EE324 Control Systems Lab

Problem Sheet 10

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1. Problem 1

Given the following state space system:

$$\begin{aligned} \dot{X} &= AX + BU \\ y &= CX + DU \end{aligned}$$

Part (i)
Taking A, B, C, D and T as follows:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 4 \\ 6 & 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 2 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 3 & 0 & 2 \end{bmatrix}$$

Initially G(s) is:

$$\begin{aligned} G_1(s) &= D + C(sI - A)^{-1}B \\ \implies G_1(s) &= \frac{2s^3 - 6s^2 - 35s + 60}{s^3 - 6s^2 - 25s + 39} \end{aligned}$$

After transforming A, B and C, I obtained the same transfer function $G_2(s) = G_1(s)$.

$$A = T^{-1}AT,\ B = T^{-1}B,\ C = CT$$

$$G_2(s) = \frac{2s^3 - 6s^2 - 35s + 60}{s^3 - 6s^2 - 25s + 39}$$

Part (ii)
Poles of $G_1(s) = G_2(s)$ were calculated as: 8.419, -3.678 and 1.259.
Eigenvalues of A were same as that of $T^{-1}AT$, which were: 8.419, -3.678 and 1.259. Therefore, here we observe that eigen values of A are the poles of $G_1(s) = G_2(s)$.

Part (iii)
Taking a proper transfer function $G_{bp}(s)$ (from part(i)) as follows,

$$G_{bp}(s) = \frac{(s+1)(s+5)}{(s+3)(s+2)}$$

I obtained the following state space realization for the chosen transfer function (using abcd()):

$$A = \begin{bmatrix} -3.6 & -4.8 \\ 0.2 & -1.4 \end{bmatrix}, B = \begin{bmatrix} 1.265 \\ 0.632 \end{bmatrix}, C = \begin{bmatrix} 0.791 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 \end{bmatrix}$$

Now, taking a strictly proper transfer function $G_{sp}(s)$ as follows,

$$G_{sp}(s) = \frac{(s+1)}{(s+3)(s+2)}$$

I obtained the following state space realization for the chosen transfer function (using abcd()):

$$A = \begin{bmatrix} -4.4 & -0.8 \\ 4.2 & -0.6 \end{bmatrix}, B = \begin{bmatrix} -1.265 \\ 0.632 \end{bmatrix}, C = \begin{bmatrix} -0.791 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

Therefore, value of D is non-zero in case of biproper transfer functions, whereas in case of strictly proper transfer function, $D = 0$. This can be understood as below.
I observed that:

$$\lim_{s \to \infty} \left(D + C(sI - A)^{-1}B\right) = D$$

And for a strictly proper transfer function, as degree of denominator is greater that numerator, hence,

$$\lim_{s \to \infty} G(s) = 0$$
$$\implies \lim_{s \to \infty} \left(D + C(sI - A)^{-1}B\right) = D$$
$$\implies D = 0$$

Scilab Code

```
clear;
clc;

s=poly(0,'s');
A = [1,2,5;
     1,3,4;
     6,1,2];
I = eye(3,3);
B = [1;
     2;
     1];
C = [1,1,3];
D = 0*eye(1,1);
T = [1,0,0;
     1,2,4;
     3,0,2];

G1 = D + C*(inv(s*I-A))*B;
evals1 = spec(A);

// Checking G(s) after modifying A, B and C
A = inv(T)*A*T;
B = inv(T)*B;
C = C*T;

G2 = D + C*(inv(s*I-A))*B;

// Eigenvalues of A
evals2 = spec(A);
poles_G = roots(G1.den);

// D value
G_bp = ((s+1)*(s+5))/((s+3)*(s+2));
```

```
G_sp = (s+1)/((s+3)*(s+2));

sysGbp = syslin('c',G_bp);
sysGsp = syslin('c',G_sp);

[A1,B1,C1,D1] = abcd(sysGbp);
[A2,B2,C2,D2] = abcd(sysGsp);
```

2. Problem 2

Given the transfer function $G(s)$:

$$G(s) = \frac{(s+3)}{(s^2+5s+4)}$$

we get state space realization as follows:

$$\begin{aligned} \dot{X} &= AX + BU \\ y &= CX + DU \\ &\textit{where,} \end{aligned}$$

$$A = \begin{bmatrix} -5 & -4 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

Now, given the transfer function $G(s)$:

$$G(s) = \frac{(s+1)}{(s^2+5s+4)}$$

we get state space realization as follows:

$$\begin{aligned} \dot{X} &= AX + BU \\ y &= CX + DU \\ &\textit{where,} \end{aligned}$$

$$A = \begin{bmatrix} -5 & -4 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

Since values in A,B,C and D were preferred to be integers and scilab's tf2ss was giving float values as entries of A,B,C, and D I switched to MATLAB.

MATLAB Code

```
clear;
clc;

syms s;
s=tf('s');

G1 = (s+3)/((s+1)*(s+4));
[n1,d1] = tfdata(G1,'v');

[A1,B1,C1,D1] = tf2ss(n1,d1)

G2 = (s+1)/(s^2+5*s+4);
[n2,d2] = tfdata(G2,'v');
[A2,B2,C2,D2] = tf2ss(n2,d2)
```

3. Problem 3

The transfer function G(s) is written as:

$$G(s) = D + C(sI - A)^{-1}B$$

Choosing A as follows:

$$A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$$

we observe that eigen values of A are $\{a_1, a_2, a_3\}$ and $(sI - A)^{-1} =$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s - a_1} & 0 & 0 \\ 0 & \frac{1}{s - a_2} & 0 \\ 0 & 0 & \frac{1}{s - a_3} \end{bmatrix}$$

Here we see that the term if the element in the k^{th} row of B or k^{th} column of C is 0, then the term $\left(\frac{1}{s-a_k}\right)$ will vanish from the product in term $C(sI - A)^{-1}B$:

$$G(s) = D + \sum_{i=1}^{3} \frac{c_i b_i}{s - a_i}$$

Hence pole a_k will no longer be a pole of $G(s)$.
Case 1: an entry in B is 0

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

The transfer function obtained for above choice is:

$$G(s) = \frac{8s - 16}{s^2 - 6s + 5}$$

which has poles at $s = 1, 5$. Thus the pole at $s = 3$ has been cancelled which is the second entry in diagonal matrix A. Corresponding to the 2^{nd}position B's entry was 0, hence the second entry in A (i.e 3) is no longer a pole of G(s).

Case 2: an entry in C is 0

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}, C = \begin{bmatrix} 0 & 4 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

The transfer function obtained for above choice is:

$$G(s) = \frac{10s - 38}{s^2 - 8s + 15}$$

which has poles at $s = 3, 5$. Thus the pole at $s = 1$ has been cancelled which is the first entry in diagonal matrix A. Corresponding to the 1^{st}position C's entry was 0, hence the first entry in A (i.e 1) is no longer a pole of G(s).

Scilab Code (used scilab as code was more or less identical to that used in q1)

```
clear;
clc;

s=poly(0,'s');
A = [1,0,0;
     0,3,0;
     0,0,5];
I = eye(3,3);
B = [2;
     0;
     6];
C = [1,4,1];
D = 0*eye(1,1);

G1 = D + C*(inv(s*I-A))*B;

A = [1,0,0;
     0,3,0;
     0,0,5];
I = eye(3,3);
B = [2;
     1;
     6];
C = [0,4,1];
D = 0*eye(1,1);

G2 = D + C*(inv(s*I-A))*B;
```

4. Problem 4

According to the given conditions we have and assuming D as 0 wlog:

$$A = \begin{bmatrix} a_1 & a_2 & 0 \\ 0 & a_3 & a_4 \\ 0 & 0 & a_5 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}, D = \begin{bmatrix} 0 \end{bmatrix}$$

The expression of $G(s)$ simplifies as:

$$G(s) = C(sI - A)^{-1}B$$

$$\implies G(s) = \frac{c_1[(s-a_3)(s-a_5)b_1 + a_2b_2(s-a_5) + a_2a_4b_3] + c_2[b_2(s-a_1)(s-a_5) + a_4b_3(s-a_1)] + b_3c_3(s-a_1)(s-a_3)}{(s-a_1)(s-a_3)(s-a_5)}$$

Here I observed that poles of the system are at $s = a_1, a_3 \ and \ a_5$, if there's no cancellation from a zero factor in numerator. There can be 3 possible cases of diagonal entries getting repeated.

Case i: If we have $a_1 = a_3$, here we have pole/zero cancellation when $b_2(a_3 - a_5) + b_3a_4 = 0$ or $a_2 = 0$.
Case ii: If we have $a_3 = a_5$,.here we have pole/zero cancellation when $c_2(a_3 - a_1) + c_1a_2 = 0$ or $a_4 = 0$.
Case iii: If we have $a_5 = a_1$, here we have pole/zero cancellation when either $a_2 = 0$ or $a_4 = 0$ or both are zero

Taking an example from each of the above cases:

Case i:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Poles and zeros of $G(s)$ are obtained as:
poles = $\{1, 1, 6\}$ and zeros = $\{0.7727, 1\}$
Therefore, pole/zero cancellation happens as denominator and numerator have a common (s-1) factor.

Case ii

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Poles and zeros of $G(s)$ are obtained as:
poles = $\{1, 6, 6\}$ and zeros = $\{0.5455, 6\}$
Therefore, pole/zero cancellation happens as denominator and numerator have a common (s-6) factor.

Case iii

$$A = \begin{bmatrix} 2 & 5 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

Poles and zeros of $G(s)$ are obtained as:
poles = $\{2, 2, 3\}$ and zeros = $\{2, 2.0455\}$

Therefore, pole/zero cancellation happens as denominator and numerator have a common (s-2) factor.

MATLAB Code (as Scilab doesn't show which were pole-zero cancelled)

```
clear;
clc;

% Case i
A = [1, 0, 0; 0, 1, 4; 0, 0, 6];
B = [1; 3; 5];
C = [1, 2, 3];
D = 0;

[n1,d1] = ss2tf(A,B,C,D);

poles_1 = roots(d1);
zeros_1 = roots(n1);



% Case ii
A = [1, 5, 0; 0, 6, 0; 0, 0, 6];
B = [1; 3; 5];
C = [1, 2, 3];
D = 0;

[n2,d2] = ss2tf(A,B,C,D);

poles_2 = roots(d2);
zeros_2 = roots(n2);

% Case iii
A = [2, 5, 0; 0, 3, 0; 0, 0, 2];
B = [1; 3; 5];
C = [1, 2, 3];
D = 0;

[n3,d3] = ss2tf(A,B,C,D);

poles_3 = roots(d3);
zeros_3 = roots(n3);
```