Assignment 1

→ lower triangular

① $Lx = b$; $L_{n\times n}$; $n = 10^5$, $b_{10^5 \times 1}$

$[L|b]$ → gaussian elemination is one way to solve this

↳ [even if we could write it well, we would need atleast two rows stored in memory for elementary row operations]

Approach - 1 : Gauss elemination.

BUT → lower △ A so we can solve this with one row

$$\begin{bmatrix} a_{11}(1) & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ \\ \end{bmatrix}$$

$b_1 \to b_1/a_{11} \to b_1$

$b_2 \to (b_2 - a_{21}b_1)/(a$

$b_3 \to (b_3 - a_{31}b_1 - a_{32}b_2)/$

⋮

∴ with just one row and b we can perform row echlon.

$\boxed{\text{Assuming we can store vector b in memory along with a row of A}}$

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Algo:

for i in range $[1, 10^5]$:

read row i

~~for j in range (1,~~

$b_i = (b_i - a_{i1}b_1 - a_{i2}b_2 \cdots a_{i,i-1}b_{i,i-1}) /$

$(a_{ii} - a_{i1} - a_{i2} \cdots - a_{i,i-1})$

end for

write b

$\therefore$ 1 write is required (b_{new} to memory)

$a_{22} - a_{21}) \Rightarrow$

$(a_{33} - a_{31} - a_{32})$

(all of this is possible because A is L)

(2) : $\begin{bmatrix} a & a & a_0 \end{bmatrix}$ 5×2 non 0

$\Rightarrow$ row 5 + α row 2

lly for 3×3 - - $n \times n$; eigen values

= a_{11}, a_{22} - - - a_{nn}

query1: elem row1, col1 (element at)

query2: elem row2, col2

⋮

" n : elem rown, coln

gives n eigen values ∴ B is correct

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① Assuming we can only store one row of A and not entire B

Algo: for i in range [1, 10^5]

read row i

for j in range [1, i-1]

read bj

bi -= aijbj

aii -= aij

del bj

end for

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$b_i = b_i / a_{ii}$

write b_i → ①

del b_i

end for

we write individual elements of $\vec{b_i}$:

⇒ n writes required (10^5 writes)

C → n×n : $C^T C = C C^T$

$(a_{ii} - a_{i1} - a_{i2} \cdots - a_{i,i-1})$

end for

write b

$\boxed{\because \text{ 1 write is required (b/w to memory)}}$

$a_{22} - a_{21}) \ldots$

$(a_{33} - a_{31} - a_{32})$

(all of this is possible because A is L)

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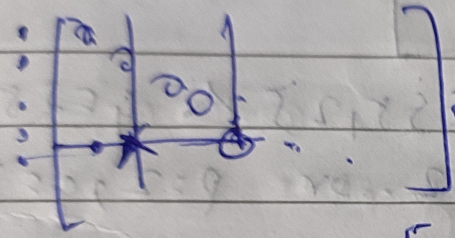
(2)



5×2 non 0

⇒ row 5 + α row 2

E ?

$$E_{ij} A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_5 + \alpha A_2 \\ \vdots \\ A_{50} \end{bmatrix} \text{ for some } \alpha$$

we want set

$\{E_{pq}\}$ s.t. $E_{pq} E_{ij} A = E_{ij} E_{pq} A$

$$\Rightarrow E_{pq} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_5 + \alpha A_2 \\ \vdots \\ A_{50} \end{bmatrix} = E_{ij} E_{pq} \begin{bmatrix} A_1 \\ \vdots \\ A_{50} \end{bmatrix}$$

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$$E_{pq}\begin{bmatrix} A_1 \\ \vdots \\ A_5+\alpha A_2 \\ \vdots \\ A_{50}\end{bmatrix} = E_{52}E_{pq}\begin{bmatrix} A_1 \\ A_2 \\ A_5 \\ \vdots \\ A_{50}\end{bmatrix}$$

clearly $p, q \in \{1, 50\} / \{5, 2\}$ is a ~~solution~~ subset of solution

as it gives

$$\text{rhs} = E_{52}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_5 \\ \vdots \\ A_{49}+\beta_{50} \\ A_{50}\end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_5+\alpha A_2 \\ \vdots \\ A_{49}+\beta A_{50} \\ A_{50}\end{bmatrix}$$

$$\& \text{ LHS} = \begin{bmatrix} A_1 \\ A_5+\alpha A_2 \\ \vdots \\ A_{49}+B_{50} \\ A_{50}\end{bmatrix} \quad \therefore \text{LHS} = \text{RHS}$$

Now; case when $p \in \{5, 2\}$ & $q \in \{5, 2\}$

ie $p = 5, q = 2$ or $p = 2, q = 5$

case 1 $p = 5, q = 2$

Q

$$\begin{pmatrix} \vdots \\ A_5+\alpha A_2 \\ \vdots \\ A_{50} \end{pmatrix} \quad \begin{pmatrix} A_2 \\ A_5 \\ \vdots \\ A_{50} \end{pmatrix}$$

Clearly $p,q \in \{1,50\} / \{5,2\}$ is a subset of solution

are at given

$$\text{rhs} = E_{52}\cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_5 \\ \vdots \\ A_{49}+\beta_{50} \\ A_{50} \end{bmatrix} = \begin{bmatrix} A_1 \\ \vdots \\ A_5+\alpha A_2 \\ \vdots \\ A_{49}+\beta A_{50} \\ A_{50} \end{bmatrix}$$

$$\& \; \text{LHS} = \begin{bmatrix} A_1 \\ A_5+\alpha A_2 \\ \vdots \\ A_{49}+B_{50} \\ A_{50} \end{bmatrix} \qquad \therefore \text{LHS} = \text{RHS}$$

Now, case when $p \in \{5,2\}$ & $q \in \{5,2\}$

ie $p=5, q=2$ or $p=2, q=5$

case 1 $p=5, q=2$

$$\text{LHS} = \begin{pmatrix} A_1 \\ \vdots \\ A_5+\alpha A_2+\beta A_2 \\ \vdots \\ A_{50} \end{pmatrix} \quad \text{RHS} = \begin{bmatrix} A_1 \\ \\ A_5+\beta A_2+\alpha A_2 \\ \\ \end{bmatrix}$$

$\therefore$ LHS = RHS

Case 2: $p=2, q=5$

$$\text{LHS} = \begin{bmatrix} A_1 \\ A_2+\beta A_5+\alpha\beta A_2 \\ A_5+\alpha A_2 \\ A_{50} \end{bmatrix} \quad \text{RHS} = A_{52} \begin{bmatrix} A_1 \\ A_2+\beta A_5 \\ A_5 \\ \vdots \end{bmatrix}$$

$$RHS = \begin{pmatrix} A_1 \\ A_2 + \beta A_5 \\ \vdots \\ A_5 + \alpha A_2 + \alpha\beta A_5 \\ \vdots \\ A_{50} \end{pmatrix}$$

$\therefore LHS \ne RHS$

$\therefore$ final soln $p, q \in \{1, 50\} / \langle 5, 2 \rangle \cup \{p = 5, q = 2\}$

b) Commutativity $E_{pq} E_{ij} A = E_{ij} E_{pq} A$ doesnt work in some cases ~~because we are affecting~~ as seen from the examples above

the row ~~of~~ that gets affected first will always be ~~diff~~ different in case of $\underbrace{E_{ij} E_{ji} A}_{X}$ and $\underbrace{E_{ji} E_{ij} A}_{Y}$

i.e ~~A_i~~ ie $X_i \ne Y_i$ & $X_j \ne Y_j$

doesn't work in some

cases ~~because we are affecting~~ ~~rate~~

as seen from the examples above

the row that gets affected first will always be different in case of $\underbrace{E_{ij} E_{ji} A}_{X}$ and $\underbrace{E_{ji} E_{ij} A}_{Y}$

i.e. ~~A_{ij}~~ i.e. $X_i != Y_i$ & $X_j != Y_j$

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(3) $\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \lambda I \right) x = 0$ $\quad \begin{matrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{matrix}$

$(a_{11}-\lambda)(a_{22}-\lambda) - a_{21}a_{12}$

$a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{21}a_{12}$

$\lambda^2 - (a_{11} a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}$

[~~Need : (A^2 has squared eigen values~~)]
[properties on product of complex number
[symmetry in some sense]

So $\lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{21}a_{12}) =$

~~use cayley hamilton~~
~~$A^2 = \alpha A + \beta = 0$~~

Since each of them is a complex number
it must be of form $\alpha \pm i\beta = x_1, x_2$

$\Rightarrow$ $\frac{-b}{a}$ = sum of roots = $\frac{a_{11}+a_{12}}{}$ = real

$\alpha^2+\beta^2$ = prod = $\frac{c}{a}$ = $a_{11}a_{22}-a_{21}a_{12}$

Lets try eigenvalue decomposition:
$\hookrightarrow$ since we have 2 distinct eigenvalues &
rank = 2; $A = PDP^{-1}$

$\Rightarrow A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$

generalising; $A^n = PD^nP^{-1}$; D = diagonal matrix of eigen values

$\Rightarrow$ $\frac{-b}{a}$ = sum of roots = $\frac{a_{11}+a_{12}}{1}$ = real

$\alpha^2+\beta^2$ $\alpha\beta$ = prod = $\frac{c}{a}$ = $a_{11}a_{22}-a_{21}a_{12}$

Lets try eigenvalue decomposition.

$\hookrightarrow$ since we have 2 distinct eigenvalues & rank = 2 ; $A = PDP^{-1}$

$\Rightarrow A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$

generalising ; $A^n = PD^nP^{-1}$; D = diagonal matrix of eigen values

$\therefore A^{50}$ has x_1^{50} & x_2^{50} as eigenvalues which are ... real.

$(\alpha+i\beta)^{50}$ is suddenly real ... ~~weird~~

$(\alpha-i\beta)^{50}$

$\cancel{(A^{50}-\lambda \psi I)x = 0}$

$\cancel{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$ $\cancel{a_{11}^2 + a_{21}a_{21}}$

ith root of unity ??

Great !!! Since we want 'one' such A, we can go with nth root of unity det of U = prod diagonal entries

proof → Q4

$z_k = \cos\left(2\pi \cdot \frac{k}{50}\right)$, $\cos^2 + \sin^2 = 1$

n root of $z^n = 1$; $z_i = \cos$ [redacted] $\frac{\pi}{50} k \pm i \sin$ [redacted] $\frac{\pi k}{50}$

$z_1 = \cos$ [redacted] $\frac{\pi}{50} \pm \sin$ [redacted] $\frac{\pi}{50}$

Since we just want one A we can directly write

A = [$\cos$ [redacted] $\frac{\pi}{50} + i \sin$ [redacted] $\frac{\pi}{50}$, 0 ; 0 , $\cos$ [redacted] $\frac{\pi}{50} - i \sin$ [redacted] $\frac{\pi}{50}$]

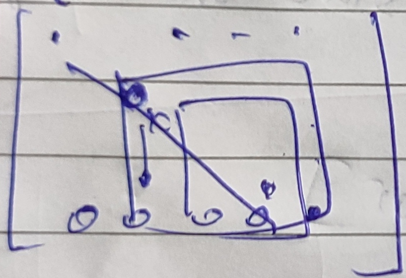
This is only possible because we do not impose real valued A

Since we just want one A we can directly write

$$A = \begin{bmatrix} \cos\frac{2\pi}{50} + i\sin\frac{2\pi}{50} & 0 \\ 0 & \cos\frac{2\pi}{50} - i\sin\frac{2\pi}{50} \end{bmatrix}$$

This is only possible because we do not impose real valued A

(4) 'equerying' ??



U, U^T

$= A$; $\det a = a_{11} \cdot \text{cof}(a_{11}) + 0$

$= a_{11}\left(a_{22} \cdot \text{cof}(a_{22}) + 0\right) + 0$ ...

iteratively; $\det A = a_{11} a_{22} \cdots a_{nn}$

i.e. prod of diagonal elements $\boxed{\text{imp}}$

also, $\prod$ eigen vals $= \det A$

Iteratively; consider 1×1 matrix $\therefore$ eigen value $= a_{11}$

Consider 2x2 U ; ∴ the eigenval = $a_{11} \cdot a_{22}$

also $\begin{bmatrix} a_{11}-\lambda & a_{12} \\ 0 & a_{22}-\lambda \end{bmatrix}$; clearly $\lambda = a_{11}$ makes det 0

∴ a_{11} is an eigen value

⇒ a_{22} is second eigen value

Illy for 3x3 - - - nxn ; eigen values
$= a_{11}, a_{22} \cdots a_{nn}$

query 1 : elem row 1, col 1 (element at)
query 2 : elem row 2, col 2
⋮
" n : elem row n, col n

gives n eigen values ∴ B is correct

write b_i → (1)
del b_i
end for

we write individual elements of b_i → :
⇒ n writes required (10^5 writes)

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5) $C \to n \times n$; $C^T C = C C^T$
$C = U \Sigma V^T$; $C^T = V \Sigma^T U^T$

$C C^T = C^T C \Rightarrow V \Sigma^T \Sigma V^T = U \Sigma \Sigma^T U^T$
$\boxed{\cancel{\text{iff } U^T U = V^T V = I}}$ ∴ $U = V$ (prove)

V, U = orthogonal matrices ; Σ = ~~m×n~~ ; $\Sigma_{ii} = \sigma_i > 0$

~~symmetric: $A^T = A$; Assume C is not symmetric~~
~~∴ $C C^T \neq C^2$?~~
~~$C = U \Sigma V^T$ $V \Sigma^T \Sigma V^T \neq U \Sigma V^T U \Sigma V^T$~~

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from sides: $C^T C = V \Sigma^T \Sigma V^T = P D P^T$
∴ eigen vectors of $C^T C = V$
||ly $C C^T = U \Sigma \Sigma^T U^T = S D S^T$
∴ eigenvectors of $C C^T = U$

but $C C^T = C^T C$ ∴ $V = U$ ∴ $C = V \Sigma V^T$

so $C^T = U^T \Sigma^T U$

$$C = U \Sigma V^T \quad := U \Sigma U^T$$

$$C^T = V \Sigma^T U^T \quad := U \Sigma^T U^T$$

$\Sigma = n \times n$ with eigen values of C diagonal

$\therefore$ transpose of diagonal matrix = matrix itself

$\therefore \Sigma^T = \Sigma \implies C = C^T$

$\therefore$ symmetric !