Neoclassical Test Section 3

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→ Uptill Now we have seen two functions of redshift z, lookback time & angular size distance r(z).

→ We'll first see neoclassical tests based on lookback time.

$$d\ell = dt = \frac{da}{\dot{a}} = -\frac{dz}{(1+z)}\frac{a}{\dot{a}}$$

$$a = \frac{a_0}{1+z} = \frac{1}{1+z}$$

$$1+z = \frac{1}{a}$$

$$dz = -\frac{da}{a^2}$$

$$\frac{dz}{1+z} = -\frac{da}{a}$$

$$d\ell = \frac{dz}{(1+z)\,H_0 E(z)}$$

radial disp. func^n ← $$\frac{d\ell}{dz} = \frac{1}{(1+z)\,H_0\,E(z)}$$

$$\left\{ E(z) = \left[\Omega(1+z)^3 + \Omega_R(1+z)^2 + \Omega_\Lambda\right]^{1/2} \right\}$$

Used in
↓

→ Given a population of objects with comoving number density $n(z)$ (number per unit volume) and cross section $\sigma(z)$ (area), what is the incremental probability dP that a line of sight will intersect one of the objects in redshift interval dz at redshift z? Questions of this form are asked frequently in the study of QSO absorption lines or pencil-beam redshift surveys. The answer is

$$dP = \sigma n\, d\ell = \sigma n_0 (1+z)^3 \cdot \frac{H_0^{-1}}{(1+z)E(z)}\,dz = \sigma n_0 \frac{H_0^{-1}(1+z)^2}{E(z)}\cdot dz$$

Optical depth for intersection of objects upto redshift z

(Kind of like the probability of seeing the object upto that particular redshift 'z')

(at high redshift)

$$\tau_i(z) = \int_0^z dP = \int_0^z \frac{\sigma n_0 H_0^{-1}(1+z)^2}{\Omega^{1/2}(1+z)^{3/2}}\,.dz = \int_0^z \frac{\sigma n_0 H_0^{-1}}{\Omega^{1/2}}(1+z)^{1/2}.dz$$

$$= \frac{\sigma n_0 H_0^{-1}}{\Omega^{1/2}}\,\frac{2}{3}\left[1+z\right]^{3/2}$$

$$= \frac{c\,\sigma n_0 H_0^{-1}}{\Omega^{1/2}}\,\frac{2}{3}(1+z)^{3/2}$$

To correct dimensionality

→ Considering our objects to be <u>ordinary galaxies</u>

$\sim h^3\ \mathrm{Mpc}^{-3}$ $\quad r_a \sim 10h^{-1}\ \mathrm{Kpc}$

→ Considering our objects ...

$n_g \sim 0.02\, h^3\, Mpc^{-3}$, $r_g \sim 10 h^{-1}$ Kpc

$\sigma = \pi r_g^2$ $(H_0 = 100h)$

$$\Rightarrow \tau_i \simeq \frac{2}{3} \times 3\times10^8 \times \frac{0.02 h^3}{10^{18}\ pc} \times 100\pi h^{-2} H_0^{-1} \times 10^6 (1+z)^{3/2} \Omega^{-1/2}$$

$(1\ pc = 3.085 \times 10^{16} m)$

$$\simeq \frac{4\pi \times 0.98 \times 10^6}{1 \cdot pc} (1+z)^{3/2} \Omega^{-1/2} \times 3.156 \times 10^7$$

$$\simeq 0.01 (1+z)^{3/2} \Omega^{-1/2}$$

Predictions

At $z = 1$

$\tau_i \sim 0.04\, \Omega^{-1/2}$

At $\tau_i = 1$ (sky covered with galaxies)

$\boxed{z_i \sim 20\, \Omega^{1/3}}$

→ Now we'll look at some neoclassical tests based on angular size distance

Brightness Magnitude of galaxies → assumption: galaxies are spheres of radius $= r_g$

Apparent brightness ← f: Energy flux received from a galaxy = Power / Area

i: surface brightness of galaxy ⇒ f / Solid Angle

$$f_g = \int_0^1 2\pi i \cos\theta \, d(\cos\theta) = \pi i \quad — (1)$$

$$L = f_g \times 4\pi r_g^2 = 4\pi^2 r_g^2 i$$

→ Due to expansion the observed brightness is $i_o = \dfrac{i}{(1+z)^4}$

→ Due to expansion

→ The angular radius of the galaxy at the observer is

$$\theta_o = \frac{r_g(1+z)}{a_o r(z)} = \frac{r_g}{a(t_{em}) r(z)} = \left(\frac{r_g a_o}{a}\right)\left(\frac{1}{a_o r(z)}\right)$$

↳ Coz the light for the object (as we see it today) was emitted at some previous time t_{em}, $a(t_{em}) = \frac{a_o}{1+z}$

→ Observed energy flux = surface brightness × solid angle galaxy subtends

$$f_o = i \times 2\pi(1-\cos\theta_o) \simeq i_o \pi \theta_o^2$$

$$\Rightarrow f_o = \left(\frac{L}{4\pi^2 r_g^2}\right)\left(\frac{1}{1+z}\right)^4 \times \frac{\pi r_g^2 (1+z)^2}{a_o^2 r^2(z)}$$

$$\boxed{f_o = \frac{L}{4\pi (a_o r)^2 (1+z)^2}}$$

→ Distance Modulus

- $m = -2.5 \log(f) + C_1$

  m → apparent magnitudes; f → received energy flux (ergs/s-cm²)

- $M = -2.5 \log(L) + C_2$

  M → Absolute magnitude; L → Intrinsic Luminosity

- The constants C_1 & C_2 are defined such that at a distance 10 pc

  $m - M = 0$

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- In a classical universe $f = \dfrac{L}{4\pi r^2}$

hence $$\boxed{m - M = 5\log(r_{\text{Mpc}}) + 25}$$

- In our cosmological model

$$f = \frac{L}{4\pi (a_0 r)^2 (1+z)^2} = \frac{L H_0^2}{4\pi (H_0 a_0 r(z))^2 (1+z)^2} \quad -(13.51)$$

$$\log f = \log L - 2\log(a_0 r(z)) - 2\log(1+z) + c$$

$$-2.5\log f = -2.5\log L + 5\log(a_0 r(z)) + 5\log(1+z) + k$$

$$m - M + k' = 5\log \underline{(a_0(1+z) r(z))}$$

Put $a_0(1+z) r(z) = \underline{10\text{ pc}}$

$$k' = 5\log(10\text{ pc})$$

$$m - M = 5\log\left[\frac{a_0 r(z)(1+z)}{10\text{ pc}}\right]$$

$$H_0 = \frac{100h \text{ Km/sec}}{\text{Mpc}}$$

$$\frac{c}{H_0} = \underline{3000\, h^{-1}\text{ Mpc}}$$

$$= 25 + 5\log\left[\frac{a_0 r(z)(1+z)}{\frac{c}{H_0}\frac{1}{3000 h^{-1}}}\right] = 25 + 5\log\left(3000\, \frac{H_0 a_0 r(z)}{(1+z)}\right) - 5\log h \quad -(13.52)$$

dimensionless

1 (is dimensionless)

$$\rightarrow f_0 = \frac{L}{4\pi (a_0 r)^2 (1+z)^2} \propto \frac{1}{\left(z - (1+q_0)\frac{z^2}{2} + \cdots\right)^2 (1+z)^2}$$

$$\rightarrow f_0 = \frac{L}{4\pi (a_0 r)^2 (1+z)^2}$$

$$\left(z - (1+q_0)\frac{z^2}{2} + \cdots\right)(1+z)$$

$$\propto \frac{1}{z^2\left(1 - (1+q_0)z + \cdots\right)\left(1 + z^2 + 2z + \cdots\right)}$$

$$\propto \frac{z^{-2}}{\left(1 + (1-q_0)z + \cdots\right)}$$

$$\propto z^{-2}\left[1 + (q_0 - 1)z + \cdots\right]$$

$\hookrightarrow$ flux starts to deviate from z^{-2} law by 25% at $z = 0.5$.

- A useful application of z-m test at this redshift would require that the galaxy luminosities be understood to better than 25%.

- But it's shown classically that this level of precision is quite difficult to achieve. We'll see get a rough idea by considering luminosity of a star.

$\rightarrow$ For star masses with lifetimes comparable to the Hubble Time (14 billion years), luminosity scales with mass as $L \propto M^{\alpha}$ ($\alpha \sim 3$). (M → mass)

mass distribution, $\frac{dN}{dM} \propto M^{-(n+1)}$ $(n \sim 1)$.

...

mass distribution, $\frac{dN}{dM} \propto M$ $(n \sim 1)$.

Lifetime $t \propto M^{-\alpha}$ $\Rightarrow$ if all stars were created at high redshift those dying at world time t would have mass $\boxed{M \propto t^{-1/\alpha}}$

→ Net luminosity of main sequence stars in a galaxy would scale with time as →

$$L \propto \int^{M} \mathcal{L}(M) \cdot dN = \int M^{\alpha} \cdot M^{-(1+n)} \cdot dM \propto M^{\alpha-n} \propto t^{-(1-n/\alpha)}$$

(considering light from luminous stars that have just evolved off main sequence)

$L \propto t^{-(1-0.2n)}$ → This is just a rough approximation but not known reliably

For Einstein-di Sitter Model ; $t \propto (1+z)^{-3/2}$ & with $n = 1$

$L \sim 1 + 1.2z$

Hence this correction term cannot be known reliably

Thus we can't use z-m test to distinguish b/w Einstein de Sitter model & a low density universe.

K- Correction

→ Eqⁿ (13.51) are the bolometric relations for the energy flux ... wavelengths). The energy flux

→ Eqⁿ (13.51) are the bolometric relations for the energy integrated over all frequencies (or wavelengths). The energy flux observed at frequency ν observed within bandwidth $\delta\nu$ was radiated at frequency $\frac{\nu a_0}{a}$ in bandwidth $\delta\nu \frac{a_0}{a}$.

$$\Rightarrow \quad f_0(\nu)\,d\nu = \frac{L(\nu a_0/a)\cdot d\nu a_0/a}{4\pi (a_0 r)^2 (1+z)}$$

→ integrating both sides for $\nu \in (0, \infty)$ gives back eqⁿ 13.51

↓

distance modulus

$$m - M = 25 + 5\log\left[3000\,(1+z)\,H_0 a_0 r(z)\right] - 5\log H + K$$

$$K = -2.5\log\left[(1+z)\,L(\nu a_0/a)/L(\nu)\right]$$

$m - M - K$ = Bolometric distance modulus in (13.52)

→ f n f r

↳ depends on both angular size distance & $\frac{dl}{dz}$

↓ Area at redshift z subtended by the field of solid angle $\delta\Omega$.

$$\boxed{\delta A = a^2 r^2 \delta\Omega}$$

↓ Linear depth of sample at redshift z to z+dz

$$\delta l = \frac{dl}{dz}\cdot \delta z = \frac{H_0^{-1}\,\delta z}{(1+z)E(z)}$$

$$\delta V = \delta A \cdot \delta l = \frac{H_0^{-1}\,\delta z}{\ldots E(z)}\,\frac{(a_0 r)^2\,\delta\Omega}{(1+z)^2}$$

$$\delta V = \delta A \cdot \delta l = \frac{H_0^{-1}\,\delta z}{(1+z)\,E(z)} \; \frac{(a_0 r)^2\,\delta\Omega}{(1+z)^2}$$

Assuming objects are conserved

$$\Rightarrow \quad n a^3 = n_0 a_0^3$$

$\downarrow$ no. density

$$n = \underline{n_0 (1+z)^3}$$

$\rightarrow$ At the time light was emitted no. density was '$\underline{n}$'.

Total Number $\quad \delta N = n \cdot \delta V$

$$\delta N = n_0 \, \frac{H_0^{-1} (a_0 r)^2 \, \delta\Omega \cdot \delta z}{E(z)}$$

$$\delta \mathcal{N} = \frac{\delta N}{\delta \Omega} = \frac{n_0 H_0^{-1} (a_0 r)^2 \, \delta z}{E(z)}$$

(No. count / steradian) increase in redshift

$$\boxed{\frac{d\mathcal{N}}{dz} = n_0 H_0^{-3} F_n(z)} \qquad F_n(z) = \underbrace{\frac{[H_0 a_0 r(z)]^2}{E(z)}}_{\text{dimensionless number}}$$

h a z

n m 1 + mp

$\rightarrow$ Suppose quasars form in associations that are not gravitationally bound, but rather expand with the general expansion of universe.

$$\frac{dl}{dz} \cdot \delta z = \delta l = a r(z) \cdot \delta\theta$$

$\hookrightarrow$ proper linear diameter

$$\delta z = \frac{a\, r(z)}{H_0^{-1}} (1+z)\, E(z) \cdot \delta\theta$$

$$\frac{1}{z}\frac{\delta z}{\delta\theta} = \frac{a_0\, r(z)\, H_0\, E(z)}{z}$$

Superluminal motion

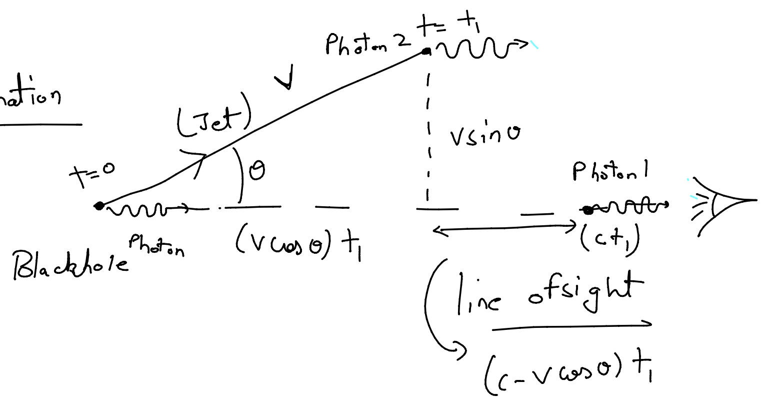
First we consider it for a non-expanding universe

Jets emitted from quasars at some angle to line of sight appear to move faster than the speed of light.

(For more info)

[M87 Black Hole Jets Move 7x Speed of Light, But How?]

Explanation



Time separation b/w two photons reaching observer $= \frac{ct_1 - v\cos\theta\, t_1}{c}$

$$\delta t_e = t_1\left(1 - \frac{v\cos\theta}{c}\right)$$

$\perp$ Distance separation b/w the two objects $\delta l_\perp = v\sin\theta\, t_1$

Rate of motion of jet $\perp$ to line of sight

$$\frac{\delta l_1}{\delta t_e} = \frac{v\sin\theta}{\left(1-\frac{v\cos\theta}{c}\right)}$$

max. at $\cos\theta = v/c$

$$\frac{\delta l_1}{\delta t_e} = \frac{v\sqrt{1-v^2/c^2}}{1-v^2/c^2}$$

$$= \frac{v}{\sqrt{1-v^2/c^2}} = \gamma v$$

$\gamma v > c$ at

$$\boxed{v > c/\sqrt{2}}$$

$\frac{\delta l_1}{t_e} > c$ at

$$\frac{v\sin\theta}{1-\frac{v\cos\theta}{c}} > c$$

$$v\sin\theta + v\cos\theta > c$$

$$\sin\theta + \cos\theta > \frac{c}{v}$$

$$\sin\left(\theta+\frac{\pi}{4}\right) > \frac{c}{\sqrt{2}\,v}$$

at $\boxed{\theta > \frac{\pi}{4} - \sin^{-1}\frac{c}{\sqrt{2}\,v}}$

Now we consider the scenario for an expanding universe

First consider this if two photons are separated by r. coordinates δr, then we calculate the difference time b/w the two photons from them to arrive at earth.

$a(t)\,dr = c - v\cos\theta$

↳ distance is small as compared to total distance travelled

$$c\int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_0^r dr$$

t_1 $\quad$ $r + dr$

1 +dt₂ $\quad$ |

$$c \int_{t_1}^{t_2+dt_2} \frac{dt}{a(t)} = \int_0^{r+dr} dr$$

$$c \int_{t_2}^{t_2+dt_2} \frac{dt}{a(t)} = dr = \frac{c - v\cos\theta}{a(t)}$$

$$\frac{c\,dt_2}{a(t_2)} = \frac{c - v\cos\theta}{a(t)} \Rightarrow dt_2 = \frac{a(t_2)}{a(t)} \frac{(c - v\cos\theta)}{c}$$

$$\boxed{dt_2 = (1+z)\left(1 - \frac{v}{c}\cos\theta\right)}$$

Observed angular distance $= \dfrac{\delta l_1}{a r(z)}$

Observed angular velocity or observed angular proper motion $= \dfrac{\left(\dfrac{\delta l_1}{a r(z)}\right)}{\delta t_2} = \dfrac{1}{a_0 r(z)} \dfrac{\delta l_1}{\delta t_e} = \dfrac{1}{a_0 r(z)} \cdot \dfrac{v\sin\theta}{\left(1 - \dfrac{v\cos\theta}{c}\right)}$

max. value

$$= \frac{\gamma v}{a_0 r(z)}$$

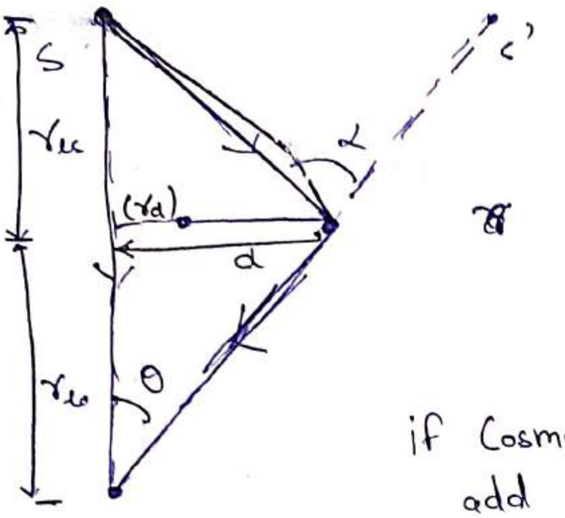
$$= \frac{H_0 \gamma v}{H_0 a_0 r(z)}$$

determines max. proper angular motion

fixed for a given redshift

Gravitational Lensing

Gravitational Lensing



$S = \theta r_{os} = \alpha r_{ls}$

$\downarrow$ Comoving coordinate length

$$d = \theta r_{ol} = \frac{\alpha r_{ol} \cdot r_{ls}}{r_{os}}$$

if Cosmologically flat model r can directly add or subtract

if not then parameter χ adds or subtracts (for radial light ($r = R_0 \sinh \chi$ or $R_0 \sin \chi$) ray $R_0 d\chi = a^{-1} dt$)

Consider the case of negative space curvature universe

Curvature parameter $\Omega_{R_0} = \frac{1}{(H_0 a_0 R_0)^2}$ → (also curvature density today)

Dimensionless quantity $y = H_0 a_0 r$

$y_{ls} = H_0 a_0 r_{ls} = H_0 a_0 R_0 \sinh(\chi_{ls}) = (\Omega_{R_0})^{-1/2} \sinh(\chi_{os} - \chi_{ol})$

$y_{ls} = (\Omega_{R_0})^{-1/2} (\sinh \chi_{os} \cosh \chi_{ol} - \sinh \chi_{ol} \cosh \chi_{os})$

$$\boxed{y_{ls} = \cancel{(\Omega_{R_0})^{-1/2}}\; y_{os}\sqrt{1+\Omega_{R_0} y_{ol}^2} - y_{ol}\sqrt{1+\Omega_{R_0} y_{os}^2}}$$

model ($\Omega_{R_0} < 0$)

↳ also applies to positive curvature model

→ For sources such as galaxies which have distributed mass of mass density ρ varying as $\frac{1}{r^2}$ the ~~lensing~~ bending angle is given by line of sight velocity dispersion.

$$\boxed{\alpha = \frac{4\pi\sigma^2}{c^2}}$$ → line of sight velocity dispersion

↳ Note that this angle doesn't depend on the distance of light ray from source (impact parameter)

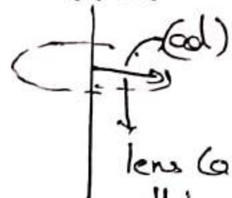
→ The condⁿ for a lensing event to produce two images is proper distance of lens from undeflected line of sight (r_d) $< d$.
(comoving)

→ Using eqⁿ (13.42) we can find lensing probability per unit of redshift

$$\frac{dP}{dz} = \frac{\sigma\, n_0 H_0^{-1} (1+z)^2}{E(z)} \quad - (13.42)$$

σ is cross section area

$$\sigma = \pi a^2 d^2 \rightarrow$$



$$d = \frac{\alpha\, r_{ol} \cdot r_{ls}}{r_{os}}$$

lens can be present anywhere inside this circle

$$d = \frac{4\pi\sigma^2}{H_0 a_0} \frac{y_{ol} \cdot y_{ls}}{y_{os}} \quad (c=1)$$

velocity dispersion

$$\sigma = \pi a^2 \cdot \frac{16\pi^2\sigma^4}{H_0^2 a_0^2} \left(\frac{y_{ol}\, y_{ls}}{y_{os}}\right)^2 = \frac{16\pi^3\sigma^4}{H_0^2} \left(\frac{y_{ol}\, y_{ls}}{y_{os}}\right)^2 \frac{1}{(1+z)^2}$$

also $n_0 = \frac{n}{(1+z)^3}$

$$\Rightarrow \frac{dP}{dz} = \frac{16\pi^3}{H_0^3} \frac{n\sigma^4}{(1+z)^3 E(z)} \left(\frac{y_{ol}\, y_{ls}}{y_{os}}\right)^2$$

if there are many lensing systems within the distance d, we can sum over each of them & get

$$\boxed{\frac{dP}{dz} = \frac{16\pi^3}{H_0^3} \frac{\sum_i n_i \sigma_i^4}{(1+z)^3 E(z)} \left(\frac{y_{ol}\, y_{ls}}{y_{os}}\right)^2} \quad - (13.71)$$

Gravitational Lensing Optical Depth

redshift ← z_s of source

$$\tau_{gl} = \int_0^{z_s} dP = \int_0^{z_s} \frac{16\pi^3}{H_0^3} \frac{\Sigma[n_i \sigma_i^4]}{(1+z)^3 E(z)} \left(\frac{y_{0l}\, y_{ls}}{y_{os}}\right)^2 \cdot dz$$

(This integral over z should be really thought of as an integral over time)

Note that n_i keeps on changing as z changes so instead we can write $\frac{n_i}{(1+z)^3} = n_{0i}$ & take it out of integral. (n_{0i} ~~mult~~ is a measure of no. density today)

$$\tau_{gl} = H_0^{-3} \Sigma[n_{0i} \sigma_i^4] \underbrace{\int_0^{z_s} 16\pi^3 \frac{dz}{E(z)} \left(\frac{y_{0l} \cdot y_{ls}}{y_{os}}\right)^2}_{\text{to be integrated} = g_1}$$

$$F_{g1}(z_s) = 16\pi^3 \int_0^{z_s} \frac{dz}{E(z)} \left(\frac{y_{0l} \cdot y_{ls}}{y_{os}}\right)^2$$

↳ (fixed for a given z_s)

From eq. (13.30) ~~y_{0l}~~ if $\Omega_R = 0$ then

$$d\,y_{0l} = \frac{dz_l}{E(z_l)} \qquad y_{ls} = y_{os} - y_{0l}$$

$$\Rightarrow F_{gl}(z_s) = 16\pi^3 \int_0^{z_s} dy_{0l} \frac{\left[y_{0l}^2 y_{os}^2 - 2y_{0l}^3 \cdot y_{os} + y_{0l}^4\right]}{y_{os}^2}$$

$$\Rightarrow \boxed{F_{g1}(z_s) = \frac{8\pi^3 y_{os}^3}{15}} \quad - (13.73)$$

Note that the lensing probability per increment of y_{0l} $\left[\frac{dP}{dy_{0l}} = \frac{dP(1+z)}{dz} = \frac{16\pi^3}{H_0^3} \sum_i n_{0i} \sigma_i^4 \left(\frac{y_{0l} \cdot y_{ls}}{y_{os}}\right)^2\right]$ is maximum at $\boxed{y_{0l} = \frac{y_{os}}{2}}$

Now we use Einstein-di Sitter Model (Matter dominated flat universe) to check orders of magnitude for the probability of lensing events.

($\Omega=1$, $\Lambda=0$) $\longrightarrow y_{os} = 2\left[1-(1+z_s)^{-1/2}\right]$

Consider a quasar at redshift $z_s = 3$ ($y_{os}=1$)

→ maximum lensing probability is at $y_{ol} = 0.5$ ($z_l = 0.3$)

→ Impact parameter $d = \dfrac{\alpha r_{ol} \cdot r_{ls}}{r_{os}} = \dfrac{\alpha r_{os}}{4}$

$$= \frac{\alpha y_{os}}{4 H_0 a_0}$$

$$d = \frac{\alpha}{2} \frac{y_{ol}}{H_0 a_0}$$

D = radii of galaxy (lens) $= a \cdot d = \dfrac{\alpha}{2} \dfrac{y_{ol}}{H_0(1+z_l)}$

→ Angular separation of images

$\alpha = 4\pi\left(\dfrac{\sigma}{c}\right)^2 = \underline{2\,\text{arc sec}}$ at $\sigma = 250\ \text{Km s}^{-1}$

(lower end of range of observed lensing events) (corresponds to large elliptic galaxies)

→ $D \simeq$ ~~$1\,\text{arcsec} \times \frac{1}{100h^{-1} \times 1.8}$~~ $4h^{-1}$ Kpc → reasonable with a flat rotation curve

→ Elliptical galaxies with dispersion velocities > $250\ \text{Km s}^{-1}$ have $n_0 \sim 0.001 h^3$ Mpc^{-3}. We want sum of $n_0\sigma^4$ but at $\sigma > 250\ \text{Km s}^{-1}$ n_0 starts to drop very fast. So we assume σ to be equal to $250\ \text{Km s}^{-1}$

$$\Rightarrow \tau_{gl} = \int dP = P = H_0^{-3}\left(0.001 h^3\ \text{Mpc}^{-3}\right)(250)^4 (\text{Km/c})^4 \cdot \frac{8\pi^3}{15}(1)$$

$$\simeq (300)^3 (250)^4 \left(\frac{1}{3\times10^5}\right)^4 \frac{8\pi^3}{15} = \underline{2.1\times10^{-4}}$$

(1 in 10000)

→ Evolution of Linear Density Perturbations

$$\rho(\vec{r},t) = \rho_b(t)\left[1+\delta(\vec{r},t)\right]$$

↳ Same ρ which appears in stress-energy tensor.

$$\cancel{\delta = \frac{\delta\rho}{\rho}} \qquad \delta(\vec{r},t) = \frac{\delta\rho(\vec{r},t)}{\rho_b(t)} \propto \frac{\dot{a}}{a}\int_0^t \frac{da}{(\dot{a})^3} \quad -(13.77)$$

$\frac{\dot a}{a} \to H_0 E(z)$

$$\delta \propto D(z) = E(z)\, G(z)$$

$$G(z) = \int_0^t \frac{da}{(\dot{a})^3} = \int_0^t \frac{da}{(H_0 E(z))^3 a^3}$$

$$a = \frac{a_0}{1+z}$$

$$da = -\frac{a_0\, dz}{(1+z)^2} = -\frac{a^2}{a_0}dz$$

$$G(z) = \int_z^\infty \frac{(1+z)\,dz}{H_0^3 E(z)^3} \Leftarrow$$

H_0^3 is a constant so we choose $\frac{5\Omega_{M0}}{2}$ as normalisation for $G(z)$

This normalisation ensures $D(z) \to \frac{1}{1+z} = \frac{a(t)}{a_0}$ as $z\to\infty$

$$\boxed{G(z) = \frac{5\Omega_{M0}}{2}\int_z^\infty \frac{(1+z)dz}{E(z)^3}}$$

~~Velocity~~ $f = \frac{\dot{D}a}{D\dot{a}} = \cancel{\frac{dD}{da}\frac{a}{D}} = \cancel{-\frac{dD}{a^2 dz}\frac{a a_0}{D}} = \cancel{-\frac{(1+z)}{D}\frac{dD}{dz}}$

$$= \frac{\frac{dD}{dz}}{D}\frac{a}{\dot{a}}\frac{dz}{dt} = -\frac{\frac{dD}{dz}}{D}\frac{a}{\dot{a}}\frac{\dot{a}a_0}{a^2} = -\frac{dD/dz}{D}(1+z)$$

$$\ln D = \ln E + \ln G \qquad \frac{dD/dz}{D} = \frac{dE/dz}{E} + \frac{1}{G}\frac{5\Omega_{M0}}{2}\frac{(1+z)}{E^3}$$

$$H = \left(\frac{\dot a}{a}\right)^{2} = H_0\, E(z) \qquad (1+z)\frac{dE}{dz} =$$

$$\cancel{2\ln\left(\frac{\dot a}{a}\right) = \ln H_0 + \ln E \Rightarrow \frac{2}{(\dot a/a)}\left[\frac{a}{a^2}\right.}$$

$$\frac{da}{dt \cdot a} = H_0\, E(z)$$

$$\ln\left(\frac{da}{dt}\right) - \ln a = \cancel{H_0} \ln H_0 + \ln E(z)$$

$$\frac{1}{da/dt}\, \frac{d^2a}{dt^2} \cdot \frac{dt}{dz} - \frac{1}{a}\frac{da}{dt}\frac{dt}{dz} = \frac{dE(z)/dz}{E(z)}$$

$$\frac{a^2}{\dot a a_0}\; \frac{a\ddot a - (\dot a)^2}{a \dot a} = \frac{dE(z)/dz}{E(z)}$$

$$(1+z)\frac{dE(z)}{dz} = \frac{a\ddot a}{(\dot a)^2} - 1$$

$$\Rightarrow \boxed{f(z) = \frac{a\ddot a}{(\dot a)^2} - 1 + \frac{S\Omega_{M_0}}{2}\,\frac{(1+z)^2}{E^2 G}}$$

$$F(0) = \frac{(\ddot a/a)}{(\dot a/a)^2} - 1 + \frac{S\Omega_{M_0}}{2}\,\frac{1}{G(0)} \quad ; \; E(0) = 1$$

$$\boxed{f(0) = \frac{\cancel{H_0}^2}{\cancel{H_0}^2}\left(\Omega_{\Lambda_0} - \frac{\Omega_{M_0}}{2}\right) - 1 + \frac{S\Omega_{M_0}}{2} \cdot \frac{1}{G(0)}}$$