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Online Supplement for "Logic-based Benders Decomposition and Binary Decision Diagram Based Approaches for Stochastic Distributed Operating Room Scheduling"

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This supplement document provides all appendices referenced in the body of the paper.

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1. Proofs

1.1. Proof of Theorem 1

THEOREM 1. *The LBBD optimality cut* (6)*, which is defined as*

$$Q^s_{hdr} \geq \bar{Q}^s_{hdr} - \sum_{p \in \hat{\mathcal{P}}_{hdr}} c_p^{cancel}(1 - x_{hdpr}),$$

is valid.

Proof. Note that the cut (6) is formulated for one OR, r, in one hospital h on one day d for scenario s, thus the following argument is formed in terms of a single tuple of (h, d, r, s) and holds for each such tuple.

To prove the validity of the cut, we need to ensure two points: the cut eliminates the current master solution, and it does not exclude any globally optimal solution.

Let us first prove that the current master solution ($\hat{u}_{hd}$, $\hat{y}_{hdr}$, $\hat{x}_{hdpr}$, $\hat{w}_p$, $\hat{Q}^s_{hdr}$) violates the cut, i.e., it will be cut off. Assume towards contradiction that it satisfies the cut. When we substitute the master solution to the cut (6), the left-hand side (LHS) of the cut becomes $\hat{Q}^s_{hdr}$, while the right-hand side (RHS) of the cut becomes $\bar{Q}^s_{hdr}$ as all $\hat{x}_{hdpr}$'s in the set $\hat{P}_{hdr}$ have the value 1. This gives us $\hat{Q}^s_{hdr} \geq \bar{Q}^s_{hdr}$. However, we know that $\hat{Q}^s_{hdr} < \bar{Q}^s_{hdr}$ because otherwise the LBBD optimality cut will not be generated. This is a contradiction, therefore the current master solution does not satisfy the cut.

Next, we need to prove that any globally optimal solution, denoted by ($\hat{u}^*_{hd}$, $\hat{y}^*_{hdr}$, $\hat{x}^*_{hdpr}$, $\hat{w}^*_p$, $\bar{Q}^{s*}_{hdr}$), is not excluded by the LBBD optimality cut that is generated from a master solution ($\hat{u}_{hd}$, $\hat{y}_{hdr}$, $\hat{x}_{hdpr}$, $\hat{w}_p$, $\hat{Q}^s_{hdr}$) using $\hat{P}_{hdr}$. Here, $(\hat{u}^*_{hd}, \hat{y}^*_{hdr}, \hat{x}^*_{hdpr}, \hat{w}^*_p)$ is obtained by solving the master problem, and $\bar{Q}^{s*}_{hdr}$ is the corresponding optimal objective value of the subproblem. For a globally optimal solution, the cancellation cost obtained by the master problem, $\hat{Q}^{s*}_{hdr}$, should match that from the subproblem, $\bar{Q}^{s*}_{hdr}$.

We discuss the different cases of globally optimal solutions below:

<u>*Case 1*</u>: If $\hat{y}^*_{hdr} = 0$, i.e. the OR r in hospital h on day d is closed, then the optimal solution must have $\hat{x}^*_{hdpr} = 0, \forall p \in \mathcal{P}$ due to (1g). This in turn yields all $z^*_p = 0$ in the optimal subproblem solution due to (5c) and thus $\bar{Q}^{s*}_{hdr} = 0$. Replacing those values in the cut, we get the following:

$$\overbrace{\bar{Q}^{s*}_{hdr}}^{=0} \geq \bar{Q}^s_{hdr} - \overbrace{\sum_{p \in \hat{\mathcal{P}}_{hdr}} c^{\text{cancel}}_p (1 - x^*_{hdpr})}^{=\sum_{p \in \hat{\mathcal{P}}_{hdr}} c^{\text{cancel}}_p} \tag{$\star$}$$

As $\bar{Q}^s_{hdr}$ is the cancellation cost for the (h, d, r, s) tuple corresponding to the patient list $\hat{\mathcal{P}}_{hdr}$, we have

$$\bar{Q}^s_{hdr} \leq \sum_{p \in \hat{\mathcal{P}}_{hdr}} c^{\text{cancel}}_p$$

which follows from the fact that the cancellation cost of an OR cannot exceed the total cancellation cost of all patients assigned to it. We can now conclude that the RHS of ($\star$)

is nonpositive. Therefore, the global optimal solution $(\hat{u}^*_{hd}, \hat{y}^*_{hdr}, \hat{x}^*_{hdpr}, \hat{w}^*_p, \bar{Q}^{s*}_{hdr})$ satisfies the LBBD optimality cut.

<u>*Case 2*</u>: If $\hat{y}^*_{hdr} = 1$, then we must have some patients assigned to the (h,d,r) tuple, otherwise we can close this OR and save the cost. At the optimal patient assignment, $\hat{x}^*_{hdpr}$, we either still have all the patients in the set $\hat{\mathcal{P}}_{hdr}$ in the (h,d,r) tuple, or some patients are no longer assigned to this (h,d,r) tuple. We further discuss those two cases separately:

<u>*Subcase a*</u>: If all the patients in $\hat{\mathcal{P}}_{hdr}$ are assigned to the current (h,d,r) tuple at an optimal solution, i.e., $\hat{x}^*_{hdpr} = 1, \forall p \in \hat{\mathcal{P}}_{hdr}$, then the corresponding cancellation cost for the current patient list, $\bar{Q}^{s*}_{hdr}$, should not be lower than the cancellation cost for $\hat{\mathcal{P}}_{hdr}$, which is $\bar{Q}^s_{hdr}$. That is, we have $\bar{Q}^{s*}_{hdr} \geq \bar{Q}^s_{hdr}$. Substitute the optimal solution into the LBBD cut:

$$\overbrace{\bar{Q}^{s*}_{hdr}}^{\geq \bar{Q}^s_{hdr}} \geq \bar{Q}^s_{hdr} - \sum_{p \in \hat{\mathcal{P}}_{hdr}} c^{\text{cancel}}_p \overbrace{(1 - \hat{x}^*_{hdpr})}^{=0}$$

which holds.

<u>*Subcase b*</u>: If at the global optimal solution only a subset of patients in $\hat{\mathcal{P}}_{hdr}$ are still assigned to the current (h,d,r), the proof is more involved. For the ease of proof, we introduce some more notations. Let us denote the patients from $\hat{\mathcal{P}}_{hdr}$ who are still assigned by set $\hat{\mathcal{P}}^{A*}_{hdr} \subset \hat{\mathcal{P}}_{hdr}$, and patients in $\hat{\mathcal{P}}_{hdr}$ who are no longer assigned as $\hat{\mathcal{P}}^{N*}_{hdr} = \hat{\mathcal{P}}_{hdr} \setminus \hat{\mathcal{P}}^{A*}_{hdr}$. Also, in the global optimal solution there may exist patients who are assigned to the current (h,d,r) but do not belong to $\hat{\mathcal{P}}_{hdr}$, we denote those patients by $\tilde{\mathcal{P}}^{A*}_{hdr} \subseteq \mathcal{P}_{hdr} \setminus \hat{\mathcal{P}}_{hdr}$. Then the set of assigned patients in the global optimal solution is $\mathcal{P}^{A*}_{hdr} = \hat{\mathcal{P}}^{A*}_{hdr} \cup \tilde{\mathcal{P}}^{A*}_{hdr}$. As noted before, the optimal cancellation cost corresponding to the assignment of $\mathcal{P}^{A*}_{hdr}$ is $\bar{Q}^{s*}_{hdr}$, while the cancellation cost corresponding to $\hat{\mathcal{P}}_{hdr}$ is $\bar{Q}^s_{hdr}$. It is also useful to find the cancellation cost when only patients in $\hat{\mathcal{P}}^{A*}_{hdr}$ are assigned to the (h,d,r,s) tuple, which we denote by $\bar{Q}^s_{hdr}(\hat{\mathcal{P}}^{A*}_{hdr})$. It is easy to see that $\bar{Q}^{s*}_{hdr} \geq \bar{Q}^s_{hdr}(\hat{\mathcal{P}}^{A*}_{hdr})$, as $\hat{\mathcal{P}}^{A*}_{hdr}$ is a subset of $\mathcal{P}^{A*}_{hdr}$.

To help illustrate the relationships between patient sets, we use the following simple example in Figure 1. There is a set of six patients $\{p_1, ..., p_6\} \in \mathcal{P}$. The first three of those patients are scheduled by the current master solution to the current (h,d,r), i.e. $\{p_1, p_2, p_3\} \in \hat{\mathcal{P}}_{hdr}$. In the figure they are colored with gray. The global optimal solution schedules the patient set $\mathcal{P}^{A*}_{hdr} = \{p_2, p_3, p_4, p_5\}$. Then according to our definition, $\hat{\mathcal{P}}^{A*}_{hdr} = \mathcal{P}^{A*}_{hdr} \cap \hat{\mathcal{P}}_{hdr} = \{p_2, p_3\}$, $\tilde{\mathcal{P}}^{A*}_{hdr} = \mathcal{P}^{A*}_{hdr} \setminus \hat{\mathcal{P}}^{A*}_{hdr} = \{p_4, p_5\}$, and $\hat{\mathcal{P}}^{N*}_{hdr} = \hat{\mathcal{P}}_{hdr} \setminus \hat{\mathcal{P}}^{A*}_{hdr} = \{p_1\}$.

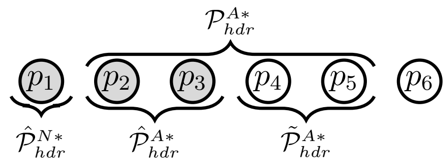

Figure 1 Illustration of relationships between patient sets

We claim that $\bar{Q}^s_{hdr}(\hat{\mathcal{P}}^{A*}_{hdr}) \geq \bar{Q}^s_{hdr} - \sum_{p\in\hat{\mathcal{P}}^{N*}_{hdr}} c^{\text{cancel}}_p$. To prove this, assume towards contradiction that it is not true. Then we have $\bar{Q}^s_{hdr}(\hat{\mathcal{P}}^{A*}_{hdr}) < \bar{Q}^s_{hdr} - \sum_{p\in\hat{\mathcal{P}}^{N*}_{hdr}} c^{\text{cancel}}_p$, meaning that if in the current OR we have all patients from $\mathcal{P}^{A*}_{hdr}$, then some patients in $\mathcal{P}^{N*}_{hdr}$ are also scheduled to this OR, the cancellation cost can increase for at most $\sum_{p\in\hat{\mathcal{P}}^{N*}_{hdr}} c^{\text{cancel}}_p$. If that is true, then the cancellation cost for the assignment $\hat{\mathcal{P}}^{A*}_{hdr} \cup \hat{\mathcal{P}}^{N*}_{hdr}$ is at most $\bar{Q}^s_{hdr}(\hat{\mathcal{P}}^{A*}_{hdr}) + \sum_{p\in\hat{\mathcal{P}}^{N*}_{hdr}} c^{\text{cancel}}_p < \bar{Q}^s_{hdr}$. However, the patient set $\hat{\mathcal{P}}^{A*}_{hdr} \cup \hat{\mathcal{P}}^{N*}_{hdr}$ is equivalent to the assignment with patients in the set $\hat{P}_{hdr}$, and its corresponding cancellation cost is exactly $\bar{Q}^s_{hdr}$. This is a contradiction.

As the lower bound of $\bar{Q}^s_{hdr}(\hat{\mathcal{P}}^{A*}_{hdr})$ is $\bar{Q}^s_{hdr} - \sum_{p\in\hat{\mathcal{P}}^{N*}_{hdr}} c^{\text{cancel}}_p$, we have the following evaluation of the LBBD cut at the global optimal optimization:

$$\bar{Q}^{s*}_{hdr} \geq \bar{Q}^s_{hdr}(\hat{\mathcal{P}}^{A*}_{hdr}) \geq \bar{Q}^s_{hdr} - \sum_{p\in\hat{\mathcal{P}}^N_{hdr}} c^{\text{cancel}}_p \overbrace{(1-\hat{x}^*_{hdp})}^{=1} - \sum_{p\in\hat{\mathcal{P}}^A_{hdr}} c^{\text{cancel}}_p \overbrace{(1-\hat{x}^*_{hdp})}^{=0}$$

which is satisfied thanks to the relations of the cut's LHS and RHS to the middle comparative term as mentioned above. □

1.2. Proof of Theorem 2

THEOREM 2. *Constraints (13), which are defined as*

$$Q^s_{hdr} \geq \left(\min_{p\in\mathcal{P}} \frac{c^{cancel}_p}{T^s_p}\right)\left(\sum_{p\in\mathcal{P}} T^s_p x_{hdpr} - B_{hd}\right) \quad \forall h\in\mathcal{H}, d\in\mathcal{D}, r\in\mathcal{R}_h, s\in\mathcal{S},$$

provide valid lower bounds (LBs) for Q^s_{hdr}.

Proof. At optimality Q^s_{hdr} equals the optimal objective value of the subproblem (5). We want to prove that the RHS of inequality (13) is either trivially true or otherwise can be obtained by relaxing the subproblem.

First, we look at the trivial case where the OR corresponding to Q^s_{hdr} is not opened. In this case, there should be no cost for cancelling as no patient is assigned in the first place. Therefore, $Q^s_{hdr} = 0$. Since in this case the RHS of the inequality becomes

$\min_{p\in\mathcal{P}}\left(\frac{c_p^{\text{cancel}}}{T_p^s}\right)(-B_{hd})<0$ as the consequence of $x_{hdpr}=0$ ($\forall p\in\mathcal{P}$), the constraint is valid.

If the OR corresponding to Q^s_{hdr} is opened in an optimal solution, there must exist at least one patient who is assigned to this OR. Given an assignment of patients, $\hat{x}_{hdpr}$ ($\forall p\in\mathcal{P}$), and the set of assigned patients, $\hat{\mathcal{P}}_{hdr}$, suppose a subset of the patients, $\hat{\mathcal{P}}^C_{hdr}\subseteq\hat{\mathcal{P}}_{hdr}$, is cancelled as dictated by the optimal solution of the subproblem (5). Then by definition we have $Q^s_{hdr}=\sum_{p\in\hat{\mathcal{P}}^C_{hdr}}c_p^{\text{cancel}}$. Due to the fact that after cancellation, the total surgery duration of accepted patients should be no more than the operating time limit, B_{hd}, we have the following:

$$
\begin{aligned}
\sum_{p\in\hat{\mathcal{P}}^C_{hdr}} c_p^{\text{cancel}} &= \sum_{p\in\hat{\mathcal{P}}^C_{hdr}} c_p^{\text{cancel}}\hat{x}_{hdpr}\\
&= \sum_{p\in\hat{\mathcal{P}}^C_{hdr}} \frac{c_p^{\text{cancel}}}{T_p^s}T_p^s\hat{x}_{hdpr}\\
&\geq \left(\min_{p\in\mathcal{P}}\frac{c_p^{\text{cancel}}}{T_p^s}\right)\sum_{p\in\hat{\mathcal{P}}^C_{hdr}} T_p^s\hat{x}_{hdpr}\\
&\geq \left(\min_{p\in\mathcal{P}}\frac{c_p^{\text{cancel}}}{T_p^s}\right)\left(\sum_{p\in\hat{\mathcal{P}}^C_{hdr}} T_p^s\hat{x}_{hdpr}+\left(\sum_{p\in\mathcal{P}\setminus\hat{\mathcal{P}}^C_{hdr}} T_p^s\hat{x}_{hdpr}-B_{hd}\right)\right)\\
&= \left(\min_{p\in\mathcal{P}}\frac{c_p^{\text{cancel}}}{T_p^s}\right)\left(\sum_{p\in\mathcal{P}} T_p^s\hat{x}_{hdpr}-B_{hd}\right)
\end{aligned}
$$

Therefore, for any assignment $\hat{x}_{hdpr}$ ($\forall p\in\mathcal{P}$), the expression $\left(\min_{p\in\mathcal{P}}\frac{c_p^{\text{cancel}}}{T_p^s}\right)\left(\sum_{p\in\mathcal{P}}T_p^s\hat{x}_{hdpr}-B_{hd}\right)$ provides a LB for Q^s_{hdr}. Substitute the fixed assignment $\hat{x}_{hdpr}$ with the variable x_{hdpr} and we get the constraint (13). $\square$

1.3. Proof of Theorem 3

THEOREM 3. *The LBBD optimality cut (16), which is defined as*

$$
\begin{aligned}
&Q_{hd}\geq\bar{Q}_{hd}\left(g_{hdj}-\sum_{p\in\hat{\mathcal{P}}_{hd}}(1-x_{hdp})\right)\\
&y_{hd}\geq(1+\hat{y}_{hd})(1-g_{hdj}),\\
&g_{hdj}\in\{0,1\}
\end{aligned}
$$

is valid, where $\bar{Q}_{hd}$ is the optimal objective value of the LBBD subproblem (15), and $\hat{y}_{hd}$ is the optimal solution of y_{hd} from the LBBD master problem (14).

Proof. For any (h,d) pair, we need to prove that any global optimal solution $(\hat{u}'_{hd}, \hat{y}'_{hd}, \hat{x}'_{hdp}, \bar{Q}'_{hd})$ is not excluded by the optimality cut. Given an optimal (DE) solution, $(\hat{u}'_{hd}, \hat{y}'_{hd}, \hat{x}'_{hdp})$ denotes the corresponding solutions for the main LBBD master decisions, and $\bar{Q}'_{hd}$ is the corresponding optimal objective of the subproblem.

Case 1: If $\hat{u}'_{hd} = 0$, then the only global optimal solution for the current h and d will be $\hat{y}'_{hd} = 0, \hat{x}'_{hdp} = 0, \bar{Q}'_{hd} = 0$. This is feasible to the optimality cut.

Case 2: If $\hat{u}'_{hd} = 1$, then we discuss the following two cases separately:

Subcase a: If $\hat{y}'_{hd} > \hat{y}_{hd}$, we are not able to find a nontrivial LB for Q_{hd} without solving another subproblem, because when there are more ORs available, the cancellation cost of the current (h,d) pair can be either zero or some nonzero value that is smaller than $\bar{Q}_{hd}$. This is why we make the cut redundant in this case: let $g_{hdj} = 0$, (16a) becomes

$$Q_{hd} \geq \bar{Q}_{hd} \left(- \sum_{p \in \hat{P}_{hd}} (1 - x_{hdp}) \right)$$

This is always true since RHS is nonpositive, so $\bar{Q}'_{hd}$ and $\hat{x}'_{hdp}$ also satisfy this inequality. (16b) is now $y_{hd} \geq (1 + \hat{y}_{hd})$, which is equivalent to $\hat{y}_{hd} > \hat{y}_{hd}$, and that is exactly the assumption of this case.

Subcase b: If $\hat{y}'_{hd} \leq \hat{y}_{hd}$, then there are further two subcases to discuss. Note that due to (16b), we always have $g_{hdj} = 1$ in this case.

Subcase b1: If in the global optimal all patients that are assigned in the current solution are still assigned, i.e. $\hat{x}'_{hdp} \geq \hat{x}_{hdp}, \forall p \in \mathcal{P}_{hd}$, then we claim that the optimal cancellation cost will not decrease from the current value $\bar{Q}_{hd}$ because there are the same number of or a smaller number of rooms, but all the current assigned patients are still assigned. To see this argument is another way, assume for contradiction that $Q_{hd} < \bar{Q}_{hd}$ in this case. Then this means we can also schedule the patients in the current patient list in the same way with a lower cancellation cost than $\bar{Q}_{hd}$, which is a contradiction. Therefore, $Q_{hd} \geq \bar{Q}_{hd}$. Let $g_{hdj} = 1$, and also replace Q_{hd} and x_{hdp} with $\hat{Q}'_{hdp}$ and $\hat{x}'_{hdp}$, (16) becomes:

$$\overbrace{\hat{Q}'_{hd}}^{\geq \bar{Q}_{hd}} \geq \bar{Q}_{hd} \left(\overbrace{g_{hdj}}^{=1} - \overbrace{\sum_{p \in \hat{P}_{hd}} (1 - \hat{x}'_{hdp})}^{=0} \right)$$

$$\overbrace{\hat{y}'_{hd}}^{\leq \hat{y}_{hd}} \geq (1+\hat{y}_{hd})(1-\overbrace{g_{hdj}}^{=1})$$

Therefore, in this case the global solution also satisfies the optimality cut.

<u>*Subcase b2*</u>: If some currently assigned patients are no longer assigned, then we cannot give a nontrivial LB for the global optimal $\bar{Q}'_{hd}$, because the cancellation cost can either be zero or be a nonzero value that is larger or smaller than $\bar{Q}_{hd}$. Thus we make the LBBD cut redundant:

$$\overbrace{\hat{Q}'_{hd}}^{\geq 0} \geq \bar{Q}_{hd}\left(\overbrace{g_{hdj}}^{=1} - \overbrace{\sum_{p\in\hat{P}_{hd}}(1-\hat{x}'_{hdp})}^{\geq 1}\right)$$

$$\overbrace{\hat{y}'_{hd}}^{\leq \hat{y}_{hd}} \geq (1+\hat{y}_{hd})(1-\overbrace{g_{hdj}}^{=1})$$

which is satisfied by the global optimal solution. $\square$

1.4. Proof of Theorem 4

THEOREM 4. *The constraints* (20)*, which are defined as*

$$Q_{hd} \geq \frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}\left(\min_{p\in\mathcal{P}}\frac{c_p^{cancel}}{T_p^s}\right)\left(\sum_{p\in\mathcal{P}}T_p^s x_{hdp} - B_{hd}y_{hd}\right) \quad \forall h\in\mathcal{H}, d\in\mathcal{D},$$

are valid for (DE).

Proof. At optimality Q_{hd} equals the optimal objective value of the subproblem (15). We want to prove that the RHS of inequality (20) is either trivially true or otherwise can be obtained by relaxing the subproblem.

First, we look at the trivia case where the (h,d) pair corresponding to Q_{hd} is not opened. In this case, there should be no cost for cancelling as no patient is assigned in the first place. Therefore, $Q_{hd} = 0$. Since in this case the RHS of the inequality becomes 0 as the consequence of $x_{hdp} = 0$ $(\forall p \in \mathcal{P})$ and $y_{hd} = 0$, the constraint is valid.

If the (h,d) pair corresponding to Q_{hd} is opened in an optimal solution, there must exist at least one patient who is assigned to this (h,d) pair. Given an assignment of patients, $\hat{x}_{hdp}$ $(\forall p \in \mathcal{P})$, the set of assigned patients, $\hat{\mathcal{P}}_{hd}$, and the number of opened ORs, $\hat{y}_{hd}$. Suppose under the scenario s a set of patients, $\hat{\mathcal{P}}^{sC}_{hdr} \subseteq \hat{\mathcal{P}}_{hd}$, is cancelled as dictated by the optimal solution of the subproblem (15). Then by definition we have $Q_{hd} =$

$\frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}\sum_{p\in\hat{\mathcal{P}}^{sC}_{hdr}} c_p^{\text{cancel}}$. Due to the fact that after cancellation, the total surgery duration of accepted patients should be no more than the total operating time limit of opened ORs, $B_{hd}\hat{y}_{hd}$, we have the following for any scenario $s\in\mathcal{S}$:

$$\begin{aligned}
\sum_{p\in\hat{\mathcal{P}}^{sC}_{hdr}} c_p^{\text{cancel}} &= \sum_{p\in\hat{\mathcal{P}}^{sC}_{hdr}} c_p^{\text{cancel}}\hat{x}_{hdp} \\
&= \sum_{p\in\hat{\mathcal{P}}^{sC}_{hdr}} \frac{c_p^{\text{cancel}}}{T_p^s} T_p^s \hat{x}_{hdp} \\
&\geq \left(\min_{p\in\mathcal{P}} \frac{c_p^{\text{cancel}}}{T_p^s}\right) \sum_{p\in\hat{\mathcal{P}}^{sC}_{hdr}} T_p^s\hat{x}_{hdp} \\
&\geq \left(\min_{p\in\mathcal{P}} \frac{c_p^{\text{cancel}}}{T_p^s}\right) \left(\sum_{p\in\hat{\mathcal{P}}^{sC}_{hdr}} T_p^s\hat{x}_{hdp} + \left(\sum_{p\in\mathcal{P}\setminus\hat{\mathcal{P}}^{sC}_{hdr}} T_p^s\hat{x}_{hdp} - B_{hd}\hat{y}_{hd}\right)\right) \\
&= \left(\min_{p\in\mathcal{P}} \frac{c_p^{\text{cancel}}}{T_p^s}\right) \left(\sum_{p\in\mathcal{P}} T_p^s\hat{x}_{hdp} - B_{hd}\hat{y}_{hd}\right)
\end{aligned}$$

Therefore, for any assignment $\hat{x}_{hdp}$ ($\forall p \in \mathcal{P}$), the expression $\frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}\left(\min_{p\in\mathcal{P}}\frac{c_p^{\text{cancel}}}{T_p^s}\right)\left(\sum_{p\in\mathcal{P}} T_p^s\hat{x}_{hdp} - B_{hd}\hat{y}_{hd}\right)$ provides a LB for Q_{hd}. Substitute the fixed assignment $\hat{x}_{hdp}$ with the variable x_{hdp} and $\hat{y}_{hd}$ with y_{hd} then we get the constraint (20). $\square$

2. Flow Chart for the Two-stage Decomposition

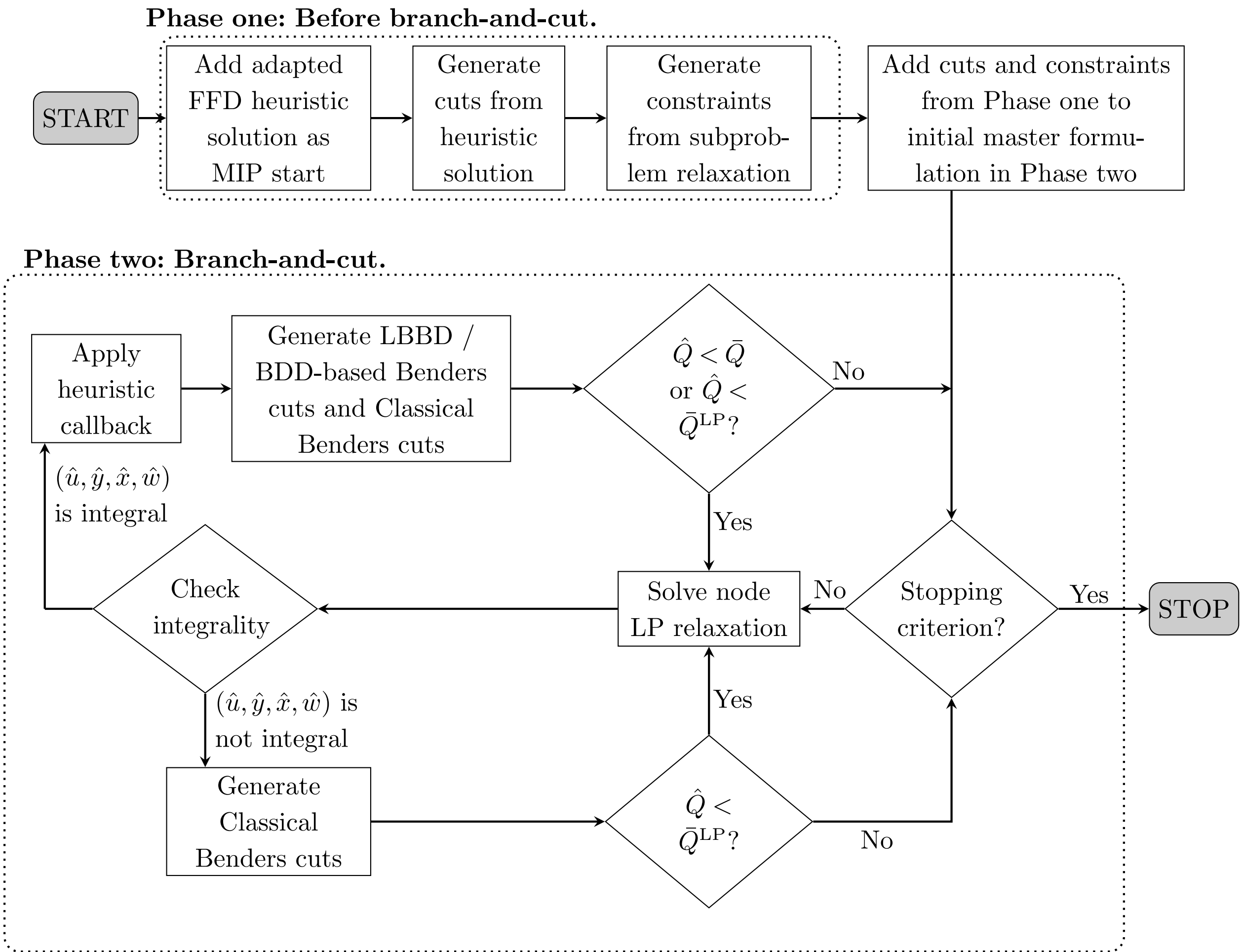

Figure 2 **Flow chart of the two-stage decomposition algorithm.**

3. Classical Benders Cuts for the Three-stage Decomposition

For the LBBD subproblem, we relax the LBBD subproblem (15). The binary variables x_{pr}, y_r, and z^s_{pr} are redefined as continuous variables. The variables in the parentheses at the end of constraints some constraints are their corresponding dual variables. We denote the relaxed LBBD subproblem by $\mathcal{Q}^{\text{LP}}_{hd}(\hat{x}_{hd\cdot}, \hat{y}_{hd}, T^s_{\cdot})$, and shorten it as $\bar{Q}^{\text{LP}}_{hd}$ later in the text:

$$\mathcal{Q}^{\text{LP}}_{hd}(\hat{x}_{hd\cdot}, \hat{y}_{hd}, T^s_{\cdot}) = \min \quad \frac{1}{|\mathcal{S}|} \sum_{s\in\mathcal{S}} \sum_{p\in\mathcal{P}} \sum_{r\in\mathcal{R}_h} c_p(x_{pr} - z^s_{pr})$$

$$\text{s.t.} \quad \sum_{r\in\mathcal{R}_h} x_{pr} = \hat{x}_{hdp} \qquad \forall p \in \mathcal{P} \qquad (\gamma_p)$$

$$
\begin{aligned}
&\sum_{p\in\mathcal{P}} T_p^s z_{pr}^s \le B_{hd} y_r && \forall r\in\mathcal{R}_h, s\in\mathcal{S} \\
&z_{pr}^s \le x_{pr} && \forall p\in\mathcal{P}, r\in\mathcal{R}_h, s\in\mathcal{S} \\
&x_{pr} \le y_r && \forall p\in\mathcal{P}, r\in\mathcal{R}_h \\
&\sum_{r\in\mathcal{R}_h} y_r \le \hat{y}_{hd} && && (\beta) \\
&y_r \le 1 && \forall r\in\mathcal{R}_h && (\delta_r) \\
&y_r, x_{pr}, z_{pr}^s \ge 0 && \forall p\in\mathcal{P}, r\in\mathcal{R}_h, s\in\mathcal{S}
\end{aligned}
$$

When $\hat{Q}_{hd} < \bar{Q}_{hd}^{\mathrm{LP}}$, the following classical Benders cuts are added to the LBBD master problem:

$$
Q_{hd} \ge \sum_{p\in\mathcal{P}} \bar{\gamma}_p x_{hdp} + \bar{\beta} y_{hd} + \sum_{r\in\mathcal{R}_h} \bar{\delta}_r
$$

where $\bar{\gamma}_p$, $\bar{\beta}$, and $\bar{\delta}_r$ are the optimal solutions of their corresponding dual variables.

For the decomposition of LBBD subproblem, we relax the variable z_{pr}^s as a continuous variable in subproblem (18). We denote the relaxed subproblem by $\theta_{sr}^{\mathrm{LP}}(\check{x}_{\cdot r}, T_\cdot^s)$, and shorten it as $\ddot{\theta}_{sr}^{\mathrm{LP}}$ later in the text:

$$
\begin{aligned}
\theta_{sr}^{\mathrm{LP}}(\check{x}_{\cdot r}, T_\cdot^s) = \min \quad & \sum_{p\in\mathcal{P}} -c_p z_{pr}^s \\
\text{s.t.} \quad & \sum_{p\in\mathcal{P}} T_p^s z_{pr}^s \le B_{hd} && && (\eta) \\
& z_{pr}^s \le \check{x}_{pr} && \forall p\in\mathcal{P} && (\iota) \\
& z_{pr}^s \in \{0,1\} && \forall p\in\mathcal{P}
\end{aligned}
$$

The corresponding classical Benders cut is:

$$
\theta_{sr} \ge \sum_{p\in\mathcal{P}} \ddot{\iota}_p x_{pr} + B_{hd}\ddot{\eta}
$$

where $\ddot{\eta}$ and $\ddot{\iota}_p$ are the optimal values for the corresponding dual variables.

4. Overall Implementation Approach for Three-stage Decomposition

In this section we first describe the LBBD decomposition, then explain the decomposition of LBBD subproblem.

LBBD decomposition:

<u>*Phase one*</u>: This phase is very similar to the phase one of two-stage decomposition in Section 5.6. We first use the adapted FFD heuristic to obtain an initial solution. This solution is added as a warm start in the commercial solver to provide a feasible solution at the start of branch-and-cut. We also generate LBBD cuts and classical Benders cuts from this solution and add them to the LBBD master problem. Next, we generated the constraints (20) from the subproblem relaxations and also add them to the LBBD master problem.

<u>*Phase two*</u>: We obtain the LBBD master problem with extra cuts and constraints from phase one and solve it with branch-and-cut. At each branch-and-bound node solve the node LP relaxation. If the objective value is greater or equal to the incumbent UB, then the current node can be pruned. Otherwise if the objective value is less than the incumbent UB, we proceed to check the integrality of $(\hat{u},\hat{y},\hat{x},\hat{w})$ in the master solution. If $(\hat{u},\hat{y},\hat{x},\hat{w})$ is integral, then solve the corresponding LBBD subproblems (15) with further decomposition (described with detail in the next paragraph). In the process of solving the LBBD subproblem, check if we need early stopping as explained in Section 6.3.2. If the solving process early stops, add LBBD cuts (19); otherwise, solve the LBBD subproblem to get the optimal objective value $\bar{Q}$. Also, solve subproblem LP relaxations and get optimal objective value $\bar{Q}^{\mathrm{LP}}$. Decide if we can insert an additional heuristic solution as described in Section 5.5.3. Also, generate LBBD cuts and classical Benders cuts. In the CPLEX lazy constraint callback, compare the master solution of $\hat{Q}$ with $\bar{Q}$ and $\bar{Q}^{\mathrm{LP}}$. If $\hat{Q} < \bar{Q}$ then add LBBD cuts; if $\hat{Q} < \bar{Q}^{\mathrm{LP}}$ add classical Benders cuts. On the other hand, if some elements in the solution $(\hat{u},\hat{y},\hat{x},\hat{w})$ are fractional, we only solve the subproblem LP relaxations, obtain $\bar{Q}^{\mathrm{LP}}$, generate the classical Benders cuts and implement them if $\hat{Q} < \bar{Q}$ within the CPLEX user cut callback. We use the same user cut management as in Section 5.6 to manage those user cuts. After cutting planes are added in the CPLEX lazy constraint callback or the user cut callback, the node LP relaxation is solved again with those additional cutting planes. We repeat this process, until the stopping criteria is met, i.e. the gap between branch-and-bound UB and LB is small enough. In our implementation we stop the algorithm when such a gap is no more than 1%.

Decomposition of LBBD subproblem:

Phase one: Use the adapted FFD heuristic to obtain an initial solution. This solution is added as a warm start in the commercial solver to provide a feasible solution at the start of branch-and-cut.

Phase two: We solve the BDD master problem with branch-and-cut. At each branch-and-bound node solve the node LP relaxation. If the objective value is greater or equal to the incumbent UB, then the current node can be pruned. Otherwise if the objective value is less than the incumbent UB, we proceed to check the integrality of $\check{x}$ in the master solution. If $\check{x}$ is integral, then solve the corresponding BDD subproblems (18) and the BDD subproblem LP relaxations to get their respective optimal objective values $\ddot{\theta}$ and $\ddot{\theta}^{\text{LP}}$. Generate BDD-based Benders cuts and classical Benders cuts. In the CPLEX lazy constraint callback, compare the master solution of $\check{\theta}$ with $\ddot{\theta}$ and $\ddot{\theta}^{\text{LP}}$. If $\check{\theta} < \ddot{\theta}$ then add BDD-based Benders cuts; if $\check{\theta} < \ddot{\theta}^{\text{LP}}$ add classical Benders cuts. On the other hand, if some elements in the solution $\check{x}$ are fractional, we only solve the subproblem LP relaxations, obtain $\ddot{\theta}^{\text{LP}}$, generate the classical Benders cuts and implement them if $\check{\theta} < \ddot{\theta}$ within the CPLEX user cut callback. The user cut management and stopping criteria are the same as in the LBBD decomposition.

5. Parameter Values for Computational Analysis

Table A.1 Parameter values

κ_1	50 dollars
κ_2	-5 dollars
κ_3	-80 dollars
κ_4	-100 dollars
Γ	500
ρ_p	Uniform distribution in {1,2,...,5}, where 1 is the least urgent 5 is the most urgent
B_{hd}	Uniform distribution [420, 480] minutes in 15-minute intervals
α_p	Uniform distribution [60, 120] days
F_{hd}	Uniform distribution [4000, 6000]
G_{hd}	Uniform distribution [1500, 2500]
c_{dp}^{sched}	$\kappa_1 \rho_p (d - \alpha_p)$
c_p^{unsched}	$\kappa_2 \rho_p (\lvert\mathcal{D}\rvert + 1 - \alpha_p)$
c_p^{cancel}	$\kappa_3 \rho_p (\lvert\mathcal{D}\rvert + 1 - \alpha_p), \forall p \in \mathcal{P} \setminus \mathcal{P}'$
c_p^{cancel}	$\kappa_4 \rho_p (\lvert\mathcal{D}\rvert + 1 - \alpha_p), \forall p \in \mathcal{P}'$
T_p^s	Truncated lognormal distribution with mean 160 minutes, standard deviation 40, and truncation range [45, 480]
ω_p	$(\alpha_p - \lvert\mathcal{D}\rvert)\rho_p$

6. Best Integer Solution Results for Algorithm Comparison

Table A.2 Comparison of Algorithms (continued): Best Integer Solution Objective Values (i.e., Upper Bounds)

instance (p-h-d-r)	MIP	2-BDD	2-LBBD	3-LBBD
10-2-3-3	-117624	-117624	-117670	-117670
25-2-3-3	-248582	-252010	-253491	-249238
10-3-5-3	-117227	-117671	-117595	-117671
25-3-5-3	-241882	-247676	-247807	-248107
50-3-5-3	-426982	-433702	-391874	-380486
75-3-5-3	-600867	-677288	-679017	-638701
10-2-3-5	-119481	-119551	-118935	-118935
25-2-3-5	-253762	-253335	-255321	-
50-2-3-5	-356952	-435996	-452404	-
75-2-3-5	-611189	-696665	-694526	-
10-3-5-5	-119555	-119537	-117611	-119588
25-3-5-5	-240490	-251945	-251931	-
50-3-5-5	-359391	-449867	-448060	-
75-3-5-5	-613869	-676269	-726670	-

Notice that for instance 10-2-3-5, both 2-BDD and 3-LBBD are solved to optimality, but they have different best integer results. This difference is caused by setting the 1% relative MIP gap in the solver.