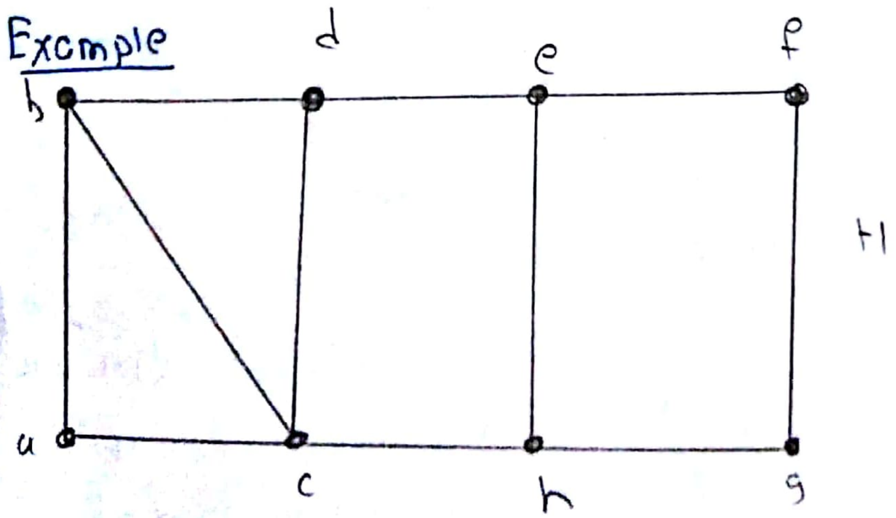
This means that there is a walk from u to v of shorter length with vertex sequence $x_0, x_1, \ldots x_{i-1}, x_j \ldots x_n$, obtained by deleting the edges corresponding to the vertex sequence $x_i, \ldots, x_{j-1}$.

- ~~The connected component of a graph G is a connected subgraph of G~~

- A connected component of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G.

A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

Example



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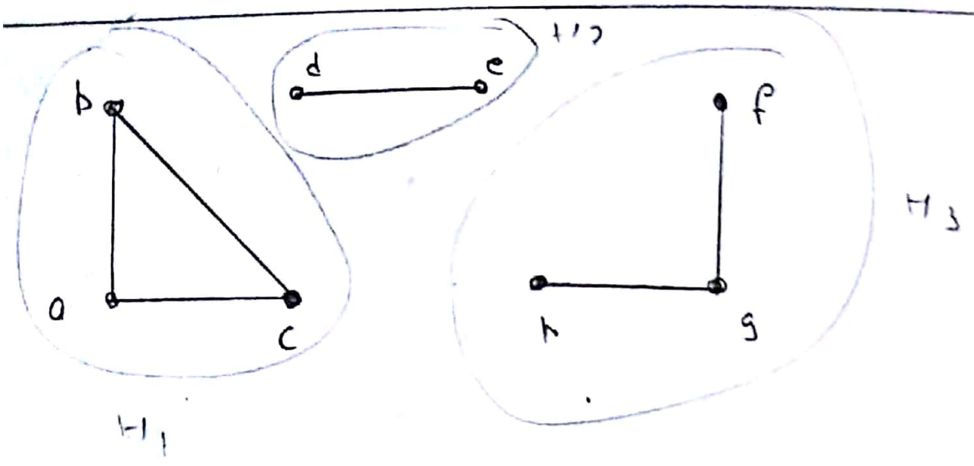
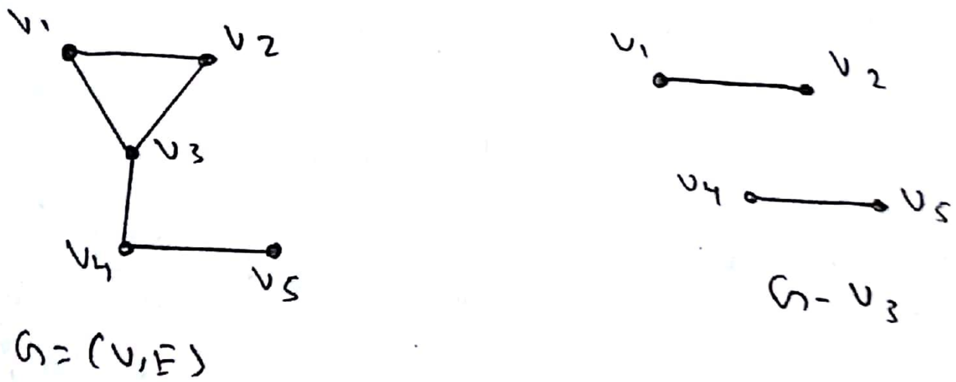


H_1, H_2 and H_3 are connected components of graphs.

<u>Cut-vertex</u> : A vertex v of a connected graph G is called a cut-vertex (or cut-point) if G-v is disconnected.

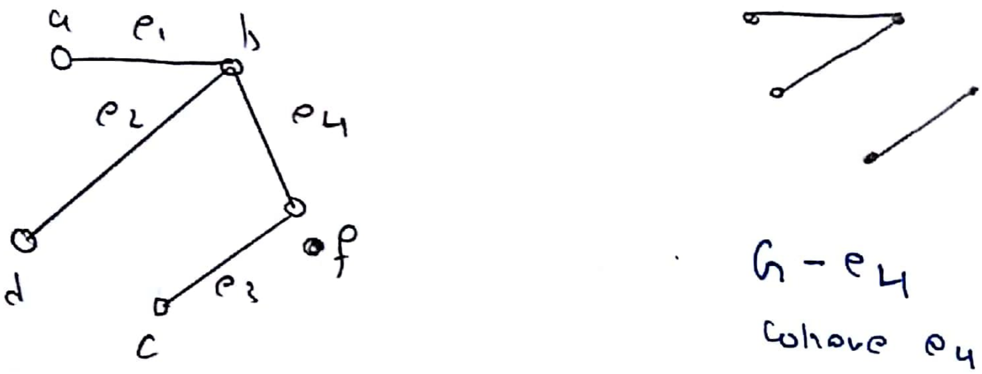
Any vertex in a graph is said to be cut vertex if graph becomes disconnected after removal of this vertex from the graph.



In above figure, after removal of vertex v_3, graph G becomes disconnected. So, v_3 is cut-vertex. Similarly, v_4 is also.

Cutedge or bridge

Any edge in graph after whose removal graphs becomes disconnected, is called bridge or cut edge. Any edge e is a bridge for G if G-e is disconnected.



$G - e_4$

where $e_4 = (b, f)$

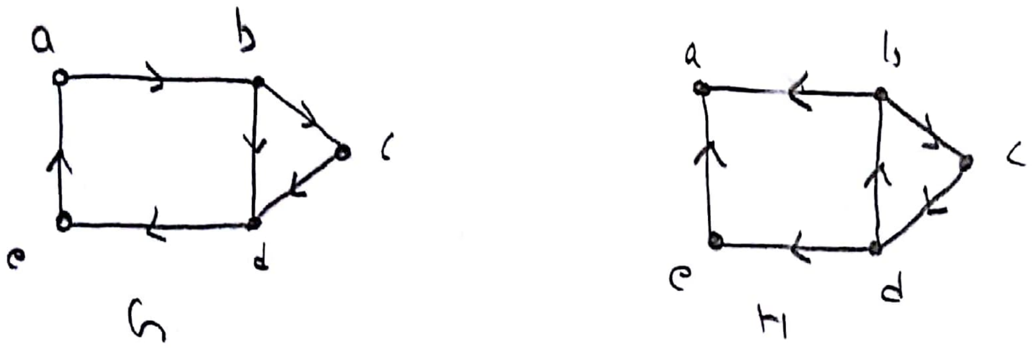
In above figure, after removal of an edge $e = \{b, f\}$ the connected graph G becomes disconnected. So, $e = \{b, f\}$ is cut-edge.

Connectedness in Directed Graphs.

- A directed graph is strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.
- A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Example: Are the directed graphs G and H, strongly connected or weakly connected?

Solution:

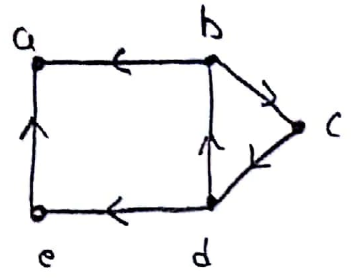


- G is strongly connected because there is a path between any two vertices in this directed graph

- Graph H is weakly connected as there is no directed path from a to b in the graph.

- The subgraphs of directed graph G that are strongly connected but not contained in larger strongly connected subgraphs, that is, the maximal strongly connected subgraphs, are called the strongly connected components or strong components of G.

Example

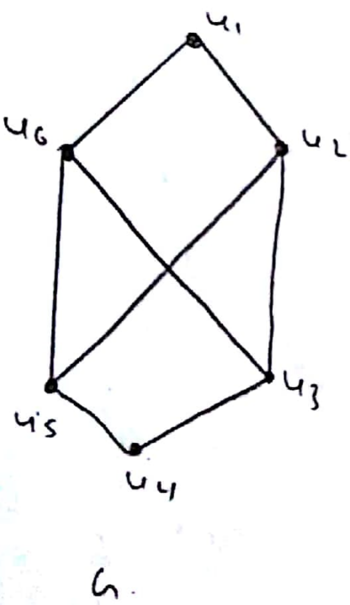


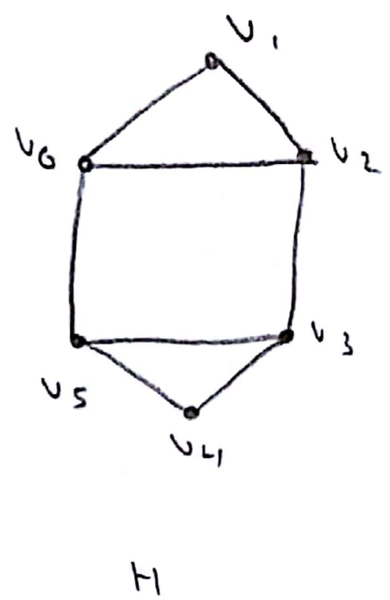
This graph are three strongly connected components

- One consisting of vertex a
- Second consisting of vertex e
- Third the graph consisting of vertices b, c and d with edges (b, c), (c, d) & (d, b).

Ex Determine whether the graphs G and H are isomorphic.

Solution:

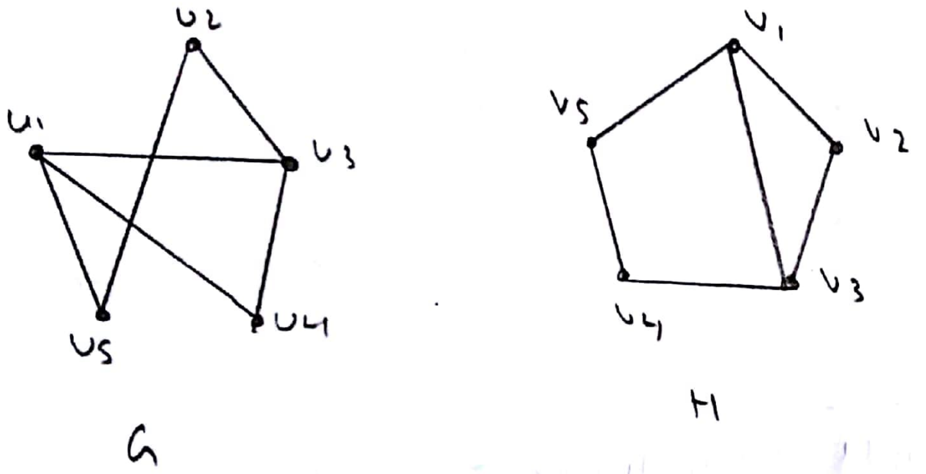




~~Solution~~ Both G and H have six vertices and eight edges. Each has four vertices of degree three, and two vertices of degree two. So the three invariants - no. of vertices, no. of edges and degree of vertices - all agree for two graphs.

However, H has a simple circuit of length three, namely, v_1, v_2, v_6, v_1 whereas G has no simple circuit of length three. Hence, G and H are not isomorphic.

• **Example:** Determine whether graphs G & H are isomorphic.



Solution

G

No. of vertex = 5

No. of edges = 6

Score = {2, 2, 2, 3, 3}

H

No. of vertex = 5

No of edges = 6

Score = {2, 2, 2, 3, 3}

Because all these invariants agree G and H may be isomorphic.

• To find the possible isomorphism, we can follow the paths that go through all the vertices so that the corresponding vertices in the two graphs have the same degree.

For example, the paths u_1, u_4, u_3, u_2, u_5 in G and v_3, v_2, v_1, v_5, v_4 in H both go through every vertex in graph; start at vertex of degree three; go to vertices of degrees two, three and two, respectively; and end at vertex of degree two.

By following these paths through the graphs, we define the mapping f with

$f(u_1) = v_3$ $\quad f(u_3) = v_1$ $\quad f(u_5) = v_4$.

$f(u_4) = v_2$ $\quad f(u_2) = v_5$

Draw adjacency matrix, to show f ~~presde~~ preserves edges and G & H are isomorphic.

EULER AND HAMILTON PATHS.

- An Euler circuit in a graph G is a simple circuit containing every edge of G.

  An Euler path in G is a simple path containing every edge of G.

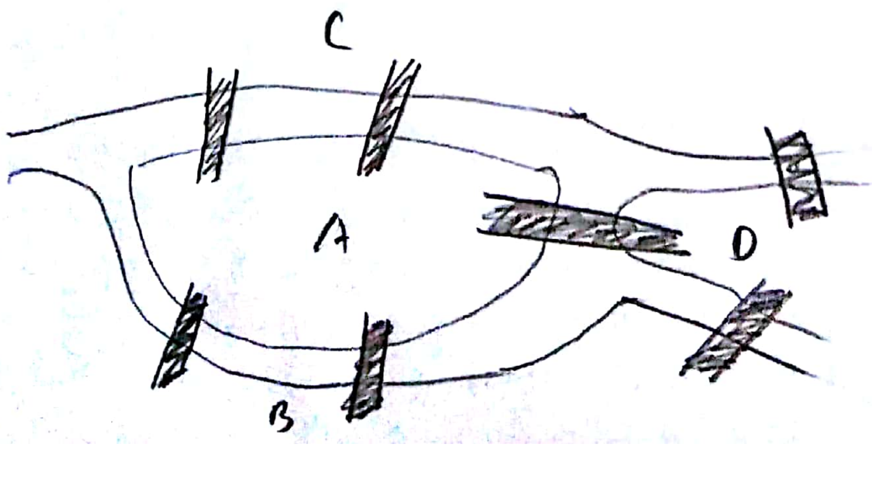
- 

Fig: The seven Bridges of Königsberg.

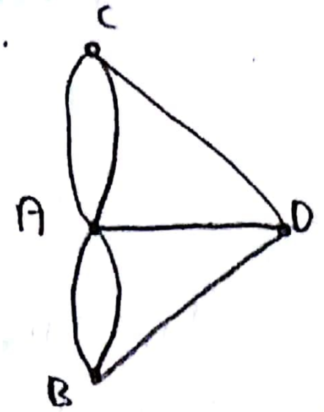
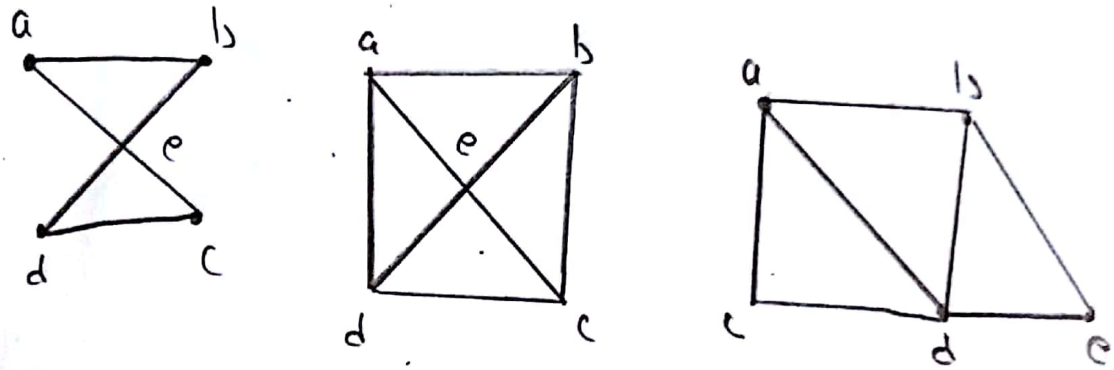


Fig: Multigraph model of the Town of Königsberg.

Example: Which of the undirected graphs in the following have an Euler circuit? Of those that do not, which have an Euler path?



G_1 G_2 G_3

Solution: The graph G_1 is an Euler circuit, for example a, e, c, d, e, a.

G_2 and G_3 ~~not~~ don't have Euler circuit.

G_3 has an Euler path. a, c, d, e, b, d a

G_2 does not have an Euler path.

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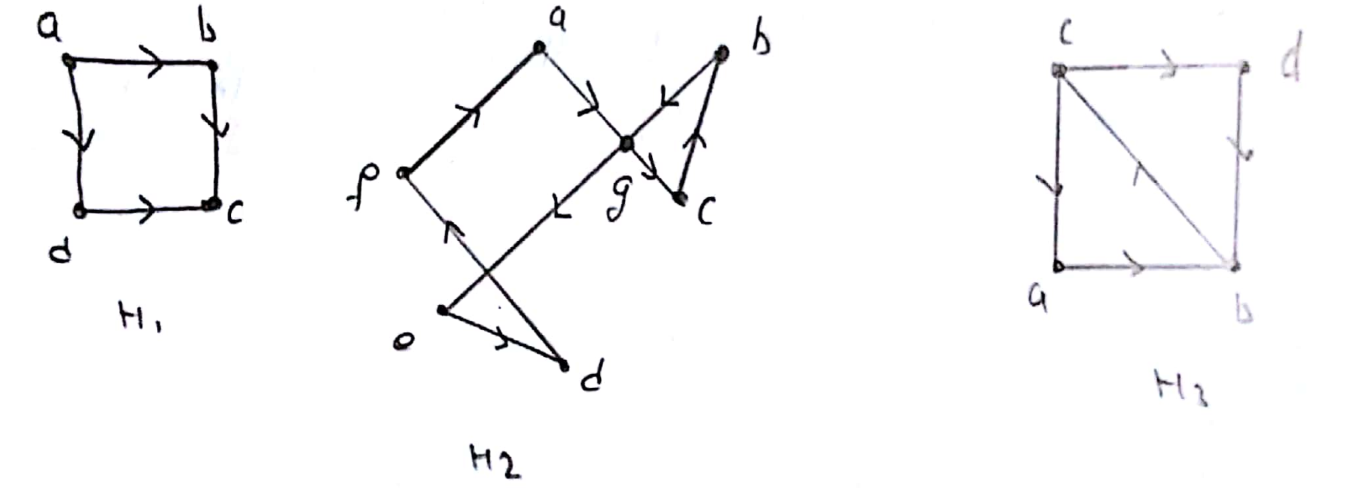
- A path or circuit is simple if it does not contain the same edge more than once.

Example: Which of the directed graphs in the following figure have an Euler circuit? Of those that do not, which have an Euler path?

Solution:



- The graph H_2 has an Euler circuit.

a g c b g e d e a

- H_1 and H_3 do not have an Euler circuit
- H_1 do not ~~has~~ have Euler path.
- H_3 do have an Euler path.

Theorem 1: A connected multigraph with at least two vertices has an Euler circuit if and only if each its vertices has an even degree.

Theorem 2: A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Example:

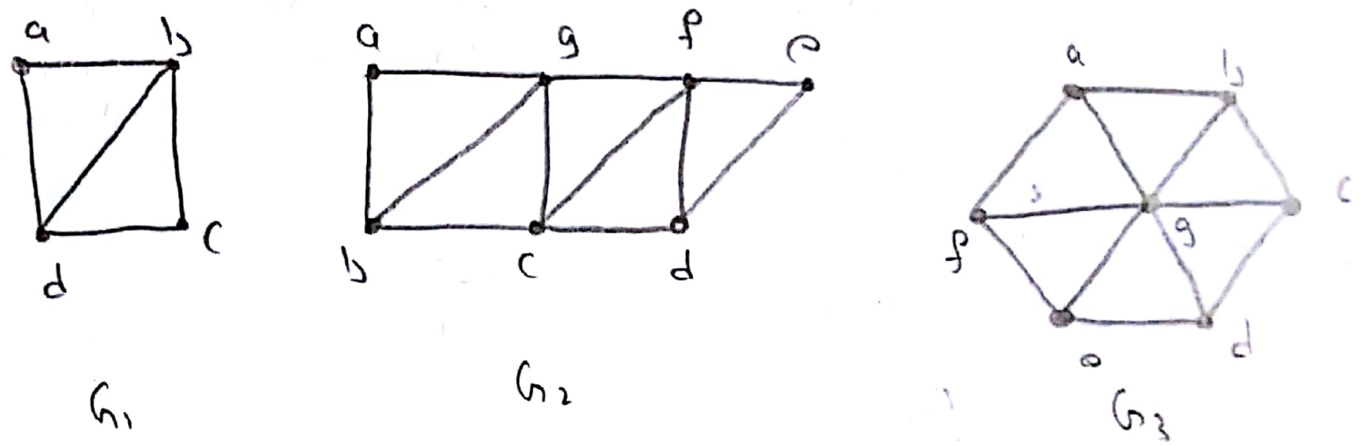


Fig: Three Undirected Graphs.

- G_1 contains exactly two vertices of odd degree, namely, b and d. Hence, it has an Euler path that must have b and d as its end points.
- G_2 has exactly two vertices of odd degree, namely b and d. So it has an Euler path.
- G_3 has no ~~Ao~~ Euler path because it has six vertices of odd degree.

Hamilton Path and Circuits:

A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path.

A simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit.

Term	Initial & Final vertex	Must include every edge	Repeated vertices allowe.
Euler circuit	Yes	Yes	Yes
Hamilton Circuit	Yes	No	No
Euler path	No	Yes	Yes
Hamilton ~~Eule~~ path	No	No	No

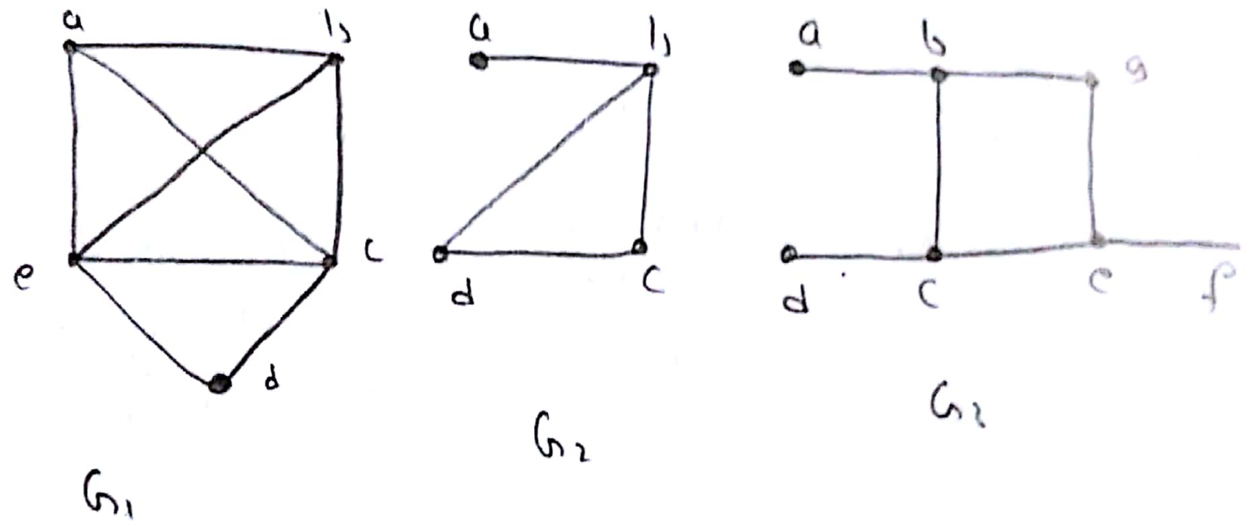
DIRAC'S THEOREM: If G is a simple graph with n vertices $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Theorem 3: ORE's Theorem: If G is a simple graph with n vertices, with $n \geq 3$ such that

$$\deg(u) + \deg(v) \geq n$$

for every pair of non adjacent vertices u and v in G, then has a Hamilton circuit.

Example:

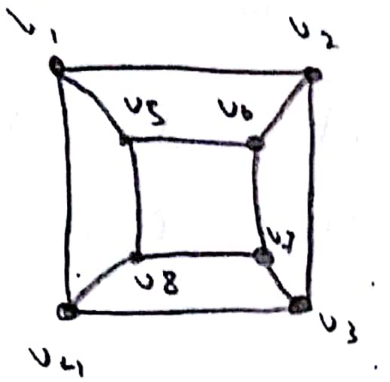


G_1 has Hamilton circuit. a, b, c, d, e, a

G_2 do not have Hamilton circuit but it does have Hamilton path, a, b, c, d.

G_3 has neither Hamilton path nor Hamilton circuit.

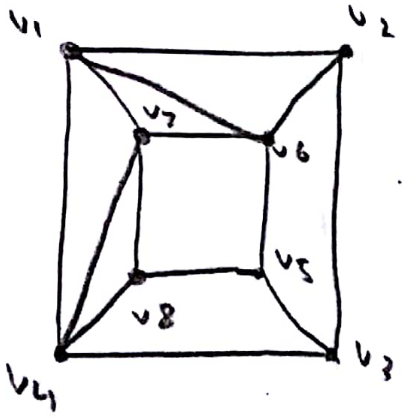
Example: Is the graph (cube) given in figure is Hamilton cycle?



Yes.

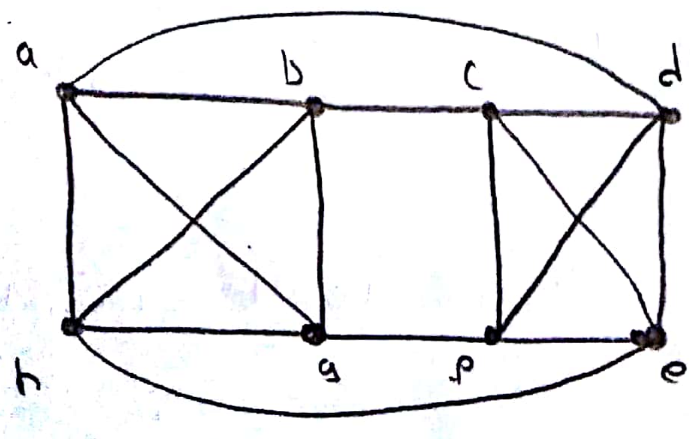
$v_1\ v_2\ v_3\ v_7\ v_6\ v_5\ v_8\ v_4\ v_1$

Example: Eulerian cycle? Hamilton cycle?



- Not Eulerian cycle as vertices have odd degrees.
- $v_1\ v_2\ v_3\ v_5\ v_6\ v_7\ v_8\ v_1$

Example: Hamilton?



By Dirac's theorem
No. of vertices = 8
each vertex has $= \frac{8}{2} = 4$ degree.

Hamilton cycle.
a, b, g, h, e, f, c, d, a

Matching

- In graph theory, a matching or independent edge set is a set of edges without common vertices.

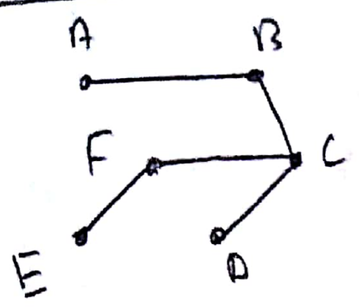
- Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges, i.e no two edges share a common vertex.

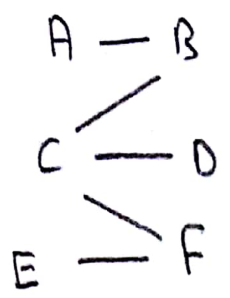
- The vertex is matched (or saturated) if it is an endpoint of one of the edges in the matching. Otherwise, the vertex is unmatched.

- A maximal matching is a matching M of a graph G with the property that if any edge not in M is added to M, it is no longer a matching, that is, M is maximal if it is not a subset of any other matching graph.

- A maximum matching (also known as maximum-cardinality matching) is a matching that contains the largest possible number of edges.

- Every maximum matching is maximal, but not every maximal matching is a maximum matching.





Matching $A-B$ $\{A,B\}$ m_1

$A-B, C-D,$ $\{\{A,B\}, \{C,D\}\}$ m_2

$A-B, C-D, E-F$ $\{\{A,B\}, \{C,D\}, \{E,F\}\}$ m_3

Also $C-B$ $\{C,B\}$ m_4

$C-B, E-F$ $\{\{C,B\}, \{E,F\}\}$ m_5

$m_1 \subseteq m_2 \subseteq m_3$ But m_3 is not subset of any other matching.

Here m_3 is maximal

~~Somilar~~

Similarly

$m_4 \subseteq m_5$ But m_5 is not subset of any other matching.

m_5 is maximal

In given graph

$|m_5| = 2$ $|m_5| < |m_3|$

$|m_3| = 3$ There is no other matching in given graph having cardinality greater than that of m_3. $\therefore$ m_3 is maximum matching for this graph.

Weighted Graph or Labelled graph:

- A weighted graph is a graph G, in which edge, e, is assigned a non-negative real number, w(e) called the weight of e. The weight of a subgraph H of G is the sum of the weights of the edges of the subgraph H.

SHORTEST-PATH PROBLEMS

Example: What is the shortest path between a and z in the weighted graph shown in the following figure?

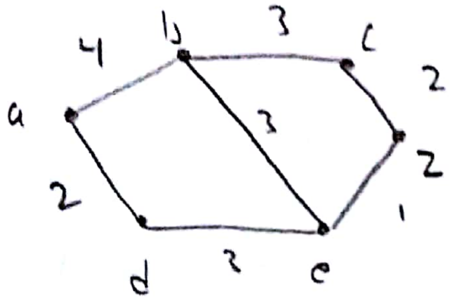


Fig: A weighted Simple Graph

- The only paths starting at 'a' that contain no vertex other than 'a' (until the terminal vertex is reached) are a,b and a,d. Because the length of a,b and a,d are 4 and 2, respectively, it ~~following~~ follows that d is the closest vertex to a.
- We can find the next closest vertex by looking at all paths that go through only a and d (until the terminal vertex is reached). The shortest such path to b is still a,b, with length 4, and the shortest path to e is a,d,e with length 5. Consequently, the next closest vertex to a is b.

To find the third closest vertex to a, we need to examine only paths that go through only a, d, and b (until the terminal vertex is reached).
There is a path of length 7 to c, namely, a, b, c and a path of length 6 to z, namely, a, d, e, z.
Consequently, z is the next closest vertex to a, and the length of a shortest path to z is 6.

Algorithm: Dijkstra's algorithm
(/'daɪkstrə/)

• Procedure: Dijkstra's algorithm (G: weighted connected simple graph, with all weights positive).

{G has vertices $a = v_0, v_1, \ldots v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}

for $i := 1$ to n

  $L(v_i) := \infty$

$L(a) := 0$

$S := \emptyset$

{the labels are now initialized so that the label of a is 0 and other labels are ∞, and S is an empty set}

while $z \notin S$

begin

  $u :=$ a vertex not in S with $L(u)$ minimal

  $S := S \cup \{u\}$

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for all vertices v not in S

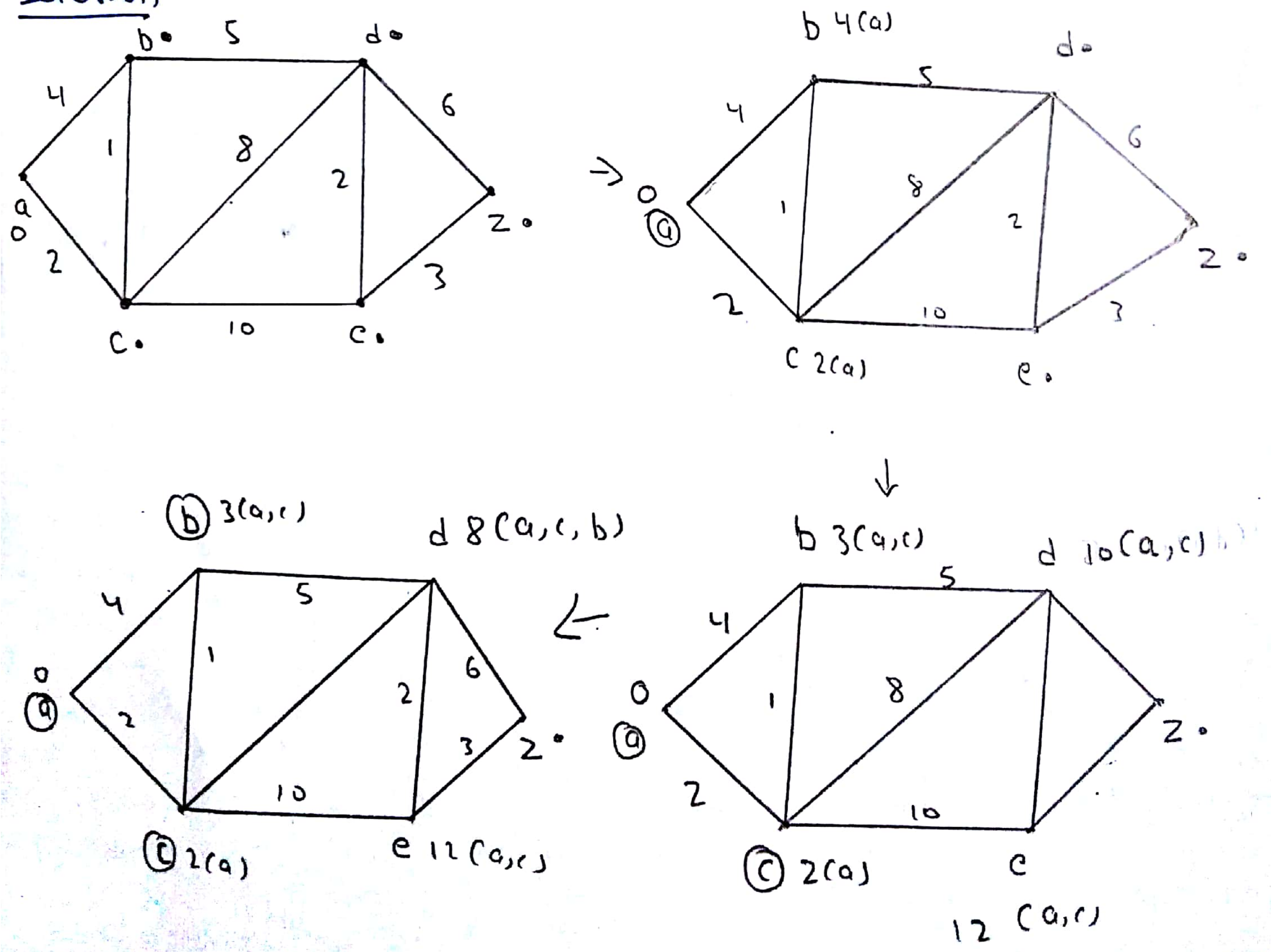
if $L(u) + w(u,v) < L(v)$ then $L(v) = L(u) + w(u,v)$

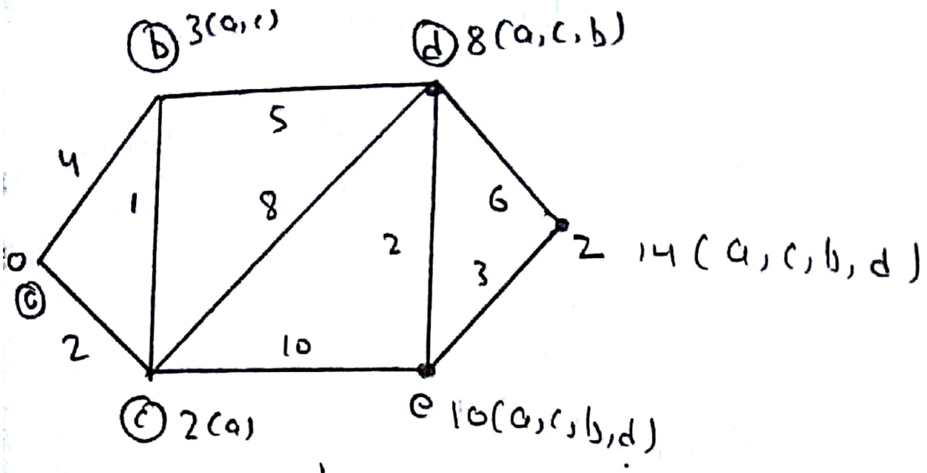
{this adds a vertex to S with minimal label and updates the labels of vertices not in S}

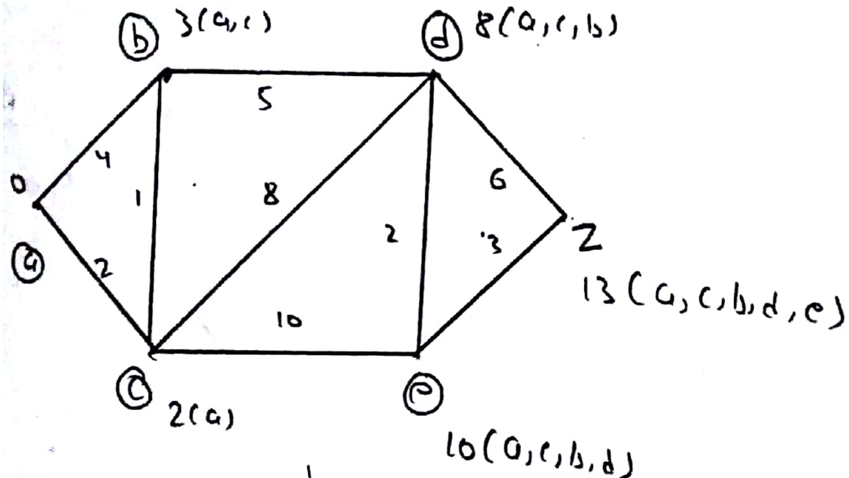
end {$L(z)$ = length of shortest path from a to z}

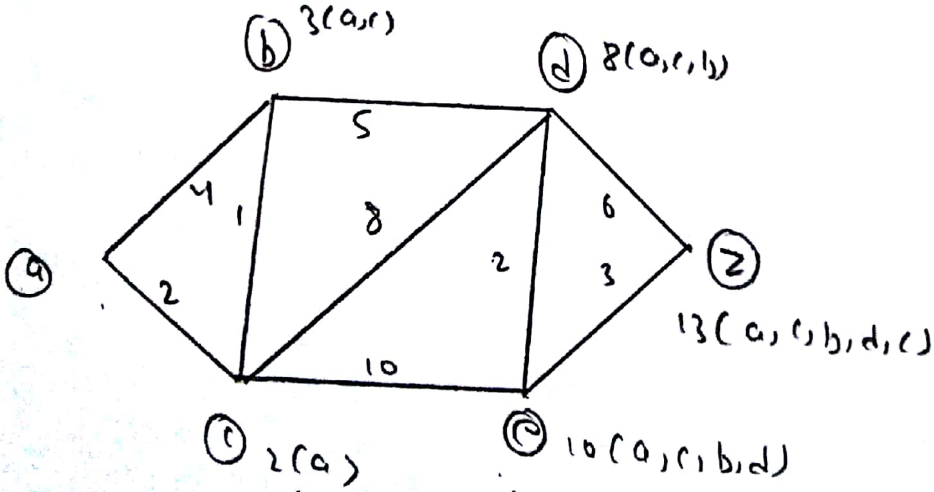
Example 2: Use Dijkstra's algorithm to find the length of a shortest path between the vertices a and z in the weighted graph displayed in the following figure.

Solution




b 3(a,c)
d 8(a,c,b)
4
5
1
8
6
2
3
a 0
z 14(a,c,b,d)
2
10
c 2(a)
e 10(a,c,b,d)


b 3(a,c)
d 8(a,c,b)
5
4
1
8
6
0
a
2
2
3
z
13(a,c,b,d,e)
10
c 2(a)
e 10(a,c,b,d)


b 3(a,c)
d 8(a,c,b)
5
4
1
8
6
2
3
a
2
z
13(a,c,b,d,e)
10
c 2(a)
e 10(a,c,b,d)

X

Traveling Salesman Problem

The travel salesman problem asks for the circuit of minimum total weighted in a weighted, complete, undirected graph that visits each vertex exactly once and returns to its starting point.
This is equivalent to asking for a Hamilton circuit with minimum total weight in the complete graph, because each vertex is visited exactly once in the circuit.

- The most straight forward way to solve an instance of the travelling salesman problem is to examine all possible Hamilton circuits and select one of minimum total length.

∴ We have to examine $\frac{(n-1)!}{2}$ circuits to solve the problem if there are n choices, or vertices in graph.

- Note that $\frac{(n-1)!}{2}$ grows extremely rapidly.
- For example, with 25 vertices a total of $\frac{24!}{2}$ different Hamilton circuit would have to be considered.

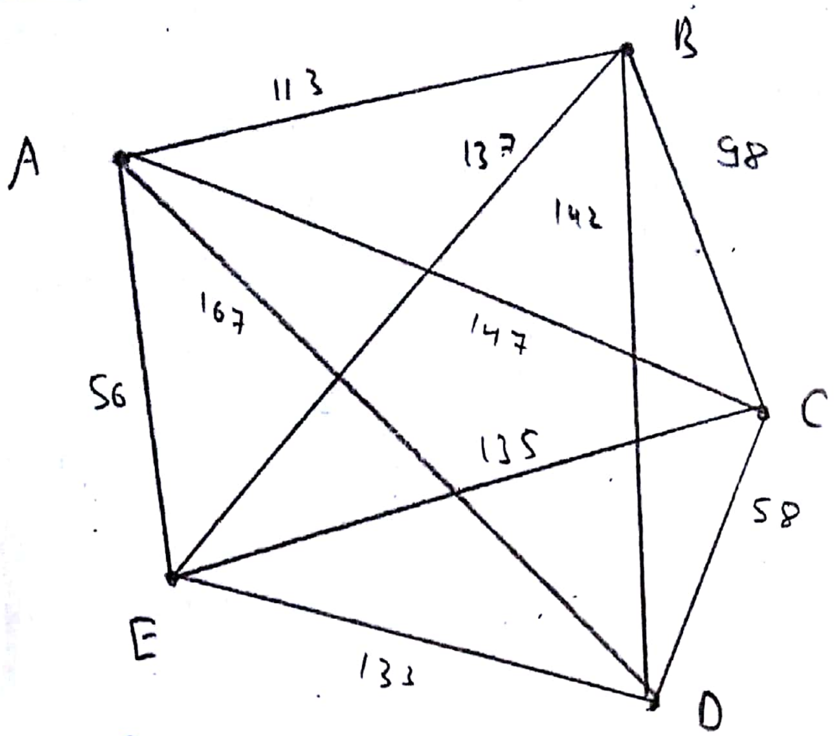
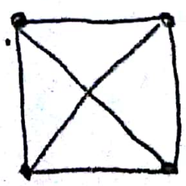


Fig. Graph. Showing distance between 5 citie.

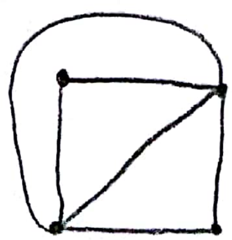
PLANAR GRAPHS

- A graph is called planar if it can be drawn is the plane without any edges crossing (where a crossing of edges is the intersection of the lines or arcs representing at a point other than their common endpoint). Such a drawing is called a planar representation of the graph.

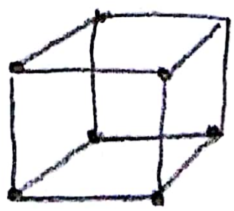
- Example:



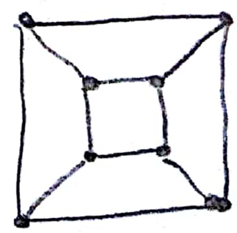
The graph K_4



K_4 drawn with No crossing



The graph Q_3



A planar representation of Q_3

- A planar representation of a graph splits the plane into regions, including an unbounded region.

→

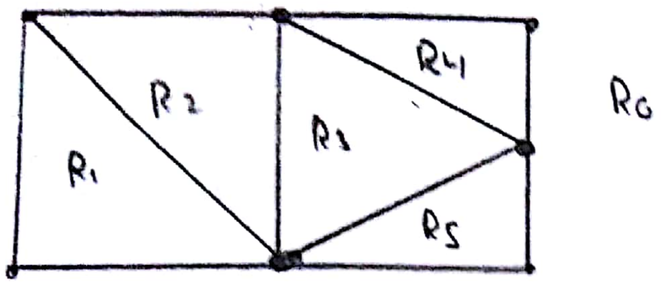


Fig. The Regions of the planar representation of a graph.

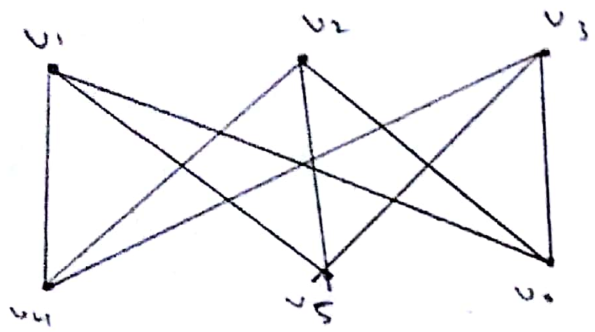
- **Euler's Formula:**
  Let G be a connected planar simple graph with 'e' edges and vertices 'v'. Let r be the number of regions in a planar representation of G. Then, $r = e - v + 2$.

- **Corollary I:** If G is a connected planar simple graph with e edges and v vertices, where $v \geq 3$, then $e \leq 3v - 6$.

- **Corollary II:** If G is a connected planar simple graph, then G has a vertex of degree not exceeding five.

- **Corollary III:** If a connected planar simple graph has e edges and vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

<u>Ex</u> Show that $K_{3,3}$ is nonplanar.



The Graph $K_{3,3}$

<u>Solution</u>

- Because $K_{3,3}$ has no circuits of length three (as it is bipartite).

  $K_{3,3}$ has 6 vertices and nine edges.

  $v = 6 \quad e = 9$

  $e \leq 2v - 4 \quad \Rightarrow$ Corollary III

  $9 \leq 2 \times 6 - 4$

  $9 \leq 8 \quad \times$

  According to Corollary III $K_{3,3}$ is not planar.

<u>GRAPH COLORING</u>:

- A <u>coloring</u> of a simple graph is the assignment of color to each vertex of the graph so that no two ~~vertices~~ adjacent vertices are assigned the same color.

- The chromatic number of a graph is the least number of colors need for a coloring of this graph. The chromatic number of a graph is denoted by $\chi(G)$.

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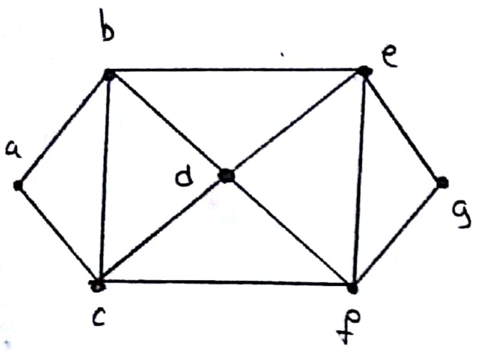
The FOUR COLOR THEOREM:

The chromatic number of a planar graph is no greater than four.

Example What are the chromatic numbers of the following graphs?



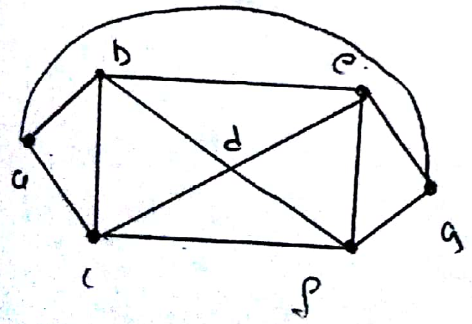
G

- The chromatic number of G is atleast three, because the vertices a, b and c must be assigned different colors.
- Let,

a → Red
b → blue
c → green

- The d can (and must) be colored in Red because it is adjacent to b (blue) and c (green).
- e can (and must) be colored in green.
- f can (and must) be colored in blue.
- g must be colored in red.

⇒



[+1]

Example: What is the chromatic number of K_n?

$\chi(K_n) = n$.

or Chromatic number of $K_n = n$.

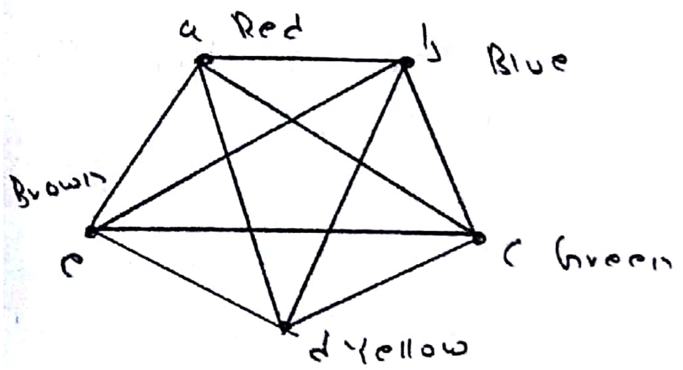


Fig: Coloring of K_5

Example: What is the chromatic number of the complete bipartite graph $K_{m,n}$, where m and n are positive integers?

Solution:

$\chi(K_{m,n}) = 2$

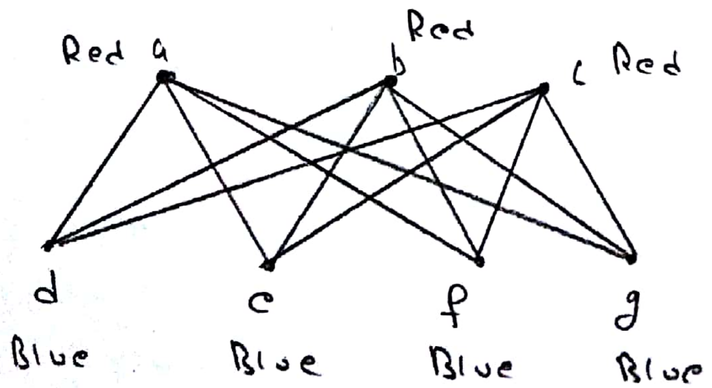


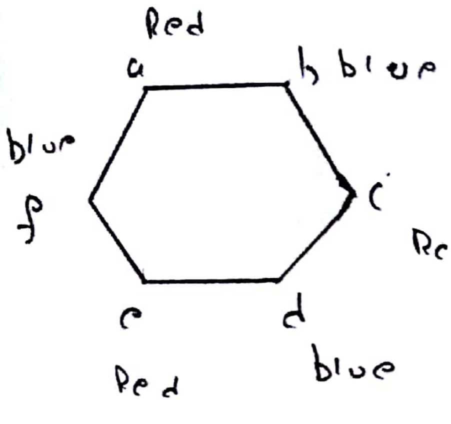
Fig: A coloring of $K_{3,4}$.

Example: What is the chromatic number of the graph C_n, where $n \geq 3$? (C_n is the cycle with n vertices).

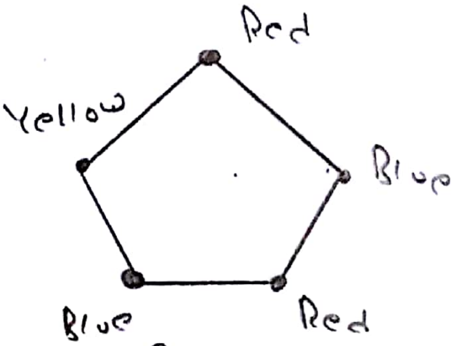
Solution:

$\chi(C_n) = 2$, if n is even positive integer $n \geq 4$.

$\chi(C_n) = 3$ if n is an odd positive integer, $n \geq 3$.



C_6



C_5

Applications of Graph Coloring:

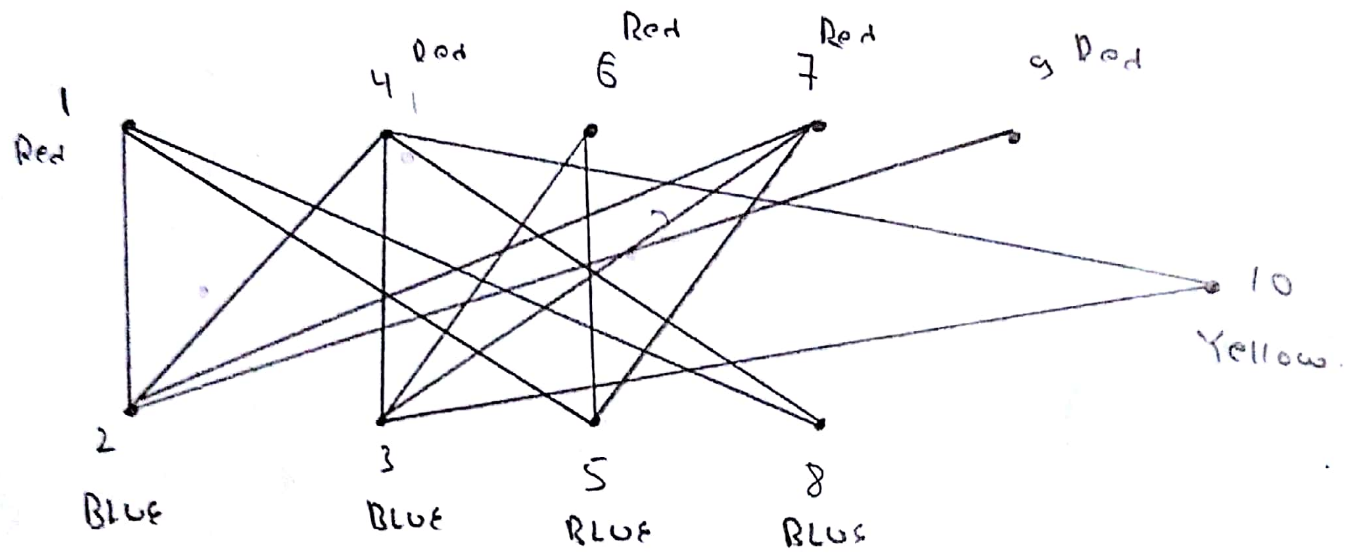
Example: Scheduling Final Exams:

How can the final exams at a university be scheduled so that no student has two exams at the same time?

Solution: This scheduling problem can be solved using a graph model, with vertices representing cources and with an edge between two vertices if there is common student in the courses they represent. Each time slot for final exam is represented by a different color.

- There are total 10 subjects
  and the pairs

{(1,2), (1,5), (1,8), (2,4), (2,9), (2,7), (3,6), (3,7), (3,10),
(4,8), (4,3), (4,10), (5,6), (5,7)}



Time period

I	1, 4, 6, 7, 9
II	2, 3, 5, 8
III	10