Neoclassical Tests

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Explanation about eq's 13.1 – 13.4

$$ds^2 = dt^2 - R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$\bar{r} \rightarrow$ comoving coordinate (dimensionless) $\{k = 0, 1, -1\}$

$R(t) \rightarrow$ scale factor

We make scale factor dimensionless

$$a(t) = \frac{R(t)}{R_0}$$

{All the subscript $_0$ values (a_0, R_0, H_0 etc.) represent the values of these quantities as of today}

& $r = R_0 \cdot \bar{r}$

In the book wherever R is written it means R_0. ← { also curvature parameter $K = \frac{k}{R_0^2}$

Note: $r = R_0\bar{r}$ is still not actual radial distance.

rather r is coordinate distance ← $r_{actual} = R(t)\bar{r} = a(t) \cdot R_0\bar{r} = a(t)\, r$.

metric

$$ds^2 = dt^2 - a^2(t)\cancel{R_0^2}\left[\frac{dr^2/\cancel{R_0^2}}{1-Kr^2} + \frac{r^2}{\cancel{R_0^2}}(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$= dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$\frac{dr^2}{1-\frac{r^2}{R_0^2}}$$

$r = R_0 \sin\chi$

$dr = R_0 \cos\chi$

$dr^2 = R_0^2 d\chi^2$

$$dr^2$$

$r = R\chi$

$dr = R_0 d\chi$

$r_{actual} = a(t)R_0\chi = R(t)\chi$

$$\frac{dr^2}{1+\frac{r^2}{R^2}}$$

$r = R_0 \sinh\chi$

$$\frac{dr^2}{1+\frac{r^2}{R^2}} = R_0^2 d\chi^2$$

$d r^2 = R_0^2 d\chi$ $\quad = a(t) R_0 \chi \quad = R(t)\chi$

Metric (13.1)

$$ds^2 = dt^2 - a^2(t) R_0^2 \left[d\chi^2 + f(\chi)^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right]$$

$$f(\chi) \rightarrow \begin{cases} \sin\chi & ; k > 0 \\ \chi & ; k = 0 \\ \sinh\chi & ; k < 0 \end{cases}$$

Friedmann eq^{n}s

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_b + 3P_b) + \frac{\Lambda}{3} \qquad -(13.2)$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho_b}{3} - \frac{K}{a^2} + \frac{\Lambda}{3}$$

$\downarrow$

Matter Radiation

$$\frac{\dot{a}}{a} = H(t)$$

$$\rho_{cr} = \frac{3H_0^2}{8\pi G}$$

Define $\Omega_0 = \frac{8\pi G}{3H_0^2(t)} \rho_0 = \frac{\rho}{\rho_{cr}}$

$$z = a(t)/a_0$$

$$\Omega = \frac{8\pi G}{3H_0^2(t)} \rho$$

$$\Omega = \Omega_0 \frac{\rho}{\rho_0} = \Omega_0 \frac{a_0^3}{a(t)^3} = \Omega_0 (1+z)^3$$

$$a(t) = \frac{R(t)}{R_0} \qquad \frac{-K}{a^2} = \frac{-k}{R_0^2 a^2} =$$

$a_0 = 1$

Define $\Omega_R = \frac{-k}{H_0^2 R_0^2 a^2} = -\frac{k}{H_0^2 R^2(t)}$

$$\Omega_R = \Omega_{R_0} \frac{a_0^2}{a^2} = \Omega_{R_0} (1+z)^2$$

Define $\Omega_\Lambda = \frac{\Lambda}{3H_0^2}$

Finally (13.2) becomes

$$H^2(t) = H_0^2 \left[\overbrace{\Omega_0 (1+z)^3 + \Omega_{R_0}(1+z)^2 + \Omega_\Lambda}^{E^2(z)} \right]$$

These are actually present values of densities

Note $\Omega_0 + \Omega_{R_0} + \Omega_\Lambda = 1$

(at $z=0$ $H^2(t) = H_0^2$)

Lookback Time

$$a(t) = a_0 \left[1 - H_0(t_0 - t) - q_0 H_0^2 \frac{(t_0-t)^2}{2} + \cdots \right] \rightarrow \text{eq } (13.6)$$

Proof $a(t_0) = a_0$

$$a(t) = a(t_0) + \left.\frac{da}{dt}\right|_{t=t_0} (t-t_0) + \left.\frac{d^2a}{dt^2}\right| \frac{(t-t_0)^2}{2} + \cdots \quad ; \quad q = \frac{-\ddot{a}_0 a_0}{\dot{a}_0^2}$$

$$= a_0 + H_0 a_0 (t-t_0) - q H_0^2 a_0 \frac{(t-t_0)^2}{2} \qquad H_0^2 = \frac{\dot{a}_0^2}{a_0^2}$$

$$= a_0 \left(1 - H_0 \underbrace{(t_0 - t)}_{\text{lookback time}} - H_0^2 q_0 \frac{(t-t_0)^2}{2} + \cdots \right) \qquad q_0 H_0^2 = \frac{-\ddot{a}_0}{a_0}$$

$$1+z = \frac{a_0}{a(t)}$$

$$H_0(t_0 - t) = 1 - \frac{a(t)}{a_0} = 1 - \frac{1}{1+z} = \frac{z}{1+z} = z(1 - z - \cdots) - \frac{H_0^2 q_0 (t-t_0)^2}{2}$$

$$\ddot{a} = \frac{-K}{a^2}$$

$$a = A(\cosh\eta - 1) \qquad \frac{\dot{a}}{a} = \frac{da/dt}{a} = \frac{da/d\eta}{dt/d\eta \cdot a} = \frac{\cancel{A}(\sinh\eta)}{B(\cosh\eta - 1)\cdot \cancel{A}(\cosh\eta - 1)}$$

$a = A(\cosh\eta - 1)$

$t = B(\sinh\eta - \eta)$

$$\frac{\dot a}{a} = \frac{da/dt}{a} = \frac{da/d\eta}{dt/d\eta \cdot a} = \frac{A(\sinh\eta)}{B(\cosh\eta - 1)\cdot A(\cosh\eta - 1)}$$

$$\ddot a = \frac{d}{dt}\left(\frac{da}{dt}\right) \qquad = \frac{\sinh\eta}{}$$

$$= \frac{d}{dt}\left(\frac{\sinh\eta}{(\cosh\eta-1)^2}\right)$$

$$= \frac{d\eta}{dt} \cdot \frac{\cosh\eta(\cosh\eta-1)^2 - \sinh\eta \times 2(\cosh\eta-1)\sinh\eta}{(\cosh\eta-1)^4}$$

$$(\cosh\eta - 1)\left[\cos^2 h\eta - \cosh\eta - 2\sin^2 h\eta\right]$$

Eq (13.12)

$$\int_0^a \frac{da}{a\cdot\left[\Omega\left(\frac{a_0}{a}\right)^4 + (1-\Omega)\left(\frac{a_0}{a}\right)^2\right]^{1/2}} = H_0 t$$

$$\int_0^a \frac{da}{\left[\Omega\left(\frac{a_0^4}{a^2}\right) + (1-\Omega)\, a_0^2\right]^{1/2}} = H_0 t$$

$$\int_0^a \frac{a\, da}{\left[\Omega(a_0^4) + (1-\Omega)\, a_0^2 \cdot a^2\right]^{1/2}} = H_0 t$$

$$\frac{}{\sqrt{(1-\Omega)\, a_0^2}} \int_0^a \frac{a\, da}{\Big[\underbrace{\frac{\Omega}{(1-\Omega)} a_0^2 + a^2}_{n}\Big]^{1/2}}$$

$$\int_{\frac{\Omega}{1-\Omega} a_0^2}^{\frac{\Omega a_0^2 + a^2}{1-\Omega}} \frac{dn}{\sqrt{n}} \qquad \frac{1}{\sqrt{(1-\Omega)\, a_0^2}}\left[\sqrt{\frac{\Omega\, a_0^2 + a^2}{1-\Omega}} - \sqrt{\frac{\Omega\, a_0^2}{1-\Omega}}\right]$$

$$= \Omega^{1/2}\left[\sqrt{1 + (1\ \ \Omega\ a^2)}\ \ 1\right]$$

$$\frac{\Omega U_{a_0}^2}{1-\Omega}$$

$$= \frac{\Omega^{1/2}}{(1-\Omega)} \left[\sqrt{1+\left(\frac{1-\Omega}{\Omega}\frac{a^2}{a_0^2}\right)} - 1 \right]$$

→ Eq. 13.21

$$\Omega (1+z_e)^3 + \Omega_R (1+z_e)^2 + \Omega \frac{(1+z_e)^3}{2} = 0$$

$$\Omega_R = \frac{-3\Omega}{2} \frac{(1+z_e)^3}{(1+z_e)^2} = -\frac{3\Omega}{2}(1+z_e)$$

$$\Omega - \frac{3\Omega}{2}(1+z_e) + \frac{\Omega (1+z_e)^3}{2} = 1$$

$$\Omega \left[\frac{(1+z_e)^3}{2} - \frac{1}{2} - \frac{3z_e}{2} \right]$$

$$\Omega \left[\frac{(z_e^3 + 3z_e^2)}{2} \right] = 1$$

$$\boxed{\Omega = \frac{2}{z_e^2(3+z_e)}}$$

Angular size distance → defined to be equal to coordinate distance r.

↳ (13.32)

Equating (13.1) to 0 (for a photon)

$$R_0 r = \int_{t_e}^{t_0} \frac{dt}{a} = \int_{a_e}^{a_0} \frac{da}{a\dot{a}} = \int_{a_e}^{a_0} \frac{da}{a^2(\dot{a}/a)} = \int_{a_e}^{a_0} \frac{da}{a^2 H_0 E(z)} = \int \frac{dz}{a_0 H_0}$$

$$z+1 = \frac{a_0}{a(t)}$$

$$z+1 = \frac{a_0}{a(t)}$$

$$dz = -\frac{a_0\, da}{a^2(t)}$$

$$da/dt = aH$$

$$ds^2 = 0 \Rightarrow \frac{dr^2}{1-Kr^2} = \frac{dt^2}{a^2(t)} = \frac{da^2}{a^4(t) H^2}$$

$$\Omega_R = \frac{-K}{H_0^2 a^2(t)}$$

if $\Omega_n = 0$ $\quad \Omega_R = (1-\Omega)$ $\quad$ (assuming $\Omega_{n_0} = 0$)

$$\frac{dr^2}{1+H_0^2 a_0^2 (1-\Omega_0) r^2} = \frac{dt^2}{a^2(t)} = \frac{da^2}{H_0^2 \left[\Omega_0 a_0^3 a + (1-\Omega_0) a_0^2 a^2\right]}$$

Substitute $\quad H_0 a_0 (1-\Omega_0)^{1/2} r = n \quad , \quad 2(\Omega_0^{-1} - 1)\frac{a}{a_0} = b-1$

$$LHS = \frac{dn^2}{\left[H_0^2 a_0^2 (1-\Omega_0)\right] (1+n^2)}$$

$$da = \frac{db \cdot a_0\ \Omega_0}{2(1-\Omega_0)}$$

$$RHS = \frac{db^2\ \cancel{a_0^2}\ \Omega_0^{\cancel{2}}}{H_0^2 \left(4(1-\Omega_0)^2 \left[\cancel{\Omega_0} a_0^{\cancel{3}} a\right]\left[1 + \frac{(1-\Omega_0)}{\Omega_0}\frac{a}{a_0}\right]\right)}$$

$$= \frac{db^2\ \Omega_0}{H_0^2 4 (1-\Omega_0)^2 \left[a\ a_0\right] \left(\frac{b+1}{2}\right)}$$

$$\overline{H_0^2 4(1-\Omega_0)^2 \left[a\ a_0\right] \left(\frac{b+1}{2}\right)}$$

$$= \frac{db^2}{H_0^2 (1-\Omega_0) a_0^2 \ (b^2-1)}$$

$$\frac{du^2}{1+u^2} = \frac{db^2}{1-b^2}$$

Subs. $u = \sinh \xi$

$b = \cosh \beta$

$$\sinh^{-1} u \Big|_0^u = -\cosh^{-1} b \Big|$$

$a = a_0$

$\uparrow$

$r = 0 \Rightarrow u = 0$

$r = r \Rightarrow u = u$

$\downarrow$

$$u = \sinh(\beta_0 - \beta) \quad \left\{ b = \cosh \beta_0 \right\} a = \quad a < a_0$$

$$u = \underline{\sqrt{b_0^2 - 1} \cdot b - b_0 \sqrt{b^2-1}}$$

$$u = H_0 a_0 (1-\Omega_0)^{1/2} r$$

$$b - 1 = 2(\Omega_0^{-1} - 1)\frac{a}{a_0}$$

$$b - 1 = \frac{2(\Omega_0^{-1} - 1)}{(1+z)}$$

$$b_0 - 1 = 2(\Omega_0^{-1} - 1)$$

$$b_0 + 1 = 2\Omega_0^{-1}$$

$$b_0^2 - 1 = 4(\Omega_0^{-2} - \Omega_0^{-1})$$

$$b = \frac{2(\Omega_0^{-1} - 1)}{1+z} + 1$$

$$= \frac{(2\Omega_0^{-1} - 1) + z}{1+z}$$

$$u = \frac{\sqrt{b_0^2 - 1}\ (b_0 + z)}{1+z} - b_0 \sqrt{\frac{(b+z)^2}{(1+z)^2} - 1}$$

$$= \frac{(b_0 + z)\sqrt{b_0^2 - 1}}{1+z} - b_0 \sqrt{\frac{b_0^2 - 1 - 2z + 2b_0 z}{(1+z)^2}}$$

$$= \frac{\sqrt{b_0 - 1}}{(1+z)} \left[(b_0 + z)\sqrt{b_0 + 1} - b_0 \sqrt{2z + b_0 + 1} \right]$$

$$b-1 = \frac{(b_0 - 1)}{(1+z)}$$

$$= \frac{\sqrt{\frac{(2-\Omega_0)-1}{\Omega_0}}}{1+z}\left[\left(\frac{2-\Omega_0}{\Omega_0}+z\right)\left(\sqrt{\frac{2}{\Omega_0}} - \frac{2-\Omega_0}{\Omega_0}\sqrt{2z+\frac{2}{\Omega_0}}\right]\right.$$

$$b+1 = \frac{(b_0+1)}{(1+z)} + \frac{2z}{1+z}$$

$$H_0 a_0 \sqrt{1-\Omega_0}\, r = \frac{\sqrt{2}\left(\sqrt{1-\Omega_0}\right)}{(1+z)\sqrt{\Omega_0}}\left[\left[\frac{(2-\Omega_0)+\Omega_0 z}{\Omega_0}\right]\left[\sqrt{\frac{2}{\Omega_0}}\right] - \frac{(2-\Omega_0)}{\Omega_0}\sqrt{\frac{2}{\Omega_0}}\left(\sqrt{\Omega_0 z+1}\right)\right]$$

$$y = H_0 a_0 r = \frac{2}{(\Omega_0)^2(1+z)}\left[(2-\Omega_0+\Omega_0 z) - (2-\Omega_0)(1+\Omega_0 z)^{1/2}\right]$$