COL351-Assignment1

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5th August 2021

1 Minimum Spanning Tree

1.1 Part a

Claim: G contains a unique minimal spanning Tree.

Proof by contradiction:
Assume for the sake of contraction that G contains two Minimum Spanning Trees T_1 and T_2. Enumerate the edges of T_1 and T_2 as follows:

$$E(T_1) = \{e_1, e_2, e_3, ..., e_m\}, where e_1 < e_2 < ... < e_m \tag{1}$$

$$E(T_2) = \{g_1, g_2, g_3, ..., g_m\}, where g_1 < g_2 < ... < g_m \tag{2}$$

Let g_k be the minimum edge in $T_2 \setminus T_1$ and Let e_k be the minimum edge in $T_1 \setminus T_2$. Without loss of generality, assume that wt(e_k) < wt(g_k).
Then, $T_2 \cup e_k$ contains a cycle C, passing through edge e_k. Let u be any edge of this cycle not in T_1. Atleast one such edge must exist because T_2 is a tree. Because $e_k \in T_1$, we can say that u $\neq e_k$ and thus, u $\in T_2 \setminus T_1$.

Now consider the spanning tree T = T_2 + $\{e_k\}$ - {u}. wt(T) = wt(T_2) + wt(e_k) - wt(u) $\leq$ wt(T_2). Now, since T_2 is a minimum spanning tree, we conclude that T is also a minimum spanning Tree.
It automatically follows that wt(e_k) = wt(u), which contradicts our assumption that all the edges of G are unique and $u \neq e_k$.

1.2 Part b

1.2.1 Algorithm

Approach:

- Assuming we have the adjacency list in the input, our algorithm uses a map which stores all the edges taken into account.
- Using the DFS function, we compute a spanning tree dfs returns a spanning tree (if graph is connected), it need not be the minimum one.
- We have a list named **NotIncluded** which stores the edges ignored by our DFS algorithm. For each such edge in **NotIncluded**, we insert it to the spanning tree, find the cycle formed (graph with n vertices and n edges will have a cycle) and removes the edge with the maximum weight.

Algorithm 1

```
procedure DFS(adjList, visited, map, root)
    visited[root] = true

    for all neighbour u of root do
        if not visited[u] then
            Set map(root, u) = 1                         ▷ include this edge in the tree
            DFS(adjList, visited, map, u)
        end if
    end for

    return map                                           ▷ edges included in the tree
end procedure
```

Algorithm 2

```
procedure Cycle_Detection(adjList, map, parent, visited, root)
    visited[root] = true

    for all neighbour u of root do
        if not visited[u] then
            parent[u] = root
            Cycle_Detection(adjList, map, parent, visited, root)
        end if

        if u != parent[root] and parent[u] != root then      ▷ Cycle Detected
            maxEdge ← Compute_Max_edge_in_Cycle              ▷ backtracing over parent list
            map(maxEdge) ← 0                                 ▷ remove the maxEdge from the tree
            return                                           ▷ End the procedure
        end if
    end for
end procedure
```

Algorithm 3

```
procedure MINIMUM SPANNING TREE(Edges, adjList, N)
    visited ← [0, 0, ... 0]                                   ▷ length N
    map
    map ← DFS(adjList, visited, map, root)      ▷ arbitrarily set any vertex to be root vertex
                                                ▷ DFS returns a Spanning tree (if connected)
    NotIncluded ← []            ▷ edges not included in the Spanning tree returned by DFS
    for all edge in Edges do
        if map(edge) != 1 then
            NotIncluded.insert(edge)
        end if
    end for

    parent ← [-1, -1, ... -1]                             ▷ stores parent of all nodes
    visited ← [0, 0, ... 0]                               ▷ marks 1 if node is visited
    for all edge in NotIncluded do
        map(edge) = 1
        Cycle_Detection(adjList, map, parent, visited, root)   ▷ deletes max edge in cycle
    end for
    return map                                                 ▷ MST of G
end procedure
```

1.2.2 Time Complexity/Space Complexity

Time complexity of the algorithm: Time complexity of DFS + Time complexity of the first for loop + Time complexity of the second for loop

- It takes O(n+m) time to run DFS on a graph. As edges are O(n), this implies, time complexity: O(n). First for loop iterates over the edge list: O(n)
- Max length of NotIncluded list can be 9 (DFS returns spanning tree and spanning tree contains n-1 edges)
- In each iteration, we run the Cycle_Detection procedure which is again O(n+m) (DFS algo with slight variation), and as m $\leqslant$ n+9, this implies, time complexity: O(n)

Overall Time Complexity O(n). Space Complexity: O(n)

1.2.3 Proof of Correctness

Claim: Largest weight edge in any cycle of Graph G is not contained in MST of G.
Proof: For simplicity, assume that all the edges have distinct weight. Suppose T be the obtained MST of G. Let e $\in E(G)\backslash E(T)$ such that $T \cup \{e\}$ forms a cycle. Let g be the largest weight edge $\in$ E(C) not equal to e. Let T' = $T\backslash\{g\} \cup \{e\}$.
wt(T') = wt(T) - wt(g) + wt(e) < wt(T), which contradicts that T is an MST of G.

Let T be the spanning tree obtained after termination of the algorithm.
Claim: Let e $\in$ E(G) $\setminus$ E(T). Let C denote the cycle formed in $T \cup \{e\}$.
Then $wt(e)$ = max(wt(g)) $\forall$ g $\in$ E(C).
Proof: We shall prove this by contradiction. Suppose $\exists$ g $\in$ E(C) such that wt(g) > wt(e) .
This creates a counter arguement because we should'nt have removed the edge e from E(T) as per our algorithm, which removes the largest weight edge in a cycle for some iteration.
Using the above claim, we can argue that the obtained Spanning Tree is minimal in nature.

2 Huffman Encoding

2.1 Part a

Fibonacci sequence follows the relation $F_{n+1} = F_n + F_{n-1}$
We consider the following algorithm for constructing Huffman tree from given Fibonacci Sequence.

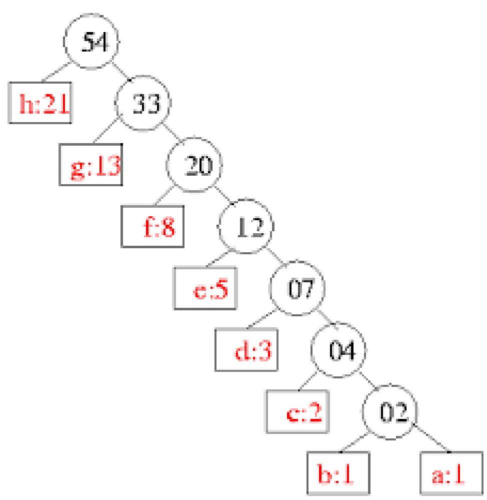
- Create leaf nodes for each character in priority queue(min heap). The value of frequency field is used to compare two nodes in min heap. Initially, the least frequent character is at root.
- Extract two nodes with the minimum frequency from the min heap.
- Create a new internal node with a frequency equal to the sum of the two nodes frequencies. Make the first extracted node as its left child and the other extracted node as its right child. Add this node to the min heap.
- Repeat the above steps till only one node is left in the min heap.

Claim:

$$\sum_{i=1}^{n} a_i < a_{n+2} \tag{3}$$

Proof:
Base case: $a_1 < a_3$
Assume it is true for n, Add a_{n+1} to both the sides, we can show that it is satisfied for n+1.

Claim: There are n-1 internal nodes, all belonging to the set $\{S_2, S_3, ... , S_n\}$, where $S_n = \sum_{i=1}^{n} a_i$.
Proof: After j^{th} iteration of algorithm, we have the following frequencies in our queue:
$\sum_{i=1}^{j+1} a_i$, a_{j+2}, a_{j+3} , ... , a_n. Using the above claim, we have shown that $\sum_{i=1}^{j+1} a_i$ and a_{j+2} are the lowest frequency nodes in our queue.
So, at the j^{th} iteration, we remove both these frequencies and add the frequency $\sum_{i=1}^{j+2} a_i$ in our queue.
The resulting huffman tree after all iterations looks like:



Therefore, the encoding of nodes with least frequencies are [(n-2)*"1" + "1"] and [(n-2)*"1" + "0"].
Other node frequencies are [(n-i)*"1" + "0"], i ranging from 3 to n.

2.2 Part b

We can formulate the given problem as :
Given a sequence of characters $\{a_1, a_2, ... , a_{2^m}\}$, where m = 16 in the given question. We have to prove that compression obtained by huffman encoding is same as fixed length encoding.

Assume for the sake of simplicity that the given sequence is present in sorted order of their frequencies, i.e.

$$f_1 \geq f_2 \geq ... \geq f_{2^m} \tag{4}$$

Claim 1 : The huffman tree obtained by the given sequence is full binary tree.
Proof: Proof by induction:

Denote,

$$F_k = \{f_1 + ... + f_{2^{m-k}*1}, f_{2^{m-k}+1} + ... + f_{2^{m-k}*2}, ..., f_{2^{m-k}*[2^k-1]+1} + ... + f_{2^m}\} \tag{5}$$

P(k) denotes the predicate such that all the nodes in F_k after 2^k / 2 iterations (where 2^k is the size of F_k) combine and generates F_{k-1} with exactly half the nodes present in F_k.

$$F_m = \{f_1, f_2, ..., f_{2^m}\} \tag{6}$$

Base case: k = m
Claim 2: For any two nodes f_n and f_{n-1}, the frequency of their parent node (i.e. $f_n + f_{n-1}$) is greater than all the nodes present in F_m.
Proof: Say f_n and f_{n-1} denote the minimum frequencies among all the nodes in F_m and f_1 be the one with maximum frequency.
Then, according to the given conditions, $f_1 < 2f_n \leqslant f_n + f_{n-1} \implies f_c < f_a + f_b \; \forall \; \{f_a, f_b, f_c\} \in F_m$
Using the above claim, we can argue that at any iteration of Huffman algorithm, $\exists$ unvisited f_i , f_j $\in F_k$ such that f_i, $f_j < f_k \; \forall \; f_k \in F_{k-1}$.

Using claim 2, We can conclude that after 2^m / 2 iterations, we visit all the frequencies from F_k and generate the set

$$F_{m-1} = \{f_1 + f_2, f_3 + f_4, ..., f_{n-1} + f_n\} \tag{7}$$

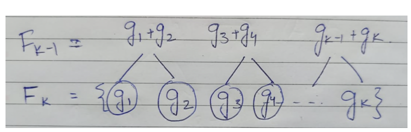
Induction step: P(k) $\implies$ P(k-1)
Say,

$$F_k = \{g_1, g_2, ..., g_k\} \tag{8}$$

Using claim 2, we can argue similarly and obtain F_{k-1} from F_k after $2^k/2$ iterations

$$F_{k-1} = \{g_1 + g_2, g_3 + g_4, ..., g_{k-1} + g_k\} \tag{9}$$

By using mathematical Induction, we claim P(1) to be true, i.e. we are left with one root node and the end when the algorithm terminates. We conclude that F_k contains nodes at kth level and number of nodes at each level i = 2^i. Thus the obtained huffman tree is complete and each f_i is present at same level. Length of encoded sequence = nlog(n), where n are total characters in the sequence, which is the same length as obtained by fixed length encoding.



3 Graduation Party of Alice

3.1 Part a

3.1.1 Algorithm

We are given n people and a list of pairs of people who know each other. Mathematically, we can formulate the problem as:
Given: A graph G with n vertices and m edges, we need to find the largest subset S(V,E) of G satisfying the following contraints:

- $\forall u \in S(V), \exists$ at least 5 v $\in S(V)$ such that (u,v) $\in S(E)$
- $\forall u \in S(V), \exists$ at least 5 w $\in S(V)$ such that (u,w) $\notin S(E)$

Algorithm 4

```
procedure ADJACENCY_LIST(Edges, N)
    adjList ← [{}, {}, ... {}]                ▷ list of unordered_set (length N)
    for i from 1 to size(Edges) do
        adjList[Edges[i][0]].insert(Edges[i][1])
        adjList[Edges[i][1]].insert(Edges[i][0])
    end for
    return adjList
end procedure
```

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Algorithm 5

```
procedure PEOPLETOINVITE(Edges, N)            ▷ People who know each other
    adjList ← ADJACENCY_LIST(Edges, N)
    RemPeople ← N
    GuestToRemove ← True
    PeopleList ← [1, 1, ....., 1]             ▷ length N

    while GuestToRemove = True do
        GuestToRemove = False
        NewlyRemoved ← []                     ▷ Will store the list of people removed in 1 iteration
        for i from 1 to N do
            if PeopleList[i] = 1 and (size(adjList[i]) < 5 or size(adjList) > RemPeople - 5) then
                GuestToRemove ← True
                PeopleList[i] ← 0
                RemPeople ← RemPeople −1      ▷ updating the size of guest list
                NewlyRemoved.insert(i)
            end if
        end for

        if GuestToRemove = True then
            for i from 1 to N do
                if PeopleList[i] = 0 then
                    continue
                end if
```

```
            for j from 1 to size(NewlyRemoved) do
                if adjList[i].find([NewlyRemoved[j]]) != adjList[i].end() then
                    adjList[i].erase(NewlyRemoved[j])                ▷ O(1) (unordered_set)
                end if
            end for
        end for
    end if

end while

GuestList ← []                                                       ▷ Empty List

for i from 1 to N do
    if PeopleList[i] = 1 then
        GuestList.insert(i)
    end if
end for

return GuestList                                                     ▷ Party Invitees
end procedure
```

3.1.2 Time Complexity/Space Complexity

Time Complexity of the algorithm = Time Complexity of the Adjacency_List procedure + Time Complexity of the While loop + Time Complexity of the for loop

- Time Complexity of the Adjacency_List procedure is O(length(Edges)) i.e. O(n^2) (considering input to be a dense graph)
- Time Complexity of the last for loop is O(n)
- While Loop
  - The while loop will run atmost n times, and, in each iteration, the first for loop runs O(n) times: n*O(n) -> O(n^2)
  - For every person added to the **NewlyRemoved** list, the algorithm runs a loop over the **adjList** list (size n) and removes the edge(if present) incident on the person in the **NewlyRemoved** list
  - The loop over the **adjList** helps to remove all such edges and it takes O(1) time (unordered_set) to remove 1 such edge
  - So, we can insert atmost n persons to the **NewlyRemoved** list, and, for each such person, we will loop over the **adjList** (size n), removal of an edge takes O(1) time
  - Therefore, Time Complexity of the while loop will be: O(n^2) + n*O(n)*O(1) -> O(n^2)
- Overall Time Complexity: O(n^2) + O(n) + O(n^2) -> **O(n^2)**

Space Complexity of the algorithm: **O(n^2)** (For storing the adjacency List), other lists can be stored in O(n)

3.1.3 Proof of Correctness

Let G be the graph at the beginning of the while loop, and, G' be the graph at the end of while loop. Let opt(G) denote the optimal subset S $\subseteq$ G (i.e. of maximum possible size) satisfying the mentioned constraints

Claim:

$$opt(G) = opt(G') \tag{10}$$

Proof:

- To obtain G' from G, we remove only those vertices or people who do not satisfy the constraints of the problem
- So, we remove only those vertices whose degree is less than 5 (we need to know atleast 5 people) or whose degree is greater than size(G(V)) - 5 (we need atleast 5 people whom we don't know)
- This implies the above algorithm always produces the largest subset of vertices or people who can be invited to the party
- Also, any vertex which is a solution for G, will also be a solution for G' i.e. it will not be removed (cause, if removed from going from G to G', then it won't be a solution for G also)
- This implies, if u $\in$ opt(G) then u $\in$ opt(G') i.e. opt(G) $\subseteq$ opt(G')
- Also, as G' is a subset of G, this implies opt(G') $\subseteq$ opt(G)
- Hence, opt(G) = opt(G')

Proof of termination: At each iteration, we are removing atleast 1 guest from Alice's invitee list (If there is no such guest, then our algo terminates). As the number of people(n) are finite, this guarantess the **termination of the algorithm**.

3.2 Part b

3.2.1 Algorithm

Given: N people and their respective age, we need to find minimum number of tables to accomodate them such that:

- Each table has a capacity of 10 people
- Age difference between members of the same table should be atmost 10

Approach: We will store the count of persons with a particular age in a list (AgeList), we will have a variable currAge which will start from MinAge (10 in this case). We will greedily choose people starting from Minage. We will increment currAge until we find non-zero value of AgeList[currAge]. Starting from currAge, we will try to go to currAge + 9 (admissible age on a table) and try to fill the table.

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Algorithm 6

```
procedure MinTables(PersonAge, N)                         ▷ Replacing n_0 with N
    Tables ← 0
    MaxAge ← 99
    MinAge ← 10
    AgeGap ← 10
    TableCap ← 10
    RemPeople ← N
    AgeList ← [0, 0, ....., 0]                  ▷ length = MaxAge - MinAge + 1
    for i from 1 to N do
        AgeList[PersonAge[i] - MinAge] ← AgeList[PersonAge[i] - MinAge] + 1
    end for

    currAge ← MinAge

    while currAge ⩽ MaxAge and RemPeople ⩾ 0 do

        while currAge ⩽ MaxAge and AgeList[currAge] = 0 do
            currAge ← currAge + 1
        end while

        if currAge > MaxAge then
            return Tables
        end if

        startAge ← currAge
        lastAge ← currAge
        peopleCount ← 0

        while peopleCount ⩽ TableCap and lastAge - startAge ⩽ AgeGap do
            peopleOfThisAge ← min(TableCap - peopleCount, AgeList[lastAge - MinAge])
            AgeList[lastAge - MinAge] ← AgeList[lastAge - MinAge]) - peopleofThisAge
            peopleCount ← peopleCount + peopleOfThisAge
            if AgeList[lastAge - MinAge] = 0 then
                lastAge ← lastAge + 1
```

```
            end if
            if lastAge > MaxAge then
                break
            end if
        end while

        RemPeople ← RemPeople - peopleCount
        Tables ← Tables + 1                          ▷ alloting a table to the chosen persons
        currAge ← lastAge
    end while

    return Tables                                    ▷ Minimum Tables
end procedure
```

3.2.2 Time Complexity/Space Complexity

Time Complexity of the algorithm = Time Complexity of the for loop + Time Complexity of the While loop

- Time Complexity of the for loop is O(n)
- While Loop
  - In each iteration, either we decrement the remaining people by 10 i.e. RemPeople ← RemPeople - 10 or we will increment the currAge by atleast 10
  - This is because, for a given table, either we fill it completely(if possible) i.e. decreasing the remaining people by 10, else, our currAge will be incremented by 10 because we would have looked at all admissible ages at this table which is [currAge, currAge + 9]
  - So, time complexity of the while loop is O((n/10) + (MaxAge - MinAge)/10) i.e. $O((n/10)+$ 9.9)
  - Therefore, Time Complexity of the while loop will be: O(n)
- Overall Time Complexity: O(n) + O(n) -> O(n)

Space Complexity of the algorithm: O(n) (considering the input space), otherwise, space complexity will be O(MaxAge - MinAge).

3.3 Proof of Correctness

Let L be the person's list at the beginning of the while loop, and, L' be the person's list at the end of the while loop. Let opt(L) denote the minimum number of tables required to satisfy the mentioned constraints

Claim:

$$opt(L) = opt(L') + 1 \tag{11}$$

Proof:

- Let k = opt(L). In a particular iteration, the algorithm accomodates only those people (atmax 10 (table capacity)) on a table which satisfy the age constraint (difference at most 10)
- After the iteration, we obtain L' and have a valid solution of k-1. This implies, opt(L') ⩽ k - 1 i.e. opt(L') ⩽ opt(L) - 1

- Now, suppose p = opt(L'). L' is obtained by removing certain number of people from L and assigning them a single table. Inserting all those elements back to L' gives us L and a valid solution for L of size p+1
- This implies, opt(L) $\leqslant$ p+1 i.e. opt(L) $\leqslant$ opt(L') + 1. This gives us opt(L') + 1 $\leqslant$ opt(L) $\leqslant$ opt(L') + 1
- Hence, opt(L) = opt(L') + 1

Proof of termination: At each iteration, we are assigning a table to atleast 1 guest. As the number of people(n) are finite, this guarantess the **termination of the algorithm**.