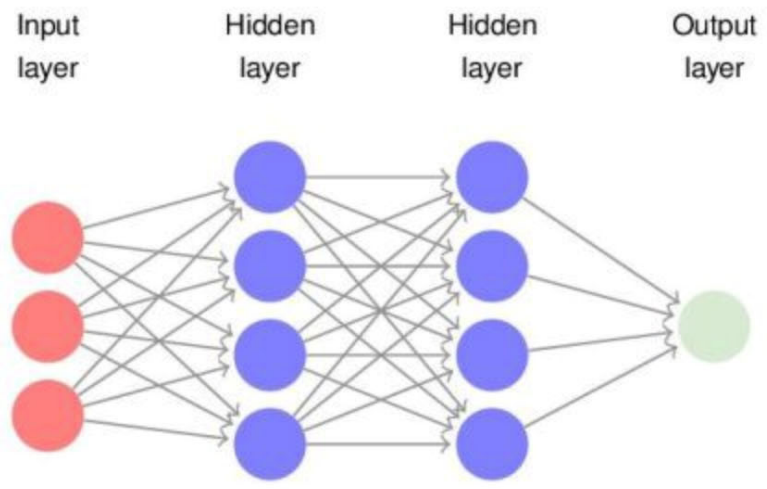
Deep Learning

Winter 2026

Neural Networks

Terminology



- Terms: Input layer, hidden layers, output layer
- A single hidden layer (of arbitrary width) can already be shown to be a **universal function approximator**
- Non-linear functions
  - are called activation functions in hidden layers
  - predict in the output layer and are used for the loss function

Feed Forward Neural Network: Basic Formulation

A feed forward neural network can be expressed by the following equation:

$$\hat{y}_k = g\left(\sum_{j=0}^{M} w_{jk} f\left(\sum_{i=0}^{N} w_{ij} x_i + b_j\right) + b_k\right)$$

Feed Forward Neural Network: Basic Formulation

number of outputs

number of hidden layer units

number of inputs per <u>sample</u>

hidden-layer activation

$$\hat{y}_k = g\left(\sum_{j=0}^{M} w_{jk} f\left(\overbrace{\sum_{i=0}^{N} w_{ij} x_i + b_j} \right) + b_k \right)$$

output bias term

input bias term

input data

input-to-hidden weight matrix
(i rows by j columns)

activation function: sigmoid, tanh(), ReLU, etc.

real-value network output

Feed Forward Neural Network: Basic Formulation

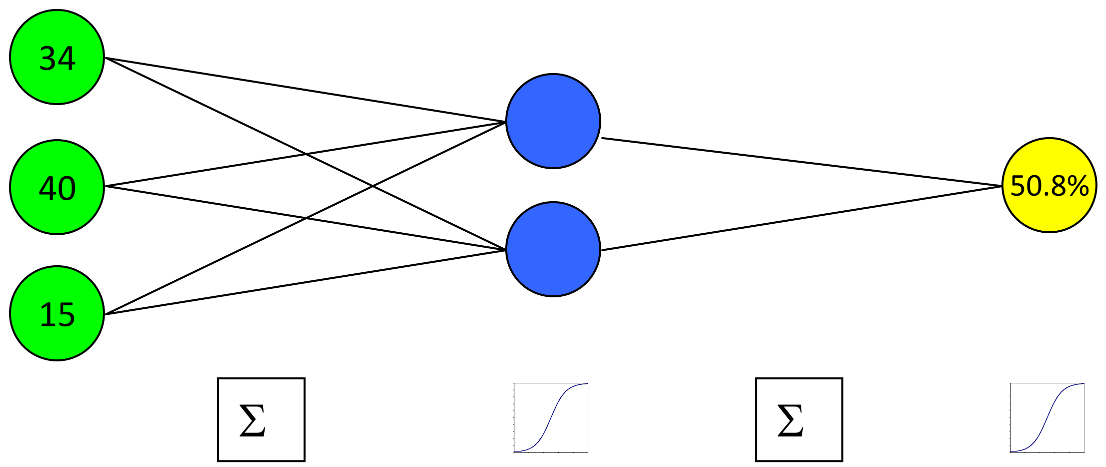
… or more compactly in Linear Algebra form:

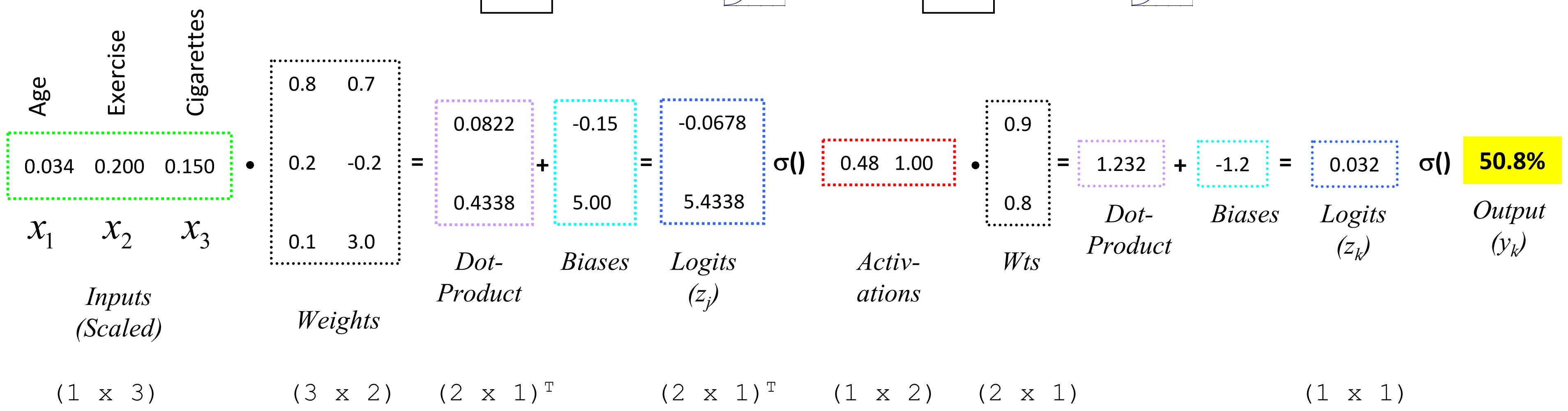
$$\hat{\mathbf{y}} = \sigma\left(\mathbf{W}_2 \cdot \tanh\left(\mathbf{W}_1 \cdot \mathbf{x} + \mathbf{b}_1\right) + \mathbf{b}_2\right)$$

$$\hat{y}_k = g\left(\sum_{j=0}^{M} w_{jk} f\left(\sum_{i=0}^{N} w_{ij} x_i + b_j\right) + b_k\right)$$

An Example

Consider a simple neural network architecture that computes the probability of being alive at the end of five years given “life style” features:

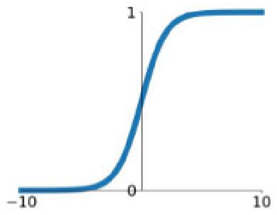




Activation Functions

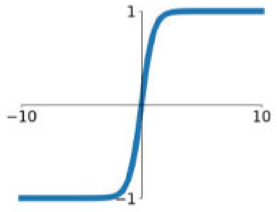
Sigmoid

$\sigma(x) = \frac{1}{1+e^{-x}}$



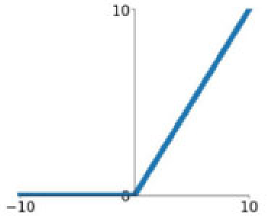
tanh

$\tanh(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$



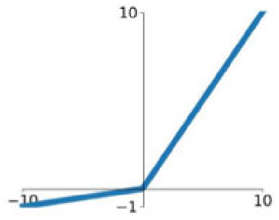
ReLU

$\max(0, x)$



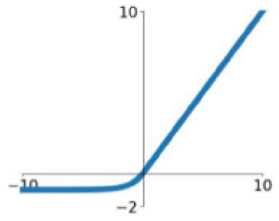
Leaky ReLU

$\max(0.1x, x)$



ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^{x} - 1) & x < 0 \end{cases}$$



Training

Step 1: The neural network is presented with a series input/output pairs $\langle \bar{x}_s, y_s \rangle \in S$ from the training set.

Step 2: The neural network is used to perform <u>inference</u> (a forward pass) to generate an estimated output $\hat{y}_s$.

Step 3: A <u>cost function</u> (see next slides) is used to quantify how well (or poorly) the neural network performed by comparing $\hat{y}_s$ and y_s .

Step 4: Back propagation (see next slides) is performed by calculating gradients with respect to each <u>trainable variable</u>, which are used to update trainable variables:

$$\hat{y}_k = g\left(\sum_{j=0}^{M} \boxed{w_{jk}} f\left(\sum_{i=0}^{N} \boxed{w_{ij}} x_i + \boxed{b_j}\right) + \boxed{b_k}\right)$$

The updates can be performed as:

- Stochastic Gradient Descent: Network <u>parameters</u> are updated for each training example, or
- Batch Gradient Descent: Gradients are accumulated over the entire training set and applied all at once.

Cost Functions

Certain cost functions are associated with typical output layers, but they are unconstrained and are influenced by the neural network architecture:

- Regression uses Mean Square Error (MSE):

$$J = \frac{1}{2}\sum_{S}\left(\hat{y}_s - y_s\right)^2$$

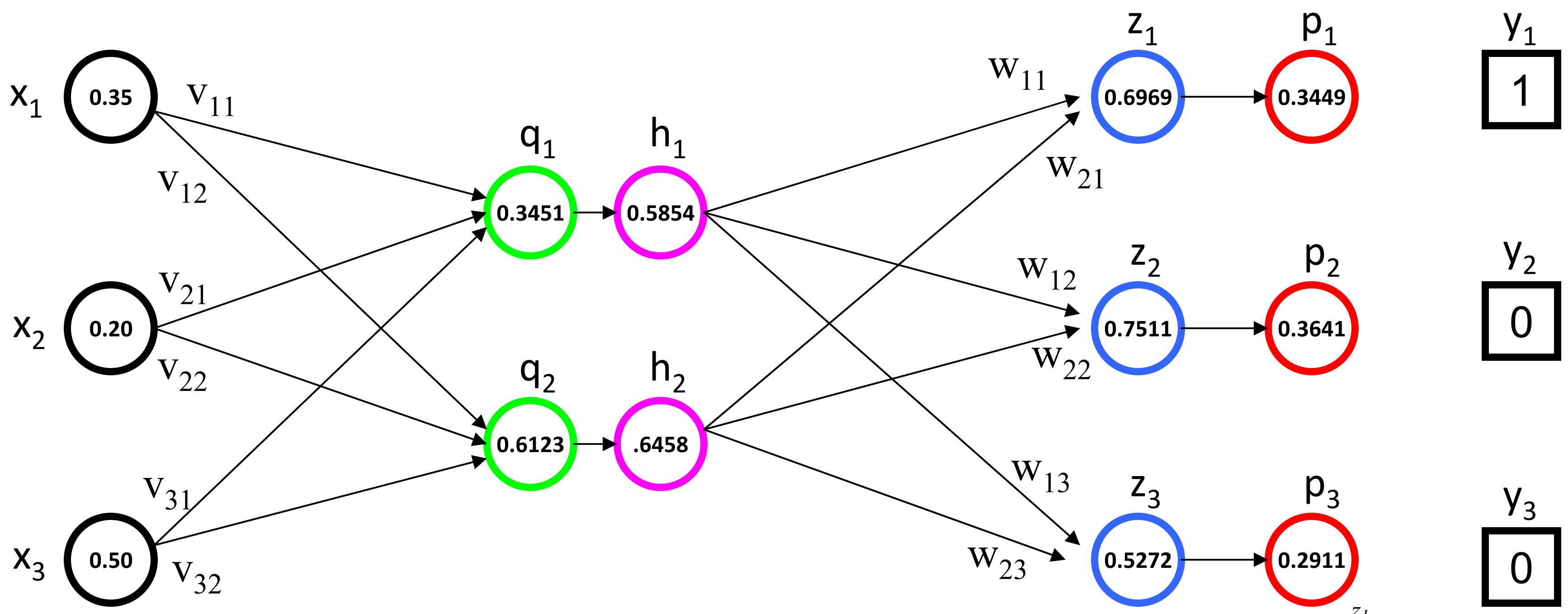
- Binary-Classification uses two-class cross-entropy:

$$J = -\left(y_s \ln(\hat{p}_s) + (1 - y_s)\ln(1 - \hat{p}_s)\right)$$

- Multi-Class Classification uses multi-class cross-entropy:

$$J = -\sum_{c \in C} y_c^{(s)} \ln(\hat{p}_c^{(s)})$$

Gradient Descent -- Architecture



$$\sigma(q) = \frac{1}{1+e^{-q}}$$

$$p_k = \frac{e^{z_k}}{\sum_{c \in C} e^{z_c}}$$

$$\begin{bmatrix} 0.3500 \\ 0.2000 \\ 0.5000 \end{bmatrix}
\quad
\begin{bmatrix} 0.4963 & 0.7682 \\ 0.0885 & 0.1320 \\ 0.3074 & 0.6341 \end{bmatrix}
\quad
\begin{bmatrix} 0.3451 \\ 0.6123 \end{bmatrix}
\quad
\begin{bmatrix} 0.5854 \\ 0.6485 \end{bmatrix}
\quad
\begin{bmatrix} 0.4901 & 0.8964 & 0.4556 \\ 0.6323 & 0.3489 & 0.4017 \end{bmatrix}
\quad
\begin{bmatrix} 0.6969 \\ 0.7511 \\ 0.5272 \end{bmatrix}
\quad
\begin{bmatrix} 34.5\% \\ 36.4\% \\ 29.1\% \end{bmatrix}
\quad
\begin{bmatrix} \textbf{Good} \\ \textbf{Neutral} \\ \textbf{Bad} \end{bmatrix}$$

Inputs X	*Weights V*	*Linear Q*	*Hidden H*	*Weights W*	*Logits Z*	*Probs P*	*Labels Y*
`(1 x 3)`	`(3 x 2)`	`(1 x 2)`	`(1 x 2)`	`(2 x 3)`	`(1 x 3)`	`(1 x 3)`	`(1 x 3)`

Gradient Descent -- Equations

Cost Function: Multi-Class Cross-Entropy

$$L = -\sum_{i=0}^{C} y_i \ln(p_i)$$

Activation Function: Sigmoid

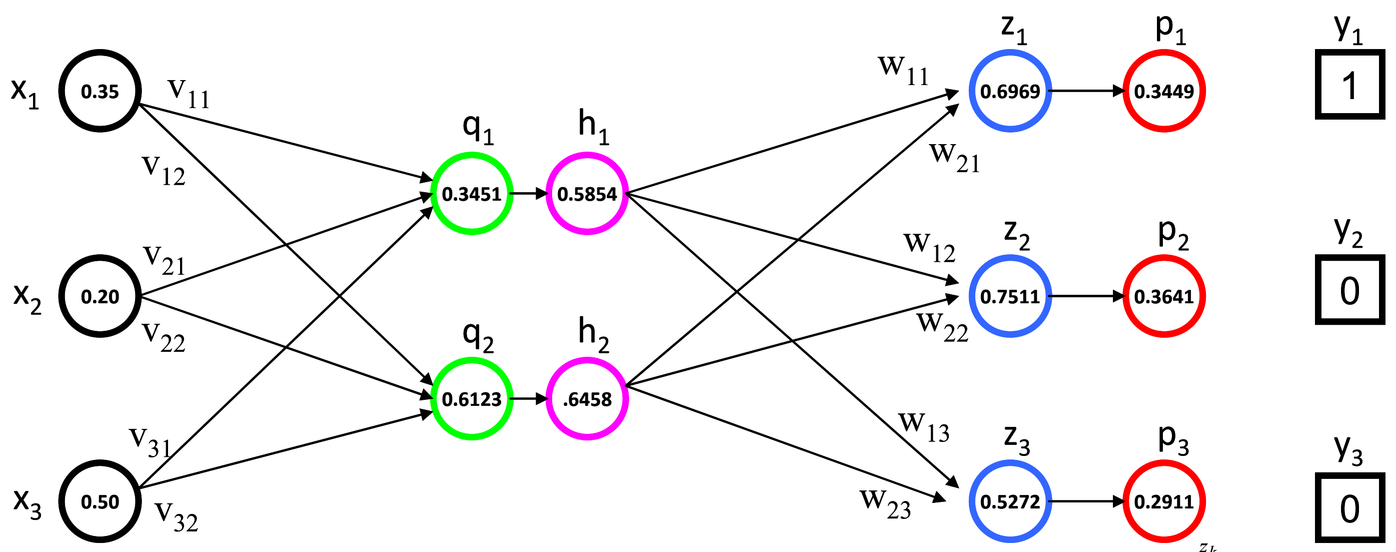
$$\sigma(q) = \frac{1}{1+e^{-q}}$$

Softmax:

$$p_k = \frac{e^{z_k}}{\sum_{c \in C} e^{z_c}}$$

$$= \frac{e^{z_1}}{e^{z_1} + e^{z_2} + e^{z_3}}$$

Gradient Descent -- Forward Pass Calculations



$$\sigma(q) = \frac{1}{1+e^{-q}}$$

$$p_k = \frac{e^{z_k}}{\sum_{c \in C} e^{z_c}}$$

$\begin{bmatrix}0.3500\\0.2000\\0.5000\end{bmatrix}$ *Inputs X* (1 x 3)

$\begin{bmatrix}0.4963 & 0.7682\\0.0885 & 0.1320\\0.3074 & 0.6341\end{bmatrix}$ *Weights V* (3 x 2)

$\begin{bmatrix}0.3451\\0.6123\end{bmatrix}$ *Linear Q* (1 x 2)

$\begin{bmatrix}0.5854\\0.6485\end{bmatrix}$ *Hidden H* (1 x 2)

$\begin{bmatrix}0.4901 & 0.8964 & 0.4556\\0.6323 & 0.3489 & 0.4017\end{bmatrix}$ *Weights W* (2 x 3)

$\begin{bmatrix}0.6969\\0.7511\\0.5272\end{bmatrix}$ *Logits Z* (1 x 3)

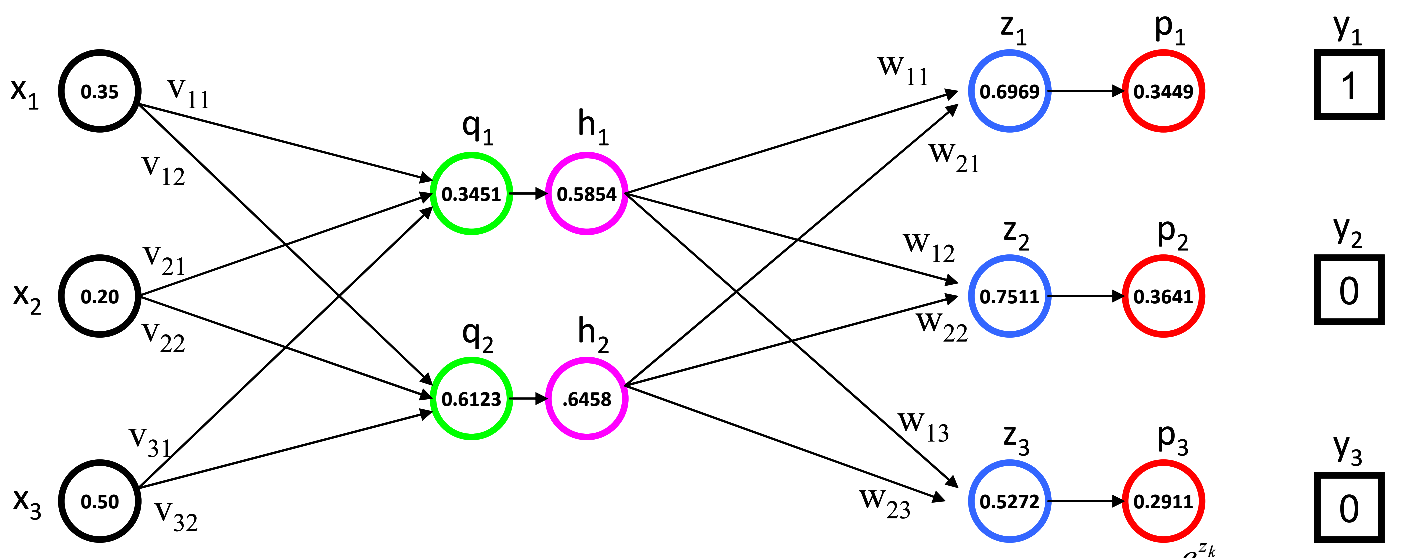
$\begin{bmatrix}34.5\%\\36.4\%\\29.1\%\end{bmatrix}$ *Probs P* (1 x 3)

$\begin{bmatrix}\textbf{Good}\\\textbf{Neutral}\\\textbf{Bad}\end{bmatrix}$ *Labels Y* (1 x 3)

$$\begin{bmatrix}0.3500 & 0.2000 & 0.500\end{bmatrix} \bullet \begin{bmatrix}0.4963 & 0.7682\\0.0885 & 0.1320\\0.3074 & 0.6341\end{bmatrix} = \begin{bmatrix}0.3451\\0.6123\end{bmatrix}$$

0.4963 X 0.3500 + 0.0885 X 0.2000 + 0.3074 X 0.500 = 0.3451

Gradient Descent -- Forward Pass Calculations



$$\sigma(q) = \frac{1}{1+e^{-q}}$$

$$p_k = \frac{e^{z_k}}{\sum_{c \in C} e^{z_c}}$$

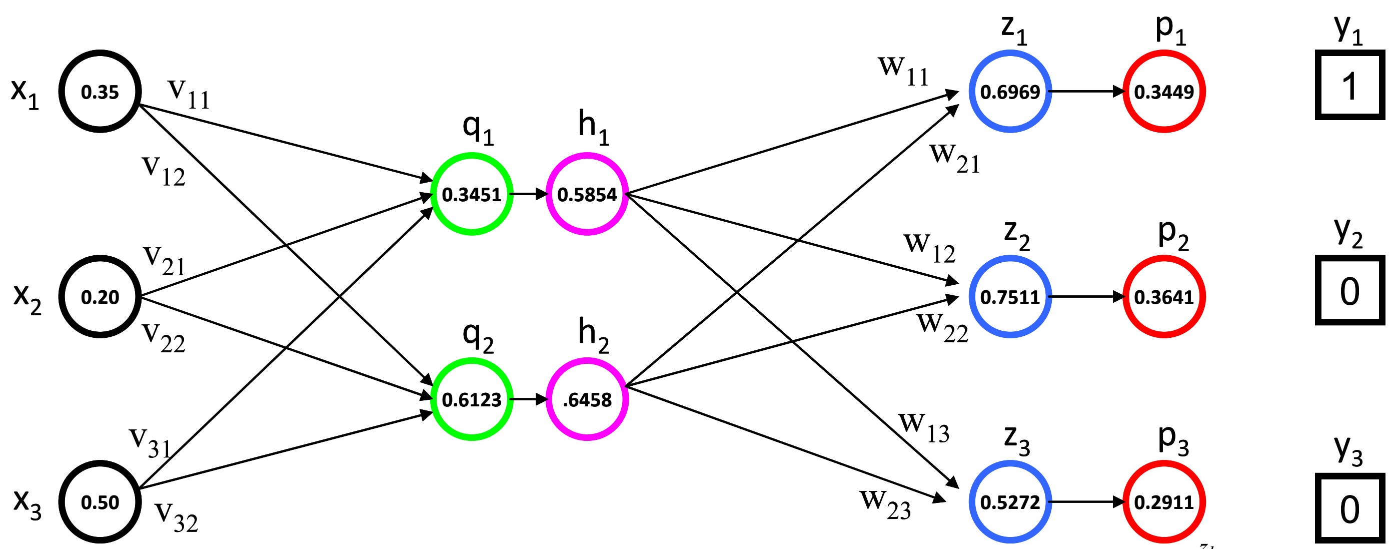
$$\begin{bmatrix} 0.3500 \\ 0.2000 \\ 0.5000 \end{bmatrix} \quad \begin{bmatrix} 0.4963 & 0.7682 \\ 0.0885 & 0.1320 \\ 0.3074 & 0.6341 \end{bmatrix} \quad \begin{bmatrix} 0.3451 \\ 0.6123 \end{bmatrix} \quad \begin{bmatrix} 0.5854 \\ 0.6485 \end{bmatrix} \quad \begin{bmatrix} 0.4901 & 0.8964 & 0.4556 \\ 0.6323 & 0.3489 & 0.4017 \end{bmatrix} \quad \begin{bmatrix} 0.6969 \\ 0.7511 \\ 0.5272 \end{bmatrix} \quad \begin{bmatrix} 34.5\% \\ 36.4\% \\ 29.1\% \end{bmatrix} \quad \begin{bmatrix} \textbf{Good} \\ \textbf{Neutral} \\ \textbf{Bad} \end{bmatrix}$$

Inputs X (1 x 3) · *Weights V* (3 x 2) · *Linear Q* (1 x 2) · *Hidden H* (1 x 2) · *Weights W* (2 x 3) · *Logits Z* (1 x 3) · *Probs P* (1 x 3) · *Labels Y* (1 x 3)

$$\sigma\left(\begin{bmatrix} 0.3451 \\ 0.6123 \end{bmatrix} \right) = \begin{bmatrix} 0.5854 \\ 0.6458 \end{bmatrix}$$

1 / (1 + exp(-0.3451) = 0.5854

Gradient Descent -- Forward Pass Calculations



$$\sigma(q) = \frac{1}{1+e^{-q}}$$

$$p_k = \frac{e^{z_k}}{\sum_{c \in C} e^{z_c}}$$

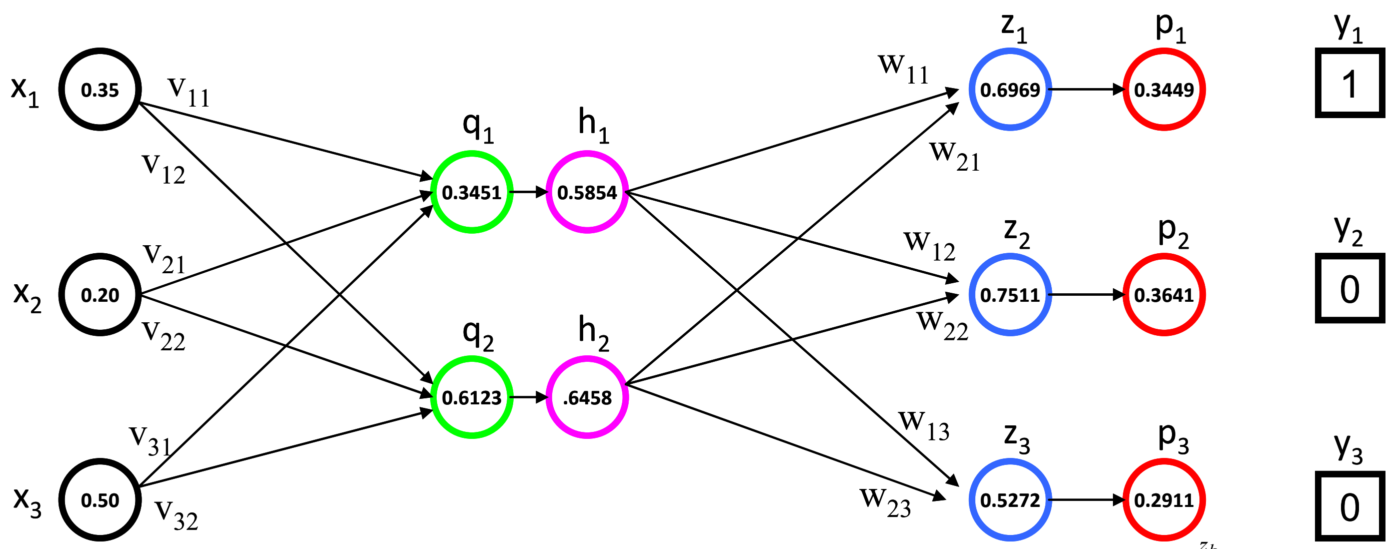
$$\begin{bmatrix} 0.3500 \\ 0.2000 \\ 0.5000 \end{bmatrix} \quad \begin{bmatrix} 0.4963 & 0.7682 \\ 0.0885 & 0.1320 \\ 0.3074 & 0.6341 \end{bmatrix} \quad \begin{bmatrix} 0.3451 \\ 0.6123 \end{bmatrix} \quad \begin{bmatrix} 0.5854 \\ 0.6485 \end{bmatrix} \quad \begin{bmatrix} 0.4901 & 0.8964 & 0.4556 \\ 0.6323 & 0.3489 & 0.4017 \end{bmatrix} \quad \begin{bmatrix} 0.6969 \\ 0.7511 \\ 0.5272 \end{bmatrix} \quad \begin{bmatrix} 34.5\% \\ 36.4\% \\ 29.1\% \end{bmatrix} \quad \begin{bmatrix} \textbf{Good} \\ \textbf{Neutral} \\ \textbf{Bad} \end{bmatrix}$$

Inputs X	*Weights V*	*Linear Q*	*Hidden H*	*Weights W*	*Logits Z*	*Probs P*	*Labels Y*
(1 x 3)	(3 x 2)	(1 x 2)	(1 x 2)	(2 x 3)	(1 x 3)	(1 x 3)	(1 x 3)

$$\begin{bmatrix} 0.5854 & 0.6485 \end{bmatrix} \bullet \begin{bmatrix} 0.4901 & 0.8964 & 0.4556 \\ 0.6323 & 0.3489 & 0.4017 \end{bmatrix} = \begin{bmatrix} 0.6969 \\ 0.7511 \\ 0.5272 \end{bmatrix}$$

0.4901 X 0.5854 + 0.6323 X 0.6458 = 0.6969

Gradient Descent -- Forward Pass Calculations



$$\sigma(q) = \frac{1}{1+e^{-q}}$$

$$p_k = \frac{e^{z_k}}{\sum_{c \in C} e^{z_c}}$$

$$\begin{array}{cccccccc}
\begin{bmatrix} 0.3500 \\ 0.2000 \\ 0.5000 \end{bmatrix} &
\begin{bmatrix} 0.4963 & 0.7682 \\ 0.0885 & 0.1320 \\ 0.3074 & 0.6341 \end{bmatrix} &
\begin{bmatrix} 0.3451 \\ 0.6123 \end{bmatrix} &
\begin{bmatrix} 0.5854 \\ 0.6485 \end{bmatrix} &
\begin{bmatrix} 0.4901 & 0.8964 & 0.4556 \\ 0.6323 & 0.3489 & 0.4017 \end{bmatrix} &
\begin{bmatrix} 0.6969 \\ 0.7511 \\ 0.5272 \end{bmatrix} &
\begin{bmatrix} 34.5\% \\ 36.4\% \\ 29.1\% \end{bmatrix} &
\begin{bmatrix} \textbf{Good} \\ \textbf{Neutral} \\ \textbf{Bad} \end{bmatrix} \\
\textit{Inputs } X & \textit{Weights } V & \textit{Linear } Q & \textit{Hidden } H & \textit{Weights } W & \textit{Logits } Z & \textit{Probs } P & \textit{Labels } Y \\
(1 \times 3) & (3 \times 2) & (1 \times 2) & (1 \times 2) & (2 \times 3) & (1 \times 3) & (1 \times 3) & (1 \times 3)
\end{array}$$

$$\text{softmax}\left(\begin{bmatrix} 0.6969 \\ 0.7511 \\ 0.5272 \end{bmatrix}\right) = \begin{bmatrix} 34.5\% \\ 36.4\% \\ 29.1\% \end{bmatrix}$$

exp(0.6969)/[exp(0.6969) + exp(0.7511) + exp(0.5272)] = 0.3449

Gradient Descent -- Some Derivatives

Cost Function: Multi-Class Cross-Entropy

$$\frac{\partial L}{\partial p_i} = -\frac{y_i}{p_i}$$

Activation Function: Sigmoid

$$\frac{\partial \sigma(q)}{\partial q} = \sigma(q)(1-\sigma(q))$$

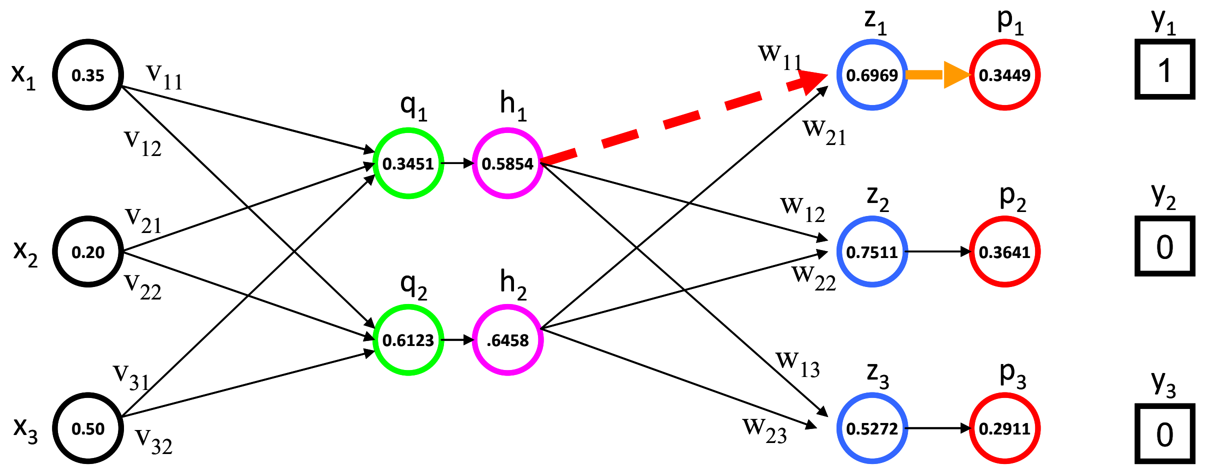
Softmax:

$$\frac{\partial \mathtt{softmax}(z_k)}{\partial z_k} = \frac{e^{z_k}(e^{z_0} + e^{z_1} + \cdots + e^{z_c} - e^{z_k})}{(e^{z_0} + e^{z_1} + \cdots + e^{z_c})^2}$$

Linear Transform:

$$\frac{\partial w \cdot x}{\partial w} = x$$

Gradient Descent -- Gradient Calculation



Chain Rule:

$$\frac{\partial f(g(x))}{\partial x} = \frac{\partial f(g(x))}{\partial g(x)} \cdot \frac{\partial g(x)}{\partial x}$$

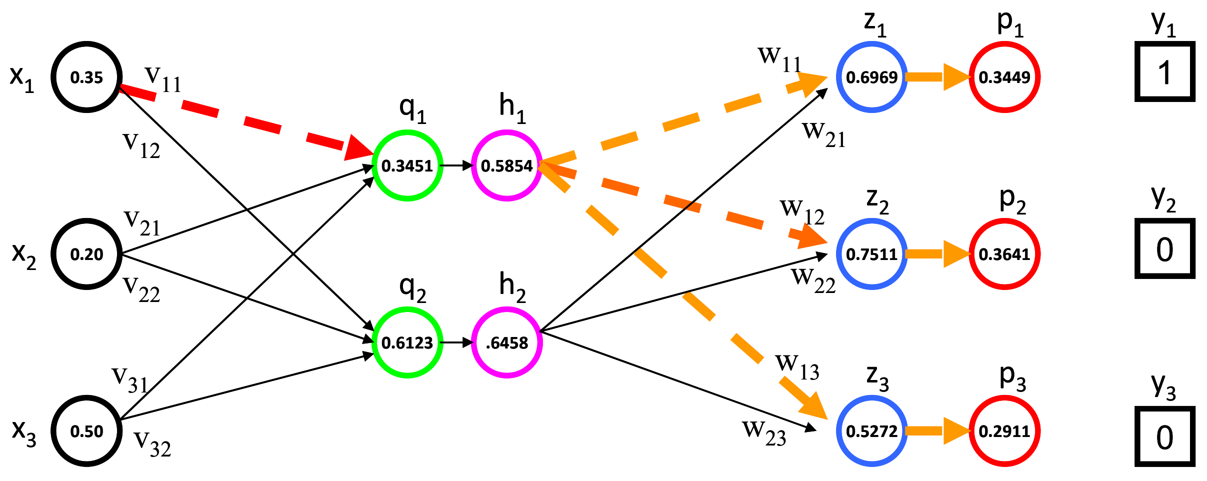
$$\frac{\partial L}{\partial w_{11}} = \frac{\partial L}{\partial p_1} \cdot \frac{\partial p_1}{\partial z_1} \cdot \frac{\partial z_1}{\partial w_{11}}$$

$$= -\frac{1}{p_1} \cdot \frac{e^{z_1} \cdot (e^{z_2} + e^{z_3})}{(e^{z_1} + e^{z_2} + e^{z_3})^2} \cdot h_1$$

$$= -\frac{1}{0.3449} \cdot \frac{e^{0.6969} \cdot (e^{0.7511} + e^{0.5272})}{(e^{0.6969} + e^{0.7511} + e^{0.5272})^2} \cdot 0.5854$$

$$= -0.3835$$

Gradient Descent -- Gradient Calculation



$$\frac{\partial L}{\partial v_{11}} = \frac{\partial L}{\partial p_1}\cdot\frac{\partial p_1}{\partial z_1}\cdot\frac{\partial z_1}{\partial h_1}\cdot\frac{\partial h_1}{\partial q_1}\cdot\frac{\partial q_1}{\partial v_{11}} + \frac{\partial L}{\partial p_1}\cdot\frac{\partial p_1}{\partial z_2}\cdot\frac{\partial z_2}{\partial h_1}\cdot\frac{\partial h_1}{\partial q_1}\cdot\frac{\partial q_1}{\partial v_{11}} + \frac{\partial L}{\partial p_1}\cdot\frac{\partial p_1}{\partial z_3}\cdot\frac{\partial z_3}{\partial h_1}\cdot\frac{\partial h_1}{\partial q_1}\cdot\frac{\partial q_1}{\partial v_{11}}$$

Gradient Descent -- Apply Gradients

```
Initialize $\theta^{(0)}$
Repeat until convergence:
    $\theta^{(t+1)} = \theta^{(t)} - \eta \nabla_{\theta^{(t)}} L(X, Y; \theta^{(t)})$
Return $\theta^{(t_{max})}$
```

$$\underset{\textit{Weights } W^{(t)}}{\begin{bmatrix} 0.4901 & 0.8964 & 0.4556 \\ 0.6323 & 0.3489 & 0.4017 \end{bmatrix}} - 0.5 \underset{\nabla_{w_{11}} L(X, Y, \theta^{(t)})}{\begin{bmatrix} -0.3835 & - & - \\ - & - & - \end{bmatrix}} = \underset{\textit{Weights } W^{(t+1)}}{\begin{bmatrix} 0.6819 & 0.8964 & 0.4556 \\ 0.6323 & 0.3489 & 0.4017 \end{bmatrix}}$$

$$\underset{\textit{Weights } V^{(t)}}{\begin{bmatrix} 0.4963 & 0.7682 \\ 0.0885 & 0.1320 \\ 0.3074 & 0.6341 \end{bmatrix}} - 0.5 \underset{\nabla_{v_{11}} L(X, Y, \theta^{(t)})}{\begin{bmatrix} & - \\ - & - \\ - & - \end{bmatrix}} = \underset{\textit{Weights } V^{(t+1)}}{\begin{bmatrix} & 0.7682 \\ 0.0885 & 0.1320 \\ 0.3074 & 0.6341 \end{bmatrix}}$$

Gradient Descent -- Additional Resources

Detailed Numerical Example:

https://medium.com/@14prakash/back-propagation-is-very-simple-who-made-it-complicated-97b794c97e5c

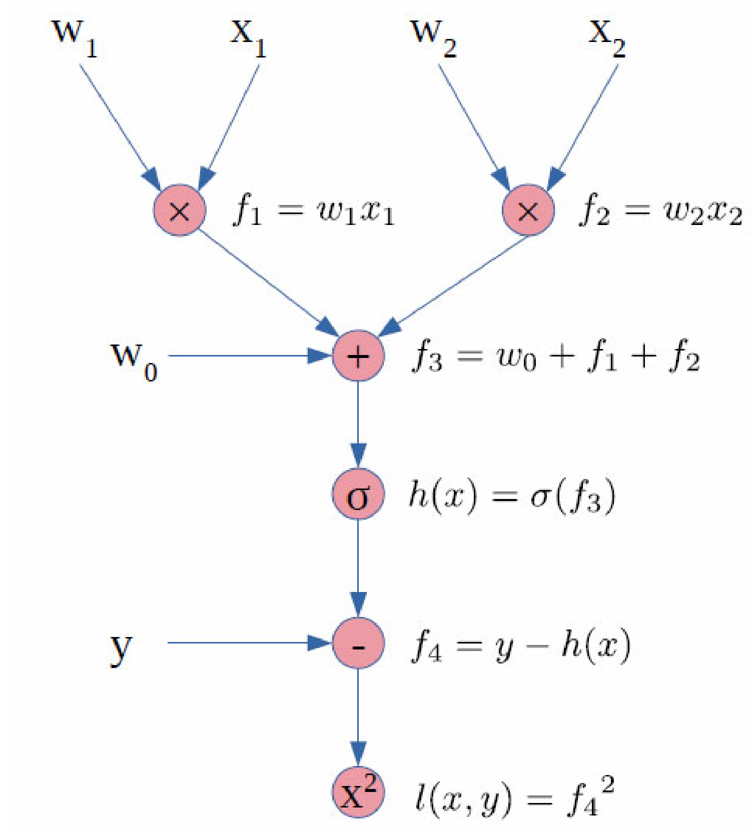
Gradient Calculation Video:

https://www.youtube.com/watch?v=lj7JYJSgpZI

Wolfram Alpha:

https://www.wolframalpha.com/

Calculation Graphs Forward Pass

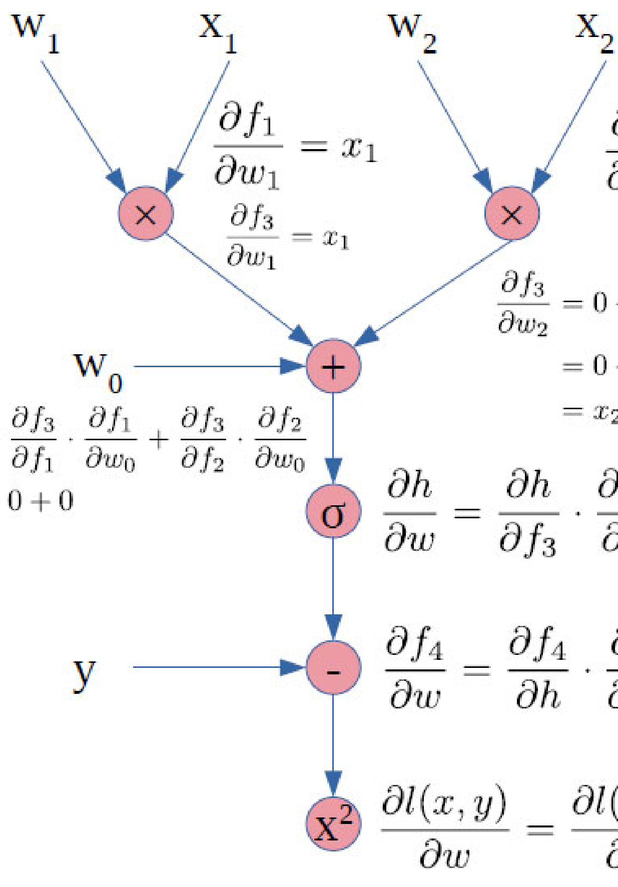


$$h(x) = \sigma\left(w_0 + w_1 x_1 + w_2 x_2\right)$$

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

$$l(x, y) = (y - h(x))^2$$

Calculation Graphs Gradients



$$\frac{\partial f_1}{\partial w_1} = x_1$$

$$\frac{\partial f_3}{\partial w_1} = x_1$$

$$\frac{\partial f_2}{\partial w_2} = x_2$$

$$\begin{aligned}\frac{\partial f_3}{\partial w_2} &= 0 + \frac{\partial f_3}{\partial f_1} \cdot \frac{\partial f_1}{\partial w_2} + \frac{\partial f_3}{\partial f_2} \cdot \frac{\partial f_2}{\partial w_2} \\ &= 0 + 1 \cdot 0 + 1 \cdot x_2 \\ &= x_2\end{aligned}$$

$$\begin{aligned}\frac{\partial f_3}{\partial w_0} &= 1 + \frac{\partial f_3}{\partial f_1} \cdot \frac{\partial f_1}{\partial w_0} + \frac{\partial f_3}{\partial f_2} \cdot \frac{\partial f_2}{\partial w_0} \\ &= 1 + 0 + 0 \\ &= 1\end{aligned}$$

$$\frac{\partial h}{\partial w} = \frac{\partial h}{\partial f_3} \cdot \frac{\partial f_3}{\partial w} = h(x)(1 - h(x)) \cdot \frac{\partial f_3}{\partial w}$$

$$\frac{\partial f_4}{\partial w} = \frac{\partial f_4}{\partial h} \cdot \frac{\partial h}{\partial w} = -1 \cdot \frac{\partial h}{\partial w}$$

$$\frac{\partial l(x,y)}{\partial w} = \frac{\partial l(x,y)}{\partial f_4} \cdot \frac{\partial f_4}{\partial w} = 2 f_4 \cdot \frac{\partial f_4}{\partial w}$$

$$h(x) = \sigma\left(w_0 + w_1 x_1 + w_2 x_2\right)$$

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

$$l(x, y) = (y - h(x))^2$$

Tunable Hyper-parameters

- Learning Rate is a "knob" that you can twist empirically
- Stochastic Gradient Descent vs. Batch Gradient Descent
- Drop Out randomly ignores certain weight dimensions during training
- Regularization minimizes a specific property of the neural network by modifying the cost function:

$$J = \frac{1}{2}\sum_{S}\left(\hat{y}_s - y_s\right)^2 + \lambda_1\left\|w_{ij}\right\|_2 + \lambda_2\left\|w_{jk}\right\|_2$$

- Scaling input values and weight initializations
- Gradient clipping and normalization
- Dynamic learning rate algorithms
- Many others