(Q8) Let $f: R \to R$ be a continuous function such that $\forall x \in R$ and $\forall t \geq 0$,

$$f(x) = f(e^t x).$$

Show that f is a constant function. (UGB-2018)

Given, $f(x) = f(e^t x)$ —— (I)

Case — I now, for $t \in [0, 1)$, i.e., for fractional values of t,

$$f(x) = f(e^f x) \quad , \quad f = t \in [0, 1) \text{ —— (II)}$$

Case — II, when $t \in [1, \infty)$,

$$f(x) = f(e^t x) = f(\infty) \ (\text{when } t \to \infty) \text{ —— (III)}$$

and since $f(x)$ and $f(e^t x)$ is continuous for all $x \in R$ and $\forall t \in [0, \infty)$,

it means at each and every point, limit exists and limit value is finite.

So, eqns (II) = eqn (III) ~~= eqn (III)~~ = $f(x)$

$$f(e^t x) = f(\infty) = f(x)$$

$$\Rightarrow f(\infty) = f(e^f x) = f(e^t x) \quad , t \geq 0, \forall x \in R$$

$\therefore$ f is a constant function.

$\sin\left(\frac{x+y}{2}\right) = 0$ and ——— (I)

$|x| + |y| = 1$ ——— (II)

so, pair of (x, y) must be satisfying both eqn (I) & (II).

$\because \sin\left(\frac{x+y}{2}\right) = 0$

$\therefore \frac{x+y}{2} = n\pi$

$\Rightarrow x+y = 2n\pi$

$\Rightarrow y = -x + 2n\pi$ ——— (III) where $n \in I$

and $|x| + |y| = 1$

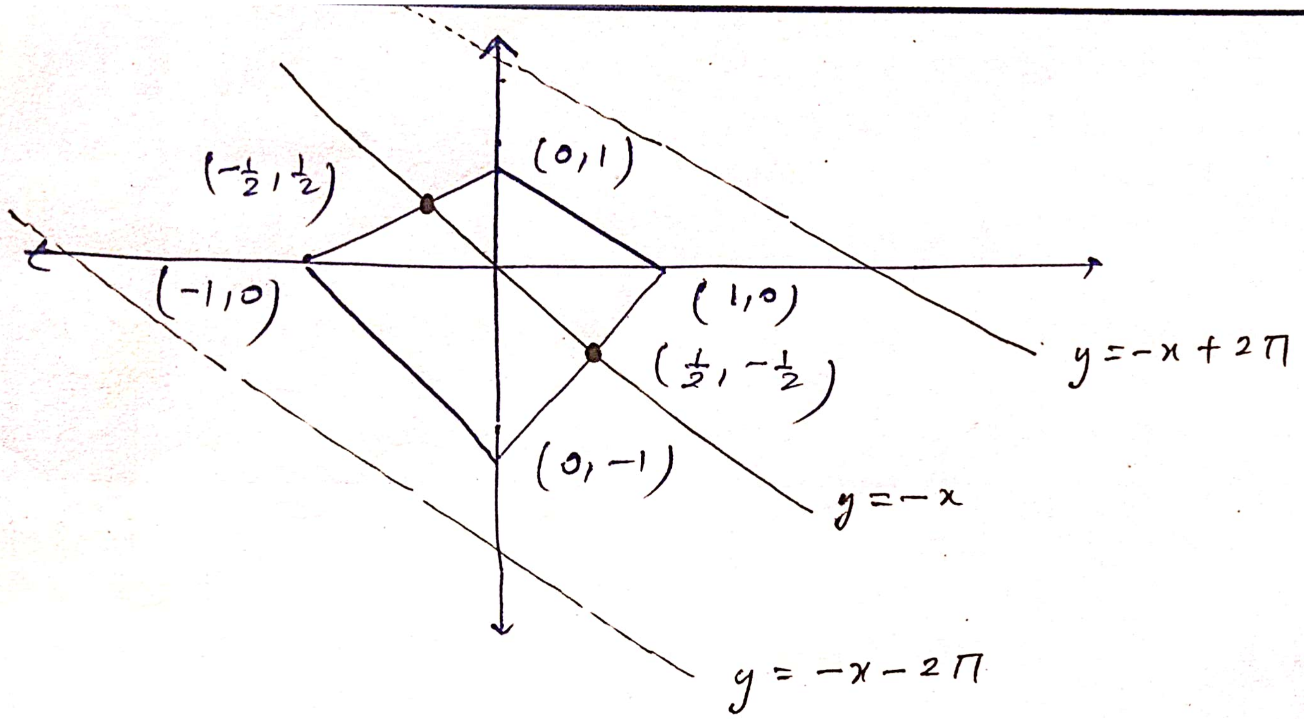
so, sketching the graph of eqn (II) and eqn (III)

eqn (II) $\longrightarrow$ $|y| = 1 - |x|$ $\begin{cases} 1^{st} \text{ quad, } y = 1 - x \\ 2^{nd} \text{ quad, } y = 1 + x \\ 3^{rd} \text{ quad, } y = -1 - x \\ 4^{th} \text{ quad, } y = x - 1 \end{cases}$

eqn (III) $\longrightarrow$ for, $n = 0$, $y = -x$

for $n = 1$, $y = -x + 2\pi$

for $n = -1$, $y = -x - 2\pi$



so, there are only two intersecting points, i.e.,

in 2nd quad $\longrightarrow$ $(-\frac{1}{2},\frac{1}{2})$

4th quad $\longrightarrow$ $(\frac{1}{2},-\frac{1}{2})$

$\}$ — mirror image about $y=x$

and $\because$ for only $|n|=1$, graph of eqn (iii) and graph of eqn (ii) do not intersect

so, ~~if |n|~~ for $|n|\geq 1$, its true that there will be no other solutions.

$\therefore$ we have only two pairs of (x,y)

i.e.; $\left(-\frac{1}{2},\frac{1}{2}\right)$ & $\left(\frac{1}{2},-\frac{1}{2}\right)$