(Q8) Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\forall x \in \mathbb{R}$ and $\forall t \geq 0$,

$$f(x) = f(e^t x).$$

Show that f is a constant function. (UGB-2018)

Given, $f(x) = f(e^t x)$ —— (I)

case — I now, for $t \in [0, 1)$, i.e., for fractional values of t,

$$f(x) = f(e^t x) \quad , \quad t = t \in [0, 1) \text{ —— (II)}$$

case — II, when $t \in [1, \infty)$,

$$f(x) = f(e^t x) = f(\infty) \quad (\text{when } t \to \infty)$$

└—— (III)

and since $f(x)$ and $f(e^t x)$ is continuous for all $x \in \mathbb{R}$ and $\forall t \in [0, \infty)$,

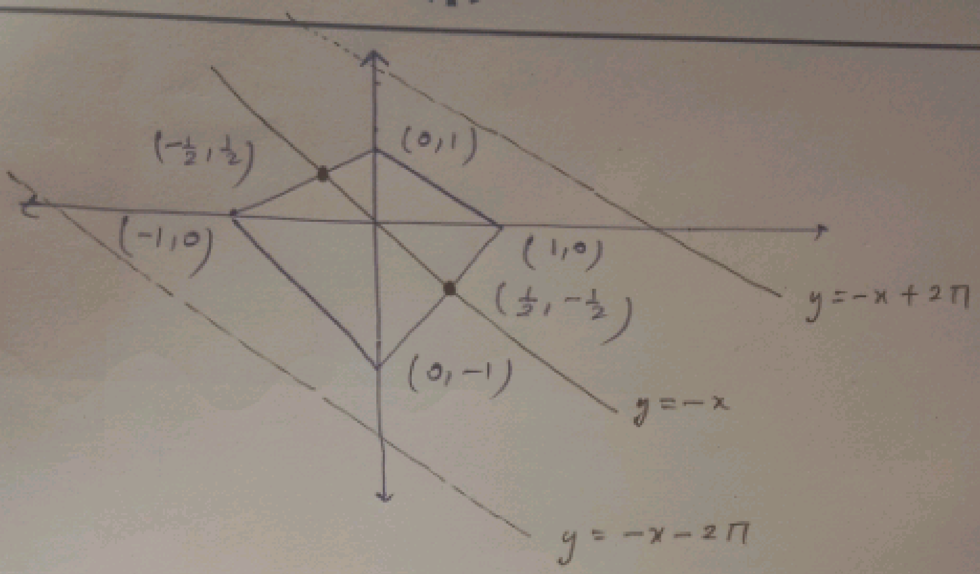
it means at each and every point, limit exists and limit value is finite.

So, eqns (II) = eqn (III) $\cancel{= \text{eqn (III)}}$ = $f(x)$

$$f(e^t x) = f(\infty) = f(x)$$

$$\Rightarrow \quad f(\infty) = f(e^t x) = f(e^t x) \quad , \; t \geq 0, \forall x \in \mathbb{R}$$

$\therefore$ f is a constant function.



so, there are only two intersecting points, i.e;

in 2^{nd} Quad ⟶ $\left(-\frac{1}{2},\frac{1}{2}\right)$

4^{th} Quad ⟶ $\left(\frac{1}{2},-\frac{1}{2}\right)$ } ← mirror image about $y=x$

and $\because$ for only $|n|=1$, graph of eqn (iii) and graph of eqn (ii) do not intersect

so, ~~if~~ for $|n|\geq 1$, its true that there will be no other solutions.

$\therefore$ we have only two pairs of (x,y)

i.e; $\left(-\frac{1}{2},\frac{1}{2}\right)$ & $\left(\frac{1}{2},-\frac{1}{2}\right)$